Hausdorff dimensions of Cantor sets in the unit circle

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Abstract. For $r > 0$, the Julia set $\mathcal{J}(z + 1 - r/z)$ is a Cantor set contained in the real line. We show that $1 - C_1/r \leq \mathrm{H.dim}\,\mathcal{J}(z + 1 - r/z) \leq 1 - C_2/r$ for r large, answering a question posed by McMullen.

1. Introduction

In [1], C. McMullen examined a one-parameter family of Cantor sets generated by reflections in three equally spaced circles. For $0 < \theta \leq \frac{2\pi}{3}$, let C_θ be a symmetric configuration of 3 circles, each orthogonal to the unit circle $S^1 \subset \mathbb{C}$ and intersecting S^1 in an arc of length θ. We write $\Gamma_\theta = \Gamma(C_\theta)$ for the group of isometries of the unit disk Δ generated by reflections in these three circles. Under this action, the hyperbolic plane is tiled under this action by copies of a fundamental hexagon with sides alternating along C_θ and along S^1. The tiles accumulate on the limit set $\Lambda_\theta \subset S^1$, which is a Cantor set for $0 < \theta < \frac{2\pi}{3}$ and the full circle for $\theta = \frac{2\pi}{3}$. In that work, McMullen derived asymptotic formulas

$$\mathrm{H.dim}(\Lambda_\theta) \simeq \frac{1}{|\log \theta|}, \qquad \text{as } \theta \to 0 \tag{1}$$

and

$$\mathrm{H.dim}(\Lambda_\theta) \simeq \sqrt{2\pi/3 - \theta}, \qquad \text{as } \theta \to 2\pi/3. \tag{2}$$

The first formula is obtained by applying the eigenvalue algorithm to an infinite Markov partition, while the second formula uses Sullivan's formula $\delta(1 - \delta) = \lambda_0(X_\theta)$ which express the Hausdorff dimension of Λ_θ in terms of the base eigenvalue of the hyperbolic Laplacian on the quotient Riemann surface $X_\theta = \Delta/\Gamma_\theta$.

1.1. Applications to Blaschke products. As a second application, we discuss the asymptotic behaviour of the Hausdorff dimension of a certain family of degree 2 Blaschke products. A finite Blaschke product F is a holomorphic self-map of the unit disk which extends to a continuous self-map of the unit circle. It can be expressed as

$$F(z) = e^{i\theta} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a_i} z}, \qquad a_i \in \mathbb{D}. \tag{3}$$

The number d is called the degree of F. The points a_i mark the locations of the zeros of F. We consider the case when $d = 2$. There are four types of degree 2 Blaschke products into four types: hyperbolic, singly-parabolic, doubly-parabolic and elliptic. To estimate dimension of Cantor sets with dimension close to 1, we contruct the system of intervals and find a cover to establish the upper bound. For the lower bound, we use Frostman's lemma.

In this paper, we introduce an unified approach to answer an open question in the Remark (p.30) in McMullen's paper [1], which is to find asymptotic formulas for the case of Blaschke products with $\mathrm{H.dim}\, J(f)$ near 1, related to the results of Theorem 3.5 and 3.6 in [1].

2. Upper bound on the Hausdorff dimension

In this section, we introduce the notion of a system of intervals, which is a convenient framework for describing Cantor sets on the unit circle and the real line. We then establish an upper bound for the Hausdorff dimension of Cantor sets with short gaps.

2.1. Systems of intervals. Let I be the interval $[0, 1]$. By a *system of intervals* in $[0, 1]$ indexed by an alphabet A, we mean a collection of intervals $I_{i_1,i_2,\ldots,i_n}$ such that for any $i_1, i_2, \ldots, i_{n-1} \in A$, the intervals $I_{i_1,i_2,\ldots,i_{n-1},i}$ with $i \in A$ are disjoint and are contained in $I_{i_1,i_2,\ldots,i_{n-1}}$.

We define the *limit set* Λ of the above system of intervals as $\bigcap_{n=1}^{\infty} \Lambda^{(n)}$ where $\Lambda^{(n)}$ is the union of the intervals of generation n. The *scaling factors*

$$0 < \rho_{i_1,i_2,\ldots,i_{n-1}i_n} < 1$$

are defined by the relation

$$|I_{i_1,i_2,\ldots,i_n}| = \rho_{i_1 i_2 i_3 \ldots i_n} \cdot \cdots \cdot \rho_{i_1 i_2 i_3} \cdot \rho_{i_1 i_2} \cdot \rho_{i_1},$$

where $|I|$ denotes the length of an interval I.

Example (Linear systems). A *linear iterated function system* (or a linear IFS) on I is determined by a collection of proper disjoint sub-intervals $I_1, I_2, \ldots,$ indexed by an alphabet A, which may be finite or countable. To each interval I_i of generation 1, we associate an orientation-preserving linear map $\phi_i : I \to I_i$. Since each I_i is strictly contained in I, every map ϕ_i is a contraction.

For a finite word $\xi = i_1 i_2 \ldots i_n$ in the alphabet A, we associate the interval $I_\xi = I_{i_1 i_2 \ldots i_n}$, which is defined as the image of the composition $\phi_{i_n} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}$. We say that I_ξ is an interval of *generation n*.

If $\omega = i_1 i_2 \dots$ is an infinite word, the intervals $I_{i_1} \supset I_{i_1 i_2} \supset I_{i_1 i_2 i_3} \supset \dots$ are nested. Since the maps $\{\phi_i\}$ are contractions, the intersection of these intervals is a single point, which we denote by x_ω. The *limit set* Λ of the IFS consists of points x_ω as ω ranges over all infinite words. Alternatively, $\Lambda = \bigcap_{n=1}^{\infty} \Lambda^{(n)}$, where $\Lambda^{(n)}$ is the union of intervals of generation n.

2.2. Tame systems. While linear iterated function systems are especially simple from a dynamical point of view, many interesting Cantor sets in the unit circle or the real line cannot be expressed as limit sets of a linear IFS. We now consider a more flexible notion. We say that a system of intervals is *tame* if there exists constants $0 < \kappa < 1$ and $M, K_1, K_2 > 0$ such that for any interval $I_{i_1 i_2 \dots i_{n-1}}$ of generation $n-1$, we have

$$K_1 \varepsilon \,\le\, 1 - \sum_{i_n \in A} \rho_{i_1, i_2, \dots, i_{n-1} i_n} \,\le\, K_2 \varepsilon \,<\, 1/2, \tag{4}$$

$$\sum_{i_n \in A} \rho^{\kappa}_{i_1, i_2, \dots, i_{n-1} i_n} \le M. \tag{5}$$

The first assumption says that the total length of the gaps is small, while the second assumption says that the lengths of the intervals do not decay too slowly. Define the *critical exponent* $\alpha^{(n)}$ of generation n as the unique real number $0 < \alpha < 1$ such that

$$\sum_{\xi \in A^n} |I_\xi|^\alpha = 1.$$

In the following theorem, we estimate the critical exponent for a tame system from above and below.

THEOREM 1. *Consider a tame systems of intervals. There exists constants $c_1, c_2 > 0$ depending only on κ, M, K_1, K_2 so that*

$$1 - c_1 \varepsilon < \alpha^{(n)} < 1 - c_2 \varepsilon, \tag{6}$$

for any $n \ge 1$. In particular, $\mathrm{H.dim}\,\Lambda \le 1 - c_2 \varepsilon$.

To prove the theorem, we first consider the special case when $n = 1$. Afterwards, we will deduce the theorem from the special case using induction.

LEMMA 1. *Let $\{\ell_i\}_{i=1}^{\infty}$ be a sequence of positive numbers in $(0,1)$ such that*

$$\sum_{i \in A} \ell_i^{\kappa} < M, \tag{7}$$

for some $0 < \kappa < 1$ and $M > 0$. Suppose that the total length of the gaps satisfies

$$K_1 \varepsilon \,\le\, 1 - \sum_{i \in A} \ell_i \,\le\, K_2 \varepsilon \,<\, 1/2, \tag{8}$$

for some constants $K_1, K_2 > 0$. *Then, the number* $0 < \alpha < 1$ *given by*

$$\sum_{i \in A} \ell_i^{\alpha} = 1$$

satisfies

$$1 - c_1\varepsilon < \alpha < 1 - c_2\varepsilon \tag{9}$$

for some constants $c_1, c_2 > 0$, *which depend on* M, K_1, K_2.

An important feature in the above the lemma is that the constants $c_1, c_2 > 0$ can be chosen to be independent of $\varepsilon > 0$.

