REGULARITY OF SOLUTION MAPS OF THE GENERALIZED SURFACE QUASI-GEOSTROPHIC EQUATIONS

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Abstract. We study the regularity properties of the data-to-solution map for the family of generalized surface quasi-geostrophic (SQG) equations

$$\partial_t\theta + (u\cdot\nabla)\theta = 0, \qquad u = -\nabla^{\perp}(-\Delta)^{-\frac{1-\beta}{2}}\theta, \quad 0\le\beta\le 1,$$

posed on $\mathbb{R}^2$. By employing the geometric formulation of the equations as an ODE on the group of exact volume-preserving diffeomorphisms endowed with a right-invariant $H^{-\beta/2}$-metric, we prove that the associated (Lagrangian) solution map, i.e. the Riemannian exponential map, is *real analytic* for all $s>2$ and $0\le\beta\le 1$. As direct consequences, we obtain that the exponential map is a local diffeomorphism near the identity and that the Cauchy problem is locally well-posed in H^s in the sense of Hadamard. We then turn to the corresponding Eulerian data-to-solution map and show that it is *nowhere locally uniformly continuous* as a map from bounded subsets of $H^s(\mathbb{R}^2)$ to $C([0,T];H^s(\mathbb{R}^2))$ for any $s>2$ and any $\beta\in[0,1]$. This sharpens earlier results obtained for small β and highlights a striking dichotomy between the analytic regularity of the Lagrangian flow and the ill-conditioning of the Eulerian formulation.

1. Introduction

Consider the Cauchy problem for the family of generalized SQG equations

$$\begin{cases}\partial_t\theta + (u\cdot\nabla)\theta = 0, & x\in\mathbb{R}^2,\ t\in\mathbb{R}\\ \theta(x,0)=\theta_0(x), \end{cases} \tag{1.1}$$

where $0\le\beta\le 1$, $\nabla^{\perp} = (-\partial_2,\partial_1)$ is the symplectic gradient and where the scalar function $\theta=\theta(t,x)$ and the velocity field $u=u(t,x)$ are related by

$$u = -\nabla^{\perp}(-\Delta)^{-1+\frac{\beta}{2}}\theta.$$

Equations of this type have been studied extensively in recent years, see e.g., [2], [5], [6], [7], [8], [10], [12], [13], [15], [17].

Our goal is to study regularity properties of the solution map of the Cauchy problem (1.1). In [13] the authors showed that the family (1.1) has a Lagrangian reformulation as an ODE system on the group of exact diffeomorphisms preserving the volume form μ of the underlying domain which can be viewed as the configuration space of the dynamical system defined by (1.1). When completed in the H^s norm with $s>2$ this space becomes a smooth Banach manifold $\mathcal{D}^s_{ex}$ whose tangent space $T_e\mathcal{D}^s_{ex}$ at the identity map $e(x)=x$ consists of divergence free vector fields characterized by single-valued stream functions. Moreover, it is also a topological group under composition of diffeomorphisms. The solution map associated with the Lagrangian reformulation of the Cauchy problem (1.1) turns out to be the (infinite-dimensional) Riemannian exponential map of the right-invariant metric defined at the identity e by the inner product

$$\langle v,w\rangle_{\dot H^{-\beta/2}} = \int \phi_v\,(-\Delta)^{\frac{2-\beta}{2}}\phi_w\,d\mu, \quad v,w\in T_e\mathcal{D}^s_{ex} \tag{1.2}$$

where $v = \nabla^{\perp}\phi_v$ and $w = \nabla^{\perp}\phi_w$ and ϕ_v, ϕ_w are the corresponding stream functions of class H^{s+1}. More precisely, we have

$$u_0 \mapsto \exp_e^{\beta} tu_0 := \gamma(t) \tag{1.3}$$

where $t \mapsto \gamma(t)$ is the curve of diffeomorphisms starting from the identity $\gamma(0) = e$ in the direction of $\dot{\gamma}(0) = u_0$. Our first goal is to prove

Theorem 1. *For $s > 2$ and any $0 \le \beta \le 1$ the (Lagrangian) solution map $u_0 \mapsto \exp_e^{\beta} u_0$ from $T_e\mathcal{D}^s_{ex}$ to $\mathcal{D}^s_{ex}$ is analytic.*

As immediate consequences we also obtain

Corollary 2. *For each $s > 2$ and $0 \le \beta \le 1$ the Riemannian exponential map is a local diffeomorphism from a neighborhood of 0 in $T_e\mathcal{D}^s_{ex}$ to a neighborhood of e in $\mathcal{D}^s_{ex}$.*

and

Corollary 3. *For each $s > 2$ and $0 \le \beta \le 1$ the generalized SQG equation in (1.1) is locally well-posed in H^s in the sense of Hadamard.*

In [4] Bourgain and Li outlined a strategy for proving nowhere uniform continuity of the solution map for equations invariant under Galilean transformations. They also suggested that a correct functional setup combining both Eulerian and Lagrangian methods when applicable can be useful in the case of equations whose velocity drifts are more singular than those of the 2D Euler equations. In [13] the authors proved that the Eulerian data-to-solution map of (1.1) is not uniformly continuous on bounded sets in H^s for all those equations which are small β perturbations of the incompressible 2D Euler equations using a continuity argument. Using the ideas of Bourgain-Li [4] and Inci [10], in this paper we will sharpen that result as follows.

Theorem 4. *Let $s > 2$. For any $0 \le \beta \le 1$ the corresponding (Eulerian) solution map $\theta_0 \mapsto \theta(t)$ is nowhere locally uniformly continuous as a map from bounded sets in H^s to $C\big([0,T]; H^s(\mathbb{R}^2)\big)$ where $T > 0$ is a local existence time.*

Theorem 4 shows that the lack of uniform continuity of the Eulerian data-to-solution map, previously observed for small values of β in [13], in fact persists for the entire family $0 \le \beta \le 1$. Together with Theorem 1 and its corollaries, this establishes a sharp contrast between the Lagrangian and Eulerian formulations of the generalized SQG equations: while the Lagrangian (geodesic) flow depends analytically on the initial data, the corresponding Eulerian evolution exhibits a strong form of ill-conditioning with respect to Sobolev perturbations.

Our approach combines the geometric framework of right-invariant metrics on the group of exact volume-preserving diffeomorphisms with perturbative constructions inspired by Bourgain–Li [4, 3] and Inci [10]. The analytic dependence of the Riemannian exponential map (Theorem 1) provides a convenient setting in which these fine regularity and instability properties can be compared within a unified Lagrangian picture.

The paper is organized as follows. In Section 2 we recall the geometric formulation of the generalized SQG equations and derive the equivalent Lagrangian system on the diffeomorphism group. Section 3 establishes the analyticity of the exponential map and proves the corresponding local well-posedness and local diffeomorphism properties. Section 4 is devoted to the proof of the non-uniform dependence result (Theorem 4). Finally, an Appendix summarizes the analytic framework in Banach spaces and collects auxiliary inequalities for fractional Sobolev norms used in the argument.

