Distributed State Space Generation of Discrete-State Stochastic Models

Gianfranco Ciardo Joshua Gluckman David Nicol

Department of Computer Science
College of William and Mary
Williamsburg, VA 23187-8795, USA
{ciardo, jgluc, nicol}@cs.wm.edu

Abstract

High-level formalisms such as stochastic Petri nets can be used to model complex systems. Analysis of logical and numerical properties of these models often requires the generation and storage of the entire underlying state space. This imposes practical limitations on the types of systems which can be modeled. Because of the vast amount of memory consumed, we investigate distributed algorithms for the generation of state space graphs. The distributed construction allows us to take advantage of the combined memory readily available on a network of workstations. The key technical problem is to find effective methods for on-the-fly partitioning, so that the state space is evenly distributed among processors. In this paper we report on the implementation of a distributed state space generator that may be linked to a number of existing system modeling tools. We discuss partitioning strategies in the context of Petri net models, and report on performance observed on a network of workstations, as well as on a distributed memory multi-computer.

Discrete-state models are a valuable tool in the representation, design, and analysis of computer and communication systems, both hardware and software. We are particularly interested in stochastic formalisms, where some probability distribution is associated with the possible events in each state, so that the model implicitly defines a stochastic process. These are then used to carry on performance, reliability, or performability studies.

Most real systems, however, exhibit complex behaviors which cannot be captured by simple models having a small or regular state space. Given the high expressive power of formalisms such as Petri nets [24, 23], queueing networks, state charts [17], and ad hoc textual languages [14], the correct logical behavior can, in principle, be modeled exactly. The timing behavior is then defined by associating a probability distribution to the duration of each activity. The resulting stochastic process can be solved by discrete-event simulation. However, if the distributions are restricted to be either exponential or geometric, the process is a continuous-time Markov chain (CTMC) or a discrete-time Markov chain (DTMC), respectively, and can be solved numerically.

We focus on the CTMC case, where, with the exception of very special circumstances such as the existence of product-form solutions [3, 26] or of extensive symmetries [5, 13], the numerical solution requires the generation and storage of the entire state space. This is the main

drawback of the numerical approach, since the size of the state space can easily be orders of magnitude larger than the main memory of a single workstation. We are interested exclusively in solving models whose underlying state-space is too large to fit in the main memory of an ordinary workstation. Even though *virtual memory* is large, state-space generation does not exhibit the locality of reference needed to avoid virtual memory thrashing; by distributing the state-space we avoid high paging costs, and accept lower inter-processor communication costs in return. For models whose state-spaces do fit in main memory, an easy way to exploit multiple workstations is to run concurrently multiple parameter settings. This however is not our concern—it makes sense to distribute the state-space principally when one has to in order to avoid paging overhead.

The amount of memory needed for the state space is normally much larger than that needed for the numerical solution. This is because, during generation, states must be represented explicitly, typically as vectors of integers, whereas integer-valued state indices suffice during the numerical solution. Another important aspect is that the computational time spent generating a state space is often comparable to the time spent solving the underlying stochastic process. Hence, it is conceivable to generate a large model's state space on a handful of processors, then transfer the encoded transition rate matrix to a single processor for the numerical solution. Assuming that storing this matrix in sparse columnwise format requires about one-tenth the storage required for the state space, this technique increases by an order of magnitude the size of models that can be solved numerically using comparable amounts of memory on a given workstation (that is, ideally without paging). To solve even larger models, the analysis of the underlying CTMC should be performed in a distributed fashion as well, otherwise the serial solution phase will become the memory bottleneck.

Our technique is suitable for local-area networks. Their ubiquitous presence makes this approach very palatable, as it effectively offers a much larger overall amount of memory and computational power to the analyst, without requiring the purchase of new hardware. We stress that logical analyses can also be parallelized (we discuss instances of these), and that the key contribution is the *capability* of solving models too large to tackle on a single workstation.

Section 1 presents the interface used to integrate an existing modeling tool with our distributed algorithm. Sections 2 and 3 discuss sequential and distributed state space exploration, respectively. Section 4 presents analysis algorithms that can be applied to the state space generated in a distributed fashion.

Our approach is not tied to a particular formalism. This greatly simplifies the parallelization of any state space-based modeling tool. In particular, we have, for now, applied the approach to the tool SPNP [10], and report the performance results in Section 5. Section 6 summarizes our work and discusses our plans for further investigation.

1 A general interface to a distributed engine

Our goal is to provide a "distributed exploration engine" which can be connected to any "discrete-state formalism front-end". Hence, the engine implementation must not depend on the type of formalism described by the front-end. While our data are obtained by using a stochastic Petri net front-end, nothing in the engine reflects this. Indeed, we are able to

integrate the engine we describe with a commercial modeling tool, BONeS Designer. Our present tool (with its capability for stationary analysis) may serve as a substitute for the transient analysis engine we have also integrated into BONeS Designer [21].

In general, we can say that the *reachability set* S, the set of states reachable from a given initial state s_0, is a subset of some structured countable set, often $\mathbb{N}^n$ for some n. The *reachability graph* (S, A) is a directed labeled graph whose nodes and arcs are the reachable states and the possible state-to-state transitions, respectively. Each arc is labeled with the identifier e of an event: $s \xrightarrow{e} s'$ means that event e causes a change of state from s to s'. If two events e_1 and e_2 can cause the same change of state, they correspond to distinct arcs in A. The model defines which events are enabled, i.e., can occur, in each state s, among the set of possible events E.

A model can then be considered as a way to define a set of functions which define the interface between the engine and the front-end. Besides reflecting good software engineering practice, this approach highly facilitates the integration of the engine to new front-ends. The state space S is implicitly described by the following functions:

- $Initial$, with no input parameters, returning the initial state s_0 for the model.
- $Enabled(s)$, returning the (finite) set of events enabled in state $s \in S$.
- $NewState(s, e)$, returning the state reached from state $s \in S$ when event $e \in Enabled(s)$ occurs.
- $Compare(s_1, s_2)$, returning the result of the comparison between two states, *SMALLER*, *EQUAL*, or *LARGER*. This function prevents the engine from having to know the structure of the state, yet it allows one to perform an efficient search for a given state in a large set of states (for example using a binary search). The only assumption is that a total order can be defined over the set of reachable states.

To define the stochastic behavior, additional functions are needed, depending on the type of stochastic process underlying the model. The simplest case is when all events have exponentially distributed durations, resulting in an underlying CTMC. Then, we only need a function

- $Rate(s, e)$, returning the rate at which event $e \in Enabled(s)$ occurs in state $s \in S$ in isolation.

In many models, however, it is useful to describe "instantaneous events" which can occur in zero time, as soon as they become enabled. In GSPNs [1], this is achieved by immediate transitions; in queueing networks, by passive resources. If a state enables an instantaneous event, "timed events" cannot occur, they are de facto disabled. We disallow infinite sequences of instantaneous events; while these subtle situations can be managed [8, 15], they usually indicate modeling errors. Then, the state space S can be partitioned into two classes, of "timed" and "instantaneous" states: S^T and S^I, where $s \in S^I$ iff it enables instantaneous events. The additional functions needed to describe this class of models are:

- $Timed(s)$, returning *TRUE* or *FALSE* according to whether s is timed or not.

- $Weight(s, e)$, returning the weight, a nonnegative real number, for the occurrence of the instantaneous event e in the (instantaneous) state s. The probability of event e in s is then obtained by normalization:

$$Prob(s, e) = \frac{Weight(s, e)}{\sum_{e' \in Enabled(s)} Weight(s, e')}$$

  This definition allows for some, but not all, of the enabled instantaneous events to have zero weight in a given state s.

In certain existing tools, this interface is inadequate because it might be much more efficient to compute the rates or probabilities for all enabled events in a given state s with a single function call. We ignore this aspect for readability's sake, but we observe that the algorithms we present are not affected in any substantial way by this choice.

