Modelling Animal Population Growth Using the Logistic Map Equation

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ABSTRACT

This report contains some important topics centered around the logistic map equation and its applications to complex systems and the chaos theory. The basic chaotic characteristics of a simple logistic map are emphasized, and how it can be used to simulate a biological population such as rabbit population. Logistic maps have a deep relationship with chaos and demonstrate the fascinating combination of stable and predictable behavior, and unstable chaotic behavior. Chaotic systems change from stable to chaotic behaviour via bifurcations. Bifurcations happen when the population stabilises or fluctuates between two or more values each year. The logistic maps bifurcation diagram has many fractal characteristics as it is contained in the middle section of the Mandelbrot set. It also demonstrates the Feigenbaum's constant, which is used to describe the self-similarity of the bifurcation plot. In this paper, the predator-prey model is investigated using the Lokta-Volterra equation. Research has shown similar characteristics between the logistic map equation and the Lokta-Volterra equation. The results from our experiment prove the same when comparing the hare-lynx population plotted over time to the bifurcation period cycles produced with different rate of growth values in the Logistic Map equation.

CCS CONCEPTS

• Computing methodologies → Symbolic and algebraic manipulation → Representation of mathematical objects → Representation of mathematical functions
• Theory of computation → Randomness, geometry and discrete structures → Computational geometry
• Mathematics of computing → Mathematical analysis → Numerical analysis → Interpolation

KEYWORDS

Logistic function, Logistic map, Fractals, Mandelbrot set, Bifurcation, Chaos Theory, Rabbit population, Feigenbaum's number, Predator-Prey model, Lokta-Volterra equation

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1)INTRODUCTION

In today's world, the word "Chaos" has some negative meaning, such as messiness, craziness, or disorder. However, in mathematics, fractals are the representations of chaos theory [7]. Chaos theory studies the behavior of dynamical systems that are highly sensitive to initial conditions. This is known as the butterfly effect [13]. A small difference in initial conditions can lead to divaricating outcomes, resulting in a chaotic system which makes it unfeasible to predict future outcomes. In simple words, if the initial condition of a system is slightly changed, then in a short time down the line, a completely different situation can be expected resulting in chaos [17]. Systems often become chaotic when there is feedback present as well [2]. There are many equations that attempt to model things that lead to chaos, such as weather systems, the flow of fluids, and populations. Population of animals in an ecosystem, especially the rabbit population often shows chaotic behaviour.

2)PROBLEM STATEMENT

Imagine an island without any contact with the exterior. Living species there have no possibility to migrate looking for a new land with affordable resources. Thus, for instance, if the island has initially a couple of rabbits, they will reproduce exponentially since rabbits are known to breed rapidly. This expansion regime of the rabbits will eventually colonize the whole island in a few generations. Hence, the island will eventually become overpopulated. However, since the island has limited food available, not all rabbits have enough food to survive. This can be modelled by a simple iterative equation called the Logistic Map which is based on the logistic function [1]. The function shows how a population grows slowly, then rapidly, before tapering off as it reaches the carrying capacity, which is simply the maximum number of individuals that a given environment can support [1]. The logistic map uses a nonlinear difference equation to map the population value at any time step to its value at the next time step [3]. The connection of the logistic map to complex and adaptive systems represent a number of interesting noteworthy features. The equation demonstrates that some deterministic systems can exhibit behavior that, under no circumstances, can be predicted. Many systems, such as weather, biological populations, fluid turbulence and economic systems exhibit chaotic behavior, the principles of which can be observed in the logistic map [9]. However, the logistic map equation is mainly used to model population growth or decline over several generations that follows a famous fractal pattern [1].

If x_n is the number of individuals in one generation, then x_{n+1} is the number in the next generation. Every generation has an average number of offspring (rate of growth), r. In theory, the population of any generation is just the number of individuals times their average number of offspring. To represent individuals that die, $1 - x_n$ is used. If the starting population and rate of reproduction is known, Equation (1) can be used to find the population at any future point [1]:

$$\boldsymbol{x_{n+1} = r x_n (1 - x_n)} \qquad (1)$$

In other words, the population level at any given time is a function of the growth rate parameter and the previous time step's population level [1].

Although this equation seems simple, it has many unique and interesting properties. One of which is that the population will always stabilize if the equation is used to look into many generations into the future [1]. However, the final value where it settles down depends on the average number of offspring. For example, if 'r' is one or less, the population will die out and become extinct as seen in Figure 1 [6].

