

# Hausdorff dimensions of Cantor sets in the unit circle

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ABSTRACT. For  $r > 0$ , the Julia set  $\mathcal{J}(z + 1 - r/z)$  is a Cantor set contained in the real line. We show that  $1 - C_1/r \leq \text{H.dim } \mathcal{J}(z + 1 - r/z) \leq 1 - C_2/r$  for  $r$  large, answering a question posed by McMullen.

## 1. Introduction

In [1], C. McMullen examined a one-parameter family of Cantor sets generated by reflections in three equally spaced circles. For  $0 < \theta \leq \frac{2\pi}{3}$ , let  $C_\theta$  be a symmetric configuration of 3 circles, each orthogonal to the unit circle  $S^1 \subset \mathbb{C}$  and intersecting  $S^1$  in an arc of length  $\theta$ . We write  $\Gamma_\theta = \Gamma(C_\theta)$  for the group of isometries of the unit disk  $\Delta$  generated by reflections in these three circles. Under this action, the hyperbolic plane is tiled under this action by copies of a fundamental hexagon with sides alternating along  $C_\theta$  and along  $S^1$ . The tiles accumulate on the limit set  $\Lambda_\theta \subset S^1$ , which is a Cantor set for  $0 < \theta < \frac{2\pi}{3}$  and the full circle for  $\theta = \frac{2\pi}{3}$ . In that work, McMullen derived asymptotic formulas

$$(1) \quad \text{H.dim}(\Lambda_\theta) \simeq \frac{1}{|\log \theta|}, \quad \text{as } \theta \rightarrow 0$$

and

$$(2) \quad \text{H.dim}(\Lambda_\theta) \simeq \sqrt{2\pi/3 - \theta}, \quad \text{as } \theta \rightarrow 2\pi/3.$$

The first formula is obtained by applying the eigenvalue algorithm to an infinite Markov partition, while the second formula uses Sullivan's formula  $\delta(1 - \delta) = \lambda_0(X_\theta)$  which express the Hausdorff dimension of  $\Lambda_\theta$  in terms of the base eigenvalue of the hyperbolic Laplacian on the quotient Riemann surface  $X_\theta = \Delta/\Gamma_\theta$ .

**1.1. Applications to Blaschke products.** As a second application, we discuss the asymptotic behaviour of the Hausdorff dimension of a certain family of degree 2 Blaschke products. A finite Blaschke product  $F$  is a holomorphic self-map of the unit disk which extends to a continuous self-map of the unit circle. It can be expressed as

$$(3) \quad F(z) = e^{i\theta} \prod_{i=1}^d \frac{z - a_i}{1 - \overline{a_i}z}, \quad a_i \in \mathbb{D}.$$

The number  $d$  is called the degree of  $F$ . The points  $a_i$  mark the locations of the zeros of  $F$ . We consider the case when  $d = 2$ . There are four types of degree 2 Blaschke products into four types: hyperbolic, singly-parabolic, doubly-parabolic and elliptic. To estimate dimension of Cantor sets with dimension close to 1, we construct the system of intervals and find a cover to establish the upper bound. For the lower bound, we use Frostman's lemma.

In this paper, we introduce an unified approach to answer an open question in the Remark (p.30) in McMullen's paper [1], which is to find asymptotic formulas for the case of Blaschke products with  $\text{H.dim } J(f)$  near 1, related to the results of Theorem 3.5 and 3.6 in [1].

## 2. Upper bound on the Hausdorff dimension

In this section, we introduce the notion of a system of intervals, which is a convenient framework for describing Cantor sets on the unit circle and the real line. We then establish an upper bound for the Hausdorff dimension of Cantor sets with short gaps.

**2.1. Systems of intervals.** Let  $I$  be the interval  $[0, 1]$ . By a *system of intervals* in  $[0, 1]$  indexed by an alphabet  $A$ , we mean a collection of intervals  $I_{i_1, i_2, \dots, i_n}$  such that for any  $i_1, i_2, \dots, i_{n-1} \in A$ , the intervals  $I_{i_1, i_2, \dots, i_{n-1}, i}$  with  $i \in A$  are disjoint and are contained in  $I_{i_1, i_2, \dots, i_{n-1}}$ .

We define the *limit set*  $\Lambda$  of the above system of intervals as  $\bigcap_{n=1}^{\infty} \Lambda^{(n)}$  where  $\Lambda^{(n)}$  is the union of the intervals of generation  $n$ . The *scaling factors*

$$0 < \rho_{i_1, i_2, \dots, i_{n-1}, i_n} < 1$$

are defined by the relation

$$|I_{i_1, i_2, \dots, i_n}| = \rho_{i_1 i_2 i_3 \dots i_n} \cdot \dots \cdot \rho_{i_1 i_2 i_3} \cdot \rho_{i_1 i_2} \cdot \rho_{i_1},$$

where  $|I|$  denotes the length of an interval  $I$ .

**EXAMPLE (Linear systems).** A *linear iterated function system* (or a linear IFS) on  $I$  is determined by a collection of proper disjoint sub-intervals  $I_1, I_2, \dots$ , indexed by an alphabet  $A$ , which may be finite or countable. To each interval  $I_i$  of generation 1, we associate an orientation-preserving linear map  $\phi_i : I \rightarrow I_i$ . Since each  $I_i$  is strictly contained in  $I$ , every map  $\phi_i$  is a contraction.

For a finite word  $\xi = i_1 i_2 \dots i_n$  in the alphabet  $A$ , we associate the interval  $I_\xi = I_{i_1 i_2 \dots i_n}$ , which is defined as the image of the composition  $\phi_{i_n} \circ \dots \circ \phi_{i_2} \circ \phi_{i_1}$ . We say that  $I_\xi$  is an interval of *generation n*.

If  $\omega = i_1 i_2 \dots$  is an infinite word, the intervals  $I_{i_1} \supset I_{i_1 i_2} \supset I_{i_1 i_2 i_3} \supset \dots$  are nested. Since the maps  $\{\phi_i\}$  are contractions, the intersection of these intervals is a single point, which we denote by  $x_\omega$ . The *limit set*  $\Lambda$  of the IFS consists of points  $x_\omega$  as  $\omega$  ranges over all infinite words. Alternatively,  $\Lambda = \bigcap_{n=1}^{\infty} \Lambda^{(n)}$ , where  $\Lambda^{(n)}$  is the union of intervals of generation  $n$ .

**2.2. Tame systems.** While linear iterated function systems are especially simple from a dynamical point of view, many interesting Cantor sets in the unit circle or the real line cannot be expressed as limit sets of a linear IFS. We now consider a more flexible notion. We say that a system of intervals is *tame* if there exists constants  $0 < \kappa < 1$  and  $M, K_1, K_2 > 0$  such that for any interval  $I_{i_1 i_2 \dots i_{n-1}}$  of generation  $n-1$ , we have

$$(4) \quad K_1 \varepsilon \leq 1 - \sum_{i_n \in A} \rho_{i_1, i_2, \dots, i_{n-1}, i_n} \leq K_2 \varepsilon < 1/2,$$

$$(5) \quad \sum_{i_n \in A} \rho_{i_1, i_2, \dots, i_{n-1}, i_n}^\kappa \leq M.$$

The first assumption says that the total length of the gaps is small, while the second assumption says that the lengths of the intervals do not decay too slowly. Define the *critical exponent*  $\alpha^{(n)}$  of generation  $n$  as the unique real number  $0 < \alpha < 1$  such that

$$\sum_{\xi \in A^n} |I_\xi|^\alpha = 1.$$

In the following theorem, we estimate the critical exponent for a tame system from above and below.

**THEOREM 1.** *Consider a tame systems of intervals. There exists constants  $c_1, c_2 > 0$  depending only on  $\kappa, M, K_1, K_2$  so that*

$$(6) \quad 1 - c_1 \varepsilon < \alpha^{(n)} < 1 - c_2 \varepsilon,$$

for any  $n \geq 1$ . In particular,  $\text{H.dim } \Lambda \leq 1 - c_2 \varepsilon$ .

To prove the theorem, we first consider the special case when  $n = 1$ . Afterwards, we will deduce the theorem from the special case using induction.