Finally, a model is used to study some quantity of interest. We assume this to mean the expected value of a stochastic reward process at some point in time, or in steady state. This process is defined by means of a reward rate function ρ defined over the state space: given a state $s \in S$, a given reward $\rho(s)$ is gained for each unit of time the model is in state s. Hence, for example, the expected steady-state reward rate of the model is

$$\sum_{s \in S} \rho(s)\pi_s,$$

where π is the steady-state probability vector. Instead of π, we might use $\sigma(t_1, t_2)$, the expected cumulative time spent in each state during the interval $(t_1, t_2]$. Then,

$$\sum_{s \in S} \rho(s)\sigma_s(t_1, t_2)$$

is the expected cumulative reward gained during this interval, and so on. In practical modeling studies, many different reward rate functions are specified over the same model, to compute different aspects, so we define

- $Reward(s, k)$, returning the value of the k-th reward rate function evaluated in state s.

2 State space exploration

In many studies, the logical analysis of the model is of interest in itself. For example, we might want to explore qualitative properties such as absence of deadlocks and livelocks, reachability (possibility of reaching states satisfying certain conditions), liveness, and so on [20, 23]. If the distributions of the durations of the timed activities have unbounded support (e.g., a geometric or exponential distribution), the timing and probabilistic behaviors do not affect the qualitative aspects of the model (e.g., which states are reachable), so they can be ignored. The only timing information used at this point is that timed events must be considered disabled whenever an instantaneous event is enabled, since they have null probability of occurring before the instantaneous event.

The state of the model is represented as a structured quantity, often of fixed size. For example, the state of Petri net is given by the number of tokens in each place (which could be

stored as a fixed-size vector of nonnegative integers), while the state of a multiclass queueing network is given by the number of customers of each class in each queue.

More complex storage schemes might be devised to save storage, often based on the existence of model invariants. For example, in a closed queueing network or in a Petri net covered by P-invariants [19], the customer populations at each queue, or the token population at each place, satisfy certain linear relationships. Sparse storage techniques can also be used to store a state, and it is possible to store an integer in just $\lceil \log k \rceil$ bits, if an upper bound k on its value is known (again invariants can be used for this purpose) [2]. We do not discuss these techniques here, since they are independent of our method and apply equally well to both sequential and distributed analysis.

Since S and (S, A) are defined only implicitly by the model, their size and characteristics might not be known a priori. In particular, S should be finite, but many formalisms of interest are Turing-equivalent, hence there is no algorithm to verify this condition in general. We are then forced to assume that S is finite, but of size known only when the state space exploration completes.

The sequential algorithm for state space exploration is shown in Fig. 1. If S_{new} is stored using a single-link list managed as a FIFO queue, the reachability graph is explored in breadth-first order. Each element in the list contains, either directly or by pointing to it, a different state. If A is not needed, the statements referring to it in Fig. 1 can be omitted. Note that the algorithm will not halt if S is infinite.

Procedure *ExploreSequential*

1. $S \leftarrow \{Initial\}$; $S_{new} \leftarrow S$; $A \leftarrow \emptyset$;
2. while $S_{new} \neq \emptyset$ do
3. select $s \in S_{new}$;
4. $S_{new} \leftarrow S_{new} \setminus \{s\}$;
5. for each $e \in Enabled(s)$ do
6. $s_{new} \leftarrow NewState(s, e)$;
7. if $s_{new} \notin S$ then
8. $S_{new} \leftarrow S_{new} \cup \{s_{new}\}$;
9. $S \leftarrow S \cup \{s_{new}\}$;
10. end if;
11. $A \leftarrow A \cup \{s \xrightarrow{e} s_{new}\}$;
12. end for;
13. end while;

Figure 1: Sequential state space exploration

2.1 Generation of the stochastic process

When the questions asked of the model refer to the timing and stochastic behavior, a stochastic process (not just a reachability graph) must be built as a result of the state space exploration.

If the underlying process is a CTMC, this means building an infinitesimal generator matrix Q, where $Q_{s,s'}$ is the rate of going from state $s \in S^T$ to state $s' \in S^T$, for $s \neq s'$. The diagonal entries of Q are defined to be the negative of the sum of the off-diagonal entries on the corresponding rows, $Q_{s,s} = -\sum_{s' \in S^T, s' \neq s} Q_{s,s'}$; in our discussion, we assume that they are stored explicitly only during the CTMC solution. It is usually more efficient to use an alternative exploration algorithm, which stores only the timed states S^T. Analogously, only a "reduced reachability graph", basically equivalent to Q, needs to be stored. Whenever an instantaneous state is found, a depth-first search is initiated, to determine the set of timed states reached. If

$$s \xrightarrow{e_0} s_1 \xrightarrow{e_1} s_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{k-1}} s_k \xrightarrow{e_k} s'$$

where $s, s' \in S^T$ and $s_1, s_2, \ldots, s_k \in S^I$, the rate of going from s to s' along this path is

$$Rate(s, e_0) \cdot \prod_{i=1}^{k} Prob(s_i, e_i)$$

and $Q_{s,s'}$ is the sum of these rates over all possible paths of this type from s to s', including paths of length one, that is, paths with no intermediate immediate states.

This is illustrated in Fig. 2. Procedure *BuildQ* is called first, which in turn uses the recursive procedure *GenerateTransitions*. For simplicity, we assume that S^T, S^T_{new}, S^I_{new}, and Q are global variables and that the initial state is timed (the algorithm can be easily modified if this is not the case). S^I_{new} is needed to recognize cycles of instantaneous events, which we consider illegal. When the execution returns to *BuildQ*, S^I_{new} is empty, that is, instantaneous states are stored only temporarily. Note that entries of Q are incremented, not set, by statement 6 in procedure *GenerateTransitions*. This is because multiple paths of instantaneous states might exist between the same source and destination.

3 Distributed state space exploration

Like the sequential algorithm, the distributed algorithm shown in Fig. 3 performs a breadth-first exploration of the state space. Each state reached is either explored locally or sent to another process. For the distributed algorithm, we then define another function in the interface:

- *Partition*(s, N), returning the identifier of the process to which state s is assigned, an integer between 0 and $N - 1$.

Assuming we have N processes running on N processors, this function partitions the state space into N classes, one assigned to each process(or), and is a critical factor affecting the performance of the distributed algorithm (see Section 5.1).

The incidence matrix of the reachability graph is kept in column-wise format: states and arcs *to* states are stored. Hence, when process i determines that state s_{new} "belongs" to a remote process $j \neq i$, it sends both s_{new} and the arc leading to it, $s \xrightarrow{e} s_{new}$, to j. As the representation of a state can be quite large, states are assigned an index. Locally (i.e., in process i), state $s \in S^i$ is identified by the index k, if s is the k-th state added to S^i. Globally (i.e., in process $j \neq i$), state s is identified by a two-component index: (k, i). State indices, rather than actual states, are stored and exchanged whenever possible.

Procedure *BuildQ*

1. $S^T \leftarrow \{Initial\}$; $S^T_{new} \leftarrow \{Initial\}$; $S^I_{new} \leftarrow \emptyset$; $Q \leftarrow 0$;
2. while $S^T_{new} \neq \emptyset$ do
3. select $s \in S^T_{new}$;
4. $S^T_{new} \leftarrow S^T_{new} \setminus \{s\}$;
5. for each $e \in Enabled(s)$ do
6. $s_{new} \leftarrow NewState(s, e)$;
7. $GenerateTransitions(s, Rate(s, e), s_{new})$;
8. end for;
9. end while;

Procedure $GenerateTransitions(t, r, t_{new})$

1. if $Timed(t_{new})$ then
2. if $t_{new} \notin S^T$ then
3. $S^T_{new} \leftarrow S^T_{new} \cup \{t_{new}\}$;
4. $S^T \leftarrow S^T \cup \{t_{new}\}$;
5. end if;
6. $Q_{t,t_{new}} \leftarrow Q_{t,t_{new}} + r$;
7. else if $t_{new} \notin S^I_{new}$ then
8. $S^I_{new} \leftarrow S^I_{new} \cup \{t_{new}\}$;
9. for each $e \in Enabled(t_{new})$ do
10. $GenerateTransitions(t, r \cdot Prob(t_{new}, e), NewState(t_{new}, e))$;
11. end for;
12. $S^I_{new} \leftarrow S^I_{new} \setminus \{t_{new}\}$;
13. else
14. error("*cycle of instantaneous events*");
15. end if;

Figure 2: Sequential generation of Q

When i sends state s_{new} to j, with the function call $SendState(j, s_{new})$, the local index of s_{new} in j is not known to i, so the actual state must be sent. However, when the arc $s \xrightarrow{e} s_{new}$ is sent, with the function call $SendArc(j, s \xrightarrow{e} s_{new})$, the global state index of s is sent, since it is known to i. Also, the destination (actual) state s_{new} does not have to be sent a second time to describe the arc, since states and arcs are always sent in pairs.

