Analysis, Pseudomonotone Operators and Applications to the Existence of Solutions of Anisotropic Elliptic Equations

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Terminology and Notation

Definitions, theorems, lemmas, corollaries and propositions are all labelled together, while remarks are labelled separately.

$\mathbb{N} = \{1, 2, 3, \dots\}$.

$\mathcal{P}(X)$: powerset of X.

sgn: $\mathbb{R} \to \{-1, 0, 1\}$ is the sign function given by

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Denote X to be a real Banach space, X^* as its dual space and X^{**} as the dual of X^*. Write $J_X : X \to X^{**}$ as the canonical embedding. We use $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ to denote the duality map.

$x_n \to x$ strongly in X denotes strong convergence in X.

$x_n \rightharpoonup x$ weakly in X denotes weak convergence in X.

$\varphi_n \overset{*}{\rightharpoonup} \varphi$ weakly* in X denotes weak* convergence in X.

$\overset{\text{cont.}}{\hookrightarrow}$ denotes continuous embedding. $\overset{\text{compact}}{\hookrightarrow}$ denotes compact embedding.

If $p \geq 1$, write p' to denote the Hölder conjugate satisfying $1/p + 1/p' = 1$.

Write $\vec{p} = (p_1, p_2, \dots, p_N) \in \mathbb{R}^N$ with $1 < p_j < \infty$ for all $1 \leq j \leq N$. Denote the harmonic mean of $p_1, p_2, \dots, p_N$ by $p = N / \sum_{j=1}^N p_j$. We call $\vec{p}$ the anisotropic exponent.

$C_c^\infty(\Omega)$: Compactly supported smooth functions on the set Ω.

$W_0^{1,\vec{p}}(\Omega)$: For Ω bounded, this is the closure of $C_c^\infty(\Omega)$ in the norm $\|u\|_{W_0^{1,\vec{p}}(\Omega)} = \sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}$.

$W^{-1,\vec{p}'}(\Omega)$: The topological dual of $W_0^{1,\vec{p}}(\Omega)$, i.e., the space of continuous linear functionals on $W_0^{1,\vec{p}}(\Omega)$.

$L^p(\Omega)$: the space of p-integrable measurable functions on Ω.

$L^p_{\text{loc}}(\Omega)$: the space of measurable functions which are p-integrable on every compact subset of Ω.

$T_k : \mathbb{R} \to \mathbb{R}$, the truncation at height k for $k > 0$ given by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

$G_k : \mathbb{R} \to \mathbb{R}$ is the function given by $G_k(s) = s - T_k(s)$. G_k satisfies $G_k = 0$ on $[-k, k]$ and $sG_k(s) \geq 0$ for all $s \in \mathbb{R}$.

χ_A: Characteristic function on the set A, with $\chi_A = 1$ on A and 0 otherwise.

$\chi_{\{f \geq g\}}$: Characteristic function on the set $\{x \in \Omega \mid f(x) \geq g(x)\}$.

$\int_\Omega d\mu$: Integral over a general measure space (Ω, σ, μ).

$\int_\Omega dx$: Integral with respect to the Lebesgue measure on $\mathbb{R}^N$ over $\Omega \subseteq \mathbb{R}^N$.

CONTENTS

Introduction

The central goal in the study of partial differential equations (PDE) is to develop general theory which allows us to obtain their solutions. This may be motivated by the *physical implications* of the solution, such as when the PDE models real world phenomena, or else we may be motivated purely by *theoretical interests*. Unfortunately, for sufficiently difficult PDE, finding closed form solutions is almost always impossible, which forces us to consider weaker, qualitative properties such as the existence, behaviour and regularity of solutions to PDE. Thankfully, the development of rigorous foundations in mathematical fields such as measure theory and functional analysis allows us to reframe PDE in an appropriate abstract setting (e.g. in the Sobolev spaces), giving us access to powerful methods and abstractions with which we can separate the issue of existence of solutions (which we will focus on) from the smoothness and regularity of solutions. The goal of this thesis is to introduce and apply this framework to prove the existence of solutions to a large class of *anisotropic* PDE, so-called since they allow anisotropy, or variation, in the regularity of the partial derivatives of solutions to the PDE. In order to familiarise readers with the prerequisite theory, we begin the thesis with three preliminary chapters, each introducing core concepts in the theory which are essential in the study of nonlinear elliptic PDE, before establishing the existence of solutions result for the class of anisotropic PDE that was studied for this thesis.

Chapter 1 introduces the methods in analysis that are essential when working with PDE. We begin by stating and proving general facts about sequences in Banach spaces, including the result on weak sequential compactness of bounded sequences in reflexive spaces. We then narrow our scope to convergences in the Lebesgue (L^p) spaces, including important connections between strong convergence, weak convergence and pointwise almost everywhere (a.e.) convergence of sequences of functions. We introduce Vitali's Convergence Theorem, a generalisation of the Dominated Convergence Theorem concerning convergence of sequences of functions in L^1. The final section is dedicated to proving the Generalised Young's Inequality, an essential tool used in the estimation of nonlinear terms. The results in this chapter form the backbone of our work in the later chapters.

Chapter 2 delves into the theory of pseudomonotone operators. We present the basics of the theory, with the main definitions, prototypes and properties of pseudomonotone operators, before finishing with the proof of the major theorem concerning the surjectivity of pseudomonotone operators on real, reflexive, separable Banach spaces. This theorem plays an essential role when we work with the PDE in Chapter 4 by allowing us to find solutions of approximate PDE after reframing them as operator equations. In general, the theory of operator equations (including monotone and pseudomonotone operators) is useful in cases, such as ours, where the wide array of *variational methods* (applied to find critical points of *energy functionals*, which directly correspond to weak solutions) in PDE do not apply.

Denote by $\vec{p} = (p_1, p_2, \ldots, p_N)$ the *anisotropy exponent* with $1 < p_j < \infty$ for all $1 \le j \le N$. Chapter 3 introduces the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$, which is, in a sense, the natural function space where one searches for solutions of anisotropic PDE with zero boundary conditions. We give the definition of $W_0^{1,\vec{p}}(\Omega)$ as the completion of the space of compactly supported smooth functions on Ω and give proofs for its main properties, including completeness, reflexivity, separability, and the anisotropic Sobolev embeddings, before finishing the chapter by proving the chain rule and the truncation property of $W_0^{1,\vec{p}}(\Omega)$. Although the properties of $W_0^{1,\vec{p}}(\Omega)$ appear to be well-known in the literature, to the best of our knowledge, this chapter is novel in explicitly extending the proofs of the isotropic case to the anisotropic case.

Let $N \ge 2$ and Ω be a bounded open subset of $\mathbb{R}^N$. In Chapter 4, we study the existence of solutions (understood in a suitable *weak sense*) of a large class of anisotropic PDE given by

$$\begin{cases} \mathcal{A}u - \mathfrak{B}u + \Phi(x, u, \nabla u) = f, \\ \Phi(x, u, \nabla u) \in L^1(\Omega), \quad f \in L^1(\Omega), \end{cases} \tag{0.0.1}$$

with homogeneous Dirichlet boundary conditions. The corresponding function space in which we consider solutions of (0.0.1) is the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$, as introduced in Chapter 3. The operator $\mathcal{A}$ is a Leray-Lions operator with prototype being the $\vec{p}$-Laplacian, an anisotropic generalisation of the well-known Laplacian and p-Laplacian operators which, in divergence form, is given by

$$\Delta_{\vec{p}} u = \sum_{j=1}^{N} \partial_j (|\partial_j u|^{p_j - 2} \partial_j u).$$

Our methods allow for broad assumptions on the operators involved in (0.0.1): Φ is a lower order term depending on the unknown function u and its gradient ∇u, the operator $\mathfrak{B}$ mapping $W_0^{1,\vec{p}}(\Omega)$ into its dual $W^{-1,\vec{p}'}(\Omega)$ satisfies certain growth and continuity conditions, and the only assumption on f is that it is in $L^1(\Omega)$. The methods by which we extract a solution of (0.0.1) relies on a two-step approximation process dealing with the main obstacles of the PDE individually, i.e., the gradient-dependent term Φ and the arbitrary input data $f \in L^1(\Omega)$. In doing so, we invoke the abstract theory of pseudomonotone operators from Chapter 2 to obtain the existence of solutions of approximations of the PDE (0.0.1), before employing extensive convergence arguments (many used directly from Chapter 1) to extract out a solution of the original PDE (0.0.1) from the set of approximate solutions. After building up the necessary theory, we conclude the thesis by proving the existence of weak solutions for the anisotropic PDE (0.0.1).

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CHAPTER 1

Methods in Analysis

1.1. Introduction

Denote by X an arbitrary real Banach space with the norm $\|\cdot\|_X$, and X^* with the operator norm $\|\cdot\|_{X^*}$ as its dual space. The classical result characterising finite-dimensional Banach spaces in terms of the compactness of the unit ball shows that there exist bounded sequences in infinite-dimensional Banach spaces which admit no strongly convergent subsequence. This poses difficulties when working with sequences in arbitrary Banach spaces, since compactness and sequential compactness are what allows us to extract information out of sequences. One of the key developments in analysis that circumvent this issue is the concept of weak convergence in Banach spaces and weak* convergence in the dual:

Definition 1.1.1 (Weak Convergence and Weak* Convergence)**.** Let X be a Banach space.

(i) A sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly in X to $x \in X$ if, for all $\varphi \in X^*$, we have

$$\langle \varphi, x_n \rangle \to \langle \varphi, x \rangle \quad \text{as } n \to \infty.$$

We denote this convergence by $x_n \rightharpoonup x$ weakly.

(ii) A sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges weakly* in X^* to $\varphi \in X^*$ if, for all $x \in X$, we have

$$\langle \varphi_n, x \rangle \to \langle \varphi, x \rangle \quad \text{as } n \to \infty.$$

We denote this convergence by $\varphi_n \rightharpoonup \varphi$ weakly*.

1.2. Convergence Principles in Banach Spaces

1.2.1. Weak and Strong Convergence. The first section presents general facts on convergences of sequences with a particular focus on weak convergence results, including showing that reflexivity is a sufficient condition for bounded sequences to be weakly compact.

The following basic results are well-known, with (iv), (v) and the two simultaneous convergence results (vi) and (vii) sourced from [13, Proposition 21.23, p. 258]. We will only give the proof of (iv), since it involves non-trivial work.

Proposition 1.2.1 (Convergence Principles)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in X.

(i) Suppose there exists $x \in X$ such that every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges strongly to x. Then $x_n \to x$ strongly in X.

(ii) Suppose every subsequence of $(x_n)_{n\in\mathbb{N}}$ has a further subsequence converging weakly to x. Then $x_n \rightharpoonup x$ weakly in X.

(iii) (ii) holds, but with 'weak convergence' replaced with 'strong convergence'.

(iv) If X is also a reflexive space, then every bounded sequence $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence.

(v) If in addition to (iv), every weakly convergent subsequence of the bounded sequence $(x_n)_{n\in\mathbb{N}}$ has the same limit $x \in X$, then $x_n \rightharpoonup x$ weakly in X.

(vi) (Simultaneous Convergence 1). Suppose we have sequences $(x_n)_{n\in\mathbb{N}}$ in X and $(\varphi_n)_{n\in\mathbb{N}}$ in X^* such that $x_n \rightharpoonup x$ weakly in X and $\varphi_n \to \varphi$ strongly in X^*. Then

$$\langle \varphi_n, x_n \rangle \to \langle \varphi, x \rangle \quad \text{as } n \to \infty. \tag{1.2.1}$$

(vii) (Simultaneous Convergence 2). Suppose X is also a reflexive space and, instead of the convergences in (vi), we have $x_n \to x$ strongly in X and $\varphi_n \rightharpoonup \varphi$ weakly in X^*. Then (1.2.1) holds.

Remark 1.2.1. The simultaneous convergence results are particularly useful when working with functions in the L^p spaces. If $1 < p < \infty$, we can apply the *Riesz Representation Theorem* (Theorem A.2.6) with Proposition 1.2.1 (vii) to show that $\int_\Omega f_k g_k \, dx \to \int_\Omega f g \, dx$ whenever $f_k \to f$ strongly in $L^p(\Omega)$ and $g_k \rightharpoonup g$ weakly in $L^{p'}(\Omega)$, with $1 < p < \infty$ and p' the Hölder conjugate of p. If we are working on a σ-finite measure space, this also extends to the case $p = 1$.

We now give the proof of Proposition 1.2.1 (iv), the method of which is inspired by [13, Theorem 21.D, p. 255] (the proof provided is only for separable Hilbert spaces), and requires auxiliary results on properties of reflexive spaces, which we do not prove, and Helley's Theorem on weak* convergence of bounded functionals, which we do prove, since it involves a (pedagogically) interesting diagonalisation argument - see [9, Helley's Theorem, p. 171].

Proposition 1.2.2 (Banach Space Properties)**.** Let X be a Banach space.

(i) (See [4, Proposition 3.20, p. 70]). If X is reflexive, then every closed subspace of X is reflexive.

(ii) (See [4, Corollary 3.21, p. 70]). X is reflexive if and only if X^* is reflexive.

(iii) (See [4, Theorem 3.26, p. 73]). If X^* is separable, then X is separable.

Lemma 1.2.1 (Helley's Theorem)**.** Let X be a separable Banach space. Then every bounded sequence in X^* has a weak*-convergent subsequence in X^*.

Proof. The key to proving the existence of a subsequence with certain properties is the classical diagonal argument. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a bounded sequence in X^*. Write $R := \sup_{n\in\mathbb{N}} \|\varphi_n\|_{X^*}$. Now, X being separable implies there exists a countable dense subset of X which we denote $(x_n)_{n\in\mathbb{N}}$. Since $(\varphi_n)_{n\in\mathbb{N}}$ is bounded, the sequence $(\langle \varphi_n, x_1\rangle)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$, so by the Bolzano-Weierstrass Theorem there exists an increasing subsequence $(\langle \varphi_{n_1}, x_1\rangle)_{n\in\mathbb{N}}$ converging to some $y_1 \in \mathbb{R}$. Since $(\langle \varphi_{n_1}, x_2\rangle)_{n\in\mathbb{N}}$ is bounded, we can extract a further subsequence $\langle \varphi_{n_2}, x_2\rangle)_{n\in\mathbb{N}}$ from $(\langle \varphi_{n_1}, x_2\rangle)_{n\in\mathbb{N}}$ converging to some $y_2 \in \mathbb{R}$. Continuing this way, we obtain a list of convergent subsequences

$$\begin{array}{cccccc}
\langle \varphi_{1_1}, x_1\rangle & \langle \varphi_{2_1}, x_1\rangle & \langle \varphi_{3_1}, x_1\rangle & \cdots \to y_1 \\
\langle \varphi_{1_2}, x_2\rangle & \langle \varphi_{2_2}, x_2\rangle & \langle \varphi_{3_2}, x_2\rangle & \cdots \to y_2 \\
\langle \varphi_{1_3}, x_3\rangle & \langle \varphi_{2_3}, x_3\rangle & \langle \varphi_{3_3}, x_3\rangle & \cdots \to y_3 \\
\vdots & \vdots & \vdots & \vdots
\end{array} \tag{1.2.2}$$

Write m_k to denote the diagonal index k_k. Taking the diagonal sequence $(\varphi_{m_k})_{k\in\mathbb{N}}$ of operators, we see that $(\langle \varphi_{m_k}, x_l\rangle)_{k\geq l}$ is a subsequence of the l-th row of the list in (1.2.2), i.e., of the sequence

$(\langle \varphi_{m_l}, x_l \rangle)_{l\in\mathbb{N}}$, and hence converges to y_l. We now show that $(\varphi_{m_k})_{k\in\mathbb{N}}$ is weak*-convergent in X^*. Fix $x \in X$ arbitrary. By density, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x - x_N\| < \varepsilon/(4R)$. Then for any $a, b \in \mathbb{N}$, we have

$$\begin{aligned}
|\langle \varphi_{m_a}, x\rangle - \langle \varphi_{m_b}, x\rangle| &\le |\langle \varphi_{m_a}, x\rangle - \langle \varphi_{m_a}, x_N\rangle| + |\langle \varphi_{m_a}, x_N\rangle - \langle \varphi_{m_b}, x_N\rangle| + |\langle \varphi_{m_b}, x_N\rangle - \langle \varphi_{m_b}, x\rangle| \\
&\le 2R\|x - x_N\| + |\langle \varphi_{m_a}, x_N\rangle - \langle \varphi_{m_b}, x_N\rangle| \\
&< \frac{\varepsilon}{2} + |\langle \varphi_{m_a}, x_N\rangle - \langle \varphi_{m_b}, x_N\rangle|.
\end{aligned}$$

Since $\langle \varphi_{m_k}, x_N\rangle \to y_l$ as $k \to \infty$, for every $a, b \in \mathbb{N}$ large enough, we have

$$|\langle \varphi_{m_a}, x_N\rangle - \langle \varphi_{m_b}, x_N\rangle| < \varepsilon/2,$$

which forces $|\langle \varphi_{m_a}, x\rangle - \langle \varphi_{m_b}, x\rangle| < \varepsilon$. This shows that $(\langle \varphi_{m_k}, x\rangle)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and hence converges in $\mathbb{R}$. Define the map $\varphi : X \to \mathbb{R}$ by $\varphi(x) = \lim_{k\to\infty}\langle \varphi_{m_k}, x\rangle$. Clearly, φ is a linear functional, with the linearity following from the linearity of φ_{m_k}. Furthermore, $\varphi : X \to \mathbb{R}$ is bounded in X^*, since

$$|\varphi(x)| = \lim_{k\to\infty} |\langle \varphi_{m_k}, x\rangle| \le R\|x\|$$

by definition of R. It follows that $\varphi_{m_k} \overset{*}{\rightharpoonup} \varphi \in X^*$ weakly* as $k \to \infty$. We conclude that $(\varphi_n)_{n\in\mathbb{N}}$ is, up to subsequence, weakly*-convergent in X^*, as required. □

Proof of Proposition 1.2.1 (iv): Suppose $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in a reflexive space X. Consider the subspace defined by

$$X_0 = \overline{\operatorname{span}\{x_n \mid n \in \mathbb{N}\}}.$$

Clearly, X_0 is a separable space, with countable dense set being the set of rational linear combinations of the x_i, and it is a closed subspace of X. Proposition 1.2.2 (i) implies X_0 is reflexive. Hence, we can canonically identify X_0 with X_0^{**} via the isometric isomorphism $J_{X_0} : X_0 \to X_0^{**}$. By Proposition 1.2.2 (ii), X_0^* is reflexive. Consider the sequence $(J_{X_0}(x_n))_{n\in\mathbb{N}}$ in X_0^{**}, which is bounded in X^{**} since $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence. Since X_0 (and hence X_0^{**}) is reflexive, we can apply Proposition 1.2.2 (iii) on X_0^{**} to show that X_0^* is reflexive. Then using Helley's Theorem (Lemma 1.2.1) on X_0^*, there exists a weak*-convergent subsequence $(J_{X_0}(x_{n_k}))_{k\in\mathbb{N}}$, with a limit that we write as $\varphi \in X_0^{**}$. By the surjectivity of J_{X_0}, there exists $x \in X_0$ such that $J_{X_0}(x) = \varphi$. Then for any $\psi \in X^*$, we have

$$\langle \psi, x_{n_k}\rangle = J_{X_0}(x_{n_k})(\psi) \to \varphi(\psi) = \langle \psi, x\rangle \quad \text{as } k \to \infty.$$

Hence, we see that $x_{n_k} \rightharpoonup x$ weakly in X, as required. □

Remark 1.2.2. It can, in fact, be shown that the weak sequential compactness of bounded sequences completely characterise reflexivity in Banach spaces: This is the *Eberlein-Šmulian Theorem* (see [4, Theorem 3.19, p. 70]).

The next set of results concern weak* convergence and the relation between the convex hull of a sequence and its weak limit. We give the proofs, since a number of them are exercises in textbooks. For clarity, we include the definition of the convex hull of a set:

Definition 1.2.1. The *convex hull* of a set Φ in the real Banach space X is defined as

$$\operatorname{conv}(\Phi) = \left\{ \sum_{i \in I} t_i f_i \mid t_i \geq 0, \quad \sum_{i \in I} t_i = 1, \quad f_i \in \Phi \right\}$$

where I is a finite index set.

Proposition 1.2.3 (Further Weak Convergence Results)**.** Let X be a Banach space. The following hold:

(i) If X is reflexive, weak*-convergence is equivalent to weak convergence. That is, $\varphi_n \overset{*}{\rightharpoonup} \varphi$ weakly* in X^* if and only if $\varphi_n \rightharpoonup \varphi$ weakly in X^*.

(ii) Suppose X is reflexive and that D is a dense subset of X. If $(\varphi_n)_{n \in \mathbb{N}}$ is a bounded sequence in X^* such that $\langle \varphi_n, v \rangle \to \langle \varphi, v \rangle$ as $n \to \infty$ for all $v \in D$, then $\varphi_n \overset{*}{\rightharpoonup} \varphi$ weakly* in X^*.

(iii) (Mazur, 1933) ([4, Corollary 3.8, p. 61]). If $x_n \rightharpoonup x$ weakly in X, then there exists a sequence $(v_n)_{n \in \mathbb{N}}$ in X with all v_n in the convex hull of $\{x_n \mid n \in \mathbb{N}\}$ and $v_n \to x$ strongly in X.

(iv) ([4, Exercise 3.4(i) and (ii), p. 80]). If $x_n \rightharpoonup x$ weakly in X, there exists a sequence $(v_n)_{n \in \mathbb{N}}$ with $v_n \in \operatorname{conv}\left(\cup_{i=n}^{\infty} \{x_i\}\right)$ such that $v_n \to x$ strongly in X.

Proof. (i): Since X is reflexive, we can naturally identify X and X^{**} by the canonical map $J_X : X \to X^{**}$ given by $x \mapsto \langle \cdot, x \rangle$. If we have a sequence $(\varphi_n)_{n \in \mathbb{N}}$ that converges weakly* to φ, then for all $\Phi \in X^{**}$, there exists $x \in X$ such that $\Phi(\varphi_n) = \langle \varphi_n, x \rangle$, which converges to $\langle \varphi, x \rangle = \Phi(\varphi)$ since φ_n converges weakly* to φ in X^*. This shows that $\varphi_n \rightharpoonup \varphi$ weakly in X^*. Conversely, if $\varphi_n \rightharpoonup \varphi$ weakly in X^*, then every element $x \in X$ corresponds to $J_X(x) \in X^{**}$, so that

$$\varphi_n(x) = J_X(x)(\varphi_n) \to J_X(x)(\varphi) = \varphi(x) \quad \text{as } n \to \infty,$$

which shows that $\varphi_n \overset{*}{\rightharpoonup} \varphi$ as $n \to \infty$.