**LEMMA 1.** *Let  $\{\ell_i\}_{i=1}^{\infty}$  be a sequence of positive numbers in  $(0, 1)$  such that*

$$(7) \quad \sum_{i \in A} \ell_i^\kappa < M,$$

for some  $0 < \kappa < 1$  and  $M > 0$ . Suppose that the total length of the gaps satisfies

$$(8) \quad K_1 \varepsilon \leq 1 - \sum_{i \in A} \ell_i \leq K_2 \varepsilon < 1/2,$$

for some constants  $K_1, K_2 > 0$ . Then, the number  $0 < \alpha < 1$  given by

$$\sum_{i \in A} \ell_i^\alpha = 1$$

satisfies

$$(9) \quad 1 - c_1 \varepsilon < \alpha < 1 - c_2 \varepsilon$$

for some constants  $c_1, c_2 > 0$ , which depend on  $M, K_1, K_2$ .

An important feature in the above the lemma is that the constants  $c_1, c_2 > 0$  can be chosen to be independent of  $\varepsilon > 0$ .

PROOF. Without loss of generality, we may assume that  $\ell_1 \geq \ell_2 \geq \ell_3 \geq \dots$  are arranged in decreasing order. By (8), we have

$$1/2 < \sum_{i=1}^{\infty} \ell_i < 1,$$

which together with (7) shows that  $\ell_1$  is bounded below by a positive constant which depends on  $M$ .

(<) To show the lower bound in (9), it is enough to find a  $c_1 > 0$  sufficiently large so that

$$(10) \quad \sum_{i \in A} \ell_i^{1-c_1 \varepsilon} > 1.$$

Since

$$\sum_{i \in A} \ell_i^{1-c_1 \varepsilon} \geq (\ell_1^{1-c_1 \varepsilon} - \ell_1) + \sum_{i \in A} \ell_i \geq (\ell_1^{1-c_1 \varepsilon} - \ell_1) + (1 - K_2 \varepsilon),$$

we need to choose  $c_1 > 0$  so that

$$\ell_1(\ell_1^{-c_1 \varepsilon} - 1) = \ell_1^{1-c_1 \varepsilon} - \ell_1 \geq K_2 \varepsilon \quad \text{or} \quad (1/\ell_1)^{c_1 \varepsilon} - 1 \geq (K_2/\ell_1) \varepsilon.$$

As  $1/\ell_1 > 1$ , when  $c_1 > 0$  is small,

$$(1/\ell_1)^{c_1 \varepsilon} = e^{c_1 \log(1/\ell_1) \varepsilon} = 1 + c_1 \log(1/\ell_1) \varepsilon + O(\varepsilon^2).$$

Therefore, (10) will be satisfied if  $c_1 > 2 \cdot \frac{K_2}{\ell_1 \log(1/\ell_1)}$  and  $\varepsilon > 0$  is sufficiently small.

(>) To show the upper bound in (9), it is enough to find a  $c_2 > 0$  sufficiently small so that

$$(11) \quad \sum_{i \in A} \ell_i^{1-c_2 \varepsilon} < 1.$$

Since  $\sum_{i \in A} \ell_i < 1 - K_1 \varepsilon$ , is enough to show that

$$(12) \quad \sum_{i \in A} \{\ell_i^{1-c_2 \varepsilon} - \ell_i\} < K_1 \varepsilon.$$

Let us write

$$\sum_{i \in A} \{\ell_i^{1-c_2 \varepsilon} - \ell_i\} = \sum_{i \in A} \ell_i^\kappa \cdot \frac{\ell_i^{1-c_2 \varepsilon} - \ell_i}{\ell_i^\kappa}.$$

Some calculus shows that

$$\frac{x^{1-c_2\varepsilon} - x}{x^\kappa} \leq C \cdot c_2\varepsilon, \quad x \in [0, 1],$$

for some constant  $C > 0$  depending on  $\kappa$ . Therefore,

$$\sum_{i \in A} \{\ell_i^{1-c_2\varepsilon} - \ell_i\} \leq C c_2 \varepsilon \sum_{i \in A} \ell_i^\kappa \leq C M c_2 \varepsilon,$$

so (12) holds for  $c_2 < \frac{K_1}{MC}$ .  $\square$

PROOF OF THEOREM 1. Without loss of generality, we assume that for any  $i_1 i_2 \dots i_{n-1} \in A^{n-1}$ , the sub-intervals of generation  $n$  are ordered by decreasing length:

$$\ell_{i_1 i_2 i_3 \dots i_{n-1} 1} \geq \ell_{i_1 i_2 i_3 \dots i_{n-1} 2} \geq \ell_{i_1 i_2 i_3 \dots i_{n-1} 3} \geq \dots$$

The assumption

$$1/2 < \sum_{k=1}^{\infty} \rho_{i_1 i_2 i_3 \dots i_{n-1} k} < 1,$$

together with (5) shows that  $\rho_{i_1 i_2 i_3 \dots i_{n-1} 1}$  is bounded below by a positive constant which depends on  $M$ .

( $<$ ) To show the lower bound in (6), it is enough to find a  $c_1 > 0$  sufficiently large so that

$$(13) \quad \sum_{i_1 i_2 \dots i_n \in A^n} \ell_{i_1 i_2 i_3 \dots i_n}^{1-c_1\varepsilon} > 1.$$

By the definition of the scaling factors, we have

$$\sum_{i_1 i_2 \dots i_n \in A^n} \ell_{i_1 i_2 i_3 \dots i_n}^{1-c_1\varepsilon} = \sum_{i_1 i_2 \dots i_{n-1}} \ell_{i_1 i_2 \dots i_{n-1}}^{1-c_1\varepsilon} \sum_{i_n} \rho_{i_1, i_2, \dots, i_{n-1}, i_n}^{1-c_1\varepsilon}.$$

In view of (10), one can find a  $c_1 > 0$  so that the inner sum is  $> 1$ . Crucially, the constant  $c_1$  only depends on  $M, K_1, K_2$  but not on  $\varepsilon$  nor on the finite word  $i_1 i_2 \dots i_{n-1}$ . Thus,

$$\sum_{i_1 i_2 \dots i_n \in A^n} \ell_{i_1 i_2 i_3 \dots i_n}^{1-c_1\varepsilon} > \sum_{i_1 i_2 \dots i_{n-1} \in A^{n-1}} \ell_{i_1 i_2 i_3 \dots i_{n-1}}^{1-c_1\varepsilon} > \sum_{i_1 i_2 \dots i_{n-2} \in A^{n-2}} \ell_{i_1 i_2 i_3 \dots i_{n-2}}^{1-c_1\varepsilon} > \dots > 1.$$

( $>$ ) To show the upper bound in (6), it is enough to find a  $c_2 > 0$  sufficiently small so that

$$(14) \quad \sum_{i_1 i_2 \dots i_n \in A^n} \ell_{i_1 i_2 i_3 \dots i_n}^{1-c_2\varepsilon} < 1.$$

By the definition of the scaling factors and (11), we have

$$\sum_{i_1 i_2 \dots i_n \in A^n} \ell_{i_1 i_2 i_3 \dots i_n}^{1-c_2\varepsilon} = \sum_{i_1 i_2 \dots i_{n-1}} \ell_{i_1 i_2 \dots i_{n-1}}^{1-c_2\varepsilon} \sum_{i_n} \rho_{i_1, i_2, \dots, i_{n-1}, i_n}^{1-c_2\varepsilon} < \sum_{i_1 i_2 \dots i_{n-1}} \ell_{i_1 i_2 \dots i_{n-1}}^{1-c_2\varepsilon}.$$

Continuing inductively, we see that

$$\sum_{i_1 i_2 \dots i_n \in A^n} \ell_{i_1 i_2 i_3 \dots i_n}^{1-c_2\varepsilon} < \sum_{i_1 i_2 \dots i_{n-1} \in A^{n-1}} \ell_{i_1 i_2 i_3 \dots i_{n-1}}^{1-c_2\varepsilon} < \sum_{i_1 i_2 \dots i_{n-2} \in A^{n-2}} \ell_{i_1 i_2 i_3 \dots i_{n-2}}^{1-c_2\varepsilon} < \dots < 1,$$

which is what we wanted to show.

Since for any  $n \geq 1$ , the collection of intervals  $I_{i_1 i_2 \dots i_n}$  of generation  $n$  covers  $\Lambda$ , and the largest interval of generation  $n$  tends to 0 as  $n \rightarrow \infty$ ,

$$\text{H. dim } \Lambda \leq \sup_{n \geq 1} \alpha^{(n)} \leq 1 - c_2 \varepsilon,$$

as desired.  $\square$

### 3. Lower bound on the Hausdorff dimension

In this section, we give a lower bound for the Hausdorff dimensions of limit sets of some special systems of intervals that can code limit sets of parabolic dynamical systems.

