ECE 369
Homework 3
Due October 1st, 2025

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List anyone you have collaborated with:

Please box your answer or make your answer bold

Problem 1

Use induction to prove:

(a) $\sum_{i=0}^{n} i^2 = \frac{n*(n+1)*(2n+1)}{6}$
(Hint: $\sum_{i=0}^{n} i^2 = 0^2 + 1^2 + 2^3 + ... + n^2$)

Proof by Induction:

Base case: Let $n = 0$.

LHS: $\sum_{i=0}^{0} i^2 = 0^2 = 0$

RHS: $\frac{0*(0+1)*(2*0+1)}{6} = \frac{0}{6} = 0$

LHS = RHS, so the base case holds.

Inductive hypothesis: Assume that for some $k \geq 0$, we have

$$\sum_{i=0}^{k} i^2 = \frac{k*(k+1)*(2k+1)}{6}$$

Inductive step: We need to prove that

$$\sum_{i=0}^{k+1} i^2 = \frac{(k+1)*(k+2)*(2(k+1)+1)}{6} = \frac{(k+1)*(k+2)*(2k+3)}{6}$$

Starting with the left-hand side:

$$\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= \sum_{i=0}^{k} i^2 + (k+1)^2 \\
&= \frac{k*(k+1)*(2k+1)}{6} + (k+1)^2 \quad \text{(by IH)} \\
&= \frac{k*(k+1)*(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k*(k+1)*(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

This completes the inductive step.

Conclusion: By mathematical induction, $\sum_{i=0}^{n} i^2 = \frac{n*(n+1)*(2n+1)}{6}$ for all $n \geq 0$. **Q.E.D.**

(b) There exists a natural number K such that for every integer $n > k$, $\frac{1}{n} \leq \frac{1}{100}$

Solution:

We claim that $\boldsymbol{K = 100}$ is the natural number that satisfies the property.

Proof:

For any integer $n \geq 100$, if we divide both sides by $n * 100$, we get:

$$n \geq 100 \Rightarrow \frac{1}{100} \geq \frac{1}{n} \Rightarrow \frac{1}{n} \leq \frac{1}{100}$$

Therefore, $K = 100$ satisfies the required property. **Q.E.D.**

Problem 2

Find the smallest integer N and prove:

$$\forall x, x \geq N \Rightarrow 2^x > 100x^2$$

Smallest Integer N:

First, we check several values to find the minimum N where $2^x > 100x^2$:

- $x = 14$: $2^{14} = 16384$, $100 * 14^2 = 19600$. False.
- $x = 15$: $2^{15} = 32768$, $100 * 15^2 = 22500$. True
- $x = 16$: $2^{16} = 65536$, $100 * 16^2 = 25600$. True

The smallest integer N is $\boldsymbol{N = 15}$.

Proof by Induction:

Base case: $x = 15$

$2^{15} = 32768$ and $100 * 15^2 = 22500$

Since $32768 > 22500$, the base case holds.

Inductive hypothesis: Assume that for some $k \geq 15$, we have $2^k > 100k^2$.

Inductive step: We need to prove that $2^{k+1} > 100(k+1)^2$.

Starting with the left-hand side:

$$2^{k+1} = 2 * 2^k > 2 * 100k^2 = 200k^2 \quad \text{(by IH)}$$

We need to show that $200k^2 > 100(k+1)^2 = 100(k^2 + 2k + 1) = 100k^2 + 200k + 100$.

$$\begin{gathered} 200k^2 > 100k^2 + 200k + 100 \\ 100k^2 > 200k + 100 \\ k^2 > 2k + 1 \\ k^2 - 2k - 1 > 0 \end{gathered}$$

Using the quadratic formula, $k^2 - 2k - 1 = 0$ when $k = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$.

So $k^2 - 2k - 1 > 0$ when $k > 1 + \sqrt{2} \approx 2.414$.

Since $k \geq 15$, this inequality holds for the inductive step.

Conclusion: By mathematical induction, $\forall x \geq 15$, $2^x > 100x^2$. Therefore, $\boldsymbol{N = 15}$. **Q.E.D.**

Problem 3

(a) Write down the steps to move N=3 disks to C.

Initial state:
- A: - 3 - 2 - 1 -
- B: - - - - - - -
- C: - - - - - - -

Step 1: Move disk 1 from A to C
- A: - 3 - 2 - - -
- B: - - - - - - -
- C: - - - 1 - - -

Step 2: Move disk 2 from A to B
- A: - 3 - - - - -
- B: - - - 2 - - -
- C: - - - 1 - - -

Step 3: Move disk 1 from C to B
- A: - 3 - - - - -
- B: - - - 2 - 1 -
- C: - - - - - - -

Step 4: Move disk 3 from A to C
- A: - - - - - - -
- B: - - - 2 - 1 -
- C: - 3 - - - - -

Step 5: Move disk 1 from B to A
- A: - - - 1 - - -
- B: - - - 2 - - -
- C: - 3 - - - - -

Step 6: Move disk 2 from B to C
- A: - - - 1 - - -
- B: - - - - - - -
- C: - 3 - 2 - - -

Step 7: Move disk 1 from A to C
- A: - - - - - - -
- B: - - - - - - -
- C: - 3 - 2 - 1 -

Final state reached in 7 steps.

(b) Prove using induction: for any positive number N, there exists a way to move N disks from pillar A to C according to the hanoi tower's game rules.

Proof by Induction:

Base case: $N = 1$

For a single disk on pillar A, we simply move it directly to pillar C. This is a valid move and completes the task in 1 step. The base case holds.

Inductive hypothesis: Assume that for some $k \geq 1$, we can move k disks from any pillar to any other pillar following the Hanoi Tower rules.

Inductive step: We need to prove that we can move $k + 1$ disks from pillar A to pillar C.

Strategy for $k + 1$ disks:

Consider the configuration with $k + 1$ disks on pillar A (disk $k + 1$ is the largest at the bottom, and disks $k, k - 1, ..., 1$ are stacked above it).

1. **Step 1:** Use the inductive hypothesis to move the top k disks from pillar A to pillar B (using pillar C as auxiliary). By the IH, this is possible.
   - Pillar A has only disk $k + 1$ (the largest)
   - Pillar B has disks $k, k - 1, ..., 1$ (properly stacked)
   - Pillar C is empty
2. **Step 2:** Move disk $k + 1$ from pillar A to pillar C. This is valid because pillar C is empty and we're only moving the topmost (and only) disk from A.
   - Pillar A is empty
   - Pillar B has disks $k, k - 1, ..., 1$
   - Pillar C has only disk $k + 1$
3. **Step 3:** Use the inductive hypothesis to move the k disks from pillar B to pillar C (using pillar A as auxiliary). By the IH, this is possible. Since disk $k + 1$ is the largest disk, all k smaller disks can be placed on top of it, maintaining the size ordering rule.
   - Pillar A is empty
   - Pillar B is empty
   - Pillar C has all $k + 1$ disks properly stacked

This completes the inductive step, showing that if we can move k disks, we can also move $k + 1$ disks.

Conclusion: By mathematical induction, for any positive integer N, there exists a valid sequence of moves to transfer N disks from pillar A to pillar C following the Hanoi Tower rules. **Q.E.D.**