Deriving the weak form of Poisson Equation from the strong form 6/3/2025

$$\frac{\partial^2 u}{\partial x^2} + f = 0$$

$u(0) = u(1) = 0$ (boundary cond.)

- The test functions V, must hav similar boundary conditions to the solution u

Multiply by test functions: $v\left(\frac{\partial^2 u}{\partial x^2} + f\right) = 0$

Integrate over domain: $\int_0^1 v\left(\frac{\partial^2 u}{\partial x^2} + f\right) dx = 0$

Integration by parts: $v\frac{\partial u}{\partial x}\Big|_0^1 - \int_0^1 \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}\,dx + \int_0^1 v f\,dx = 0$

to get rid of the 2nd order derivative

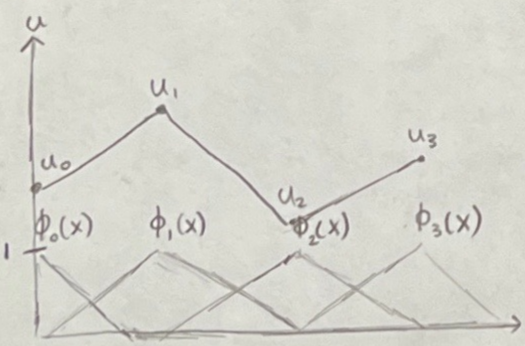
Enforce boundary conditions on test functions to eliminate terms:

$v(0) = v(1) = 0$

$\therefore \quad v\frac{\partial u}{\partial x}\Big|_0^1 = 0 \rightarrow \boxed{\int_0^1 \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}\,dx = \int_0^1 v\cdot f\,dx}$ weak form

Deriving a linear system for finite elements (from the weak form) (Galerkin projection) 6/4/2025

- Need to find a u that satisfies some simplifying constraints (continuous & piecewise)
  - represent u as a piecewise linear function and thus it becomes a linear combination of basis/shape functions
- Let's say there's 3 elements ($\therefore$ 4 nodal values) and u is the solution curve



- These basis functions ϕ_i sort of represent the discretization of space; they are equal to 1 at the point in space where the element resides, 0 elsewhere, but linearly interpolates with neighboring elements.
- With these basis functions, we can then define an approximation of the solution curve u

$$u(x) = u_0\phi_0(x) + u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x)$$

$$u(x) = \sum_{i=0}^{n} u_i\,\phi_i(x)$$

u_i are scaler values that represent points on the solution curve

$\phi_i(x)$ are basis functions

- The next step is to then enforce constraints on the test functions $(v(x))$ to only be in the space of the basis functions so that we can use it in the weak forms

($\forall$ = for any)

$$\forall\ v:\ v(0) = v(1) = 0 \quad \rightarrow \quad v(x) = \phi_i(x)$$ ← the set of our basis functions

- Reasoning: if you enforce the weak form for only ϕ_i, then you also enforce it for any linear combination of ϕ_i

- So, replace $v(x)$ with $\phi_i(x)$ in the weak form:

$$\int_0^1 \frac{\partial \phi_i}{\partial x} \frac{\partial u}{\partial x}\, dx = \int_0^1 \phi_i f\, dx$$

$$\int_0^1 \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial}{\partial x}\left(\sum_{j=1}^{n-1} u_j \phi_j(x)\right) dx = \int_0^1 \phi_i(x) \cdot f\, dx$$

$$\sum_{j=1}^{n-1} \left(u_j \underbrace{\int_0^1 \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial \phi_j}{\partial x}\, dx}_{a_{ij}} \right) = \underbrace{\int_0^1 \phi_i(x) \cdot f\, dx}_{b_i}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n-1,n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n-1,1} & \cdots & \cdots & a_{n-1,n-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