**THEOREM 2.** *Suppose  $\Lambda$  is a system of intervals where each interval has countably many children, ordered by the positive integers. Assume that*

- (1) *There exists constants  $N_\varepsilon > 0$  and  $c_1 > 0$  so that for any interval  $I_{i_1 i_2 \dots i_{n-1}}$  in the system,*

$$(15) \quad \sum_{i=1}^{N_\varepsilon} |\rho_{i_1 i_2 \dots i_n}|^{1-c_1 \varepsilon} \geq 1.$$

- (2) *There exists a constant  $\theta_\varepsilon > 0$  so that for any two adjacent intervals  $I_n, I'_n \subset I_{n-1}$  of generation  $n$  contained in an interval of generation  $n-1$ ,*

$$\text{dist}(I_n, I'_n) \geq \theta_\varepsilon \cdot \max(|I_n|, |I'_n|).$$

*Then, there exists a probability measure  $\mu$  supported on  $\Lambda$  so that  $\mu(J) \lesssim |J|^{1-c_1 \varepsilon}$  for any interval  $J$  centered at a point of  $\Lambda$ .*

In view of Frostman's theorem, the above theorem says that  $\text{H. dim } \Lambda \geq 1 - c_1 \varepsilon$ .

**PROOF.** *Step 0.* We build the measure  $\mu$  as the weak-\* limit of a sequence of probability measures  $\mu_n$  which satisfy the following two conditions:

- Each measure  $\mu_n$  is a constant multiple of the Lebesgue measure on each interval  $I$  of generation  $n$ :

$$\mu_n = \sum_I a_I \cdot dm|_I.$$

- Furthermore, the measures  $\mu_n$  stabilize in the following sense: for any interval  $I$  of generation  $n$ , we have

$$(16) \quad \mu_n(I) = \mu_{n+1}(I) = \mu_{n+2}(I) = \dots.$$

To define the measure  $\mu_1$ , we specify it on intervals of generation 1:

$$(17) \quad \mu_1(I_i) = \begin{cases} \frac{|I_i|^{1-c_1 \varepsilon}}{\sum_{j=1}^{N_\varepsilon} |I_j|^{1-c_1 \varepsilon}}, & i = 1, 2, \dots, N_\varepsilon, \\ 0, & i > N_\varepsilon. \end{cases}$$

More generally, we define the measure  $\mu_n$  so that

$$(18) \quad \mu_n(I_{i_1 i_2 \dots i_{n-1} i_n}) = \mu_{n-1}(I_{i_1 \dots i_{n-1}}) \cdot \frac{|I_{i_1 i_2 \dots i_{n-1} i_n}|^{1-c_1\varepsilon}}{\sum_{j=1}^{N_\varepsilon} |I_{i_1 i_2 \dots i_{n-1} j}|^{1-c_1\varepsilon}}, \quad \text{for } i_n = 1, 2, \dots, N_\varepsilon$$

and  $\mu_n(I_{i_1 i_2 \dots i_{n-1} i_n}) = 0$  if  $i_n > N_\varepsilon$ . Notice that if  $I_{i_1 \dots i_{n-1}}$  is an interval of generation  $n-1$ , then

$$\sum_{i_n} \mu_n(I_{i_1 \dots i_{n-1} i_n}) = \sum_{i_n} \mu_{n-1}(I_{i_1 \dots i_{n-1}}) \frac{|I_{i_1 i_2 \dots i_{n-1} i_n}|^{1-c_1\varepsilon}}{\sum_{j=1}^{N_\varepsilon} |I_{i_1 i_2 \dots i_{n-1} j}|^{1-c_1\varepsilon}} = \mu_{n-1}(I_{i_1 \dots i_{n-1}}),$$

so the stability condition (16) is satisfied.

*Step 1.* In this step, we verify that the Frostman condition  $\mu_1(J) \lesssim |J|^\alpha$  holds for the measure  $\mu_1$ . We consider several cases:

**Case IA.**  $J = I_{i_1}$ . By definition of  $\mu_1$ , we have  $\mu_1(I_{i_1}) \lesssim |I_{i_1}|^{1-c_1\varepsilon}$  for any interval  $I_{i_1}$ .

**Case IB.**  $J \subset I_{i_1}$ . Since  $I_{i_1}$  satisfies the Frostman condition by Case 1A,

$$\mu_1(J) = \mu_1(I_{i_1}) \frac{|J|}{|I_{i_1}|} \lesssim |I_{i_1}|^{1-c_1\varepsilon} \frac{|J|}{|I_{i_1}|} \lesssim |J|^{1-c_1\varepsilon}.$$

**Case IC.** We denote the left endpoint of an interval  $I$  by  $I^{\text{left}}$  and the right endpoint by  $I^{\text{right}}$ . We now verify the Frostman condition for the intervals  $J_{m,n} = [I_m^{\text{left}}, I_n^{\text{right}}]$  with  $1 \leq m < n \leq N_\varepsilon$ :

$$\mu_1(J_{m,n}) = \sum_{j=m}^n \mu_1(I_j) \leq \sum_{j=m}^n |I_j|^{1-c_1\varepsilon} \leq N_\varepsilon |J_{m,n}|^{1-c_1\varepsilon}.$$

**Case ID.** We now consider the case when  $J = [J^{\text{left}}, J^{\text{right}}]$  is an interval in  $[0, 1]$ , contained in  $J_{m,n}$  for some  $m < n$  and intersecting both  $I_m$  and  $I_n$ . By Assumption (2), the length of the interval  $[I_m^{\text{left}}, J^{\text{left}}]$  is bounded by a constant (depending on  $\varepsilon$ ) times the length of the gap between  $I_m$  and  $I_{m+1}$ , while the length of the interval  $[J^{\text{right}}, I_n^{\text{right}}]$  is bounded by a constant (depending on  $\varepsilon$ ) times the length of the gap between  $I_{n-1}$  and  $I_n$ . Consequently,  $|J| \simeq |J_{m,n}|$  and

$$\mu(J) \leq \mu(J_{m,n}) \lesssim |J_{m,n}|^{1-c_1\varepsilon} \simeq |J|^{1-c_1\varepsilon}.$$

*Step 2.* We now inductively check the Frostman condition for the measures  $\mu_N$  with  $N > 1$ . We consider the following cases:

- A.  $J = I_{i_1 \dots i_{N-1} i_N}$ ,
- B.  $J \subset I_{i_1 \dots i_N}$ ,
- C.  $J = J_{m,n} = [I_{i_1 \dots i_{N-1} m}^{\text{left}}, I_{i_1 \dots i_{N-1} n}^{\text{right}}]$ ,
- D.  $J \subset J_{m,n} = [I_{i_1 \dots i_{N-1} m}^{\text{left}}, I_{i_1 \dots i_{N-1} n}^{\text{right}}]$ .

**Case IIA.** We now check the Frostman condition for the intervals  $I_{i_1 i_2 \dots i_n}$  of generation  $n$ . We may assume  $1 \leq i_n \leq N_\varepsilon$ , otherwise  $\mu_n(I_{i_1 i_2 \dots i_n}) = 0$  and the Frostman condition is trivially satisfied. We need to check that

$$\mu_n(I_{i_1 i_2 \dots i_{n-1} i_n}) \lesssim |I_{i_1 i_2 \dots i_{n-1} i_n}|^{1-c_1\varepsilon}$$

or

$$\mu_{n-1}(I_{i_1 \dots i_{n-1}}) \frac{|I_{i_1 i_2 \dots i_{n-1} i_n}|^{1-c_1\varepsilon}}{\sum_{j=1}^{N_\varepsilon} |I_{i_1 i_2 \dots i_{n-1} j}|^{1-c_1\varepsilon}} \lesssim |I_{i_1 i_2 \dots i_{n-1}}|^{1-c_1\varepsilon} \frac{|I_{i_1 i_2 \dots i_{n-1} i_n}|^{1-c_1\varepsilon}}{|I_{i_1 i_2 \dots i_{n-1}}|^{1-c_1\varepsilon}}.$$

However, the last equation holds by induction and (15).