Proof. Without loss of generality, we may assume that $\ell_1 \geq \ell_2 \geq \ell_3 \geq \dots$ are arranged in decreasing order. By (8), we have

$$1/2 < \sum_{i=1}^{\infty} \ell_i < 1,$$

which together with (7) shows that ℓ_1 is bounded below by a positive constant which depends on M.

($<$) To show the lower bound in (9), it is enough to find a $c_1 > 0$ sufficiently large so that

$$\sum_{i \in A} \ell_i^{1-c_1\varepsilon} > 1. \tag{10}$$

Since

$$\sum_{i \in A} \ell_i^{1-c_1\varepsilon} \;\geq\; \left(\ell_1^{1-c_1\varepsilon} - \ell_1\right) + \sum_{i \in A} \ell_i \;\geq\; \left(\ell_1^{1-c_1\varepsilon} - \ell_1\right) + (1 - K_2\varepsilon),$$

we need to choose $c_1 > 0$ so that

$$\ell_1\left(\ell_1^{-c_1\varepsilon} - 1\right) = \ell_1^{1-c_1\varepsilon} - \ell_1 \geq K_2\varepsilon \qquad \text{or} \qquad (1/\ell_1)^{c_1\varepsilon} - 1 \geq (K_2/\ell_1)\varepsilon.$$

As $1/\ell_1 > 1$, when $c_1 > 0$ is small,

$$(1/\ell_1)^{c_1\varepsilon} \;=\; e^{c_1 \log(1/\ell_1)\varepsilon} \;=\; 1 + c_1 \log(1/\ell_1)\varepsilon + O(\varepsilon^2).$$

Therefore, (10) will be satisfied if $c_1 > 2 \cdot \frac{K_2}{\ell_1 \log(1/\ell_1)}$ and $\varepsilon > 0$ is sufficiently small.

($>$) To show the upper bound in (9), it is enough to find a $c_2 > 0$ sufficiently small so that

$$\sum_{i \in A} \ell_i^{1-c_2\varepsilon} < 1. \tag{11}$$

Since $\sum_{i \in A} \ell_i < 1 - K_1\varepsilon$, is enough to show that

$$\sum_{i \in A} \left\{\ell_i^{1-c_2\varepsilon} - \ell_i\right\} < K_1\varepsilon. \tag{12}$$

Let us write

$$\sum_{i \in A} \left\{\ell_i^{1-c_2\varepsilon} - \ell_i\right\} = \sum_{i \in A} \ell_i^{\kappa} \cdot \frac{\ell_i^{1-c_2\varepsilon} - \ell_i}{\ell_i^{\kappa}}.$$

Some calculus shows that
$$\frac{x^{1-c_2\varepsilon}-x}{x^\kappa} \le C\cdot c_2\varepsilon, \qquad x\in[0,1],$$
for some constant $C>0$ depending on κ. Therefore,
$$\sum_{i\in A}\left\{\ell_i^{1-c_2\varepsilon}-\ell_i\right\} \le Cc_2\varepsilon\sum_{i\in A}\ell_i^\kappa \le CMc_2\varepsilon,$$
so (12) holds for $c_2<\frac{K_1}{MC}$. □

PROOF OF THEOREM 1. Without loss of generality, we assume that for any $i_1i_2\dots i_{n-1}\in A^{n-1}$, the sub-intervals of generation n are ordered by decreasing length:
$$\ell_{i_1i_2i_3\dots i_{n-1}1} \ge \ell_{i_1i_2i_3\dots i_{n-1}2} \ge \ell_{i_1i_2i_3\dots i_{n-1}3} \ge \dots$$
The assumption
$$1/2<\sum_{k=1}^{\infty}\rho_{i_1i_2i_3\dots i_{n-1}k}<1,$$
together with (5) shows that $\rho_{i_1i_2i_3\dots i_{n-1}1}$ is bounded below by a positive constant which depends on M.

($<$) To show the lower bound in (6), it is enough to find a $c_1>0$ sufficiently large so that
$$\sum_{i_1i_2\dots i_n\in A^n}\ell_{i_1i_2i_3\dots i_n}^{1-c_1\varepsilon}>1. \tag{13}$$
By the definition of the scaling factors, we have
$$\sum_{i_1i_2\dots i_n\in A^n}\ell_{i_1i_2i_3\dots i_n}^{1-c_1\varepsilon} = \sum_{i_1i_2\dots i_{n-1}}\ell_{i_1i_2\dots i_{n-1}}^{1-c_1\varepsilon}\sum_{i_n}\rho_{i_1,i_2,\dots,i_{n-1}i_n}^{1-c_1\varepsilon}.$$
In view of (10), one can find a $c_1>0$ so that the inner sum is >1. Crucially, the constant c_1 only depends on M,K_1,K_2 but not on ε nor on the finite word $i_1i_2\dots i_{n-1}$. Thus,
$$\sum_{i_1i_2\dots i_n\in A^n}\ell_{i_1i_2i_3\dots i_n}^{1-c_1\varepsilon} > \sum_{i_1i_2\dots i_{n-1}\in A^{n-1}}\ell_{i_1i_2i_3\dots i_{n-1}}^{1-c_1\varepsilon} > \sum_{i_1i_2\dots i_{n-2}\in A^{n-2}}\ell_{i_1i_2i_3\dots i_{n-2}}^{1-c_1\varepsilon} > \dots > 1.$$
($>$) To show the upper bound in (6), it is enough to find a $c_2>0$ sufficiently small so that
$$\sum_{i_1i_2\dots i_n\in A^n}\ell_{i_1i_2i_3\dots i_n}^{1-c_2\varepsilon}<1. \tag{14}$$
By the definition of the scaling factors and (11), we have
$$\sum_{i_1i_2\dots i_n\in A^n}\ell_{i_1i_2i_3\dots i_n}^{1-c_2\varepsilon} = \sum_{i_1i_2\dots i_{n-1}}\ell_{i_1i_2\dots i_{n-1}}^{1-c_2\varepsilon}\sum_{i_n}\rho_{i_1,i_2,\dots,i_{n-1}i_n}^{1-c_2\varepsilon} < \sum_{i_1i_2\dots i_{n-1}}\ell_{i_1i_2\dots i_{n-1}}^{1-c_2\varepsilon}.$$
Continuing inductively, we see that
$$\sum_{i_1i_2\dots i_n\in A^n}\ell_{i_1i_2i_3\dots i_n}^{1-c_2\varepsilon} < \sum_{i_1i_2\dots i_{n-1}\in A^{n-1}}\ell_{i_1i_2i_3\dots i_{n-1}}^{1-c_2\varepsilon} < \sum_{i_1i_2\dots i_{n-2}\in A^{n-2}}\ell_{i_1i_2i_3\dots i_{n-2}}^{1-c_2\varepsilon} < \dots < 1,$$
which is what we wanted to show.

Since for any $n \geq 1$, the collection of intervals $I_{i_1 i_2 \ldots i_n}$ of generation n covers Λ, and the largest interval of generation n tends to 0 as $n \to \infty$,

$$\mathrm{H.}\dim \Lambda \;\leq\; \sup_{n \geq 1} \alpha^{(n)} \;\leq\; 1 - c_2 \varepsilon,$$

as desired. $\square$

3. Lower bound on the Hausdorff dimension

In this section, we give a lower bound for the Hausdorff dimensions of limit sets of some special systems of intervals that can code limit sets of parabolic dynamical systems.

THEOREM 2. *Suppose Λ is a system of intervals where each interval has countably many children, ordered by the positive integers. Assume that*

(1) *There exists constants $N_\varepsilon > 0$ and $c_1 > 0$ so that for any interval $I_{i_1 i_2 \ldots i_{n-1}}$ in the system,*

$$\sum_{i=1}^{N_\varepsilon} |\rho_{i_1 i_2 \ldots i_n}|^{1 - c_1 \varepsilon} \geq 1. \tag{15}$$

(2) *There exists a constant $\theta_\varepsilon > 0$ so that for any two adjacent intervals $I_n, I'_n \subset I_{n-1}$ of generation n contained in an interval of generation $n-1$,*

$$\mathrm{dist}(I_n, I'_n) \geq \theta_\varepsilon \cdot \max(|I_n|, |I'_n|).$$

Then, there exists a probability measure μ supported on Λ so that $\mu(J) \lesssim |J|^{1-c_1\varepsilon}$ for any interval J centered at a point of Λ.

In view of Frostman's theorem, the above theorem says that $\mathrm{H.}\dim \Lambda \geq 1 - c_1 \varepsilon$.