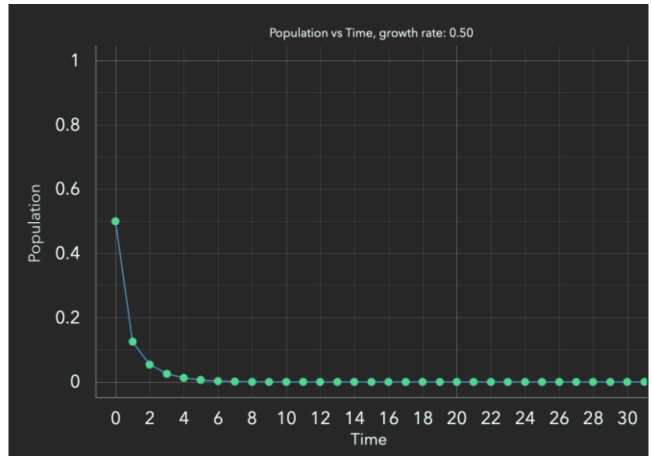

Figure 1: Population vs. Time graph when rate of growth is 0.5 and initial condition is 0.5 [10].

If r is two or three, the final population stabilizes somewhere near where it started (approaches a constant value) as seen in Figure 2 [6].

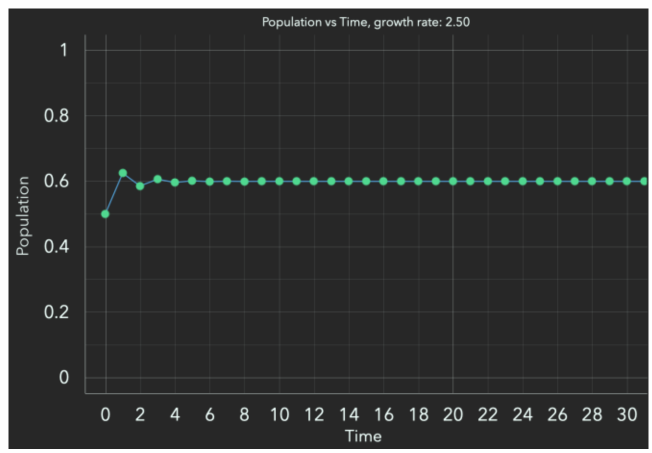

Figure 2: Population vs. Time graph when rate of growth is 2.5 and initial condition is 0.5 [10].

But if r is larger than 3 and less than 3.5, it stabilizes at two values instead of one as seen in Figure 3. From generation to generation, the population fluctuates between two extremes when given any initial condition. This characteristic change in behaviour is called Bifurcation [6].

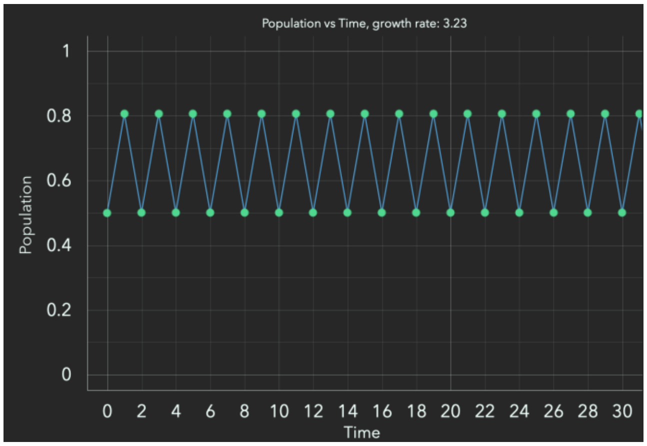

Figure 3: Population vs. Time graph when rate of growth is 3.23 and initial condition is 0.5 [10].

If r is increased slightly, the population will approach oscillations among 4 values (as seen in Figure 4), then 8, 16, 32, etc [6].