**Cases IIB, IIC, IID.** These cases can be handled similarly to Cases IB, IC and ID respectively.

Having verified the Frostman condition for an arbitrary interval  $J \subset [0, 1]$ , the proof is complete.  $\square$

#### 4. Dimensions of Schottky groups

For  $x, y \in \mathbb{R}$ , we write  $\gamma(x, y)$  for the hyperbolic geodesic joining  $x$  and  $y$ . For  $\varepsilon > 0$ , let  $\Gamma_\varepsilon$  be the Fuchsian group acting on the upper half-plane generated by reflections in the hyperbolic geodesics  $C_0 = \gamma(0, 1)$ ,  $C_L = \gamma(\varepsilon, 1 - \varepsilon/2)$  and the semicircle  $C_R = \gamma(1 + \varepsilon/2, 1)$ . In this section, we apply the techniques developed in Sections 2 and 3 to show:

**THEOREM 3.** As  $\varepsilon \rightarrow 0^+$ ,

$$1 - \text{H. dim } \Lambda_\varepsilon \simeq \sqrt{\varepsilon}.$$

**4.1. Fundamental domains.** In this section, we describe limit sets of Schottky groups generated by three equally-spaced circles in terms of systems of intervals and recover the upper bound of McMullen's dimension estimate. Recall from the introduction that  $\Gamma = \langle R_{c_1}, R_{c_2}, R_{c_3} \rangle$  is the group of isometries of the unit disk  $\Delta$  generated by reflections in the three circles. The elements in  $\Gamma$  can be written as the product of the generators' elements. There is a correspondence between the orbit of 0 and the dual tree. The root vertex is at 0. The other vertices in  $\{\Gamma 0\}$  are labeled by the path from 0 along the tree. There are two natural systems of intervals which one can associate with the limit set  $\Lambda_\theta$ . The hyperbolic system every interval has 2 children. For our purposes, the hyperbolic system described above is not adequate, because it does not satisfy the estimate of Theorem 2. However, in the parabolic system the way describe the interval of generation 1 is a little bit different. Let  $I$  be  $[0, 1]$  and its children  $I_k$ , for  $k \in \mathbb{Z} \setminus \{0\}$ . For  $k > 0$ , we have words of this kind  $\{1L^{k-1}R\}$ . For  $k < 0$ , we have words of the type  $\{1R^{k-1}L\}$ . This defines the interval of generation 1. For the generation 2,  $I_{k-1} = 1L^{k-1}R$  children are of the type  $I_{k-1,m} = 1L^{k-1}R^mL$  for  $m > 1$ . The first interval  $[0, 1]$ , the children is indexed by  $k \in \mathbb{Z} \setminus \{0\}$ . The intervals  $I_{k_1, k_2, \dots, k_n}$  of generation  $n$  with  $k_1 \in \mathbb{Z} \setminus \{0\}$  and  $k_2, k_3, \dots, k_n > 1$ .

**4.2. Systems of intervals.** The words  $\Gamma$  of length  $n$  has the form  $\{1, 2, 3\} \times \{L, R\}^{n-1}$ , with the convention that the empty word has the length 0. In the hyperbolic system, the generation of an element is just equal to  $n$ , which is the number of letters described that element. However, in the parabolic system the way describe the interval of generation 1 is a little bit different. Let  $I$  be  $[0, 1]$  and its children  $I_k$ , for  $k \in \mathbb{Z} \setminus \{0\}$ . For  $k > 0$ , we have words of this kind  $\{1L^{k-1}R\}$ . For  $k < 0$ , we have words of the type  $\{1R^{k-1}L\}$ . This defines the interval of generation 1. For the generation 2,  $I_{k-1} = 1L^{k-1}R$  children are of the type  $I_{k-1,m} = 1L^{k-1}R^mL$  for  $m > 1$ . The first interval  $[0, 1]$ , the children is indexed by  $k \in \mathbb{Z} \setminus \{0\}$ . The intervals  $I_{k_1, k_2, \dots, k_n}$  of generation  $n$  with  $k_1 \in \mathbb{Z} \setminus \{0\}$  and  $k_2, k_3, \dots, k_n > 1$ .

In view of Theorem 2, to prove the lower bound for the Hausdorff dimension of  $\Lambda(\Gamma_\varepsilon)$  in Theorem 3, it is enough to show:

LEMMA 2. *There exists a constant  $c_1 > 0$  independent of  $\varepsilon$  such that*

$$\sum_{i=1}^{\infty} \frac{|I_{j_1 j_2 \dots j_{n-1} i}|^{1-c_1 \sqrt{\varepsilon}}}{|I_{j_1 j_2 \dots j_{n-1}}|^{1-c_1 \sqrt{\varepsilon}}} > 1, \quad \text{for any } j_1, j_2, \dots, j_{n-1} \geq 1.$$

*In fact, one can choose  $N_\varepsilon \geq 1$  sufficiently large so that*

$$\sum_{i=1}^{N_\varepsilon} \frac{|I_{j_1 j_2 \dots j_{n-1} i}|^{1-c_1 \sqrt{\varepsilon}}}{|I_{j_1 j_2 \dots j_{n-1}}|^{1-c_1 \sqrt{\varepsilon}}} > 1,$$

*where  $N_\varepsilon$  depends only on  $\varepsilon$ .*

To deduce the upper bound  $1 - \text{H. dim } \Lambda_\varepsilon \lesssim \sqrt{\varepsilon}$  from Theorem 3, we prove the following lemma

LEMMA 3. *For any  $\delta > 0$ ,*

$$(19) \quad |I_i^{(\varepsilon)}| < C/n^{2-\delta},$$

*for every sufficiently small  $\varepsilon > 0$ .*

COROLLARY 1. *For any  $1/2 < \kappa < 1$ , there exists an  $M > 0$  so that*

$$(20) \quad \sum_{i \in \mathbb{N}} |I_i^{(\varepsilon)}|^\kappa < M,$$

*for every sufficiently small  $\varepsilon > 0$ .*

LEMMA 4 (Uniform polynomial decay). *For any  $\delta > 0$ , there exists a constant  $C > 0$  such that for all sufficiently small  $\varepsilon > 0$ , all generations  $n \geq 1$ , and all words  $w = j_1 j_2 \dots j_{n-1}$  (including the empty word for  $n = 1$ ), the lengths of the children intervals satisfy:*

$$\frac{|I_{wi}|}{|I_w|} \leq \frac{C}{i^{2-\delta}},$$

*for all  $i \geq 1$ . In particular, for the first generation ( $w$  empty), we have:*

$$|I_i| \leq \frac{C}{i^{2-\delta}}.$$

*Moreover, the constant  $C$  is independent of  $\varepsilon$ ,  $n$ , and the choice of the parent word  $w$ .*

**4.3. Background on the Möbius transformation  $m_1$ .** Let  $m_1 \in \text{Aut } \mathbb{H}$  be the Möbius transformation which maps the fundamental domain  $\Delta_{\text{root}}^*$  to  $\Delta_{\text{root}}$  and satisfies  $m_1(C) = C_L$ . It is not difficult to see that  $m_1$  permutes the tiles in the partition of  $\mathbb{H}$ . Inspection shows that  $m_1$  maps

$$\dots \rightarrow \Delta_{L^2}^* \rightarrow \Delta_L^* \rightarrow \Delta_\emptyset^* \rightarrow \Delta_\emptyset \rightarrow \Delta_L \rightarrow \Delta_{L^2} \rightarrow \dots$$

Consequently,  $m_1$  is a hyperbolic Möbius transformation with an attracting fixed point at

$$a = a_\varepsilon = \lim_{k \rightarrow \infty} \Delta_{0,L^k} \in (0, 1)$$

and a repelling fixed point at

$$r = r_\varepsilon = \lim_{k \rightarrow \infty} \Delta_{-1,R^k} \in (-1, 0).$$

Since  $m_1$  maps 0 to  $\varepsilon$ , 1 to  $\frac{1-\varepsilon}{2}$  and  $-\frac{\varepsilon}{1-2\varepsilon}$  to 0, it is given by

$$(21) \quad m_1(z) = \frac{(1-\varepsilon)^2 z + \varepsilon(1-\varepsilon)}{(1+\varepsilon)z + 1-\varepsilon}.$$

Differentiating, we get

$$(22) \quad m'_1(z) = \frac{(1-\varepsilon)(1-3\varepsilon)}{((1+\varepsilon)z + 1-\varepsilon)^2}.$$

Some arithmetic shows that

$$(23) \quad r = \frac{-\varepsilon(1-\varepsilon) - \sqrt{\varepsilon(1-\varepsilon)(4+5\varepsilon-\varepsilon^2)}}{2(1+\varepsilon)} \sim -\sqrt{\varepsilon},$$

$$(24) \quad a = \frac{-\varepsilon(1-\varepsilon) + \sqrt{\varepsilon(1-\varepsilon)(4+5\varepsilon-\varepsilon^2)}}{2(1+\varepsilon)} \sim \sqrt{\varepsilon}$$

and

$$(25) \quad \begin{aligned} m'_1(a) &= \frac{4(1-\varepsilon)(1-3\varepsilon)}{(2-3\varepsilon+\varepsilon^2+s)^2}, \\ m'_1(r) &= \frac{4(1-\varepsilon)(1-3\varepsilon)}{(2-3\varepsilon+\varepsilon^2-s)^2}. \end{aligned}$$

For small  $\varepsilon > 0$ , we have

$$(26) \quad m'_1(a) \sim 1 - 2\sqrt{\varepsilon} + O(\varepsilon), \quad m'_1(r) \sim 1 + 2\sqrt{\varepsilon} + O(\varepsilon)$$

and

$$(27) \quad \lambda_\varepsilon = m'_1(r_\varepsilon) \approx 1 + 2\sqrt{\varepsilon},$$

as  $\varepsilon \rightarrow 0$ .