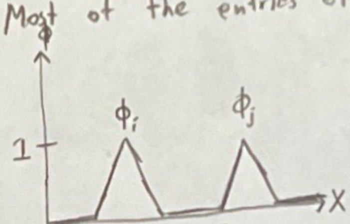
- Now, to convert to a system of linear equations, we only need two equations because of boundary conditions (only two unknowns)
- The summation can be taken out of the derivative and integral because $\frac{d}{dx}[a+b] = \frac{d}{dx}a + \frac{d}{dx}b$, etc
- u_j can also be taken out since it is constant over x
- We could now use a system solver to find the nodal values u_j for u, but we still need to compute the integrals for the LHS matrix (stiffness matrix) and the RHS column vector (force vector)

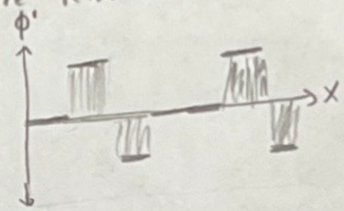
Computing the integrals for the matrices

$$a_{ij} = \int_0^1 \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial \phi_j}{\partial x}\, dx$$

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & \ddots & & 0 & 0 & 0 \\ 0 & & \ddots & & 0 & 0 \\ 0 & 0 & & \ddots & & 0 \\ 0 & 0 & 0 & & \ddots & \\ 0 & 0 & 0 & 0 & & a_{n-1,n-1} \end{bmatrix}$$

- Most of the entries of the RHS matrix are going to be 0





- if ϕ_i and ϕ_j do not correspond to neighboring elements then the value of the integral will equal zero; there needs to be an overlap of the basis functions

- The entries solely being on the diagonal is only because the problem is defined in 1D and the elements are ordered from left to right. Nonetheless, usually, the majority of the entries will be 0.

- Now we have to compute the integrals when there is overlap between the basis functions

- For the rest of the LHS matrix entries where the basis functions actually overlap

Case 0: $|i-j| > 1 \rightarrow a_{ij} = 0$ (as written previously)

Case 1: $i = j$ (center diagonal)

$$a_{ii} = \int_0^1 \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial \phi_i}{\partial x}\, dx = \int_0^1 \left(\frac{\partial \phi_i}{\partial x}\right)^2 dx$$

- X_i represents the x-pos corresponding to u_i; $u(x_i) = u_i$

$$= \int_{X_{i-1}}^{X_i} \left(\frac{\partial \phi_i}{\partial x}\right)^2 dx + \int_{X_i}^{X_{i+1}} \left(\frac{\partial \phi_i}{\partial x}\right)^2 dx$$

- We can split into two integrals and ignore everything outside of the interval $[X_{i-1}, X_{i+1}]$ because $\phi_i(x) = 0$ outside the domain of each element

$$= \int_{X_{i-1}}^{X_i} \left(\frac{1}{X_i - X_{i-1}}\right)^2 dx + \int_{X_i}^{X_{i+1}} \left(\frac{1}{X_{i+1} - X_i}\right)^2 dx$$

- Basis functions are linear so the $\frac{\partial}{\partial x}$ can be found using $\frac{\text{rise}}{\text{run}}$

$$= \boxed{\frac{1}{X_i - X_{i-1}} + \frac{1}{X_{i+1} - X_i}}$$

- When integrating, X_i is constant so the whole integrand comes straight out of the integral.

Case 2: $i = j+1$ or $i = j-1$ (upper and lower diagonals)

$$a_{i\,i-1} = \int_0^1 \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_{i-1}}{\partial x}\, dx$$

$$= \int_{X_{i-1}}^{X_i} \left(\frac{1}{X_i - X_{i-1}}\right)\left(-\frac{1}{X_i - X_{i-1}}\right) dx$$

$$= \int_{X_{i-1}}^{X_i} -\left(\frac{1}{X_i - X_{i-1}}\right)^2 dx = \boxed{-\frac{1}{X_i - X_{i-1}}}$$

- Computing the integrals for the RHS column vector

For simplification, we are currently assuming that $f(x) = 1$ (so, $\frac{\partial^2 u}{\partial x^2} = -1$)

$$b_i = \int_0^1 \phi_i\, dx$$

$$= \boxed{\frac{X_{i+1} - X_{i-1}}{2}}$$

- The basis functions are really just triangles with base $= X_{i+1} - X_{i-1}$ and height $= 1$. The integral gives the area, which for a triangle is $A = \frac{1}{2}bh$

- Now that all of the integrals have been computed, you can solve the linear system. Make sure to replace X_i with the positions of the nodal values u_i.