The functions *ReceiveStates* and *ReceiveArcs* return all the states and arcs sent to process i since the last time they were called, respectively. Each state x in the set returned by *ReceiveStates* has a corresponding arc $(k_1, j) \xrightarrow{e} x$ in the set returned by *ReceiveArcs*. The set S^i is then searched. If $x \in S^i$, its previously assigned index is retrieved, otherwise x is added to S^i and a new index is generated for x. If k_2 is the local index of x, an arc $(k_1, j) \xrightarrow{e} (k_2, i)$ is added to A^i. These are high-level descriptions; the actual implementation

Procedure $ExploreDistributed_i$

1. if $Partition(Initial, N) = i$ then $S^i \leftarrow \{Initial\}$; else $S^i \leftarrow \emptyset$; end if;
2. $S^i_{new} \leftarrow S^i$; $A^i \leftarrow \emptyset$;
3. while "*not received terminate message*" do
4. while $S^i_{new} \neq \emptyset$ do
5. select $s \in S^i_{new}$;
6. $S^i_{new} \leftarrow S^i_{new} \setminus \{s\}$;
7. for each $e \in Enabled(s)$ do
8. $s_{new} \leftarrow NewState(s, e)$;
9. $j = Partition(s_{new}, N)$;
10. if $j \neq i$ then
11. $SendState(j, s_{new})$;
12. $SendArc(j, s \xrightarrow{e} s_{new})$;
13. else
14. if $s_{new} \notin S^i$ then
15. $S^i_{new} \leftarrow S^i_{new} \cup \{s_{new}\}$;
16. $S^i \leftarrow S^i \cup \{s_{new}\}$;
17. end if;
18. $A^i \leftarrow A^i \cup \{s \xrightarrow{e} s_{new}\}$;
19. end if;
20. end for;
21. end while;
22. $S^i_{rec} \leftarrow ReceiveStates \setminus S^i$;
23. $S^i \leftarrow S^i \cup S^i_{rec}$;
24. $S^i_{new} \leftarrow S^i_{new} \cup S^i_{rec}$;
25. $A^i \leftarrow A^i \cup ReceiveArcs$;
26. end while;

Figure 3: Distributed state space exploration for process i using N processes

details for the communication mechanism are beyond the scope of this presentation.

In the sequential algorithm, the choice between storing the incidence matrix of the reachability graph in row-wise or column-wise format is irrelevant. For the distributed version, however, a row-wise format would require a more complex protocol. With row-wise storage, the entry $s_1 \xrightarrow{e} s_2$ is stored by process i as $(k_1, i) \xrightarrow{e} (k_2, j)$, if s_1 is assigned index (k_1, i) and s_2 is assigned index (k_2, j). However, i does not know the local index k_2 of s_2, so it must send s_2 to j, and wait for (k_2, j) in return. Only then i can complete the storage of the entry in the incidence matrix. With the column-wise format we use, i simply sends the pair s_2 and $(k_1, i) \xrightarrow{e} s_2$ to j, without having to wait for further information from j, since it is up to j to fill-in the value for the arc destination.

The communication complexity of the distributed state space algorithm is then one actual state (s_{new}), one state index (of s), and one event identifier (of e), for each "cross-arc" (an arc from a state in S^i to a state in S^j, $i \neq j$).

When process i finishes exploring its local states (S^i_{new} is empty), it waits for more states and arcs from other processes. When all processes have finished their local work and are waiting to receive a message, the distributed state space exploration has completed. Detecting termination is a well-known problem with many solutions. In the workstation network data we present, we used the circulating probe algorithm described by Dijkstra et al. [12]. We have since made the engine portable by using MPI [16] as the communication mechanism, and in that context employ the scalable "Non-committal barrier" described by Nicol [22].

3.1 Distributed generation of the stochastic process

For brevity's sake, we do not present the pseudo-code for the distributed generation of the underlying stochastic process, obtained by merging the algorithm for the distributed generation of the state space of Fig. 3 with the elimination of the immediate states used in the algorithm of Fig. 2. Only timed states are assigned to a particular process using the partitioning function. Immediate states are managed in the process that generates them, and then discarded after all the timed states reachable from them have been explored. Storing the immediate states together with the timed ones is a reasonable alternative (see [8, 4] for the tradeoffs involved in storing these states permanently), but is probably less appropriate if the paramount goal is to minimize storage requirements.

At the end, process i contains the states $S^{T,i} = \{s \in S^T : Partition(s, N) = i\}$ and the entries of Q corresponding to arcs reaching these states, $Q_{\bullet,S^{T,i}}$.

Before concluding this section, we discuss a few implementation issues. Communication between processes is accomplished through message passing. Flow control is provided by acknowledged messages: a sender does not continue until the receipt of its last A messages has been acknowledged (A is a tuning parameter whose value depends on the buffer space available for messages). This avoids deadlocks that can otherwise occur when buffer space is exhausted.

Because of the potentially high number of states sent to another process, each state/arc pair is buffered in the sender. The buffer size is a compilation parameter. As the size increases, more states and arcs can fit into a single message, and fewer, although larger, messages are exchanged. This reduces one type of overhead, but it also increases the likelihood that a process j remains idle waiting for states to be imported, while some other process i delays sending states that j should explore because the buffer is not full. It is desirable to use an "appropriate" buffer size, but this choice is highly model-dependent, so we do not discuss this implementation issue further.

4 Distributed analysis of the model

Once the state space is built, analysis can proceed. We present two types of state space-based analysis. The first type analyzes logical properties of the state space; a number of questions can be answered in a distributed fashion using the distributed state space. The second type

is numerical analysis; we presently perform this sequentially, but point out some unexpected ramifications of our distributed generation of the state space. In the following, we use the following symbols:

- n and η are the total number of tangible states and arcs stored; $n = |S^T|$ is the dimension of Q and η is the number of nonzero entries in Q: $\eta = |\{(s_1, s_2) : Q_{s_1,s_2} > 0\}|$.
- n^i and η^i are the analogous quantities for process i in the distributed algorithm: $n^i = |S^{T,i}|$ and $\eta^i = |\{(s_1, s_2) : Q_{s_1,s_2} > 0, s_2 \in S^{T,i}\}|$. Furthermore, we define $\eta^{j,i}$ to be the number of entries stored by i corresponding to events originating from states in j: $\eta^{j,i} = |\{(s_1, s_2) : Q_{s_1,s_2} > 0, s_1 \in S^{T,j} \wedge s_2 \in S^{T,i}\}|$. Hence, $\sum_{i=0}^{N-1} n^i = n$, $\sum_{i=0}^{N-1} \eta^i = \eta$, and $\sum_{j=0}^{N-1} \eta^{j,i} = \eta^i$.

4.1 Distributed computation of logical properties

The notion of reachability pervades logical analysis of state spaces: "is it possible to reach some state s_1 from s_0?". As we will see, solution to this problem permits one to address higher-level questions. For instance, to determine whether there are any transient states, it is sufficient to test whether the initial state is transient, that is, whether there exists a state $s \in S^T$ which does not reach s_0. This is equivalent to determining whether there is a state s unreachable from s_0 in the "reverse reachability graph", obtained by reversing the direction of all arcs. A simple breadth-first search algorithm that "touches" all states reachable from s_0 in the reverse graph can be used for this purpose—if any state remains untouched, s_0 is transient. An efficient implementation requires a row-wise storage of the incidence matrix of the reverse graph, and this is a further reason to use a column-wise format for the storage of the incidence matrix of the original graph, since one is the transpose of the other. Note that the state space generation process is itself a breadth-first search, and that the algorithm for testing whether s_0 is transient is essentially the same.