Proof of (ii): Let D be the dense subset of X, $x \in X$ be arbitrary and $(\varphi_n)_{n \in \mathbb{N}}$ a bounded sequence in X^* such that $\langle \varphi_n, v \rangle \to \langle \varphi, v \rangle$ for all $v \in D$. By density, there exists a sequence $(v_n)_{n \in \mathbb{N}} \subseteq D$ such that $v_n \to x$ as $n \to \infty$. The triangle inequality gives us, for some fixed v_k,

$$\begin{aligned} |\langle \varphi_n, x \rangle - \langle \varphi, x \rangle| &\leq |\langle \varphi_n, x \rangle - \langle \varphi_n, v_k \rangle| + |\langle \varphi_n, v_k \rangle - \langle \varphi, v_k \rangle| + |\langle \varphi, v_k \rangle - \langle \varphi, x \rangle| \\ &\leq \|\varphi_n\|_{X^*} \|x - v_k\|_X + |\langle \varphi_n, v_k \rangle - \langle \varphi, v_k \rangle| + \|\varphi\|_{X^*} \|v_k - x\|_X. \end{aligned}$$

Since $v_k \to x$ strongly in X and $\langle \varphi_n, v_k \rangle \to \langle \varphi, v_k \rangle$ as $n \to \infty$ for any $k \in \mathbb{N}$, an $\varepsilon/3$ argument shows that $\langle \varphi_n, x \rangle \to \langle \varphi, x \rangle$ for all $x \in X$, so that $\varphi_n \overset{*}{\rightharpoonup} \varphi$ weakly* in X^*.

Proof of (iii): See the proof in [4, Corollary 3.8] and the definition of the weak topology. In particular, we need the fact that the weak closure of a convex hull is its closure in the norm of the space, which we assume for the next proof.

Proof of (iv): For all $N \in \mathbb{N}$, let $G_N = \operatorname{conv}(\cup_{i=N}^{\infty} \{x_i\})$. Since $x_n \rightharpoonup x$ weakly in X, x is in the weak closure $\overline{G_N}^{\sigma(X,X^*)}$ for all $N \in \mathbb{N}$, but recalling that G_N is convex, we must have x in the strong closure $\overline{G_N}$ for every $N \in \mathbb{N}$ as well. By Mazur's Lemma (Proposition 1.2.3 (iii)), there exist sequences $(y_i^N)_{i \in \mathbb{N}} \subseteq G_N$ such that $y_i^N \to x$ as $i \to \infty$. We pick a subsequence $(y_{i_k}^k)_{k \in \mathbb{N}}$ such that $\|y_{i_k}^k - x\| < 1/k$. Then each $y_{i_k}^k$ is in G_k for all $k \in \mathbb{N}$ and $(y_{i_k}^k)_{k \in \mathbb{N}}$ converges strongly to x in X, as we wanted. $\square$

1.2.2. Compact Operators and Weak Convergence. To finish this section, we state a result connecting weak convergence with strong convergence via compact operators.

Definition 1.2.2. Let X, Y be Banach spaces. A linear operator $T : X \to Y$ is said to be *compact* if, for all bounded sequences $(x_n)_{n\in\mathbb{N}}$ in X, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in Y.

Proposition 1.2.4 (Compact Operators and Weakly Convergent Sequences)**.** Suppose X, Y are Banach spaces and $T : X \to Y$ is a compact, linear operator. If $(u_n)_{n\in\mathbb{N}}$ is sequence in X such that $u_n \rightharpoonup u$ weakly in X, then $Tu_n \to Tu$ strongly in Y.

Remark 1.2.3. This property is especially important in light of the compact embeddings which relate Sobolev spaces with the L^p spaces. Indeed, a standard argument involves extracting a weakly convergent subsequence in a Sobolev space, finding the form of its weak limit in some manner (see e.g. Proposition 1.3.2), then upgrading this weak convergence into strong convergence via compact embedding into an L^p space. This naturally leads us to consider convergence properties of sequences in the L^p spaces in the next section.

1.3. Convergence in L^p Spaces

In this section, we write (Ω, σ, μ) to denote an arbitrary measure space. It is a fact that for $1 \le p < \infty$ the space $L^p(\Omega)$ is complete with respect to the usual norm $f \mapsto \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}$, and $L^\infty(\Omega)$ is complete with respect to the norm $f \mapsto \operatorname{ess\,sup}_\Omega |f|$. Furthermore, for $1 < p < \infty$, the L^p spaces are reflexive, with the dual space being identified with $L^{p'}(\Omega)$ with p' the Hölder conjugate, as a result of the Riesz Representation Theorem; it is also known that if (Ω, σ, μ) is σ-finite, then in the case $p = 1$, the dual of $L^1(\Omega)$ is $L^\infty(\Omega)$ (Theorem A.2.6). We now state a classical proposition which will be useful in the chapters to come.

Proposition 1.3.1 (Pointwise Bound and Convergence of L^p Subsequences)**.** Suppose $1 \le p \le \infty$, and $f_n \to f$ in $L^p(\Omega)$. Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and a function $g \in L^p(\Omega)$ such that $f_{n_k} \to f$ a.e. in Ω and $|f_{n_k}| \le g$ a.e. in Ω for all $k \in \mathbb{N}$.

Proof. See [4, Theorem 4.9, p. 94] for a reference. □

We conclude this section with the proof of the result which allows us to find the form of a sequence of function's weak limit if we also know its pointwise limit. This result, along with Proposition 1.3.1 and the weak sequential compactness of bounded sequences, form the foundation of the toolbox of convergence results we can apply when working with limits and convergences in function spaces, as we will see in later chapters. The result is presented as [4, Exercise 4.16, p. 123]:

Proposition 1.3.2 (Weak Limit Coincides with Pointwise Convergence Limit)**.** Let Ω be a measure space. Let $1 < p < \infty$ and $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in $L^p(\Omega)$ and suppose $f_n \to f$ a.e. in Ω. Then $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ as $n \to \infty$.

Proof. Since $(f_n)_{n\in\mathbb{N}}$ is bounded in the reflexive space $L^p(\Omega)$, Proposition 1.2.1 (iv) gives a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ converging weakly in $L^p(\Omega)$ to some $\tilde{f}$. We show that if $f_n \to \tilde{f}$ weakly in $L^p(\Omega)$ and $f_n \to f$ a.e. in Ω, then $f = \tilde{f}$ a.e. on Ω. From Proposition 1.2.3 (iv), there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $L^p(\Omega)$ such that $g_n \in \operatorname{conv}\big(\cup_{i\geq n}\{f_i\}\big)$ and $g_n \to \tilde{f}$ strongly in $L^p(\Omega)$. In fact, this forces $g_n \to f$ a.e. in Ω. Indeed, since $g_n \in \operatorname{conv}\big(\cup_{i\geq n}\{f_i\}\big)$, we can write $g_n = \sum_{i\geq n} t_i^n f_i$ with $t_i^n \geq 0$ non-zero for finitely many i and $\sum_{i\geq n} t_i^n = 1$. Furthermore, since $f_n \to f$ a.e., for any fixed $\varepsilon > 0$ and for almost every fixed $x \in \Omega$, there exists $N \in \mathbb{N}$ such that for all $k > N$, we have $|f_k(x) - f(x)| < \varepsilon$. Then for almost every $x \in \Omega$, using the triangle inequality,

$$|f(x) - g_k(x)| = |f(x) - \sum_{i\geq k} t_i^k f_i(x)| \leq \sum_{i\geq k} t_i^k |f_i(x) - f(x)| < \sum_{i\geq k} t_i^k \varepsilon = \varepsilon \quad \text{for every } k > N.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $g_n \to f$ a.e. in Ω as $n \to \infty$, so $f = \tilde{f}$ a.e., as we wanted. By a standard 'uniqueness of weak limits argument' (see Proposition 1.2.1 (v)), we conclude that $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ as $n \to \infty$, as required. ☐

1.4. Vitali's Convergence Theorem

In this section, we state Vitali's Convergence Theorem on finite measure spaces, a result which generalises the dominating function requirement of the Dominated Convergence Theorem to pointwise a.e. convergence along with a 'uniformly integrable' condition. Along the way, we give useful sufficient conditions for the uniform integrability of sets of functions. The source of the statements of this section is [9, Section 4.6].

Definition 1.4.1. Let (Ω, σ, μ) be a measure space. Let $\Phi \subseteq L^1(\Omega)$ be a set of integrable functions. We say Φ is *uniformly integrable* if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all measurable sets $E \subseteq \Omega$, we have

$$f \in \Phi \quad \text{and} \quad \mu(E) < \delta \implies \int_E |f|\, d\mu < \varepsilon.$$

If Φ contains a single function $f \in L^1(\Omega)$, then we say f satisfies the *uniform integrability condition*, or f is *uniformly integrable*.[1]

We want to show that there are natural examples of sets of functions that are uniformly integrable. However, we will first need a technical result by the name of the Borel-Cantelli Lemma:

Lemma 1.4.1 (Borel-Cantelli)**.** Let $\{E_n\}_{n\in\mathbb{N}}$ be a countable collection of measurable sets in the measure space (Ω, μ). Suppose $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Then $\mu(\cap_{i=1}^{\infty} \cup_{n=i}^{\infty} E_n) = 0$.

With the Borel-Cantelli Lemma, we can now prove the following:

Proposition 1.4.1. Let (Ω, σ, μ) be a measure space and let $f \in L^1(\Omega)$. Then f satisfies the uniform integrability condition: for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable set $E \subseteq \Omega$, we have

$$\mu(E) < \delta \implies \int_E |f| d\mu < \varepsilon.$$

Proof. Suppose by contradiction that there exist $\varepsilon > 0$ and measurable sets $A_k \subseteq \Omega$, $k \in \mathbb{N}$, such that $\mu(A_k) < 1/2^k$ and $\int_{A_k} |f|\, d\mu \geq \varepsilon$ for all $k \in \mathbb{N}$. Define $A := \limsup_{k\to\infty} A_k = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$. Since

[1] This is also sometimes called *equi-integrability*.

$\sum_{k=1}^{\infty} \mu(A_k) < 1 < \infty$, the Borel-Cantelli Lemma implies that $\mu(A) = 0$. Now let $\nu : \mathcal{P}(\Omega) \to \mathbb{R}$ be defined by $\nu(E) = \int_E |f| \, d\mu$. Since $f \in L^1(\Omega)$, this is a well-defined density measure on Ω. By the continuity property of measures, we have

$$\nu(A) = \nu\left(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k\right) = \lim_{n\to\infty} \nu\left(\cup_{k=n}^{\infty} A_k\right) \geq \nu\left(A_n\right) \quad \text{for all } n \in \mathbb{N}. \tag{1.4.1}$$

We now recall, by assumption, that $\nu(A_n) = \int_{A_n} |f| \, d\mu \geq \varepsilon$, whence we conclude $\nu(A) \geq \varepsilon$ from (1.4.1). But this is a contradiction, since $\mu(A)$ has measure zero, so any integral over A is zero as well. This concludes the proof. □

Corollary 1.4.1. Let (Ω, σ, μ) be a measure space. Suppose $f_n \to f$ in $L^1(\Omega)$. Then the set $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable.

Proof. Fix an arbitrary $\varepsilon > 0$. Since $f_n \to f$ in $L^1(\Omega)$, there exists $N \in \mathbb{N}$ such that for any measurable $E \subseteq \Omega$, we have

$$\int_E |f_n| - |f| \, d\mu \leq \int_E |f_n - f| \, d\mu < \frac{\varepsilon}{2},$$

so that $\int_E |f_n| \, d\mu < \int_E |f| \, d\mu + \varepsilon/2$ for all $n > N$. We note further that $f_n \in L^1(\Omega)$ for $n = 1, 2, \ldots, N$. By Proposition 1.4.1, for each f_n with $n \leq N$ we can find $\delta_n > 0$ such that $\int_\Omega |f_n| \, d\mu < \varepsilon$. Now, since $f \in L^1(\Omega)$, there exists $\delta_{N+1} > 0$ such that whenever $\mu(E) < \delta_{N+1}$, we have $\int_E |f| \, d\mu < \varepsilon/2$. Defining $\delta := \min\{\delta_1, \delta_2, \ldots, \delta_{N+1}\}$, we get that for any measurable $E \subseteq \Omega$,

$$\mu(E) < \delta \implies \int_\Omega |f_n| \, d\mu < \varepsilon \text{ for all } n \in \mathbb{N}$$

so $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable over Ω, as required. □

Remark 1.4.1. From Definition 1.4.1, it is not difficult to see that any *finite* linear combination of uniformly integrable functions is again uniformly integrable. It follows that we can apply Proposition 1.4.1 and Corollary 1.4.1 for an easy-to-check sufficient condition for uniform integrability.

The key result of this chapter is Vitali's Convergence Theorem on finite measure spaces, which we will state next. This theorem is particularly relevant since in Chapter 4, we will be considering a class of anisotropic PDE on a **bounded**, open set, which is of finite measure, and hence allows us to prove convergence of integrals without having to find a dominating L^1 function. A proof may be found in [9, Section 4.6, p. 94].

Theorem 1.4.1 (Vitali's Convergence Theorem)**.** Let (Ω, μ) be a finite measure space. Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^1(\Omega)$ and f is a measurable function on Ω satisfying

(i) $f_n \to f$ a.e. in Ω.
(ii) The set $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable on Ω

Then $f \in L^1(\Omega)$ and, moreover,

$$\int_\Omega f_n \, d\mu \to \int_\Omega f \, d\mu \quad \text{as } n \to \infty.$$

Remark 1.4.2. It immediately follows that we can modify the requirements of Vitali's Theorem to accommodate convergence in $L^p(\Omega)$ for $1 \leq p < \infty$. Indeed, if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^p(\Omega)$

such that $f_n \to f$ a.e. in Ω with $f \in L^p(\Omega)$, then we know $|f_n - f|^p \to 0$ a.e. in Ω, so it suffices to show that $\{|f_n - f|^p\}_{n\in\mathbb{N}}$ is uniformly integrable to obtain the convergence $f_n \to f$ in $L^p(\Omega)$. For fixed $1 \leq p < \infty$, we have $|f_n - f|^p \leq C_p(|f_n|^p + |f|^p)$ for some constant $C_p > 0$; since $f \in L^p(\Omega)$ implies $|f|^p \in L^1(\Omega)$, we can apply Proposition 1.4.1 to see that it is sufficient to show that $\{|f_n|^p\}_{n\in\mathbb{N}}$ is uniformly integrable, from which we can conclude that $f_n \to f$ in $L^p(\Omega)$. We will often rely on this remark when using Vitali's Convergence Theorem without explicitly referencing this fact.

1.5. Generalised Young's Inequality

When working in non-linear analysis, one often needs to estimate the products of non-negative numbers - for example, when finding an upper bound for the norm of a nonlinear term in a PDE. One useful tool for such estimates is Young's inequality on products, which states that for $a, b > 0$ and Hölder conjugates $p, q > 1$ satisfying $1/p + 1/q = 1$, we have $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, resulting from the concavity of the log function. One useful application is to introduce an arbitrary $\varepsilon > 0$ factor, so that we obtain the extended inequality $ab \leq \frac{\varepsilon a^p}{p} + \frac{b^q}{\varepsilon^{q/p} q}$. In the next theorem, we give a generalisation of this application, useful in estimating the product of N terms while affording a degree of freedom on the size of the estimate for all but one term. Only the statement is given in [3, Lemma 2.4, p. 8]. We give an original proof.

Theorem 1.5.1 (Generalised Young's Inequality)**.** Let $N \geq 2$ be an integer and $\beta_1, \beta_2, \dots, \beta_N$ be N positive real numbers. Suppose $1 < R_k < \infty$ for $R_k \in \mathbb{R}$ and $1 \leq k \leq N-1$ such that $\sum_{k=1}^{N-1}(1/R_k) < 1$. Then for every fixed $\delta > 0$, there exists a positive constant C_δ depending on δ such that

$$\Pi_{k=1}^{N} \beta_k \leq \delta \sum_{k=1}^{N-1} \beta_k^{R_k} + C_\delta \beta_N^{R_N} \tag{1.5.1}$$

where $R_N = \left(1 - \sum_{k=1}^{N-1} R_k^{-1}\right)^{-1}$.

Proof. The proof uses induction on N, the number of terms in the product. For $N = 2$, we want to show that for every $\delta > 0$ and $1 < R_1 < \infty$, we can find a constant C_δ such that $\beta_1\beta_2 \leq \delta\beta_1^{R_1} + C_\delta\beta_2^{R_2}$ with $R_2 = \left(1 - R_1^{-1}\right)^{-1}$. Write $\beta_1\beta_2 = (\hat{\delta}\beta_1)(\beta_2\hat{\delta}^{-1})$ where $\hat{\delta} > 0$ is an arbitrary constant. Apply Young's inequality with $p = R_1$. Then $q = \left(1 - R_1^{-1}\right)^{-1} = R_2$, and we get

$$\beta_1\beta_2 = (\hat{\delta}\beta_1)(\beta_2\hat{\delta}^{-1}) \leq \frac{(\hat{\delta}\beta_1)^{R_1}}{R_1} + (1 - R_1^{-1})\left(\beta_2\hat{\delta}^{-1}\right)^{R_2} = \frac{\hat{\delta}^{R_1}}{R_1}\beta_1^{R_1} + \frac{1 - R_1^{-1}}{\hat{\delta}^{R_2}}\beta_2^{R_2}.$$

Picking $\hat{\delta} = \left(R_1\delta\right)^{1/R_1}$ gives us the required inequality, with

$$C_\delta = \frac{1 - R_1^{-1}}{(R_1\delta)^{\frac{R_2}{R_1}}} > 0$$

which is a constant depending only on δ. Now assume that inequality (1.5.1) holds for an arbitrary $N \geq 2$. We want to show the same inequality holds in $N + 1$ variables. Suppose we have

$1 < R_1, R_2, \ldots, R_N < \infty$ with $\sum_{k=1}^{N} 1/R_k < 1$, a fixed $\delta > 0$, and $N+1$ positive real numbers given by $\beta_1, \beta_2, \ldots$ and β_{N+1}. Write

$$S_N = \frac{1}{1 - \sum_{k=1}^{N-1} R_k^{-1}} \quad \text{and} \quad R_{N+1} = \frac{1}{1 - \sum_{k=1}^{N} R_k^{-1}}$$

Then applying the inductive hypothesis onto the product $(\beta_1 \ldots \beta_{N-1})(\beta_N \beta_{N+1})$, we obtain

$$\beta_1 \ldots \beta_N \beta_{N+1} \leq \delta \sum_{k=1}^{N-1} \beta_k^{R_k} + C_\delta (\beta_N \beta_{N+1})^{S_N}. \tag{1.5.2}$$

We will rewrite the remaining term in the sum and apply the original Young's inequality. Notice that

$$C_\delta (\beta_N \beta_{N+1})^{S_N} = \left(C_\delta \varepsilon \beta_N^{S_N} \right) \left(\beta_{N+1}^{S_N} \varepsilon^{-1} \right)$$

where $\epsilon > 0$ is an arbitrary constant that we will pick appropriately later. Applying Young's inequality, we get

$$\left(C_\delta \varepsilon \beta_N^{S_N} \right) \left(\beta_{N+1}^{S_N} \varepsilon^{-1} \right) \leq \frac{\left(C_\delta \varepsilon \beta_N^{S_N} \right)^p}{p} + \frac{\left(\beta_{N+1}^{S_N} \varepsilon^{-1} \right)^q}{q} \tag{1.5.3}$$

for any $p, q > 1$ with $p^{-1} + q^{-1} = 1$. Let $p = \frac{R_N}{S_N}$. We show $p > 1$. Indeed, from the assumptions, we have

$$\begin{aligned} \sum_{k=1}^{N} R_k^{-1} < 1 &\implies \sum_{k=1}^{N-1} R_k^{-1} < 1 - R_N^{-1} \\ &\implies R_N^{-1} < 1 - \sum_{k=1}^{N-1} R_k^{-1} \\ &\implies 1 < \left(1 - \sum_{k=1}^{N-1} R_k^{-1} \right) R_N = \frac{R_N}{S_N} = p. \end{aligned}$$

Hence, this is a valid choice of p. With some calculation, the Hölder conjugate of p is given by

$$q = \frac{1}{1 - p^{-1}} = \frac{R_N (1 - \sum_{k=1}^{N-1} R_k^{-1})}{R_N (1 - \sum_{k=1}^{N-1} R_k^{-1}) - 1}.$$

Now, by choice of p, we have $p S_N = R_N$ and

$$q S_N = \frac{q}{1 - \sum_{k=1}^{N-1} R_k^{-1}} = \frac{R_N}{R_N (1 - \sum_{k=1}^{N-1} R_k^{-1}) - 1} = \frac{1}{1 - \sum_{k=1}^{N} R_k^{-1}} = R_{N+1}.$$

For $\delta > 0$ given, we pick $\varepsilon = \frac{(\delta p)^{1/p}}{C_\delta}$. Substituting p, q and ε into expression (1.5.3), we obtain

$$C_\delta (\beta_N \beta_{N+1})^{S_N} \leq \delta \beta_N^{R_N} + D_\delta \beta_{N+1}^{R_{N+1}}, \tag{1.5.4}$$

where $D_\delta = \frac{1}{q \varepsilon^q}$ is a constant depending on δ. This completes the proof, since substituting (1.5.4) into (1.5.2) gives us exactly the inductive step. $\square$

Chapter 2

Pseudomonotone Operators

In this chapter, we introduce the theory of pseudomonotone operators, which have applications in the methods of showing existence of solutions for nonlinear elliptic PDE. A general theme in the field is to reframe the PDE as an equation in some abstract space, on which we can apply powerful theory from functional analysis or other areas of mathematics. Examples of this framework in action include applications of functional analysis to finding critical points of *energy functionals* attached to certain PDE (as part of a wide array of *variational methods*), and applications of the Lax-Milgram Theorem to solving linear PDE. In our case, we will be using the following idea: it is often fruitful to rewrite a given PDE in operator form:

$$Au = b, \quad u \in X, \quad b \in X^* \tag{2.0.1}$$

for $A : X \to X^*$ and a Banach space X, such that solutions of the abstract operator equation (2.0.1) correspond to weak solutions of the PDE. If one can show that A is surjective, then certainly, the corresponding PDE admits a weak solution. Indeed, *monotone* operators are a class of well-studied operators which, under some additional assumptions, satisfy such a surjectivity theorem (see [14, Theorem 26.A, p. 557]). Pseudomonotone operators are a weaker form of the monotone condition, allowing for their appearance in a broader class of PDE compared to monotone operators (as we will see in the treatment of a class of anisotropic elliptic PDE in Chapter 4). The major result of this chapter (Theorem 2.3.2) shows that pseudomonotone operators satisfy an equivalent surjectivity result.

Throughout Chapter 2, we always consider X to be a **real, reflexive Banach space**. The main source that we use is [14, Chapter 27]; we also give some useful supplementary texts - see [6, Chapter 6] and [8, Section 8.5].

2.1. Definitions

We begin with the basic definitions, including pseudomonotonicity, coercivity and some modes of continuity. A comprehensive overview of the common definitions in the field may be found in [14, Definition 27.1, p. 583] and [6, Definition 6.1, p. 270].

Definition 2.1.1. Let X be a real reflexive Banach space. An operator $A : X \to X^*$

(i) is *pseudomonotone* if

$$u_n \rightharpoonup u \text{ weakly in } X \text{ and } \limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0$$
$$\implies \langle Au, u - w\rangle \le \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle \quad \text{for all } w \in X.$$

This is the *pseudomonotonicity condition*.