2. Technical preliminaries and Lagrangian reformulation

Our first task is to rewrite the SQG system (1.1) in the form which is convenient for our subsequent analysis. Let $S_{\beta,k} = R_k(-\Delta)^{-\frac{1-\beta}{2}}$ where $R_k = \partial_k(-\Delta)^{-1/2}$, $(k = 1, 2)$ are the standard Riesz transforms, so that $u = -S_\beta^\perp \theta$ with components

$$u_1 = S_{\beta,2}\theta \quad \text{and} \quad u_2 = -S_{\beta,1}\theta. \tag{2.1}$$

From (2.1) and the first of the equations in (1.1), we obtain

$$\partial_t u_1 + u \cdot \nabla u_1 = [u \cdot \nabla, S_{\beta,2}]\theta \tag{2.2}$$

and similarly

$$\partial_t u_2 + u \cdot \nabla u_2 = -[u \cdot \nabla, S_{\beta,1}]\theta. \tag{2.3}$$

Eliminating θ with the help of

$$R_1 u_2 - R_2 u_1 = \big(R_1 S_{\beta 1} + R_2 S_{\beta,2}\big)\theta = (-\Delta)^{-\frac{1-\beta}{2}}\theta \tag{2.4}$$

leads to the equation for the velocity field in Eulerian coordinates

$$\partial_t u + u \cdot \nabla u = \begin{pmatrix} [u \cdot \nabla, -S_{\beta,2}] \\ [u \cdot \nabla, S_{\beta,1}] \end{pmatrix} (-\Delta)^{\frac{1-\beta}{2}} (R_2 u_1 - R_1 u_2) =: q(u, u). \tag{2.5}$$

Equation (2.5) together with the initial condition

$$u(0) = u_0 \tag{2.6}$$

is an equivalent formulation of the Cauchy problem (1.1). That the latter implies the former follows directly from the above derivation. On the other hand, if u is a solution of the Cauchy problem (2.5)-(2.6) then θ given by (2.4) satisfies the first equation in (1.1). In fact, applying appropriate Riesz transforms to (2.2) and (2.3) and substituting into (2.4) after differentiating with respect to t variable gives

$$\begin{aligned}(-\Delta)^{-\frac{1-\beta}{2}}\partial_t\theta &= -\big(R_1 S_{\beta,1} + R_2 S_{\beta,2}\big) u \cdot \nabla\theta \\ &= (-\Delta)^{-\frac{1-\beta}{2}} u \cdot \nabla\theta.\end{aligned} \tag{2.7}$$

The equivalence of the two formulations is now a consequence of the following

Lemma 5. *Let $s > 2$ and $T > 0$. If $u \in C([0, T]; H^s)$ is a solution to (2.5) with $\operatorname{div} u_0 = 0$, then $\operatorname{div} u(t) = 0$ for all $0 \le t \le T$.*

Proof. We adapt here an argument from [10]. From (2.5) we have

$$u(t) = u_0 + \int_0^t \Big(q\big(u(s), u(s)\big) - u(s) \cdot \nabla u(s) \Big)\, ds$$

for $0 \le t \le T$. Let $\Phi(t) = R_1 u_1(t) + R_2 u_2(t)$. Note that as u_0 is divergence free, we have $\Phi(0) = 0$. From (2.2) and (2.3) we have

$$\begin{aligned}\partial_t \Phi = R_1\Big([u \cdot \nabla, S_{\beta,2}](-\Delta)^{\frac{1-\beta}{2}}(R_2 u_1 - R_1 u_2)\Big) - R_2\Big([u \cdot \nabla, S_{\beta,1}](-\Delta)^{\frac{1-\beta}{2}}(R_2 u_1 - R_1 u_2)\Big) \\ - R_1(u \cdot \nabla u_1) - R_2(u \cdot \nabla u_2).\end{aligned}$$

We consider these above terms on the right hand side separately. We have

$$\begin{aligned}R_1\Big([u\cdot\nabla, S_{\beta,2}](-\Delta)^{\frac{1-\beta}{2}}(R_2u_1-R_1u_2)\Big) =& R_1u_1R_2^2\partial_1u_1 - R_1S_{\beta,2}u_1R_2(-\Delta)^{\frac{1-\beta}{2}}\partial_1u_1\\ &+R_1u_2R_2^2\partial_2u_1 - R_1S_{\beta,2}u_2R_2(-\Delta)^{\frac{1-\beta}{2}}\partial_2u_1\\ &-R_1u_1R_1R_2\partial_1u_2 + R_1S_{\beta,2}u_1R_1(-\Delta)^{\frac{1-\beta}{2}}\partial_1u_2\\ &-R_1u_2R_1\mathcal{R}_2\partial_2u_2 + R_1S_{\beta,2}u_2R_1(-\Delta)^{\frac{1-\beta}{2}}\partial_2u_2.\end{aligned}$$

Similarly we have

$$\begin{aligned}R_2\Big([u\cdot\nabla, S_{\beta,1}](-\Delta)^{\frac{1-\beta}{2}}(R_2u_1-R_1u_2)\Big) =& R_2u_1R_1R_2\partial_1u_1 - S_{\beta,1}R_2u_1R_2(-\Delta)^{\frac{1-\beta}{2}}\partial_1u_1\\ &+R_2u_2R_1R_2\partial_2u_1 - S_{\beta,1}R_2u_2R_2(-\Delta)^{\frac{1-\beta}{2}}\partial_2u_1\\ &-R_2u_1R_1^2\partial_1u_2 + S_{\beta,1}R_2u_1R_1(-\Delta)^{\frac{1-\beta}{2}}\partial_1u_2\\ &-R_2u_2R_1^2\partial_2u_2 + S_{\beta,1}R_2u_2R_1(-\Delta)^{\frac{1-\beta}{2}}\partial_2u_2\end{aligned}$$

and

$$\begin{aligned}-R_1(u\cdot\nabla u_1)-R_2(u\cdot\nabla u_2) = &-R_1u_1(-R_1^2-R_2^2)\partial_1u_1 - R_1u_2(-R_1^2-R_2^2)\partial_2u_1\\ &-R_2u_1(-R_1^2-R_2^2)\partial_1u_2 - R_2u_2(-R_1^2-R_2^2)\partial_2u_2,\end{aligned}$$

since $R_1^2+R_2^2=-id$.

Combining all these identities and rearranging the terms gives

$$\begin{aligned}\partial_t\Phi =& R_1(u_1R_1^2\partial_1u_1)+R_1(u_2R_1^2\partial_2u_1)+R_1(u_1R_1R_2\partial_1u_2)+R_1(u_2R_1R_2\partial_2u_2)\\ &+R_2(u_1R_1R_2\partial_1u_1)+R_2(u_2R_1R_2\partial_2u_1)+R_2(u_1R_2^2\partial_1u_2)+R_2(u_2R_2^2\partial_2u_2)\\ =& R_1(u_1\cdot R_1\partial_1\Phi)+R_1(u_2\cdot R_1\partial_2\Phi)+R_2(u_1\cdot R_2\partial_1\Phi)+R_2(u_2\cdot R_2\partial_2\Phi).\end{aligned} \tag{2.8}$$

Therefore, multiplying by Φ and integrating by parts, yields

$$\begin{aligned}\frac{1}{2}\partial_t\|\Phi\|_{L^2}^2 &= \big\langle R_1(u_1\cdot R_1\partial_1\Phi)+R_1(u_2\cdot R_1\partial_2\Phi)+R_2(u_1\cdot R_2\partial_1\Phi)+R_2(u_2\cdot R_2\partial_2\Phi),\Phi\big\rangle_{L^2}\\ &=\sum_{k=1}^2\int R_k(u_1\cdot R_k\partial_1\Phi)\cdot\Phi\,dx+\sum_{k=1}^2\int R_k(u_2\cdot R_k\partial_2\Phi)\cdot\Phi\,dx\\ &=\frac{1}{2}\sum_{k=1}^2\int \partial_1u_1(R_k\Phi)^2\,dx+\frac{1}{2}\sum_{k=1}^2\int\partial_2u_2(R_k\Phi)^2\,dx\\ &\le C\|Du\|_\infty\Big(\|R_1\Phi\|_{L^2}^2+\|R_2\Phi\|_{L^2}^2\Big)\\ &\le C\sup_{[0,T]}\|u(t)\|_{H^s}\|\Phi(t)\|_{L^2}^2\end{aligned}$$

where in the last step we used the Sobolev lemma. Since $\Phi(0)=0$, applying Gronwall's inequality completes the proof. □

The above arguments can be summarized as follows.