PROOF. *Step 0.* We build the measure μ as the weak-* limit of a sequence of probability measures μ_n which satisfy the following two conditions:

- Each measure μ_n is a constant multiple of the Lebesgue measure on each interval I of generation n:
$$\mu_n = \sum_I a_I \cdot dm|_I.$$
- Furthermore, the measures μ_n stabilize in the following sense: for any interval I of generation n, we have
$$\mu_n(I) = \mu_{n+1}(I) = \mu_{n+2}(I) = \cdots . \tag{16}$$

To define the measure μ_1, we specify it on intervals of generation 1:

$$\mu_1(I_i) = \begin{cases} \frac{|I_i|^{1-c_1\varepsilon}}{\sum_{j=1}^{N_\varepsilon} |I_j|^{1-c_1\varepsilon}}, & i = 1, 2, \ldots, N_\varepsilon, \\ 0, & i > N_\varepsilon. \end{cases} \tag{17}$$

More generally, we define the measure μ_n so that

$$\mu_n(I_{i_1 i_2 \ldots i_{n-1} i_n}) = \mu_{n-1}(I_{i_1 \ldots i_{n-1}}) \cdot \frac{|I_{i_1 i_2 \ldots i_{n-1} i_n}|^{1-c_1\varepsilon}}{\sum_{j=1}^{N_\varepsilon} |I_{i_1 i_2 \ldots i_{n-1} j}|^{1-c_1\varepsilon}}, \qquad \text{for } i_n = 1, 2, \ldots, N_\varepsilon \tag{18}$$

and $\mu_n(I_{i_1 i_2 \ldots i_{n-1} i_n}) = 0$ if $i_n > N_\varepsilon$. Notice that if $I_{i_1 \ldots i_{n-1}}$ is an interval of generation $n-1$, then

$$\sum_{i_n} \mu_n(I_{i_1 \ldots i_{n-1} i_n}) \;=\; \sum_{i_n} \mu_{n-1}(I_{i_1 \ldots i_{n-1}}) \frac{|I_{i_1 i_2 \ldots i_{n-1} i_n}|^{1-c_1\varepsilon}}{\sum_{j=1}^{N_\varepsilon} |I_{i_1 i_2 \ldots i_{n-1} j}|^{1-c_1\varepsilon}} \;=\; \mu_{n-1}(I_{i_1 \ldots i_{n-1}}),$$

so the stability condition (16) is satisfied.

Step 1. In this step, we verify that the Frostman condition $\mu_1(J) \lesssim |J|^\alpha$ holds for the measure μ_1. We consider several cases:

Case IA. $J = I_{i_1}$. By definition of μ_1, we have $\mu_1(I_{i_1}) \lesssim |I_{i_1}|^{1-c_1\varepsilon}$ for any interval I_{i_1}.

Case IB. $J \subset I_{i_1}$. Since I_{i_1} satisfies the Frostman condition by Case 1A,

$$\mu_1(J) \;=\; \mu_1(I_{i_1}) \frac{|J|}{|I_{i_1}|} \;\lesssim\; |I_{i_1}|^{1-c_1\varepsilon} \frac{|J|}{|I_{i_1}|} \;\lesssim\; |J|^{1-c_1\varepsilon}.$$

Switch m, n to p, q, since n is reserved for the generation.

Case IC. We denote the left endpoint of an interval I by I^{left} and the right endpoint by I^{right}. We now verify the Frostman condition for the intervals $J_{m,n} = [I_m^{\text{left}}, I_n^{\text{right}}]$ with $1 \le m < n \le N_\varepsilon$:

$$\mu_1(J_{m,n}) \;=\; \sum_{j=m}^{n} \mu_1(I_j) \;\le\; \sum_{j=m}^{n} |I_j|^{1-c_1\varepsilon} \;\le\; N_\varepsilon\, |J_{m,n}|^{1-c_1\varepsilon}.$$

In this section, we seem to be ordering the children from left to right instead of by length. Technically, the children can be ordered by $\mathbb{Z}$ instead of by $\mathbb{N}$.

Case ID. We now consider the case when $J = [J^{\text{left}}, J^{\text{right}}]$ is an interval in $[0,1]$, contained in $J_{m,n}$ for some $m < n$ and intersecting both I_m and I_n. By Assumption (2), the length of the interval $[I_m^{\text{left}}, J^{\text{left}}]$ is bounded by a constant (depending on ε) times the length of the gap between I_m and I_{m+1}, while the length of the interval $[I_n^{\text{right}}, J^{\text{right}}]$ is bounded by a constant (depending on ε) times the length of the gap between I_{n-1} and I_n. Consequently, $|J| \simeq |J_{m,n}|$ and

$$\mu(J) \;\le\; \mu(J_{m,n}) \;\lesssim\; |J_{m,n}|^{1-c_1\varepsilon} \;\simeq\; |J|^{1-c_1\varepsilon}.$$

Step 2. We now inductively check the Frostman condition for the measures μ_N with $N > 1$. We consider the following cases:

A. $J = I_{i_1 \ldots i_{N-1} i_N}$,
B. $J \subset I_{i_1 \ldots i_N}$,
C. $J = J_{m,n} = [I^{\text{left}}_{i_1 \ldots i_{N-1} m}, I^{\text{right}}_{i_1 \ldots i_{N-1} n}]$,
D. $J \subset J_{m,n} = [I^{\text{left}}_{i_1 \ldots i_{N-1} m}, I^{\text{right}}_{i_1 \ldots i_{N-1} n}]$.

Case IIA. We now check the Frostman condition for the intervals $I_{i_1 i_2 \ldots i_n}$ of generation n. We may assume $1 \le i_n \le N_\varepsilon$, otherwise $\mu_n(I_{i_1 i_2 \ldots i_n}) = 0$ and the Frostman condition is trivially satisfied. We need to check that

$$\mu_n(I_{i_1 i_2 \ldots i_{n-1} i_n}) \lesssim |I_{i_1 i_2 \ldots i_{n-1} i_n}|^{1-c_1\varepsilon}$$

or

$$\mu_{n-1}(I_{i_1 \ldots i_{n-1}}) \frac{|I_{i_1 i_2 \ldots i_{n-1} i_n}|^{1-c_1\varepsilon}}{\sum_{j=1}^{N_\varepsilon} |I_{i_1 i_2 \ldots i_{n-1} j}|^{1-c_1\varepsilon}} \lesssim |I_{i_1 i_2 \ldots i_{n-1}}|^{1-c_1\varepsilon} \frac{|I_{i_1 i_2 \ldots i_{n-1} i_n}|^{1-c_1\varepsilon}}{|I_{i_1 i_2 \ldots i_{n-1}}|^{1-c_1\varepsilon}}.$$

However, the last equation holds by induction and (15).

Cases IIB, IIC, IID. These cases can be handled similarly to Cases IB, IC and ID respectively.

Having verified the Frostman condition for an arbitrary interval $J \subset [0,1]$, the proof is complete. $\square$

4. Dimensions of Schottky groups

For $x, y \in \mathbb{R}$, we write $\gamma(x,y)$ for the hyperbolic geodesic joining x and y. For $\varepsilon > 0$, let Γ_ε be the Fuchsian group acting on the upper half-plane generated by reflections in the hyperbolic geodesics $C_0 = \gamma(0,1)$, $C_L = \gamma(\varepsilon, 1-\varepsilon/2)$ and the semicircle $C_R = \gamma(1+\varepsilon/2, 1)$. In this section, we apply the techniques developed in Sections 2 and 3 to show:

Theorem 3. *As $\varepsilon \to 0^+$,*

$$1 - \mathrm{H.}\dim \Lambda_\varepsilon \simeq \sqrt{\varepsilon}.$$

4.1. Fundamental domains. In this section, we describe limit sets of Schottky groups generated by three equally-spaced circles in terms of systems of intervals and recover the upper bound of McMullen's dimension estimate. Recall from the introduction that $\Gamma = \langle R_{c_1}, R_{c_2}, R_{c_3} \rangle$ is the group of isometries of the unit disk Δ generated by reflections in the three circles. The elements in Γ can be written as the product of the generators' elements. There is a correspondence between the orbit of 0 and the dual tree. The root vertex is at 0. The other vertices in $\{\Gamma 0\}$ are labeled by the path from 0 along the tree. There are two natural systems of intervals which one can associate with the limit set Λ_θ. The hyperbolic system every interval has 2 children. For our purposes, the hyperbolic system described above is not adequate, because it does not satisfy the estimate of Theorem 2. However, in the parabolic system the way describe the interval of generation 1 is a little bit different. Let I be $[0,1]$ and its children I_k, for $k \in \mathbb{Z} \setminus \{0\}$. For $k > 0$, we have words of this kind $\{1L^{k-1}R\}$. For $k < 0$, we have words of the type $\{1R^{k-1}L\}$. This defines the interval of generation 1. For the generation 2, $I_{k-1} = 1L^{k-1}R$ children are of the type $I_{k-1,m} = 1L^{k-1}R^m L$ for $m > 1$. The first interval $[0,1]$, the children is indexed by $k \in \mathbb{Z} \setminus \{0\}$. The intervals $I_{k_1,k_2,\ldots,k_n}$ of generation n with $k_1 \in \mathbb{Z} \setminus \{0\}$ and $k_2, k_3, \ldots, k_n > 1$.