(ii) is *coercive* if, for $u \in X$,

$$\frac{\langle Au, u\rangle}{\|u\|_X} \to \infty \text{ as } \|u\|_X \to \infty.$$

(iii) satisfies *Condition* (M) if for some $b \in X^*$,

$$(*) \qquad u_n \rightharpoonup u \text{ weakly in } X, \quad Au_n \rightharpoonup b \text{ weakly in } X^*, \quad \limsup_{n\to\infty}\langle Au_n, u_n\rangle \le \langle b, u\rangle,$$
$$\implies Au = b.$$

If we also have $\langle Au_n, u_n\rangle \to \langle Au, u\rangle$ as $n \to \infty$, then we say A satisfies Condition $(\hat{M})$.

(iv) is *demicontinuous* if

$$u_n \to u \text{ strongly in } X \implies Au_n \rightharpoonup Au \text{ weakly in } X^*.$$

(v) is *strongly continuous* if

$$u_n \rightharpoonup u \text{ weakly in } X \implies Au_n \to Au \text{ strongly in } X^*.$$

All definitions except Condition $(\hat{M})$ can be found in [14], but we note that our Condition $(\hat{M})$ also appears in [8, Definition 8.3 (ii), p. 232] as 'sense 1 pseudomonotone operators'. We originally added in Condition $(\hat{M})$ as a strengthened Condition (M) which is satisfied naturally by pseudomonotone operators (see Proposition 2.2.3), whereas Condition $(\hat{M})$ with an additional bounded assumption implies pseudomonotonicity (see Proposition 2.2.5).

2.2. Properties of Pseudomonotone Operators

In a sense that we will see in the coming section, the pseudomonotonicity condition is "intermediate" to many of the properties given in Definition 2.1.1. We provide the proofs for the implications which we will explicitly use in the rest of the thesis. Our main source is [14, Chapter 27], with many proofs rewritten and made more explicit. First, we give sufficient conditions for the pseudomonotonicity condition.

Proposition 2.2.1. Every strongly continuous operator $\mathcal{A} : X \to X^*$ is pseudomonotone.

Proof. Suppose $\mathcal{A}$ is strongly continuous and that there exists a weakly convergent sequence $u_n \rightharpoonup u$ weakly in X. To show A is pseudomonotone, we claim that we can prove

$$\lim_{n\to\infty}\langle Au_n, u_n - w\rangle = \langle Au, u - w\rangle \quad \text{for all } w \in X \tag{2.2.1}$$

from which the pseudomonotonicity condition follows trivially. Indeed, since A is strongly continuous, we have Au_n converges strongly to Au in X^*. Fix an arbitrary $w \in X$. By the triangle inequality, we have

$$|\langle Au_n, u_n - w\rangle - \langle Au, u - w\rangle| \le \|Au_n - Au\|_{X^*}\|u_n - w\|_X + |\langle Au, u_n - u\rangle|$$

and note that this upper bound converges to 0 as $n \to \infty$, since $Au_n \to Au$ strongly in X^* and $u_n \rightharpoonup u$ weakly in X also implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in the norm of X. Since w was arbitrary, we conclude the convergence in (2.2.1) holds, so A is pseudomonotone. □

We now give a small lemma which will streamline the proof that pseudomonotonicity is closed under addition.

Lemma 2.2.1. Suppose $u_n \rightharpoonup u$ weakly in X and $A : X \to X^*$ is pseudomonotone. Then

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \geq 0.$$

Proof. Suppose by contradiction that we have such a sequence $(u_n)_{n\in\mathbb{N}}$ weakly converging to u in X such that $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle < 0$. Then by letting $w = u$ in the definition of pseudomonotonicity, we have

$$0 = \langle Au, u - u\rangle \leq \liminf_{n\to\infty}\langle Au_n, u_n - u\rangle \leq \limsup_{n\to\infty}\langle Au_n, u_n - u\rangle < 0,$$

which is a contradiction. Hence, we must have $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \geq 0$. ☐

Proposition 2.2.2 (Closure under Addition)**.** If $A, B : X \to X^*$ are pseudomonotone operators, then the sum $A + B$ is a pseudomonotone operator.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence weakly converging to u in X and suppose

$$\limsup_{n\to\infty}\langle Au_n + Bu_n, u_n - u\rangle \leq 0. \tag{2.2.2}$$

We claim that the conditions imply

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0 \quad \text{and} \quad \limsup_{n\to\infty}\langle Bu_n, u_n - u\rangle \leq 0. \tag{2.2.3}$$

Suppose otherwise. Without loss of generality, we can assume $\langle Au_n, u_n - u\rangle > 0$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that

$$\langle Au_{n_k}, u_{n_k} - u\rangle \to c \in (0, \infty] \quad \text{as } k \to \infty$$

This, together with (2.2.2), forces

$$\limsup_{k\to\infty}\langle Bu_{n_k}, u_{n_k} - u\rangle \leq -c < 0, \tag{2.2.4}$$

which is a contradiction with Lemma 2.2.1 applied to the operator B and the sequence $(u_{n_k})_{k\in\mathbb{N}}$. We conclude that (2.2.3) holds, whence it follows that both A and B satisfy the pseudomonotonicity condition on $(u_n)_{n\in\mathbb{N}}$. Hence, for all $w \in X$, we have

$$\langle Au, u - w\rangle \leq \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle \quad \text{and} \quad \langle Bu, u - w\rangle \leq \liminf_{n\to\infty}\langle Bu_n, u_n - w\rangle,$$

whence we obtain the inequalities

$$\langle Au + Bu, u - w\rangle \leq \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle + \liminf_{n\to\infty}\langle Bu_n, u_n - w\rangle \leq \liminf_{n\to\infty}\langle Au_n + Bu_n, u_n - w\rangle.$$

It follows that the operator $A + B$ is pseudomonotone. ☐

The following propositions give some necessary conditions for pseudomonotone operators.

Proposition 2.2.3. Any pseudomonotone operator $A : X \to X^*$ satisfies Condition $(\hat{M})$.

Proof. Suppose we have a sequence $(u_n)_{n\in\mathbb{N}}$ in X such that $u_n \rightharpoonup u$ weakly in X, $Au_n \rightharpoonup b$ weakly in X^* and $\limsup_{n\to\infty}\langle Au_n, u_n\rangle \leq \langle b, u\rangle$. We first show that $Au = b$. To that end, we want to prove that $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$. Indeed, since X is reflexive, by Proposition 1.2.3 (i), weak convergence and weak* convergence are equivalence, so we have $Au_n \overset{*}{\rightharpoonup} b$ weakly* in X^*. Then $\langle Au_n, u\rangle \to \langle b, u\rangle$, so by applying Proposition A.2.1, we have

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle = \limsup_{n\to\infty}\langle Au_n, u_n\rangle - \langle b, u\rangle \leq 0.$$

We conclude that A satisfies the pseudomonotonicity condition on $(u_n)_{n\in\mathbb{N}}$, which implies

$$\langle Au, u - w\rangle \leq \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle \quad \text{for all } w \in X. \tag{2.2.5}$$

Weak convergence and characterisation of the limit inferior as the minimal subsequential limit gives

$$\begin{aligned} &\langle Au, u - w\rangle \leq \langle b, u - w\rangle \\ \implies &\langle Au - b, u - w\rangle \leq 0 \quad \text{for all } w \in X. \end{aligned}$$

Let $w = u \pm \chi$ for any non-zero $\chi \in X$. Then

$$\langle Au - b, \pm\chi\rangle \leq 0 \quad \text{for all } \chi \in X. \tag{2.2.6}$$

By the linearity of the operator $Au - b$, (2.2.6) forces $\langle Au - b, \chi\rangle = 0$ for all $\chi \in X$, whence we obtain $Au = b$ in X^*. This shows that Condition (M) is satisfied. Now, we know that

$$\limsup_{n\to\infty}\langle Au_n, u_n\rangle \leq \langle b, u\rangle = \langle Au, u\rangle.$$

By picking $w = 0$ in (2.2.5), we obtain $\langle Au, u\rangle \leq \liminf_{n\to\infty}\langle Au_n, u_n\rangle$, which, along with the upper bound on $\limsup_{n\to\infty}\langle Au_n, u_n\rangle$, forces the convergence $\lim_{n\to\infty}\langle Au_n, u_n\rangle = \langle Au, u\rangle$. It follows that A satisfies Condition $(\hat{M})$, as we wanted. □

Proposition 2.2.4. Suppose $A : X \to X^*$ is a bounded pseudomonotone operator. Then A is demicontinuous.

Proof. Assume we have $u_n \to u$ strongly in X. Since convergent sequences are bounded and A is a bounded operator, the sequence $(Au_n)_{n\in\mathbb{N}}$ is also bounded in X^*. Since X is reflexive, Proposition 1.2.1 (iv) implies the existence of a subsequence $(u_{n'})$ such that $Au_{n'} \rightharpoonup b$ weakly in X^* for some $b \in X^*$. Since A is a bounded operator, we have

$$\begin{aligned} |\langle Au_{n'}, u_{n'}\rangle - \langle b, u\rangle| &\leq |\langle Au_{n'}, u_{n'}\rangle - \langle Au_{n'}, u\rangle| + |\langle Au_{n'}, u\rangle - \langle b, u\rangle| \\ &\leq \|Au_{n'}\|_{X^*}\|u_{n'} - u\| + |\langle Au_{n'} - b, u\rangle| \to 0 \quad \text{as } n' \to \infty. \end{aligned}$$

Hence, $\langle Au_{n'}, u_{n'}\rangle \to \langle b, u\rangle$ as $n' \to \infty$. From Proposition 2.2.3, we know that A satisfies Condition (M), which implies that $Au = b$. Since the above argument can be repeated for any weakly convergent subsequence of $(Au_n)_{n\in\mathbb{N}}$ and weak limits are unique, we see that every weakly convergent subsequence of Au_n has the same limit. We conclude from Proposition 1.2.3 that $Au_n \rightharpoonup b = Au$ weakly in X^* as $n \to \infty$. This shows exactly that A is demicontinuous, as we wanted. □

In practical use, such as PDE research, the direct definition for pseudomonotone operators is unwieldy and difficult to apply. To obtain a usable sufficient condition, we recall that pseudomonotone operators satisfy Condition $(\hat{M})$. As it turns out, bounded operators satisfying Condition $(\hat{M})$ is enough to imply pseudomonotonicity. We were motivated to prove this result by the proof of pseudomonotonicity presented in [1, Appendix, p. 185]; it does not appear in [14].

Proposition 2.2.5 (Sufficient Condition for Pseudomonotonicity)**.** Let $A : X \to X^*$ be a bounded operator satisfying Condition $(\hat{M})$. Then A is pseudomonotone.

Proof. Let $A : X \to X^*$ be a bounded operator satisfying Condition $(\hat{M})$. Suppose we have $v_n \rightharpoonup v$ weakly in X and $\limsup_{n\to\infty}\langle Av_n, v_n - v\rangle \leq 0$. Fix any $w \in X$. Then there exists a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ on which

$$\lim_{k\to\infty}\langle Av_{n_k}, v_{n_k} - w\rangle = \liminf_{n\to\infty}\langle Av_n, v_n - w\rangle.$$

Since $(v_{n_k})_{k\in\mathbb{N}}$ is weakly convergent and hence bounded, the sequence $(Av_{n_k})_{k\in\mathbb{N}}$ is bounded in X^* (which is reflexive by Proposition 1.2.2 (ii)) and hence admits a subsequence $(Av_{n'_k})_{k\in\mathbb{N}}$ weakly converging to some $b \in X^*$. Then

$$\limsup_{k\to\infty}\langle Av_{n'_k}, v_{n'_k} - v\rangle = \limsup_{k\to\infty}\langle Av_{n'_k}, v_{n'_k}\rangle - \langle b, v\rangle \leq \limsup_{k\to\infty}\langle Av_n, v_n - v\rangle \leq 0$$

so that $\limsup_{k\to\infty}\langle Av_{n'_k}, v_{n'_k}\rangle \leq \langle b, v\rangle$. Applying Condition $(\hat{M})$ to the sequence $(v_{n'_k})_{k\in\mathbb{N}}$, we conclude that $Av = b$ and $\langle Av_{n'_k}, v_{n'_k}\rangle \to \langle Av, v\rangle$ as $n \to \infty$. Hence, we have the equalities

$$\begin{aligned}\liminf_{n\to\infty}\langle Av_n, v_n - w\rangle &= \lim_{k\to\infty}\langle Av_{n_k}, v_{n_k} - w\rangle \\ &= \lim_{k\to\infty}\langle Av_{n'_k}, v_{n'_k} - w\rangle \\ &= \langle Av, v - w\rangle.\end{aligned}$$

Since the choice of $w \in X$ was arbitrary, we conclude that A is pseudomonotone, as required. □

2.3. Surjectivity of Pseudomonotone Operators

We now state a useful fixed point theorem:

Theorem 2.3.1 (Brouwer, (1912))**.** Let M be a non-empty, convex and compact subset of $\mathbb{R}^N$. Suppose $f : \mathbb{R}^N \to M$ is a continuous mapping. Then f admits a fixed point in M.

The Brouwer Fixed Point Theorem is given as [12, Proposition 2.6, p. 51]. As a topological property of continuous maps, a nice proof may also be given with some knowledge of algebraic topology by considering the homology groups $H_n(M)$. The Brouwer Fixed Point Theorem allows us to prove an important existence principle concerning systems of equations in $\mathbb{R}^N$ with a "coercive" boundary condition.

Corollary 2.3.1 (Existence Principle)**.** Let $R > 0$ be a fixed constant and $\|\cdot\|$ be any norm on $\mathbb{R}^N$. Consider continuous functions $g_k : \overline{B}(0, R) \to \mathbb{R}$ for $k = 1, 2, \dots, N$ and suppose that the following boundary condition is satisfied:

$$\sum_{i=1}^{N} g_i(x)x_i \geq 0 \quad \text{for all } x \text{ with } \|x\| = R$$

where $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$. Then the system of equations

$$g_1(x) = 0, \quad g_2(x) = 0, \quad \dots \quad g_N(x) = 0$$

has a solution $\xi \in \mathbb{R}^N$ with $\|\xi\| \leq R$.

Proof. Write $g = (g_1, g_2, \ldots, g_N)$. Suppose by contradiction that $g(x) \neq 0$ for all $x \in \overline{B}(0, R)$. Then the function $f : \overline{B}(0, R) \to \overline{B}(0, R)$ given by

$$f(x) = -\frac{Rg(x)}{\|g(x)\|}$$

is well-defined and is continuous by the continuity of each g_k. Furthermore, the set $\overline{B}(0, R)$ is non-empty, compact and convex in $\mathbb{R}^N$, so applying the Brouwer Fixed Point Theorem, f has a fixed point ξ in $\overline{B}(0, R)$. Then

$$\xi = \frac{-Rg(\xi)}{\|g(\xi)\|},$$

so by taking norms, we see that $\|\xi\| = R$. We also see that

$$\sum_{k=1}^{N} g_k(\xi)\xi_k = -R \sum_{k=1}^{N} \frac{g_k^2(\xi)}{\|g(\xi)\|} < 0,$$

a contradiction with the coercivity condition. Hence, g has a solution in $\overline{B}(0, R) \subseteq \mathbb{R}^N$, as required. □

We are now in a position to prove the main theorem of this chapter. This is presented as [14, Theorem 27.A, p. 589] - however, in the text, the proof is given as a corollary of a number of previous results; in our proof, we explicitly provide the *Galerkin method* that is used to prove the result, illuminating the process.

Theorem 2.3.2 (Brézis (1968))**.** Suppose X is a real, separable and reflexive Banach space with basis $\{w_k\}_{k\in\mathbb{N}}$ and that $A : X \to X^*$ is a pseudomonotone, bounded and coercive operator. Then for every $b \in X^*$, the equation

$$Au = b, \quad u \in X \tag{2.3.1}$$

has a solution in X.

Proof. The driving force behind the proof of Theorem 2.3.2 is the *Galerkin method*, where a problem set in an infinite dimensional space can be resolved by solving a sequence of problems in finite dimensional subspaces then taking the limit. For a more comprehensive overview, we refer to [8, Chapter 4].

Step 1: Define $X_n := \operatorname{span}\{w_k \mid 1 \leq k \leq n\}$. We note that $\cup_{n\in\mathbb{N}} X_n = X$. The idea is to obtain a sequence of solutions using Corollary 2.3.1 to the system of equations defined by

$$\langle Au_n, w_k\rangle = \langle b, w_k\rangle, \quad u_n \in X_n, \quad k = 1, 2, \ldots, n \tag{2.3.2}$$

for any $n \in \mathbb{N}$, show that this sequence is bounded in the reflexive space X, then extract a weakly convergent subsequence which converges to a solution of (2.3.1). To that end, set

$$g(u) = \langle Au - b, u\rangle, \quad g_k(u) = \langle Au - b, w_k\rangle. \tag{2.3.3}$$

From (2.3.2), we then have a system of equations (the *Galerkin equations*) indexed by k:

$$g_k(u_n) = 0, \quad \text{for } u_n \in X_n, \quad k = 1, 2, \ldots, n \tag{2.3.4}$$

Since the set $\{w_k\}_{1\leq k\leq n}$ forms a basis of X_n, we can write $u_n = \sum_{k=1}^n c_{kn} w_k$ for constants $c_{kn} \in \mathbb{R}$. Hence, we can view (2.3.4) as a system of nonlinear equations with respect to $(c_{1n}, c_{2n}, \dots, c_{nn})$. We now claim the following: the functions

$$g_k : \mathbb{R}^n \to \mathbb{R}, \quad g_k(c_{11}, \dots, c_{1n}) = \langle Au - b, w_k\rangle$$

are continuous for $1 \leq k \leq n$. Indeed, suppose we have some $x^i \in R^n$ with

$$x^i = (x_1^i, x_2^i, \dots, x_n^i) \to (x_1, x_2, \dots, x_n) = x \quad \text{as } i \to \infty$$

and write $y^i = \sum_{k=1}^n x_k^i w_k$. Then $y^i \to y$ strongly in X as $i \to \infty$, where $y = \sum_{k=1}^n x_k w_k$, and $(Ay^i)_{i\in\mathbb{N}}$ is a bounded sequence in X^*. Since X^* is reflexive, up to subsequence we have $Ay^i \rightharpoonup \varphi$ weakly in X^* for some $\varphi \in X^*$. On this subsequence, the boundedness of $(Ay^i)_{i\in\mathbb{N}}$ in X^* gives us

$$\langle Ay^i, y^i - y\rangle \to 0 \quad \text{as } i \to \infty$$

which implies that

$$\lim_{i\to\infty}\langle Ay^i, y^i\rangle \to \langle \varphi, y\rangle \quad \text{as } i \to \infty.$$

Since A is pseudomonotone, A satisfies Condition $(\hat{M})$ by Proposition 2.2.3. Applying this condition to the subsequence $(y^i)_{i\in\mathbb{N}}$, we conclude that $Ay = \varphi$. Hence,

$$g_k(x^i) = \langle Ay^i - b, w_k\rangle \to \langle Ay - b, w_k\rangle = g_k(x)$$

and so $g_k : \mathbb{R}^n \to \mathbb{R}$ is continuous for all $1 \leq k \leq n$, proving our claim.
Now, from the coercivity condition on the operator A, we have

$$\frac{g(u)}{\|u\|} = \frac{\langle Au - b, u\rangle}{\|u\|} \to \infty \quad \text{as } \|u\| \to \infty.$$

Hence, there exists $R > 0$ such that

$$g(u) > 0 \quad \text{for all } \|u\| \geq R. \tag{2.3.5}$$

Furthermore, it follows from (2.3.5) that for any $u_n \in X_n$ with $\|u_n\| = R$, we have

$$g(u_n) = \langle Au_n - b, u_n\rangle = \sum_{k=1}^n c_{kn}\langle Au_n - b, w_k\rangle = \sum_{k=1}^n c_{kn} g_k(u_n) > 0.$$

We can now apply Corollary 2.3.1 to see that there exists a solution $u_n \in X_n$ to the Galerkin equation (2.3.4) for every $n \in \mathbb{N}$.

Step 2: A priori estimates. We give upper bound estimates to the set of solutions of (2.3.4), so that we can apply Proposition 1.2.1 (iv) to this sequence. If $u_n \in X_n$ is such that $\|u_n\| \geq R$, then by (2.3.5) we have

$$\sum_{k=1}^n g_k(u_n) c_{kn} = g(u_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Hence, it follows from (2.3.5) that any solution $u_n \in X_n$ of (2.3.4) satisfies $\|u_n\| \leq R$ for all $n \in \mathbb{N}$. Furthermore, if $u \in X$ is a solution of (2.3.1), then $g(u) = 0$ and the same reasoning gives $\|u\| \leq R$.

Step 3: From Step 1, we have generated a sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in X_n$ being solutions of the Galerkin equation (2.3.4) satisfying $\|u_n\| \leq R$. Now, $(u_n)_{n\in\mathbb{N}}$ satisfies (2.3.4) implies that for all $n \in \mathbb{N}$,

$$g_k(u_n) = \langle Au_n - b, w_k\rangle = 0 \quad \text{for all } k \in \mathbb{N}. \tag{2.3.6}$$

Since $\{w_j\}_{j\in\mathbb{N}}$ form a countable basis of X, we conclude that for all $v \in \operatorname{span}\{w_j \mid j \in \mathbb{N}\}$,

$$g_k(u_n) = \langle Au_n - b, v\rangle \to 0 \quad \text{as } n \to \infty.$$

By assumption, A is a bounded operator, so $(Au_n)_{n\in\mathbb{N}}$ is a bounded sequence in X^*. Hence, Proposition 1.2.3 (ii) implies that $(Au_n - b) \rightharpoonup 0$ weakly in X^*, i.e.

$$Au_n \rightharpoonup b \text{ weakly in X}^* \quad \text{as } n \to \infty. \tag{2.3.7}$$

Since $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in a reflexive space, there exists a subsequence $(u_{n'})$ of $(u_n)_{n\in\mathbb{N}}$ converging weakly to some $u \in X$. Now, A being pseudomonotone implies that A satisfies Condition (M) by Proposition 2.2.3. Recalling that $(u_n)_{n\in\mathbb{N}}$ satisfies (2.3.6), we expand using linearity and the fact that $\{w_k \in X\}_{k\in\mathbb{N}}$ form a basis of X to get

$$\begin{aligned}
&\langle Au_{n'} - b, u_{n'}\rangle = \sum_{k=1}^{n} c_{kn'}\langle Au_{n'} - b, w_k\rangle = 0\\
&\implies \langle Au_{n'}, u_{n'}\rangle = \langle b, u_{n'}\rangle \to \langle b, u\rangle \quad \text{by weak convergence of } u_{n'} \text{ in X as } n \to \infty.
\end{aligned}$$

Since the hypotheses of Condition $(\hat{M})$ are now trivially satisfied on $(u_{n'})_{n\in\mathbb{N}}$, we conclude that $Au = b$, as required. $\square$

CHAPTER 3

The Anisotropic Sobolev Space $W_0^{1,\vec{p}}(\Omega)$

3.1. Construction of $W_0^{1,\vec{p}}(\Omega)$

Unless otherwise stated, throughout this chapter, we write Ω to be an arbitrary open set in $\mathbb{R}^N$ for $N \geq 2$ and $\vec{p} = (p_1, p_2, \dots, p_N)$ to be the anisotropy exponent with $1 < p_j < \infty$ for all $1 \leq j \leq N$. We introduce the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$, starting with its definition, before proving the anisotropic Sobolev embedding results and other useful analytic properties of $W_0^{1,\vec{p}}(\Omega)$. In a sense, $W_0^{1,\vec{p}}(\Omega)$ is the natural function space where solutions of anisotropic PDE with homogeneous Dirichlet boundary conditions live, as we will see by the construction of $W_0^{1,\vec{p}}(\Omega)$. A direct benefit of working with $W_0^{1,\vec{p}}(\Omega)$ is the added flexibility allowing the partial derivatives $\partial_j u$ to be in different L^p spaces depending on the derivative, compared to the usual isotropic $W_0^{1,p}(\Omega)$ spaces.