A sequential breadth-first search algorithm requires $O(\eta)$ operations. A distributed implementation requires the same number of operations, but also $O(\chi)$ communication, where

$$\chi = \sum_{i,j \in \{0,\ldots,N-1\}, i \neq j} \eta^{j,i}$$

is the number of cross-arcs.

The communication cost is then of the same order of complexity as for the state space generation, although now only state indices, not the actual states, are sent between processes.

Many other important questions about the behavior of a system can be answered in a distributed way using the information in the reachability set and graph. The following lists a few classical problems ([20, 23] discuss them in the case of Petri nets, but they are relevant in many areas, including operating systems theory and correctness analysis of distributed software).

- Reachability: a condition c is reachable if there is a state $s \in S$ satisfying c. Each process i can simply test for this every time it adds a new state to S^i, that is, this question can be answered without further examining the reachability graph. Deadlocks are just a special case: a deadlock is an absorbing state, that is, a state which does not enable any event.

- Livelock: a livelock is a set of states L, $1 < |L| < |S|$ such that, once L is entered, no state in $L - S$ can be reached. Formally, the reachability graph must contain a strongly connected component with two or more nodes and no outgoing arcs (no way to leave the component). This is equivalent to testing whether the initial state is transient in a modified reachability graph without absorbing states (these can be easily tagged during state space generation).

- Liveness: an event e is not live if there is a state $s \in S$ such that, once s is entered, e can never become enabled. If we define S_e to be the set of states where e is enabled, liveness can be established by checking that every state in S can reach a state in S_e. The same breadth-first algorithm used to determine whether the initial state is transient can be adopted; the only difference is that the search in the reverse graph proceeds from the set of states S_e, not from s_0 alone.

- Conditional reachability: we are sometimes interested in determining whether there exist two states s_1, satisfying condition c_1, and s_2 satisfying condition c_2, such that s_1 reaches s_2. If the entire graph is strongly connected, this is equivalent to asking whether conditions c_1 and c_2 can be satisfied. Otherwise, a modified version of distributed breadth-first search algorithm used to determine whether s_0 is transient can be employed. Instead of starting from s_0, we start from a set of states $S_2 = \{s : c_2 \text{ is satisfied in } s\}$, and we determine the set of states R reachable from S_2 in the reverse graph. Our goal is to find a state in R satisfying s_1.

We stress that logical analysis can be a prerequisite to a numerical study. For example, the choice of the algorithm used for the steady-state solution of the CTMC depends on the logical properties of the state space. If (S, A) is strongly connected, the CTMC is ergodic, and the initial probability distribution is irrelevant. In this case, it is a good idea to use the Gauss-Seidel or Successive-Over-Relaxation (SOR) methods [28, 25] and choose an initial iterate likely to reduce the number of iterations required for convergence. Common choices are the uniform distribution, $\pi_s = 1/|S|$, or one proportional to the expected sojourn time in each state, $\pi_s = c/Q_{s,s}$, where c is a normalization constant.

If the CTMC is not ergodic, that is, if s_0 is transient, this could be an indication of a problem, either in the model, or in the system being modeled. If instead this reflects the intended behavior, the state space S can be partitioned into a set of transient states plus one or more recurrent classes. For finite CTMCs, state classification is strictly a logical "graph-property", that is, it does not rely on the actual numerical values of the rates but only on their presence. If we are interested in the steady-state solution, it is much more efficient to study the transient states first, computing the accumulated reward in each of them until reaching a recurrent state. The initial probability vector corresponding to the initial state must be used: $\pi_{s_0} = 1$, $\pi_s = 0$ for all other states $s \in S$. Then, we can obtain the probability of reaching the recurrent classes, and solve each independently [7].

4.2 Numerical solution of the underlying stochastic process

In the current implementation, the numerical solution of the CTMC is centralized. While this prevents us from obtaining good speedups, it is important to remember that our immediate

goal is to increase the size of models we can solve. As pointed out earlier, the large difference in memory requirements for the generation and solution phases means we can solve models on one processor that are one order of magnitude larger than we can generate on one processor.

After building Q and testing for a transient initial state, each process sends its portion of Q to a solver process where the numerical solution is performed. This is reasonable given our current target level of parallelism, up to a dozen workstations. A distributed numerical solution will be required to further increase the level of parallelism (e.g., 100 processors).

The "solution" sought for the CTMC is often the steady-state probability vector π satisfying $\pi Q = 0$, if Q is ergodic, or the sojourn times in each transient state until absorption, or the transient instantaneous or cumulative probability in each state. In any case, the solution is given by a real vector v of size n.

Vector v is used to compute the expected reward earned by the model. Rewards are a function of individual states; to compute the reward for state s one must generally have available the full state representation of s. Consequently, after computing v serially, we distribute it back to the processors holding the full state space description. Recall now that the compact representation of a state identifies the processor that owns it. It is straightforward then to return each component v^i to process i, for $i \in \{0, \ldots, N-1\}$. It is then possible to compute the expected value of a measure, $m = \sum_{s \in S^T} Reward(s) \cdot v_s$ in a distributed way. Process i computes $m^i = \sum_{s \in S^{T,i}} Reward(s) \cdot v_s$, and a master process combines these subresults as $m = \sum_{i=0}^{N-1} m^i$. Note that several measures are usually computed for a given v, corresponding to multiple reward functions.

An interesting observation should be made at this point. It is well known that the ordering of the variables (states) can affect the speed of convergence for iterative methods such as Gauss-Seidel and Successive-Over-Relaxation (SOR) [28, 25]. We indeed experienced this phenomenon when studying the number of iterations required by a sequential SOR implementation. In our first implementation, all the states in $S^{T,i}$ were ordered before those in $S^{T,i+1}$, $i \in \{0, 1, \ldots, N-1\}$, while the sequential state space exploration results in a breadth-first order, starting from s_0. The ordering from the distributed implementation regularly required more iterations, even if the SOR implementation was exactly the same.

We conclude that the natural breadth-first order by which states are generated and indexed in the sequential implementation is a better choice. To verify this, we sorted the states in the solver process according to a breadth-first order: if the distance from s_0 to s_i is less than the distance from s_0 to s_j, s_i is assigned an index smaller than s_j. This does not necessarily achieve exactly the same order as in the sequential implementation (since multiple total orders are compatible with the above partial order), but it does result in approximately the same number of iterations in the two implementations. We believe that state ordering will become an issue in a distributed implementation, where it requires reshuffling states, and the corresponding columns of Q, among the N processes. This is in not the case if we use the power or the Jacobi methods [27], which have slower convergence rates but are unaffected by state ordering, or if we were interested in performing a transient analysis of the CTMC, using the uniformization method [18].

The partition heuristic might affect the convergence of a distributed solution in other ways as well. We have not yet considered these aspects in detail, but it is clear that the actual numerical values of the rates of the state-to-state transitions, rather than the mere existence or absence of an arc, will need to be taken into account in this case.

For example, the idea of decomposability [11] is based on finding a block partition of the transition matrix where the entries of the off-diagonal blocks are orders of magnitude smaller than those in the diagonal blocks. This ensures that, after entering a block, the stochastic process reaches an "approximate steady-state" before moving to a different block. If the partition heuristic is such that each class corresponds to one or more blocks having this property, then several attractive iterative methods will be appropriate, since most of the iterations will occur within a single class, while only a few global iterations requiring the exchange of data across processors will be needed.

5 Results

We consider the model of a flexible manufacturing system (FMS) shown in Fig. 4. We omit a description of this SPN, since we are focusing on a comparison of the sequential and distributed algorithms for its analysis. The interested reader can consult [9] for a detailed presentation of its behavior and the meaning of its places and transitions. For this discussion, it is sufficient to observe that, as the number k of initial tokens in the three places $P1$, $P2$, and $P3$ increases, the number of states n and arcs η increases sharply (see Table 1). The first partitioning function used (to be described) assigns states to processors depending on the markings in places $P1$, $P2$, and $P3$.