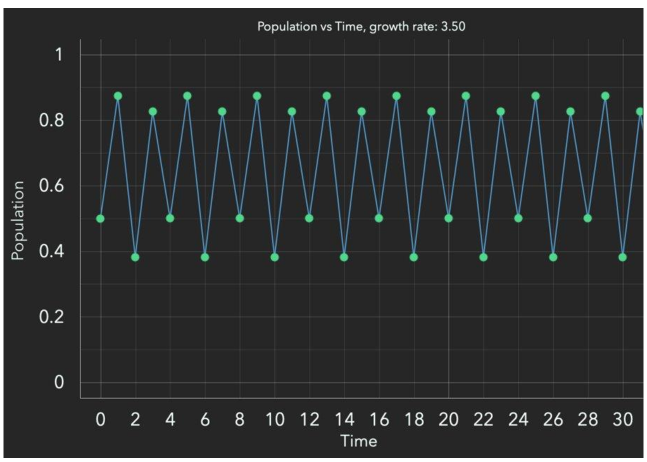

Figure 4: Population vs. Time graph when rate of growth is 3.5 and initial condition is 0.5 [10].

However, if r is greater than 3.57, the behaviour becomes more interesting. As seen in Figure 5, the system becomes aperiodic and chaotic [6].

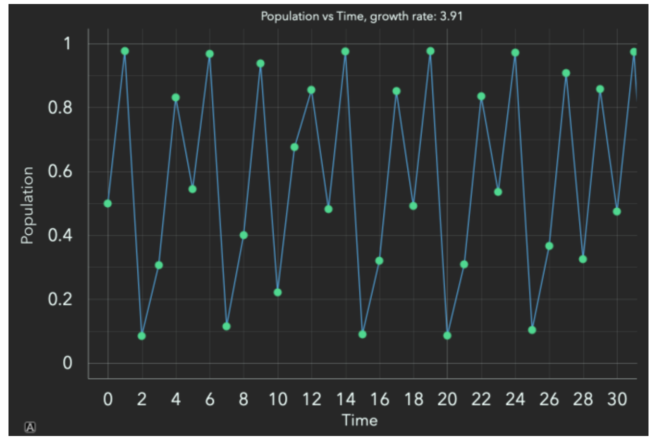

Figure 5: Population vs. Time graph when rate of growth is 3.91 and initial condition is 0.5 [10].

As a result, increasing the number of offspring increases the number of rotating extremes leading to an infinite number of possible population sizes. Thus, for any high average number of offspring, the population of any generation becomes random. At this point, the behaviour becomes chaotic and unpredictable [1]. This simply means that a chaotic system oscillates forever, never repeating itself or settling into a steady state behaviour. It never hits the same point twice and its structure has a fractal form, meaning the same pattern exists at every scale no matter how much you zoom into it [7]. From this, it is noted that complex behaviour can arise out of very simple rules for any system. One way of illustrating the complexity of situation is by observing the bifurcation diagram shown in Figure 6.

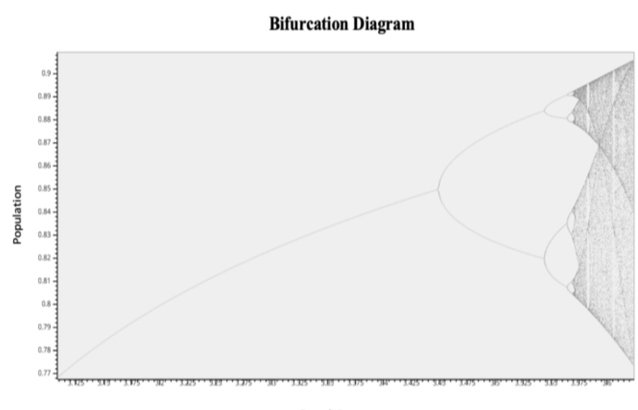

Figure 6: The bifurcation diagram shows the vertical slices corresponding to each growth rate. Each vertical slice depicts the population values that the logistic map settles towards [10].

As you can see in the bifurcation plot above, the first split into having two states happens at $r = 3$. But if we keep increasing r little by little, the plot continues to split and produces more doubling of stable states. The splitting of stable states is call bifurcation [17]. Hence, increasing the rate of growth, a cascade of period doubling events leads to a deterministic chaos. In other words, if the parameter r is increased, x_n goes through bifurcations of period 2, 4, 8, 16, and the chaos [5]. At this point, a fractal, the Mandelbrot Set is formed. The bifurcation diagram is contained in the Mandelbrot set as shown in Figure 7 [7].