To begin, we need a way to generalise pointwise differentiation to the broader class of locally integrable functions. We will always consider real valued functions on an open subset Ω of $\mathbb{R}^N$, and we will write $W^{-1,\vec{p}'}(\Omega)$ to denote the dual of $W_0^{1,\vec{p}}(\Omega)$, where p' is the Hölder conjugate of p.

Definition 3.1.1 (Weak Derivative)**.** Let $u \in L^1_{\text{loc}}(\Omega)$. Denote by $x_1, x_2, \dots, x_N$ the canonical unit vectors of $\mathbb{R}^N$. For $1 \leq j \leq N$, the *weak partial derivative in the direction* x_j of u is a function $v \in L^1_{\text{loc}}(\Omega)$ satisfying

$$\int_\Omega u(\partial_j \varphi)\,dx = -\int_\Omega v\varphi\,dx \quad \text{for all } \varphi \in C_c^1(\Omega).$$

If no such v exists, then u is not weakly differentiable with respect to x_j. If u is differentiable in the classical sense, then the *integration by parts* formula (Theorem A.2.5) shows that u is weakly differentiable, with the weak derivative coinciding with the classical derivative.

If $u \in L^1_{\text{loc}}(\Omega)$ has a weak partial derivative v in the direction x_j, then we would like to write v as $\partial_j u$. In fact, this is permissible, since weak derivatives are uniquely determined up to sets of measure 0 on Ω:

Proposition 3.1.1 (Uniqueness of Weak Derivatives)**.** If $u \in L^1_{\text{loc}}(\Omega)$ has two weak partial derivatives $v_1, v_2 \in L^1_{\text{loc}}(\Omega)$ in the direction x_j, then $v_1 = v_2$ a.e. in Ω.

Proof. If two such functions $v_1, v_2 \in L^1_{\text{loc}}(\Omega)$ exist, by definition they satisfy

$$-\int_\Omega u\partial_j\varphi\,dx = \int_\Omega v_1\varphi\,dx = \int_\Omega v_2\varphi\,dx \quad \text{for all } \varphi \in C_c^1(\Omega).$$

Then we have

$$\int_\Omega (v_1 - v_2)\varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^1(\Omega).$$

Now, [4, Corollary 4.24, p. 110] implies then that $v_1 = v_2$ a.e. in Ω, as required. □

Definition 3.1.2. Denote by p the harmonic mean of $p_1, p_2, \ldots, p_N$ with $p = \frac{N}{\sum_{j=1}^N 1/p_j}$. The Sobolev space $W^{1,\vec{p}}(\Omega)$ is defined as the space of functions $u \in L^p(\Omega)$ with weak partial derivatives $\partial_j u$ in $L^{p_j}(\Omega)$ for all $1 \le j \le N$. We can equip the Sobolev space $W^{1,\vec{p}}(\Omega)$ with the norm

$$\|u\|_{W^{1,\vec{p}}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}. \tag{3.1.1}$$

Definition 3.1.3. The anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$ is defined to be the closure of $C_c^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,\vec{p}}(\Omega)}$.

In Chapter 4, we will be working with an anisotropic PDE with homogeneous boundary conditions. Hence, for the remainder of this section, we will be focusing on properties of $W_0^{1,\vec{p}}(\Omega)$.

Remark 3.1.1. We first remark that similar definitions for $W^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$ may be found in [1], [3] and [5] - however, the construction of $W_0^{1,\vec{p}}(\Omega)$ is usually summarised by taking the "completion of $C_c^\infty(\Omega)$" with respects to a certain norm (as defined in Theorem 3.3.2). As we shall see, this is equivalent to the way we construct $W_0^{1,\vec{p}}(\Omega)$ in this chapter if Ω is bounded, but this can only be made rigorous after proving the Sobolev embeddings (see Corollary 3.3.2).

As a second remark, we could equivalently have defined $W_0^{1,\vec{p}}(\Omega)$ as the closure of the (larger) space $C_c^1(\Omega)$ in $W^{1,\vec{p}}(\Omega)$. Indeed, suppose there exists a sequence $(u_n)_{n\in\mathbb{N}} \in C_c^1(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$ and $\partial_j u_n \to \partial_j u$ in $L^{p_j}(\Omega)$ as $n \to \infty$. We show that $u \in W_0^{1,\vec{p}}(\Omega)$. Fix $\varepsilon > 0$ arbitrary. Take $(\rho_k)_{k\in\mathbb{N}}$ a sequence of mollifiers as in Proposition A.2.5 and fix $\eta \in \mathbb{N}$ so that for all $n \ge \eta$,

$$\|u_n - u\|_{L^p(\Omega)} < \frac{\varepsilon}{2} \quad \text{and} \quad \|\partial_j u_n - \partial_j u\|_{L^{p_j}(\Omega)} < \frac{\varepsilon}{2} \quad \text{for all } 1 \le j \le N.$$

For a function f on Ω, denote by $\overline{f}$ the extension of f to $\mathbb{R}^N$ by 0 outside of Ω. By properties of mollifiers and convolutions, we know that $\rho_k * \overline{u_\eta} \to \overline{u_\eta}$ in $L^p(\mathbb{R}^N)$ and $\rho_k * \overline{\partial_j u_\eta} \to \overline{\partial_j u_\eta}$ in $L^{p_j}(\mathbb{R}^N)$ as $k \to \infty$ for all $1 \le j \le N$, and that each of these convolutions are in $C_c^\infty(\mathbb{R}^N)$. Hence, there exists k_ε such that

$$\|\rho_{k_\varepsilon} * \overline{u_\eta} - \overline{u_\eta}\|_{L^p(\mathbb{R}^N)} < \frac{\varepsilon}{2} \quad \text{and} \quad \|\rho_{k_\varepsilon} * \overline{\partial_j u_\eta} - \overline{\partial_j u_\eta}\|_{L^{p_j}(\mathbb{R}^N)} < \frac{\varepsilon}{2} \text{ for all } 1 \le j \le N.$$

We still need to restrict $\rho_{k_\varepsilon} * \overline{u_\eta}$ and $\rho_{k_\varepsilon} * \overline{\partial_j u_\eta}$ to have compact support on Ω, but since we have

$$\operatorname{supp}(f * g) \subseteq \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}$$

for all $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$ for $1 \le p \le \infty$ (see Proposition A.2.4), we can take k_ε large enough so that

$$\operatorname{supp}(\rho_{k_\varepsilon} * \overline{u_\eta}) \subseteq \overline{B_{1/k_\varepsilon}(0) + \operatorname{supp}(u_\eta)} \subsetneq \Omega,$$

$$\operatorname{supp}(\rho_{k_\varepsilon} * \overline{\partial_j u_\eta}) \subseteq \overline{B_{1/k_\varepsilon}(0) + \operatorname{supp}(\partial_j u_\eta)} \subsetneq \Omega \text{ for all } 1 \le j \le N.$$

Then by the triangle inequality, we have

$$\|\rho_{k_\varepsilon} * \overline{u_\eta} - u\|_{L^p(\Omega)} \leq \|\rho_{k_\varepsilon} * \overline{u_\eta} - u_\eta\|_{L^p(\Omega)} + \|u_\eta - u\|_{L^p(\Omega)} < \varepsilon.$$

Hence, we can arbitrarily approximate u with a sequence of $C_c^\infty(\Omega)$ functions, so $u \in W_0^{1,\vec{p}}(\Omega)$.

3.2. Properties of $W_0^{1,\vec{p}}(\Omega)$.

We now give useful properties of $W_0^{1,\vec{p}}(\Omega)$. No source (that we found) has proved these results in detail for $W_0^{1,\vec{p}}(\Omega)$. For our purposes, we have adapted proofs of similar properties of the isotropic Sobolev spaces $W^{1,p}(\Omega)$ - see in particular, [4, Proposition 9.5, p. 270] and [7, Lemma 1.25, p. 21] for the chain rule, and [7, Section 1.11] for the truncation property.

Proposition 3.2.1. Let Ω be an open set in $\mathbb{R}^N$ and $\vec{p} = (p_j)_{1\leq j\leq N}$ with $1 < p_j < \infty$. The Sobolev space $W_0^{1,\vec{p}}(\Omega)$ is a real vector space that is complete, reflexive and separable.

Proof. The space $W_0^{1,\vec{p}}(\Omega)$ is clearly a real vector space. For the other properties, since $W_0^{1,\vec{p}}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$, it suffices to show that $W^{1,\vec{p}}(\Omega)$ satisfies completeness, reflexivity and separability, then use that $W_0^{1,\vec{p}}(\Omega)$ is a closed metric subspace of $W^{1,\vec{p}}(\Omega)$.

Banach: Suppose $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,\vec{p}}(\Omega)$. This implies that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$ and $(\partial_j u_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $L^{p_j}(\Omega)$ for each $1 \leq j \leq N$, and hence there exists $u \in L^p(\Omega)$ with $u_n \to u$ in $L^p(\Omega)$ and $f_j \in L^{p_j}(\Omega)$ with $\partial_j u_n \to f_j$ strongly in $L^{p_j}(\Omega)$. It remains to show that $f_j = \partial_j u$. Indeed, by definition of the weak derivative, for every $\varphi \in C_c^1(\Omega)$,

$$\int_\Omega f_j\varphi\, dx \leftarrow \int_\Omega (\partial_j u_n)\varphi\, dx = -\int_\Omega u_n(\partial_j\varphi)\, dx \to -\int_\Omega u(\partial_j\varphi)\, dx \quad \text{as } n\to\infty$$

where the left convergence holds by considering $\varphi \in L^{p_j'}(\Omega)$, the right convergence by considering $\partial_j\varphi \in L^\infty(\Omega)$, and applying Hölder's inequality in both cases. Since this holds for every $\varphi \in C_c^1(\Omega)$, we have $f_j = \partial_j u$. It follows that $u \in L^p(\Omega)$ with weak derivatives $\partial_j u \in L^{p_j}(\Omega)$ for all $1 \leq j \leq N$, so $u \in W^{1,\vec{p}}(\Omega)$, and we have $u_n \to u$ strongly in $W^{1,\vec{p}}(\Omega)$. Hence, $W^{1,\vec{p}}(\Omega)$ is complete.

Reflexive: Let $E = L^p(\Omega) \times \Pi_{j=1}^N (L^{p_j}(\Omega))$ be equipped with the product norm $\|\cdot\|_E = \|\cdot\|_{L^p(\Omega)} + \sum_{j=1}^N \|\cdot\|_{L^{p_j}(\Omega)}$ and let $T : W^{1,\vec{p}}(\Omega) \to E$ be the mapping given by $T : u \mapsto (u, \partial_1 u, \partial_2 u, \dots, \partial_N u)$. By definition of the $W^{1,\vec{p}}(\Omega)$ norm and $\|\cdot\|_E$, the map T is norm-preserving, and hence is an isometry into its image $T(W^{1,\vec{p}}(\Omega))$.

We show that $T(W^{1,\vec{p}}(\Omega))$ is a closed subspace of E. Suppose $(Tu_n)_{n\in\mathbb{N}}$ is a convergent sequence in $L^p(\Omega) \times \Pi_{j=1}^N (L^{p_j}(\Omega))$. This forces the convergence $u_n \to u$ in $L^p(\Omega)$ and $\partial_j u_n \to \partial_j u$ in $L^{p_j}(\Omega)$ for all $1 \leq j \leq N$ for some $u \in W^{1,\vec{p}}(\Omega)$, since $W^{1,\vec{p}}(\Omega)$ is complete. Then we have $Tu_n \to Tu$ in $\|\cdot\|_E$, so $T(W^{1,\vec{p}}(\Omega))$ is closed in $L^p(\Omega) \times \Pi_{j=1}^N (L^{p_j}(\Omega))$. By Proposition 1.2.2 (i), $T(W^{1,\vec{p}}(\Omega))$ is reflexive. Since all isometries are homeomorphisms into their image and hence preserve reflexivity, we conclude $W^{1,\vec{p}}(\Omega)$ is reflexive.

Separable: Recalling that the (product of) L^p spaces are separable for $1 \le p < \infty$ (see [4, Theorem 4.10, p. 95] and that subspaces of separable metric spaces are separable, the isometry T also shows that $W_0^{1,\vec{p}}(\Omega)$ is separable.

Since $W_0^{1,\vec{p}}(\Omega)$ is a closed metric subspace of $W^{1,\vec{p}}(\Omega)$, it satisfies all three properties from $W^{1,\vec{p}}(\Omega)$, finishing the proof. □

3.2.1. The Chain Rule and the Truncation Property.

Theorem 3.2.1 (Chain Rule)**.** Suppose $G \in C^1(\mathbb{R})$ with $G(0) = 0$ and $|G'(s)| \le M$ for all $s \in \mathbb{R}$ and some $M \ge 0$. Then for all $u \in W_0^{1,\vec{p}}(\Omega)$, we have

$$G \circ u \in W_0^{1,\vec{p}}(\Omega) \quad \text{and} \quad \partial_j(G \circ u) = (G' \circ u)\partial_j u \quad \text{for all } 1 \le j \le N$$

Proof. Let $u \in W_0^{1,\vec{p}}(\Omega)$. By definition of $W_0^{1,\vec{p}}(\Omega)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of elements in $C_c^\infty(\Omega)$ with $u_n \to u$ in $W^{1,\vec{p}}(\Omega)$. Then $G \circ u_n \in C_c^1(\Omega)$, and integration by parts and the smooth chain rule gives us

$$\int_\Omega (G \circ u_n)\partial_j \varphi \, dx = -\int_\Omega (G' \circ u_n)(\partial_j u_n)\varphi \, dx \quad \text{for all } \varphi \in C_c^1(\Omega). \tag{3.2.1}$$

Fix $\varphi \in C_c^1(\Omega)$ and j with $1 \le j \le N$.

Claim 1: We have $(G \circ u_n)\partial_j \varphi \to (G \circ u)\partial_j \varphi$ in $L^1(\Omega)$ as $n \to \infty$.

Note that for all $t, t_0 \in \mathbb{R}$, the Mean Value Theorem gives us c between t and t_0 such that

$$\left| \frac{G(t) - G(t_0)}{t - t_0} \right| \le |G'(c)| \le M. \tag{3.2.2}$$

In particular, by the assumption $G(0) = 0$ and letting $t_0 = 0$ and , we obtain $|G \circ u| \le M|u|$ a.e. on Ω, and so

$$\int_\Omega |(G \circ u)\partial_j \varphi| \, dx \le M \int_\Omega |u||\partial_j \varphi| \, dx \le M \|u\|_{L^p(\Omega)} \|\partial_j \varphi\|_{L^{p'}(\Omega)}$$

where we have used Hölder's inequality along with $\partial_j \varphi \in L^{p'}(\Omega)$ by compactness of the support of $\partial_j \varphi$. This shows that $(G \circ u)\partial_j \varphi \in L^1(\Omega)$. To see the convergence of Claim 1, we first use (3.2.2) to obtain the inequality

$$|G \circ u_n - G \circ u| \le M|u_n - u| \quad \text{a.e. on } \Omega. \tag{3.2.3}$$

It follows that

$$\begin{aligned} \int_\Omega |(G \circ u_n)\partial_j \varphi - (G \circ u)\partial_j \varphi| \, dx &= \int_\Omega |G \circ u_n - G \circ u||\partial_j \varphi| \, dx \\ &\le M \int_\Omega |u_n - u||\partial_j \varphi| \, dx \\ &\le M \|u_n - u\|_{L^p(\Omega)} \|\partial_j \varphi\|_{L^{p'}(\Omega)} \to 0 \text{ as } n \to \infty \end{aligned}$$

where we have used Hölder's inequality with the right-hand side vanishing in the limit since $u_n \to u$ in the $W^{1,\vec{p}}(\Omega)$ norm, and $\partial_j \varphi$ has compact support in Ω. This proves Claim 1.

Claim 2: We have $(G'\circ u_n)(\partial_j u_n)\varphi \to (G'\circ u)(\partial_j u)\varphi$ in $L^1(\Omega)$ as $n \to \infty$.

Using the triangle inequality, we can bound the term $\int_\Omega |(G'\circ u_n)(\partial_j u_n)\varphi - (G'\circ u)(\partial_j u)\varphi|\,dx$ from above by

$$\underbrace{\|\varphi(G'\circ u_n)\|_{L^{p_j'}(\Omega)}\|\partial_j u_n - \partial_j u\|_{L^{p_j}(\Omega)}}_{(A)} + \underbrace{\int_\Omega |\varphi\partial_j u||G'\circ u_n - G'\circ u|\,dx}_{(B)}. \tag{3.2.4}$$

Now, expression (A) vanishes as $n \to \infty$, which we can see by the inequality

$$\|\varphi(G'\circ u_n)\|_{L^{p_j'}(\Omega)}\|\partial_j u_n - \partial_j u\|_{L^{p_j}(\Omega)} \leq M\|\varphi\|_{L^{p_j'}(\Omega)}\|\partial_j u_n - \partial_j u\|_{L^{p_j}(\Omega)} \to 0 \text{ as } n \to \infty. \tag{3.2.5}$$

The limit in (3.2.5) holds since $\partial_j u_n \to \partial_j u$ in $L^{p_j}(\Omega)$ by the convergence $u_n \to u$ in $W^{1,\vec{p}}(\Omega)$. For the convergence of expression (B), we note the following. Since $u_n \to u$ in $L^p(\Omega)$, by Proposition 1.3.1, every subsequence $(u_{n_k})_{k\in\mathbb{N}}$ has a further subsequence $(u_{n_k'})_{k\in\mathbb{N}}$ converging to u pointwise a.e., which implies $G'\circ u_{n_k'} \to G'\circ u$ by the continuity of G'. Furthermore, since G' is bounded, we have

$$|\varphi\partial_j u||G'\circ u_{n_k'} - G'\circ u| \leq 2M|\varphi||\partial_j u|$$

where $|\varphi||\partial_j u| \in L^1(\Omega)$ since $\varphi \in C_c^1(\Omega) \subseteq L^{p_j'}(\Omega)$ and $\partial_j u \in L^{p_j}(\Omega)$. Hence, by the Dominated Convergence Theorem,

$$\int_\Omega |\varphi\partial_j u||G'\circ u_{n_k'} - G'\circ u|\,dx \to 0 \text{ as } k \to \infty \tag{3.2.6}$$

which forces the convergence

$$\int_\Omega |\varphi\partial_j u||G'\circ u_n - G'\circ u|\,dx \to 0 \text{ as } n \to \infty \tag{3.2.7}$$

by Proposition 1.2.1 (ii). Combining (3.2.5) and (3.2.7) and taking the limit as $n \to \infty$ in (3.2.4) completes the proof of Claim 2. Using the results of both claims and taking the limit as $n \to \infty$ in (3.2.1), we obtain

$$\int_\Omega (G\circ u)\partial_j\varphi\,dx = -\int_\Omega (G'\circ u)(\partial_j u)\varphi\,dx$$

to conclude that the weak derivative $\partial_j(G\circ u)$ is given by $(G'\circ u)\partial_j u$.

It remains to show that there is a sequence of C_c^1 functions converging to $G\circ u$ in the $W^{1,\vec{p}}$ norm. Indeed, we show that $G\circ u_n \to G\circ u$ strongly in $L^p(\Omega)$ and $\partial_j(G\circ u_n) \to \partial_j(G\circ u)$ strongly in $L^{p_j}(\Omega)$ as $n \to \infty$. Using (3.2.3),

$$\left(\int_\Omega |G\circ u_n - G\circ u|^p\,dx\right)^{1/p} \leq M\left(\int_\Omega |u_n - u|^p\,dx\right)^{1/p} \to 0 \text{ as } n \to \infty,$$

which implies $G\circ u_n \to G\circ u$ in $L^p(\Omega)$. For the convergence of derivatives, we use a similar argument to the convergence of (3.2.7). Indeed, for each $1 \leq j \leq N$,

$$\|\partial_j(G\circ u_n) - \partial_j(G\circ u)\|_{L^{p_j}(\Omega)} \leq \|(G'\circ u_n)\partial_j u - (G'\circ u_n)\partial_j u_n\|_{L^{p_j}(\Omega)} + \|(G'\circ u_n)\partial_j u - (G'\circ u)\partial_j u\|_{L^{p_j}(\Omega)}$$

with

$$\|G'\circ u_n\partial_j u - G'\circ u_n\partial_j u_n\|_{L^{p_j}(\Omega)} \leq M\|\partial_j u_n - \partial_j u\|_{L^{p_j}(\Omega)} \to 0 \text{ as } n \to \infty.$$

For the term $\|(G'\circ u_n)\partial_j u-(G'\circ u)\partial_j u\|_{L^{p_j}(\Omega)}$, we note that out of every subsequence $(u_{n_k})_{k\in\mathbb{N}}$ we can extract a further subsequence such that $(G'\circ u_{n'_k})\partial_j u \to (G'\circ u)\partial_j u$ a.e. on Ω. Along with the pointwise bound

$$|(G'\circ u_n - G'\circ u)\partial_j u|^{p_j} \le (2M)^{p_j}|\partial_j u|^{p_j} \in L^1(\Omega),$$

we conclude by Dominated Convergence Theorem that on this subsequence, we have the convergence

$$\|(G'\circ u_{n'_k})\partial_j u-(G'\circ u)\partial_j u\|_{L^{p_j}(\Omega)} \to 0 \text{ as } k\to\infty. \tag{3.2.8}$$

Hence, we conclude $(G'\circ u_n)\partial_j u \to (G'\circ u)\partial_j u$ in $L^{p_j}(\Omega)$. Combining (3.2.8) with the convergence of $G\circ u_n$ to $G\circ u$ in $L^p(\Omega)$, we see that $G\circ u_n \to G\circ u$ in $W_0^{1,\vec{p}}(\Omega)$. This shows that $G\circ u\in W_0^{1,\vec{p}}(\Omega)$. □

Remark 3.2.1. We remark that the chain rule also holds if G' is not bounded, as long as we also require $u\in L^\infty(\Omega)$. We omit the details of the proof, although we will use this form of the chain rule in Chapter 4.

With the anisotropic chain rule established, we now seek to prove the truncation property of $W_0^{1,\vec{p}}(\Omega)$, in the sense that if $u\in W_0^{1,\vec{p}}(\Omega)$, the truncations of u at level k given by

$$T_k(u)=\begin{cases} k\frac{u}{|u|} & \text{if } |u|\ge k,\\ u & \text{if } |u|<k,\end{cases}$$

are in $W_0^{1,\vec{p}}(\Omega)$. Truncated functions play an important role when we attempt to obtain convergence results for solutions of PDE in Chapter 4.