The key performance metrics we present are speedup, and effectiveness of distribution. We emphasize again that our principle goal in this paper is to show how to solve models one order of magnitude larger than normally possible, using approximately ten processors. Speedup is not so much the issue as functionality. However, we recognize that to achieve another order of magnitude functionality in problem size, we will have be sure that temporal parallelism is also effectively exploited. By showing that problems can be solved on distributed platforms, we demonstrate our functional ability to exploit distributed memory. By simultaneously demonstrating that our method achieves speedups, we demonstrate its potential for larger-scale problems.

The largest case listed ($k = 6$) exceeded the storage capacity of a single workstation, yet it was solvable (in approximately 40 minutes) by generating the state space using five processors. The subsequent numerical solution of the stochastic process, a CTMC with a transition rate matrix having 4.2 million nonzero entries, could then be run on a single workstation with no paging. To assess how well the generation process parallelizes, we consider speedup on problems small enough to solve on one processor, hence for all cases except $k = 6$. Fig. 5 shows the speedup obtained by running our distributed algorithm on a set of homogeneous workstations (SPARCstation 10 class) communicating over a 10Mbs Ethernet, as a function of the initial number of tokens k. The plot on the left refers to the timing collected for the generation of the state space alone, while the one on the right refers to the timing for the entire solution process. This includes the numerical steady-state solution performed on a single processor and the distributed computation of a single measure, the expected total number of tokens in $P1$, $P2$, or $P3$.

The numerical solution of the CTMC using SOR required approximately 16%, 18%, 25%, and 44% of the entire process for the cases $k = 2, 3, 4$, and 5, respectively. In other words, the percentage of time consumed by SOR increases with k but, for the values of k considered,

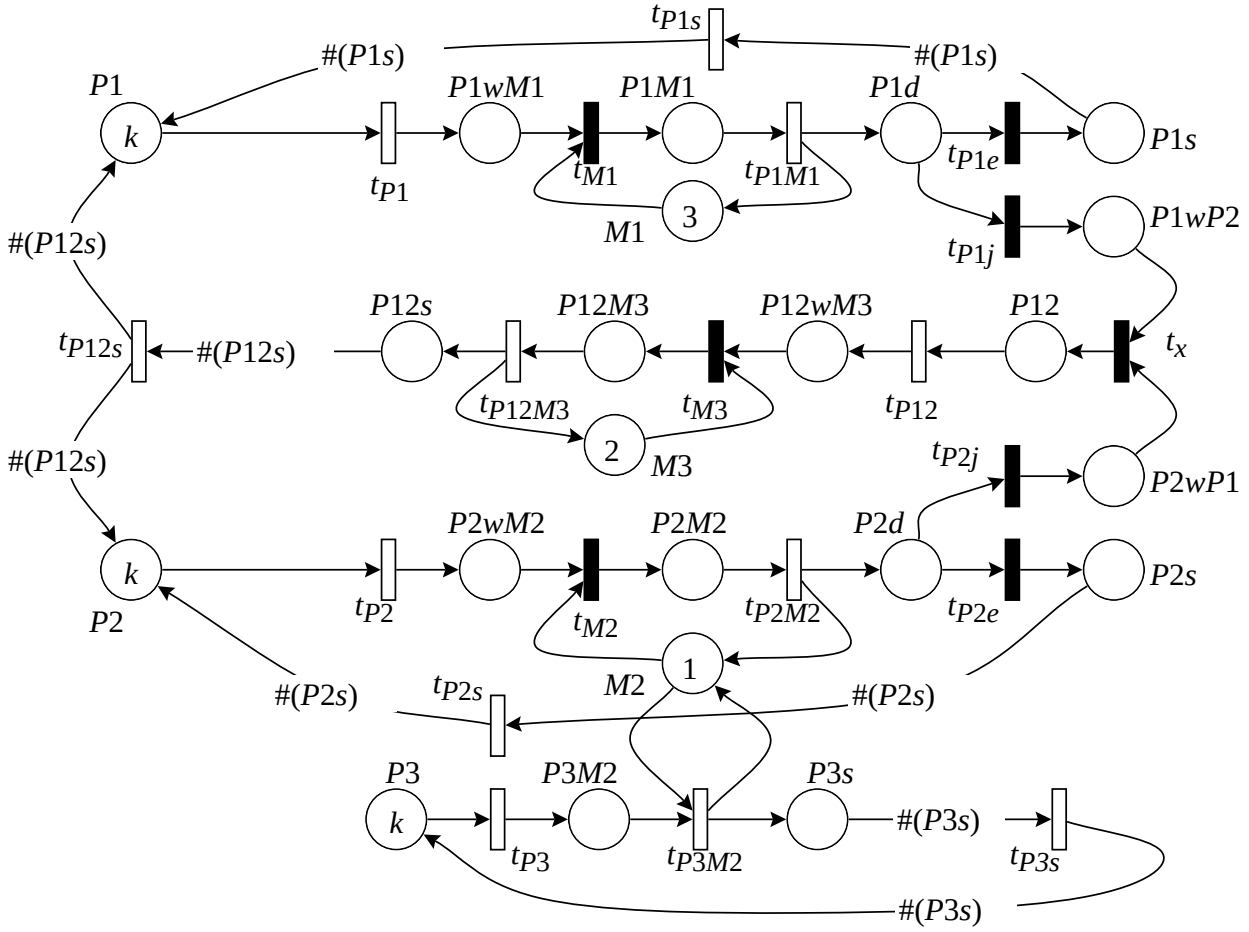


Figure 4: The FMS stochastic Petri net.

most time is still spent in the state space generation phase. This stresses the appropriateness of our approach, but also the need for a distributed numerical solution before tackling much larger problems than the ones presented.

The speedup with N processes is obtained by dividing the runtime of the sequential solution ($N = 1$) by the runtime of the distributed solution with N processes (defined as the maximum processor runtime). The runtime of a process includes the time spent executing user or system instructions and the time spent communicating or waiting for states to be imported, if $N > 1$. Speedups are calculated only for model problems where the serial solution ran without paging (other than to load, of course). Speedup of the generation phase thus measures the relative cost of the communication overhead during that phase.

Since our motivation is exploiting distributed memory, the case of highest interest is that of $k = 5$, the largest problem we could solve sequentially. Here we see evidence of a favorable computation to communication balance (as well as good load balance), since speedup increases almost linearly in the number of processors. The complete serial solution required nearly an hour of computation.

We also ported our distributed engine to an IBM SP-2 multiprocessor. Fig. 6 shows the

