Some Types of Sets which are not Sequentially Normally Compact

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Abstract. Sequential normal compactness (SNC for short) is a crucial concept in variational analysis. This paper investigates some specific cases in which a set does not have the SNC property. We show that the homeomorphic image of an open set in a finite dimensional normed space via a Fréchet differentiable mapping into an infinite dimensional Banach space is non-SNC at any point satisfying a regularity assumption. An analogue of the result is valid for closed sets and nondifferentiable mappings. In addition, we give a detailed analysis of the pathological set suggested by J.M. Borwein, S. Fitzpatrick and R. Girgensohn (2003), which has a non-closed Mordukhovich normal cone. The difference between the Bouligand-Severi tangent cone and the weak Bouligand-Severi tangent cone is also considered.

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1   Introduction

Variational analysis is a relatively new direction in nonlinear analysis. It has numerous applications in optimization theory and theory of equilibrium problems. The basic notions in variational analysis include tangent cones and normal cones sets, subdifferentials of extended-real-valued functions, derivatives and coderivatives of multifunctions, Aumann integrals of multifunctions (understood in the sense of integrals of measurable selections); see e.g. [10], [2], [19], [11], [5], [8]. The monograph of Penot [18] is one of the newest books on variational analysis.

The two-volume book of Mordukhovich [16] has presented in a systematical way his approach in variational analysis. The major distinct point between the theory of generalized differentiation of Mordukhovich and the other theories of generalized differentiation is that instead of the primal space setting and the concepts of tangent cones, he uses a dual space setting and notions of normal cones, accepting that (limiting) normal cones to closed sets can be not only non-convex but also non-closed. The main concepts of Mordukhovich's theory are the limiting normal cone (now also called

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the Mordukhovich normal cone), normal coderivative of multifunctions, limiting subdifferential of extended-real-valued functions, and second-order limiting subdifferential.

As far as we understand, the most important theoretical results and the principal applications of the theory constructed by Mordukhovich and his collaborators in the period 1976–2006 have been described in [16].

In the theory of generalized differentiation of Mordukhovich, sequential normal compactness (SNC for short) of sets in infinite dimensional Banach spaces, introduced in [17], plays a significant role. Throughout the book [16], SNC properties appear as qualification conditions in many circumstances: coderivative/subdifferential calculus rules, optimality conditions, stability and sensitivity analysis of variational systems, mean value theorems, etc.

The above notion of SNC is an extension of the notion of compactly epi-Lipschitzian (CEL) property of sets which was introduced by Borwein and Strojwas [7]. The SNC property is weaker than the CEL property and it is easier for using.

By [16, Theorem 1.21], a subset of a Banach space is sequentially normally compact (also abbreviated by SNC) at a given point only if the codimension of the closed affine hull of its intersection with any neighborhood of the point is finite. This necessary condition for the SNC property is not a sufficient one. Some years ago, the second author of this paper raised a question about *finding conditions for a homeomorphic image of an open interval of real numbers via a single-valued mapping into a Banach space to be SNC at a given point.* Note that the image set may satisfy the finite condition property, because the mapping can be twisted. Koucerek [15] first gave a nontrivial solution to the above question.

The first aim of this paper is to revisit the result of Koucerek [15] and show that the homeomorphic image of an open set in a finite dimensional normed space via a Fréchet differentiable mapping into an infinite dimensional Banach space is not SNC at any point satisfying a regularity assumption. Our second aim is to obtain an analogue of this result for closed sets and nondifferentiable mappings. Our third aim is to give a detailed analysis of the special set (a "fan") suggested by Borwein, Fitzpatrick, and Girgensohn [4] (see also [16, Example 1.7]), which has a non-closed Mordukhovich normal cone. It turned out that giving an elementary and complete proof for the non-closedness property of the Mordukhovich normal cone is not an easy task. Interestingly, pathological set is also not SNC at the point in question. As an addition to the variety of sets not having the SNC property, we reconsider the spiral set given by Gould and Tolle [13] (see also [3]), who showed that sometimes the Bouligand-Severi tangent cone can be strictly smaller than the weak Bouligand-Severi tangent cone.

The organization of the present paper as follows. After giving some preliminaries in Section 2, we study the SNC property of the images of finite dimensional open set via differentiable mappings in Section 3. The SNC property of the images of finite dimensional closed set via non-differentiable mappings is discussed in Section 4. A detailed analysis of the "fan" suggested in [4] (also discussed in [16, Example 1.7]) in Section 5. The spiral set from [13] and [3] is considered in Section 6.

2 Preliminaries

In this section we recall some necessary concepts and results from set-valued analysis and variational analysis. For further references one can find in [2], [17] and in [16, Chaps. 1 and 3, Vol. I]. Throughout this section, it will be tacitly assumed that X is an Asplund space. We denote its norm by $\|\cdot\|$, the dual space by X^*, and the duality pairing by $\langle \cdot, \cdot \rangle$. Setting $F : X \rightrightarrows X^*$ is a set-valued

mapping, the *sequential Painlevé-Kuratowski upper limit* of $F(x)$ when x approaches $\bar{x} \in X$, with respect to the norm topology in X and the weak* topology in X^*, is given by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \left\{ x^* \in X^* \mid \exists\, x_k \to \bar{x}, x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \ \forall k \in \mathbb{N} \right\}.$$

If $\Omega \subset X$ is a given subset, the notation $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ and $x \in \Omega$.

Definition 2.1. (See [16, Vol. I, p. 4]) Let Ω be a nonempty subset of X.

(i) For a given $x \in \Omega$ and $\varepsilon \geq 0$, one defines *the set of ε-normals* of Ω at x by

$$\widehat{N}_\varepsilon(x; \Omega) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq \varepsilon \right\}. \tag{2.1}$$

The set $\widehat{N}(x; \Omega) := \widehat{N}_0(x; \Omega)$ is called the *Fréchet normal cone* of Ω at x. If x is not in Ω, we follow the convention that $\widehat{N}_\varepsilon(x; \Omega) := \emptyset$ for all $\varepsilon \geq 0$.

(ii) If $\bar{x} \in \Omega$, the set

$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{x \to \bar{x},\, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x; \Omega), \tag{2.2}$$

is called the *Mordukhovich normal cone* or the *limiting normal cone* of Ω at $\bar{x}$. One puts $N(\bar{x}; \Omega) := \emptyset$ if $\bar{x} \notin \Omega$.

It is obvious that $\widehat{N}(x; \Omega) \subset N(x; \Omega)$ for all $\Omega \subset X$ and $x \in \Omega$. We say that Ω is *normally regular* at x if $\widehat{N}(x; \Omega) = N(x; \Omega)$ for $x \in \Omega$ (see [16, Def. 1.4]).

We adhere closely to the following notion and the development in [16].

Definition 2.2. (See [16, Def. 1.20, Vol. I, p. 27]) A set $\Omega \subset X$ is called *sequentially normally compact* (SNC for short) at $\bar{x} \in \Omega$ if for all sequences $(\varepsilon_k, x_k, x_k^*) \in [0, \infty) \times \Omega \times X^*$ satisfying

$$\varepsilon_k \downarrow 0,\ x_k \to \bar{x},\ x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega),\ \text{and } x_k^* \xrightarrow{w^*} 0,$$

then $\|x_k^*\| \to 0$ when $k \to \infty$.

Definition 2.3. Let X be a Banach space, $\Omega \subset X$. *Affine hull* of Ω, denoted by $\operatorname{aff} \Omega$, is the smallest affine set containing Ω. The closure of $\operatorname{aff} \Omega$ in X is called the *closed affine hull* of Ω and denoted by $\overline{\operatorname{aff}} \Omega$.

One has

$$\operatorname{aff} \Omega = \left\{ \sum_{i=1}^\ell \alpha_i x_i \mid x_i \in \Omega,\ \alpha_i \in \mathbb{R},\ \sum_{i=1}^\ell \alpha_i = 1,\ \ell \in \mathbb{N} \right\}.$$

For arbitrary $x \in \overline{\operatorname{aff}} \Omega$, the set $\overline{\operatorname{aff}} \Omega - x$ is the closed linear subspace of X.