Applying Dirichlet Boundary Conditions to Poisson's Equation in 1D 6/6/2025

$$\frac{\partial^2 u}{\partial x^2} + f(x) = 0$$

$f(x) = 1$

$u(0) = \alpha, \ u(1) = \beta$

$$\int_0^1 \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial x} dx = \int_0^1 v \cdot f \, dx$$

$$\int_0^1 \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial}{\partial x}\left(\sum_{j=1}^{n-1} (u_j \phi_j) + \alpha \phi_0 + \beta \phi_n \right) dx = \int_0^1 \phi_i \cdot f \, dx$$

$$\sum_{j=1}^{n-1} \Big(\underbrace{u_j}_{u_i} \underbrace{\int_0^1 \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial \phi_j}{\partial x} dx}_{a_{ij}} \Big) + \underbrace{\alpha \int_0^1 \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial \phi_0}{\partial x} dx}_{C_{left}} + \underbrace{\beta \int_0^1 \frac{\partial \phi_i}{\partial x} \cdot \frac{\partial \phi_n}{\partial x} dx}_{C_{right}} = \underbrace{\int_0^1 \phi_i \cdot f \, dx}_{b_i}$$

Linear system of equations: $a_{ij} u_i + \alpha C_{left} + \beta C_{right} = b_i$

- C_{left} and C_{right} are both column vectors that contain known values and thus can be subtracted to the right hand side.

Applying Neumann Boundary Conditions to Poisson's Equation in 1D 6/8/2025

$$\frac{\partial^2 u}{\partial x^2} + f(x) = 0 \qquad f(x) = 1 \qquad u(0) = \alpha \qquad \left.\frac{\partial u}{\partial x}\right|_{x=1} = \beta$$

- Neumann boundaries require a change in the weak form
- Also, the entire boundary cannot be Neumann, otherwise there is nothing to constrain the solutio

$$\int_0^1 v \left(\frac{\partial^2 u}{\partial x^2} + f \right) dx = 0$$

$$\left. v \frac{\partial u}{\partial x} \right|_0^1 - \int_0^1 \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dx + \int_0^1 v f \, dx = 0$$

$$\int_0^1 \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dx = \int_0^1 v f \, dx + \beta v(1)$$

- Since $v(0) = 0$ (Dirichlet BC is enforced strongly), only the term $v(1) \left.\frac{\partial u}{\partial x}\right|_{x=1}$ remains
- We do not need to manage Dirichlet BC in the weak form (since they give known values) and they only pop up during the conversion to a linear syste

$$\sum_{j=1}^{n} u_j \int_0^1 \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dx + \alpha \int_0^1 \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_0}{\partial x} dx = \int_0^1 v f \, dx + \beta \phi_i(1)$$

- both $\alpha \int_0^1 \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_0}{\partial x} dx$ and $\beta \phi_i(1)$ are column vectors
- note that $\phi_i(1)$ should equal $v(1)$

Linear system: $a_{ij} u_i = b_i - \alpha C_{dirichlet} + \beta C_{neumann}$

How to compute integrals in general using Gaussian Quadrature   6/8/2025

- How do we compute integrals like $\int_0^1 \phi_i f \, dx$ for any arbitrary function f?

