ECE 369
Homework 2
Due September 24th, 2025

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List anyone you have collaborated with:

In this exam, you may use the following inference rules:

Name	**Abbreviation**	**Rule**
modus ponens	mp	$\frac{P \to Q \quad P}{Q}$
modus tollens	mt	$\frac{P \to Q \quad \neg Q}{\neg P}$
simplification	simpl	$\frac{P \wedge Q}{P}$
addition	add	$\frac{P}{P \vee Q}$
conjunction	conj	$\frac{P \quad Q}{P \wedge Q}$
hypothetical syllogism	hs	$\frac{P \to Q \quad Q \to R}{P \to R}$
disjunctive syllogism	ds	$\frac{P \vee Q \quad \neg P}{Q}$
disjunction elimination	de	$\frac{(P \vee Q) \quad P \to R \quad Q \to R}{R}$

And you may use the following equivalences ($\alpha \Leftrightarrow \beta$ should be read as "α is equivalent to β"):

Name	**Abbreviation**	**Equivalence**
Double negation	dn	$P \Leftrightarrow \neg\neg P$
Contrapositive	cp	$P \to Q \Leftrightarrow \neg Q \to \neg P$
Implication	impl	$P \to Q \Leftrightarrow \neg P \vee Q$
Exportation	exp	$P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$
DeMorgan's Laws	dm	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
Associativity	assoc	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$
Commutativity	comm	$P \wedge Q \Leftrightarrow Q \wedge P$ $P \vee Q \Leftrightarrow Q \vee P$
Distributivity	dist	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
Self-reference	self	$P \Leftrightarrow P \wedge P$ $P \Leftrightarrow P \vee P$

When you use the deduction theorem, introduce your assumption by labeling it “hyp.” and discharge the assumption (turn it into an implication) by labeling it “X–Y, ded” where line X is where you introduced the assumption (the left hand side of the implication), and line Y is where you concluded the right-hand side of the implication. Drawing the vertical line is mandatory.

You may also use the following predicate logic equivalences:

Name	**Abbreviation**	**Equivalence**
DeMorgan’s Law (1)	dm	$\neg\forall x.\alpha \Leftrightarrow \exists x.\neg\alpha$
DeMorgan’s Law (2)	dm	$\neg\exists x.\alpha \Leftrightarrow \forall x.\neg\alpha$

And the following predicate logic inference rules. Remember that $\alpha[a \mapsto b]$ means “replace all occurrences of a in α with b.”

Name	**Abbr.**	**Rule**	**Rules for application**
universal instantiation	ui	$\forall x.\alpha$ $\alpha[x \mapsto c]$	c can be a constant or a variable, but if it is a variable, it cannot be one that is re-mapped inside α
existential generalization	eg	α $\exists x.\alpha[c \mapsto x]$	x cannot already appear inside α. It should be a “fresh” variable.
existential instantiation	ei	$\exists x.\alpha$ $\alpha[x \mapsto c]$	c should be a fresh constant: it shouldn’t be in any of the premises, and should not have appeared anywhere else in the proof.
universal generalization	ug	α $\forall x.\alpha[c \mapsto x]$	c should not be constrained in the proof in any way—it should be possible for c to be talking about any object in the domain.

Problem 1

Translate these English statements into logical formulas. One formula per sentence.

(a)

(1) All integers are neither even or odd.
(2) For all integers, if it is even, it is a multiple of 2.
(3) For all integers, if it is a multiple of 4, it cannot be odd.
(4) For all integers, if it is a multiple of 12, it must also be a multiple of 3.
(5) There exist some integers that are a multiple of 3 but is not odd.

