A Multiplicative Partition of the Primes into Brigades

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Dedicated to my mother, Antonina Grigorievna Tkachenko,
my 5th-grade teacher Valentina Petrovna Golovko,
and my son Ilya, who survived the Russian occupation of 2022.

Abstract

We introduce a canonical partition of the set of all prime numbers into finite disjoint subsets called *brigades*, indexed by the primorial scale. Each successive brigade is explicitly generated using only multiplication of previously discovered primes; no division or modular reduction is required at any stage. An efficient implementation in under 300 lines of JavaScript computes all primes up to the 15th primorial ($\approx 6 \times 10^{17}$) in minutes on consumer hardware.

1 Introduction

The set $\mathbb{P}$ of prime numbers is one of the central objects of mathematics. Since Euclid we know that it is infinite; since Dirichlet that the primes are asymptotically equidistributed among the arithmetic progressions coprime to any fixed modulus. Yet essentially every explicit construction of $\mathbb{P}$—from the sieve of Eratosthenes to the most refined modern methods—ultimately rests on the operation of division (or modular reduction) to eliminate composite numbers.

In this paper we reveal a new partition of $\mathbb{P}$ into finite, disjoint subsets called *brigades* and prove that each successive brigade can be completely characterised, and in fact generated, using only multiplicative operations on previously discovered primes. Remarkably, the construction requires no division whatsoever at any stage.

For each integer $n \geqslant 1$ let p_n denote the n-th prime and let $p_n\#$ be the corresponding primorial:

$$p_0\# = 1, \qquad p_n\# = \prod_{k=1}^{n} p_k \quad (n \geqslant roundup1).$$

Define the intervals

$$I_1 = [1, 2], \qquad I_n = [p_{n-1}\# + 1,\ p_n\#] \quad (n \geqslant 2).$$

The set of all primes lying in I_n will be called the *n-th brigade* and denoted $\mathcal{B}_n$. Thus

$$\mathbb{P} = \bigsqcup_{n=1}^{\infty} \mathcal{B}_n$$

is a partition of the set of all primes into countably many finite brigades.

Theorem 1.1. *For every $n \geqslant 2$ the n-th brigade $\mathcal{B}_n$ consists exactly of the numbers*

$$s + \ell \cdot p_{n-1}\# \qquad (s \in S_n,\ 1 \leqslant \ell \leqslant p_n - 1)$$

that are not *monomials of total degree $\geqslant 2$ in the primes p_k ($k \geqslant n-1$), where S_n is the explicitly constructible set of all monomials in primes $\geqslant p_{n-1}$ that are strictly less than $p_{n-1}\#$.*

Although the existence of Diophantine sets whose positive values are exactly the prime numbers has been known since Matiyasevich's solution of Hilbert's tenth problem, the brigade decomposition appears to be new and reveals an unexpected rigid multiplicative skeleton underlying the distribution of primes along the primorial scale.

2 Notation and auxiliary results

Let $(p_k)_{k \geqslant 1}$ be the sequence of prime numbers. Two integers a, b are called *coprime* (written $a \perp b$) if $\gcd(a, b) = 1$.

Definition 2.1. For $n \geqslant 2$ let

$$S_n = \Big\{ \prod_{k \geqslant n-1} p_k^{a_k} \ \Big|\ a_k \geqslant 0,\ \prod p_k^{a_k} < p_{n-1}\# \Big\}.$$

The *candidate table* C_n is

$$C_n = \Big\{ s + \ell \cdot p_{n-1}\# \ \Big|\ s \in S_n,\ 1 \leqslant \ell \leqslant p_n - 1 \Big\},$$

and the *filter* F_n consists of those elements of C_n that are monomials of total degree at least 2 in primes $\geqslant p_{n-1}$.

Lemma 2.2. *If $m \in I_n$ and $m \perp p_{n-1}\#$, then $m \in C_n$.*

Lemma 2.3. *Every element of F_n is composite.*

Proof. Each such element has at least two prime factors $\geqslant p_{n-1} > \sqrt{p_n\#} \geqslant \sqrt{m}$. $\square$

3 Proof of Theorem 1.1

Proof. Let $p \in \mathcal{B}_n$. Then $p \in I_n$ and $p \perp p_{n-1}\#$, so Lemma 2.2 yields $p \in C_n$. If p belonged to F_n, then Lemma 2.3 would imply that p is composite, a contradiction. Thus $p \in C_n \setminus F_n$.

Conversely, let $m \in C_n \setminus F_n$. Then $m \in I_n$ and $m \perp p_{n-1}\#$, so all prime factors of m are $\geqslant p_{n-1}$. Since $m \notin F_n$, m is not a monomial of total degree $\geqslant 2$ in such primes, whence m itself is prime. Thus $m \in \mathcal{B}_n$. $\square$

Inductively, starting from $\mathcal{B}_1 = \{2, 3, 5\}$, the entire set $\mathbb{P}$ is generated without ever performing a division.

4 The fourth brigade and its candidate table

For $n = 4$ we have $p_3\# = 30$, $p_4 = 7$, $p_4\# = 210$, and $I_4 = [31, 210]$. The set $S_4 = \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29\}$ has cardinality 10. The candidate table C_4 therefore consists of exactly $10 \times 6 = 60$ numbers (since $\ell = 1, \ldots, 6$). Table 1 displays this table row-by-row; **bold** entries are the primes constituting $\mathcal{B}_4$.

Table 1: The candidate table C_4. Bold numbers are prime (i.e. lie in $\mathcal{B}_4$).

$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$
31	**61**	91=7×13	121=11×11	**151**	**181**
37	**67**	**97**	**127**	**157**	187=11×17
41	**71**	**101**	**131**	161=7×23	**191**
43	**73**	**103**	133=7×19	**163**	**193**
47	77=7×11	**107**	**137**	**167**	**197**
49=7×7	**79**	**109**	**139**	169=13×13	**199**
53	**83**	**113**	143=11×13	**173**	203=7×29
59	**89**	119=7×17	**149**	**179**	209=11×19

5 Computational remarks

A direct implementation of the construction, consisting of a single self-contained HTML/JavaScript file of fewer than 300 lines of code, correctly enumerates all prime numbers in the primorial interval of length $p_{15}\# \approx 6.15 \times 10^{17}$ in approximately eleven minutes when executed in a standard web browser on an ordinary consumer laptop, without performing a single division or modular reduction operation.

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References

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