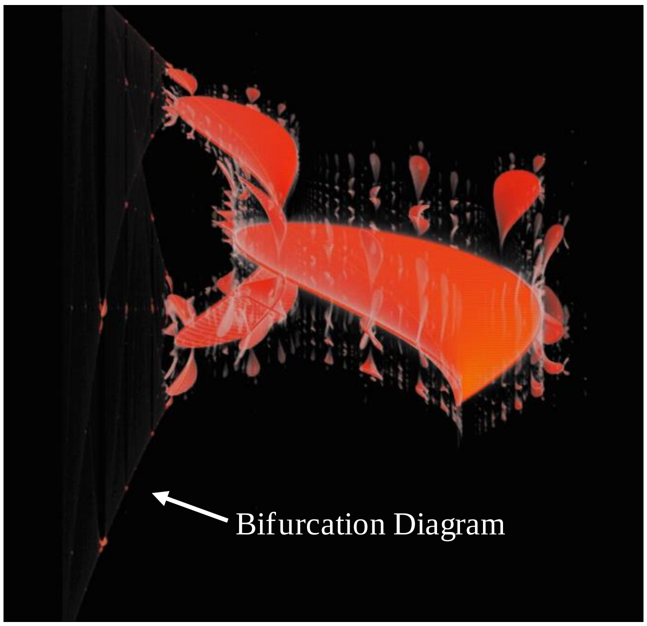

Figure 7: The bifurcation diagram is a part of the Mandelbrot set [10].

The ratio between the lengths of two successive bifurcation intervals approaches the Feigenbaum constant $\delta \approx 4.66920$. This is a universal constant that keeps reappearing in systems like the logistic map and is intimately connected to chaotic systems [9]. As seen in Figure 8, it is important to note that the plot splits from 1 to 2, 2 to 4, 4 to 8, and so on. This behavior is an example of a period-doubling cascade.

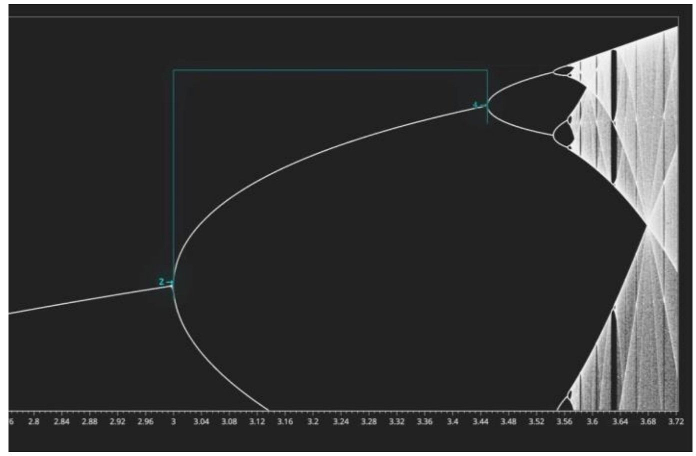

Figure 8: A close-up of the logistic map's bifurcation diagram showing how δ is defined [10].

3)RELATED WORKS

Verhulst Equation:

The logistic map can be modified in order to describe how the numbers of predators and prey can oscillate to generate the Mandelbrot set. Figure 9 shows the oscillation that occurs between rabbits and cayotes. This fluctuation is natural because when the population of rabbits is high, then there will be a sudden increase in the population of coyotes [7]. This will lead to a decline in the number of rabbits, and then a drop in the number of coyotes, as their food supply becomes scarce. This cycle will continue to oscillate the populations. However, sometimes there can be unpredictable population downfalls, and the length of the time between the cycles can also change unpredictably [7].

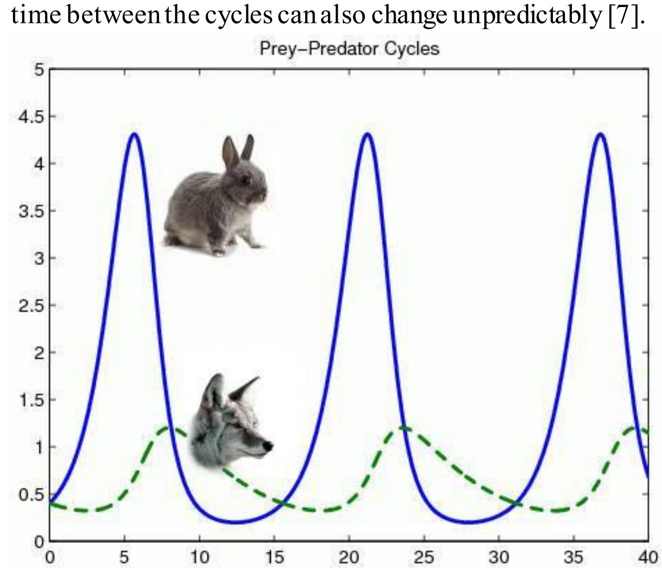

Figure 9: Population of Rabbits vs. Coyotes [7]