Corollary 6. *Let $s>2$, $T>0$ and $\theta_0\in H^s$. If $\theta\in C([0,T];H^s(\mathbb{R}^2))$ is a solution to (1.1) with $\theta(0)=\theta_0$ then $u=\big(R_2(-\Delta)^{-\frac{1-\beta}{2}}\theta,-R_1(-\Delta)^{-\frac{1-\beta}{2}}\theta\big)$ is a solution to (2.5) on $[0,T]$. On the other hand, if $u\in C([0,T];H^s)$ is a solution to (2.5) with $u(0)=\big(R_2(-\Delta)^{-\frac{1-\beta}{2}}\theta_0,-R_1(-\Delta)^{-\frac{1-\beta}{2}}\theta_0\big)$, then $\theta=(-\Delta)^{\frac{1-\beta}{2}}(R_1u_2-R_2u_1)$ is a solution to (1.1) on $[0,T]$.*

Finally, we are ready to proceed with the Lagrangian formulation of the Cauchy problem (2.5)-(2.6). To this end we employ the flow equation, namely

$$\frac{d\gamma}{dt}(t,x) = u\big(t,\gamma(t,x)\big), \quad \gamma(0,x) = x \tag{2.9}$$

and rewrite the velocity equation (2.5) as a first order system on the tangent bundle $T\mathcal{D}^s_{ex}$ of the group $\mathcal{D}^s_{ex}$, that is

$$\partial_t \begin{pmatrix} \gamma \\ \dot\gamma \end{pmatrix} = \begin{pmatrix} \dot\gamma \\ q(\dot\gamma\circ\gamma^{-1}, \dot\gamma\circ\gamma^{-1})\circ\gamma \end{pmatrix} =: F(\gamma,\dot\gamma) \tag{2.10}$$

subject to the initial conditions

$$\gamma(0) = e \quad \text{and} \quad \dot\gamma(0) = u_0 \tag{2.11}$$

where $F : U \to H^s \times H^s$ is defined in some open neighborhood U of the point $(e, 0)$ in $\mathcal{D}^s_{ex} \times H^s$. The flow $t \mapsto \gamma(t)$ of the velocity field u describes of course the geodesic path in $\mathcal{D}^s_{ex}$ starting from e in the direction u_0, see (1.3) above.

3. Local well-posedness

In this section we prove Theorem 1 and Corollary 3. We will adapt the method used by Shnirelman [14]. For the definition and relevant constructions involving analytic maps in the infinite dimensional setting see the Appendix.

3.1. **Proof of Theorem 1.** Given u_0 in $T_e\mathcal{D}^s_{\mu,ex}$ let $\gamma(t)$ be the geodesic in the diffeomorphism group with $\gamma(0) = e$ and $\dot\gamma(0) = u_0$ and let θ_0 be the corresponding initial condition in (1.1). Observe that for any $t \geq 0$ we have the conservation law

$$\theta(t,\gamma(t,x)) = \theta_0(x). \tag{3.1}$$

From (2.9), using (1.1) and (3.1), we obtain

$$\frac{\partial\gamma(t,x)}{\partial t} = \Big\{\mathrm{curl}^{-1}\,(-\Delta)^{\frac{\beta}{2}}\,\big(\theta_0\circ\gamma^{-1}(t)\big)\Big\}\circ\gamma(t,x), \tag{3.2}$$

where $\mathrm{curl}^{-1} = -\nabla^\perp(-\Delta)^{-1}$ is the inverse of the curl operator on the space H^s_0 denote the space of Sobolev H^s divergence free vector fields with mean zero.[1]

We will construct a local chart near the identity and use Proposition A.1 and Proposition A.2 to show that the exponential map is locally analytic between Banach spaces.

Lemma 7. *Let $\varepsilon > 0$ be sufficiently small. If $v \in H^s_0$ with $\|v\|_s < \varepsilon$ then there is a unique (up to an additive constant) function $\varphi_v \in H^{s+1}$ such that the map*

$$x \to \xi_v(x) = x + v(x) + \nabla\varphi_v(x) \tag{3.3}$$

is an exact volume-preserving H^s diffeomorphism. Furthermore, the map $v \mapsto v + \nabla\varphi_v$ is analytic as a map from H^s_0 to H^s.

Proof. Since the map $x \mapsto x + v(x)$ does not in general preserve the volume form μ, we need to adjust it by adding a gradient term $\nabla\varphi_v(x)$ which can be done with the help of the Hodge decomposition. Call the adjusted map $\xi_v = e + v + \nabla\varphi_v$ satisfying $\det(J(\xi_v)) = 1$. So ξ_v is an element of the subgroup $\mathcal{D}^s_{\mu,ex}$ near the identity and we have

$$\det\begin{pmatrix} 1 + \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial^2\varphi_v}{\partial x_1^2} & \dfrac{\partial v_1}{\partial x_2} + \dfrac{\partial^2\varphi_v}{\partial x_1\partial x_2} \\ \dfrac{\partial v_2}{\partial x_1} + \dfrac{\partial^2\varphi_v}{\partial x_2\partial x_1} & 1 + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial^2\varphi_v}{\partial x_2^2} \end{pmatrix} = 1. \tag{3.4}$$

[1]Note that H^s_0 corresponds to $T_e\mathcal{D}^s_{\mu,ex}$ but in the case of $\mathbb{R}^2$ we need appropriate vanishing conditions at infinity.

Expanding the (3.4) and rewriting it we obtain an equation for the Laplacian of φ_v, that is

$$\Delta\varphi_v = P(Dv, D^2\varphi_v). \tag{3.5}$$

where P is a polynomial of degree 2 in Dv and $D^2\varphi$, namely

$$\begin{aligned} P\big(Dv, D^2\varphi\big) = -\Big(&\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} + \frac{\partial v_1}{\partial x_1}\frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial v_2}{\partial x_2}\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_1^2}\frac{\partial^2\varphi}{\partial x_2^2}\Big) \\ &+ \frac{\partial v_2}{\partial x_1}\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\frac{\partial^2\varphi}{\partial x_1\partial x_2} + \frac{\partial v_1}{\partial x_2}\frac{\partial^2\varphi}{\partial x_2\partial x_1} + \Big(\frac{\partial^2\varphi}{\partial x_1\partial x_2}\Big)^2. \end{aligned} \tag{3.6}$$

Let $Q(v,\varphi_v) = \Delta^{-1}P(Dv, D^2\varphi_v)$. Observe that Q is analytic in both variables as a map from $H^s_0 \times H^{s+1}$ to H^{s+1} and, moreover, Q and its Gâteaux derivatives satisfy the inequalities

$$\|Q(v,\varphi)\|_{s+1} \le C\big(\|v\|_s^2 + \|\varphi\|_{s+1}^2\big) \tag{3.7}$$

and

$$\|\partial_v Q(v,\varphi)\|_{s+1} + \|\partial_\varphi Q(v,\varphi)\|_{s+1} \le C(\|v\|_s + \|\varphi\|_{s+1}), \tag{3.8}$$

with some constant $C > 0$.

Let $\varepsilon < (2C)^{-1}$. For any v in H^s_0 with $\|v\|_s < \varepsilon$, if $\|\varphi\|_{s+1} \le \varepsilon$ then from (3.7) we have

$$\|Q(v,\varphi)\|_{s+1} \le C(\varepsilon^2 + \varepsilon^2) = 2C\varepsilon^2 < \varepsilon \tag{3.9}$$

which implies that $\varphi \to Q(v,\varphi)$ maps the ε-ball B^{s+1}_ε centered at the origin in H^{s+1} to itself.

Furthermore, from (3.8), for any φ_1, φ_2 in B^{s+1}_ε we have

$$\begin{aligned} \|Q(v,\varphi_1) - Q(v,\varphi_2)\|_{s+1} &\le \max_{0\le\lambda\le1} \|\nabla Q\big(v, \lambda\varphi_1 + (1-\lambda)\varphi_2\big)\|\|\varphi_1 - \varphi_2\|_{s+1} \\ &\le 2C\varepsilon\|\varphi_1 - \varphi_2\|_{s+1} \\ &< \|\varphi_1 - \varphi_2\|_{s+1}. \end{aligned} \tag{3.10}$$

Thefore, there is a ball of sufficiently small radius such that for any v belongs to this ball, there exists a unique solution $\varphi = \varphi_v$ of (3.5) by the contraction mapping argument.

Since Q is analytic, by the Analytic Implicit Function Theorem A.2, there exists a unique analytic solution φ_v which shows that the map $v \mapsto \xi_v - e$ is analytic as a map from H^s_0 to H^s.