Use these predicates (Please fill in the blanks):
Odd(x): Number x is odd.
Even(x):

$\boxed{\textbf{Number x is even}}$

Divisible(a, b):

$\boxed{\textbf{Number a is divisible by b}}$

Solutions:

(1) $\boxed{\forall x(\neg \textbf{ Even}(x) \vee \neg \textbf{ Odd}(x))}$

(2) $\boxed{\forall x(\textbf{Even}(x) \to \textbf{Divisible}(x, 2))}$

(3) $\boxed{\forall x(\textbf{Divisible}(x, 4) \to \neg \textbf{ Odd}(x))}$

(4) $\boxed{\forall x(\textbf{Divisible}(x, 12) \to \textbf{Divisible}(x, 3))}$

(5) $\boxed{\exists x(\textbf{Divisible}(x, 3) \wedge \neg \textbf{ Odd}(x))}$

(b)

(1) A positive integer is a prime if and only if there does not exist an integer that is its factor (i.e., it is divisible by this factor) beside 1 and itself.
(2) A composite number is an integer that has at least one positive factor besides 1 and itself.
(3) A composite number is an integer that has at least one factor that is between 1 and itself.
(4) An integer greater than 1 is either a prime or a composite.
(5) There exists a positive integer that is neither a prime nor a composite number.

Use these predicates (Please fill in the blanks):
P(x): x is a prime number.
C(x): x is a composite number.
Div(a,b):

$\boxed{\textbf{a is divisible by b (b is a factor of a)}}$

Solutions:

(1) $\boxed{\forall x((x > 0) \rightarrow (P(x) \leftrightarrow \neg\exists y((y \neq 1 \wedge y \neq x) \wedge \mathrm{Div}(x, y))))}$

(2) $\boxed{\forall x(C(x) \leftrightarrow \exists y(y > 0 \wedge y \neq 1 \wedge y \neq x \wedge \mathrm{Div}(x, y)))}$

(3) $\boxed{\forall x(C(x) \leftrightarrow \exists y(1 < y < x \wedge \mathrm{Div}(x, y)))}$

(4) $\boxed{\forall x((x > 1) \rightarrow (P(x) \vee C(x)))}$

(5) $\boxed{\exists x(x > 0 \wedge \neg P(x) \wedge \neg C(x))}$

(c)

(1) There is a movie star who is richer than anyone else.
(2) Anyone who is richer than other person pays more taxes than the other person does.
(3) Therefore, there is a movie star who pays more taxes than anyone else.

Use these predicates (Please fill in):
Movie(x): x is a movie star.
R(x,y):

$\boxed{\textbf{x is richer than y}}$

T(x,y):

$\boxed{\textbf{x pays more taxes than y}}$

(Careful: You need to use $\neg(x = y)$ to indicate the statements are applied to someone other than the person themself.)

Solutions:

(1) $\boxed{\exists x(\text{Movie}(x) \wedge \forall y(\neg(x = y) \rightarrow R(x, y)))}$

(2) $\boxed{\forall x \forall y((\neg(x = y) \wedge R(x, y)) \rightarrow T(x, y))}$

(3) $\boxed{\exists x(\text{Movie}(x) \wedge \forall y(\neg(x = y) \rightarrow T(x, y)))}$

(d) Prove the statement in (c).

Proof:

1.	$\exists x(\text{Movie}(x) \wedge \forall y(\neg(x = y) \to R(x, y)))$	Premise
2.	$\forall x \forall y((\neg(x = y) \wedge R(x, y)) \to T(x, y))$	Premise
3.	$\text{Movie}(a) \wedge \forall y(\neg(a = y) \to R(a, y))$	1, ei
4.	$\text{Movie}(a)$	3, simpl
5.	$\forall y(\neg(a = y) \to R(a, y))$	3, simpl
6.	$\forall y((\neg(a = y) \wedge R(a, y)) \to T(a, y))$	2, ui
7.	$\mid \neg(a = b)$	hyp.
8.	$\mid \neg(a = b) \to R(a, b)$	5, ui
9.	$\mid R(a, b)$	7, 8, mp
10.	$\mid \neg(a = b) \wedge R(a, b)$	7, 9, conj
11.	$\mid (\neg(a = b) \wedge R(a, b)) \to T(a, b)$	6, ui
12.	$\mid T(a, b)$	10, 11, mp
13.	$\neg(a = b) \to T(a, b)$	7-12, ded
14.	$\forall y(\neg(a = y) \to T(a, y))$	13, ug
15.	$\text{Movie}(a) \wedge \forall y(\neg(a = y) \to T(a, y))$	4, 14, conj
16.	$\boldsymbol{\exists x(\text{Movie}(x) \wedge \forall y(\neg(x = y) \to T(x, y)))}$	15, eg

Problem 2

Prove the statement using deduction. For full credit, draw the vertical lines correctly for deduction scoping.

