The authors showed that, empirically, the model fit is not overly sensitive to the choice of distributions among sensible alternatives. It is not clear, however, that this is always true at the extreme percentiles of the distributions. Important to the practitioner, in any case, is that the parameters keep some operational meaning. The models chosen by the authors accomplish this. All of the competing models have numerous parameters, leading to difficulties in the maximum likelihood procedure. Castillo and Hadi (1995) considered this a serious flaw in the estimation of the S-N curve and derived simpler regression-type estimation alternatives.

From Table 3, we might infer that the extreme correlation of the maximum likelihood estimates (MLE's) might be due to overparameterization in the fatigue-limit model. Although the authors carefully address the problems associated with such correlations, it made the task of computing MLE's a difficult one for us and makes awkward any discussion of the joint distribution of the model-parameter estimates. One consequence of the model's lack of fit can be seen in the S-N plots (Figs. 1 and 10) for both examples, along with the plots of the residuals (Figs. 9 and 16).

For the laminate panel data, the model seems to fit the data better at the higher stress levels (280 MPa and above) compared to the lowest stress level (270 MPa). This cannot be reflected in the overall tests of model fit. Although it is not stated in the article (or the corresponding references), we might assume that the fit of the model is important at low (regular-use) stresses. This assumption is analogous to those made in accelerated life tests, for example.

Overall, the guidelines laid out by Pascual and Meeker are thorough, succinct, and carefully explained. The article entices statisticians with an interesting problem—estimating a random fatigue limit in an S-N curve. It also invites material scientists and engineers to apply a sensible algorithm to characterize and verify the S-N curve. We enjoyed reading the article and found the examples especially motivating.

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Discussion

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In an extremely readable article, Pascual and Meeker (PM) have developed a new random fatigue-limit model for use in analyzing fatigue data. The authors use a random fatigue limit to model the curvature that often exists in fatigue curves as well as to capture the inverse relationship between the standard deviation of fatigue life and stress. Using maximum likelihood (ML) methods, they fit the model to two different datasets and thoroughly examine the fits using a variety of diagnostic probability and residual plots as well as goodness-of-fit techniques. These two examples clearly illustrate the potential widespread application of the model, and we compliment the authors for an interesting and useful article.

Because the fatigue limit associated with each specimen is considered to be a random effect, it occurred to us that hierarchical Bayes (HB) might represent a useful alternative way to model and analyze such data. We now describe this approach and present the results for the same laminate-panel fatigue data considered by PM. We also present some important advantages and disadvantages of this alternative approach in Section 2.

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1. AN ALTERNATIVE HIERARCHICAL BAYES APPROACH

Using the PM notation, a directed graph for the random fatigue-limit model is shown in Figure 1. A solid arrow indicates a stochastic dependency, and a dashed arrow indicates a logical function. The two undirected dashed lines represent logical range constraints resulting from the censored data and the restriction that $\gamma < \mathrm{s}$ for uncensored observations.

For illustration, we consider the laminate-panel fatigue data in conjunction with the normal-normal model. Our HB approach proceeds as follows. We first specify prior distributions for the five parameters outside the "specimen (observation) i" box in Figure 1. Inference on these five unknown parameters requires that we obtain samples from the joint posterior $\pi(\beta_0, \beta_1, \sigma^2, \mu_\gamma, \sigma_\gamma^2|\mathbf{Y})$, which of course also depends on the observed stresses s as well as knowledge of which observations have been censored (denoted by **Y.cen**). We then use empirical summary statistics from these samples to draw the desired inferences. To calculate samples from this joint posterior, we must successively sample from the full conditional distributions; that is, we must successively sample from the conditional distribution of each stochastic node given all the others in the graph.

Consider the joint prior distribution $\pi(\beta_0, \beta_1, \sigma^2, \mu_\gamma, \sigma_\gamma^2)$. Our initial calculations suggest that, for the "standard" noninformative prior $\pi(\beta_0, \beta_1, \sigma^2, \mu_\gamma, \sigma_\gamma^2) \propto 1/(\sigma^2\sigma_\gamma^2)$, the posterior is, in fact, unbounded if $v_i = \mu_\gamma$ for all i and $\sigma_\gamma \to 0$. In this case, the posterior distributions on the model parameters are highly irregular as well. A similar situation also occurs when $\sigma = 0$ and all observations are predicted exactly by inclusion of the random effects γ_i. In both of these cases we suspect that this posterior is improper and places infinite mass at the modes mentioned previously. Similarly, certain other relationships between the parameters may also lead to an improper posterior distribution. A similar problem exists for the PM likelihood approach—the likelihood function has unbounded modes at $\sigma^2 = 0$ (when the random fatigue limits are permitted to provide an exact fit to the data) and $\sigma_\gamma^2 = 0$.