  $g(x) = \phi_i(x) f(x)$ where $f(x)$ could

- If we want to compute these integrals element by element we have to map the element interval $[x_j, x_{j+1}]$ to the reference interval $[-1, 1]$ used by most quadrature rules

$$\int_{x_j}^{x_{j+1}} g(x)\,dx = \sum_{k=1}^{m} g\left(\xi_k\left(\frac{x_{j+1}-x_j}{2}\right) + \left(\frac{x_j + x_{j+1}}{2}\right)\right) W_k \left(\frac{x_{j+1}-x_j}{2}\right)$$

$\frac{dx}{d\xi}$ is the Jacobian in 1D which needs to be multiplied because we are performing a change of variable in this integral

- To find good Gaussian points for the 1D $[-1, 1]$ reference element, there is a table on the Wikipedia page for Gaussian Quadrature.

- For triangular elements we follow a similar strategy. First, convert from the triangular element to the reference triangle by using an affine transformation matrix.

Element Triangle (T)	Reference Triangle ($\hat{T}$)
$v_1 = (x_1, y_1)$	$(0, 0)$
$v_2 = (x_2, y_2)$	$(1, 0)$
$v_3 = (x_3, y_3)$	$(0, 1)$

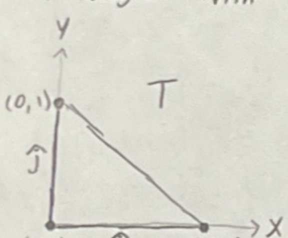
- To make a transformation where $\hat{T} \to T$, $\phi(\xi, \eta)$

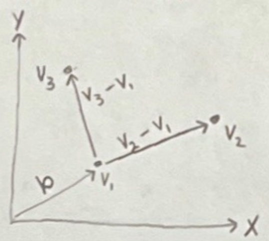
$$\phi(\xi, \eta) = A \cdot \begin{bmatrix} \xi \\ \eta \end{bmatrix} + b$$

- We want v_1 to correspond to $(0,0)$

  $\therefore b = v_1$

- $(1,0)$ and $(0,1)$ are sort of like the basis vectors for the reference triangle with their origin at $(0,0)$. So, we need to describe the corresponding basis vectors for the element triangle with their origin at v_1.





$\hat{i} \to v_2 - v_1$

$\hat{j} \to v_3 - v_1$

which means $A = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}$

so, $\phi(\xi, \eta) = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

component-wise that becomes:

$$\phi_x(\xi, \eta) = x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta$$

$$\phi_y(\xi, \eta) = y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta$$

- But we also need to find the Jacobian determinant to account for the scaling of areas under the affine transformation matrix.

$$|\det J| = \begin{vmatrix} \frac{\partial \phi_x}{\partial \xi} & \frac{\partial \phi_x}{\partial \eta} \\ \frac{\partial \phi_y}{\partial \xi} & \frac{\partial \phi_y}{\partial \eta} \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$$

- Now we can write the Gaussian Quadrature approximation.

$$\iint_T f(x,y)\,dx\,dy = \iint_{\hat{T}} f(\phi_x(\xi,\eta),\ \phi_y(\xi,\eta))\,|\det J|\,d\xi\,d\eta$$

$$= |\det J| \sum_{k=1}^{m} W_k f(\phi_x(\xi_k,\eta_k),\ \phi_y(\xi_k,\eta_k))$$

m: number of Gaussian points

W_k: weight of the k^{th} Gaussian point

ξ_k: x-coordinate of the k^{th} Gaussian point on the reference triangle

η_k: y-coordinate of the k^{th} Gaussian point on the reference triangle

Assembling the matrices element by element 6/9/2025

- Since most of the basis functions do not overlap, many of the integrals will evaluate to 0. Instead of computing the integrals for each pair of i and j for every single entry of the matrix, we only have to compute them for each pair that is associated with a specific element.

- For example, in the 1D case, each element is associated with 2 nodal values, so therefore there will be 4 possible pairs of i and j for which we need to evaluate integrals for. Once we do that, we can simply add the integral to the corresponding matrix entry.

- In the 2D case with triangular elements, we follow a similar strategy only this time each element is associated with 3 nodal values meaning there are 9 possible pairs and thus 9 integrals to evaluate and add to the matrix.