(a) $A \to (B \to (C \to D)) \vdash C \to (A \to (B \to D))$

Proof:

1.	$A \to (B \to (C \to D))$	Premise
2.	$\mid C$	hyp.
3.	$\mid \mid A$	hyp.
4.	$\mid \mid B \to (C \to D)$	1, 3, mp
5.	$\mid \mid \mid B$	hyp.
6.	$\mid \mid \mid C \to D$	4, 5, mp
7.	$\mid \mid \mid D$	2, 6, mp
8.	$\mid \mid B \to D$	5-7, ded
9.	$\mid A \to (B \to D)$	3-8, ded
10.	$\boldsymbol{C \to (A \to (B \to D))}$	2-9, ded

(b) $A \to (B \to (\neg C \wedge \neg D)) \vdash D \to (A \to \neg B)$

Proof:

1.	$A \to (B \to (\neg C \wedge \neg D))$	Premise
2.	$\mid D$	hyp.
3.	$\mid \mid A$	hyp.
4.	$\mid \mid B \to (\neg C \wedge \neg D)$	1, 3, mp
5.	$\mid \mid \neg B \vee (\neg C \wedge \neg D)$	4, impl
6.	$\mid \mid \neg B \vee \neg D$	5, simpl (on disjunct)
7.	$\mid \mid \neg\neg D$	2, dn
8.	$\mid \mid \neg B$	6, 7, ds
9.	$\mid A \to \neg B$	3-8, ded
10.	$\boldsymbol{D \to (A \to \neg B)}$	2-9, ded

Problem 3

Prove the following statements using predicate logic.

(a) $\{\neg\exists x(M(x) \wedge P(x)), \forall x(S(x) \rightarrow M(x))\} \vdash \forall x(S(x) \rightarrow \neg P(x))$
(Hint: One of the "perfect" syllogism Aristotle has proposed)

Proof:

1.	$\neg\exists x(M(x) \wedge P(x))$	Premise
2.	$\forall x(S(x) \rightarrow M(x))$	Premise
3.	$\forall x\neg(M(x) \wedge P(x))$	1, dm
4.	$\forall x(\neg M(x) \vee \neg P(x))$	3, dm
5.	$\forall x(M(x) \rightarrow \neg P(x))$	4, impl
6.	$S(a) \rightarrow M(a)$	2, ui
7.	$M(a) \rightarrow \neg P(a)$	5, ui
8.	$S(a) \rightarrow \neg P(a)$	6, 7, hs
9.	$\boldsymbol{\forall x(S(x) \rightarrow \neg P(x))}$	8, ug

(b) $\{\forall x(F(x) \rightarrow \exists y(C(y) \wedge O(x,y))), \forall x(D(x) \rightarrow \forall y(C(y) \rightarrow \neg O(x,y)))\} \vdash \forall x(F(x) \rightarrow \neg D(x))$
(Hint: You may use two deductions in the proof sequence. This formal logical argument can be interpreted as "Every farmer owns a cow. No dentist owns a cow. Therefore, no farmer is a dentist.")