For  $n \in \mathbb{Z}$ , we write  $m_n(z) := m_1^{\circ n}(z)$ . The following lemma will be useful when applying Koebe's distortion theorem:

LEMMA 5. *For any  $n \geq 1$ , the pole of  $m_n$  lies on  $(-\infty, 0)$ . In particular,  $m_n$  is holomorphic on the right half-plane  $\{\operatorname{Re} z > 0\}$ .*

PROOF. Let  $P_n$  be the location of the pole of  $m_n$ . Since  $m_n$  is an automorphism of the upper half-plane, it is symmetric with respect to the real axis, and therefore,  $P_n \in \mathbb{R}$ . As  $m_1$  is decreasing on  $\mathbb{R} \setminus [r, a]$ , its pole  $P_1$  belongs to  $(-\infty, r)$ . The same reasoning shows that  $P_n = m_1^{-(n-1)}(P_1)$  also belongs to  $(-\infty, r)$ .  $\square$

**4.4. Estimates on sizes of intervals and gaps in generation 1.** By a *gap* of generation 1, we mean a connected component of the complement  $[0, 1] \setminus \bigcup_{k=1}^{\infty} I_k$ . We label the gaps of generation 1 by  $E_0, E_1, E_2, E_3, \dots$  and  $E_{\infty}$ , where  $E_k$  is the gap between the intervals  $I_k$  and  $I_{k+1}$ ,  $E_0 = [1 - \varepsilon, 1]$  is the gap on the right which contains the point 1 and  $E_{\infty} = [0, a]$  is the gap on the left which contains the point 0. The following lemma describes the sizes of the intervals  $I_i$  and the gaps  $E_i$  of generation 1:

LEMMA 6. *For any  $0 < i < \infty$ , we have  $|E_i| \simeq \varepsilon |I_i|$ , while  $|E_0| \simeq \varepsilon$ ,  $|E_{\infty}| \simeq \sqrt{\varepsilon}$ .*

PROOF. The estimate  $|E_{\infty}| \simeq \sqrt{\varepsilon}$  is clear, while the estimate  $|E_0| \simeq \varepsilon$  follows from (25). Since  $I_1 = [\frac{1+\varepsilon}{2}, 1 - \varepsilon]$  and  $E_1 = [\frac{(1-\varepsilon)^2}{3-5\varepsilon}, \frac{1+\varepsilon}{2}]$ , we have  $|I_1| \simeq 1$  and  $|E_1| \simeq \varepsilon$  respectively. Notice that the intervals  $I_1$  and  $E_1$  lie inside the ball  $B(1, \frac{4}{5})$ . Since  $m_{n-1}$  is conformal on the  $B(1, \frac{4}{5})$  by Lemma 5, by Koebe's distortion theorem,

$$\frac{|E_n|}{|I_n|} \simeq \frac{|E_1|}{|I_1|} \simeq \varepsilon$$

as desired.  $\square$

We will need the following elementary estimate:

LEMMA 7. *There exists a constant  $m > 0$  so that*

$$x^{1-\delta} > x + m\delta x$$

for any  $0 < x < 3/4$  and  $0 < \delta < 1/2$ .

PROOF. Let  $f(x) := x^{1-\delta} - x$ . For  $1/4 < x < 3/4$  and  $0 < \delta < 1/2$ , we have

$$(28) \quad (4/3)^{\delta} \leq x^{-\delta} \leq 4^{\delta}.$$

Consequently,  $x^{-\delta} \geq (4/3)^{\delta} = e^{\delta \log(4/3)} = 1 + \log(4/3)\delta + o(\delta)$  and

$$x^{1-\delta} - x = x(x^{-\delta} - 1) \geq \frac{1}{4}(\log(4/3) + o(\delta))\delta.$$

The lemma now follows from a compactness argument. Similarly, we have

$$(29) \quad x^{-\delta} = e^{-\delta \log(x)} = 1 - \delta \log(x) + o(\delta \log(x))$$

We obtain

$$\begin{aligned} x^{1-\delta} - x - m\delta x &= x(x^{-\delta} - 1 - m\delta) \\ &= x(-\delta \log x + o(\delta \log(x)) - m\delta) \\ &\geq 0 \end{aligned}$$

for  $\delta$  is small enough.  $\square$

LEMMA 8. *There exists a constant  $c_1 > 0$ , which does not depend on  $\varepsilon$ , such that*

$$\sum_{i=1}^{\infty} |I_i|^{1-c_1\sqrt{\varepsilon}} > 1.$$

Consequently, there exists a pair of constants  $c_1, N_\varepsilon > 0$  such that

$$\sum_{i=1}^{N_\varepsilon} |I_i|^{1-c_1\sqrt{\varepsilon}} > 1,$$

where  $N_\varepsilon$  may depend on  $\varepsilon > 0$ .

PROOF. Since the union of the intervals and gaps covers  $[0, 1]$ ,

$$\sum_{i \in \mathbb{N}} (|I_i| + |E_i|) + (|E_0| + |E_\infty|) = 1.$$

By Lemmas 6 and 7, we can choose  $c_1 > 0$  sufficiently large so that

$$|I_i|^{1-c_1\sqrt{\varepsilon}} \geq |I_i| + mc_1\sqrt{\varepsilon} \cdot |I_i| \geq |I_i| + |E_i|.$$

Summing over  $i$  and using Lemma 7, we get

$$\begin{aligned} \sum_{i \in \mathbb{N}} |I_i|^{1-c_1\sqrt{\varepsilon}} &\geq |I_1|^{1-c_1\sqrt{\varepsilon}} + \sum_{2 \leq i < \infty} (|I_i| + |E_i|) \\ &= 1 + (|I_1|^{1-c_1\sqrt{\varepsilon}} - |I_1|) - (|E_0| + |E_1| + |E_\infty|) \\ &> 1 + mc_1\sqrt{\varepsilon} - (|E_0| + |E_1| + |E_\infty|) \\ &> 1 + mc_1\sqrt{\varepsilon} - c\sqrt{\varepsilon} \\ &> 1, \end{aligned}$$

provided that  $c_1 > \frac{c}{m}$ .  $\square$

#### 4.5. Lemma 2.

PROOF OF LEMMA 2. Recall that we are working with the parabolic system of intervals associated with the Schottky group  $\Gamma_\varepsilon$ . The intervals are labeled as  $I_{j_1, j_2, \dots, j_n}$  for  $j_k \geq 1$ , and the scaling factors are defined by

$$|I_{j_1, \dots, j_n}| = \rho_{j_1, \dots, j_n} \cdot |I_{j_1, \dots, j_{n-1}}| \cdots \rho_{j_1}.$$

The goal is to verify that this system satisfies the conditions of Theorem 1, which requires the system to be *tame*. Specifically, we recall that we need to check

(4) There exist constants  $K_1, K_2 > 0$  such that for any word  $w = j_1 \dots j_{n-1}$ ,

$$K_1\sqrt{\varepsilon} \leq 1 - \sum_{i \in \mathbb{N}} \rho_{wi} \leq K_2\sqrt{\varepsilon} < \frac{1}{2}.$$

(5) There exist constants  $0 < \kappa < 1$  and  $M > 0$  such that for any word  $w$ ,

$$\sum_{i \in \mathbb{N}} \rho_{wi}^\kappa \leq M.$$

These conditions must hold uniformly for all generations and for all sufficiently small  $\varepsilon > 0$ .