The following equation can be used to predict the number of rabbits in the following generation.

$$P_{n+1} = P_n + CP_n\left(1 - \frac{P_n}{K}\right) \quad (2)$$

Equation (2), known as the Verhulst Equation, adds two additional terms to the logistic map. In order to determine the population at the next generation, the current population must be known which is denoted by P_n. The rate of growth is the factor 'C' in the equation. The value 'K' is known as the carrying capacity of the environment. It is the maximum number of animals the ecosystem can sustain. When P_n/K is small, it can be noted that the population is much less than the carrying capacity [16].

Lotka-Volterra Equation:

The Predator-Prey Equations (3) and (4) also known as the Lotka-Volterra equations, are first order nonlinear differential equations. They are used to describe the dynamics of biological systems in which two species interact [4]. The populations change through time given the following equations:

Prey Equation:

$$\frac{dx}{dt} = \alpha x - \beta xy \quad (3)$$

Predator Equation:

$$\frac{dy}{dt} = \delta xy - \gamma y \quad (4)$$

Where,

x is the number of prey (rabbits)

y is the number of predators (foxes)

dy/dt and dx/dt represent the instantaneous growth of rates for the two populations

α, β, δ and γ are positive real parameters describing the interaction of two species.

Since differential equations are used, the solution is continuous which indicates that the generations of both the predator and prey are constantly overlapping. In order to use the Lotka-Volterra equations, a number of assumptions are made about the environment and evolution of the predator and prey populations [4].

1. The prey population finds sufficient food at all times.
2. The food supply of the predator population depends entirely on the size of the prey population.
3. The rate of change of population is proportional to its size.
4. During the process, the environment does not change in favor of one species, and genetic adaption is inconsequential.
5. Predators have limitless appetite.

Since it is assumed that the prey has unlimited food supply, an exponential growth occurs which is represented in the equation by the term αx. The rate of change of population is proportion to its size is represented by βxy. In the predator equation, the term δxy represents the growth of the population and γy denotes the loss rate of the predators due to either natural death or migration [4].

The Lotka-Volterra equations can produce various solutions. A linearization of the equations produces a solution similar to simple harmonic motion with the population of predators following that of prey by 90° in the cycle [15]. This can be seen in Figure 10.

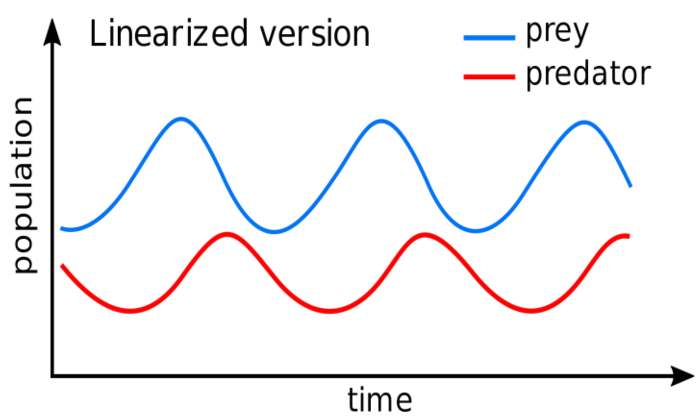

Figure 10: Linearization of the Lotka-Volterra Equations [15]

A simple solution would be to plot the progression for two species over time.

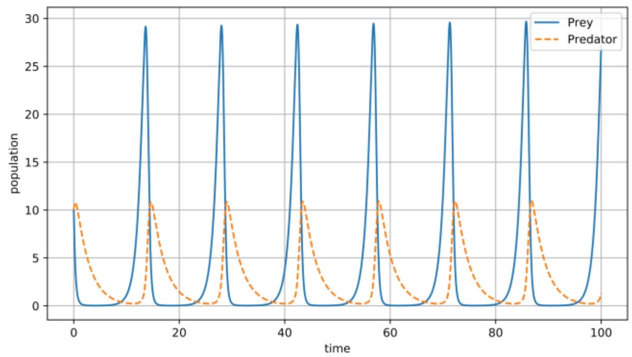

Figure 11: Population Dynamics for Two Species over Time [15]

Figure 11 shows the progression of two species, with an initial condition of 10 and growth rate of the prey is considered to be 1.1 and 0.4 for the predator. The death rate is 0.1 for the prey and 0.4 for the predator.