4.2. Systems of intervals. The words Γ of length n has the form $\{1,2,3\} \times \{L,R\}^{n-1}$, with the convention that the empty word has the length 0. In the hyperbolic system, the generation of an element is just equal to n, which is the number of letters described that element. However, in the parabolic system the way describe the interval of generation 1 is a little bit different. Let I be $[0,1]$ and its children I_k, for $k \in \mathbb{Z} \setminus \{0\}$. For $k > 0$, we have words of this kind $\{1L^{k-1}R\}$. For $k < 0$, we have words of the type $\{1R^{k-1}L\}$. This defines the interval of generation 1. For the generation 2, $I_{k-1} = 1L^{k-1}R$ children are of the type $I_{k-1,m} = 1L^{k-1}R^mL$ for $m > 1$. The first interval $[0,1]$, the children is indexed by $k \in \mathbb{Z} \setminus \{0\}$. The intervals $I_{k_1,k_2,\ldots,k_n}$ of generation n with $k_1 \in \mathbb{Z} \setminus \{0\}$ and $k_2, k_3, \ldots, k_n > 1$.

In view of Theorem 2, to prove the lower bound for the Hausdorff dimension of $\Lambda(\Gamma_\varepsilon)$ in Theorem 3, it is enough to show:

Lemma 2. *There exists a constant $c_1 > 0$ independent of ε such that*

$$\sum_{i=1}^{\infty} \frac{|I_{j_1 j_2 \ldots j_{n-1} i}|^{1-c_1\sqrt{\varepsilon}}}{|I_{j_1 j_2 \ldots j_{n-1}}|^{1-c_1\sqrt{\varepsilon}}} > 1, \qquad \textit{for any } j_1, j_2, \ldots, j_{n-1} \geq 1.$$

In fact, one can choose $N_\varepsilon \geq 1$ sufficiently large so that

$$\sum_{i=1}^{N_\varepsilon} \frac{|I_{j_1 j_2 \ldots j_{n-1} i}|^{1-c_1\sqrt{\varepsilon}}}{|I_{j_1 j_2 \ldots j_{n-1}}|^{1-c_1\sqrt{\varepsilon}}} > 1,$$

where N_ε depends only on ε.

To deduce the upper bound $1 - \mathrm{H.}\dim \Lambda_\varepsilon \lesssim \sqrt{\varepsilon}$ from Theorem 3, we prove the following lemma: You need to verify the assumptions (4) and (5) of the theorem. The lemma below in its current form may appear later in the paper, but here you need a statement valid for all generations. Also note you have to check two assumptions, not one.

Lemma 3. *For any $\delta > 0$,*

$$|I_i^{(\varepsilon)}| < C/n^{2-\delta}, \tag{19}$$

for every sufficiently small $\varepsilon > 0$.

Corollary 1. *For any $1/2 < \kappa < 1$, there exists an $M > 0$ so that*

$$\sum_{i \in \mathbb{N}} \left|I_i^{(\varepsilon)}\right|^\kappa < M, \tag{20}$$

for every sufficiently small $\varepsilon > 0$.

Lemma 4 (Uniform polynomial decay). *For any $\delta > 0$, there exists a constant $C > 0$ such that for all sufficiently small $\varepsilon > 0$, all generations $n \geq 1$, and all words $w = j_1 j_2 \ldots j_{n-1}$ (including the empty word for $n = 1$), the lengths of the children intervals satisfy:*

$$\frac{|I_{wi}|}{|I_w|} \leq \frac{C}{i^{2-\delta}},$$

for all $i \geq 1$. In particular, for the first generation (w empty), we have:

$$|I_i| \leq \frac{C}{i^{2-\delta}}.$$

Moreover, the constant C is independent of ε, n, and the choice of the parent word w.

4.3. Background on the Möbius transformation m_1. Let $m_1 \in \operatorname{Aut} \mathbb{H}$ be the Möbius transformation which maps the fundamental domain $\triangle^*_{\text{root}}$ to $\triangle_{\text{root}}$ and satisfies $m_1(C) = C_L$. It is not difficult to see that m_1 permutes the tiles in the partition of $\mathbb{H}$. Inspection shows that m_1 maps

$$\dots \to \triangle^*_{L^2} \to \triangle^*_L \to \triangle^*_\emptyset \to \triangle_\emptyset \to \triangle_L \to \triangle_{L^2} \to \dots$$

Consequently, m_1 is a hyperbolic Möbius transformation with an attracting fixed point at

$$a = a_\varepsilon = \lim_{k\to\infty} \triangle_{0,L^k} \in (0,1)$$

and a repelling fixed point at

$$r = r_\varepsilon = \lim_{k\to\infty} \triangle_{-1,R^k} \in (-1,0).$$

Since m_1 maps 0 to ε, 1 to $\frac{1-\varepsilon}{2}$ and $-\frac{\varepsilon}{1-2\varepsilon}$ to 0, it is given by

$$m_1(z) = \frac{(1-\varepsilon)^2 z + \varepsilon(1-\varepsilon)}{(1+\varepsilon)z + 1 - \varepsilon}. \tag{21}$$

Differentiating, we get

$$m_1'(z) = \frac{(1-\varepsilon)(1-3\varepsilon)}{\big((1+\varepsilon)z + 1 - \varepsilon\big)^2}. \tag{22}$$

Some arithmetic shows that

$$r = \frac{-\varepsilon(1-\varepsilon) - \sqrt{\varepsilon(1-\varepsilon)(4+5\varepsilon-\varepsilon^2)}}{2(1+\varepsilon)} \sim -\sqrt{\varepsilon}, \tag{23}$$

$$a = \frac{-\varepsilon(1-\varepsilon) + \sqrt{\varepsilon(1-\varepsilon)(4+5\varepsilon-\varepsilon^2)}}{2(1+\varepsilon)} \sim \sqrt{\varepsilon} \tag{24}$$

and

$$\begin{aligned} m_1'(a) &= \frac{4(1-\varepsilon)(1-3\varepsilon)}{(2-3\varepsilon+\varepsilon^2+s)^2}, \\ m_1'(r) &= \frac{4(1-\varepsilon)(1-3\varepsilon)}{(2-3\varepsilon+\varepsilon^2-s)^2}. \end{aligned} \tag{25}$$

For small $\varepsilon > 0$, we have

$$m_1'(a) \sim 1 - 2\sqrt{\varepsilon} + O(\varepsilon), \qquad m_1'(r) \sim 1 + 2\sqrt{\varepsilon} + O(\varepsilon) \tag{26}$$

and

$$\lambda_\varepsilon = m_1'(r_\varepsilon) \approx 1 + 2\sqrt{\varepsilon}, \tag{27}$$

as $\varepsilon \to 0$.

For $n \in \mathbb{Z}$, we write $m_n(z) := m_1^{\circ n}(z)$. The following lemma will be useful when applying Koebe's distortion theorem:

Lemma 5. *For any $n \geq 1$, the pole of m_n lies on $(-\infty, 0)$. In particular, m_n is holomorphic on the right half-plane $\{\mathrm{Re}\, z > 0\}$.*

Proof. Let P_n be the location of the pole of m_n. Since m_n is an automorphism of the upper half-plane, it is symmetric with respect to the real axis, and therefore, $P_n \in \mathbb{R}$. As m_1 is decreasing on $\mathbb{R} \setminus [r, a]$, its pole P_1 belongs to $(-\infty, r)$. The same reasoning shows that $P_n = m_1^{-(n-1)}(P_1)$ also belongs to $(-\infty, r)$. $\square$

4.4. Estimates on sizes of intervals and gaps in generation 1. By a *gap* of generation 1, we mean a connected component of the complement $[0,1] \setminus \bigcup_{k=1}^{\infty} I_k$. We label the gaps of generation 1 by $E_0, E_1, E_2, E_3, \dots$ and E_∞, where E_k is the gap between the intervals I_k and I_{k+1}, $E_0 = [1-\varepsilon, 1]$ is the gap on the right which contains the point 1 and $E_\infty = [0, a]$ is the gap on the left which contains the point 0. The following lemma describes the sizes of the intervals I_i and the gaps E_i of generation 1:

....