For a fixed parent interval  $I_w$ , the children intervals are  $I_{wi}$  for  $i \geq 1$ . The total gap within  $I_w$  is the complement of the union of these children:

$$\text{Gap}(I_w) = I_w \setminus \bigcup_{i \geq 1} I_{wi}.$$

This gap consists of

- (i) The left-end gap  $E_{w\infty}$  (adjacent to the left endpoint of  $I_w$ ),
- (ii) The right-end gap  $E_{w0}$  (adjacent to the right endpoint of  $I_w$ ),
- (iii) The internal gaps  $E_{wi}$  between  $I_{wi}$  and  $I_{w(i+1)}$  for  $i \geq 1$ .

Thus, we have the total gap length

$$|\text{Gap}(I_w)| = |E_{w\infty}| + |E_{w0}| + \sum_{i \geq 1} |E_{wi}|.$$

We have the relative gap

$$1 - \sum_{i \geq 1} \rho_{wi} = \frac{|\text{Gap}(I_w)|}{|I_w|}.$$

From Lemma 6 and Lemma 9, we have the following asymptotic estimates (uniform in  $w$  and  $\varepsilon$ )

$$(30) \quad |E_{w0}| \simeq \sqrt{\varepsilon}|I_w|, \quad |E_{w\infty}| \simeq \sqrt{\varepsilon}|I_w|, \quad |E_{wi}| \simeq \varepsilon|I_{wi}| \text{ for } i \geq 1.$$

Also, from Lemma ??, we have the scaling factors satisfy

$$\rho_{wi} = \frac{|I_{wi}|}{|I_w|} \simeq |I_i|,$$

where  $|I_i|$  are the lengths of the first-generation intervals.

Using the above, then we have

$$1 - \sum_{i \geq 1} \rho_{wi} = \frac{|E_{w0}| + |E_{w\infty}| + \sum_{i \geq 1} |E_{wi}|}{|I_w|} \simeq 2\sqrt{\varepsilon} + \varepsilon \sum_{i \geq 1} \rho_{wi}.$$

Since the children intervals are disjoint and contained in  $I_w$ , we have  $\sum_{i \geq 1} \rho_{wi} \leq 1$ . Thus,

$$\varepsilon \sum_{i \geq 1} \rho_{wi} \leq \varepsilon.$$

For small  $\varepsilon$ , the term  $2\sqrt{\varepsilon}$  dominates, so we have

$$1 - \sum_{i \geq 1} \rho_{wi} \simeq 2\sqrt{\varepsilon}.$$

More precisely, there exist constants  $K_1, K_2 > 0$  such that for all sufficiently small  $\varepsilon$  and all words  $w$ :

$$K_1\sqrt{\varepsilon} \leq 1 - \sum_{i \geq 1} \rho_{wi} \leq K_2\sqrt{\varepsilon}.$$

Also, since  $\sqrt{\varepsilon} < 1/2$  for small  $\varepsilon$ , condition (4) is satisfied.

### Condition (5)

From Lemma ??,  $\rho_{wi} \simeq |I_i|$  uniformly in  $w$  and  $\varepsilon$ . Thus, there exist constants  $C_1, C_2 > 0$  such that:

$$C_1|I_i| \leq \rho_{wi} \leq C_2|I_i|.$$

Therefore, for any  $\kappa > 0$ :

$$\sum_{i \geq 1} \rho_{wi}^\kappa \leq C_2^\kappa \sum_{i \geq 1} |I_i|^\kappa.$$

So, it suffices to show that  $\sum_{i \geq 1} |I_i|^\kappa$  is bounded uniformly in  $\varepsilon$ .

Lemma 3 states that for any  $\delta > 0$ , there exists  $C > 0$  such that for all sufficiently small  $\varepsilon$ :

$$|I_i| \leq \frac{C}{i^{2-\delta}}.$$

This is a polynomial decay, independent of  $\varepsilon$ .

Choose  $\kappa < 1$  such that  $\kappa(2 - \delta) > 1$ .

Then we have

$$\sum_{i \geq 1} |I_i|^\kappa \leq C^\kappa \sum_{i \geq 1} \frac{1}{i^{\kappa(2-\delta)}} < \infty,$$

since the series  $\sum i^{-p}$  converges for  $p > 1$ .

Let  $M' = C^\kappa \sum_{i \geq 1} i^{-\kappa(2-\delta)}$ . Then:

$$\sum_{i \geq 1} \rho_{wi}^\kappa \leq C_2^\kappa M' = M.$$

This bound  $M$  is independent of  $w$  and  $\varepsilon$ , so condition (5) is satisfied. □

**4.6. Estimates on sizes of intervals and gaps in generation  $n$ .** In this section, we extend the estimates from generation 1 to higher generations. For a word  $w = j_1 j_2 \dots j_{n-1}$ , the interval  $I_w$  has children intervals  $I_{wi}$  for  $i \geq 1$ , and gaps  $E_{wi}$  for  $i = 0, 1, 2, \dots, \infty$ , where  $E_{w0}$  is the gap on the right end of  $I_w$ ,  $E_{w\infty}$  on the left end, and for  $i \geq 1$ ,  $E_{wi}$  is the gap between  $I_{wi}$  and  $I_{w(i+1)}$ .

LEMMA 9. For  $0 < i_n < \infty$ , we have

$$|E_{i_1, i_2, \dots, i_n}| \simeq \varepsilon |I_{i_1, i_2, \dots, i_n}|.$$

We also have

$$|E_{i_1, i_2, \dots, 0}| \simeq \varepsilon |I_{i_1, i_2, \dots, i_n}| \quad \text{and} \quad |E_{i_1, i_2, \dots, \infty}| \simeq \sqrt{\varepsilon} |I_{i_1, i_2, \dots, i_n}|$$

and

$$\frac{|I_{i_1, i_2, \dots, i_n}|}{|I_{i_1, i_2, \dots, i_{n-1}}|} \simeq |I_{i_n}|.$$

PROOF. We show that

$$\frac{|E_{i_1, i_2, \dots, i_n}|}{|I_{i_1, i_2, \dots, i_n}|} \simeq \frac{|E_{i_1}|}{|I_{i_1}|}.$$

We consider the Möbius transformation  $M_{i_1 \dots i_{n-1}} : I_{i_n} \rightarrow I_{i_1, \dots, i_{n-1}, i_n}$ ,  $E_{i_n} \rightarrow E_{i_1, \dots, i_{n-1}, i_n}$  and generally  $I_{j_1 \dots j_k} \rightarrow I_{i_1 \dots i_{n-1}, j_1 \dots j_k}$  which maps the ball with diameter  $1 - 2\varepsilon$  to the ball of diameter  $|I_{i_1, \dots, i_{n-1}}|$ . Both  $I_{i_n}$  and  $E_{i_n}$  are well-contained in the ball of radius  $1 - 2\varepsilon$ . We want to show that  $M_{i_1 \dots i_{n-1}}$  is holomorphic on  $\theta B_n$  ( $\theta > 1$ ) lie inside the ball of radius  $|I_{i_1, \dots, i_{n-1}}|$ .  $M_2 : I_{j_1 j_2 \dots j_n} \rightarrow I_{2j_1 \dots j_n}$ . We need to show the pole of  $M_2$  lies outside the ball of generation 1. We need to show  $M_2$  has uniformly bounded distortion on each interval  $I_{j_1 \dots j_n}$ . We need to show there exists  $\theta > 1$  such that  $\theta I_{j_1 \dots j_n} \subset \theta I_{j_1} \subset B$ . We have  $\lim_{n \rightarrow \infty} \frac{|I_n|}{|I_{n+1}|} = m'_2(a)$ . There exists  $\rho \in (0, r_B)$  such that  $B(\frac{1}{2}, r_B) \supset B(\frac{1}{2}, \rho) \supset \cup_{n=1}^{\infty} I_n$ . We have  $M_2$  restrict to the ball  $B(\frac{1}{2}, r_B)$  of generation 1 is holomorphic and has no pole in this ball so it has bounded distortion on this ball.

We denote the Möbius maps:  $M_1 : I_n \rightarrow I_{1n}$  and  $M_2 : I_n \rightarrow I_{2n}$ , ...,  $M_k : I_n \rightarrow I_{kn}$  are holomorphic because the images of the balls are contained in the compact plane and the poles are outside these balls. More generally, for any  $M_{i_1, i_2, i_3, \dots, i_n}$  maps the ball  $B$  to the ball  $B_{i_1, i_2, i_3, \dots, i_n}$  and any intervals of the generation  $k$   $I_{j_1, j_2, \dots, j_k}$  maps to  $I_{i_1, i_2, i_3, \dots, i_n, j_1, j_2, \dots, j_k}$ . All those Möbius transformations are holomorphic at least on the ball  $B$  and we can apply Koebe's distortion theorem, which gives us the second statement.