Furthermore, the solutions can also be plotted parametrically as orbits in phase space, without representing time. One axis represents the number of prey and the other axis represents the number of predators for all times [12]. This corresponds to eliminating time from the two differential equations, which then produces a single differential equation (5):

$$\frac{dy}{dx} = \frac{y}{x}\frac{\delta x - \gamma}{\beta y - \alpha} \tag{5}$$

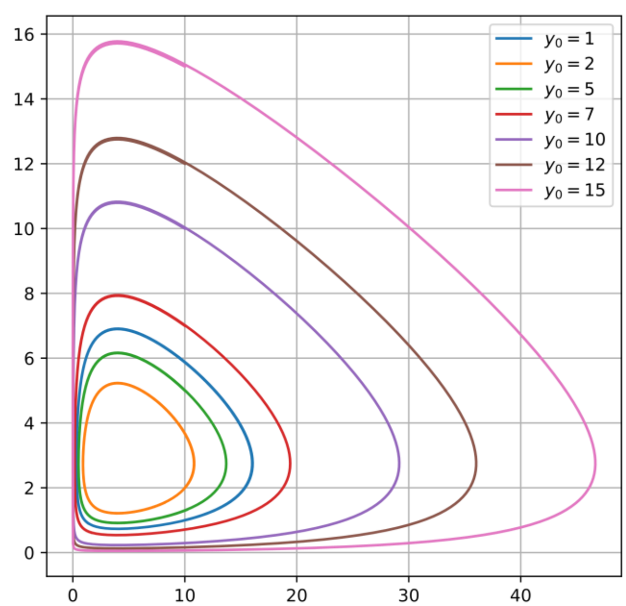

Figure 12: Phase-space Plot with Various Initial Conditions of the Predator Population [12]

Figure 12 shows the phase space plot for the predator and prey relation with various initial conditions of the predator population. However, these graphs are not ideal for real-life situations due to fluctuations.

4)METHODOLOGY

In order to investigate the logistic map equation and its use for modelling population growth, research on the topic was done first. A video by Veritasium called "This Equation Will Change How You See the World (The Logistic Map)" was used as a baseline for the project as it discusses many applications of the logistic map equation and shows how it can be used to model population growth over time. After the initial research was completed, Python code was used in order to plot the logistic map using different parameters. Different properties of the equation were observed by plotting the logistic map using different growth rates. Additionally, the logistic map's bifurcation diagram was recreated in order to further investigate the periods of stability within the logistic map equation and at what growth rate values those periods occur. The logistic map was further investigated by altering the equation. An additional power to the restricting term of the logistic map (1-x) was added. Finally, the similarities seen between the logistic map equation and the Lokta-Volterra predator prey equations were investigated by plotting hare and lynx populations over time. Plotting these populations shows the cyclical nature of the Lokta-Volterra equation which is similar to the cycles observed in the periods of stability of the logistic map equation. This showed that the logistic map equation can be used as a reductionist model of the Lokta-Volterra equations when used to model population growth over time.

5)EXPERIMENTS & RESULTS

Python's NumPy and matplotlib libraries were used in order to investigate the logistic map equation. A function was created for the equation and its many properties were investigated in order to determine how the equation can be used to model a rabbit population. Assuming a theoretical maximum population value of 1, the current year's population and next year's population are constrained between 0 and 1. This means if the maximum possible population is 100,000 then a result value of 0.5 for next year's population from the logistic map would correspond to a population size of 50,000. In order to see how different growth rates, r, effect the result of the logistic map, the population next year (xn) vs the population this year (xo) was plotted which can be seen in Figure 13 and 14. In Figure 13, when an r value of 2 is used, an xo value of 0.5 will result in an xn value of 0.5 as well.