□

This lemma defines a local analytic chart at the identity e in $\mathcal{D}^s_{\mu,ex}$, namely

$$U \ni v \mapsto \xi_v = e + v + \nabla\varphi_v \in \mathcal{U}$$

where U and $\mathcal{U}$ are open neighborhood of 0 in $T_e\mathcal{D}^s_{\mu,ex} \simeq H^s_0$ and of e in $\mathcal{D}^s_{\mu,ex}$.

Next, we rewrite the equation (3.2). First, consider the space

$$H^s_{0,\gamma} := \Big\{U = u\circ\gamma : u \in H^s,\ \operatorname{div} u = 0 \ \text{ and } \int_M u\,d\mu = 0\Big\},$$

which is the right translation of H^s_0 by $\gamma \in \mathcal{D}^s_{\mu,ex}$. To see that $H^s_{0,\gamma}$ depends analytically on γ we proceed as follows. First, observe that by the change of variables formula and the fact that γ preserves the volume form μ we have

$$0 = \int_M u\,d\mu = \int_{\gamma(M)} \gamma^*(U\circ\gamma^{-1}\,d\mu) = \int_M U\,d\mu.$$

Next, we have

$$0 = \operatorname{div} u = \operatorname{div}(U \circ \gamma^{-1}) = \sum_k \partial_k (U^k \circ \gamma^{-1})$$

$$= \sum_{k,l} \partial_l U^k \circ \gamma^{-1} \cdot \partial_k (\gamma^{-1})^l =: L_\gamma(U). \tag{3.11}$$

Observe that L_γ defined by the right hand side of (3.11) is a linear first order differential operator in U. Since $\gamma^{-1} \circ \gamma = e$ we have $D\gamma^{-1} \circ \gamma \cdot D\gamma = Id$ and since $\det(D\gamma) = 1$ the coefficients of L_γ depend linearly on the entries of $D\gamma$. Furthermore, the composition with γ^{-1} is an analytic map (see e.g. Whittlesey [16]).

We can now describe the space $H^s_{0,\gamma}$ in terms of U and γ as

$$H^s_{0,\gamma} := \Big\{ U : L_\gamma(U) = 0 \text{ and } \int U \, d\mu = 0 \Big\} \tag{3.12}$$

and L_γ depends on γ analytically it follows that the dependence of the subspace $H^s_{0,\gamma}$ on γ is analytic as well.

Next, using the conservation law (3.1) and the equation expressing the drift velocity u in terms of θ in (1.1) we have

$$U = u \circ \gamma = \Big\{ \operatorname{curl}^{-1} (-\Delta)^{\frac{\beta}{2}} (\theta_0 \circ \gamma^{-1}) \Big\} \circ \gamma =: F(\gamma, \theta_0) \tag{3.13}$$

and we observe that the operator $F(\gamma, \theta_0)$ defined by the right hand side of (3.13) depends analytically on both γ and θ_0. The analyticity of $F(\gamma, \theta_0)$ in γ follows from the inclusion $F(\gamma, \theta_0) \in H^s_{0,\gamma}$, which ensures that the inversion mapping $\gamma \mapsto \gamma^{-1}$ is analytic. Consequently, the composition $\gamma \mapsto \theta_0 \circ \gamma^{-1}$ is also analytic. Alternatively, we can argue by calculating the Gâteaux derivative of the map $\gamma \mapsto \theta_0 \circ \gamma$ from $\mathcal{D}^s_{\mu,ex} \to H^{s-1}$

$$\frac{d}{dt} (\theta_0 \circ (\gamma + th)^{-1}) \Big|_{t=0} = \nabla \theta_0 ((\gamma + th)^{-1}) \cdot \frac{d}{dt} (\gamma + th)^{-1}) \Big|_{t=0} \tag{3.14}$$

$$= -\nabla \theta_0 (\gamma^{-1}) \cdot (D\gamma^{-1} \cdot h \circ \gamma^{-1}) \tag{3.15}$$

and observe that we have

$$\left\| \nabla \theta_0 \circ \gamma^{-1} \cdot \left[D\gamma^{-1} \cdot h \circ \gamma^{-1} \right] \right\|_{H^{s-1}} \leq C \left\| \theta_0 \right\|_{H^{s-1}} \left\| \gamma \right\|_{H^s}.$$

As a result, analyticity is preserved under the linear operator $\operatorname{curl}^{-1} (-\Delta)^{\frac{\beta}{2}}$ and the right composition with γ.

Our task is thus equivalent to showing that the solution of the following system in $\mathcal{D}^s_{\mu,ex} \times H^{s-1+\beta}$ depends analytically on u_0

$$\begin{cases} \dfrac{\partial \gamma}{\partial t} = F(\gamma, \theta_0) \\ \gamma(0) = e \end{cases} \tag{3.16}$$

where $F : \mathcal{D}^s_{\mu,ex} \times H^{s-1+\beta} \to H^{s-1+\beta}$ is analytic in both variables. Applying Proposition A.1 in Appendix we obtain the solution $\gamma(t) = \exp^\beta_e tu_0$ with $0 \leq t \leq T$ for some $T > 0$ and $u_0 = -\nabla^\perp (-\Delta)^{-1+\frac{\beta}{2}} \theta_0$, which depends analytically on both t and θ_0. This completes the proof of Theorem 1.

3.2. **Proof of Corollary 3.** We start with the following lemma

Lemma 8. *Let $s > 2$ and $T > 0$. Assume that γ is a solution of (2.10) on $[0, T]$ for the initial values $\gamma(0) = e$ and $\dot\gamma(0) = u_0 \in H^s$. Then u given by*

$$u(t) := \dot\gamma(t) \circ \gamma(t)^{-1}$$

is a solution to (2.5).

Proof. We need to prove

$$u(t) = u_0 + \int_0^t q(u(s), u(s)) - (u(s) \cdot \nabla)u(s)\, ds \quad \forall\, 0 \le t \le T$$

Note that by Theorem 1 we have $\gamma \in C^\infty([0, T]; \mathcal{D}^s_{\mu,ex})$. Therefore by Sobolev's imbedding lemma and the properties of the composition $u = \dot\gamma \circ \gamma^{-1} \in C^1([0, T] \times \mathbb{R}^2; \mathbb{R}^2)$ (see e.g. [11]). Thus we have pointwise

$$\ddot\gamma = \big(u_t + (u \cdot \nabla)u\big) \circ \gamma$$

And from (2.10) we conclude

$$(u_t + (u \cdot \nabla)u) \circ \gamma = q(u, u) \circ \gamma$$

or composing on the right with γ^{-1} we have

$$u_t + (u \cdot \nabla)u = q(u, u).$$

Rewriting this last equation in the integral form we get

$$u(t) = u_0 + \int_0^t q(u(s), u(s)) - (u(s) \cdot \nabla)u(s)\, ds$$

for any $0 \le t \le T$. Since $s > 2$ by the algebra property of H^{s-1} and Sobolev's imbedding lemma, the result follows. ∎

Lemma 9. *Let $s > 2$ and $T > 0$. If $u \in C([0, T]; H^s)$ is a solution to (2.5) then its flow $t \mapsto \gamma(t)$ is a solution to (2.10).*

Proof. We know that given u there is a unique $\gamma \in C^1([0, T]; \mathcal{D}^s_{\mu,ex})$ with

$$\dot\gamma = u \circ \gamma \quad \text{with} \quad \gamma(0) = e. \tag{3.17}$$

From the integral relation $u(t) = u_0 + \int_0^t \big(q(u, u) - (u \cdot \nabla)u\big)\, ds$ we see that $u \in C^1([0, T] \times \mathbb{R}^2; \mathbb{R}^2)$. Taking the derivative in t of the differential equation in (3.17) we get pointwise

$$\ddot\gamma = \big(u_t + (u \cdot \nabla)u\big) \circ \gamma = q(u, u) \circ \gamma$$

and consequently

$$\gamma(t) = \dot\gamma(0) + \int_0^t q(\dot\gamma \circ \gamma^{-1}, \dot\gamma \circ \gamma^{-1}) \circ \gamma\, ds \quad \forall\, 0 \le t \le T.$$

Therefore $\gamma \in C^1([0, T]; \mathcal{D}^s_{\mu,ex})$ and γ is a solution to (2.10). ∎

Recall from previous sections that solutions of (2.10) with initial conditions $\gamma(0) = e$ and $\dot\gamma(0) = u_0 \in H^s$ can be described by the Riemannian exponential map on $\mathcal{D}^s_{\mu,ex}$ (see (1.2) and (1.3)). From Theorem 1 we know that $\exp^\beta_e$ is real analytic.