- Note that since we compute the integrals over the domain of a specific element, we need to accumulate the values in the matrix entries to get the full value of the integral. This is shown mathematically:

$$A_{ij} = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\,dx$$

- The basis functions are non-zero on a domain that is larger than one single element, so that is why accumulation across all elements is necessary.

Solving Unsteady PDEs with Finite Element Method: Heat Equation 6/9/2025

- Follow a similar strategy to Poisson's Equation to convert from strong to weak form and to a linear system.

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad u(0,t) = u(1,t) = 0 \quad \text{across the interval } [0, 1]$$

$$v \frac{\partial u}{\partial t} = \kappa v \frac{\partial^2 u}{\partial x^2}$$

$$\int_0^1 v \frac{\partial u}{\partial t} dx = \kappa \int_0^1 v \frac{\partial^2 u}{\partial x^2} dx$$

$$\int_0^1 v \frac{\partial u}{\partial t} dx = \kappa \left(v \frac{\partial u}{\partial x}\Big|_0^1 - \int_0^1 \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dx \right)$$

- $v(0) = v(1) = 0$ to match constraints of u, so the boundary term goes to 0

$$\frac{\partial}{\partial t} \int_0^1 v \cdot u \, dx = -\kappa \int_0^1 \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dx$$

$$\sum_{j=1}^{n-1} \underbrace{\frac{\partial u_j}{\partial t}}_{\frac{du}{dt}} \underbrace{\int_0^1 \phi_i \phi_j \, dx}_{M_{ij}} = -\kappa \underbrace{\sum_{j=1}^{n-1} u_j}_{u} \underbrace{\int_0^1 \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dx}_{A_{ij}}$$

$$M \frac{du}{dt} = -\kappa A u$$

- Now we can use the finite difference method to discretize the time domain. You can either use explicit or implicit (forward or backward) Euler method, but both ways require the inversion of a matrix of equivalent size, so the implicit method is more often used.

Explicit: $\frac{du}{dt} \approx \frac{u_{n+1} - u_n}{\Delta t}$

Implicit: $\frac{du}{dt} \approx \frac{u_n - u_{n-1}}{\Delta t}$

- The underlined portions to the left indicate the matrices which must be inverted to solve the systems explicitly and implicitly respectively.

Explicit Scheme:

$$M \frac{u_{n+1} - u_n}{\Delta t} = -\kappa A u_n$$

$$\underline{M} u_{n+1} = M u_n - \kappa \Delta t A u_n$$

Implicit Scheme:

$$M \frac{u_n - u_{n-1}}{\Delta t} = -\kappa A u_n$$

$$\frac{M u_n}{\Delta t} + \kappa A u_n = \frac{M u_{n-1}}{\Delta t}$$

$$\underline{\left(\frac{M}{\Delta t} + \kappa A \right)} u_n = \frac{M u_{n-1}}{\Delta t}$$

Deriving the weak form of the Poisson Equation in 2D 6/9/2025

$$\nabla \cdot \nabla u + f = 0 \quad \text{with} \quad f = 1 \quad \text{and} \quad u\big|_{\partial\Omega} = 0$$

$$\int_\Omega v(\nabla \cdot \nabla u)\, dx + \int_\Omega v f\, dx = 0$$

$$\int_\Omega \nabla v \cdot \nabla u\, dx + \int_{\partial\Omega} \vec{n}\cdot(\overbrace{v}^{v \text{ is 0 along the boundary due to Dirichlet BC}}\nabla u)\, dx + \int_\Omega v f\, dx = 0$$

$$\boxed{\int_\Omega \nabla v \cdot \nabla u\, dx = \int_\Omega v f\, dx}$$

- We can use the Divergence Theorem and Green's First Identity to expand this integral.