Lemma 3.2.1. If u and v are functions in $W_0^{1,\vec{p}}(\Omega)$, then the following functions are also in $W_0^{1,\vec{p}}(\Omega)$:

$$\begin{gathered} |u|,\quad u^+=\max\{u,0\},\quad u^-=-\min\{u,0\},\\ \max(u,v)\quad \text{and}\quad \min(u,v)\end{gathered} \tag{3.2.9}$$

Proof. The idea of the proof is that if we can create a sequence $f_n\in C^1(\mathbb{R})$ such that $f_n(t)\to|t|$ as $n\to\infty$, $f_n(0)=0$ and $f'\in L^\infty(\Omega)$, then we can apply the chain rule and Remark 3.2.1 to show that $f_n\circ u\in W_0^{1,\vec{p}}(\Omega)$ with $f_n\circ u\to|u|$ in $W^{1,\vec{p}}(\Omega)$, so that $|u|\in W_0^{1,\vec{p}}(\Omega)$. To that end, consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ with $f_n:\mathbb{R}\to\mathbb{R}$ defined by

$$f_n(t)=\sqrt{t^2+\frac{1}{n^2}}-\frac{1}{n}\quad\text{for all } n\in\mathbb{N},\quad t\in\mathbb{R}.$$

Calculations show that for all $n\in\mathbb{N}$,

$$f_n(0)=0,\quad \lim_{n\to\infty}f_n(t)=|t|,\quad |f_n'(t)|=\left|\frac{t}{\sqrt{t^2+\frac{1}{n^2}}}\right|<1,\quad \lim_{n\to\infty}f_n'(t)=\operatorname{sgn}(t). \tag{3.2.10}$$

By the anisotropic chain rule (Theorem 3.2.1), for all $u\in W_0^{1,\vec{p}}(\Omega)$, we have $f_n\circ u\in W_0^{1,\vec{p}}(\Omega)$. We now show that the weak derivative of $|u|$ exists and is given by $g=\operatorname{sgn}(u)\partial_j u\in L^{p_j}(\Omega)$. To that end, fix any $\varphi\in C_c^1(\Omega)$ and j with $1\le j\le N$. We claim the following convergences hold:

$$\int_\Omega |u|\partial_j\varphi\,dx=\lim_{n\to\infty}\int_\Omega (f_n\circ u)\partial_j\varphi\,dx,\qquad \int_\Omega g\varphi\,dx=\lim_{n\to\infty}\int_\Omega f_n'(u)(\partial_j u)\varphi\,dx. \tag{3.2.11}$$

Indeed, we have $(f_n \circ u)\partial_j \varphi \to |u|\partial_j\varphi$ a.e. in Ω as $n \to 0$, and $(f_n \circ u)\partial_j u$ is dominated by $\|\partial_j\varphi\|_{L^\infty(\Omega)}|u|$ which is integrable on Ω. Furthermore, $f_n'(u)\partial_j u \to g$ pointwise a.e. in Ω by the limit in (3.2.10), and $|f_n'(u)(\partial_j u)\varphi| \le |(\partial_j u)\varphi| \in L^1(\Omega)$. By the Dominated Convergence Theorem, we conclude the limits in (3.2.11) hold. We immediately obtain the equalities

$$\int_\Omega |u|\partial_j\varphi\, x = \lim_{n\to\infty}\int_\Omega (f_n \circ u)\partial_j\varphi\, dx = -\lim_{n\to\infty}\int_\Omega f_n'(u)(\partial_j u)\varphi\, dx = -\int_\Omega g\varphi\, dx,$$

which shows that the weak derivative $\partial_j|u|$ exists and is given by g. Repeating the Dominated Convergence Theorem arguments for Claims 1 and 2, we can further show that $\partial_j(f_n \circ u) \to \partial_j|u|$ in $L^{p_j}(\Omega)$ as $n \to \infty$ for each $1 \le j \le N$, and $f_n \circ u \to |u|$ in $W^{1,\vec{p}}(\Omega)$. Since $W_0^{1,\vec{p}}(\Omega)$ is a closed subspace of $W^{1,\vec{p}}(\Omega)$, we have $|u| \in W_0^{1,\vec{p}}(\Omega)$. Finally, by noting that

$$u^+ = \frac{1}{2}(u + |u|), \quad u^- = \frac{1}{2}(u - |u|),$$

$$\max(u, v) = \frac{1}{2}(u + v + |u - v|) \quad \text{and} \quad \min(u, v) = \frac{1}{2}(u + v - |u - v|),$$

we conclude that each of the functions in (3.2.9) are in $W_0^{1,\vec{p}}(\Omega)$. □

Theorem 3.2.2. Let $u \in W_0^{1,\vec{p}}(\Omega)$ and $k > 0$ be arbitrary. Then $T_k(u)$, the truncation of u at level k, belongs to $W_0^{1,\vec{p}}(\Omega)$ and moreover, the partial weak derivative $\partial_j(T_k(u))$ for $1 \le j \le N$ is given by

$$\partial_j(T_k(u)) = \chi_{\{|u|\le k\}}\partial_j u = \begin{cases} \partial_j u & \text{if } |u| \le k, \\ 0 & \text{if } |u| > k. \end{cases}$$

Proof. Fix $k > 0$. First, we notice that

$$G_k(t) + T_k(t) = t \text{ for all } t \in \mathbb{R}, \tag{3.2.12}$$

where $G_k : \mathbb{R} \to \mathbb{R}$ is defined by

$$G_k(t) = \begin{cases} 0 & \text{if } |t| \le k, \\ t - k & \text{if } t > k, \\ t + k & \text{if } t < -k. \end{cases}$$

In this form, we cannot claim that $G_k \in W_0^{1,\vec{p}}(\Omega)$ via Theorem 3.2.1 since $G_k \notin C^1(\mathbb{R})$. To circumvent this, we perform an "allowed correction" by introducing the C^1 function $H_k : \mathbb{R} \to \mathbb{R}$, which satisfies $H_k \in C^1(\mathbb{R})$, $H_k(0) = 0$ and

$$H_k(t) > 0 \text{ if } 0 < \mathrm{t} < \mathrm{k}, \quad H_k(t) > 0 \text{ if } -k < t < 0 \text{ and } H_k(t) = G_k(t) \text{ if } |t| \ge k. \tag{3.2.13}$$

Take, for example, the function $H_k(t) = \frac{t^3 - k^2 t}{2k^2}$ if $|t| < k$ and $H_k(t) = G_k(t)$ for $|t| \ge k$. We can apply Theorem 3.2.1 to see that $H_k(u) \in W_0^{1,\vec{p}}(\Omega)$. Furthermore, by Lemma 3.2.1, we have $u^+ := \max\{0, u\} \in W_0^{1,\vec{p}}(\Omega)$, so that $\max\{0, H_k(u^+)\} \in W_0^{1,\vec{p}}(\Omega)$. By a similar argument, we must also have $u^- = \min\{0, u\} \in W_0^{1,\vec{p}}(\Omega)$ and $\min\{0, H_k(u^-)\} \in W_0^{1,\vec{p}}(\Omega)$. Since for all $t \in \mathbb{R}$ we have

$$\min\{0, H_k(t^-)\} + \max\{0, H_k(t^+)\} = G_k(t) \text{ where}$$

$$t^- = \begin{cases} 0 & \text{if } t \ge 0, \\ t & \text{if } t < 0, \end{cases} \quad \text{and} \quad t^+ = \begin{cases} 0 & \text{if } t \le 0, \\ t & \text{if } t > 0, \end{cases}$$

we conclude that $G_k(u) = \min\{0, H_k(u^-)\} + \max\{0, H_k(u^+)\} \in W_0^{1,\vec{p}}(\Omega)$. Finally, we have $T_k(u) = u - G_k(u) \in W_0^{1,\vec{p}}(\Omega)$ from (3.2.12). This finishes the proof. □

Remark 3.2.2. In the proof of Theorem 3.2.2, it is more convenient to show that the "reverse truncation" functions G_k is in $W_0^{1,\vec{p}}(\Omega)$, rather than work with T_k directly. Indeed, although we can express T_k as a composition of functions in Lemma 3.2.1 via the formula $T_k(u) = \max\{-k, \min\{u, k\}\}$, this is not helpful, since the constant functions $-k$ and k are not in $W_0^{1,\vec{p}}(\Omega)$.

3.3. Sobolev Embeddings

An essential property of isotropic Sobolev spaces is that they admit natural continuous and compact embeddings into the L^p spaces. We prove that these embeddings hold in the anisotropic case as well. In this section, we require

$$p^* = \frac{Np}{N-p} \quad \text{with} \quad \sum_{j=1}^{N} 1/p_j > 1. \tag{3.3.1}$$

along with the assumptions, unless otherwise stated, that Ω is an arbitrary open set of $\mathbb{R}^N$ and $\vec{p} = (p_1, p_2, \dots, p_N)$ with $1 < p_j < \infty$. The requirement on $\sum_{j=1}^N 1/p_j$ forces $p^* > 0$.

The key result that allows us to prove the anisotropic Sobolev embeddings is Troisi's Inequality, from [10]:

Theorem 3.3.1 (Troisi's Inequality)**.** There exists a positive constant S depending only on N and $\vec{p}$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq S\Pi_{j=1}^N \|\partial_j u\|_{L^{p_j}(\mathbb{R}^N)}^{1/N} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^N). \tag{3.3.2}$$

Corollary 3.3.1. For all $u \in W_0^{1,\vec{p}}(\Omega)$, we have

$$\|u\|_{L^{p^*}(\Omega)} \leq S\Pi_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}^{1/N} \leq \frac{S}{N} \sum_{j=1}^{N} \|\partial_j u\|_{L^{p_j}(\Omega)}. \tag{3.3.3}$$

Proof. The second inequality follows by applying the arithmetic-geometric inequality to the middle term of (3.3.3), so it suffices to consider the first inequality. Since $W_0^{1,\vec{p}}(\Omega)$ is defined to be the closure of $C_c^\infty(\Omega)$ with respect to the norm on $W^{1,\vec{p}}(\Omega)$, for any $u \in W_0^{1,\vec{p}}(\Omega)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \to u$ strongly in $W_0^{1,\vec{p}}(\Omega)$, and for each $n \in \mathbb{N}$, u_n satisfies the inequality

$$\|u_n\|_{L^{p^*}(\Omega)} = \|u_n\|_{L^{p^*}(\mathbb{R}^N)} \leq S\Pi_{j=1}^N \|\partial_j u_n\|_{L^{p_j}(\mathbb{R}^N)}^{1/N} = S\Pi_{j=1}^N \|\partial_j u_n\|_{L^{p_j}(\Omega)}^{1/N}. \tag{3.3.4}$$

Note that whenever we take a norm in $\mathbb{R}^N$, we are implicitly extending all functions by 0 outside of Ω. The equalities follow since u_n have compact support on Ω, so $u_n|_{\mathbb{R}^N\setminus\Omega} = 0$. Since $u_n \to u$ in $W_0^{1,\vec{p}}(\Omega)$, we have

$$\|u_n - u\|_{W^{1,\vec{p}}(\Omega)} = \|u_n - u\|_{L^p(\Omega)} + \sum_{j=1}^{N} \|\partial_j u_n - \partial_j u\|_{L^{p_j}(\Omega)} \to 0 \quad \text{as } n \to \infty. \tag{3.3.5}$$

Hence, each individual term in the sum must vanish in the limit, so the reverse triangle inequality gives us

$$\left| \|\partial_j u_n\|_{L^{p_j}(\Omega)} - \|\partial_j u\|_{L^{p_j}(\Omega)} \right| \leq \|\partial_j u_n - \partial_j u\|_{L^{p_j}(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

That is, $\|\partial_j u_n\|_{L^{p_j}(\Omega)} \to \|\partial_j u\|_{L^{p_j}(\Omega)}$ as $n \to \infty$. Next, we show that $u_n \to u$ in $L^{p^*}(\Omega)$ as $n \to \infty$. For any $n, m \in \mathbb{N}$, Troisi's Inequality gives us

$$\begin{aligned} \|u_n - u_m\|_{L^{p^*}(\Omega)} = \|u_n - u_m\|_{L^{p^*}(\mathbb{R}^N)} &\leq \mathcal{S}\Pi_{j=1}^N \|\partial_j u_n - \partial_j u_m\|_{L^{p_j}(\mathbb{R}^N)}^{1/N} \\ &\leq \frac{\mathcal{S}}{N} \sum_{j=1}^N \|\partial_j u_n - \partial_j u_m\|_{L^{p_j}(\mathbb{R}^N)} = \frac{\mathcal{S}}{N} \sum_{j=1}^N \|\partial_j u_n - \partial_j u_m\|_{L^{p_j}(\Omega)}. \end{aligned}$$

Since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W_0^{1,\vec{p}}(\Omega)$, we see that $(u_n)_{n\in\mathbb{N}}$ is Cauchy in $L^{p^*}(\Omega)$ and by completeness, $u_n \to w$ for some $w \in L^{p^*}(\Omega)$. By Proposition 1.3.2, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging a.e. to w in Ω. Since $u_n \to u$ in $L^p(\Omega)$, we also have $u_{n_k} \to u$ in $L^p(\Omega)$, and hence a subsequence $(u_{n_k'})_{k\in\mathbb{N}}$ converging a.e. to u in D. Since we already know $u_{n_k} \to w$ a.e. in Ω, this forces $u = w$. It follows that $\|u_n\|_{L^{p^*}(\Omega)} \to \|u\|_{L^{p^*}(\Omega)}$ as $n \to \infty$, so by taking limits on both sides of (3.3.4), we conclude the result in (3.3.3), as required. □

Corollary 3.3.2 (Sobolev Embeddings)**.** If Ω is a bounded open set of $\mathbb{R}^N$, then we have the embeddings $W_0^{1,\vec{p}}(\Omega) \overset{\text{cont.}}{\hookrightarrow} L^s(\Omega)$ for $s \in [1, p^*]$ and $W_0^{1,\vec{p}}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^s(\Omega)$ for $s \in [1, p^*)$.

Proof. Note that Ω is of finite measure, so Hölder's inequality gives us $L^p(\Omega) \overset{\text{cont.}}{\hookrightarrow} L^q(\Omega)$ whenever $1 \leq q \leq p$. In particular, for all $s \in [1, p^*]$, if $u \in W_0^{1,\vec{p}}(\Omega)$, then by Corollary 3.3.1, there exists a constant $C_s > 0$ depending on s such that

$$\|u\|_{L^s(\Omega)} \leq C_s \|u\|_{L^{p^*}(\Omega)} \leq C_s \frac{\mathcal{S}}{N} \|u\|_{W_0^{1,\vec{p}}(\Omega)}$$

whence the continuous embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^s(\Omega)$ holds. Due to constraints in the size of this thesis, we do not give the proof of the compact embedding, although for interested readers, the proof in the isotropic case is given in [4, Corollary 9.14, p. 284]. □

The Sobolev embedding theorems are at the core of working with functions in Sobolev spaces. To finish this chapter, we see as a consequence that whenever Ω is a bounded open set, functions in $W^{1,\vec{p}}(\Omega)$ are essentially defined by their weak derivatives:

Theorem 3.3.2. Let Ω be a bounded open subset of $\mathbb{R}^N$ and $\vec{p} = (p_1, p_2, \dots, p_N)$ satisfying the assumptions of (3.3.1). Then there exists a constant $C > 0$ such that for every $u \in W_0^{1,\vec{p}}(\Omega)$,

$$\sum_{j=1}^N \|u\|_{L^{p_j}(\Omega)} \leq \|u\|_{L^p(\Omega)} + \sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)} \leq C \sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}.$$

In particular, the expression $\|u\|_{W_0^{1,\vec{p}}(\Omega)} = \sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}$ defines an equivalent norm to $\|\cdot\|_{W^{1,\vec{p}}(\Omega)}$, which we will henceforth use as the norm on $W_0^{1,\vec{p}}(\Omega)$.

CHAPTER 4

Anisotropic Elliptic PDE with Gradient-Dependent Term and L^1 Data

4.1. Introduction

Anisotropic PDE have become a topic of much interest in both theoretical and applied research, where the added flexibility is useful in modelling real-world phenomena; their use in image recovery is well documented (for example, see [11]). In theoretical research, as we are interested in, one focus area is the study of existence of solutions of anisotropic PDE. In this chapter, we will draw upon the results of Chapters 1, 2, and 3 to follow the paper [3] and present the methods and proofs therein, with some techniques adapted from [1] as well (which will be cited when appropriate).

We are considering the following nonlinear homogeneous Dirichlet problem

$$\begin{cases} \mathcal{A}u - \mathfrak{B}u + \Phi(x, u, \nabla u) = f \quad \text{ in } \Omega, \\ u \in W_0^{1,\vec{p}}(\Omega), \quad \Phi(x, u, \nabla u) \in L^1(\Omega), \\ \mathcal{A}, \mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega), \quad f \in L^1(\Omega) \end{cases} \tag{4.1.1}$$

for a bounded open set $\Omega \subseteq \mathbb{R}^N$, where $\mathcal{A}$ is a Leray-Lions type operator in divergence form given by

$$\mathcal{A}u = -\nabla \cdot \mathbf{A}(x, u, \nabla u) = -\sum_{j=1}^{N} \partial_j (A_j(x, u, \nabla u)).$$

for functions $A_j : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $\mathfrak{B}$ is an operator satisfying certain growth conditions and continuity conditions, Φ is a lower order gradient dependent term and $f \in L^1(\Omega)$ is an arbitrary input datum. We are searching for solutions u satisfying (4.1.1) in a suitable weak sense (which we will define later) in the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$, as introduced in Chapter 3.

The difficulty and novelty of this problem is in the presence of a lower order gradient dependent Φ term and the input data f with no regularity, which necessitates an approach using the theory of pseudomonotone operators (as expounded upon in Chapter 2) and solving a sequence of approximate PDEs. The solution $u \in W_0^{1,\vec{p}}(\Omega)$ must also force $\Phi(x, u, \nabla u)$ to be in $L^1(\Omega)$, in order for our notion of 'weak solution' to make sense; in a sense, the growth conditions that we enforce on Φ - in particular, the presence of the anisotropic growth terms - effectively *forces* $\Phi(x, u, \nabla u)$ to be in $L^1(\Omega)$, as opposed to any other L^p space.

We will now give the specific assumptions on all the operators involved in (4.1.1), constructed to admit interesting examples while giving sufficient properties so that the PDE actually admits solutions.

In this chapter, we always write the following:

$$\vec{p} = (p_1, p_2, \dots, p_N) \in \mathbb{R}^N \text{ with } 1 < p_1 \le p_2 \le \cdots \le p_N < \infty, \quad p = \frac{N}{\sum_{j=1}^N 1/p_j} \quad \text{and} \quad p^* = \frac{Np}{N-p}.$$

In order for p^* to be positive, we always require $\sum_{j=1}^N 1/p_j > 1$.

4.1.1. The Assumptions. For $1 \le j \le N$, we assume the functions $A_j(x,t,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\Phi(x,t,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions (i.e., they are measurable on Ω for every $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in the last two variables t, ξ for a.e. $x \in \Omega$) satisfying the following conditions:

$$\begin{aligned} &\sum_{j=1}^N A_j(x,t,\xi)\xi_j \ge \nu_0 \sum_{j=1}^N |\xi_j|^{p_j}, \quad \text{(coercivity)} \\ &\sum_{j=1}^N \left(A_j(x,t,\xi) - A_j(x,t,\hat{\xi})\right)\left(\xi_j - \hat{\xi}_j\right) > 0 \text{ if } \xi \ne \hat{\xi} \quad \text{(monotonicity)} \end{aligned} \tag{4.1.2}$$

for ν_0 a positive real constant. We assume that the following growth conditions hold for a.e. $x \in \Omega$ and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$:

$$\begin{aligned} &|A_j(x,t,\xi)| \le \nu \left(\eta_j(x) + |t|^{p^*/p'_j} + \left(\sum_{l=1}^N |\xi_l|^{p_l} \right)^{1/p'_j} \right) \text{ for all } 1 \le j \le N, \\ &|\Phi(x,t,\xi)| \le \zeta(|t|) \left(\sum_{j=1}^N |\xi_j|^{p_j} + c(x) \right) \end{aligned} \tag{4.1.3}$$

where $\nu > 0$ is a positive real constant, η_j is a non-negative function in $L^{p'_j}(\Omega)$ for $1 \le j \le N$, $c \in L^1(\Omega)$ is a non-negative function and $\zeta : [0,\infty) \to [0,\infty)$ is a continuous, non-negative and non-decreasing function. With these assumptions on A_j for all $1 \le j \le N$, the operator $\mathcal{A} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is a Leray-Lions[1] operator. We assume that Φ satisfies a sign condition for a.e. $x \in \Omega$ and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$, given by

$$\Phi(x,t,\xi)\, t \ge 0 \quad \text{(sign-condition)}. \tag{4.1.4}$$

Whenever the input data $f \in L^1(\Omega)$ is non-zero, we assume that there exist real constants $\tau > 0$ and $\gamma > 0$ such that for a.e. $x \in \Omega$ and for all $\xi \in \times\mathbb{R}^N$, we have

$$|\Phi(x,t,\xi)| \ge \gamma \sum_{j=1}^N |\xi_j|^{p_j} \quad \text{for all } |t| \ge \tau. \tag{4.1.5}$$

Finally, the operator $\mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ satisfies a growth condition (P_1) and a continuity condition (P_2):

[1]'Leray-Lions' operators are a class of operators with specific assumptions that force them to be bounded, coercive and pseudomonotone. A useful resource on Leray-Lions type operators is [8, Section 8.6].

- (P_1): There exists constants $\mathfrak{C} > 0$, $s \in [1, p^*)$, $\mathfrak{a}_0 \geq 0$, with $\mathfrak{b} \in (0, p_1 - 1)$ if $\mathfrak{a}_0 > 0$ and $\mathfrak{b} \in (0, p_1/p')$ if $\mathfrak{a}_0 = 0$ such that for all $u, v \in W_0^{1,\vec{p}}(\Omega)$, we have

$$|\langle \mathfrak{B}u, v\rangle| \leq \mathfrak{C}\left(1 + \|u\|^{\mathfrak{b}}_{W_0^{1,\vec{p}}(\Omega)}\right)\left(\mathfrak{a}_0\|v\|_{W_0^{1,\vec{p}}(\Omega)} + \|v\|_{L^s(\Omega)}\right).$$

- (P_2): If $u_l \rightharpoonup u$ and $v_l \rightharpoonup v$ weakly in $W_0^{1,\vec{p}}(\Omega)$ as $l \to \infty$, then we have

$$\lim_{l\to\infty}\langle \mathfrak{B}u_l, v_l\rangle = \langle \mathfrak{B}u, v\rangle.$$

Example 4.1.1. The above assumptions are naturally satisfied when considering the following model case involving the $\vec{p}$-Laplacian operator:

$$\begin{cases} -\sum_{j=1}^N \partial_j\left(|\partial_j u|^{p_j-2}\partial_j u\right) + \Phi(x, u, \nabla u) = h + f \text{ in } \Omega, \\ u \in W_0^{1,\vec{p}}(\Omega), \quad \Phi(x, u, \nabla u) = \left(\sum_{j=1}^N |\partial_j u|^{p_j} + 1\right)|u|^{m-2}u \end{cases}$$

for $m > 1$, $h \in W^{-1,\vec{p}'}(\Omega)$ and $f \in L^1(\Omega)$ are arbitrary.