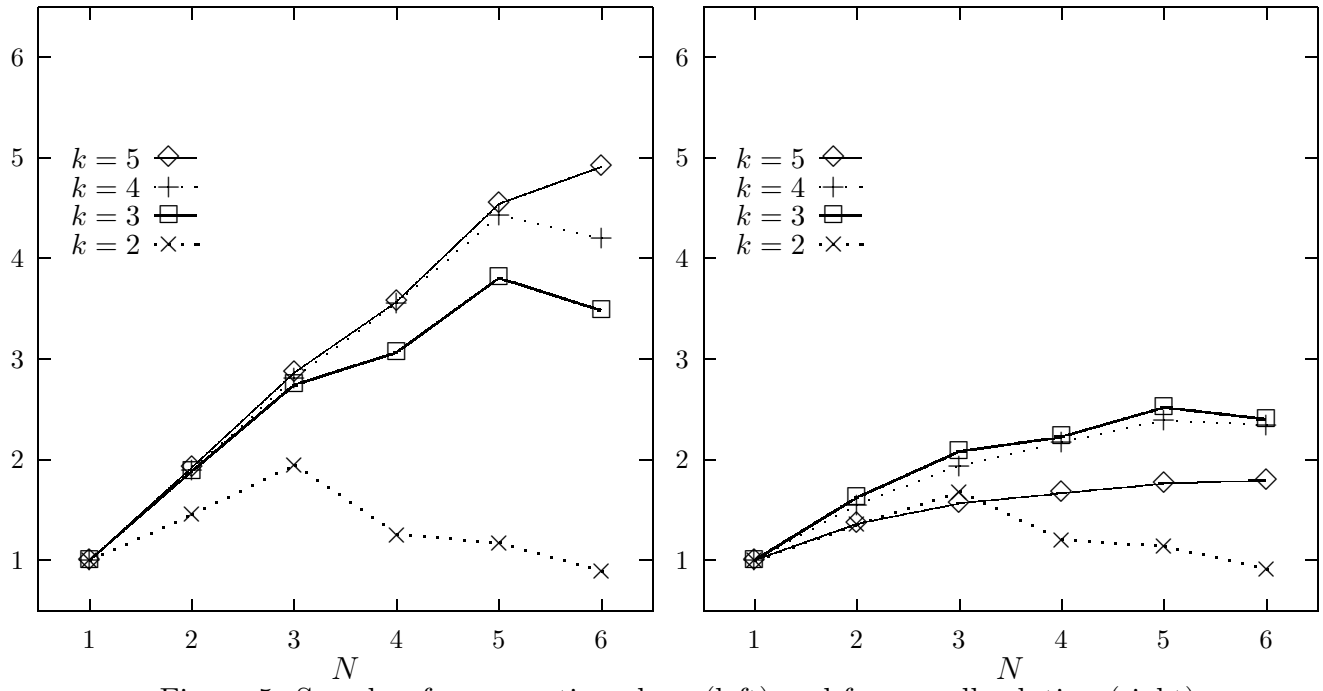


Figure 5: Speedup for generation alone (left) and for overall solution (right).

k	n	η
1	54	155
2	810	3,699
3	6,520	37,394
4	35,910	237,120
5	152,712	1,111,482
6	537,768	4,205,670

Table 1: Size of the tangible reachability set and graph as a function of k.

speedup for the state space generation on the same FMS SPN, for the case $k = 5$, as a function of N. We achieved a speedup of 11.35 when $N = 16$. The serial state space generation requires 14 minutes on a single processor. We also experimented with larger values of k, and various machine sizes. With $k = 6$ there are 537,768 states. The time needed to generate the state space varied from three hours using 2 SP-2 processors (the smallest configuration able to solve the problem), to 18 minutes using 32 processors. Using 32 processors we were able to generate the $k = 7$ state space (1,639,440 states) in 51 minutes, and the $k = 8$ state space (4,459,455 states) in four hours.

The conclusion we may draw from this data is that the algorithm works well, and makes possible the generation of state spaces that are much larger than those usually considered tractable. However, it is clear that parallelizing the solution phase is necessary, both so that the solution vector can be accessed with paging, and to shorten the otherwise intolerably long time to solve a model of large size.

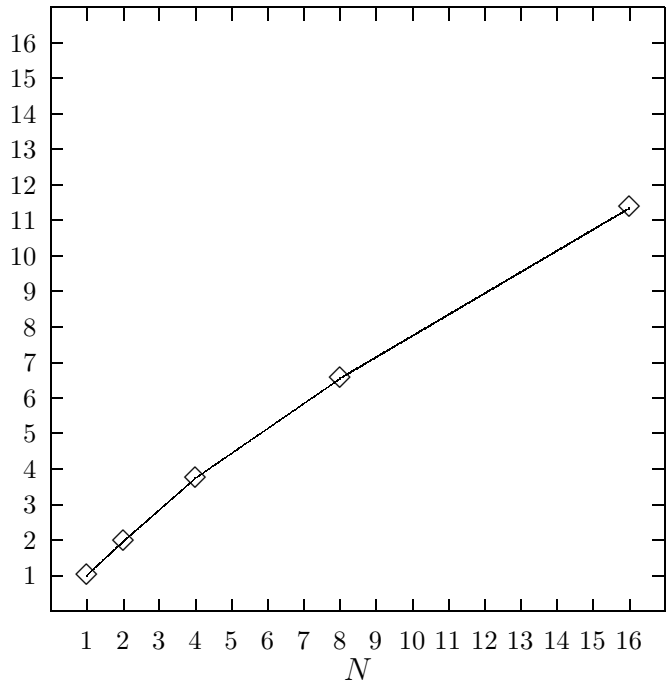


Figure 6: Speedup on the SP-2 ($k = 5$).

5.1 Choosing a good partitioning function

Choice of a good partitioning function is critical. It must provide both *locality* (if possible), and *balance*. Locality means that, in general, most of a state's descendents are assigned to the same processor of the parent state. Locality reduces communication overhead. Spatial balance means that each processor is assigned approximately the same number of states; contrast this with temporal balance which additionally calls for each processor to be busy most of the time. Spatial balance is sufficient if problem-solving capacity is a concern, whereas temporal balance is required to achieve good speedups.

The modeling formalism may provide clues to locality. The SPN reported in this paper is a good example. The marking of a Petri net with set of places $\mathcal{P}$ is a vector $m \in \mathbb{N}^{|\mathcal{P}|}$ describing the number of tokens m_p in each place $p \in \mathcal{P}$. When a transition fires, typically only a small number of components of this vector change. Thus, if we define the partitioning function based on the number of tokens in only a small fixed subset $\mathcal{P}'$ of all places (we call this a *control set*), any transition firing that does not involve any place in $\mathcal{P}'$ corresponds to a state transition between two markings assigned to the same processor. In other words, the firing of a Petri net transition might correspond to a cross-arc only if it changes the submarking m' of m corresponding to the control set $\mathcal{P}'$.

To achieve spatial balance we first need to choose the control set $\mathcal{P}'$ so that the number of possible submarkings for it (the combinations of tokens in each place of the control set for each reachable marking) is reasonably large with respect to the number of processors N (e.g., at least ten times larger):

$$\left|\{m' \in \mathbb{N}^{|\mathcal{P}'|} : \exists m \in S, \forall p \in \mathcal{P}', m'_p = m_p\}\right| > 10N.$$

Unfortunately, in the worst case one needs to generate the state space to discover just what that range is; we must therefore rely upon the user's intuition about the model to provide this property. Finally, given $\mathcal{P}'$, we assign a marking m to a processor by applying a hashing function to the submarking m':

$$f : \mathbb{N}^{|\mathcal{P}'|} \to \{0, \ldots, N-1\}.$$

The idea of using a hashing function for this purpose is quite natural (e.g., [6]), but our use of a control set and the study on the characteristics of a good hashing function we present in this section are, we believe, new.

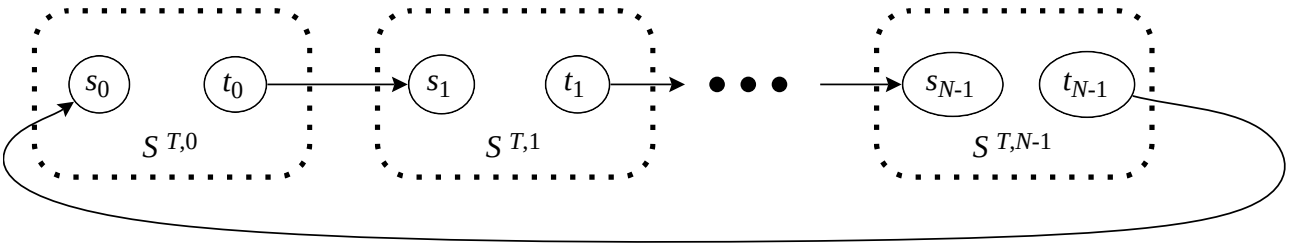


Figure 7: An extreme case for the distributed algorithm.

To further illustrate the difference between spatial and temporal balance, consider the extreme case of Fig. 7, where only one arc, $t_i \rightharpoonup s_{(i+1) \bmod N}$, connects $S^{T,i}$ to $S^{T,(i+1) \bmod N}$, and $s_0 \in S^{T,0}$. If state t_i is the last one examined in each $S^{T,i}$, the distributed algorithm will run sequentially, even if the states might be evenly allocated onto the N processes, and the number of cross arcs is certainly minimum: $\chi = 6$. On the other hand, even in this unfortunate situation, the distributed algorithm would still have an advantage over the sequential one, since communication overhead would be negligible, and the entire amount on the N processors would be available.