Proof of Corollary 3. Take $\theta_0 \in H^s(\mathbb{R}^2)$ and for $u_0 = (S_{\beta,2}\theta_0, -S_{\beta,1}\theta_0)$ let

$$\gamma(t) = \exp^\beta_e(tu_0) \quad \text{and} \quad u(t) = \dot\gamma(t) \circ \gamma(t)^{-1}$$

for $0 \le t \le T$ where $T > 0$ comes from the Fundamental theorem of ODE as in the previous proof. Since $\gamma(t) \in \mathcal{D}^s_{\mu,ex}$ where $s > 2$, we know that $\gamma(t)$ is a curve of C^1 diffeomorphisms by the Sobolev embedding theorem. Therefore, by properties of the composition map – see e.g. [9], [11] – this implies that $u \in C([0,T];H^s)$ for some $T>0$.

Next, set

$$\theta(t) = (-\Delta)^{\frac{1-\beta}{2}}\Big(R_1 u_2(t) - R_2 u_1(t)\Big)$$

and observe that θ solves (1.1) by Lemma 8 and Corollary 6. By Theorem 1, $\exp_e^\beta tu_0$ is analytic in u_0, hence (locally) Lipschitz continuous. In addition, the map $\theta_0 \mapsto u_0 = -\nabla^\perp(-\Delta)^{-1+\frac{\beta}{2}}\theta_0$ is linear and bounded in H^s. Combining with the continuity of the composition map $\theta_0 \mapsto \theta_0\circ\gamma^{-1}$, we obtain that the dependence on the initial condition θ_0 is continuous.

Uniqueness of solutions follows directly from the Fundamental ODE theorem. Indeed, suppose two solutions $\theta_1(t)$ and $\theta_2(t)$ exists for the same initial data θ_0. The associated Lagrangian flows $\gamma_1(t)$ and $\gamma_2(t)$ must satisfy $\gamma_1(t)=\gamma_2(t)$ for all time as long as they defined due to uniqueness of geodesics in Theorem 1. Hence, $\theta_1(t) = \theta_0(\gamma_1^{-1}(t)) = \theta_0(\gamma_2^{-1}(t)) = \theta_2(t)$. □

Remark 10. *Alternatively, we can prove Corollary 3 by directly showing that the right hand side $F(\gamma,\theta_0)$ is locally Lipschitz with respect to both variables γ and θ_0. This can be done by showing that $F(\gamma,\theta_0)$ has a bounded Gâteaux derivative in the open neighborhood $\mathcal{U}$ of the identity e in $\mathcal{D}^s_{\mu,ex}$ and then apply the mean value estimate to deduce the desired property of $F(\gamma,\theta_0)$.*

Proof of Corollary 2. As before let $\gamma(t) = \exp_e^\beta tu_0$ with $u_0 \in T_eD^s_{\mu,ex}$. A standard calculation (as in finite-dimensional Riemannian geometry) gives

$$d\exp_e^\beta(0)u_0 = \dot\gamma(0) = u_0 \tag{3.18}$$

which combined with the Inverse Function Theorem yields the result. □

4. Non-uniform dependence of the solution map for $0 \le \beta \le 1$.

In the following we fix the local existence time (see e.g., Corollary 3) and let $\mathcal{O}\subset H^s(\mathbb{R}^2)$ denote the set of those initial conditions θ_0 for which the solution of (1.1) exists at least up to time $T=1$. To prove Theorem 4 we use the following lemma.

Lemma 11. *There is a dense subset $\mathcal{S}\subseteq\mathcal{O}$ consisting of functions with compact support such that for each function $\theta_0\in\mathcal{S}$ we can find $x_*\in\mathbb{R}^2$ and $\theta_*\in H^s(\mathbb{R}^2)$ satisfying*

$$B(x_*,2)\cap\operatorname{supp}\theta_0 = \emptyset \quad \text{and} \quad \big(d\exp_e^\beta(u_0)w_*\big)(x_*) \neq 0 \tag{4.1}$$

where

$$u_0 = \Big(S_{\beta,2}\theta_0, -S_{\beta,1}\theta_0\Big) \quad \text{and} \quad w_* = \Big(S_{\beta,2}\theta_*, -S_{\beta,1}\theta_*\Big)$$

and $\exp_e^\beta$ is the Riemannian exponential map in (1.3).

Proof. We can assume that the initial data θ_0 has compact support since such functions are dense in $H^s(\mathbb{R}^2)$. Let $x_*\in\mathbb{R}^2$ be any point whose distance from $\operatorname{supp}\theta_0$ is at least 2. Expressing $S_{\beta,k}$ as a principal value integral

$$S_{\beta,k}\theta(x) = \frac{1}{\sqrt{2\pi}}\,p.v.\int_{\mathbb{R}^2}\frac{x_k-y_k}{|x-y|^{2-\beta}}\,\theta(y)\,dy \qquad (k=1,2)$$

we see that it is always possible to choose a smooth positive bump function θ_* supported near x_* such that $w_*(x_*) = \big(S_{\beta,2}\theta_*(x_*), -S_{\beta,1}\theta_*(x_*)\big)\neq 0$.

Next, consider the derivative of the exponential map evaluated at x_* and w_* above

$$t\mapsto \epsilon(t) = \big(d\exp_e^\beta(tu_0)w_*\big)(x_*) \tag{4.2}$$

as an analytic function of the time variable. From (3.18) (cf. the proof of Cor. 2) we have that $d\exp_e^\beta(0) = \mathrm{id}$ which implies that $\epsilon(t)$ is not identically zero and, consequently, we can choose a sequence of points in the interval $(0,1)$ with $t_n\nearrow 1$ such that $\epsilon(t_n)\neq 0$ for $n=1,2\ldots$. Letting now $\mathcal{S}$ be the set which consists of the functions of the form $t_n\theta_0$ where θ_0 has compact support we see that both conditions in (4.1) are trivially satisfied and the lemma follows. □

The following inequalities apply to functions whose compact supports do not overlap. For any $s \geq 0$ there exists a constant $C > 0$ such that given any x, y in $\mathbb{R}^2$ satisfying $0 < r = |x-y|/4 < 1$, the inequality

$$\|f_1 + f_2\|_{H^s} \geq C(\|f_1\|_{H^s} + \|f_2\|_{H^s}) \tag{4.3}$$

holds for all functions $f_1, f_2 \in H^s(\mathbb{R}^2)$ where f_1 is supported in the ball $B(x,r)$ centered at x and f_2 is supported in the ball $B(y,r)$ centered at y (See Lemma A.4 in the Appendix for the proof.).

We next proceed with the proof of the main theorem.