The lemma follows from Lemma 6. □

We are now ready to prove Lemma 2 and therefore establish the lower bound in Theorem 3:

PROOF OF LEMMA 2. Since the union of the intervals and gaps covers  $[0, 1]$ ,

$$\sum_{i \in \mathbb{N}} \left( \frac{|I_{j_1 j_2 \dots j_{n-1}, i}|}{|I_{j_1 j_2 \dots j_{n-1}}|} + \frac{|E_{j_1 j_2 \dots j_{n-1}, i}|}{|I_{j_1 j_2 \dots j_{n-1}}|} \right) + \left( \frac{|E_{j_1 j_2 \dots j_{n-1}, 0}|}{|I_{j_1 j_2 \dots j_{n-1}}|} + \frac{|E_{j_1 j_2 \dots j_{n-1}, \infty}|}{|I_{j_1 j_2 \dots j_{n-1}}|} \right) = 1.$$

By Lemmas 6 and 7, we can choose  $c_1 > 0$  sufficiently large so that

$$\frac{|I_{j_1 j_2 \dots j_{n-1}, i}|^{1-c_1\sqrt{\varepsilon}}}{|I_{j_1 j_2 \dots j_{n-1}}|^{1-c_1\sqrt{\varepsilon}}} \geq \frac{|I_{j_1 j_2 \dots j_{n-1}, i}|}{|I_{j_1 j_2 \dots j_{n-1}}|} + mc_1\sqrt{\varepsilon} \cdot \frac{|I_{j_1 j_2 \dots j_{n-1}, i}|}{|I_{j_1 j_2 \dots j_{n-1}}|} \geq \frac{|I_{j_1 j_2 \dots j_{n-1}, i}|}{|I_{j_1 j_2 \dots j_{n-1}}|} + \frac{|E_{j_1 j_2 \dots j_{n-1}, i}|}{|I_{j_1 j_2 \dots j_{n-1}}|}.$$

Using Lemma 9, we have

$$(31) \quad \sum_{i \in \mathbb{N}} \frac{|I_{j_1 j_2 \dots j_{n-1} i}|^{1-c_1 \sqrt{\varepsilon}}}{|I_{j_1 j_2 \dots j_{n-1}}|^{1-c_1 \sqrt{\varepsilon}}} \geq \frac{|I_{j_1 j_2 \dots j_{n-1} 1}|^{1-c_1 \sqrt{\varepsilon}}}{|I_{j_1 j_2 \dots j_{n-1}}|^{1-c_1 \sqrt{\varepsilon}}} + \sum_{2 \leq i < \infty} \left( \frac{|I_{j_1 j_2 \dots j_{n-1} i}|}{|I_{j_1 j_2 \dots j_{n-1}}|} + \frac{|E_{j_1 j_2 \dots j_{n-1} i}|}{|I_{j_1 j_2 \dots j_{n-1}}|} \right)$$

$$(32) \quad = 1 + \left( \frac{|I_{j_1 j_2 \dots j_{n-1} 1}|^{1-c_1 \sqrt{\varepsilon}}}{|I_{j_1 j_2 \dots j_{n-1}}|^{1-c_1 \sqrt{\varepsilon}}} - \frac{|I_{j_1 j_2 \dots j_{n-1} 1}|}{|I_{j_1 j_2 \dots j_{n-1}}|} \right)$$

$$(33) \quad - \left( \frac{|E_{j_1 j_2 \dots j_{n-1} 0}|}{|I_{j_1 j_2 \dots j_{n-1}}|} + \frac{|E_{j_1 j_2 \dots j_{n-1} 1}|}{|I_{j_1 j_2 \dots j_{n-1}}|} + \frac{|E_{j_1 j_2 \dots j_{n-1} \infty}|}{|I_{j_1 j_2 \dots j_{n-1}}|} \right)$$

$$(34) \quad > 1 + m(c_1 \sqrt{\varepsilon}) - \left( \frac{|E_{j_1 j_2 \dots j_{n-1} 0}|}{|I_{j_1 j_2 \dots j_{n-1}}|} + \frac{|E_{j_1 j_2 \dots j_{n-1} 1}|}{|I_{j_1 j_2 \dots j_{n-1}}|} + \frac{|E_{j_1 j_2 \dots j_{n-1} \infty}|}{|I_{j_1 j_2 \dots j_{n-1}}|} \right)$$

$$(35) \quad > 1 + m(c_1 \sqrt{\varepsilon}) - c\sqrt{\varepsilon}$$

$$(36) \quad > 1,$$

provided that  $c_1 > \frac{c}{m}$ .  $\square$

MORE DETAILS PROOF OF LEMMA 2 . Fix a word  $w = j_1 j_2 \dots j_{n-1}$ . Consider the sum

$$(37) \quad S = \sum_{i=1}^{\infty} \left| \frac{I_{wi}}{I_w} \right|^{1-c_1 \sqrt{\varepsilon}} = \sum_{i=1}^{\infty} \left( \frac{|I_{wi}|}{|I_w|} \right)^{1-c_1 \sqrt{\varepsilon}}.$$

By Lemma ??, we have  $\left| \frac{I_{wi}}{I_w} \right| \sim |I_i|$ , so there exist constants  $C_1, C_2 > 0$  such that

$$(38) \quad C_1 |I_i| \leq |I_{wi}| \leq C_2 |I_i|.$$

Therefore, we have

$$(39) \quad \left( \frac{|I_{wi}|}{|I_w|} \right)^{1-c_1 \sqrt{\varepsilon}} \geq C_1^{1-c_1 \sqrt{\varepsilon}} \left( \frac{|I_{wi}|}{|I_w|} \right)^{1-c_1 \sqrt{\varepsilon}}.$$

For small  $\varepsilon$ ,  $C_1^{1-c_1 \sqrt{\varepsilon}}$  is bounded away from 0. Now, from Lemma 7b, there exists  $m > 0$  such that for  $x = |I_i|$  and  $\delta = c_1 \sqrt{\varepsilon}$ , we have

$$(40) \quad |I_i|^{1-c_1 \sqrt{\varepsilon}} \geq |I_i| + mc_1 \sqrt{\varepsilon} |I_i|.$$

By Lemma 6,  $|E_i| \sim \sqrt{\varepsilon} |I_i|$ , so we can choose  $c_1$  large enough so that

$$(41) \quad |I_i|^{1-c_1 \sqrt{\varepsilon}} > |I_i| + |E_i|.$$

Hence, we have

$$(42) \quad \left( \frac{|I_{wi}|}{|I_w|} \right)^{1-c_1 \sqrt{\varepsilon}} \geq C'_1 (|I_i| + |E_i|) \geq C'_1 \left( \left| \frac{I_{wi}}{I_w} \right| + \left| \frac{E_{wi}}{I_w} \right| \right),$$

where the last inequality uses Lemma ?? . Now, summing over  $i \geq 2$ , and handling  $i = 1$  separately, we get

$$(43) \quad S \geq \left( \frac{|I_{w1}|}{|I_w|} \right)^{1-c_1 \sqrt{\varepsilon}} + \sum_{i=2}^{\infty} C'_1 \left( \left| \frac{I_{wi}}{I_w} \right| + \left| \frac{E_{wi}}{I_w} \right| \right).$$

As in the proof of Lemma 8, we have

$$(44) \quad \sum_{i=2}^{\infty} \left( \left| \frac{I_{wi}}{I_w} \right| + \left| \frac{E_{wi}}{I_w} \right| \right) = 1 - |E_{w0}| + |E_{w\infty}| + |E_{w1}| - |E_{w1}|.$$

Also, by Lemma 9,  $|E_{w0}|, |E_{w\infty}|, |E_{w1}| \sim \sqrt{\varepsilon}|I_w|$ , so their sum is  $O(\sqrt{\varepsilon})|I_w|$ . Moreover,

$$(45) \quad \left( \frac{|I_w|}{|I_w|} \right)^{1-c_1\sqrt{\varepsilon}} - \frac{|I_w|}{|I_w|} \geq mc_1\sqrt{\varepsilon} \geq m'c_1\sqrt{\varepsilon},$$

since  $|I_w| \sim |I_1|$  is bounded away from 0. Therefore,

$$(46) \quad S \geq 1 - O(\sqrt{\varepsilon}) + m'c_1\sqrt{\varepsilon} = 1 + \sqrt{\varepsilon}(m'c_1 - O(1)).$$

Choosing  $c_1$  sufficiently large, we obtain  $S > 1$ . Moreover, by taking a finite sum up to  $N_\varepsilon$  large enough, we can ensure the finite sum also exceeds 1. This completes the proof of Lemma 2.  $\square$