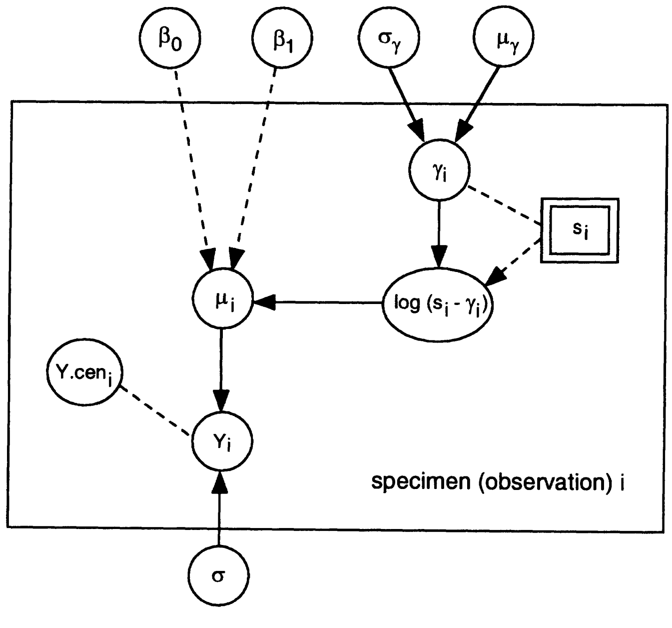

Figure 1. Directed Graph for the Hierarchical Bayes Analysis of the Random Fatigue-Limit Model.

Table 1. Hierarchical Bayes Laminate Model Parameter Estimates

Parameter	Mean	Median	Standard error	95% interval
β_0	28.201	27.979	2.610	(23.701, 34.438)
β_1	−4.723	−4.686	.472	(−5.833, −3.898)
σ	.467	.473	.052	(.341, .556)
μ_γ	5.396	5.400	.039	(5.301, 5.460)
σ_γ	.011	.009	.006	(.004, .026)

To ensure a proper posterior distribution, we consider the following informative prior distribution. The marginal prior on μ_γ was taken to be $\mathrm{N}(\mu_0, \sigma_0)$. Based on the stress levels chosen in the experiment, we empirically choose $\mu_0 = \log(250) = 5.52$. We have also chosen the prior standard deviation to be approximately 50 on the original scale, which implies that $\sigma_0 = .5[\log(300) - \log(200)] = .20$.

The informative marginal prior on σ_γ^2 is taken to be an inverse gamma $\mathrm{IG}(\alpha_0, \lambda_0)$ distribution. Examination of the functional equation producing the mean fatigue life μ indicates that random errors in the fatigue limit affect the predictions of mean life by a factor of

$$\exp\{\beta_1 \log[s - \exp(v)]\} = [s - \exp(\mu_\gamma + \text{error})]^{\beta_1}.$$

For β_1 near $-5, s = 300$ and $\mu_\gamma = \log(250)$, which is its prior mean, the change in mean fatigue life associated with a random error of .01 is

$$[300 - 250\exp(.01)]^{-5}/(300 - 250)^{-5},$$

or approximately 23%. Assuming that such changes in mean represent a two-standard-deviation change beyond that from a zero random effect leads to our assumption that the prior standard deviation on the random fatigue limit should be about .005, thus implying a prior variance of the order of magnitude of .000025.

Taking $\alpha_0 = 3$ ensures that the IG prior has a finite variance. Furthermore, setting $\lambda_0 = .0001$ implies a prior mode for σ_γ^2 of $\lambda_0/(\alpha_0 + 1) = .000025$, a prior mean of $\lambda_0/(\alpha_0 - 1) = .00005$, and a prior standard deviation of $\lambda_0/[(\alpha_0 - 1)(\alpha_0 - 2)^{.5}] = .00005$. Because this distribution is relatively heavy-tailed, it places substantial mass around values of σ_γ^2 beyond the mean. At the same time, the small positive value of λ_0 eliminates a singularity in the posterior distribution that occurs in the likelihood when all random effects are identically 0.

The marginal prior on σ^2 was likewise assumed to be an informative $\mathrm{IG}(\xi_0, \eta_0)$ distribution. An analysis of vari-

Table 2. Correlations Between the Hierarchical Bayes Model Parameters for the Laminate Data

	β_0	β_1	σ	μ_γ	σ_γ
β_0	1.000	−.999	.107	−.983	−.119
β_1		1.000	−.103	.974	.114
σ			1.000	−.113	−.724
μ_γ				1.000	.131
σ_γ					1.000

ance of the logarithm of fatigue life was performed treating stress as a factor variable and excluding censored observations. This resulted in a preliminary estimate of σ^2, unadjusted for random effects, of $(.45)^2$ with 110 df. Based on this analysis, we took $\xi_0 = 1$ and $\eta_0 = .405$. Although this IG prior has neither a mean nor a variance, it has a mode at the desired value of $(.45)^2$.