The model case corresponds to taking the functions $A_j(x, t, \xi) = |\xi_j|^{p_j-2}\xi_j$, the operator $\mathfrak{B}$ as the constant operator $\mathfrak{B}u = h \in W^{-1,\vec{p}'}(\Omega)$ for all $u \in W_0^{1,\vec{p}}(\Omega)$, and $\Phi(x, t, \xi) = (\sum_{j=1}^N |\xi_j|^{p_j} + 1)|t|^{m-2}t$. First, we show that the conditions for $\mathcal{A}$ are met. For the coercivity condition, we have

$$\sum_{j=1}^N A_j(x, t, \xi)\xi_j = \sum_{j=1}^N |\xi_j|^{p_j},$$

so we can take $\nu_0 = 1$, satisfying the first requirement in (4.1.2). The monotonicity condition follows quickly as well, since we need

$$\sum_{j=1}^N (A_j(x, t, \xi) - A_j(x, t, \hat{\xi}))(\xi_j - \hat{\xi}_j) = \sum_{j=1}^N (|\xi_j|^{p_j-2}\xi_j - |\hat{\xi}_j|^{p_j-2}\hat{\xi}_j)(\xi_j - \hat{\xi}_j) > 0 \text{ if } \xi \neq \hat{\xi},$$

and it suffices to notice that, by the strict monotonicity of the function $x \mapsto |x|^{p_j-2}x$,

$$\xi_j > \hat{\xi}_j \implies |\xi_j|^{p_j-2}\xi_j > |\hat{\xi}_j|^{p_j-2}\hat{\xi}_j.$$

Since we can swap ξ_j and $\hat{\xi}_j$, we see that the sign of $\xi_j - \hat{\xi}_j$ and $|\xi_j|^{p_j-2}\xi_j - |\hat{\xi}_j|^{p_j-2}\hat{\xi}_j$ is the same, so monotonicity follows. Now, for the growth condition on A_j, note that if a, a' are Hölder conjugates, we have $a/a' = a - 1$, so that for all $1 \leq j \leq N$, we have

$$|A_j(x, t, \xi)| = |\xi_j|^{p_j-1} = \left(|\xi_j|^{p_j}\right)^{1/p_j'} \leq \left(\sum_{l=1}^N |\xi_l|^{p_l}\right)^{1/p_j'}$$

and so the upper bound in (4.1.3) is satisfied by taking η_j as the zero function on Ω and $\nu = 1$. We now consider the assumptions on Φ. The sign condition (4.1.4) holds, since

$$\Phi(x, t, \xi)\, t = \left(\sum_{j=1}^N |\xi_j|^{p_j} + 1\right)|t|^m \geq 0.$$

Furthermore, we have $|\Phi(x,t,\xi)| = \left(\sum_{j=1}^N |\xi_j|^{p_j} + 1\right)|t|^{m-1}$, so by taking $\zeta(s) = s^{m-1}$ for $s \in [0,\infty)$ and c as the constant function 1, we see that the growth condition is satisfied. As for the lower bound condition on Φ as in (4.1.5), we have, for any $\tau > 0$, a.e. $x \in \Omega$ and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$\left(\sum_{j=1}^N |\xi_j|^{p_j} + 1\right)|t|^{m-1} \geq \tau^{m-1}\left(\sum_{j=1}^N |\xi_j|^{p_j}\right) \text{ whenever } |t| \geq \tau$$

so (4.1.5) holds by taking $\tau = \gamma = 1$. Finally, for $\mathfrak{B}$, we see that $|\langle \mathfrak{B}u, v\rangle| \leq \|h\|_{W^{-1,\vec{p}'}(\Omega)}\|v\|_{W_0^{1,\vec{p}}(\Omega)}$, so we can take $\mathfrak{a}_0 = \mathfrak{b} = 1$ and $\mathfrak{C} = \|h\|_{W^{-1,\vec{p}'}(\Omega)}$ to satisfy (P_1). Property (P_2) is trivially satisfied, since if $u_l \rightharpoonup u$ weakly and $v_l \rightharpoonup v$ weakly in $W_0^{1,\vec{p}}(\Omega)$, then

$$\langle \mathfrak{B}u_l, v_l\rangle = \langle h, v_l\rangle \to \langle h, v\rangle \text{ as } l \to \infty.$$

This finishes the example.

4.1.2. Existence of Solutions.

We give a suitable definition for a function in $W_0^{1,\vec{p}}(\Omega)$ to be a weak solution of (4.1.1):

Definition 4.1.1. Let $f \in L^1(\Omega)$. Under the assumptions of Section 4.1.1, a function $u \in W_0^{1,\vec{p}}(\Omega)$ is a weak solution of (4.1.1) if $\Phi(x,u,\nabla u) \in L^1(\Omega)$ and u satisfies the integral equation

$$\sum_{j=1}^N \int_\Omega A_j(x,u,\nabla u)\partial_j v\,dx + \int_\Omega \Phi(x,u,\nabla u)v - \langle \mathfrak{B}u, v\rangle = \int_\Omega fv\,dx \text{ for all } v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

To ensure that $\Phi(x,u,\nabla u)v$ and fv are integrable, we require v to be in $L^\infty(\Omega)$. The aim of this final chapter is to show that there exists a solution to (4.1.1), which satisfies the equation in a suitable weak sense:

Theorem 4.1.1 (Existence Theorem)**.** Let $f \in L^1(\Omega)$, and suppose the assumptions (4.1.2), (4.1.3), (4.1.4) and (4.1.5) hold, with $\mathfrak{B}$ satisfying properties (P_1) and (P_2). Then the PDE

$$\begin{cases} \mathcal{A}u - \mathfrak{B}u + \Phi(x,u,\nabla u) = f & \text{in } \Omega, \\ u \in W_0^{1,\vec{p}}(\Omega), \quad \Phi(x,u,\nabla u) \in L^1(\Omega), \end{cases}$$

admits at least one weak solution $u \in W_0^{1,\vec{p}}(\Omega)$ such that $\Phi(x,u,\nabla u) \in L^1(\Omega)$.

As stated previously, the main difficulties in working with this PDE is the low summability of the input data $f \in L^1(\Omega)$ and the term $\Phi(x,u,\nabla u)$ which depends on the gradient ∇u and whose growth is unrestricted with respect to $|u|$. It is known that in the isotropic case, we can circumvent similar obstacles by introducing sequences of simpler, approximate PDE, using the abstract theory of pseudomonotone operators to produce solutions of these approximate PDE, then taking the limit as $\varepsilon \to 0$ (see [2, Theorem 1.1 and Theorem 4.1, p. 145, 157]). By taking the limit as $\varepsilon \to 0$, it is possible to extract a solution of the original PDE from the sequence of approximate solutions. In our case, due to the presence of both Φ and f, we need to perform a double approximation: first, Φ is replaced by bounded terms Φ_ε with f set to 0, then the input data f is approximated by a sequence of L^∞ functions f_ε. In the next section, we put this plan into action.

4.2. The Approximate Problem

For every $\varepsilon > 0$, define the function $\Phi_\varepsilon : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by

$$\Phi_\varepsilon(x,t,\xi) := \frac{\Phi(x,t,\xi)}{1+\varepsilon|\Phi(x,t,\xi)|}$$

and replace Φ with Φ_ε and $f = 0$ in (4.1.1). We record here useful properties of Φ_ε: for a.e. $x \in \Omega$ and every $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$, we have

$$\Phi_\varepsilon(x,t,\xi)\,t \geq 0, \quad |\Phi_\varepsilon(x,t,\xi)| \leq \min\left\{\frac{1}{\varepsilon}, |\Phi(x,t,\xi)|\right\}. \tag{4.2.1}$$

Fix $\varepsilon > 0$. We are now considering the PDE

$$\begin{cases} \mathcal{A}u_\varepsilon - \mathfrak{B}u_\varepsilon + \Phi_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) = 0 & \text{in } \Omega, \\ u_\varepsilon \in W_0^{1,\vec{p}}(\Omega), \end{cases} \tag{4.2.2}$$

and we understand a solution of (4.2.2) in the weak sense:

Definition 4.2.1. A function $u_\varepsilon \in W_0^{1,\vec{p}}(\Omega)$ is a weak solution of (4.2.2) if it satisfies the following integral equation:

$$\sum_{j=1}^N \int_\Omega A_j(x,u_\varepsilon,\nabla u_\varepsilon)\partial_j v\,dx + \int_\Omega \Phi_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)v\,dx - \langle \mathfrak{B}u_\varepsilon, v\rangle = 0 \text{ for all } v \in W_0^{1,\vec{p}}(\Omega). \tag{4.2.3}$$

We do not need $v \in L^\infty(\Omega)$ unlike in Definition 4.1.1, since Φ_ε is in $L^\infty(\Omega)$ and $f = 0$.

The aim of this section is to prove the following existence theorem for solutions of the bounded approximate PDE:

Theorem 4.2.1. Under the assumptions of Section 4.1.1 (barring the lower bound on Φ in (4.1.5) since $f = 0$), the bounded approximate PDE (4.2.2) has a weak solution in $W_0^{1,\vec{p}}(\Omega)$ for every $\varepsilon > 0$.

In this approximated form, we can reframe the weak solution condition (4.2.3) as an operator equation from $W_0^{1,\vec{p}}(\Omega)$ to its dual. Indeed, for $\varepsilon > 0$, we define the operators $\mathcal{A} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ and $\zeta_\varepsilon : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ as follows:

$$\langle \zeta_\varepsilon u, v\rangle = \int_\Omega \Phi_\varepsilon(x,u,\nabla u)v\,dx, \quad \langle \mathcal{A}u, v\rangle = \sum_{j=1}^N \int_\Omega A_j(x,u,\nabla u)\partial_j v\,dx \quad \text{for every } u,v \in W_0^{1,\vec{p}}(\Omega).$$

In this formulation, a weak solution $u_\varepsilon \in W_0^{1,\vec{p}}(\Omega)$ of (4.2.2) must satisfy $(\mathcal{A}+\zeta_\varepsilon-\mathfrak{B})u_\varepsilon = 0_{W^{-1,\vec{p}'}(\Omega)}$. This naturally leads us to consider applying the theory of pseudomonotone operators in Chapter 3 to this operator equation. In particular, we will be using the surjectivity result of Theorem 2.3.2. Since $W_0^{1,\vec{p}}(\Omega)$ is a real, reflexive, separable Banach space (see Chapter 3, Proposition 3.2.1), we only need to prove $\mathcal{A}+\zeta_\varepsilon-\mathfrak{B}$ is a bounded, coercive and pseudomonotone operator, which will be done via a number of lemmas. However, before we begin, we give two necessary auxiliary propositions. For $u \in W_0^{1,\vec{p}}(\Omega)$ and $1 \leq j \leq N$, write

$$\hat{A}_j(u)(x) = A_j(x,u(x),\nabla u(x)) \quad \text{and} \quad \hat{\Phi}_\varepsilon(u)(x) = \Phi_\varepsilon(x,u(x),\nabla u(x)).$$

Proposition 4.2.1. For each $1 \leq j \leq N$ and every $\varepsilon > 0$, the maps

$$\hat{A}_j : W_0^{1,\vec{p}}(\Omega) \to L^{p_j'}(\Omega) \quad \text{and} \quad \hat{\Phi}_\varepsilon : W_0^{1,\vec{p}}(\Omega) \to L^{(p^*)'}(\Omega)$$

are well-defined and are continuous.

Proof. Fix some j with $1 \leq j \leq N$ and $\varepsilon > 0$. Note the following inequality: for every $m > 0$, there exists a constant $M > 0$ such that for every $a, b \in [0, \infty)$, we have $(a+b)^m \leq M(a^m + b^m)$. For $M > 1$, this inequality results from the convexity of the function $t \mapsto t^m$, with the constant $M = 2^m$. Applying this inequality, we have

$$\begin{aligned} |\hat{A}_j(u)|^{p_j'} &= \leq C\left(\eta_j + |u|^{p^*/p_j'} + \left(\sum_{l=1}^N |\partial_l u|^{p_l}\right)^{1/p_j'}\right)^{p_j'} \\ &\leq C_1\left((\eta_j)^{p_j'} + |u|^{p^*} + \left(\sum_{l=1}^N |\partial_l u|^{p_l}\right)\right), \end{aligned} \tag{4.2.4}$$

for some constant $C_1 > 0$. Note that this upper bound is in $L^1(\Omega)$ since $\eta_j \in L^{p_j'}(\Omega)$ and $u \in W_0^{1,\vec{p}}(\Omega) \overset{\text{cont.}}{\hookrightarrow} L^{p^*}(\Omega)$. Furthermore, we note that $|\Phi_\varepsilon(x,t,\xi)| \leq 1/\varepsilon$ for a.e. $x \in \Omega$ and every $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$, so clearly $\hat{\Phi}_\varepsilon(u) \in L^{(p^*)'}(\Omega)$. This shows that both $\hat{A}_j : W_0^{1,\vec{p}}(\Omega) \to L^{p_j'}(\Omega)$ and $\hat{\Phi}_\varepsilon : W_0^{1,\vec{p}}(\Omega) \to L^{(p^*)'}(\Omega)$ are well-defined.

To prove the continuity, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that $u_n \to u$ in $W_0^{1,\vec{p}}(\Omega)$. Then on a subsequence $(u_{n'})$ we have $u_{n'} \to u$ and $\nabla u_{n'} \to \nabla u$ a.e. in Ω. Now, Φ_ε and A_j are Carathéodory functions, so $\hat{\Phi}_\varepsilon(u_{n'}) \to \hat{\Phi}_\varepsilon(u)$ and $\hat{A}_j(u_{n'}) \to \hat{\Phi}_\varepsilon(u)$ a.e. in Ω. Since $\hat{\Phi}_\varepsilon$ is bounded on Ω and (4.2.4) shows $\hat{A}_j(u_{n'})$ is bounded in $L^{p_j'}(\Omega)$, we can apply the Dominated Convergence Theorem to get

$$\hat{\Phi}_\varepsilon(u_{n'}) \to \hat{\Phi}_\varepsilon(u) \text{ in } L^{(p^*)'}(\Omega) \quad \text{and} \quad \hat{A}_j(u_{n'}) \to \hat{A}_j(u) \text{ in } L^{p_j'}(\Omega) \text{ as } n \to \infty. \tag{4.2.5}$$

By a uniqueness of limits argument (see Proposition 1.2.1 (iii)), we conclude that the convergences in (4.2.5) extend to the total sequence $(\hat{\Phi}_\varepsilon(u_n))_{n\in\mathbb{N}}$ and $(\hat{A}_j(u_n))_{n\in\mathbb{N}}$. It follows that the operators $\hat{A}_j : W_0^{1,\vec{p}}(\Omega) \to L^{p_j'}(\Omega)$ and $\hat{\Phi}_\varepsilon : W_0^{1,\vec{p}}(\Omega) \to L^{(p^*)'}(\Omega)$ are continuous, as required. ☐

Proposition 4.2.2. (Adapted from [1, Lemma 3.2, p. 174]) Suppose that the growth, coercivity and monotonocity conditions hold for $A_j(x,t,\xi)$ as in (4.1.2) and (4.1.3). Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,\vec{p}}(\Omega)$ and

$$\lim_{n\to\infty} \sum_{j=1}^N \int_\Omega (A_j(x,u_n,\nabla u_n) - A_j(x,u_n,\nabla u))(\partial_j u_n - \partial_j u)\,dx = 0.$$

Then $\nabla u_n \to \nabla u$ a.e. in Ω as $n \to \infty$.

Proof. Write $D_n := (A_j(x,u_n,\nabla u_n) - A_j(x,u_n,\nabla u))(\partial_j u_n - \partial_j u)$. By the monotonicity condition, $D_n \geq 0$ a.e. in Ω, and by the assumption, we have $D_n \to 0$ in $L^1(\Omega)$ as $n \to \infty$. This convergence implies on a subsequence we have $D_n \to 0$ a.e. on Ω. By the embedding $W_0^{1,\vec{p}}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^p(\Omega)$ (Corollary 3.3.2) and the weak convergence $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$, we have $u_n \to u$ strongly in $L^p(\Omega)$, so there exists a further subsequence such that $u_n \to u$ a.e. in Ω. Hence, we can define a subset $Z \subseteq \Omega$ with $|Z| = 0$ and for every $x \in \Omega \setminus Z$, we have

$$u(x) < +\infty, \quad |\nabla u(x)| < \infty, \quad |\eta_j(x)| < \infty,$$
$$u_n(x) \to u(x) \quad \text{and} \quad D_n(x) \to 0 \quad \text{as } n \to \infty.$$

Fix $x \in \Omega$. By applying the Generalised Young's Inequality (Theorem 1.5.1) to the terms $\left(\sum_{l=1}^N |\partial_j u_n(x)|^{p_l}\right)^{1/p_j'}$ for each $1 \le j \le N$, for some $\delta > 0$ fixed there exists $C_\delta > 0$ such that

$$\sum_{j=1}^N \left(\sum_{l=1}^N |\partial_j u_n(x)|^{p_l}\right)^{1/p_j'} \le \delta \sum_{l=1}^N |\partial_j u_n(x)|^{p_l} + C_\delta.$$

Furthermore, since $(u_n(x))_{n\in\mathbb{N}}$ is convergent, $|u_n(x)|$ is bounded uniformly from below by a constant depending on x. We can use these facts to expand the definition of $D_n(x)$, while also using the growth conditions and the coercivity conditions on A_j to simplify:

(4.2.6)

$$\begin{aligned}
D_n(x) &= \sum_{j=1}^N (A_j(x, u_n, \nabla u_n) - A_j(x, u_n, \nabla u))(\partial_j u_n - \partial_j u) \\
&= \sum_{j=1}^N (A_j(x, u_n, \nabla u_n)\partial_j u_n + A_j(x, u_n \nabla u)\partial_j u - A_j(x, u_n, \nabla u_n)\partial_j u - A_j(x, u_n, \nabla u)\partial_j u_n) \\
&\ge \nu_0 \sum_{j=1}^N |\partial_j u_n|^{p_j} + \nu_0 \sum_{j=1}^N |\partial_j u|^{p_j} - \nu \sum_{j=1}^N \left(\eta_j(x) + |u_n|^{p^*/p_j'} + \left(\sum_{l=1}^N |\partial_l u_n|^{p_l}\right)^{1/p_j'}\right)|\partial_j u| \\
&\qquad - \nu \sum_{j=1}^N \left(\eta_j(x) + |u_n|^{p^*/p_j'} + \left(\sum_{l=1}^N |\partial_l u|^{p_l}\right)^{1/p_j'}\right)|\partial_j u_n| \\
&\ge^{(*)} \nu_0 \sum_{j=1}^N |\partial_j u_n|^{p_j} - g(x)(1 + \delta \sum_{j=1}^N |\partial_j u_n|^{p_j} + \sum_{j=1}^N |\partial_j u_n|).
\end{aligned}$$

where $g(x)$ is a positive constant depending on x (to get $(*)$, we expand the product and factorise out all terms that are independent on n; we also use that $u_n(x)$ is convergent). By the Generalised Young's Inequality, we can again find a constant $K_\delta > 0$ such that $|\partial_j u_n| \le \delta |\partial_j u_n|^{p_j} + K_\delta$. By picking $\delta > 0$ small and recalling that $D_n(x) \to 0$ as $n \to \infty$, we have $(|\partial_j u_n(x)|^{p_j})_{n\in\mathbb{N}}$ is bounded for all $1 \le j \le N$, which forces $(\partial_j u_n(x))_{n\in\mathbb{N}}$ to be bounded. Write $\xi^* = (\xi_j^*)_{1\le j\le N}$ where ξ_j^* is a limit point of $(\partial_j u_n(x))_{n\in\mathbb{N}}$ with $|\xi_j^*| < \infty$ for $1 \le j \le N$. Since A_j is a Carathéodory function and hence continuous in its last two variables, we have

$$\begin{aligned}
0 = \lim_{n\to\infty} D_n(x) &= \lim_{n\to\infty} (A_j(x, u, \nabla u_n) - A_j(x, u_n, \nabla))(\partial_j u_n(x) - \partial_j u(x)) \\
&= (A_j(x, u, \xi^*) - A_j(x, u, \nabla u))(\xi_j^* - \partial_j u(x))
\end{aligned}$$

By the monotonocity condition, this forces $\xi^* = \nabla u(x)$, and since this is the only possible limit point of $\nabla u_n(x)$ for $x \in \Omega$ arbitrary, we conclude that $\nabla u_n \to \nabla u$ a.e. in Ω, as we wanted. □

We now begin our program of proving that $\mathcal{A} + \zeta_\varepsilon - \mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}}(\Omega)$ is a bounded, coercive, pseudomonotone operator.

Lemma 4.2.1. The operator $\mathcal{A} + \zeta_\epsilon : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is a continuous, coercive operator and maps bounded sets into bounded sets.

Continuity: Recall the operator norm

$$\|(\mathcal{A} + \zeta_\varepsilon)u\|_{W^{-1,\vec{p}'}(\Omega)} = \sup_{\|v\|_{W_0^{1,\vec{p}}(\Omega)} \le 1} |\langle (\mathcal{A} + \zeta_\varepsilon)u, v\rangle|. \tag{4.2.7}$$

For every $u_1, u_2 \in W_0^{1,\vec{p}}(\Omega)$, we can apply the triangle inequality and Hölder's inequality to see that $\|(\mathcal{A} + \zeta_\varepsilon)u_1 - (\mathcal{A} + \zeta_\varepsilon)u_2\|_{W^{-1,\vec{p}'}(\Omega)}$ is bounded from above by

$$\begin{aligned} &\sup_{\|v\|_{W_0^{1,\vec{p}}(\Omega)} \le 1} \int_\Omega \sum_{j=1}^N \left(|\hat{A}_j(u_1) - \hat{A}_j(u_2)||\partial_j v|\right) + |\hat{\Phi}_\varepsilon(u_1) - \hat{\Phi}_\varepsilon(u_2)||v|\, dx \\ &\le \left(\sum_{j=1}^N \|\hat{A}_j(u_1) - \hat{A}_j(u_2)\|_{L^{p_j'}(\Omega)} \right) + C\|\hat{\Phi}_\varepsilon(u_1) - \hat{\Phi}_\varepsilon(u_2)\|_{L^{(p^*)'}(\Omega)} \end{aligned} \tag{4.2.8}$$

with $C > 0$ a constant from the embedding $W_0^{1,\vec{p}}(\Omega) \overset{\text{cont.}}{\hookrightarrow} L^{p^*}(\Omega)$; we have also used $\|\partial_j v\|_{L^{p_j}(\Omega)} \le 1$ if v is in the closed unit ball in $W_0^{1,\vec{p}}(\Omega)$. Now, if $u_n \to u$ in $W_0^{1,\vec{p}}(\Omega)$, then $\hat{A}_j(u_n) \to \hat{A}_j(u)$ in $L^{p_j'}(\Omega)$ and $\hat{\Phi}_\varepsilon(u_n) \to \hat{\Phi}_\varepsilon(u)$ in $L^{(p^*)'}(\Omega)$ by Proposition 4.2.1. It then follows from (4.2.8) that $\mathcal{A} + \zeta_\varepsilon : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is continuous, as required.