LEMMA 6. *For any $0 < i < \infty$, we have $|E_i| \simeq \varepsilon |I_i|$, while $|E_0| \simeq \varepsilon$, $|E_\infty| \simeq \sqrt{\varepsilon}$.*

PROOF. The estimate $|E_\infty| \simeq \sqrt{\varepsilon}$ is clear, while the estimate $|E_0| \simeq \varepsilon$ follows from (25). Since $I_1 = [\frac{1+\varepsilon}{2}, 1-\varepsilon]$ and $E_1 = \big[\frac{(1-\varepsilon)^2}{3-5\varepsilon}, \frac{1+\varepsilon}{2}\big]$, we have $|I_1| \simeq 1$ and $|E_1| \simeq \varepsilon$ respectively. Notice that the intervals I_1 and E_1 lie inside the ball $B(1, \frac{4}{5})$. Since m_{n-1} is conformal on the $B(1, \frac{4}{5})$ by Lemma 5, by Koebe's distortion theorem,

$$\frac{|E_n|}{|I_n|} \simeq \frac{|E_1|}{|I_1|} \simeq \varepsilon$$

as desired. □

We will need the following elementary estimate:

LEMMA 7. *There exists a constant $m > 0$ so that*

$$x^{1-\delta} > x + m\delta x$$

for any $0 < x < 3/4$ and $0 < \delta < 1/2$.

PROOF. Let $f(x) := x^{1-\delta} - x$. For $1/4 < x < 3/4$ and $0 < \delta < 1/2$, we have

$$(4/3)^\delta \le x^{-\delta} \le 4^\delta. \tag{28}$$

Consequently, $x^{-\delta} \ge (4/3)^\delta = e^{\delta \log(4/3)} = 1 + \log(4/3)\delta + o(\delta)$ and

$$x^{1-\delta} - x = x(x^{-\delta} - 1) \ge \frac{1}{4}\big(\log(4/3) + o(\delta)\big)\delta.$$

The lemma now follows from a compactness argument. Similarly, we have

$$x^{-\delta} = e^{-\delta \log(x)} = 1 - \delta \log(x) + o(\delta \log(x)) \tag{29}$$

We obtain

$$\begin{aligned} x^{1-\delta} - x - m\delta x &= x(x^{-\delta} - 1 - m\delta) \\ &= x(-\delta \log x + o(\delta \log(x)) - m\delta) \\ &\ge 0 \end{aligned}$$

for δ is small enough. □

LEMMA 8. *There exists a constant $c_1 > 0$, which does not depend on ε, such that*

$$\sum_{i=1}^{\infty} |I_i|^{1-c_1\sqrt{\varepsilon}} > 1.$$

Consequently, there exists a pair of constants $c_1, N_\varepsilon > 0$ such that

$$\sum_{i=1}^{N_\varepsilon} |I_i|^{1-c_1\sqrt{\varepsilon}} > 1,$$

where N_ε may depend on $\varepsilon > 0$.

PROOF. Since the union of the intervals and gaps covers $[0, 1]$,

$$\sum_{i\in\mathbb{N}} (|I_i| + |E_i|) + (|E_0| + |E_\infty|) = 1.$$

By Lemmas 6 and 7, we can choose $c_1 > 0$ sufficiently large so that

$$|I_i|^{1-c_1\sqrt{\varepsilon}} \ \geq\ |I_i| + mc_1\sqrt{\varepsilon}\cdot|I_i| \ \geq\ |I_i| + |E_i|.$$

Summing over i and using Lemma 7, we get

$$\begin{aligned}
\sum_{i\in\mathbb{N}} |I_i|^{1-c_1\sqrt{\varepsilon}} &\geq |I_1|^{1-c_1\sqrt{\varepsilon}} + \sum_{2\leq i<\infty} (|I_i| + |E_i|) \\
&= 1 + (|I_1|^{1-c_1\sqrt{\varepsilon}} - |I_1|) - (|E_0| + |E_1| + |E_\infty|) \\
&> 1 + mc_1\sqrt{\varepsilon} - (|E_0| + |E_1| + |E_\infty|) \\
&> 1 + mc_1\sqrt{\varepsilon} - c\sqrt{\varepsilon} \\
&> 1,
\end{aligned}$$

provided that $c_1 > \frac{c}{m}$. □

4.5. Lemma 2.

PROOF OF LEMMA 2. Recall that we are working with the parabolic system of intervals associated with the Schottky group Γ_ε. The intervals are labeled as $I_{j_1,j_2,\dots,j_n}$ for $j_k \geq 1$, and the scaling factors are defined by

$$|I_{j_1,\dots,j_n}| = \rho_{j_1,\dots,j_n} \cdot |I_{j_1,\dots,j_{n-1}}| \cdots \rho_{j_1}.$$

The goal is to verify that this system satisfies the conditions of Theorem 1, which requires the system to be *tame*. Specifically, we recall that we need to check

(4) There exist constants $K_1, K_2 > 0$ such that for any word $w = j_1 \dots j_{n-1}$,

$$K_1\sqrt{\varepsilon} \leq 1 - \sum_{i\in\mathbb{N}} \rho_{wi} \leq K_2\sqrt{\varepsilon} < \frac{1}{2}.$$

(5) There exist constants $0 < \kappa < 1$ and $M > 0$ such that for any word w,

$$\sum_{i\in\mathbb{N}} \rho_{wi}^{\kappa} \leq M.$$

These conditions must hold uniformly for all generations and for all sufficiently small $\varepsilon > 0$.

For a fixed parent interval I_w, the children intervals are I_{wi} for $i \geq 1$. The total gap within I_w is the complement of the union of these children:

$$\mathrm{Gap}(I_w) = I_w \setminus \bigcup_{i\geq 1} I_{wi}.$$

This gap consists of

(i) The left-end gap $E_{w\infty}$ (adjacent to the left endpoint of I_w),

(ii) The right-end gap E_{w0} (adjacent to the right endpoint of I_w),

(iii) The internal gaps E_{wi} between I_{wi} and $I_{w(i+1)}$ for $i \geq 1$.

Thus, we have the total gap length

$$|\mathrm{Gap}(I_w)| = |E_{w\infty}| + |E_{w0}| + \sum_{i\geq 1} |E_{wi}|.$$

We have the relative gap

$$1 - \sum_{i\geq 1} \rho_{wi} = \frac{|\mathrm{Gap}(I_w)|}{|I_w|}.$$

From Lemma 6 and Lemma 9, we have the following asymptotic estimates (uniform in w and ε)

$$|E_{w0}| \simeq \sqrt{\varepsilon}|I_w|, \quad |E_{w\infty}| \simeq \sqrt{\varepsilon}|I_w|, \quad |E_{wi}| \simeq \varepsilon|I_{wi}| \text{ for } i \geq 1. \tag{30}$$

Also, from Lemma **??**, we have the scaling factors satisfy

$$\rho_{wi} = \frac{|I_{wi}|}{|I_w|} \simeq |I_i|,$$

where $|I_i|$ are the lengths of the first-generation intervals.

Using the above, then we have

$$1 - \sum_{i\geq 1} \rho_{wi} = \frac{|E_{w0}| + |E_{w\infty}| + \sum_{i\geq 1} |E_{wi}|}{|I_w|} \simeq 2\sqrt{\varepsilon} + \varepsilon \sum_{i\geq 1} \rho_{wi}.$$

Since the children intervals are disjoint and contained in I_w, we have $\sum_{i\geq 1} \rho_{wi} \leq 1$. Thus,

$$\varepsilon \sum_{i\geq 1} \rho_{wi} \leq \varepsilon.$$

For small ε, the term $2\sqrt{\varepsilon}$ dominates, so we have

$$1 - \sum_{i\geq 1} \rho_{wi} \simeq 2\sqrt{\varepsilon}.$$

More precisely, there exist constants $K_1, K_2 > 0$ such that for all sufficiently small ε and all words w:

$$K_1\sqrt{\varepsilon} \leq 1 - \sum_{i\geq 1} \rho_{wi} \leq K_2\sqrt{\varepsilon}.$$

Also, since $\sqrt{\varepsilon} < 1/2$ for small ε, condition (4) is satisfied.

Condition (5)

From Lemma **??**, $\rho_{wi} \simeq |I_i|$ uniformly in w and ε. Thus, there exist constants $C_1, C_2 > 0$ such that:

$$C_1|I_i| \leq \rho_{wi} \leq C_2|I_i|.$$

Therefore, for any $\kappa > 0$:

$$\sum_{i\geq 1} \rho_{wi}^{\kappa} \leq C_2^{\kappa} \sum_{i\geq 1} |I_i|^{\kappa}.$$

So, it suffices to show that $\sum_{i\geq 1} |I_i|^{\kappa}$ is bounded uniformly in ε.