Where g is a scalar field and $\vec{h}$ is a vector field, $\vec{n}$ is the vector normal to the boundary $\partial\Omega$

$$\int_\Omega \nabla\cdot(g\vec{h})\, dx = \int_{\partial\Omega} \vec{n}\cdot(g\vec{h})\, ds$$

Green's First Identity (product rule but for vector calculus) states that:

$$\nabla\cdot(g\vec{h}) = \vec{h}\cdot\nabla g + g\nabla\cdot\vec{h}$$

$$\int_\Omega h\cdot\nabla g\, dx + \int_\Omega g\nabla\cdot\vec{h}\, dx = \int_{\partial\Omega} \vec{n}\cdot(g\vec{h})\, ds$$

So now we can say that:

$$\vec{h} = \nabla u$$
$$g = v$$

Converting to a linear system and computing integrals in 2D 6/9/2025

$$\sum_{j=1}^{n-1} u_j \int_\Omega \nabla\phi_i \cdot \nabla\phi_j\, dx = \int_\Omega \phi_i f\, dx$$

- We already know how we can assemble and solve these linear systems in general, but the problem arises here that we do not know how to compute the gradients $\nabla\phi_i$ and $\nabla\phi_j$. To find these we can set up another linear system.

$$\nabla\phi_i \cdot (X_i - X_j) = \phi_i(X_i) - \overbrace{\phi_i(X_j)}^{\text{only true if } i \neq j!} = 1$$

(this is because ϕ_i is linear across the element)

$$\Rightarrow \begin{bmatrix} X_1 - X_2 & Y_1 - Y_2 \\ X_1 - X_3 & Y_1 - Y_3 \end{bmatrix} \begin{bmatrix} \nabla_x \phi_1 \\ \nabla_y \phi_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- This system is for the basis function where $\phi_1(X_1, Y_1) = 1$. So to solve for $\nabla\phi_1$ you simply have to solve this system

$$\nabla\phi_1 = \langle \nabla_x \phi_1, \nabla_y \phi_1 \rangle$$

- The points (X_2, Y_2) and (X_3, Y_3) have to be points adjacent to (X_1, Y_1) in order for the system to remain true.

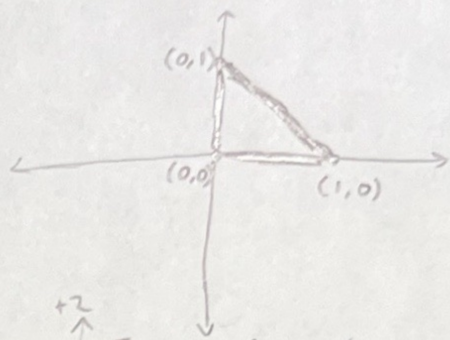
- Then to compute $\int_\Omega \phi_i f\, dx$, you can use the 2D Gaussian Quadrature outlined in previous notes

Assembling the Global Stiffness Matrix for 2D with Linear Basis Functions 6/16/2025

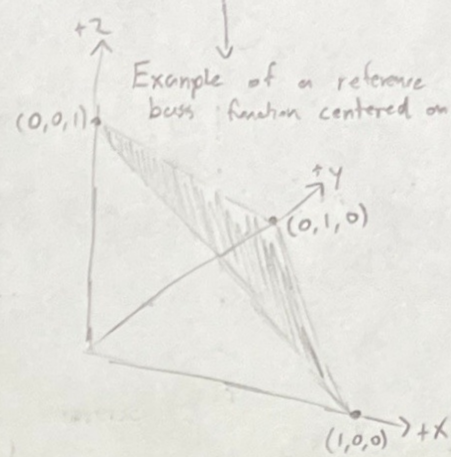
- We can evaluate the integral $\int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, d\Omega$ by transforming the reference triangle and its corresponding reference basis functions:

$$\hat{\phi}_1 = 1 - \xi - \eta \qquad \hat{\phi}_2 = \xi \qquad \hat{\phi}_3 = \eta$$