Definition 2.4. The *codimension* of $\overline{\operatorname{aff}} \Omega$ is defined by the dimension of the quotient space $X/(\overline{\operatorname{aff}} \Omega - x)$.

Definition 2.5. The *relative interior* of $\Omega \subset X$, denoted by $\operatorname{ri}\Omega$, is the interior of Ω with respect to the induced topology of $\overline{\operatorname{aff}}\,\Omega$.

$$\operatorname{ri}\Omega = \{x \in \Omega \mid \exists\, \varepsilon > 0,\ B(x,\varepsilon) \bigcap \overline{\operatorname{aff}}\,\Omega \subseteq \Omega\}.$$

Theorem 2.6. (Finite codimension of SNC sets; see [16, Theorem 1.21, Vol. I, p. 27]) *The set $\Omega \subset X$ is SNC at $\bar{x} \in \Omega$ only if for all neighborhoods U of $\bar{x}$, $\operatorname{codim}\overline{\operatorname{aff}}\,(\Omega \cap U) < \infty$. Especially, a singleton set in X is SNC if and only if the dimension of X is finite. Furthermore, if Ω is a convex set and $\operatorname{ri}\Omega \neq \emptyset$, the SNC property at each $\bar{x} \in \Omega$ is equivalent to the condition $\operatorname{codim}\overline{\operatorname{aff}}\,\Omega < \infty$.*

Let Ω be a nonempty set in a normed space X. Assuming that $\bar{x} \in \Omega$.

Definition 2.7. (First order tangent cones; see [16, Vol. I, p. 13]) The set

$$T(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{t\downarrow 0} \frac{\Omega - \bar{x}}{t}, \tag{2.3}$$

where "$\operatorname{Lim\,sup}$" is calculated with respect to the norm topology of X, is called the *Bouligand-Severi tangent cone* to Ω at $\bar{x}$. If the "$\operatorname{Lim\,sup}$" in the formula (2.3) is computed with respect to the weak topology of X, then the obtained set, denoted by $T_w(\bar{x};\Omega)$, is called the *weak tangent cone* to Ω at $\bar{x}$.

Let X, Y be normed spaces, $F : X \rightrightarrows Y$ be a set-valued mapping.

The following concepts of derivative are based on geometric structures, that is, the tangent cone of the graph of the set-valued mapping F which is considered at a given point. Counting on the concepts of derivative, one can characterize the convexity and the convexity corresponding to cone of the graphs of set-valued mappings through the monotonicity and the monotonicity corresponding to cone of the family of derivative mappings $\{DF_z(\cdot)\}_{z \in \operatorname{gph}F}$. Similar to the classical analysis, one can also base on the following concepts of derivative to give open mapping theorem, inverse and implicit function theorems for set-valued mappings. These notions of derivative are used to establish the necessary and sufficient conditions for extreme points in the Optimization Theory and Vector Optimization Theory as well.

The following definition of the contingent derivative was introduced by J.-P. Aubin [1] in 1981.

Definition 2.8. (See [2, Def. 5.1.1, p. 182]) The *contingent derivative (Bouligand derivative)* $DF_{\bar{z}}(\cdot) : X \rightrightarrows Y$ of F at $\bar{z} = (\bar{x}, \bar{y}) \in \operatorname{gph}F$ is the set-valued mapping whose graph coincides with the Bouligand-Severi tangent cone $T(\bar{z}; \operatorname{gph}F)$, that is,

$$DF_{\bar{z}}(u) := \left\{v \in Y \mid (u,v) \in T(\bar{z}; \operatorname{gph}F)\right\}, \quad \forall u \in X.$$

If $F(x) = \{f(x)\}$ for all $x \in X$, where $f : X \to Y$ is single-valued mapping, then one can write $Df_{\bar{x}}(\cdot)$ instead of $DF_{(\bar{x}, f(\bar{x}))}(\cdot)$.

If $T(\bar{z}; \operatorname{gph}F)$ is replaced by the weak Bouligand-Severi tangent cone $T_w(\bar{z}; \operatorname{gph}F)$ in the above definition, then one has the notion of the weak contingent derivative as following one.

Definition 2.9. The *weak contingent derivative (weak Bouligand derivative)* $D^w F_{\overline{z}}(\cdot) : X \rightrightarrows Y$ of F at $\overline{z} = (\overline{x}, \overline{y}) \in \mathrm{gph}F$ is the set-valued mapping whose graph agrees with the weak Bouligand-Severi tangent cone $T_w(\overline{z}; \mathrm{gph}F)$, that is,

$$D^w F_{\overline{z}}(u) := \big\{ v \in Y \mid (u, v) \in T_w(\overline{z}; \mathrm{gph}F) \big\}, \quad \forall u \in X.$$

Likewise, if $F(x) = \{f(x)\}$ for all $x \in X$, where $f : X \to Y$ is single-valued mapping, then we write $D^w f_{\overline{x}}(\cdot)$ instead of $D^w F_{(\overline{x}, f(\overline{x}))}(\cdot)$.

3   The image of a finite dimensional open set

In this section, we present two theorems related to the non-SNC property of sets. The first theorem, belongs to P. Koucerek, was proven as the solution for an exercise in one master course in Kaohsiung, Taiwan. Base on the Koucerek's proof, we expand the result into the more generalized case where we use an open set in $\mathbb{R}^n$ instead of an open interval in $\mathbb{R}$.

Theorem 3.1. (Koucerek [15]) *Let X be an infinite dimensional Banach space, (a, b) is a nonempty open interval of $\mathbb{R}$, and $f : (a, b) \to X$ is a Fréchet differentiable function. Suppose $\overline{t} \in (a, b)$ with $\nabla f(\overline{t}) \neq 0$, and $\Omega := \{f(t) \mid t \in (a, b)\}$. If $f : (a, b) \to \Omega$, where Ω is considered with respect to the induced topology, is a homeomorphism, then Ω is non-SNC at $\overline{x} := f(\overline{t})$.*

Proof.   To verify the claim the set Ω is non-SNC at $\overline{x}$, we will indicate a sequence $\{(x_m, x_m^*)\} \subset \Omega \times X^*$ such that

$$x_m \to \overline{x}, \quad x_m^* \in \widehat{N}(x_m; \Omega), \quad x_m^* \xrightarrow{w} 0,$$

but $\|x_m^*\| \not\to 0$ as $m \to \infty$. Due to the definition of the Fréchet normal cone, $x^* \in \widehat{N}(\overline{x}; \Omega)$ if and only if

$$0 \geq \limsup_{x \xrightarrow{\Omega} \overline{x}} \frac{\langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|}. \tag{3.1}$$

Since $f : (a, b) \to \Omega$ is a homeomorphism, where Ω is considered with respect to the induced topology, $x \xrightarrow{\Omega} \overline{x}$ if and only if $f^{-1}(x) \to f^{-1}(\overline{x})$. Hence, since $\nabla f(\overline{t}).1 \neq 0$ we can rewrite (3.1) in the equivalent form as follow

$$
\begin{aligned}
0 \geq{} & \limsup_{t \to \overline{t}} \frac{\langle x^*, f(t) - f(\overline{t}) \rangle}{\|f(t) - f(\overline{t})\|} \\
={} & \limsup_{t \to \overline{t}} \frac{\langle x^*, \nabla f(\overline{t})(t - \overline{t}) + o(\|t - \overline{t}\|) \rangle}{\|\nabla f(\overline{t})(t - \overline{t}) + o(\|t - \overline{t}\|)\|} \\
={} & \frac{|\langle x^*, \nabla f(\overline{t}).1 \rangle|}{\|\nabla f(\overline{t}).1\|}.
\end{aligned}
$$

It proves that $x^* \in \widehat{N}(\overline{x}; \Omega)$ if and only if $\langle x^*, \nabla f(\overline{t}).1 \rangle = 0$. By Josefson-Nissenzweig's theorem [12, Chapter XII, p. 219], one can find a sequence $\{y_m^*\} \subset X^*$, $y_m^* \xrightarrow{w^*} 0$ as $m \to \infty$ and $\|y_m^*\| = 1$ for all m. We fix $m \in \mathbb{N}$. Considering a case that $\langle y_m^*, \nabla f(\overline{t}).1 \rangle \neq 0$ for infinite indexes m. Then, by considering a subsequence, we can suppose that $\langle y_m^*, \nabla f(\overline{t}).1 \rangle \neq 0$ for all m. We choose an arbitrary $m \in \mathbb{N}$. Since $y_k^* \xrightarrow{w^*} 0$ as $k \to \infty$, then there exists $k_m \in \mathbb{N}$, $k_m > m$, such that

$$|\langle y_{k_m}^*, \nabla f(\overline{t}).1 \rangle| \leq 2^{-1} |\langle y_m^*, \nabla f(\overline{t}).1 \rangle|. \tag{3.2}$$