**4.7. Upper bound for the Hausdorff dimension.** To prove (19), we use Theorem 8.2 from [1]:

THEOREM 4 (McMullen). *Let  $f_n \rightarrow f$  on a neighbourhood of  $z = \infty$  where*

$$\begin{aligned} f_n(z) &= \lambda_n z + 1 + O(1/z), \\ f(z) &= z + 1 + O(1/z), \end{aligned}$$

*where the attracting multipliers  $\lambda_n > 1$  converge to 1 as  $n \rightarrow \infty$ . Then, for any  $\eta > 0$ , there are  $(1 + \eta)$ -quasiconformal maps  $\phi_n \rightarrow \phi$  defined near  $\infty$  and conjugating  $f_n \rightarrow f$  to  $T_n \rightarrow T$ , where*

$$\begin{aligned} T_n(z) &= \lambda_n z + 1, \\ T(z) &= z + 1. \end{aligned}$$

We will also need Mori's Theorem, which states that quasiconformal mappings are Hölder continuous. Here is a precise statement:

THEOREM 5 (Mori's Theorem). *Let  $U \subset \mathbb{C}$  be a domain. For any compact set  $F \subset U$  and any quasiconformal map  $\phi : U \rightarrow \mathbb{C}$ , there exists a constant  $C > 0$  such that*

$$|\phi(x) - \phi(y)| \leq C(F, \phi) |x - y|^{1/K}.$$

*More generally, if  $\phi_n \rightarrow \phi$  uniformly on compact subsets of  $U$ , then the constant is uniform in  $n$ :*

$$|\phi_n(x) - \phi_n(y)| \leq C(F) |x - y|^{1/K} \quad \text{for all } n = 1, 2, \dots$$

**4.8. Applying McMullen's conjugacy theorem.** To apply Theorem 4, we conjugate so that the attracting fixed point moves to  $\infty$ . Define

$$H(z) = \frac{1}{z - a_\varepsilon},$$

and let

$$F_\varepsilon(z) = H \circ m_1 \circ H^{-1}(z).$$

Then  $F_\varepsilon(\infty) = \infty$ , and the multiplier at  $\infty$  is

$$m'_1(a_\varepsilon) = \frac{1}{\lambda_\varepsilon} \sim 1 - 2\sqrt{\varepsilon} \rightarrow 1.$$

Setting  $\lambda'_\varepsilon = 1/\varepsilon$ , we get

$$F_\varepsilon(z) = \lambda'_\varepsilon z + 1 + O(1/z), \quad \text{with } \lambda'_\varepsilon \rightarrow 1.$$

By Theorem 4, for any  $\eta > 0$ , there exist  $(1+\eta)$ -quasiconformal maps  $\phi_\varepsilon$  defined near  $\infty$ , conjugating  $F_\varepsilon$  to

$$T_\varepsilon(z) = \lambda'_\varepsilon z + 1,$$

and such that  $\phi_\varepsilon \rightarrow \phi$ , which conjugates the limiting map to  $T(z) = z + 1$ .

The map  $T(z) = z + 1$  is parabolic:

$$|E_n| \sim \frac{C}{n^2} \quad \text{as } n \rightarrow \infty.$$

By Mori's Theorem, since  $\phi_\varepsilon$  is  $(1+\eta)$ -quasiconformal, it is Hölder continuous with exponent  $\frac{1}{1+\eta}$ . Hence, for some constant  $C > 0$ ,

$$|I_i^{(\varepsilon)}| = |\phi_\varepsilon(E_n)| \leq C|E_n|^{1/(1+\eta)} \leq C' \cdot \frac{1}{n^{2/(1+\eta)}}.$$

Given any  $\delta > 0$ , choose  $\eta > 0$  such that

$$\frac{2}{1+\eta} > 2 - \delta.$$

Then,

$$|I_i^{(\varepsilon)}| < \frac{C''}{n^{2-\delta}},$$

which is (19) This completes the proof of Lemma 10.

...

By Theorem 4, the maps  $F_\varepsilon$  are conjugated to parabolic maps via  $(1+\eta)$ -quasiconformal maps  $\varphi_\varepsilon$ . The gaps of the parabolic Julia set decay like  $1/n^2$ , and by Mori's Theorem, the images under  $\varphi_\varepsilon$  decay like  $1/n^{2/(1+\eta)}$ .

....

Now, we consider  $F_\varepsilon(z) = H(z) \circ m_1(z) \circ H(z)^{-1}$  where  $H(z) = \frac{1}{z + \sqrt{\varepsilon(1-\varepsilon)}}$ .

## 5. Dimensions of Julia sets in the unit circle

In this section, we give a lower bound for the Hausdorff dimensions of certain Julia sets contained in the unit circle.

**THEOREM 6.** *Let  $J(F_\varepsilon)$  be the Julia set of*

$$F_\varepsilon = \frac{z^2 + 1/3 + \varepsilon}{1 + (1/3 + \varepsilon)z^2}$$

*can be represented as a nice system of intervals. Therefore,*

$$1 - \text{H. dim } J(F_\varepsilon) \asymp \sqrt{\varepsilon}.$$

The proof of the above theorem relies on the two lemmas below.

**LEMMA 10** (Parabolic estimate). *There exists a  $0 < \lambda < 1$  (probably,  $\lambda = 1/2$ ) such that*

$$\ell(I_n) \lesssim n^{-\lambda} \ell(I_1) \asymp n^{-\lambda}$$

and

$$\ell(E_n) \lesssim n^{-\lambda} \ell(E_0) \asymp n^{-\lambda} \sqrt{\varepsilon}.$$

**LEMMA 11** (Conformal elevator). *For any  $i_1, i_2, \dots, i_{n-1}$ ,*

$$\frac{\ell(I_{i_1 i_2 \dots i_{n-1} k})}{\ell(I_{i_1 i_2 \dots i_{n-1}})} \asymp \frac{\ell(I_k)}{\ell(I)} \asymp \ell(I_k)$$

and

$$\frac{\ell(E_{i_1 i_2 \dots i_{n-1} k})}{\ell(E_{i_1 i_2 \dots i_{n-1}})} \asymp \frac{\ell(E_k)}{\ell(I)} \asymp \ell(E_k).$$

**PROOF.** Koebe's distortion theorem.  $\square$

The interval  $I_{-1}$  gets mapped bijectively onto  $A_+ = I_1 \cup E_1 \cup I_2 \cup E_2 \cup \dots$ . This allows us to write

$$I_{-1} = I_{-1,1} \cup E_{-1,1} \cup I_{-1,2} \cup E_{-1,2} \cup \dots,$$

where  $I_{-1,k} := F_\varepsilon^{-1}(I_k)$  and  $E_{-1,k} := F_\varepsilon^{-1}(E_k)$ . We claim that

$$0 < C_1 < \frac{\ell(I_{-1,k})}{\ell(I_k)} < C_2,$$

where the constants  $C_1, C_2$  are independent of  $\varepsilon$ . To see this, notice that as  $\varepsilon \rightarrow 0$ ,  $I_{-1}(\varepsilon) \rightarrow [-1, q] \subset \partial\mathbb{D}$  and  $A_+(\varepsilon) \rightarrow [1, -1] \subset \partial\mathbb{D}$ . Here  $q$  is some point on the circle, with  $F_0(q) = -1$ .

Since

$$C_1 < |F'_0(z)| < C_2$$

on  $[-1, q]$ , the same holds for  $F'_\varepsilon$ , perhaps after making  $C_1 > 0$  a bit smaller and  $C_2$  a little larger.

We use

$$\ell(I_k) = \int_{I_{-1,k}} |F'_\varepsilon(z)| |dz|$$

...

For any  $\varepsilon$ , the critical set of  $F_\varepsilon = \{0, \infty\}$ . The post-critical set  $\mathcal{P}(F_\varepsilon)$  is contained in the positive real ray  $[0, \infty]$ . Therefore, by Koebe's distortion theorem,

$$\frac{\ell(I_{i_1 i_2 i_3 \dots i_{n-1} k})}{\ell(I_{i_1 i_2 i_3 \dots i_{n-1}})} \asymp \frac{\ell(I_{-1, k})}{\ell(I_{-1})} \asymp \ell(I_k).$$

### References

1. Curtis T. McMullen, *Hausdorff dimension and conformal dynamics, iii: Computation of dimension*, American Journal of Mathematics **120** (1998), 691 – 721. [1](#), [2](#), [17](#)

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