Proof of Theorem 4. Without loss of generality we assume that $T = 1$. Given any θ_0 in $\mathcal{O}$ we aim to show that there exists $R_* > 0$ such that for all R in the interval $(0, R_*)$ the data-to-solution map of (1.1) is not uniformly continuous in $\mathcal{O} \cap B^s(\theta_0, R)$, where $B^s(\theta_0, R)$ stands for the open ball in H^s centered at θ_0 of radius R. To this end we will construct two sequences of initial data $\theta_0^{(1,n)}$ and $\theta_0^{(2,n)}$ where $n = 1, 2\ldots$ confined in $B^s(\theta_0, R)$ and satisfying

$$\lim_{n\to\infty} \left\|\theta_0^{(1,n)} - \theta_0^{(2,n)}\right\|_{H^s} = 0 \tag{4.4}$$

and

$$\limsup_{n} \left\|\theta_0^{(1,n)} \circ (\gamma^{(1,n)})^{-1} - \theta_0^{(2,n)} \circ (\gamma^{(2,n)})^{-1}\right\|_{H^s} > 0 \tag{4.5}$$

where $t \to \gamma^{(i,n)}$ are the flow maps of the velocity fields corresponding to

$$u_0^{(i,n)} = -\nabla^\perp(-\Delta)^{-1+\frac{\beta}{2}}\theta_0^{(i,n)}, \qquad \text{where } \; n = 1, 2\ldots \; \text{ and } \; i = 1, 2.$$

In what follows it will be convenient to introduce the notation $E_e^\beta(\theta) = \exp_e^\beta u$. From Lemma 11 we can find a nonzero H^s vector field w_* and a point x_* in $\mathbb{R}^2$ located far away from the support of θ_0 such that

$$\left|\big(dE_e^\beta(\theta_0)(w_*)\big)(x_*)\right| \geq \kappa_*\|w_*\|_{H^s} \tag{4.6}$$

for some constant $\kappa_* > 0$.

Furthermore, recall that by continuity of the composition operator and the Sobolev lemma there is a constant $C > 0$ such that

$$\frac{1}{C}\|f\|_{H^s} \leq \|f \circ \gamma^{-1}\|_{H^s} \leq C\|f\|_{H^s} \tag{4.7}$$

for any $f \in H^s(\mathbb{R}^2)$ and any transformation γ in $E_e^\beta(B^s(\theta_0, R_1))$ and where $R_1 > 0$ is chosen so that $B^s(\theta_0, R_1) \subset \mathcal{O}$.

Let $\gamma_0 = E_e^\beta(\theta_0)$ and observe that it follows from (4.6) that for some sufficiently small $r_* > 0$ the distance between the images under γ_0 of $\operatorname{supp}\theta_0$ and of the closed ball $\overline{B(x_*, r_*)}$ is strictly positive, namely

$$d = \Big(\gamma_0(\theta_0), \gamma_0\big(\overline{B(x_*, r_*)}\big)\Big) > 0.$$

Introduce the following sets

$$\mathcal{K}_1 = \big\{x \in \mathbb{R}^2 \mid x \text{ is at most } d/4 \text{ away from } \gamma_0(\operatorname{supp}\theta_0)\big\} \tag{4.8}$$

and

$$\mathcal{K}_2 = \big\{x \in \mathbb{R}^2 \mid x \text{ is at most } d/4 \text{ away from } \gamma_0(\overline{B(x_*, r_*)})\big\}. \tag{4.9}$$

By selecting $R_2 \in (0, R_1)$ we can arrange things so that for every $\gamma, \gamma' \in E_e^\beta\big(B^s(\theta_0, R_2)\big)$, we have the following estimates

$$|\gamma(x) - \gamma(y)| \lesssim \|D\gamma\|_\infty |x-y| \lesssim L|x-y|, \quad \forall x, y \in \mathbb{R}^2 \tag{4.10}$$

where $L > 0$ is a constant and

$$\|\gamma - \gamma'\|_{L^\infty} \le \min\{1, d/4\} \tag{4.11}$$

which follow by combining the mean value estimate, the Sobolev lemma and the triangle inequality. In particular, this ensures that

$$\gamma(\operatorname{supp}\theta_0) \subseteq \mathcal{K}_1 \quad \text{and} \quad \gamma\big(\overline{B(x_*, 1)}\big) \subseteq \mathcal{K}_2$$

for any $\gamma \in E_e^\beta\big(B^s(\theta_0, R_2)\big)$.

Next, consider the Taylor expansion with the integral remainder

$$E_e^\beta(\theta + h) = E_e^\beta(\theta) + dE_e^\beta(\theta)h + \int_0^1 (1-t) d^2 E_e^\beta(\theta + th)(h, h)\, dt \tag{4.12}$$

where $h \in H^s(\mathbb{R}^2)$. Choose $R_3 \in (0, R_2)$ and observe that there exists $M > 0$ such that

$$\|d^2 E_e^\beta(\theta)(h_1, h_2)\|_{H^s} \le M \|h_1\|_{H^s} \|h_2\|_{H^s}$$

and

$$\|d^2 E_e^\beta(\theta_1)(h_1, h_2) - d^2 E_e^\beta(\theta_2)(h_1, h_2)\|_{H^s} \le M\|\theta_1 - \theta_2\|_{H^s} \|h_1\|_{H^s} \|h_2\|_{H^s}$$

for all $\theta, \theta_1, \theta_2 \in B^s(\theta_0, R_3)$ and $h_1, h_2 \in H^s(\mathbb{R}^2)$. Next, we select R_* within the interval $(0, R_3)$ such that it satisfies the condition

$$\max\{CMR_*^2, CMR_*\} < \kappa_*/8. \tag{4.13}$$

Now, we choose $R \in (0, R_*)$ and construct two sequences $(\theta_0^{(1,n)})_{n\ge1}$ and $(\theta_0^{(2,n)})_{n\ge1}$ as follows. The initial sequence is defined as

$$\theta_0^{(1,n)} = \theta_0 + \vartheta^{(n)}$$

where $\vartheta^{(n)}$ in $H^s(\mathbb{R}^2)$ is chosen arbitrarily such that $\|\vartheta^{(n)}\|_{H^s} = R/2$ and its support is contained within $B(x_*, r_n)$, where

$$r_n = \frac{\kappa_*}{8Ln} \|w_*\|_{H^s}\,.$$

Therefore, while the total mass of $\vartheta^{(n)}$ remains unchanged, its support gradually contracts. The second sequence is obtained by perturbing the first one with $w_*^{(n)} := w_*/n$ which introduces a shift in the supports. This results in

$$\theta_0^{(2,n)} = \theta_0^{(1,n)} + w_*^{(n)} = \theta_0 + \vartheta^{(n)} + w_*^{(n)}. \tag{4.14}$$

Taking N sufficiently large we have

$$\theta_0^{(1,n)}, \theta_0^{(2,n)} \in B^s(\theta_0, R) \quad \text{and} \quad r_n \le 1,\ \forall n \ge N.$$

By construction we have

$$\lim_{n\to\infty} \|\theta_0^{(1,n)} - \theta_0^{(2,n)}\|_{H^s} = \lim_{n\to\infty} \|w_*^{(n)}\|_{H^s} = 0.$$

For $n \ge N$, let $\gamma^{(1,n)} = E_e^\beta(\theta_0^{(1,n)})$ and $\gamma^{(2,n)} = E_e^\beta(\theta_0^{(2,n)})$, we get

$$\begin{aligned}
&\|\theta_0^{(1,n)} \circ (\gamma^{(1,n)})^{-1} - \theta_0^{(2,n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s} \\
&\qquad \ge \|\theta_0^{(1,n)} \circ (\gamma^{(1,n)})^{-1} - \theta_0^{(1,n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s} - \|w_*^{(n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s}
\end{aligned}$$

and by (4.7) we have $\limsup_{n\to\infty} \|w_*^{(n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s} = 0$. Moreover, observe that

$$\begin{aligned}
&\|\theta_0^{(1,n)} \circ (\gamma^{(1,n)})^{-1} - \theta_0^{(1,n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s} \\
&= \|(\theta_0 \circ (\gamma^{(1,n)})^{-1} - \theta_0 \circ (\gamma^{(2,n)})^{-1}) + (\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1} - \vartheta^{(n)} \circ (\gamma^{(2,n)})^{-1})\|_{H^s} \\
&\geq C\|(\theta_0 \circ (\gamma^{(1,n)})^{-1} - \theta_0 \circ (\gamma^{(2,n)})^{-1})\|_s + C\|(\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1} - \vartheta^{(n)} \circ (\gamma^{(2,n)})^{-1})\|_{H^s}.
\end{aligned}$$