Putting $\alpha_m = \langle y_m^*, \nabla f(\bar{t}).1 \rangle$ and $\beta_m = \langle y_{k_m}^*, \nabla f(\bar{t}).1 \rangle$. Due to (3.2), $|\beta_m| \leq 2^{-1}|\alpha_m|$. Furthermore, for $\gamma_m := \|\alpha_m y_{k_m}^* - \beta_m y_m^*\|$ one has

$$\gamma_m \geq |\alpha_m| \, \|y_{k_m}^*\| - |\beta_m| \, \|y_m^*\| = |\alpha_m| - |\beta_m|$$
$$\geq 2^{-1}|\alpha_m| > 0.$$

We put $x_m^* = \gamma_m^{-1}(\alpha_m y_{k_m}^* - \beta_m y_m^*)$, one obtains

$$\|x_m^*\| = \|\gamma_m^{-1}(\alpha_m y_{k_m}^* - \beta_m y_m^*)\| = \gamma_m^{-1}\|\alpha_m y_{k_m}^* - \beta_m y_m^*\| = 1.$$

On the other hand, since $\gamma_m \geq 2^{-1}|\alpha_m|$ and $|\beta_m| \leq 2^{-1}|\alpha_m|$, we have

$$\left|\gamma_m^{-1}\alpha_m\right| \leq 2, \quad \left|\gamma_m^{-1}\beta_m\right| \leq 1. \tag{3.3}$$

Due to (3.3), because $y_m^* \xrightarrow{w^*} 0$, for each $x \in X$ we have

$$\langle x_m^*, x \rangle = \gamma_m^{-1}\alpha_m \langle y_{k_m}^*, x \rangle - \gamma_m^{-1}\beta_m \langle y_m^*, x \rangle \longrightarrow 0$$

as $m \to \infty$, because $k_m \to \infty$ as $m \to \infty$. This yields $x_m^* \xrightarrow{w^*} 0$. Because of above chosen α_m and β_m, one has

$$\langle x_m^*, \nabla f(\bar{t}).1 \rangle = \langle \gamma_m^{-1}(\alpha_m y_{k_m}^* - \beta_m y_m^*), \nabla f(\bar{t}).1 \rangle$$
$$= \gamma_m^{-1}(\langle y_m^*, \nabla f(\bar{t}).1 \rangle y_{k_m}^* - \langle y_{k_m}^*, \nabla f(\bar{t}).1 \rangle y_m^*), \nabla f(\bar{t}).1 \rangle)$$
$$= 0.$$

Hence, for $x^m := \bar{x}$ we have $x_m^* \in \widehat{N}(x^m; \Omega)$ for all m. Because $\|x_m^*\| = 1$ for all m, we follows that Ω is non-SNC at $\bar{x}$. Consider the case that $\langle y_m^*, \nabla f(\bar{t}).1 \rangle \neq 0$ for finite indexes m. By considering a subsequence, we can suppose that $\langle y_m^*, \nabla f(\bar{t}).1 \rangle = 0$ for all m. Therefore, $y_m^* \in \widehat{N}(\bar{x}; \Omega)$ for all m. Since $y_m^* \xrightarrow{w^*} 0$ and $\|y_m^*\| = 1$ for all m, one still concludes that Ω is non-SNC at $\bar{x}$. $\qquad\square$

Theorem 3.2. *Let X be an infinite dimensional Banach space, $U \subset \mathbb{R}^n$ is an open set, $f : U \to X$ is a Fréchet differentiable mapping. Let $\bar{u} \in U$ such that $\ker \nabla f(\bar{u}) = \{0\}$, and $\Omega := \{f(u) \mid u \in U\}$. If $f : U \to \Omega$, where Ω is considered with respect to the induced topology, is a homeomorphism, then Ω is non-SNC at $\bar{x} = f(\bar{u})$.*

Proof. Upon the proof of Theorem 3.1, to prove the claim the set Ω is non-SNC at $\bar{x}$, we will indicate a sequence $\{(x_m, x_m^*)\} \subset \Omega \times X^*$ such that

$$x_m \to \bar{x}, \quad x_m^* \in \widehat{N}(x_m; \Omega), \quad x_m^* \xrightarrow{w^*} 0,$$

but $\|x_m^*\| \nrightarrow 0$ as $m \to \infty$. By the definition of the Fréchet normal cone, $x^* \in \widehat{N}(\bar{x}; \Omega)$ if and only if

$$0 \geq \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|}. \tag{3.4}$$

Since $f : U \to \Omega$ is a homeomorphism, where Ω is considered with respect to the induced topology, $x \xrightarrow{\Omega} \bar{x}$ if and only if $f^{-1}(x) \to f^{-1}(\bar{x})$. Hence, we can rewrite (3.4) in the equivalent form as follow

$$0 \geq \limsup_{u \to \bar{u}} \frac{\langle x^*, f(u) - f(\bar{u}) \rangle}{\|f(u) - f(\bar{u})\|}$$
$$= \limsup_{u \to \bar{u}} \frac{\langle x^*, \nabla f(\bar{u})(u - \bar{u}) + o(\|u - \bar{u}\|) \rangle}{\|\nabla f(\bar{u})(u - \bar{u}) + o(\|u - \bar{u}\|)\|}. \tag{3.5}$$

One calls j-th in $\mathbb{R}^n$ is e_j. Suppose that the coordinates of $\bar{u}$ are $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$. Considering an index $i \in \{1, \ldots, n\}$. Substitute $u = (u_1, u_2, \cdots, u_n)$, where $u_j = \bar{u}_j$ for all $j \neq i$, in (3.5) and note that $\nabla f(\bar{u})(e_i) \neq 0$ since $\ker \nabla f(\bar{u}) = \{0\}$, one obtains

$$0 \geq \limsup_{u_i \to \bar{u}_i} \frac{\langle x^*, (u_i - \bar{u}_i)\nabla f(\bar{u})(e_i) + o(|u_i - \bar{u}_i|)\rangle}{||(u_i - \bar{u}_i)\nabla f(\bar{u})(e_i) + o(|u_i - \bar{u}_i|)||} = \frac{|\langle x^*, \nabla f(\bar{u})(e_i)\rangle|}{||\nabla f(\bar{u})(e_i)||}.$$

Therefore,

$$\langle x^*, \nabla f(\bar{u})(e_i)\rangle = 0 \quad \forall i \in \{1, 2, \ldots, n\}. \tag{3.6}$$

We see that (3.6) is the necessary condition for $x^* \in \widehat{N}(\bar{x}; \Omega)$. Now let us prove that the necessary condition is also a sufficient condition. Assume that (3.6) satisfied. Then we have

$$\begin{aligned}
\limsup_{u \to \bar{u}} \frac{\langle x^*, f(u) - f(\bar{u})\rangle}{||f(u) - f(\bar{u})||} &= \limsup_{u \to \bar{u}} \frac{\langle x^*, \nabla f(\bar{u})(u - \bar{u}) + o(||u - \bar{u}||)\rangle}{||\nabla f(\bar{u})(u - \bar{u}) + o(||u - \bar{u}||)||} \\
&= \limsup_{u \to \bar{u}} \frac{\langle x^*, (u_1 - \bar{u}_1)\nabla f(\bar{u})(e_1) + \ldots + (u_n - \bar{u}_n)\nabla f(\bar{u})(e_n) + o(||u - \bar{u}||)\rangle}{||\nabla f(\bar{u})(u - \bar{u}) + o(||u - \bar{u}||)||} \\
&= \limsup_{u \to \bar{u}} \frac{\langle x^*, o(||u - \bar{u}||)\rangle}{||\nabla f(\bar{u})(u - \bar{u}) + o(||u - \bar{u}||)||} \\
&= \limsup_{u \to \bar{u}} \frac{\langle x^*, ||u - \bar{u}||^{-1}o(||u - \bar{u}||)\rangle}{||\nabla f(\bar{u})\left(\frac{u - \bar{u}}{||u - \bar{u}||}\right) + ||u - \bar{u}||^{-1}o(||u - \bar{u}||)||} \\
&= 0.
\end{aligned}$$