Reference Triangle



- Where (ξ, η) corresponds to (x, y) on the reference triangle. Each reference basis function defines a plane in 3D space.
- The gradients of these linear reference basis functions can trivially computed (constant since they are planes) and then transformed by multiplying by the inverse of the Jacobian matrix. (see notes page 5)



$$\nabla\hat{\phi}_1 = \langle -1, -1 \rangle \qquad \nabla\hat{\phi}_2 = \langle 1, 0 \rangle \qquad \nabla\hat{\phi}_3 = \langle 0, 1 \rangle$$

$$(\xi, \eta) \rightarrow (x, y) \text{ with } A\begin{bmatrix} \xi \\ \eta \end{bmatrix} + b$$

$$\therefore \; J = A$$

$$\nabla\phi_1 = \nabla\hat{\phi}_1 \cdot J^{-1} \rightarrow \nabla\phi_1^T = (J^{-1})^T (\nabla\hat{\phi}_1)^T$$

- This can get compacted nicely with matrix multiplication

$$\begin{bmatrix} \nabla_x\phi_1 & \nabla_x\phi_2 & \nabla_x\phi_3 \\ \nabla_y\phi_1 & \nabla_y\phi_2 & \nabla_y\phi_3 \end{bmatrix} = (J^{-1})^T \begin{bmatrix} \nabla_x\hat{\phi}_1 & \nabla_x\hat{\phi}_2 & \nabla_x\hat{\phi}_3 \\ \nabla_y\hat{\phi}_1 & \nabla_y\hat{\phi}_2 & \nabla_y\hat{\phi}_3 \end{bmatrix}$$

$$\begin{bmatrix} \nabla_x\phi_1 & \nabla_y\phi_1 \\ \nabla_x\phi_2 & \nabla_y\phi_2 \\ \nabla_x\phi_3 & \nabla_y\phi_3 \end{bmatrix} \begin{bmatrix} \nabla_x\phi_1 & \nabla_x\phi_2 & \nabla_x\phi_3 \\ \nabla_y\phi_1 & \nabla_y\phi_2 & \nabla_y\phi_3 \end{bmatrix} = \begin{bmatrix} \nabla\phi_1\cdot\nabla\phi_1 & \nabla\phi_1\cdot\nabla\phi_2 & \nabla\phi_1\cdot\nabla\phi_3 \\ \nabla\phi_2\cdot\nabla\phi_1 & \nabla\phi_2\cdot\nabla\phi_2 & \nabla\phi_2\cdot\nabla\phi_3 \\ \nabla\phi_3\cdot\nabla\phi_1 & \nabla\phi_3\cdot\nabla\phi_2 & \nabla\phi_3\cdot\nabla\phi_3 \end{bmatrix}$$

- From there, multiplying the matrix by the area of the triangle will compute the integral for 9 pairwise basis functions associated with an element. Then, it is just matter of mapping the indices of this local stiffness matrix to the global one.

Assembling the Global Mass Matrix for Linear Basis Functions in 2D

6/16/2025

$$\int_{\Omega} \phi_i \phi_j \, d\Omega = \iint_{T_k} \phi_i \phi_j \, dx\, dy = |\det J_k| \iint_{\hat{T}} \hat{\phi}_i \hat{\phi}_j \, d\xi \, d\eta$$

where i goes from $1 \to 3$ and j goes from $1 \to 3$

$$M_{local} = |\det J_k| \begin{bmatrix} \iint_{\hat{T}} \hat{\phi}_1 \hat{\phi}_1 \, d\xi \, d\eta & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & \iint_{\hat{T}} \hat{\phi}_3 \hat{\phi}_3 \, d\xi \, d\eta \end{bmatrix}$$

$$\hat{\phi}_1 = 1 - \xi - \eta$$

$$\hat{\phi}_2 = \xi$$

$$\hat{\phi}_3 = \eta$$

$$M_{local} = |\det J_k| \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{bmatrix}$$

- $|\det J_k|$ ends up being double the area of the element triangle

$$M_{local} = \frac{A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

- Then, just like the local stiffness matrix, map the local indices to the global indices