Bounded: It is sufficient to show that $\mathcal{A} + \zeta_\epsilon$ maps a ball in $W_0^{1,\vec{p}}(\Omega)$ centred at 0, to a bounded set in $W^{-1,\vec{p}'}(\Omega)$. To that end, let $r > 0$ and let $u \in W_0^{1,\vec{p}}(\Omega)$ with $\|u\|_{W_0^{1,\vec{p}}(\Omega)} < r$. We want to bound $(\mathcal{A} + \zeta_\varepsilon)u \in W^{-1,\vec{p}'}(\Omega)$ in the operator norm (4.2.7). Notice that by the triangle inequality, for all $v \in W_0^{1,\vec{p}}(\Omega)$, we have

$$\left|\langle (A + \zeta_\varepsilon)u, v\rangle\right| \le \sum_{j=1}^N \int_\Omega \left|A_j(x, u, \nabla u)\partial_j v\right| dx + \int_\Omega \left|\Phi_\varepsilon(x, u, \nabla u)v\right| dx. \tag{4.2.9}$$

By restricting to $v \in W_0^{1,\vec{p}}(\Omega)$ with $\|v\|_{W_0^{1,\vec{p}}(\Omega)} \le 1$, we have

$$\int_\Omega \left|\Phi_\varepsilon(x, u, \nabla u)v\right| dx \le \frac{1}{\varepsilon}\int_\Omega |v|\, dx = \frac{1}{\varepsilon}\|v\|_{L^1(\Omega)} \le \frac{C}{\varepsilon}\|v\|_{W_0^{1,\vec{p}}(\Omega)} \le \frac{C}{\varepsilon} \tag{4.2.10}$$

where the constant C comes from the continuous embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^1(\Omega)$. For the other terms in (4.2.9), use the growth condition on A_j and Hölder's inequality to obtain, for all $u, v \in W_0^{1,\vec{p}}(\Omega)$,

$$\begin{aligned} \sum_{j=1}^N \int_\Omega \left|A_j(x, u, \nabla u)\partial_j v\right| dx &\le \sum_{j=1}^N \nu \int_\Omega \left(\eta_j(x) + |u|^{p^*/p_j'} + \left(\sum_{l=1}^N \left|\partial_l u\right|^{p_l} \right)^{1/p_j'} \right) |\partial_j v|\, dx \\ &\le \sum_{j=1}^N \nu \|\partial_j v\|_{L^{p_j}(\Omega)} \left(\|\eta_j\|_{L^{p_j'}(\Omega)} + \|u\|_{L^{p^*}(\Omega)}^{p^*/p_j'} + \left(\|u\|_{W_0^{1,\vec{p}}(\Omega)} \right)^{1/p_j'} \right). \end{aligned} \tag{4.2.11}$$

When the norms of u and v in $W_0^{1,\vec{p}}(\Omega)$ are bounded, the upper bound in (4.2.11) is bounded by a constant since $\eta_j \in L^{p_j'}(\Omega)$ for all $1 \leq j \leq N$ and $W_0^{1,\vec{p}}(\Omega) \overset{\text{cont.}}{\hookrightarrow} L^{p^*}(\Omega)$. Using (4.2.10) and (4.2.11) in (4.2.9), we conclude that $\mathcal{A} + \zeta_\varepsilon : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is a bounded operator, as we wanted.

Coercive: Recall the definition of coercivity. We have the expressions

$$\langle (\mathcal{A} + \zeta_\varepsilon)u, u\rangle = \sum_{j=1}^N \int_\Omega A_j(x, u, \nabla u)\partial_j u \, dx + \int_\Omega \underbrace{\Phi_\varepsilon(x, u, \nabla u)u}_{\geq 0 \text{ from sign condition}} \, dx,$$

$$\geq \nu_0 \sum_{j=1}^N \int_\Omega |\partial_j u|^{p_j} \, dx = \nu_0 \sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}^{p_j}$$

for a fixed constant ν_0 from the coercivity condition on A_j. Hence, it suffices to show that

$$\frac{\sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}^{p_j}}{\|u\|_{W_0^{1,\vec{p}}(\Omega)}} = \frac{\sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}^{p_j}}{\sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}} \to \infty \text{ as } \|u\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty.$$

The following proof formalises the intuitive idea that the powers of the numerator dominate those in the denominator using the Generalised Young's Inequality. Suppose by contradiction that there exists a constant $M > 0$ such that for some sequence $(u_n)_{n\in\mathbb{N}}$ in $W_0^{1,\vec{p}}(\Omega)$ with $\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty$,

$$\frac{\sum_{j=1}^N \|\partial_j u_n\|_{L^{p_j}(\Omega)}^{p_j}}{\sum_{j=1}^N \|\partial_j u_n\|_{L^{p_j}(\Omega)}} \leq M. \tag{4.2.12}$$

For each term $\|\partial_j u_n\|_{L^{p_j}(\Omega)}$ in the denominator, notice that for any fixed $\delta > 0$ and for all $1 \leq j \leq N$, there exists a constant $C_{\delta,j} > 0$ such that

$$\|\partial_j u_n\|_{L^{p_j}(\Omega)} = 1\|\partial_j u_n\|_{L^{p_j}(\Omega)} \leq \delta \|\partial_j u_n\|_{L^{p_j}(\Omega)}^{p_j} + C_{\delta,j},$$

by applying the Generalised Young's Inequality (Theorem 1.5.1) to control the term $\|\partial_j u_n\|_{L^{p_j}(\Omega)}$ and lose control over the constant. This is permissible since $1 < p_1 := \min_{1\leq j\leq N} p_j$. Defining $C_\delta := \max_{1\leq j\leq N}\{C_{\delta,j}\}$ and rearranging (4.2.12), we get

$$\sum_{j=1}^N (1 - M\delta)\|\partial_j u_n\|_{L^{p_j}(\Omega)}^{p_j} \leq MNC_\delta.$$

Picking $0 < \delta < 1/M$, we get the contradiction that $\sum_{j=1}^N \|\partial_j u_n\|_{L^{p_j}(\Omega)}^{p_j}$ is bounded for all $n \in \mathbb{N}$, which is impossible since by assumption $\|u_n\|_{W_{1,\vec{p}}(\Omega)} \to \infty$, which implies that at least one of $\|\partial_j u_n\|_{L^{p_j}(\Omega)} \to \infty$ as $n \to \infty$. Hence, $\mathcal{A} + \zeta_\varepsilon : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is a coercive operator.

Lemma 4.2.2. The operator $\mathcal{A} + \zeta_\epsilon : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is pseudomonotone.

Proof. (Adapted from [1, Appendix, p. 185]; it was originally stated in [3] that $\mathcal{A}+\zeta_\varepsilon$ is a *monotone* operator, but it has since been corrected). We have already proven $\mathcal{A} + \zeta_\varepsilon : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is a bounded operator. By Proposition 2.2.5, to show that $\mathcal{A} + \zeta_\varepsilon$ is pseudomonotone, it suffices to show that it satisfies Condition $(\hat{M})$. Hence, it is sufficient to assume that there exists $u \in W_0^{1,\vec{p}}(\Omega)$, a sequence $(u_l)_{l\in\mathbb{N}}$ in $W_0^{1,\vec{p}}(\Omega)$ and $\chi \in W^{-1,\vec{p}'}(\Omega)$ such that

$$\begin{cases} u_l \rightharpoonup u \text{ weakly in } W_0^{1,\vec{p}}(\Omega), \\ (\mathcal{A}+\zeta_\varepsilon)u_l \rightharpoonup \chi \text{ weakly in } W^{-1,\vec{p}'}(\Omega), \\ \limsup_{l\to\infty}\langle(\mathcal{A}+\zeta_\varepsilon)u_l, u_l\rangle \le \langle \chi, u\rangle, \end{cases} \tag{4.2.13}$$

and prove

$$\chi = (\mathcal{A}+\zeta_\varepsilon)u \quad \text{and} \quad \langle(\mathcal{A}+\zeta_\varepsilon)u_l, u_l\rangle \to \langle \chi, u\rangle \text{ as } l\to\infty. \tag{4.2.14}$$

where the convergence is only up to a subsequence of $(u_l)_{l\in\mathbb{N}}$. Since $W_0^{1,\vec{p}}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^p(\Omega)$ for $s\in[1,p^*)$, we can apply Proposition 1.2.4 to see that up to subsequence $u_l \to u$ in $L^p(\Omega)$ strongly and, by Proposition 1.3.1, $u_l \to u$ a.e. in Ω. From the growth condition in (4.1.3) and the calculations in (4.2.11), since $(u_l)_l$ is a bounded sequence in $W_0^{1,\vec{p}}(\Omega)$, the sequence $(A_j(x,u_l,\nabla u_l))_l$ is bounded in $L^{p_j'}(\Omega)$ for each $1\le j\le N$. Then we can extract a further subsequence, renamed $(u_l)_l$, such that for all $1\le j\le N$, there exists $\varphi_j \in L^{p_j'}(\Omega)$ satisfying

$$A_j(x,u_l,\nabla u_l) \rightharpoonup \varphi_j \text{ weakly in } L^{p_j'}(\Omega) \text{ as } k\to\infty. \tag{4.2.15}$$

Furthermore, we notice that $|\hat{\Phi}_\varepsilon(u_l)| \le 1/\varepsilon$ on Ω and hence $\hat{\Phi}_\varepsilon(u_l)$ is uniformly bounded in $L^\infty(\Omega)$. Since Ω is a bounded set of $\mathbb{R}^N$, we have the continuous embedding $L^\infty(\Omega) \hookrightarrow L^{p'}(\Omega)$, which forces $(\hat{\Phi}_\varepsilon(u_l))_l$ to be a bounded sequence in $L^{p'}(\Omega)$. Hence, there exists $\eta \in L^{p'}(\Omega)$ such that, up to subsequence, $\hat{\Phi}_\varepsilon(u_l) \rightharpoonup \eta$ weakly in $L^{p'}(\Omega)$. It follows from Proposition 1.2.1 (vii) (simultaneous convergence) that the following convergences hold:

$$\begin{aligned} \lim_{l\to\infty}\sum_{j=1}^N \int_\Omega A_j(x,u_l,\nabla u_l)\partial_j v\,dx &= \sum_{j=1}^N \int_\Omega \varphi_j \partial_j v\,dx, \\ \lim_{l\to\infty}\sum_{j=1}^N \int_\Omega \Phi_\varepsilon(x,u_l,\nabla u_l) v\,dx &= \int_\Omega \eta v\,dx. \end{aligned} \tag{4.2.16}$$

From here, we will show (4.2.14) in steps. In Step 1, we will prove that

$$\lim_{l\to\infty}\sum_{j=1}^N \int_\Omega A_j(x,u_l,\nabla u_l)\partial_j u_l\,dx = \sum_{j=1}^N \int_\Omega \varphi_j \partial_j u\,dx. \tag{4.2.17}$$

which is sufficient to obtain the convergence requirement in (4.2.14). In Step 2, we show that $\chi = (\mathcal{A}+\zeta_\varepsilon)u$, from which the pseudomonotonicity of $\mathcal{A}+\zeta_\varepsilon$ follows.

Step 1. To show that the limit in (4.2.17) holds, it suffices to show that the limit inferior and superior of $\sum_{j=1}^N \int_\Omega A_j(x,u_l,\nabla u_l)\partial_j u_l\,dx$ are equal as $l\to\infty$. For all $v\in W_0^{1,\vec{p}}(\Omega)$, using (4.2.15) and (4.2.16), we have

$$\langle \chi, v\rangle = \lim_{l\to\infty}\langle(\mathcal{A}+\zeta_\varepsilon)u_l, v\rangle = \sum_{j=1}^N \int_\Omega \varphi_j\partial_j v\,dx + \int_\Omega \eta v\,dx. \tag{4.2.18}$$

Hence, by using that $\hat{\Phi}_\varepsilon(u_l) \rightharpoonup \eta$ weakly in $L^{p'}(\Omega)$ and $u_l \to u$ strongly in $L^p(\Omega)$ as $l \to \infty$, along with Proposition 1.2.1 (vii), we conclude that

$$\begin{aligned}
&\limsup_{l\to\infty}\left(\sum_{j=1}^N \int_\Omega A_j(x,u_l,\nabla u_l)\partial_j u_l\,dx + \int_\Omega \Phi_\varepsilon(x,u_l,\nabla u_l)u_l\,dx\right)\\
&= \limsup_{l\to\infty}\left(\sum_{j=1}^N \int_\Omega A_j(x,u_l,\nabla u_l)\partial_j u_l\,dx\right) + \int_\Omega \eta u\,dx\\
&= \limsup_{l\to\infty}\langle(\mathcal{A}+\zeta_\varepsilon)u_l,u_l\rangle \leq^{(*)} \langle\chi,u\rangle = \sum_{j=1}^N \int_\Omega \varphi_j\partial_j u\,dx + \int_\Omega \eta u\,dx,
\end{aligned}$$

where in $(*)$ we used the assumption (4.2.13). Finally, this implies

$$\limsup_{l\to\infty}\sum_{j=1}^N \int_\Omega A_j(x,u_l,\nabla u_l)\partial_j u_l\,dx \leq \sum_{j=1}^N \int_\Omega \varphi_j\partial_j u\,dx \tag{4.2.19}$$

by using (4.2.18) and rearranging. Now, by the monotonicity condition (4.1.2), we have, whenever $\partial_j u_l \neq \partial_j u$,

$$\sum_{j=1}^N \int_\Omega \left(A_j(x,u_l,\nabla u_l) - A_j(x,u_l,\nabla u)\right)(\partial_j u_l - \partial_j u)\,dx > 0.$$

Expanding the integrand, we get a strict lower bound for $\sum_{j=1}^N \int_\Omega A_j(x,u_l,\nabla u_l)\partial_j u_l\,dx$ of the form

$$\underbrace{-\sum_{j=1}^N \int_\Omega A_j(x,u_l,\nabla u)\partial_j u\,dx}_{(A)} + \sum_{j=1}^N \int_\Omega A_j(x,u_l,\nabla u_l)\partial_j u\,dx + \underbrace{\sum_{j=1}^N \int_\Omega A_j(x,u_l,\nabla u)\partial_j u_l\,dx}_{(B)}. \tag{4.2.20}$$

We show (A) and (B) both converge to the same limit as $l \to \infty$ - specifically, the limit is $\sum_{j=1}^N \int_\Omega A_j(x,u,\nabla u)\partial_j u\,dx$. We first prove that $A_j(x,u_l,\nabla u) \to A_j(x,u,\nabla u)$ in $L^{p_j'}(\Omega)$ for all $1 \leq j \leq N$. Fix any such j. The idea is to apply Vitali's Theorem. Since $u_l \to u$ a.e. in Ω and A_j is a Carathéodory function, we get $|A_j(x,u_l,\nabla u)|^{p_j} \to |A_j(x,u,\nabla u)|^{p_j}$ a.e. in Ω. We only need to show that $\{|A_j(x,u_l,\nabla u)|^{p_j}\}_{l\in\mathbb{N}}$ is uniformly integrable. Using the growth conditions (4.1.3) and the trick (4.2.4) to bring the power down, we obtain

$$|A_j(x,u_l,\nabla u)|^{p_j} \leq C\left(\eta_j^{p_j'}(x) + |u_l|^{p^*} + \sum_{i=1}^N |\partial_i u|^{p_i}\right) \tag{4.2.21}$$

for some constant $C > 0$. We recall the following: η_j is a function in $L^{p_j'}(\Omega)$, $(u_l)_{l\in\mathbb{N}}$ is a bounded sequence in $L^{p^*}(\Omega)$ from the Sobolev embeddings and $(u_l)_{l\in\mathbb{N}}$ converges weakly (and hence is bounded) in $W_0^{1,\vec{p}}(\Omega)$. Applying these facts to (4.2.21) shows that $|A_j(x,u_l,\nabla u)|^{p_j}$ can be bounded from above by an integrable function independent of l. By Proposition 1.4.1, we conclude that $\{|A_j(x,u_l,\nabla u)|^{p_j}\}_{l\in\mathbb{N}}$ is uniformly integrable. Hence, we have $A_j(x,u_l,\nabla u) \to A_j(x,u,\nabla u)$ in $L^{p_j'}(\Omega)$ by Vitali's Theorem, and it follows that $\sum_{j=1}^N A_j(x,u_l,\nabla u)\partial_j u$ converges to $\sum_{j=1}^N A_j(x,u,\nabla u)\partial_j u$ in $L^1(\Omega)$.

Next, we find the limit of $\sum_{j=1}^{N} A_j(x, u_l, \nabla u)\partial_j u_l$ in $L^1(\Omega)$, and we do so by showing that $\partial_j u_l \rightharpoonup \partial_j u$ weakly in $L^{p_j}(\Omega)$. Since $u_l \rightharpoonup u$ weakly in $W_0^{1,\vec{p}}(\Omega)$, the sequence $(u_l)_{l\in\mathbb{N}}$ is bounded in $W_0^{1,\vec{p}}(\Omega)$, which forces $(\partial_j u_l)_{l\in\mathbb{N}}$ to be bounded in $L^{p_j}(\Omega)$. Then up to subsequence, $\partial_j u_l \rightharpoonup f_j$ weakly in $L^{p_j}(\Omega)$ for some $f_j \in L^{p_j}(\Omega)$. By definition of the weak derivative, for all $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega f_j \varphi \, dx \leftarrow \int_\Omega (\partial_j u_l)\varphi \, dx = -\int_\Omega u_l \partial_j \varphi \, dx \to -\int_\Omega u \partial_j \varphi \, dx = \int_\Omega (\partial_j u)\varphi \, dx.$$

Hence, we have $\partial_j u_l \rightharpoonup \partial_j u$ weakly in $L^{p_j}(\Omega)$, so we know that $\sum_{j=1}^{N} A_j(x, u_l, \nabla u)\partial_j u$ converges in $L^1(\Omega)$ to $\sum_{j=1}^{N} A_j(x, u, \nabla u)\partial_j u$, as we wanted. Then by taking $\liminf_{l\to\infty}$ with (4.2.20), we obtain

$$\liminf_{l\to\infty} \sum_{j=1}^{N} \int_\Omega A_j(x, u_l, \nabla u_l)\partial_j u_l \, dx \geq \sum_{j=1}^{N} \int_\Omega \varphi_j \partial_j u \, dx.$$

This lower bound on the lim inf coupled with the upper bound on the lim sup in (4.2.19) proves the convergence in (4.2.17). This finishes Step 1.

Using the convergences in (4.2.16) and (4.2.17), we get

$$\lim_{l\to\infty} \langle (\mathcal{A} + \zeta_\varepsilon)u_l, u_l\rangle = \sum_{j=1}^{N} \int_\Omega \varphi_j \partial_j u \, dx + \int_\Omega \eta u \, dx = \langle \chi, u\rangle.$$

which is exactly what we needed for the convergent requirement in (4.2.14).

Step 2. In this step, we prove that χ must be $(\mathcal{A} + \zeta_\varepsilon)u \in W^{-1,\vec{p}}(\Omega)$. Since we proved that $A_j(x, u_l, \nabla u) \to A_j(x, u, \nabla u)$ in $L^{p_j'}(\Omega)$, we have the limit

$$\lim_{l\to\infty} \sum_{j=1}^{N} \int_\Omega \big(A_j(x, u_l, \nabla u_l) - A_j(x, u_l, \nabla u)\big)\big(\partial_j u_l - \partial_j u\big) = 0.$$

By Proposition 4.2.2, this shows that, up to subsequence, we have $\nabla u_l \to u$ and $u_l \to u$ a.e. in Ω. Hence, we obtain $\hat{A}_j(u_l) \to \hat{A}_j(u)$ a.e. in Ω, by the Carathéodory property of A_j. Since pointwise limits and weak limits coincide by Proposition 1.3.2, we must have $\hat{A}_j(u_l) \rightharpoonup \hat{A}_j(u)$ weakly in $L^{p_j'}(\Omega)$ (recall that we labelled the weak limit of $\hat{A}_j(u_l)$ as φ_j previously). Furthermore, we have $\hat{\Phi}_\varepsilon(u_l) \to \Phi_\varepsilon(u)$ in $L^{p'}(\Omega)$. This follows immediately from the Dominated Convergence Theorem, since we have $\hat{\Phi}(u_l) \to \hat{\Phi}(u)$ a.e. on Ω and $\Phi_\varepsilon \in L^\infty(\Omega)$.

Combining the convergence $\hat{\Phi}(u_l) \to \hat{\Phi}(u)$ in $L^{p'}(\Omega)$ and $\hat{A}_j(u_l) \to \hat{A}_j(u)$ in $L^{p_j'}(\Omega)$, for arbitrary $v \in W_0^{1,\vec{p}}(\Omega)$, we have

$$\langle \chi, v\rangle = \lim_{l\to\infty} \langle \big(\mathcal{A} + \zeta_\varepsilon\big)u_l, v\rangle = \sum_{j=1}^{N} \int_\Omega A_j(x, u, \nabla u)\partial_j v \, dx + \int_\Omega \Phi_\varepsilon(x, u, \nabla u) v \, dx.$$

That is, $\chi = \big(\mathcal{A} + \zeta_\varepsilon\big)u$. By Proposition 2.2.5, this proves that $\mathcal{A} + \zeta_\varepsilon$ is a pseudomonotone operator, as we wanted. □

Lemma 4.2.3. Suppose $\mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is an operator satisfying (P_1) and (P_2). Then the operator $-\mathfrak{B}$ is bounded and pseudomonotone.

Bounded: It suffices to show that $\mathfrak{B}$ is bounded, since this implies $-\mathfrak{B}$ is bounded. By Property (P_1), for any $u, v \in W_0^{1,\vec{p}}(\Omega)$, we have

$$|\langle \mathfrak{B}u, v\rangle| \leq \mathfrak{C}\left(1 + \|u\|^{\mathfrak{b}}_{W_0^{1,\vec{p}}(\Omega)}\right)\left(\mathfrak{a}_0\|v\|_{W_0^{1,\vec{p}}(\Omega)} + \|v\|_{L^s(\Omega)}\right).$$

By the continuous embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^s(\Omega)$, this can be simplified to

$$|\langle \mathfrak{B}u, v\rangle| \leq \mathfrak{C}_1\left(1 + \|u\|^{\mathfrak{b}}_{W_0^{1,\vec{p}}(\Omega)}\right)\|v\|_{W_0^{1,\vec{p}}(\Omega)} \tag{4.2.22}$$

for some constant $\mathfrak{C}_1 > 0$. We note that this will hold whether $\mathfrak{a}_0 = 0$ or $\mathfrak{a}_0 > 0$. Hence, assuming that u and v are both bounded in $W_0^{1,\vec{p}}(\Omega)$, we see that $\mathfrak{B}$ is bounded in the operator norm.