This follows (3.5). Hence, the assertion states that "$x^* \in \widehat{N}(\bar{x}; \Omega)$ if and only if (3.6) satisfied" is proved. By Josefson-Nissenzweig's Theorem [12, Chapter XII, p. 219], one can find a sequence $\{y_m^*\} \subset X^*$, $y_m^* \xrightarrow{w^*} 0$ as $m \to \infty$ and $||y_m^*|| = 1$ for all m. Fixing $m \in \mathbb{N}$. If $\langle y_m^*, \nabla f(\bar{u})(e_1)\rangle \neq 0$ for infinite indexes m then, by considering a subsequence, one can assume that $\langle y_m^*, \nabla f(\bar{u})(e_1)\rangle \neq 0$ for all m. Taking arbitrary $m \in \mathbb{N}$. Because $y_k^* \xrightarrow{w^*} 0$ as $k \to \infty$, then there exists $k_m \in \mathbb{N}$, $k_m > m$, satisfying

$$|\langle y_{k_m}^*, \nabla f(\bar{u})(e_1)\rangle| \leq 2^{-1}|\langle y_m^*, \nabla f(\bar{u}).(e_1)\rangle|. \tag{3.7}$$

One puts $\alpha_m = \langle y_m^*, \nabla f(\bar{u})(e_1)\rangle$ and $\beta_m = \langle y_{k_m}^*, \nabla f(\bar{u})(e_1)\rangle$. Since (3.7), $|\beta_m| \leq 2^{-1}|\alpha_m|$. In addition, with $\gamma_m := ||\alpha_m y_{k_m}^* - \beta_m y_m^*||$, by estimations in the previous theorem, we have $\gamma_m > 0$. One puts $x_{m,1}^* = \gamma_m^{-1}(\alpha_m y_{k_m}^* - \beta_m y_m^*)$ and notes that $||x_{m,1}^*|| = 1$. On the other hand, similar to the proof of the previous theorem, (3.3) holds true. Due to (3.3), since $y_m^* \xrightarrow{w^*} 0$ then for each $x \in X$ one has

$$\langle x_{m,1}^*, x\rangle = \gamma_m^{-1}\alpha_m\langle y_{k_m}^*, x\rangle - \gamma_m^{-1}\beta_m\langle y_m^*, x\rangle \longrightarrow 0$$

as $m \to \infty$, since $k_m \to \infty$ as $m \to \infty$. This deduces that $x_{m,1}^* \xrightarrow{w^*} 0$. Because of above way of chosen α_m and β_m, one obtains $\langle x_{m,1}^*, \nabla f(\bar{u})(e_1)\rangle = 0$. If $\langle y_m^*, \nabla f(\bar{u})(e_1)\rangle \neq 0$ just for finite indexes m. By considering a subsequence, one can suppose that $\langle y_m^*, \nabla f(\bar{u})(e_1)\rangle = 0$ for all m. We put $x_{m,1}^* := y_m^*$. If $\langle x_{m,1}^*, \nabla f(\bar{u})(e_2)\rangle \neq 0$ for infinite indexes m then, by considering a subsequence, one can suppose that $\langle x_{m,1}^*, \nabla f(\bar{u})(e_2)\rangle \neq 0$ for all m. One takes arbitrary $m \in \mathbb{N}$. Since $x_{k,1}^* \xrightarrow{w^*} 0$ as $k \to \infty$, then there exists $k_m \in \mathbb{N}$, $k_m > m$, satisfying

$$|\langle x_{k_m,1}^*, \nabla f(\bar{u})(e_2)\rangle| \leq 2^{-1}|\langle x_{m,1}^*, \nabla f(\bar{u}).(e_2)\rangle|. \tag{3.8}$$

Putting $\alpha_m = \langle x_{m,1}^*, \nabla f(\bar{u})(e_2)\rangle$ and $\beta_m = \langle x_{k_m,1}^*, \nabla f(\bar{u})(e_2)\rangle$. Due to (3.8), $|\beta_m| \leq 2^{-1}|\alpha_m|$. Additionally, with $\gamma_m := ||\alpha_m x_{k_m,1}^* - \beta_m x_{m,1}^*||$, one has $\gamma_m > 0$. We put $x_{m,2}^* = \gamma_m^{-1}(\alpha_m x_{k_m,1}^* -$

$\beta_m x^*_{m,1}$) and note that $\|x^*_{m,2}\| = 1$. On the other hand, like in the proof of the previous theorem (3.3) holds true. Due to (3.3), because $x^*_{m,1} \xrightarrow{w^*} 0$ then for each $x \in X$ we have

$$\langle x^*_{m,2}, x \rangle = \gamma_m^{-1} \alpha_m \langle x^*_{k_m,1}, x \rangle - \gamma_m^{-1} \beta_m \langle x^*_{m,1}, x \rangle \to 0$$

when $m \to \infty$, since $k_m \to \infty$ when $m \to \infty$. This proves that $x^*_{m,2} \xrightarrow{w^*} 0$. Due to the above chosen α_m and β_m, one has $\langle x^*_{m,2}, \nabla f(\bar{u})(e_2) \rangle = 0$. Besides, since $\langle x^*_{m,1}, \nabla f(\bar{u})(e_1) \rangle = 0$ for all m, then from the formula of the definition of $x^*_{m,2}$ we have $\langle x^*_{m,2}, \nabla f(\bar{u})(e_1) \rangle = 0$ for all m. If $\langle x^*_{m,1}, \nabla f(\bar{u})(e_2) \rangle \neq 0$ for finite indexes m. By considering subsequences, we can assume that $\langle x^*_{m,1}, \nabla f(\bar{u})(e_2) \rangle = 0$ for all m. One puts $x^*_{m,2} := x^*_{m,1}$. Here we also have $\langle x^*_{m,2}, \nabla f(\bar{u})(e_1) \rangle = 0$ for all m. Similarly, continuing the above process, we consider respectively unit vectors $e_3, e_4, \ldots, e_n$ and construct the sequence of vectors $\{x^*_{m,3}\}, \ldots, \{x^*_{m,n}\}$ in the dual space X^* such that the norm of each vector equals 1 and each sequence weak star converges to 0. Since $\langle x^*_{m,2}, \nabla f(\bar{u})(e_i) \rangle = 0$, $i = 1, 2$, for all m, the construction of $\{x^*_{m,3}\}$ guarantees that $\langle x^*_{m,3}, \nabla f(\bar{u})(e_i) \rangle = 0$, $i = 1, 2, 3$, for all m. For the last sequence of vectors, one has $\langle x^*_{m,n}, \nabla f(\bar{u})(e_i) \rangle = 0$, $i = 1, 2, \ldots, n$, for all m. Therefore, applying the assertion "$x^* \in \widehat{N}(\bar{x}; \Omega)$ if and only if (3.6) satisfied" is proved above, we conclude that $\{x^*_{m,n}\} \subset \widehat{N}(\bar{x}; \Omega)$. Hence, for $x^m := \bar{x}$, then one has $x^*_{m,n} \in \widehat{N}(x^m; \Omega)$ for all m. Because $x^*_{m,n} \xrightarrow{w^*} 0$ and $\|x^*_{m,n}\| = 1$ for all m, one concludes that Ω is non-SNC at $\bar{x}$. $\square$

4 The image of finite dimensional closed sets

We now investigate the SNC property of the image of a finite dimensional closed set without using the Fréchet derivative. The next theorem complements the results obtained in the preceding section.