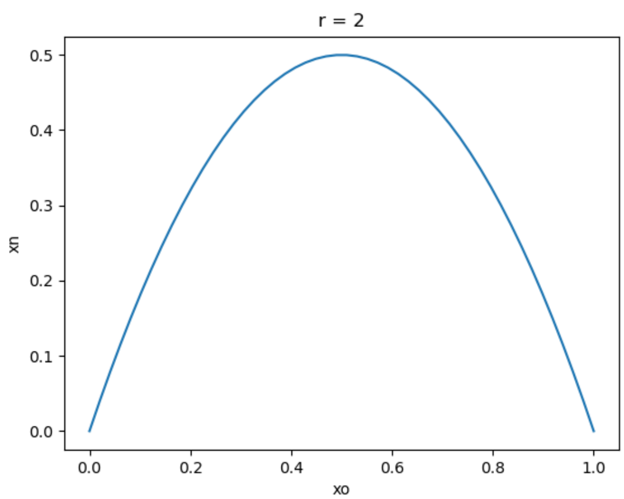

Figure 13: xn vs xo with a rate of 2

This demonstrates the behaviour that no matter the initial conditions of the population, if the rate is equal to 2 the population will always stabilize at 0.5. For example, if the initial population is 0.8, the population the following year will drop to 0.3 and then continue to rise until it stabilizes at 0.5. This changes dramatically with different r values. When rate is equal to 3, a current population of 0.5 will result in a population in the following year of 0.75 as seen in Figure 14.

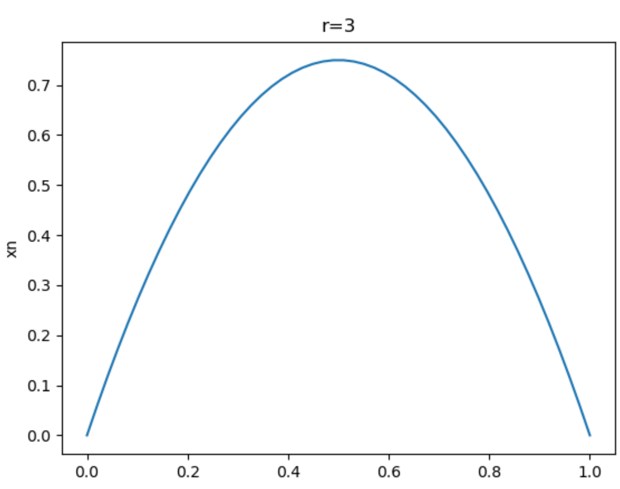

Figure 14: xn vs xo with rate of 3

This demonstrates the behaviour that when the r value reaches 3 the population will no longer stabilize at one value but will continuously oscillate between multiple values. These properties are reflected in the logistic map's bifurcation diagram where the first split is seen at an r value of 3. In order to fully investigate the behaviour of the logistic map, it's bifurcation diagram was reproduced and plotted using matplotlib. This was done by creating a linspace of 1000 values between 0 and 4, which was used as the different rate values for the x axis of the bifurcation diagram. An initial x value of 0.5 was used and constrained between 0 and 1 for the y axis. The logistic map equation was iterated 500 times in order to produce the bifurcation diagram. The result can be seen in Figure 15 below and the many properties of the logistic map at each r value can be seen.