Lemma 3 states that for any $\delta > 0$, there exists $C > 0$ such that for all sufficiently small ε:

$$|I_i| \leq \frac{C}{i^{2-\delta}}.$$

This is a polynomial decay, independent of ε.

Choose $\kappa < 1$ such that $\kappa(2-\delta) > 1$.

Then we have

$$\sum_{i\geq 1} |I_i|^{\kappa} \leq C^{\kappa} \sum_{i\geq 1} \frac{1}{i^{\kappa(2-\delta)}} < \infty,$$

since the series $\sum i^{-p}$ converges for $p > 1$.

Let $M' = C^{\kappa} \sum_{i\geq 1} i^{-\kappa(2-\delta)}$. Then:

$$\sum_{i\geq 1} \rho_{wi}^{\kappa} \leq C_2^{\kappa} M' = M.$$

This bound M is independent of w and ε, so condition (5) is satisfied. □

4.6. Estimates on sizes of intervals and gaps in generation n. In this section, we extend the estimates from generation 1 to higher generations. For a word $w = j_1 j_2 \ldots j_{n-1}$, the interval I_w has children intervals I_{wi} for $i \geq 1$, and gaps E_{wi} for $i = 0, 1, 2, \ldots, \infty$, where E_{w0} is the gap on the right end of I_w, $E_{w\infty}$ on the left end, and for $i \geq 1$, E_{wi} is the gap between I_{wi} and $I_{w(i+1)}$.

LEMMA 9. *For $0 < i_n < \infty$, we have*

$$|E_{i_1,i_2,\ldots,i_n}| \simeq \varepsilon |I_{i_1,i_2,\ldots,i_n}|.$$

We also have

$$|E_{i_1,i_2,\ldots,0}| \simeq \varepsilon |I_{i_1,i_2,\ldots,i_n}| \qquad \textit{and} \qquad |E_{i_1,i_2,\ldots,\infty}| \simeq \sqrt{\varepsilon}\, |I_{i_1,i_2,\ldots,i_n}|$$

and

$$\frac{|I_{i_1,i_2,\ldots,i_n}|}{|I_{i_1,i_2,\ldots,i_{n-1}}|} \simeq |I_{i_n}|.$$

Proof. We show that

$$\frac{|E_{i_1,i_2,\ldots,i_n}|}{|I_{i_1,i_2,\ldots,i_n}|} \simeq \frac{|E_{i_1}|}{|I_{i_1}|}.$$

We consider the Möbius transformation $M_{i_1\ldots i_{n-1}} : I_{i_n} \to I_{i_1,\ldots,i_{n-1}i_n}$, $E_{i_n} \to E_{i_1,\ldots,i_{n-1}i_n}$ and generally $I_{j_1\ldots j_k} \to I_{i_1\ldots i_{n-1}j_1\ldots j_k}$ which maps the ball with diameter $1-2\varepsilon$ to the ball of diameter $|I_{i_1,\ldots,i_{n-1}}|$. Both I_{i_n} and E_{i_n} are well-contained in the ball of radius $1-2\varepsilon$. We want to show that $M_{i_1\ldots i_{n-1}}$ is holomorphic on θB_n ($\theta > 1$) lie inside the ball of radius $|I_{i_1,\ldots,i_{n-1}}|$. $M_2 : I_{j_1j_2\ldots j_n} \to I_{2j_1\ldots j_n}$. We need to show the pole of M_2 lies outside the ball of generation 1. We need to show M_2 has uniformly bounded distortion on each interval $I_{j_1\ldots j_n}$. We need to show there exists $\theta > 1$ such that $\theta I_{j_1\ldots j_n} \subset \theta I_{j_1} \subset B$. We have $\lim_{n\to\infty} \frac{|I_n|}{|I_{n+1}|} = m_2'(a)$. There exists $\rho \in (0, r_B)$ such that $B(\frac{1}{2}, r_B) \supset B(\frac{1}{2}, \rho) \supset \cup_{n=1}^{\infty} I_n$. We have M_2 restrict to the ball $B(\frac{1}{2}, r_B)$ of generation 1 is holomorphic and has no pole in this ball so it has bounded distortion on this ball.

We denote the Möbius maps: $M_1 : I_n \to I_{1n}$ and $M_2 : I_n \to I_{2n}$, ..., $M_k : I_n \to I_{kn}$ are holomorphic because the images of the balls are contained in the compact plane and the poles are outside these balls. More generally, for any $M_{i_1,i_2,i_3,\ldots,i_n}$ maps the ball B to the ball $B_{i_1,i_2,i_3,\ldots,i_n}$ and any intervals of the generation k $I_{j_1,j_2,\ldots,j_k}$ maps to $I_{i_1,i_2,i_3,\ldots,i_n,j_1,j_2,\ldots,j_k}$. All those Mobius transformations are holomorphic at least on the ball B and we can apply Koebe's distortion theorem, which gives us the second statement.

??? The lemma follows from Lemma 6. □

We are now ready to prove Lemma 2 and therefore stablish the lower bound in Theorem 3:

Proof of Lemma 2. Since the union of the intervals and gaps covers $[0,1]$,

$$\sum_{i\in\mathbb{N}} \left(\frac{|I_{j_1j_2\ldots j_{n-1}i}|}{|I_{j_1j_2\ldots j_{n-1}}|} + \frac{|E_{j_1j_2\ldots j_{n-1}i}|}{|I_{j_1j_2\ldots j_{n-1}}|} \right) + \left(\frac{|E_{j_1j_2\ldots j_{n-1}0}|}{|I_{j_1j_2\ldots j_{n-1}}|} + \frac{|E_{j_1j_2\ldots j_{n-1}\infty}|}{|I_{j_1j_2\ldots j_{n-1}}|} \right) = 1.$$

By Lemmas 6 and 7, we can choose $c_1 > 0$ sufficiently large so that

$$\frac{|I_{j_1j_2\ldots j_{n-1}i}|^{1-c_1\sqrt{\varepsilon}}}{|I_{j_1j_2\ldots j_{n-1}}|^{1-c_1\sqrt{\varepsilon}}} \geq \frac{|I_{j_1j_2\ldots j_{n-1}i}|}{|I_{j_1j_2\ldots j_{n-1}}|} + mc_1\sqrt{\varepsilon} \cdot \frac{|I_{j_1j_2\ldots j_{n-1}i}|}{|I_{j_1j_2\ldots j_{n-1}}|} \geq \frac{|I_{j_1j_2\ldots j_{n-1}i}|}{|I_{j_1j_2\ldots j_{n-1}}|} + \frac{|E_{j_1j_2\ldots j_{n-1}i}|}{|I_{j_1j_2\ldots j_{n-1}}|}.$$

Using Lemma 9, we have

$$\sum_{i\in\mathbb{N}} \frac{|I_{j_1j_2\ldots j_{n-1}i}|^{1-c_1\sqrt{\varepsilon}}}{|I_{j_1j_2\ldots j_{n-1}}|^{1-c_1\sqrt{\varepsilon}}} \geq \frac{|I_{j_1j_2\ldots j_{n-1}1}|^{1-c_1\sqrt{\varepsilon}}}{|I_{j_1j_2\ldots j_{n-1}}|^{1-c_1\sqrt{\varepsilon}}} + \sum_{2\leq i<\infty} \left(\frac{|I_{j_1j_2\ldots j_{n-1}i}|}{|I_{j_1j_2\ldots j_{n-1}}|} + \frac{|E_{j_1j_2\ldots j_{n-1}i}|}{|I_{j_1j_2\ldots j_{n-1}}|} \right) \tag{31}$$

$$= 1 + \left(\frac{|I_{j_1j_2\ldots j_{n-1}1}|^{1-c_1\sqrt{\varepsilon}}}{|I_{j_1j_2\ldots j_{n-1}}|^{1-c_1\sqrt{\varepsilon}}} - \frac{|I_{j_1j_2\ldots j_{n-1}1}|}{|I_{j_1j_2\ldots j_{n-1}}|} \right) \tag{32}$$

$$- \left(\frac{|E_{j_1j_2\ldots j_{n-1}0}|}{|I_{j_1j_2\ldots j_{n-1}}|} + \frac{|E_{j_1j_2\ldots j_{n-1}1}|}{|I_{j_1j_2\ldots j_{n-1}}|} + \frac{|E_{j_1j_2\ldots j_{n-1}\infty}|}{|I_{j_1j_2\ldots j_{n-1}}|} \right) \tag{33}$$

$$> 1 + m(c_1\sqrt{\varepsilon}) - \left(\frac{|E_{j_1j_2\ldots j_{n-1}0}|}{|I_{j_1j_2\ldots j_{n-1}}|} + \frac{|E_{j_1j_2\ldots j_{n-1}1}|}{|I_{j_1j_2\ldots j_{n-1}}|} + \frac{|E_{j_1j_2\ldots j_{n-1}\infty}|}{|I_{j_1j_2\ldots j_{n-1}}|} \right) \tag{34}$$

$$> 1 + m(c_1\sqrt{\varepsilon}) - c\sqrt{\varepsilon} \tag{35}$$

$$> 1, \tag{36}$$

provided that $c_1 > \frac{c}{m}$. □

More details proof of Lemma 2 . Fix a word $w = j_1j_2\ldots j_{n-1}$. Consider the sum

$$S = \sum_{i=1}^{\infty} \left| \frac{I_{wi}}{I_w} \right|^{1-c_1\sqrt{\varepsilon}} = \sum_{i=1}^{\infty} \left(\frac{|I_{wi}|}{|I_w|} \right)^{1-c_1\sqrt{\varepsilon}}. \tag{37}$$