Proof:

1.	$\forall x(F(x) \to \exists y(C(y) \land O(x,y)))$	Premise
2.	$\forall x(D(x) \to \forall y(C(y) \to \neg O(x,y)))$	Premise
3.	$\mid F(a)$	hyp.
4.	$\mid F(a) \to \exists y(C(y) \land O(a,y))$	1, ui
5.	$\mid \exists y(C(y) \land O(a,y))$	3, 4, mp
6.	$\mid C(b) \land O(a,b)$	5, ei
7.	$\mid C(b)$	6, simpl
8.	$\mid O(a,b)$	6, simpl
9.	$\mid \mid D(a)$	hyp.
10.	$\mid \mid D(a) \to \forall y(C(y) \to \neg O(a,y))$	2, ui
11.	$\mid \mid \forall y(C(y) \to \neg O(a,y))$	9, 10, mp
12.	$\mid \mid C(b) \to \neg O(a,b)$	11, ui
13.	$\mid \mid \neg O(a,b)$	7, 12, mp
14.	$\mid \mid O(a,b) \land \neg O(a,b)$	8, 13, conj
15.	$\mid \mid \bot$	14, contradiction
16.	$\mid \neg D(a)$	9-15, ded (by contradiction)
17.	$F(a) \to \neg D(a)$	3-16, ded
18.	$\boldsymbol{\forall x(F(x) \to \neg D(x))}$	17, ug

Problem 4

Prove the following claims by contradiction. You will not get credit if you prove using a proof style other than contradiction!

For full credit, clearly write down your definitions and predicates first.

(a) There is no smallest, positive, rational number.

Definitions:

- Let $\mathbb{Q}^+$ denote the set of positive rational numbers
- A rational number can be expressed as $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$

Proof by Contradiction:

Assume for contradiction that there exists a smallest positive rational number r.

So we have: $r \in \mathbb{Q}^+$ and $\forall y \in \mathbb{Q}^+(r \leq y)$

Consider the number $\frac{r}{2}$. Since r is a positive rational number:

- $r > 0$
- $r = \frac{p}{q}$ for some integers $p > 0, q > 0$
- Therefore $\frac{r}{2} = \frac{p}{2q}$ which is also a positive rational number

We observe:

- $\frac{r}{2} > 0$ (since $r > 0$)
- $\frac{r}{2} < r$ (since dividing by 2 reduces a positive number)
- $\frac{r}{2} \in \mathbb{Q}^+$ (positive rationals are closed under division by positive integers)

But this contradicts our assumption that r is the smallest positive rational number, since we found $\frac{r}{2} \in \mathbb{Q}^+$ with $\frac{r}{2} < r$.

Therefore, there is no smallest positive rational number. **Q.E.D.**

(b) Consider the roots of the quadratic polynomial $px^2 + qx + r$, where p, q, and r are all > 0. If $q > 2\sqrt{p \cdot r}$, the roots of this polynomial cannot be complex.

Hint: Let the root be $\boldsymbol{x = (a + bi)}$ which leads to $\boldsymbol{px^2 + qx + r = 0}$

Proof by Contradiction:

Assume for contradiction that when $p, q, r > 0$ and $q > 2\sqrt{pr}$, the polynomial has complex roots.

Let one root be $x = a + bi$ where $b \neq 0$ (non-real). Since coefficients are real, the other root must be the conjugate: $x = a - bi$

By Vieta's formulas:
- Sum of roots: $(a + bi) + (a - bi) = 2a = -\frac{q}{p}$
- Product of roots: $(a + bi)(a - bi) = a^2 + b^2 = \frac{r}{p}$

From the sum: $a = -\frac{q}{2p}$

Since $q > 0$ and $p > 0$: $a < 0$

From the product: $a^2 + b^2 = \frac{r}{p}$

Substituting $a = -\frac{q}{2p}$: $\frac{q^2}{4p^2} + b^2 = \frac{r}{p}$

Therefore: $b^2 = \frac{r}{p} - \frac{q^2}{4p^2} = \frac{4pr - q^2}{4p^2}$

Since b is real (part of a complex number), we need $b^2 \geq 0$: $\frac{4pr - q^2}{4p^2} \geq 0$

This requires: $4pr - q^2 \geq 0$, which means $q^2 \leq 4pr$, or $q \leq 2\sqrt{pr}$

But this contradicts our given condition that $q > 2\sqrt{pr}$.

Therefore, when $q > 2\sqrt{pr}$, the roots cannot be complex. **Q.E.D.**