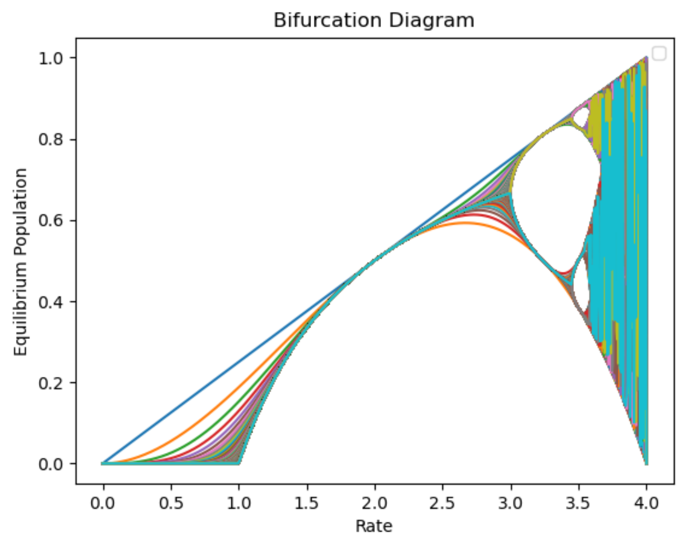

Figure 15: Bifurcation Diagram Reproduced Using Python

For r values less than 3, the population converges at a single point, however once 3 is reached the bifurcation diagram splits in two demonstrating the values of which the population is oscillating between. At an r value of 3.5 the bifurcation diagram splits again corresponding with the four values the population is oscillating between. Additional modifications to the equation were made in order to investigate the effect of raising the limiting term of the logistic map (1-x) to the power of different values. To see the effect of changing the value of the power term, (1-x) was raised to the power of 100 values evenly spaced from 0.1 to 10. The population next year vs the population this year was plotted using all these variations of the logistic map and the effect this had on the equation can be seen in Figure 16.

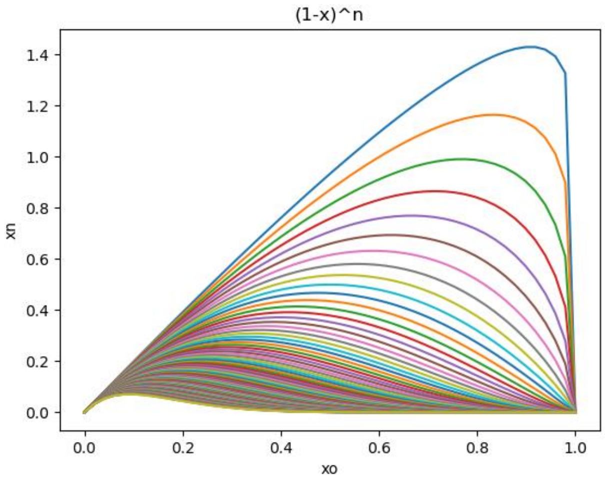

Figure 16: xn vs xo With Modified Logistic Map Equation

As discussed earlier, the Lokta-Volterra predator prey equations are similarly useful in modelling population growth over time and model the relationship that predator and prey populations have rather than isolating a single species. In order to confirm this model, fur trade data from the Hudson's Bay Company was used in order to estimate the population of both snowshoe hares and lynx. These population estimations were plotted using matplotlib between the years 1845 and 1937. Plotting these populations shows the effect the population of one of these animals has on the other as the snowshoe hare is prey to the predatory lynx. As seen in Figure 17 this estimation of lynx and snowshoe hare populations over time closely follows the population dynamics for two species over time shown in Figure 11. The logistic map can therefore be used as a reductionist model of the Lokta-Volterra equations as it shows similar properties however simplifies the model by only focusing on the population of one species rather than two who influence each other continually.

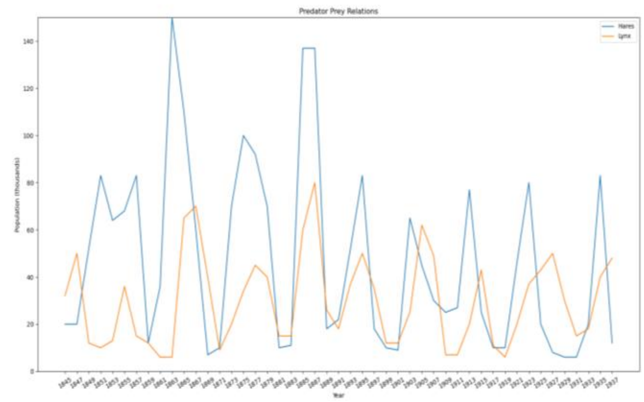

Figure 17: Snowshoe Hare and Lynx Populations Over Time

6)CONCLUSION

In conclusion, chaotic characteristics in a logistic map were analyzed. Chaos theory studies the behavior of dynamical systems that are highly sensitive to initial conditions and often become chaotic when there is feedback present as well. Population of animals in an ecosystem were explored, especially the rabbit population which often shows chaotic behavior. The logistic map equation was used to observe the different growth rates and the bifurcation diagram was recreated in order to further investigate the periods of stability within the logistic map equation. It was noted that when the growth rate value, r, reaches 3, the population oscillates between multiple values and does not stabilize at a certain value. Furthermore, the Lokta-Volterra equation were used to describe the dynamics of biological systems in which two species interact. To verify this model, population of snowshoe hares and lynx was used to see the relationship between the predator and prey. Results from the experiment prove the similarities between the logistic map and Lokta-Volterra equations.

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