Theorem 4.1. *Let Ω be a closed set in $\mathbb{R}^n$, $u \in \Omega$. Let X be an infinite dimensional Banach space, and $f : \mathbb{R}^n \to X$ a mapping satisfying the conditions:*

(i) *The directional derivative $f'(u; v)$ of f at u exists for every $v \in T(u; \Omega)$, where $T(u; \Omega)$ denotes the Bouligand tangent cone of Ω at u; and the space $\mathrm{aff}\,[f'(u; v) \mid v \in T(u; \Omega)]$ is of finite dimension.*

(ii) *f is locally Lipschitz around u, and there exist $\alpha > 0$ and a neighborhood U of u with*

$$\|f(u'') - f(u')\| \geq \alpha \|u'' - u'\|, \quad \forall u'', u' \in U; \tag{4.1}$$

(iii) *$f : \Omega \to f(\Omega)$ is a homeomorphism.*

Then $f(\Omega)$ is not SNC at the point $x = f(u)$.

Proof. Suppose that $u \in \Omega$ and the assumption (i)-(iii) are satisfied. We must show that $M := f(\Omega)$ is not SNC at $x := f(u) \in M$. Put $L = \mathrm{aff}\,\{f'(u; v) \mid v \in T(u; \Omega)\}$ and note that L is a linear subspace of X. Taking any $x^* \in L^\perp$, where $L^\perp = \{w^* \in X^* \mid \langle w^*, w \rangle = 0 \forall w \in L\}$, let us show that $x^* \in \widehat{N}(x; M)$. The last inclusion means that

$$\limsup_{x' \xrightarrow{M} x} \frac{\langle x^*, x' - x \rangle}{\|x' - x\|} \leq 0. \tag{4.2}$$

Choose a sequence $\{x_k\} \subset M$ such that $x_k \to x$, $x_k \neq x$ for all k, and

$$\limsup_{x' \xrightarrow{M} x} \frac{\langle x^*, x' - x \rangle}{\|x' - x\|} = \lim_{k \to \infty} \frac{\langle x^*, x_k - x \rangle}{\|x_k - x\|}. \tag{4.3}$$

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If (4.2) fails then, setting $u_k = f^{-1}(x_k)$, we have

$$\lim_{k\to\infty} \frac{\langle x^*, x_k - x\rangle}{||x_k - x||} = \lim_{k\to\infty}\left\langle x^*, \frac{f(u_k) - f(u)}{||f(u_k) - f(u)||}\right\rangle > 0. \tag{4.4}$$

Because $f : \Omega \to M$ is a homeomorphism by (iii), u_k converges to $u = f^{-1}(x)$ as $k \to \infty$. Since $x_k \neq x$, $u_k \neq u$ for all k, hence we may suppose that $v_k := \dfrac{u_k - u}{||u_k - u||} \longrightarrow v \in \mathbb{R}^n$ as $k \to \infty$. Putting $\tau_k = ||u_k - u||$, one has $\tau_k \to 0^+$ and $u_k = u + \tau_k v_k \in \Omega$ for all k. As $v_k \to v$ and $u_k \in \Omega$ for all k, we have $v \in T(u; \Omega)$. By the hypothesis (ii), there exist $\ell > 0$ and a neighborhood U of u such that

$$||f(u'') - f(u')|| \leq \ell||u'' - u'|| \quad \forall u',\ u'' \in U \cap \Omega. \tag{4.5}$$

By (4.1) and (4.5),

$$
\begin{aligned}
\lim_{k\to\infty}\left\langle x^*, \frac{f(u_k) - f(u)}{||f(u_k) - f(u)||}\right\rangle &= \lim_{k\to\infty}\left[\frac{||u_k - u||}{||f(u_k) - f(u)||}\left\langle x^*, \frac{f(u_k) - f(u)}{||u_k - u||}\right\rangle\right] \\
&= \lim_{k\to\infty}\left[\frac{||u_k - u||}{||f(u_k) - f(u)||}\left\langle x^*, \frac{f(u + \tau_k v_k) - f(u + \tau_k v) + f(u + \tau_k v) - f(u)}{||u_k - u||}\right\rangle\right] \\
&\leq \lim_{k\to\infty}\left\{\frac{||u_k - u||}{||f(u_k) - f(u)||}\left[\left|\left\langle x^*, \frac{f(u + \tau_k v_k) - f(u + \tau_k v)}{||u_k - u||}\right\rangle\right| + \left|\left\langle x^*, \frac{f(u + \tau_k v) - f(u)}{||u_k - u||}\right\rangle\right|\right]\right\} \\
&\leq \lim_{k\to\infty}\left\{\frac{||u_k - u||}{\alpha||u_k - u||}\left[\frac{\tau_k \cdot \ell||x^*|| \cdot ||v_k - v||}{||u_k - u||} + \left|\left\langle x^*, \frac{f(u + \tau_k v) - f(u)}{||u_k - u||}\right\rangle\right|\right]\right\} \\
&\leq \lim_{k\to\infty}\left\{\frac{1}{\alpha}\left[\ell||x^*|| \cdot ||v_k - v|| + \left|\left\langle x^*, \frac{f(u + \tau_k v) - f(u)}{\tau_k}\right\rangle\right|\right]\right\}.
\end{aligned}
\tag{4.6}
$$

Since $v_k \to v$ as $k \to \infty$, $\lim_{\tau_k \downarrow 0}\dfrac{f(u + \tau_k v) - f(u)}{\tau_k} = f'(u; v) \in L$, and $x^* \in L^\perp$, the last expression in (4.6) equals 0. Hence (4.6) contradicts (4.4). We have thus proved that $L^\perp \subset \widehat{N}(x; M)$. Since $\dim L^\perp = \infty$ by hypothesis (i), using the Josefson-Nissenzweig's theorem [12, p. 219] one can find a sequence $\{x_m^*\} \subset L^\perp \subset \widehat{N}(x; M)$, $x_m^* \xrightarrow{w^*} 0$ as $m \to \infty$ and $||x_m^*|| = 1$ for all m. The set $f(\Omega)$ is not SNC at the point $x = f(u)$.

$\square$

5 A non-SNC set with a non-closed Mordukhovich normal cone

If Ω is a closed subset in a separable Hilbert space, then $N(0; \Omega)$ may not be closed with respect to the weak topology. Moreover, it may not be closed with respect to the norm topology. The below example is given by J.M. Borwein, S. Fitzpatrick and R. Girgensohn (see [4] and [16, Vol. I, p. 11]).

Let $H = \ell_2(\mathbb{N})$ and

$$\Omega := \{s(e_1 - je_j) + t(je_1 - e_k) \mid k > j > 1;\ s, t \geq 0\} \cup \{te_1 \mid t \geq 0\}, \tag{5.1}$$

where $e_1, e_2, \ldots, e_n, \ldots$ is an orthonormal basis in ℓ_2. Then we have the following assertions:

(i) Ω is the closed nonconvex cone;

(ii) Mordukhovich normal cone $N(0; \Omega)$ is not closed;

(iii) Ω is not SNC at $\bar{x} = 0$;

(iv) $T(0; \Omega) = \Omega$.

Firstly, we prove the assertion (i). It is clear that the set $\{te_1 \mid t \geq 0\}$ is a closed cone. Suppose that
$$x^m = s_m(e_1 + j_m e_{j_m}) + t_m(j_m e_1 - e_{k_m}), \quad k_m > j_m > 1,$$
for $m = 1, 2, \ldots$, are the elements of Ω satisfying $x^m \to x$ when $m \to \infty$. We have

$$x^m = (s_m + t_m j_m)e_1 - s_m j_m e_{j_m} - t_m e_{k_m}. \tag{5.2}$$

Because $\{e_1, e_{j_m}, e_{k_m}\}$ is the orthogonal set for $k_m > j_m > 1$, one has

$$\|x^m\|^2 = (s_m + t_m j_m)^2 + (s_m j_m)^2 + t_m^2. \tag{5.3}$$

Since $x^m \to x$, then $\|x^m\| \to \|x\|$. Therefore, the sequence $\{\|x^m\|\}$ is bounded. So, from (5.3) we imply that $\{s_m + t_m j_m\}$, $\{s_m j_m\}$, and $\{t_m\}$ are bounded. By choosing a subsequence (if necessary), without loss of generality one can assume that $\{s_m + t_m j_m\}$, $\{s_m j_m\}$, and $\{t_m\}$ are convergent sequences. Considering the following cases:

CASE 1: $x = 0$. Since $0 \in \Omega$, it is obvious that $x \in \Omega$.