By Lemma ??, we have $\left|\frac{I_{wi}}{I_w}\right| \sim |I_i|$, so there exist constants $C_1, C_2 > 0$ such that

$$C_1|I_i| \leq |I_{wi}| \leq C_2|I_i|. \tag{38}$$

Therefore, we have

$$\left(\frac{|I_{wi}|}{|I_w|} \right)^{1-c_1\sqrt{\varepsilon}} \geq C_1^{1-c_1\sqrt{\varepsilon}} \left(\frac{|I_{wi}|}{|I_w|} \right)^{1-c_1\sqrt{\varepsilon}}. \tag{39}$$

For small ε, $C_1^{1-c_1\sqrt{\varepsilon}}$ is bounded away from 0. Now, from Lemma 7b, there exists $m > 0$ such that for $x = |I_i|$ and $\delta = c_1\sqrt{\varepsilon}$, we have

$$|I_i|^{1-c_1\sqrt{\varepsilon}} \geq |I_i| + mc_1\sqrt{\varepsilon}|I_i|. \tag{40}$$

By Lemma 6, $|E_i| \sim \sqrt{\varepsilon}|I_i|$, so we can choose c_1 large enough so that

$$|I_i|^{1-c_1\sqrt{\varepsilon}} > |I_i| + |E_i|. \tag{41}$$

Hence, we have

$$\left(\frac{|I_{wi}|}{|I_w|} \right)^{1-c_1\sqrt{\varepsilon}} \geq C_1'(|I_i| + |E_i|) \geq C_1' \left(\left| \frac{I_{wi}}{I_w} \right| + \left| \frac{E_{wi}}{I_w} \right| \right), \tag{42}$$

where the last inequality uses Lemma ??. Now, summing over $i \geq 2$, and handling $i = 1$ separately, we get

$$S \geq \left(\frac{|I_{w1}|}{|I_w|} \right)^{1-c_1\sqrt{\varepsilon}} + \sum_{i=2}^{\infty} C_1' \left(\left| \frac{I_{wi}}{I_w} \right| + \left| \frac{E_{wi}}{I_w} \right| \right). \tag{43}$$

As in the proof of Lemma 8, we have

$$\sum_{i=2}^{\infty}\left(\left|\frac{I_{wi}}{I_w}\right|+\left|\frac{E_{wi}}{I_w}\right|\right)=1-|E_{w0}|+|E_{w\infty}|+|E_{w1}|-|E_{w1}|. \tag{44}$$

Also, by Lemma 9, $|E_{w0}|, |E_{w\infty}|, |E_{w1}| \sim \sqrt{\varepsilon}|I_w|$, so their sum is $O(\sqrt{\varepsilon})|I_w|$. Moreover,

$$\left(\frac{|I_w|}{|I_w|}\right)^{1-c_1\sqrt{\varepsilon}}-\frac{|I_w|}{|I_w|}\geq mc_1\sqrt{\varepsilon}\geq m'c_1\sqrt{\varepsilon}, \tag{45}$$

since $|I_w| \sim |I_1|$ is bounded away from 0. Therefore,

$$S\geq 1-O(\sqrt{\varepsilon})+m'c_1\sqrt{\varepsilon}=1+\sqrt{\varepsilon}(m'c_1-O(1)). \tag{46}$$

Choosing c_1 sufficiently large, we obtain $S > 1$. Moreover, by taking a finite sum up to N_ε large enough, we can ensure the finite sum also exceeds 1. This completes the proof of Lemma 2. □

4.7. Upper bound for the Hausdorff dimension. To prove (19), we use Theorem 8.2 from [1]:

THEOREM 4 (McMullen). *Let $f_n \to f$ on a neighbourhood of $z=\infty$ where*

$$f_n(z)=\lambda_n z+1+O(1/z),$$
$$f(z)=z+1+O(1/z),$$

where the attracting multipliers $\lambda_n > 1$ converge to 1 as $n \to \infty$. Then, for any $\eta > 0$, there are $(1+\eta)$-quasiconformal maps $\phi_n \to \phi$ defined near ∞ and conjugating $f_n \to f$ to $T_n \to T$, where

$$T_n(z)=\lambda_n z+1,$$
$$T(z)=z+1.$$

We will also need Mori's Theorem, which states that quasiconformal mappings are Hölder continuous. Here is a precise statement:

THEOREM 5 (Mori's Theorem). *Let $U \subset \mathbb{C}$ be a domain. For any compact set $F \subset U$ and any quasiconformal map $\phi : U \to \mathbb{C}$, there exists a constant $C > 0$ such that*

$$|\phi(x)-\phi(y)|\leq C(F,\phi)\,|x-y|^{1/K}.$$

More generally, if $\phi_n \to \phi$ uniformly on compact subsets of U, then the constant is uniform in n:

$$|\phi_n(x)-\phi_n(y)|\leq C(F)|x-y|^{1/K} \quad \textit{for all} \quad n=1,2,\ldots$$

4.8. ??? ...

First approach Recall that in the context of the Schottky group Γ_ε acting on the upper half-plane $\mathbb{H}$, we can explicitly compute the sizes of the intervals I_n by conjugating the Möbius transformations to linear maps. Here's the step-by-step derivation:

The group Γ_ε is generated by reflections in:

(1) The vertical geodesics $V_0 = \{iy \mid y > 0\}$ and $V_1 = \{1 + iy \mid y > 0\}$.
(2) The semicircle C_ε centered at $1/2$ with radius $1/2 - \varepsilon$.

The key Möbius transformation m_1 maps the fundamental domain $\triangle_{-1,\mathrm{root}}$ to $\triangle_{0,\mathrm{root}}$ and satisfies:

$$m_1(V_0) = C_\varepsilon. \tag{47}$$

This is a hyperbolic Möbius transformation with fixed points at $a \in (0,1)$ and $-a \in (-1,0)$.

To simplify calculations, conjugate m_1 to a linear map via:

$$\phi(z) = \frac{z-a}{z+a}, \quad \phi^{-1}(w) = \frac{a(1+w)}{1-w}. \tag{48}$$

Under this conjugation, m_1 becomes:

$$\widetilde{m}_1(w) = \phi \circ m_1 \circ \phi^{-1}(w) = \lambda w, \tag{49}$$

where $\lambda > 1$ is the dilation factor. The inverse map is:

$$\widetilde{m}_1^{-1}(w) = \lambda^{-1} w. \tag{50}$$

The intervals I_n are the projections of the tiles $\triangle_{0,R^n}$ to $\mathbb{R}$. Under ϕ, these map to:

$$\phi(I_n) = [\lambda^{-n} c_1, \lambda^{-n} c_2], \tag{51}$$

for constants c_1, c_2. The length scales as:

$$|\phi(I_n)| = \lambda^{-n} |c_2 - c_1|. \tag{52}$$

Reverting to I_n:

$$I_n = \phi^{-1}([\lambda^{-n} c_1, \lambda^{-n} c_2]) = \left[\frac{a(1+\lambda^{-n} c_1)}{1-\lambda^{-n} c_1}, \frac{a(1+\lambda^{-n} c_2)}{1-\lambda^{-n} c_2} \right]. \tag{53}$$

For small ε, the dilation factor λ is close to 1. Taylor-expanding $\lambda = 1 + \delta$ ($\delta \sim \sqrt{\varepsilon}$):

$$\lambda^{-n} \approx e^{-n\delta} \approx 1 - n\delta + \frac{n^2\delta^2}{2}. \tag{54}$$