The last inequality follows from the reverse triangle inequality (4.3) and the fact that the support of the terms $(\theta_0 \circ (\gamma^{(1,n)})^{-1}$ and $\theta_0 \circ (\gamma^{(2,n)})^{-1})$ is contained in $\mathcal{K}_1$, which implies that their difference $(\theta_0 \circ (\gamma^{(1,n)})^{-1} - \theta_0 \circ (\gamma^{(2,n)})^{-1})$ is also supported in $\mathcal{K}_1$. Similarly, the functions $(\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1}$ and $\vartheta^{(n)} \circ (\gamma^{(2,n)})^{-1})$ are supported in $\mathcal{K}_2$, meaning that their difference $(\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1} - \vartheta^{(n)} \circ (\gamma^{(2,n)})^{-1})$ is also supported in $\mathcal{K}_2$. Therefore, it suffices to show

$$\limsup_{n\to\infty} \|\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1} - \vartheta^{(n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s} > 0.$$

Again, this can be done by showing that the supports of the functions $\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1}$ and $\vartheta^{(n)} \circ (\gamma^{(2,n)})^{-1}$ are disjoint so that we can apply (4.3)

$$\begin{aligned}
\|\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1} - \vartheta^{(n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s} &\geq C\Big(\|\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1}\|_s + \|\vartheta^{(n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s}\Big) && (4.15)\\
&\geq C\|\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1}\|_{H^s}, && (4.16)
\end{aligned}$$

and $\|\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1}\|_s$ has a universal positive lower bound using the first of the inequalities in (4.7).

Now, we estimate the quantity $|\gamma^{(1,n)}(x_*) - \gamma^{(2,n)}(x_*)|$. We have

$$\begin{aligned}
\gamma^{(1,n)} &= E_e^\beta(\theta_0 + \vartheta^{(n)}) \\
&= E_e^\beta(\theta_0) + dE_e^\beta(\theta_0)(\vartheta^{(n)}) + \int_0^1 (1-t) d^2 E_e^\beta(\theta_0 + t\vartheta^{(n)})(\vartheta^{(n)}, \vartheta^{(n)})\, dt
\end{aligned}$$

and

$$\begin{aligned}
\gamma^{(2,n)} &= E_e^\beta(\theta_0 + \vartheta^{(n)} + w_*^{(n)}) && (4.17)\\
&= E_e^\beta(\theta_0) + dE_e^\beta(\theta_0)(\vartheta^{(n)} + w_*^{(n)}) \\
&\quad + \int_0^1 (1-t) d^2 E_e^\beta(\theta_0 + t(\vartheta^{(n)} + w_*^{(n)}))(\vartheta^{(n)} + w_*^{(n)}, \vartheta^{(n)} + w_*^{(n)})\, dt.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\gamma^{(1,n)} - \gamma^{(2,n)} = -dE_e^\beta(\theta_0)(w_*^{(n)}) &- \underbrace{2\int_0^1 (1-t) d^2 E_e^\beta(\theta_0 + t(\vartheta^{(n)} + w_*^{(n)}))(w_*^{(n)}, \vartheta^{(n)})\, dt}_{J_1} \\
&- \underbrace{\int_0^1 (1-t) d^2 E_e^\beta(\theta_0 + t(\vartheta^{(n)} + w_*^{(n)}))(w_*^{(n)}, w_*^{(n)})\, dt}_{J_2} \\
&+ \underbrace{\int_0^1 (1-t) \Big(d^2 E_e^\beta(\theta_0 + t\vartheta^{(n)})(\vartheta^{(n)}, \vartheta^{(n)}) - d^2 E_e^\beta(\theta_0 + t(\vartheta^{(n)} + w_*^{(n)}))(\vartheta^{(n)}, \vartheta^{(n)})\Big)\, dt}_{J_3}.
\end{aligned}$$

Applying the previously derived bounds for the second derivatives, we obtain the following estimates:

$$\|J_1\|_{H^s} \leq 2K\|w_*^{(n)}\|_{H^s}\|\vartheta^{(n)}\|_{H^s} = \frac{MR}{n}\|w_*\|_{H^s},$$

and

$$\|J_2\|_{H^s} \leq \frac{M}{2n^2}\|w_*^{(n)}\|_{H^s}^2 \leq \frac{MR}{n}\|w_*\|_{H^s}$$

and

$$\|J_3\|_{H^s} \leq \int_0^1 (1-t)M\|w_*^{(n)}\|_{H^s}\|\vartheta^{(n)}\|_{H^s}^2 dt = \frac{M}{2}\|w_*^{(n)}\|_{H^s}\|\vartheta^{(n)}\|_{H^s}^2 \leq \frac{MR^2}{4n}\|w_*\|_{H^s}$$

where the bound for $\|J_2\|_{H^s}$ holds for $n \geq N$ by increasing N if necessary. Thus, by the Sobolev lemma and the choice for R_* in (4.13), we obtain

$$\begin{aligned}|J_1(x_*)| + |J_2(x_*)| + |J_3(x_*)| &\leq \frac{CMR}{n}\|w_*\|_{H^s} + \frac{CMR}{n}\|w_*\|_{H^s} + \frac{CMR^2}{4n}\|w_*\|_{H^s} \\ &\leq \frac{\kappa_*}{2n}\|w_*\|_{H^s}.\end{aligned}$$

Using this inequality we find

$$\begin{aligned}\left|\gamma^{(1,n)}(x_*) - \gamma^{(2,n)}(x_*)\right| &\geq \left|dE_e^\beta(\theta_0)(w_*^{(n)})(x_*)\right| - \frac{\kappa_*}{2n}\|w_*\|_{H^s} \\ &\geq \frac{1}{n}\kappa_*\|w_*\|_{H^s} - \frac{\kappa_*}{2n}\|w_*\|_{H^s} \\ &= \frac{\kappa_*}{2n}\|w_*\|_{H^s}.\end{aligned}$$

By the Lipschitz property of $\gamma^{(1,n)}, \gamma^{(2,n)}$ we have

$$\gamma^{(1,n)}(B_{r_n}(x_*)) \subseteq B(\gamma^{(1,n)}(x_*), \frac{\kappa_*}{8n}\|w_*\|_{H^s}) \quad \text{and} \quad \gamma^{(2,n)}(B_{r_n}(x_*)) \subseteq B(\gamma^{(2,n)}(x_*), \frac{\kappa_*}{8n}\|w_*\|_{H^s}).$$

This means $\vartheta^{(n)}\circ(\gamma^{(1,n)})^{-1}$ is supported in $B(\gamma^{(1,n)}(x_*), \frac{\kappa_*}{8n}\|w_*\|_{H^s})$ and $\vartheta^{(n)}\circ(\gamma^{(2,n)})^{-1}$ is supported in $B(\gamma^{(2,n)}(x_*), \frac{\kappa_*}{8n}\|w_*\|_{H^s})$. Since the distance between the centers of support is larger than $\frac{\kappa*}{2n}\|v\|_{H^s}$ and the radii of the supports are $\frac{\kappa_*}{8n}\|w_*\|_{H^s}$, we can apply (4.3). Thus, combining with (4.7), we obtain

$$\begin{aligned}\|\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1} - \vartheta^{(n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s} &\geq C'(\|\vartheta^{(n)} \circ (\gamma^{(1,n)})^{-1}\|_{H^s} + \|\vartheta^{(n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s}) \\ &\geq \frac{C'}{C}R/2\end{aligned}$$

and therefore we have

$$\limsup_{n\to\infty} \|\theta_0^{(1,n)} \circ (\gamma^{(1,n)})^{-1} - \theta_0^{(2,n)} \circ (\gamma^{(2,n)})^{-1}\|_{H^s} \geq \overline{C}R$$

with $\overline{C}$ independent of $R \in (0, R_*)$ whereas $\lim_{n\to\infty} \|\theta_0^{(1,n)} - \theta_0^{(2,n)}\|_{H^s} = 0$. This holds for every R within the range $(0, R_*)$ and the proof is completed. $\square$

Appendix A. Analyticity in real Banach spaces

The notion of analyticity can be extended to the setting of maps between infinite dimensional spaces. Let E and F be Banach spaces over $\mathbb{C}$ and let $U \subset X$ be an open subset. A map $f : U \to F$ is analytic if for each $x \in U$, $h \in E$ and $y^* \in F^*$ the function $z \to y^*\big(f(x+zh)\big)$ is an analytic function of the complex variable z when $|z|$ is sufficiently small. Consequently, $y^*\big(f(x+zh)\big)$ can be represented locally as a power series in the z variable and the standard Cauchy estimates apply.