CASE 2: $x \neq 0$. Suppose that $t_m \to t$ when $m \to \infty$, we consider the following subcases:

(a) $t = 0$.

   Situation 1: $s_m \to 0$ when $m \to \infty$.

   - If $s_m j_m \to 0$ when $m \to \infty$, then (5.2) follows $x^m \to \alpha e_1$, where $\alpha \geq 0$ is a limit point of the sequence $\{s_m + t_m j_m\}$ when $m \to \infty$. So, $x = \alpha e_1 \in \Omega$.

   - If $s_m j_m \to \delta > 0$, then the sequence $\{s_m j_m e_{j_m}\}_{m \geq 0}$ is not a Cauchy sequence since

   $$\|s_{m_1} j_{m_1} e_{j_{m_1}} - s_{m_2} j_{m_2} e_{j_{m_2}}\| = \sqrt{(s_{m_1} j_{m_1})^2 + (s_{m_2} j_{m_2})^2} \to \delta\sqrt{2},$$

   when $m_1 \to \infty$ and $m_2 \to \infty$.

   Situation 2: $s_m \to s$, $s > 0$.

   It follows that $\{s_m\}$, $\{t_m\}$, $\{j_m\}$ are bounded. Since $\{j_m\} \subseteq \mathbb{N}^*$, we can assume that $j_m \equiv j$. Let $m \to \infty$, since $t = 0$ the formula (5.2) implies $x = se_1 - sje_j$ or $x = s(e_1 - je_j)$. It means that $x \in \Omega$.

(b) $t > 0$.

   Then, the sequences $\{s_m\}$, $\{t_m\}$, $\{j_m\}$ are bounded. Without loss of generality, we can assume that $s_m \to s$, $t_m \to t$, $j_m \equiv j$ when $m \to \infty$. The formula (5.2) follows that

   $$e_{k_m} = t_m^{-1}[-x^m + (s_m + t_m j_m)e_1 - s_m j_m e_j] \to t^{-1}[-x + (s + tj)e_1 - sje_j]$$

   when $m \to \infty$. Because $\{e_{k_m}\}$ is a subsequence of the orthonormal system $\{e_1, e_2, \ldots\}$ in ℓ_2, there exits m_0 such that $e_{k_m} = e_{k_{m_0}}$ for all $m \geq m_0$. Indeed, let $\varepsilon = \dfrac{\sqrt{2}}{2}$, since $\{e_{k_m}\}$ is a Cauchy sequence, there exits m_0 satisfying

   $$\|e_{k_m} - e_{k_{m'}}\| \leq \varepsilon \quad \forall m, \ m' \geq m_0,$$

$$||e_{k_m} - e_{k_{m_0}}|| \leq \frac{\sqrt{2}}{2} \qquad \forall m \geq m_0.$$

Observe that $||e_{k_m} - e_{k_{m_0}}|| = \sqrt{2}$ for $k_m \neq k_{m_0}$, contradict to $||e_{k_m} - e_{k_{m_0}}|| \leq \frac{\sqrt{2}}{2}$.

We have already proven that there is m_0 such that $e_{k_m} = e_{k_{m_0}}$ for all $m \geq m_0$.

Taking the limit of (5.2) when $m \to \infty$, one has

$$x = (s + tj)e_1 - sje_j - te_{k_{m_0}} = s(e_1 - je_j) + t(je_1 - e_{k_{m_0}}) \in \Omega.$$

Therefore, we have shown that the set Ω is bounded.

Secondly, we prove (ii). We will show that

$$e_1 + j^{-1}e_j + je_k \in \widehat{N}(u_{jk}; \Omega), \tag{5.4}$$

where $u_{jk} = \frac{j}{k}e_1 - \frac{1}{k}e_k \in \Omega$ for all $k > j > 1$. It means that

$$\limsup_{u \xrightarrow{\Omega} u_{jk}} \frac{\langle e_1 + j^{-1}e_j + je_k, u - u_{jk}\rangle}{||u - u_{jk}||} \leq 0. \tag{5.5}$$

Choosing the sequence $\{u_m\} \subset \Omega$, $u_m \to u_{jk}$ such that

$$\lim_{m \to \infty} \frac{\langle e_1 + j^{-1}e_j + je_k, u_m - u_{jk}\rangle}{||u_m - u_{jk}||} = \limsup_{u \xrightarrow{\Omega} u_{jk}} \frac{\langle e_1 + j^{-1}e_j + je_k, u - u_{jk}\rangle}{||u - u_{jk}||}. \tag{5.6}$$

We have $\Omega = \Omega_0 \cup \Omega_1$, where

$$\Omega_0 := \{te_1 \mid t \geq 0\},$$
$$\Omega_1 := \{s(e_1 - je_j) + t(je_1 - e_k) \mid k > j > 1, \ s, t \geq 0\}.$$

Using the Dirichlet discipline, by extracting a subsequence, one can suppose that $u_m \in \Omega_0$ for all m, or $u_m \in \Omega_1$ for all m.

CASE 1: $\{u_m\} \subset \Omega_0$. Then one has $u_m = t_m e_1$ for $t_m \geq 0$. Since $u_m \to u_{jk}$, we have $u_{jk} = \bar{t}e_1$ for $\bar{t} \geq 0$. Therefore, $\frac{j}{k}e_1 - \frac{1}{k}e_k = \bar{t}e_1$. By identifying two sides of the equation, one implies that $\frac{j}{k} = \bar{t}$, and $\frac{-1}{k} = 0$, a contradiction.

CASE 2: $\{u_m\} \subset \Omega_1$. Then

$$u_m = s_m(e_1 - j_m e_{j_m}) + t_m(j_m e_1 - e_{k_m}), \ s_m \geq 0, \ t_m \geq 0, \ k_m > j_m > 1.$$

We have

$$u_m = (s_m + t_m j_m)e_1 - s_m j_m e_{j_m} - t_m e_{k_m} \longrightarrow u_{jk} \quad \text{when } m \to \infty. \tag{5.7}$$

It follows that

$$||u_m||^2 = (s_m + t_m j_m)^2 + (s_m j_m)^2 + t_m^2 \longrightarrow ||u_{jk}|| = (\frac{j}{k})^2 + \frac{1}{k^2} \tag{5.8}$$

as $m \to \infty$. Therefore, the sequences $\{s_m + t_m j_m\}$, $\{s_m j_m\}$, $\{t_m\}$ are bounded. By extracting a sequence (if necessary), one can suppose that three above sequences are convergent, i.e., $s_m + t_m j_m \to \beta$, $s_m j_m \to \alpha$, $t_m \to t$, where $\beta \geq 0$, $\alpha \geq 0$, $t \geq 0$. We consider the following subcases:

(A) $t = 0$.

If $\alpha = 0$ then from (5.7) we have $u_m \to \beta e_1$. Since $u_m \to u_{jk}$, then $\beta e_1 = \frac{j}{k}e_1 - \frac{1}{k}e_k$. Thus $\beta = \frac{j}{k}$ and $\frac{-1}{k} = 0$, a contradiction. Now we suppose that $\alpha > 0$. Consider m_1, m_2 where $1 < m_1 < m_2$, we have

$$\|u_{m_1} - u_{m_2}\| \geq \|[(s_{m_1} + t_{m_1}j_{m_1}) - (s_{m_2} + t_{m_2}j_{m_2})]e_1 - s_{m_1}j_{m_1}e_{j_{m_1}}$$
$$+ s_{m_2}j_{m_2}e_{j_{m_2}}\| - \|t_{m_1}e_{k_{m_1}} - t_{m_2}e_{k_{m_2}}\|,$$

or

$$\|u_{m_1} - u_{m_2}\| \geq [((s_{m_1} + t_{m_1}j_{m_1}) - (s_{m_2} + t_{m_2}j_{m_2}))^2 + (s_{m_1}j_{m_1})^2$$
$$+ (s_{m_2}j_{m_2})^2]^{\frac{1}{2}} - \|t_{m_1}e_{k_{m_1}} - t_{m_2}e_{k_{m_2}}\|.$$

Because $\lim\limits_{m_1,\, m_2 \to \infty} \|u_{m_1} - u_{m_2}\| = 0$, the last inequality follows that $0 \geq \sqrt{2}\alpha$, a contradiction.

(B) $t > 0$.