Using the linear approximation for ϕ^{-1} near $w = 0$ (i.e., $z \approx a$):

$$|I_n| \approx \left| \frac{2a\lambda^{-n}(c_2 - c_1)}{(1-\lambda^{-n} c_1)(1-\lambda^{-n} c_2)} \right| \approx 2a|c_2 - c_1| \lambda^{-n}. \tag{55}$$

Substituting $\lambda = 1 + \delta$:

$$|I_n| \approx 2a|c_2 - c_1| e^{-n\delta} \approx 2a|c_2 - c_1|(1 - n\delta). \tag{56}$$

For $\delta \sim \sqrt{\varepsilon}$, this gives exponential decay:

$$|I_n| \sim e^{-n\sqrt{\varepsilon}}. \tag{57}$$

The gaps E_n between I_n and I_{n+1} satisfy:

$$|E_n| \approx |I_n| - |I_{n+1}| \approx 2a|c_2 - c_1|\delta e^{-n\delta} \sim \sqrt{\varepsilon}e^{-n\sqrt{\varepsilon}}. \tag{58}$$

This confirms $|E_n| \sim \varepsilon|I_n|$. The critical exponent α satisfies:

$$\sum_{n=1}^{\infty} |I_n|^\alpha \approx \sum_{n=1}^{\infty} e^{-\alpha n\sqrt{\varepsilon}} < \infty. \tag{59}$$

This sum converges if $\alpha\sqrt{\varepsilon} > 0$, but we refine this using the gap sizes. From Lemma 8:

$$|I_n|^{1-c_1\sqrt{\varepsilon}} \geq |I_n| + mc_1\sqrt{\varepsilon}|I_n| \geq |I_n| + |E_n|. \tag{60}$$

Summing over n and using $\sum(|I_n| + |E_n|) = 1 - |E_\infty|$, we obtain:

$$\sum |I_n|^{1-c_1\sqrt{\varepsilon}} > 1, \tag{61}$$

implying $\alpha > 1 - c_1\sqrt{\varepsilon}$. Hence, by conjugating the Möbius transformations to linear maps, we explicitly derive:

$$|I_n| \sim e^{-n\sqrt{\varepsilon}}, \quad |E_n| \sim \sqrt{\varepsilon}e^{-n\sqrt{\varepsilon}}. \tag{62}$$

The Hausdorff dimension satisfies:

$$1 - \dim_H \Lambda_\varepsilon \simeq \sqrt{\varepsilon}, \tag{63}$$

with constants independent of ε.

4.9. Applying McMullen's conjguacy theorem. To apply Theorem 4, we conjugate so that the attracting fixed point moves to ∞. Define

$$H(z) = \frac{1}{z - a_\varepsilon},$$

and let

$$F_\varepsilon(z) = H \circ m_1 \circ H^{-1}(z).$$

Then $F_\varepsilon(\infty) = \infty$, and the multiplier at ∞ is

$$m_1'(a_\varepsilon) = \frac{1}{\lambda_\varepsilon} \sim 1 - 2\sqrt{\varepsilon} \to 1.$$

Setting $\lambda_\varepsilon' = 1/\varepsilon$, we get

$$F_\varepsilon(z) = \lambda_\varepsilon' z + 1 + O(1/z), \qquad \text{with } \lambda_\varepsilon' \to 1.$$

By Theorem 4, for any $\eta > 0$, there exist $(1+\eta)$-quasiconformal maps ϕ_ε defined near ∞, conjugating F_ε to

$$T_\varepsilon(z) = \lambda'_\varepsilon z + 1,$$

and such that $\phi_\varepsilon \to \phi$, which conjugates the limiting map to $T(z) = z + 1$.

The map $T(z) = z + 1$ is parabolic:

$$|E_n| \sim \frac{C}{n^2} \quad \text{as } n \to \infty.$$

By Mori's Theorem, since ϕ_ε is $(1+\eta)$-quasiconformal, it is Hölder continuous with exponent $\frac{1}{1+\eta}$. Hence, for some constant $C > 0$,

$$|I_i^{(\varepsilon)}| = |\phi_\varepsilon(E_n)| \leq C|E_n|^{1/(1+\eta)} \leq C' \cdot \frac{1}{n^{2/(1+\eta)}}.$$

Given any $\delta > 0$, choose $\eta > 0$ such that

$$\frac{2}{1+\eta} > 2 - \delta.$$

Then,

$$|I_i^{(\varepsilon)}| < \frac{C''}{n^{2-\delta}},$$

which is (19) This completes the proof of Lemma 10.

...

By Theorem 4, the maps F_ε are conjugated to parabolic maps via $(1+\eta)$-quasiconformal maps φ_ε. The gaps of the parabolic Julia set decay like $1/n^2$, and by Mori's Theorem, the images under φ_ε decay like $1/n^{2/(1+\eta)}$.

....

Now, we consider $F_\varepsilon(z) = H(z) \circ m_1(z) \circ H(z)^{-1}$ where $H(z) = \frac{1}{z+\sqrt{\varepsilon(1-\varepsilon)}}$.

5. Dimensions of Julia sets in the unit circle

In this section, we give a lower bound for the Hausdorff dimensions of certain Julia sets contained in the unit circle.

THEOREM 6. *Let $J(F_\varepsilon)$ be the Julia set of*

$$F_\varepsilon = \frac{z^2 + 1/3 + \varepsilon}{1 + (1/3 + \varepsilon)z^2}$$

can be represented as a nice system of intervals. Therefore,

$$1 - \mathrm{H.\,dim}\, J(F_\varepsilon) \asymp \sqrt{\varepsilon}.$$

You need to describe the system of intervals.

The proof of the above theorem relies on the two lemmas below.

LEMMA 10 (Parabolic estimate). *There exists a $0 < \lambda < 1$ (probably, $\lambda = 1/2$) such that*

$$\ell(I_n) \lesssim n^{-\lambda} \ell(I_1) \asymp n^{-\lambda}$$

and

$$\ell(E_n) \lesssim n^{-\lambda} \ell(E_0) \asymp n^{-\lambda}\sqrt{\varepsilon}.$$

LEMMA 11 (Conformal elevator). *For any $i_1, i_2, \ldots i_{n-1}$,*

$$\frac{\ell(I_{i_1 i_2 \ldots i_{n-1} k})}{\ell(I_{i_1 i_2 \ldots i_{n-1}})} \asymp \frac{\ell(I_k)}{\ell(I)} \asymp \ell(I_k)$$

and

$$\frac{\ell(E_{i_1 i_2 \ldots i_{n-1} k})}{\ell(I_{i_1 i_2 \ldots i_{n-1}})} \asymp \frac{\ell(E_k)}{\ell(I)} \asymp \ell(E_k).$$

PROOF. Koebe's distortion theorem. $\square$

The interval I_{-1} gets mapped bijectively onto $A_+ = I_1 \cup E_1 \cup I_2 \cup E_2 \cup \ldots$. This allows us to write

$$I_{-1} = I_{-1,1} \cup E_{-1,1} \cup I_{-1,2} \cup E_{-1,2} \cup \ldots,$$

where $I_{-1,k} := F_\varepsilon^{-1}(I_k)$ and $E_{-1,k} := F_\varepsilon^{-1}(E_k)$. We claim that

$$0 < C_1 < \frac{\ell(I_{-1,k})}{\ell(I_k)} < C_2,$$

where the constants C_1, C_2 are independent of ε. To see this, notice that as $\varepsilon \to 0$, $I_{-1}(\varepsilon) \to [-1, q] \subset \partial\mathbb{D}$ and $A_+(\varepsilon) \to [1, -1] \subset \partial\mathbb{D}$. Here q is some point on the circle, with $F_0(q) = -1$.

Since

$$C_1 < |F_0'(z)| < C_2$$

on $[-1, q]$, the same holds for F_ε', perhaps after making $C_1 > 0$ a bit smaller and C_2 a little larger. We use

$$\ell(I_k) = \int_{I_{-1,k}} |F_\varepsilon'(z)|\, |dz|$$

...

For any ε, the critical set of $F_\varepsilon = \{0, \infty\}$. The post-critical set $\mathcal{P}(F_\varepsilon)$ is contained in the positive real ray $[0, \infty]$. Therefore, by Koebe's distortion theorem,

$$\frac{\ell(I_{i_1 i_2 i_3 \ldots i_{n-1} k})}{\ell(I_{i_1 i_2 i_3 \ldots i_{n-1}})} \asymp \frac{\ell(I_{-1,k})}{\ell(I_{-1})} \asymp \ell(I_k).$$

References

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