For the problem whose performance we studied, $\{P1, P2, P3\}$ is the control set. The hashing function is

$$(\#_{P1} + q \cdot \#_{P2} + q^2 \cdot \#_{P3}) \bmod N$$

where q is a prime number (1013, in our case) and $\#_p$ indicates the number of tokens in place p (in a given marking). The bar chart on the top right in Fig. 8 shows the distribution of states using this partitioning function with $N = 6$ processes. The columns are labelled by the corresponding processor index, but we chose to sort them in increasing order according to the number of states assigned to each processor, since the actual index of a processor is not relevant to assess the uniformity of the state distribution.

A good tool to decide the quality of the partitioning function is then the matrix of the numbers of edges cut by the partition, $\eta^{i,j}$. Fig. 8 describes these values for four different choices of the partitioning function. The parameters considered are six processes ($N = 6$) and five tokens initially in $P1$, $P2$, and $P3$ ($k = 5$), resulting in 152,712 states and 1,111,482 arcs in the graph representing the possible transitions between timed states. These values are obviously independent of the partitioning function chosen.

A simple hashing on a few components of the state description achieves a reasonably uniform allocation of states to processes. Using *all* the components might not be a good idea for several reasons:

- If linear invariants exist relating the values of the components, and if these values are simply summed, the partitioning function might not achieve a good "random" spread. For example, in a closed queueing network, the choice "sum of the number of customers in each queue mod N" would allocate all the states to the same process, a bad choice. Petri nets also often exhibit this type of invariants.

- Multiplying the components by some power of a large prime number, as we did, eliminates most problems due to the existence of invariants. However, in this case, if every component is factored in the computation of the partitioning function, most, if not all, arcs will be cross-arcs, because every event changes one or more components of the state.

Hence, it is best to use only a few components of the state in the definition of the partitioning function. Any event which changes only the values of the other components is then guaranteed not to generate any cross-arcs (e.g., t_{P12M3} in Fig. 4, with the first partitioning function).

If a particular structure is desired for the partition of Q, an appropriate partitioning function might be employed. The third choice in Fig. 8,

$$(\#_{P1wM1} + \#_{P1M1} + \#_{P2wM2} + \#_{P2M2} + \#_{P3M2}) \bmod N$$

for example, results in a block-tridiagonal structure for Q, except for two blocks, in the upper-right and lower-left corners, due to the wrap-around nature of the modulo operator. This happens because we intentionally chose the control set so that any event can change by at most one the number of tokens in a place of the control set. Such a sparsity pattern might be very desirable, depending on the type of communication available between the workstations. If they were connected in a circular fashion with bidirectional links, they could potentially be all communicating at the same time, since each workstation only needs to exchange data with its two neighbors.

We observe that the difference between the first and second partitioning function in Fig. 8 is just in the multiplication by powers of 1013 in the first case, resulting in a 10% reduction in the number of cross arcs, while the state distribution is substantially similar, if we ignore the actual processor identifiers. The price paid to obtain the tridiagonal structure with the third partitioning function is instead a higher number of cross arcs (20% more than for the first partitioning function). Interestingly enough, the second and third partitioning functions result in exactly the same state distribution. This is due to the existence of invariants, ensuring, for example, that the number of states where $\#_{P1} = m$ or $\#_{P1wM1} + \#_{P1M1} = m$ is exactly the same, and so on. Finally, the fourth function minimizes the number of cross arcs (fewer than a quarter of the arcs are cross arcs), and also achieves a good distribution of states. The clearly recognizable patterns in the matrix and in the state distributions are due to the independence of the rest of SPN from the number of tokens in $P3$ and $P3M2$. There are $\binom{5+3-1}{5} = 21$ ways to distribute $k = 5$ tokens over three places. For every such combination, exactly 7,272 combinations of tokens in the other places exist (7,272 $\times$ 21 = 152,712, the total number of states). Hence, the value of $|S^{T,i}|$ is easily determined once we know how

many of the 21 combinations correspond to process i. The hashing enforced by the expression $(\#_{P3} + \#_{P3M2} \cdot 1013) \bmod N$ results in three combinations assigned to processes 0, 2, and 4 ($|S^{T,0}| = |S^{T,2}| = |S^{T,4}| = 3 \times 7{,}272 = 21{,}816$), and four combinations assigned to processes 1, 3, and 5 ($|S^{T,1}| = |S^{T,3}| = |S^{T,5}| = 4 \times 7{,}272 = 29{,}088$).

Clearly, much work remains to be done in this area, but the good news is that even just moderately informed choices, such as the first two in Fig. 8 still achieve our goals of locality, spatial balance, and temporal balance.

6 Conclusion and future work

We have demonstrated the feasibility of distributing the state space generation phase of discrete state stochastic system analysis using only a small network of workstations. The approach exploits the memories of multiple workstations, allowing one to build and perform logical analyses of state spaces too large for a single processor. We stress that our approach provides a distributed algorithm which is independent of the particular user-level formalism adopted for the specification of the model.

To improve the applicability and usefulness of our approach, two aspects need to explored further. First, a centralized numerical solution is appropriate when using up to a dozen or so workstations, or when a machine particularly suited for numerical computation and equipped with a substantial amount of memory is available. However, our approach is a natural candidate for a completely distributed implementation, so we intend to explore the rich area of distributed solutions of linear systems, and implement some of the most appropriate techniques. This becomes a necessity if we hope to scale up to a much larger number of workstations. Fortunately, parallel versions of the techniques typically used to solve these systems are well known, and are effective. We have demonstrated here that the state-space generation phase can itself be effectively parallelized and have every hope that a fully distributed implementation will extend our ability to solve large models even further.

Second, the efficiency of our approach is highly sensitive to the partition heuristics used. Hence, we plan to investigate algorithms to derive a "good" partition from an automatic structural analysis of the model, that is, before starting to generate the state space. This is doubly important because the specification of the heuristics, in addition to being a critical factor, is a new burden put upon the user with respect to the sequential solution.

7 Acknowledgments

The authors were supported in part by the National Aeronautics and Space Administration under NASA Contracts No. NAS1-19480 and NAG-1-1132.

References

[1] M. Ajmone Marsan, G. Balbo, and G. Conte. 1984. A class of Generalized Stochastic Petri Nets for the performance evaluation of multiprocessor systems, *ACM Trans. Comp. Syst.*, 2(2), 93–122.

[2] G. Balbo, G. Chiola, G. Franceschinis, and G. Molinari Roet. 1987. On the efficient construction of the tangible reachability graph of generalized stochastic Petri nets, in *Proc. 2nd Int. Workshop on Petri Nets and Performance Models (PNPM'87)*, 136–145. IEEE Comp. Soc. Press.

[3] F. Baskett, K. M. Chandy, R. R. Muntz, and F. Palacios-Gomez. 1975. Open, Closed, and Mixed networks of queues with different classes of customers, *J. ACM*, 22(2), 335–381.

[4] A. Blakemore. 1989. The cost of eliminating vanishing markings form generalized stochastic Petri nets, in *Proc. 3rd Int. Workshop on Petri Nets and Performance Models (PNPM'89)*, 85–92. IEEE Comp. Soc. Press.

[5] P. Buchholz. 1992. Hierarchical Markovian models – Symmetries and Reduction, in *Modelling Techniques and Tools for Computer Performance Evaluation*. Elsevier Science Publishers B.V. (North-Holland).

[6] S. Caselli, G. Conte, and P. Marenzoni. 1995. Parallel state space exploration for GSPN models, in *Application and Theory of Petri Nets 1995, Lecture Notes in Computer Science 935 (Proc. 16th Int. Conf. on Applications and Theory of Petri Nets, Turin, Italy)*, G. De Michelis and M. Diaz (eds.), 181–200. Springer-Verlag.

[7] G. Ciardo, A. Blakemore, P. F. J. Chimento, J. K. Muppala, and K. S. Trivedi. 1993. Automated generation and analysis of Markov reward models using Stochastic Reward Nets, in *Linear Algebra, Markov Chains, and Queueing Models*, C. Meyer and R. J. Plemmons (eds.), volume 48 of *IMA Volumes in Mathematics and its Applications*, Springer-Verlag, 145–191.