The formula (5.8) follows that the sequences $\{s_m\}$, $\{t_m\}$, $\{j_m\}$ are bounded. Since $\{j_m\} \subset \mathbb{N}$, we can suppose that $j_m \equiv j_*$ for all m. Besides, one can assume that $s_m \to s$ as $m \to \infty$. We have

$$u_m = (s_m + t_m j_*)e_1 - s_m j_* e_{j_*} - t_m e_{k_m}. \tag{5.9}$$

Because the first and second terms on the left hand side tend to βe_1 and $sj_* e_{j_*}$ respectively as $m \to \infty$, the third term must also be convergent. Since $t_m \to t > 0$ as $m \to \infty$, we imply that e_{k_m} is a constant vector when m large enough. Without loss of generality, one can suppose that $e_{k_m} = e_{k_*}$ for all m. Since $j_m < k_m$ for all m, one has $j_* < k_*$. Since $u_m \to u_{jk}$ as $m \to \infty$, we have

$$\lim_{m \to \infty} u_m = (s + tj_*)e_1 - sj_* e_{j_*} - te_{k_*} = \frac{j}{k}e_1 - \frac{1}{k}e_k. \tag{5.10}$$

If $j_* \geq k$, then $k_* > j_* \geq k > 1$. The second equality in (5.10) follows that e_{k_*} linearly depends on e_1, e_{j_*}, and e_k, a contradiction. So, $j_* < k$. If $k_* \neq k$, then since $k_* \notin \{1, j_*, k\}$, the second equality in (5.10) implies a contradiction. Therefore, $k_* = k$. Thus, one has

$$(s + tj_*)e_1 - sj_* e_{j_*} - te_k = \frac{j}{k}e_1 - \frac{1}{k}e_k. \tag{5.11}$$

Since $j_* \notin \{1, k\}$, one has $sj_* = 0$, that is $s = 0$. So, the equality (5.11) becomes

$$tj_* e_1 - te_k = \frac{j}{k}e_1 - \frac{1}{k}e_k.$$

Because the vectors e_1 and e_k are linearly independent, this equality deduces that $t = k^{-1}$ and $j_* = j$. From the above results, the equality (5.9) turns into

$$u_m = (s_m + t_m j)e_1 - s_m j e_j - t_m e_k.$$

Since $\{e_1,\, e_j,\, e_k\}$ is a orthogonal system, one has

$$\langle e_1 + j^{-1}e_j + je_k, u_m - u_{jk}\rangle$$
$$= \langle e_1 + j^{-1}e_j + je_k, (s_m + t_m j - \frac{j}{k})e_1 - s_m j e_j + (\frac{1}{k} - t_m)e_k\rangle$$
$$= 0.$$

Combining with (5.5) and (5.6), one deduces that

$$u_{jk}^* := e_1 + j^{-1}e_j + je_k$$

is an element of $\widehat{N}(u_{jk}; \Omega)$.

Fixing j, we obtain $u_{jk}^* \xrightarrow{w} e_1 + j^{-1}e_j$ as $k \to \infty$. Since $u_{jk} = \frac{j}{k}e_1 - \frac{1}{k}e_k \to 0$, we have $x_j^* := e_1 + j^{-1}e_j$ is an element of $N(0; \Omega)$. Because $x_j^* \to e_1$ (w.r.t. the norm topology) as $j \to \infty$, to prove that the Mordukhovich normal cone $N(0; \Omega)$ is not closed with respect to norm topology, we must show that $e_1 \notin N(0; \Omega)$. By contradiction, we assume that there exist $x_m \xrightarrow{\Omega} 0$, $\varepsilon_m \downarrow 0$, $x_m^* \in \widehat{N}_{\varepsilon_m}(x_m; \Omega)$, and $x_m^* \xrightarrow{w} e_1$. For each m, one has

$$\varepsilon_m \geq \limsup_{u \xrightarrow{\Omega} x_m} \langle x_m^*, \frac{u - x_m}{||u - x_m||} \rangle. \tag{5.12}$$

Using the expression $\Omega = \Omega_1 \cup \Omega_0$, where $\Omega_0 = \{te_1 \mid t \geq 0\}$, $\Omega_1 = \{s(e_1 - je_j) + t(je_1 - e_k) \mid k > j > 1, \ s, t \geq 0\}$, and applying Dirichlet discipline for the sequence $\{x_m\}$, by considering a subsequence of $\{x_m\}$ (if necessary) we have the two following cases.

CASE 1: $\{x_m\} \subset \Omega_0$.

Then for each given m, one has $x_m = t_m e_1$, $t_m \geq 0$. Let $u = x_m + re_1$, $r > 0$ in the right hand side of (5.12), we have

$$\varepsilon_m \geq \limsup_{r \to 0^+} \langle x_m^*, \frac{re_1}{||re_1||} \rangle = \langle x_m^*, e_1 \rangle. \tag{5.13}$$

On the other hand, since $x_m^* \xrightarrow{w} e_1$ as $m \to \infty$,

$$\langle x_m^*, e_1 \rangle \to 1 \ \text{as} \ m \to \infty. \tag{5.14}$$

Since (5.14) and $\varepsilon_m \downarrow 0$, we imply that (5.13) can only hold for finite indexes m, a contradiction. Hence the case $\{x_m\} \subset \Omega_0$ cannot happen.

CASE 2: $\{x_m\} \subset \Omega_1$.

Suppose that $x_m = s(e_1 - je_j) + t(je_1 - e_k)$, where $s = s_m \geq 0$, $j = j_m > 1$, $t = t_m \geq 0$, and $k = k_m > j_m$.

+) Let $u = x_m + r(je_1 - e_k)$, $r > 0$ in the right hand side of (5.12), we deduce

$$\varepsilon_m \geq \limsup_{r \to 0^+} \langle x_m^*, \frac{r(je_1 - e_k)}{||r(je_1 - e_k)||} \rangle = \langle x_m^*, \frac{je_1 - e_k}{||je_1 - e_k||} \rangle.$$

Thus,

$$\langle x_m^*, je_1 - e_k \rangle \leq \varepsilon_m ||je_1 - e_k||$$
$$\Leftrightarrow j^{-1} \langle x_m^*, je_1 - e_k \rangle \leq \varepsilon_m ||e_1 - j^{-1}e_k||$$
$$\Leftrightarrow \langle x_m^*, e_1 - j^{-1}e_k \rangle \leq \varepsilon_m ||e_1 - j^{-1}e_k||.$$

Because of the last inequality, we have

$$\langle x_m^*, e_1 - j^{-1}e_k \rangle \leq \varepsilon_m \sqrt{1 + j^{-2}} \tag{5.15}$$

$$\Leftrightarrow \langle x_m^*, e_1 \rangle \leq \langle x_m^*, j_m^{-1}e_{k_m} \rangle + \varepsilon_m \sqrt{1 + j_m^{-2}}. \tag{5.16}$$

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Letting $m \to \infty$ and using the property $x_m^* \xrightarrow{w} e_1$, one gets

$$1 \leq \liminf_{m \to \infty} \langle x_m^*, j_m^{-1} e_{k_m} \rangle. \tag{5.17}$$

+) Let $u = x_m + r(e_1 - je_j)$, $r > 0$ in the right hand side of (5.12), we have

$$\varepsilon_m \geq \limsup_{r \to 0^+} \langle x_m^*, \frac{r(e_1 - je_j)}{||r(e_1 - je_j)||} \rangle = \langle x_m^*, \frac{e_1 - je_j}{||e_1 - je_j||} \rangle.$$

Therefore, one deduces

$$\langle x_m^*, e_1 - je_j \rangle \leq \varepsilon_m ||e_1 - je_j||$$
$$\Leftrightarrow \langle x_m^*, e_1 \rangle - \langle x_m^*, je_j \rangle \leq \varepsilon_m ||e_1 - je_j||.$$

Thus,

$$\langle x_m^*, e_1 \rangle \leq \langle x_m^*, j_m e_{j_m} \rangle + \varepsilon_m \sqrt{1 + j_m^2}. \tag{5.18}$$

If j_m is not bounded, then we have $\lim_{m \to \infty} \langle x_m^*, j_m^{-1} e_{k_m} \rangle = 0$; contradicts to (5.17). Hence the sequence $\{j_m\}$ is finite. By extracting a subsequence, we can assume that $j_m \equiv j_*$ for all m, where $j_* > 1$ is a fixed index. So, from (5.18) one has

$$\langle x_m^*, e_1 \rangle \leq \langle x_m^*, j_* e_{j_*} \rangle + \varepsilon_m \sqrt{1 + j_*^2}. \tag{5.19}$$

Since $x_m^* \xrightarrow{w} e_1$, we have $\langle x_m^*, j_* e_{j_*} \rangle \to \langle e_1, j_* e_{j_*} \rangle = 0$ as $m \to \infty$. Taking the limit of (5.19) as $m \to \infty$, we obtain $1 \leq 0$, a contradiction. So, the case $\{x_m\} \subset \Omega_1$ cannot hold as well.