Furthermore, if f is locally bounded then f is complex analytic in U if and only if f is Gâteaux differentiable at each point in U.

If the Gâteaux derivative $\partial_h f(x)$ is linear in $h \in E$ and continuous in $x \in U$ then it coincides with the standard Fréchet differential $df(x_0)h$ and f is Fréchet differentiable at x_0.

The fundamental theorem of ordinary differential equations and the implicit function theorem in the analytic setting of Banach spaces are well known. Proofs of both of the propositions below can be found for example in [18] or in the paper of Shnirelman [14].

Proposition A.1 (Fundamental theorem of ODE)**.** *Let X, Y be complex Banach spaces. Consider the equation*

$$\frac{d}{dt}\xi(t) = F(\xi(t), \vartheta), \quad \xi(0) = \xi_0, \tag{A.1}$$

where $t \to \xi(t) \in X$, $\vartheta \in Y$ and $F : X \times Y \to X$ is analytic in a neighborhood of $(\xi_0, \vartheta_0) \in X \times Y$. There exist $r > 0$ and $T > 0$ such that

(i) if $\|\vartheta - \vartheta_0\|_Y < r$, then there exists a unique solution $\xi(t, \vartheta)$ of (A.1) for $|t| < T$;

(ii) the map $(t, \vartheta) \mapsto \xi(t, \vartheta)$ from $(-T, T) \times B_r(\vartheta_0)$ to X is analytic, where $B_r(\vartheta_0)$ denotes the open ball centered at ϑ_0 of radius r.

Proof. See [14], Theorem 2.2 or [18], Chapter 4. □

Proposition A.2. *Let E, F, G be complex Banach spaces, and let $\Phi : E \times F \to G$ be an analytic mapping defined in a neighborhood of a point $(x_0, y_0) \in E \times F$, with $\Phi(x_0, y_0) = z_0$. Suppose that the partial derivative $\frac{\partial \Phi}{\partial x}(x_0, y_0)$ is an invertible linear operator whose image is all of G. Then:*

1. There exists a radius $r > 0$ such that for any $y \in F$ and $z \in G$ satisfying $\|y - y_0\|_F \le r$ and $\|z - z_0\|_G \le r$, the equation $\Phi(x, y) = z$ has a unique solution $x(y, z)$ in a neighborhood of x_0.

2. The function $x(y, z)$ is analytic with respect to (y, z).

A.1. **Inequalities for fractional Sobolev functions.** In this subsection we will establish inequalities of the form

$$\|f + g\| \ge C(\|f\|_s + \|g\|_s)$$

for functions f, g with disjoint support. For fractional s this causes some difficulties as the norm $\|\cdot\|_s$ is defined in a non-local way. For fixed supports we have

Lemma A.3. *Let $s \in \mathbb{R}$. There is a constant $C > 0$ such that for all $f, g \in C_c^\infty(\mathbb{R})$ with* $\operatorname{supp} f \subseteq (-3, -1)$ *and* $\operatorname{supp} g \subseteq (1, 3)$ *we have*

$$\|f + g\|_s^2 \ge C(\|f\|_s^2 + \|g\|_s^2)$$

Proof. We take $\gamma, \psi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\gamma \subseteq (-3.5, -0.5)$ and $\operatorname{supp}\psi \subseteq (0.5, 3.5)$ such that $\gamma|_{(-3,-1)} \equiv 1$ and $\psi|_{(1,3)} \equiv 1$. We then have

$$\|f\|_s = \|\gamma(f + g)\|_s \le C_1\|f + g\|_s$$

and similarly

$$\|g\|_s = \|\psi(f + g)\|_s \le C_2\|f + g\|_s$$

giving the desired result. □

In the following we will use the fact that the H^s-norm is equivalent to the homogeneous $\dot{H}^s$-norm if we restrict ourselves to functions with support in a fixed compact $K \subseteq \mathbb{R}$ (see e.g. [1] p. 39). Recall

$$\|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi$$

We often also use $f^\lambda(x) := f(x/\lambda)$ for which we have the following scaling property

$$\|f^\lambda\|_{\dot{H}^s}^2 = \lambda^{1-2s}\|f\|_{\dot{H}^s}^2$$

We have

Lemma A.4. *Let $s \geq 0$. Then there is a constant $C > 0$ with the following property: For x, y in $\mathbb{R}$ with $0 < r := |x-y|/4 < 1$ we have*

$$\|f+g\|_s^2 \geq C(\|f\|_s^2 + \|g\|_s^2)$$

for all functions $f, g \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \subseteq (x-r, x+r)$, $\operatorname{supp} g \subseteq (y-r, y+r)$

Proof. We use the homogeneous norm. Now scaling with $\lambda = (4r)^{-1}$ gives a situation as in Lemma A.3. We have

$$\|f+g\|_{\dot{H}^s}^2 = \lambda_n^{2s-1}\|f^\lambda + g^\lambda\|_{\dot{H}^s}^2$$

Now by Lemma A.3 we then get

$$\|f+g\|_{\dot{H}^s}^2 \geq C\lambda^{2s-1}\left(\|f^\lambda\|_{\dot{H}^s}^2 + \|g^\lambda\|_{\dot{H}^s}^2\right)$$

Scaling back gives

$$\|f+g\|_{\dot{H}^s}^2 \geq C(\|f\|_{\dot{H}^s}^2 + \|g\|_{\dot{H}^s}^2)$$

This establishes the lemma. □

We will encounter Lemma A.4 also for some negative values of s. In these cases we will use

Lemma A.5. *Let $s < 0$ and the same situation as in Lemma A.4. Then we have*

$$\|f+g\|_s^2 \geq C(\|f\|_s^2 + \|g\|_s^2)$$

for all functions $f, g \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \subseteq (x-r, x+r)$, $\operatorname{supp} g \subseteq (y-r, y+r)$

Proof. We claim that for functions with support in some fixed compact set $K \subseteq \mathbb{R}$ the homogeneous norm

$$\|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi$$

is equivalent to the non-homegeneous norm $\|\cdot\|_s$. One then can argues as in Lemma A.4 by scaling to a fixed situation as in Lemma A.3. So it remains to show the equivalence of the norms. We clearly have $\|\cdot\|_s \leq \|\cdot\|_{\dot{H}^s}$ since

$$\int_{\mathbb{R}} (1+\xi^2)^s|\hat{f}(\xi)|^2 d\xi \leq \int_{\mathbb{R}} \xi^{2s}|\hat{f}(\xi)|^2 d\xi$$

For the other direction we use the dual definition of the Sobolev norm

$$\|f\|_s = \sup_{\|g\|_{-s}\leq 1} |\langle f, g\rangle|$$

and analogously for the homogeneous norm. Taking $\psi \in C_c^\infty(\mathbb{R})$ with $\psi = 1$ on K we have for f with support in K

$$\|f\|_{\dot{H}^s} = \sup_{\|g\|_{\dot{H}^{-s}}\leq 1} |\langle f, \psi\cdot g\rangle|$$

Now note that we have equivalence of the norms $\|\cdot\|_{-s}$ and $\|\cdot\|_{\dot{H}^{-s}}$ for functions with support in some fixed compact. Therefore

$$\begin{aligned}
\|f\|_{\dot{H}^s} &= \sup_{\|g\|_{\dot{H}^s}\leq 1} |\langle f, \psi\cdot g\rangle| \leq \sup_{g, \|\psi\cdot g\|_{-s}\leq C_1} |\langle f, \psi\cdot g\rangle| \\
&\leq C_1 \sup_{\|g\|_{-s}\leq 1} |\langle f, g\rangle| = \|f\|_{-s}
\end{aligned}$$

showing the equivalence. □

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