[8] G. Ciardo, J. K. Muppala, and K. S. Trivedi. 1991. On the solution of GSPN reward models, *Perf. Eval.*, 12(4), 237–253.

[9] G. Ciardo and K. S. Trivedi. 1993. A decomposition approach for stochastic reward net models, *Perf. Eval.*, 18(1), 37–59.

[10] G. Ciardo, K. S. Trivedi, and J. K. Muppala. 1989. SPNP: Stochastic Petri net package, in *Proc. 3rd Int. Workshop on Petri Nets and Performance Models (PNPM'89)*, 142–151. IEEE Comp. Soc. Press.

[11] P. J. Courtois. 1977. *Decomposability: Queueing and Computer System Applications*, Academic Press, New York, NY.

[12] E. W. Dijkstra, W. Feijen, and A. Van Gasteren. 1983. Derivation of a termination detection algorithm for a distributed computation, *Inf. Proc. Letters*, 16, 217–219.

[13] S. Donatelli. 1991. Superposed Stochastic Automata: a class of stochastic Petri nets amenable to parallel solution, in *Proc. 4th Int. Workshop on Petri Nets and Performance Models (PNPM'91)*, 54–63. IEEE Comp. Soc. Press.

[14] A. Goyal, W. C. Carter, E. de Souza e Silva, S. S. Lavenberg, and K. S. Trivedi. 1986. The System Availability Estimator, in *Proc. 16th Int. Symp. on Fault-Tolerant Computing*, 84–89. CS Press.

[15] W. K. Grassmann and Y. Wang. 1995. Immediate events in Markov chains, in *Computations with Markov Chains*, W. J. Stewart (ed.), Kluwer, Boston, MA, 163–176.

[16] W. Gropp, E. Lusk, and A. Skjellum. 1994. *Using MPI*, MIT Press, Cambridge, MA.

[17] D. Harel. 1988. On visual formalisms, *Comm. of ACM*, 31(5), 512–530.

[18] A. Jensen. 1953. Markoff chains as an aid in the study of Markoff processes, *Skand. Aktuarietidskr.*, 36, 87–91.

[19] J. Martinez and M. Silva. 1981. A simple and fast algorithm to obtain all invariants of a generalised Petri net, in *Proc. 2nd European Workshop on Application and Theory of Petri Nets*, 411–422.

[20] T. Murata. 1989. Petri Nets: properties, analysis and applications, *Proc. of the IEEE*, 77(4), 541–579.

[21] D. Nicol, D. Palumbo, and M. Ulrey. 1995. A graphical tool for reliability and failure-mode-effects analysis, in *Proc. of the 1995 Reliability and Maintainability Conference*, 74–81.

[22] D. M. Nicol. 1995. Non-commital barrier synchronization, *Parallel Computing*, 21, 529–549.

[23] J. L. Peterson. 1981. *Petri Net Theory and the Modeling of Systems*, Prentice-Hall.

[24] C. Petri. *Kommunikation mit Automaten*. PhD thesis, University of Bonn, Bonn, West Germany, 1962.

[25] S. Pissanetzky. 1984. *Sparse Matrix Technology*, Academic Press.

[26] M. Sereno and G. Balbo. 1993. Computational algorithms for product form solution stochastic Petri nets, in *Proc. 5th Int. Workshop on Petri Nets and Performance Models (PNPM'93)*, 98–107. IEEE Comp. Soc. Press.

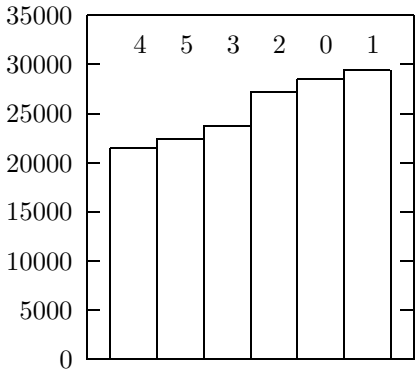
[27] W. J. Stewart. 1994. *Introduction to the Numerical Solution of Markov Chains*, Princeton University Press.

[28] R. S. Varga. 1962. *Matrix Iterative Analysis*, Prentice-Hall.

	0	**1**	**2**	**3**	**4**	**5**
0	104,265	31,123	10,844	9,345	9,376	38,528
1	47,925	107,937	30,878	11,797	10,062	9,588
2	7,640	47,830	97,875	28,272	11,235	9,027
3	6,729	6,103	42,542	83,385	25,288	9,698
4	7,750	5,694	5,541	36,516	73,881	24,239
5	27,516	7,962	6,411	6,494	33,967	78,219

$(\#_{P1} + \#_{P2} \cdot 1013 + \#_{P3} \cdot 1013^2) \bmod N$

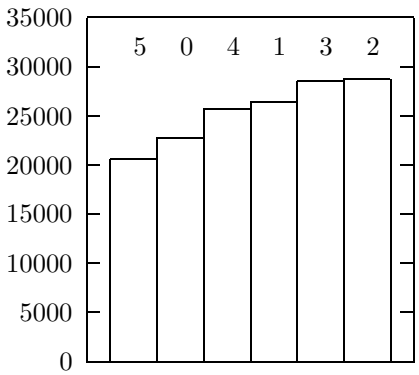
$\chi = 565,920$ (50.9% of η)



	0	**1**	**2**	**3**	**4**	**5**
0	73,269	18,070	16,619	6,427	5,065	37,889
1	43,422	87,870	20,373	19,878	8,433	7,345
2	7,671	51,855	96,147	22,236	22,286	10,052
3	9,413	6,723	56,546	94,080	22,644	22,731
4	19,805	7,587	5,150	55,090	82,395	21,368
5	17,349	15,108	5,356	3,464	47,782	63,984

$(\#_{P1} + \#_{P2} + \#_{P3}) \bmod N$

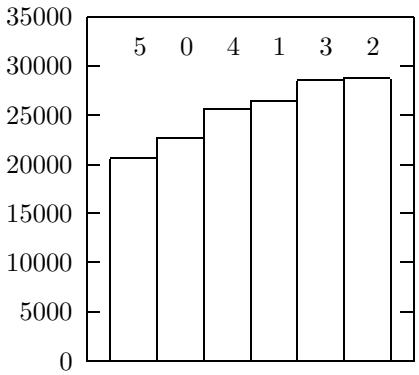
$\chi = 613,737$ (55.2% of η)



	0	**1**	**2**	**3**	**4**	**5**
0	63,282	41,130	0	0	0	48,253
1	59,677	76,956	49,782	0	0	0
2	0	71,701	84,660	54,948	0	0
3	0	0	77,209	82,662	54,012	0
4	0	0	0	74,185	71,694	47,148
5	35,940	0	0	0	63,715	54,528

$(\#_{P1wM1} + \#_{P1M1} + \#_{P2wM2} + \#_{P2M2} + \#_{P3M2}) \bmod N$

$\chi = 677,700$ (61.0% of η)



	0	**1**	**2**	**3**	**4**	**5**
0	120,906	13,536	0	7,272	14,544	7,272
1	0	161,208	9,396	7,272	0	29,088
2	14,544	0	120,906	20,808	0	7,272
3	0	21,816	0	161,208	9,396	14,544
4	0	7,272	14,544	0	120,906	20,808
5	9,396	7,272	0	29,088	0	161,208

$(\#_{P3} + \#_{P3M2} \cdot 1013) \bmod N$

$\chi = 265,140$ (23.9% of η)

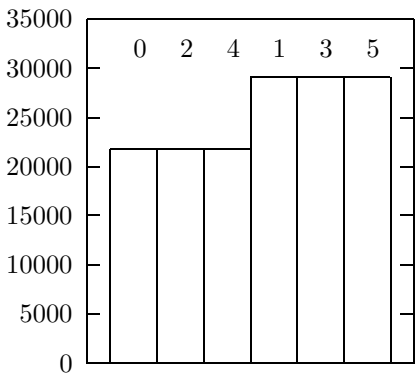


Figure 8: Number of arcs from process i to process j, $\eta^{i,j}$ ($N = 6$ and $k = 5$).