We have already proven that $N(0; \Omega)$ is not a closed cone.

To prove the assertion (iii), we will show that there exists a sequence $\{(x_k, x_k^*)\} \subset \Omega \times X^*$ such that

$$x_k \to 0, \quad x_k^* \in \widehat{N}(x_k; \Omega), \quad \text{and} \quad x_k^* \xrightarrow{w} 0, \tag{5.20}$$

but $||x_k^*|| \nrightarrow 0$ as $k \to \infty$. Assume that $k > 1$ is an arbitrary square integer. Putting $j = \sqrt{k}$, $x_k = \frac{j}{k} e_1 - \frac{1}{k} e_k$, and $x_k^* = j^{-1} e_1 + j^{-2} e_j + e_k$. Since $1 < j < k$ and the property (5.4), which is established in the proof of the assertion (ii), one has

$$x_k^* = j^{-1} e_1 + j^{-2} e_j + e_k = j^{-1}(e_1 + j^{-1} e_j + je_k) \in \widehat{N}(x_k; \Omega).$$

On the one hand, since $x_k = \frac{1}{\sqrt{k}} e_1 - \frac{1}{k} e_k$, then $x_k \to 0$ when $k \to \infty$ and k is a square integer. On the other hand, we have $x_k^* = \frac{1}{\sqrt{k}} e_1 + \frac{1}{k} e_{\sqrt{k}} + e_k \xrightarrow{w} 0$. Hence, the sequence $\{(x_k, x_k^*)\} \subset \Omega \times X^*$, where k is a integer square number, satisfies those conditions at (5.20). Since

$$||x_k^*|| = ||\frac{1}{\sqrt{k}} e_1 + \frac{1}{k} e_{\sqrt{k}} + e_k|| = \left(\frac{1}{k} + \frac{1}{k^2} + 1 \right)^{1/2} \geq 1$$

for all integer square numbers k, we constructed the sequence $\{(x_k, x_k^*)\} \subset \Omega \times X^*$ with the desired property. The claim (iii) has already been proven.

To verify the claim (iv), we will show that $\Omega \subset T(0; \Omega)$ and $\Omega \supset T(0; \Omega)$, respectively. Take an arbitrary $v \in \Omega$. We put $t_k = k^{-1}$ and $v^k = v$ for all k, one has $x_k = t_k v^k \in \Omega$ because Ω is a cone. This implies that $v \in T(0; \Omega)$. The next step, we take $v \in T(0; \Omega)$. Then, by the definition of the Bouligand-Severi tangent cone, there is a $t_k \to 0^+$ and $v^k \to v$ such that $t_k v^k \in \Omega$ for all k. Because Ω is a cone, the inclusion $t_k v^k \in \Omega$ follows $v^k \in \Omega$. By the closed property of Ω in the assertion (i) and the property $v^k \to v$, we imply that $v \in \Omega$. We have proven that $T(0; \Omega) = \Omega$.

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6 Bouligand tangent cones of a discrete non-SNC spiral set

The following example is an another non-SNC example, where the Bouligand-Severi tangent cone is not coincided with the weak tangent cone. This set was given in [13] and [3].

Let $X = \ell_2$ and $\{e_1, e_2, \cdots\}$ be its orthonormal basis. Consider the set

$$A = \{n^{-1}(e_1 + e_n) \mid n \geq 2\} \cup \{0\}.$$

We have the following assertions:

(a) The Bouligand-Severi tangent cone $T(0; A)$ is not agreed with the weak tangent cone $T_w(0; A)$,

(b) The set A is non-SNC at 0.

To prove the claim (a), one shows that $T(0; A) = \{0\}$ and $T_w(0; A) \neq \{0\}$, respectively. It is clear that $0 \in T(0; A)$. If $T(0; A) \neq \{0\}$ then there exists $v \in T(0; A)$ such that $v \neq 0$. By the definition of Bouligand-Severi tangent cone, one can find $t_k \to 0^+$ and $v^k \to v$ such that $t_k v^k \in A$ for all k. Then, for each k, there exists the index $n_k \in \mathbb{N}$, $n_k \geq 2$ such that

$$t_k v^k = n_k^{-1}(e_{n_k} + e_1) = (n_k^{-1}, 0, \ldots, 0, n_k^{-1}, 0, \ldots) \longrightarrow (0, \ldots, 0, \ldots). \tag{6.1}$$

If $\{n_k\}$ is not bounded, then by extracting a subsequence one can assume that $n_k \to \infty$. Thus, from (6.1), one has $t_k v^k \to 0$, that is $v = 0$; a contradiction. If $\{n_k\}$ is bounded, then without loss generality, one can suppose that $n_k = n$, $n \geq 2$ for all k. If $t_k v^k = n^{-1}(e_n + e_1)$, then (6.1) follows that

$$(n^{-1}, 0, \ldots, 0, n^{-1}, 0, \ldots) = (0, \ldots, 0, \ldots),$$

a contradiction. Therefore, one has $T(0; A) = \{0\}$. In order to prove $T_w(0; A) \neq \{0\}$, we will indicate that $e_1 \in T_w(0; A)$. By the definition,

$$T_w(0; A) = \{v \in X \mid \exists t_k \downarrow 0, \ \exists v^k \xrightarrow{w} v, \ t_k v^k \in A, \ \forall k\}. \tag{6.2}$$

For each $k \in \mathbb{N}$, $k \geq 2$ we put $t_k = k^{-1}$ and $v^k = e^1 + e^k$. For $u \in X^* \equiv \ell^2$, $u = (u_1, u_2, \ldots)$ one has $\langle u, v^k \rangle = u_1 + u_k \to u_1$ as $k \to \infty$. So, $v^k \xrightarrow{w} e^1$. Since $t_k v^k = k^{-1}(e^1 + e^k) \in A$ for all $k \geq 2$, (6.2) follows $e^1 \in T_w(0; A)$.

To prove the claim (b), first we will prove that for all $k \geq 2$,

$$\exists \delta_k > 0 \ \text{such that} \ A \cap B(x_k, \delta_k) = \{x_k\}, \tag{6.3}$$

where $x_k := k^{-1}(e_1 + e_k)$. For each $m \geq k + 1$, one has

$$\|x_m - x_k\|^2 = \|k^{-1}(e_1 + e_k) - m^{-1}(e_1 + e_m)\|^2 = (k^{-1} - m^{-1})^2 + k^{-2} + m^{-2}.$$

Hence, $\|x_m - x_k\| \geq k^{-1}$ for all $m \geq k + 1$. Putting

$$\delta_k = \min\left\{k^{-1}, \|x_1 - x_k\|, \ldots, \|x_{k-1} - x_k\|\right\},$$

we easily see that (6.3) is satisfied for all k. Next, we have $\widehat{N}(x_k; A) = X^* \equiv \ell_2$ for all $k \geq 2$. Choosing $e_k \in \widehat{N}(x_k; A)$, $k \geq 2$. Since $e_k \xrightarrow{w} 0$ when $k \to \infty$ and $\|e_k\| = 1$ for all $k \geq 2$, one concludes that the set A is non-SNC at 0.

7 Concluding remarks

This paper studies the SNC property of a given set under certain conditions, the necessary condition for the SNC property given by B.S. Mordukhovich, where a set has finite codimension. This is a crucial signal to check the SNC property of a given point. However, there are some sets that are non-SNC that it satisfies the finite codimension. These sets are usually very twisted and have complicated structures.

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