Finally, marginal uniform priors were assumed for β_0 and β_1, the regression parameters. Although the preceding choice of priors is rather arbitrary, this choice bounds the marginal posteriors on σ_γ^2 and σ^2 away from 0.

To simulate samples from the corresponding posterior distribution, we used a modified Gibbs/Metropolis–Hastings Markov-chain Monte Carlo sampling technique (Gilks, Richardson, and Spiegelhalter 1996). After 5,000 burn-in iterations, 20,000 simulated parameter values for these five parameters were recorded every 50 iterations in a chain run for 10^6 iterations. Total computational time for this simulation required less than 30 minutes on a midpriced Unix workstation.

Table 1 contains the posterior means, medians, standard deviations, and 95% credible intervals for the five parameters. By comparing these estimates with those in Tables 1 and 2 of PM we see that the HB posterior mean estimate of σ is 50% larger than PM's ML estimate, but the HB mean estimate of σ_γ is 50% smaller than the ML estimate. The HB 95% credible intervals for both σ and σ_γ are significantly shorter than the corresponding likelihood ratio confidence intervals, and the standard errors of the HB estimates of β_0, β_1, and μ_γ are also smaller than those of PM. This may be due to the use of large-sample asymptotic results by PM and/or the possible effect of the informative prior.

Table 2 gives the correlations between the HB model parameters for the laminate data. The extremely high correlations between β_0, β_1, and μ_γ compare favorably with those in Table 3 of PM, but the remaining correlations are all smaller (in absolute value) than those reported by PM.

Figure 2 displays fitted values under the HB model. The centermost line in this figure is an estimate of the posterior median fatigue life as a function of the stress level. The two lines immediately surrounding the median estimate represent a 90% posterior credible interval for the mean fatigue life as a function of stress. The most extreme lines provide a 90% posterior predictive region for future fatigue-life observations as a function of stress level. These lines were all easily obtained by recording appropriate sample data on fatigue life and mean fatigue life while executing the Markov-chain simulation.

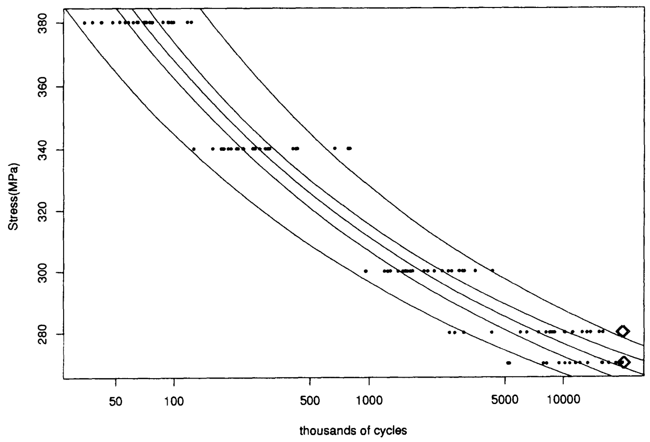

Figure 2. Log-Linear S-N Plot for the Laminate Panel Data with Hierarchical Bayes Estimates of the Median Fatigue Life, 90% Posterior Credible Interval on the Mean Fatigue Life, and 90% Prediction Limits for Future Fatigue Life Observations.

2. CONCLUSIONS

There are several advantages in using an HB approach for modeling and analyzing random fatigue-limit data. The primary advantage is the ability to use prior information in the analysis beyond that contained in the sample data. New materials, whose fatigue-life properties are of interest, are often evolutionary developments of existing materials whose mechanical properties are well known. These existing properties can provide the basis for informative prior distributions.

Furthermore, the posterior distribution provides great flexibility with which posterior inferences can be summarized. For example, the 90% posterior predictive limits in Figure 2 are easily obtained as a simple function of the simulated data. This simple, straightforward approach is in contrast to the more complicated methods that must often be used when computing lower confidence bounds on fatigue-life distribution quantiles. In addition, posterior credible intervals are direct probability statements about a parameter, but confidence intervals do not have this desirable interpretation. We also note here that the posterior predictive limits in Figure 2 were nearly invariant to several different choices of informative and noninformative marginal prior distributions on σ^2 that we considered.

Because all the information regarding the parameters is contained in the posterior distribution, there is no need to resort to asymptotic approximations for calculating such quantities as standard errors and interval estimates. In the HB approach these are readily empirically estimated from the simulated posterior data to any desired degree of precision. There is also no need for numerical integration of the likelihood components as required by PM.

Although not illustrated here, there is another important advantage of an HB approach. Model validation and checking are easily accomplished within the HB paradigm. Gelman, Carlin, Stern, and Rubin (1995, chap. 6) illustrated various ways to accomplish this validation.

The primary disadvantage of the HB analysis of the random fatigue-limit model is that a suitable prior distribution that guarantees a proper posterior must be identified. As we have observed, this is sometimes not a trivial task.

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