Pseudomonotone: To show $-\mathfrak{B}$ is pseudomonotone, we prove that $\mathfrak{B}$ is strongly continuous, which would imply that $-\mathfrak{B}$ is strongly continuous, and apply Proposition 2.2.1. Let $(u_l)_{l\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ with $u_l \rightharpoonup u$ weakly in $W_0^{1,\vec{p}}(\Omega)$. We prove $\mathfrak{B}u_l \to \mathfrak{B}u$ strongly in $W^{-1,\vec{p}'}(\Omega)$. To that end, suppose by contradiction that there exists a constant $\delta > 0$ such that for a subsequence $\{u_{l'}\}$,

$$\|\mathfrak{B}u_{l'} - \mathfrak{B}u\|_{W^{-1,\vec{p}'}(\Omega)} = \sup_{\|v\|_{W_0^{1,\vec{p}}(\Omega)}\leq 1} \langle \mathfrak{B}u_{l'} - Bu, v\rangle > \delta \text{ for all } l \in \mathbb{N}. \tag{4.2.23}$$

Then there exists a sequence $\{v_n\}_{n\in\mathbb{N}}$ in $W_0^{1,\vec{p}}(\Omega)$ such that this supremum is attained. We can take a subsequence, relabelled as $\{v_{l'}\}_{l'}$, such that $\|v_{l'}\|_l \leq 1$ and $\langle \mathfrak{B}u_{l'} - Bu, v_{l'}\rangle > \delta$. Since $(v_{l'})$ is bounded in $W_0^{1,\vec{p}}(\Omega)$, we can extract a further subsequence (relabelled $\{v_{l'}\}$), which is weakly convergent to some $w \in W_0^{1,\vec{p}}(\Omega)$. Using Property (P_2), we see that

$$\langle \mathfrak{B}u_{l'} - \mathfrak{B}u, v_{l'}\rangle = \langle \mathfrak{B}u_{l'}, v_{l'}\rangle - \langle \mathfrak{B}u, v_{l'}\rangle \to \langle \mathfrak{B}u, w\rangle - \langle \mathfrak{B}u, w\rangle = 0 \text{ as } l \to \infty.$$

This is a contradiction with (4.2.23), so $\mathfrak{B}$ is strongly continuous, which, by Proposition 2.2.1, implies $\mathfrak{B}$ is pseudomonotone, as required. □

Lemma 4.2.4. The operator $\mathcal{A} + \zeta_\varepsilon - \mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is a coercive, pseudomonotone operator which maps bounded sets to bounded sets.

Proof. *Bounded and Pseudomonotone*: Both properties follow from Lemmas 4.2.1, 4.2.2 and 4.2.3, since the sum of bounded operators is bounded, and the sum of pseudomonotone operators is pseudomonotone (see Proposition 2.2.2).

Coercive: Recalling the definition of coercivity, we want to show the following:

$$\frac{\langle \mathcal{A}u + \zeta_\varepsilon u - \mathfrak{B}u, u\rangle}{\|u\|_{W_0^{1,\vec{p}}(\Omega)}} \to \infty \text{ as } \|u\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty. \tag{4.2.24}$$

First, let $\mathfrak{a}_0 > 0$. Suppose by contradiction that there exists some sequence $(u_n)_{n\in\mathbb{N}}$ in $W_0^{1,\vec{p}}(\Omega)$ with $\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty$ as $n \to \infty$ and some $M > 0$ such that

$$\frac{\langle \mathcal{A}u_n + \zeta_\varepsilon u_n - \mathfrak{B}u_n, u_n\rangle}{\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}} \leq M$$

for all $n \in \mathbb{N}$. This part of the proof follows closely with the coercivity proof of Lemma 4.2.1. Since by the sign condition $\langle \zeta_\varepsilon u, u\rangle \geq 0$ for all $\varepsilon > 0$ and $u \in W_0^{1,\vec{p}}(\Omega)$, we have

$$\nu_0 \sum_{j=1}^N \|u_n\|_{L^{p_j}(\Omega)}^{p_j} - \mathfrak{C}_1\left(\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} + \|u_n\|_{W_0^{1,\vec{p}}(\Omega)}^{\mathfrak{b}+1}\right) \leq \langle \mathcal{A}u_n - \mathfrak{B}u_n, u_n\rangle \leq M\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} \tag{4.2.25}$$

where we have also applied the growth condition (4.1.3) of A_j and the upper bound (4.2.22) for the left inequality. By rearranging, we get

$$\nu_0 \sum_{j=1}^N \|u_n\|_{L^{p_j}(\Omega)}^{p_j} - \mathfrak{C}_2\left(\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} + \|u_n\|_{W_0^{1,\vec{p}}(\Omega)}^{\mathfrak{b}+1}\right) \leq 0$$

for some constant $\mathfrak{C}_2 > \mathfrak{C}_1 + M$. We have the following inequality by expanding the definition of the norm in $W_0^{1,\vec{p}}(\Omega)$ and using the trick in (4.2.4) to bring the power $\mathfrak{b}+1$ down:

$$\nu_0 \sum_{j=1}^N \|u_n\|_{L^{p_j}(\Omega)}^{p_j} - \mathfrak{C}_3\left(\sum_{j=1}^N \|\partial_j u_n\|_{L^{p_j}(\Omega)} + \sum_{j=1}^N \|\partial_j u_n\|_{L^{p_j}(\Omega)}^{\mathfrak{b}+1}\right) \leq 0$$

for some constant $\mathfrak{C}_3 > 0$ depending on $\mathfrak{b}+1$ and $\mathfrak{C}_2$. Since $\mathfrak{b}+1 < p_j$ for all $1 \leq j \leq N$, we can follow the proof of coercivity in Lemma 4.2.1 and apply the Generalised Young's Inequality on each $1\|\partial_j u_n\|_{L^{p_j}(\Omega)}$ and $\|\partial_j u_n\|_{L^{p_j}(\Omega)}^{\mathfrak{b}+1}$ term. This gives us, for an arbitrarily chosen $\delta > 0$ and C_δ large depending on δ, the inequality $(\nu_0 - \mathfrak{C}_3\delta)\sum_{j=1}^N \|u_n\|_{L^{p_j}(\Omega)}^{p_j} \leq C_\delta$. By picking $\delta < \nu_0/\mathfrak{C}_3$, we get a contradiction, since we assumed $\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty$ as $n \to \infty$. Hence, we conclude (4.2.24), as required.

Now suppose $\mathfrak{a}_0 = 0$. Since we do not have $b < p_1 - 1$, the same strategy does not work. Instead, we bound $|\langle \mathfrak{B}u, u\rangle|$. (4.2.22) along with the result of Remark 3.3.1 gives us the inequalities

$$\begin{aligned}
|\langle Bu, u\rangle| &\leq \mathfrak{C}_1\left(\|u\|_{W^{1,\vec{p}}(\Omega)} + \|u\|_{W_0^{1,\vec{p}}(\Omega)}^{\mathfrak{b}}\|u\|_{W^{1,\vec{p}}(\Omega)}\right)\\
&\leq C\left(\|u\|_{W^{1,\vec{p}}(\Omega)} + \|u\|_{W_0^{1,\vec{p}}(\Omega)}^{\mathfrak{b}}\Pi_{k=1}^N \|\partial_k u\|_{L^{p_k}(\Omega)}^{1/N}\right) \quad (*)\\
&\leq C\left(\|u\|_{W^{1,\vec{p}}(\Omega)} + \sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}^{\mathfrak{b}}\Pi_{k=1}^N \|\partial_k u\|_{L^{p_k}(\Omega)}^{1/N}\right) \quad (**)
\end{aligned}$$

where in $(*)$ we used Corollary 3.3.1 and in $(**)$ we used the trick from (4.2.4) to bring down the power $\mathfrak{b}$. Note that we are successively renaming the constant as some large $C > 0$ to avoid cumbersome notation. We now apply the Generalised Young's Inequality to the product to get, for arbitrary $\delta > 0$ and the constant $C_{\delta_j} > 0$,

$$\begin{aligned}
\|\partial_j u\|_{L^{p_j}(\Omega)}^{\mathfrak{b}}\Pi_{k=1}^N \|\partial_k u\|_{L^{p_k}(\Omega)}^{1/N} &= \|\partial_1 u\|_{L^{p_1}(\Omega)}^{1/N} \cdots \|\partial_j u\|_{L^{p_j}(\Omega)}^{\mathfrak{b}+1/N} \cdots \|\partial_N u\|_{L^{p_N}(\Omega)}^{1/N}\\
&\leq \delta \sum_{k\neq j} \|\partial_k u\|_{L^{p_k}(\Omega)}^{p_k} + C_{\delta_j}\|\partial_j u\|_{L^{p_j}(\Omega)}^{\alpha_j}.
\end{aligned} \tag{4.2.26}$$

To get to this upper bound, we needed to choose each R_k (as defined in Theorem 1.5.1) to be $R_k = Np_k$ for $k \neq j$. Then α_j (similarly, see Theorem 1.5.1) is given by

$$\alpha_j = \left(\mathfrak{b} + \frac{1}{N}\right)\left(1 - \sum_{k\neq j}\frac{1}{Np_k}\right)^{-1}$$

which gives $\alpha_j = p' p_j(N\mathfrak{b}+1)/(Np_j + p')$. In particular, it is not hard to see that $\alpha_j < p_j$ for all $1 \le j \le N$. Substituting inequality (4.2.26) into (∗∗), we obtain

$$|\langle Bu, u\rangle| \le C\left(\|u\|_{W_0^{1,\vec{p}}(\Omega)} + \sum_{j=1}^{N}\left(\delta\sum_{k\neq j}\|\partial_k u\|^{p_k}_{L^{p_k}(\Omega)} + C_{\delta_j}\|\partial_j u\|^{\alpha_j}_{L^{p_j}(\Omega)}\right)\right). \tag{4.2.27}$$

Then substituting this inequality into (4.2.24), we get a lower bound for $\langle \mathcal{A}u + \zeta_\varepsilon u - \mathfrak{B}u, u\rangle$ of the form

$$\nu_0\sum_{j=1}^{N}\|\partial_j u\|^{p_j}_{L^{p_j}(\Omega)} - C\left(\|u\|_{W_0^{1,\vec{p}}(\Omega)} + \sum_{j=1}^{N}\left(\delta\sum_{k\neq j}\|\partial_k u\|^{p_k}_{L^{p_k}(\Omega)} + C_{\delta_j}\|\partial_j u\|^{\alpha_j}_{L^{p_j}(\Omega)}\right)\right). \tag{4.2.28}$$

The fact that $\alpha_j < p_j$ allows us to apply the Generalised Young's Inequality on the terms $C_{\delta_j}\|\partial_j u\|^{\alpha_j}_{L^{p_j}(\Omega)}$ and $\|u\|_{W_0^{1,\vec{p}}(\Omega)}$ for each $1 \le j \le N$, to obtain, for $\tilde{\delta} > 0$ arbitrarily chosen,

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} + \sum_{j=1}^{N} C_{\delta_j}\|\partial_j u\|^{\alpha_j}_{L^{p_j}(\Omega)} \le C_{\tilde{\delta},\delta} + \tilde{\delta}\sum_{j=1}^{N}\|\partial_j u\|^{p_j}_{L^{p_j}(\Omega)}$$

for $C_{\tilde{\delta},\delta} > 0$ depending only on $\tilde{\delta}$ and C_{δ_j}. Returning to (4.2.28), it suffices to choose δ and $\tilde{\delta}$ small enough so that the coefficient of $\sum_{j=1}^{N}\|\partial_j u\|^{p_j}_{L^{p_j}(\Omega)}$ is positive, whence the coercivity of $\mathcal{A} + \zeta_\varepsilon - \mathfrak{B}$ follows, as required. This finishes the proof of Lemma 4.2.4. □

Proof of Theorem 4.2.1: Fix $\varepsilon > 0$. We have shown that the operator $\mathcal{A} + \zeta_\varepsilon - \mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is a bounded, coercive and pseudomonotone operator on the real, separable, reflexive Banach space $W_0^{1,\vec{p}}(\Omega)$. By Theorem 2.3.2, there exists $u_\varepsilon \in W_0^{1,\vec{p}}(\Omega)$ such that $(\mathcal{A} + \zeta_\varepsilon - \mathfrak{B})u = 0_{W^{-1,\vec{p}'}(\Omega)}$. By the way we defined the operators $\mathcal{A}$ and ζ_ε, this is exactly a weak solution $u_\varepsilon \in W_0^{1,\vec{p}}(\Omega)$ of the bounded approximate PDE (4.2.2), as we wanted.

4.3. Approximate Anisotropic Equation with Zero Input Data

4.3.1. A priori estimates. The aim of the next section is to obtain a weak solution of the original PDE as in Definition 4.1.1 with $f = 0$, from the set of solutions $\{u_\varepsilon\}_\varepsilon$ to the approximated PDE. To do so, we must find a priori estimates on the set $(u_\varepsilon)_\varepsilon$ in $W_0^{1,\vec{p}}(\Omega)$. The importance of this step cannot be understated. Indeed, by showing that the sequence of solutions $\{u_\varepsilon\}_\varepsilon$ is bounded, the results of Chapter 1 on reflexive spaces and weak compactness of bounded sequences can be applied to extract information out of the set of approximate solutions $\{u_\varepsilon\}_\varepsilon$.

Lemma 4.3.1. Let $\{u_\varepsilon\}_\varepsilon$ be the set of solutions of the bounded approximate problem (4.2.2), indexed by the constant $\varepsilon > 0$. The following statements hold:

(i) There exists a constant $C > 0$, independent of ε, such that for all $\varepsilon > 0$,

$$\|u_\varepsilon\|_{W_0^{1,\vec{p}}(\Omega)} + \int_\Omega \hat{\Phi}_\varepsilon(u_\varepsilon)u_\varepsilon\,dx \le C. \tag{4.3.1}$$

In particular, by the sign condition on Φ, the set $\{u_\varepsilon\}_\varepsilon$ is uniformly bounded with respect to ε in the $W_0^{1,\vec{p}}(\Omega)$ norm, and $\{\hat{\Phi}_\varepsilon(u_\varepsilon)u_\varepsilon\}_\varepsilon$ is uniformly bounded in $L^1(\Omega)$.

(ii) There exists $U \in W_0^{1,\vec{p}}(\Omega)$ such that, up to a subsequence of $\{u_\varepsilon\}_\varepsilon$, the following convergences hold:

$$u_\varepsilon \rightharpoonup U \text{ weakly in } W_0^{1,\vec{p}}(\Omega) \quad \text{and} \quad u_\varepsilon \to U \text{ a.e. in } \Omega \quad \text{as } \varepsilon \to 0 \tag{4.3.2}$$

where by $\varepsilon \to 0$ we mean a sequence of real numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$ as $n \to \infty$.

Proof. (i): Since u_ε is a weak solution of (4.1.1), it satisfies the integral equation (4.2.3). Combined with the inequality (4.2.22) and the coercivity condition (4.1.2), we have

$$\begin{aligned} \nu_0 \sum_{j=1}^{N} \|\partial_j u_\varepsilon\|_{L^{p_j}(\Omega)}^{p_j} + \int_\Omega \hat{\Phi}_\varepsilon(u_\varepsilon)u_\varepsilon \, dx &\le \sum_{j=1}^{N} \int_\Omega \hat{A}_j(u_\varepsilon)\partial_j u_\varepsilon \, dx + \int_\Omega \hat{\Phi}_\varepsilon(u_\varepsilon)u_\varepsilon \, dx \\ &= \langle \mathfrak{B}u_\varepsilon, u_\varepsilon\rangle \le \mathfrak{C}_1 \left(\|u_\varepsilon\|_{W_0^{1,\vec{p}}(\Omega)} + \|u_\varepsilon\|_{W_0^{1,\vec{p}}(\Omega)}^{\mathfrak{b}+1} \right) \end{aligned} \tag{4.3.3}$$

Again, there will be two cases, with $\mathfrak{a}_0 = 0$ (where we need to prove $|\langle \mathfrak{B}u_\varepsilon, u_\varepsilon\rangle|$ is bounded) and $\mathfrak{a}_0 > 0$ (where we use the fact that $\mathfrak{b} + 1 < p_j$ for all $1 \le j \le N$) with Young's Inequality). By following the proofs of the coercivity of the operator $\mathcal{A} + \zeta_\varepsilon - \mathfrak{B}$ in Lemmas 4.2.1 and 4.2.4, we have that

$$\nu_0 \sum_{j=1}^{N} \|\partial_j u_\varepsilon\|_{L^{p_j}(\Omega)}^{p_j} \, dx + \int_\Omega \hat{\Phi}_\varepsilon(u_\varepsilon)u_\varepsilon \, dx \text{ is bounded from above by a constant.}$$

This implies (4.3.1), since $\sum_{j=1}^{N} \|\partial_j u_\varepsilon\|_{L^{p_j}(\Omega)}^{p_j}$ being bounded implies $\sum_{j=1}^{N} \|\partial_j u_\varepsilon\|_{L^{p_j}(\Omega)}$ is bounded. This finishes (i).

Proof of (ii): The existence of a subsequence on which $u_\varepsilon \rightharpoonup U$ weakly in $W_0^{1,\vec{p}}(\Omega)$ is guaranteed by the reflexivity of $W_0^{1,\vec{p}}(\Omega)$. We recall Proposition 1.2.4: On this subsequence, the compactness of the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^s(\Omega)$ for $s \in [1, p^*)$ means that $u_\varepsilon \to U$ strongly in $L^s(\Omega)$. Proposition 1.2.3 (iv) then implies there exists a subsequence of $(u_\varepsilon)_\varepsilon$ which converges a.e. in Ω to U, whence the result follows. □

Remark 4.3.1. From now on, whenever we refer to the "sequence" $(u_\varepsilon)_\varepsilon$ and the limit $\varepsilon \to 0$, we refer to a sequence $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ from the set $\{u_\varepsilon\}_\varepsilon$ satisfying the convergences in (4.3.2) with $\varepsilon_n \to 0$ as $n \to \infty$. Furthermore, it is standard practice that whenever we extract a further subsequence, we relabel that subsequence as $(u_\varepsilon)_\varepsilon$ again, often without explicitly referencing this fact, to avoid cumbersome notation.

In the ideal scenario, we would be able to get the strong convergence of $u_\varepsilon \to U$ in $W_0^{1,\vec{p}}(\Omega)$. If this holds, one particular consequence is that, up to subsequence, we would have $\nabla u_\varepsilon \to \nabla U$ a.e. in Ω, and gives us more options to find uniformly integrable upper bounds to apply Vitali's Theorem on. Although this is possible to prove with effort (see [3, Section 5.1]), we do not take this route. The reason is that in order to introduce the arbitrary input data $f \in L^1(\Omega)$, we need to perform a second approximation, for which the same method showing $u_\varepsilon \to U$ strongly in $W_0^{1,\vec{p}}(\Omega)$ does not apply. In order to unify the two approximations (see comments leading up to Corollary 4.4.1 and in the appendix with Remark A.1.1), the following results aim to prove that we can get "strong convergence up to truncations" in $W_0^{1,\vec{p}}(\Omega)$ and still obtain a.e. convergence of ∇u_ε in Ω.

Proposition 4.3.1. Suppose $(u_\varepsilon)_\varepsilon$ is a sequence in $W_0^{1,\vec{p}}(\Omega)$ weakly converging to $u \in W_0^{1,\vec{p}}(\Omega)$. For fixed $k > 0$, define

$$D_{\varepsilon,k}(x) = \sum_{j=1}^{N} \left[A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon)) - A_j(x, u_\varepsilon, \nabla T_k(u))\right] (\partial_j T_k(u_\varepsilon) - \partial_j T_k(u)).$$

If $D_{\varepsilon,k} \to 0$ in $L^1(\Omega)$ as $\varepsilon \to 0$, then $\nabla T_k(u_\varepsilon) \to \nabla T_k(u)$ a.e. in Ω as $n \to \infty$.

Proof. Repeat the proof of Proposition 4.2.2, replacing all instances of ∇u_ε and ∇u with $\nabla T_k(u_\varepsilon)$ and $\nabla T_k(u)$ respectively. □

We wish to apply this proposition with $(u_\varepsilon)_\varepsilon$ being the set of solutions to the approximate PDE. To do so, we need to show that $D_{\varepsilon,k}$ converges to 0 in $L^1(\Omega)$, which is the subject of the next lemma.

Lemma 4.3.2. Let $(u_\varepsilon)_\varepsilon$ be the sequence of weak solutions of the bounded approximate problem (4.2.2). Whenever $k \geq 1$ is an integer, there exists a subsequence of $\{u_\varepsilon\}_\varepsilon$ (also denoted $\{u_\varepsilon\}_\varepsilon$), such that

$$D_{\varepsilon,k} \to 0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.3.4}$$

Proof. We first note that the monotonicity condition (4.1.2) on A_j forces $D_{\varepsilon,k} \geq 0$ a.e. on Ω and $\liminf_{\varepsilon\to 0} \int_\Omega D_{\varepsilon,k} dx \geq 0$. Hence, to prove that $D_{\varepsilon,k} \to 0$ in $L^1(\Omega)$, it suffices to show that $\limsup_{\varepsilon\to 0} \int_\Omega D_{\varepsilon,k}\, dx = 0$. Notice that we can write

$$\int_\Omega D_{\varepsilon,k}\, dx = \int_\omega D_{\varepsilon,k} \chi_{\{|u_\varepsilon| \geq k\}}\, dx + \int_\Omega D_{\varepsilon,k} \chi_{\{|u_\varepsilon| < k\}}\, dx. \tag{4.3.5}$$

We find bounds on each separate integral in (4.3.5).[2]

Step 1: In the first part, we show

$$\int_\omega D_{\varepsilon,k} \chi_{\{u_\varepsilon \geq k\}}\, dx \to 0 \text{ as } \varepsilon \to 0. \tag{4.3.6}$$

Indeed, by definition,

$$\int_\omega D_{\varepsilon,k} \chi_{\{u_\varepsilon \geq k\}}\, dx = \int_\Omega \sum_{j=1}^{N} \left[A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon)) - A_j(x, u_\varepsilon, \nabla T_k(U))\right] (\partial_j T_k(u_\varepsilon) - \partial_j T_k(U)) \chi_{\{u_\varepsilon \geq k\}}\, dx.$$

Fix some j with $1 \leq j \leq N$. Clearly, we have $\partial_j T_k(u_\varepsilon) \chi_{\{|u_\varepsilon| \geq k\}} = 0$ (see Theorem 3.2.2). Hence, we can simplify:

$$(\partial_j T_k(u_\varepsilon) - \partial_j T_k(U)))\chi_{\{|u_\varepsilon| \geq k\}} = -\partial_j (T_k(U))\chi_{\{|u_\varepsilon| \geq k\}} = -\partial_j U \chi_{\{|u_\varepsilon| \geq k\} \cap \{|U| < k\}}.$$

From Lemma 4.3.1, there exists a subsequence of $(u_\varepsilon)_\varepsilon$ such that $u_\varepsilon \to U$ a.e. in Ω as $\varepsilon \to 0$. It follows then that on this subsequence,

$$\chi_{\{|u_\varepsilon| \geq k\} \cap \{|U| < k\}} \to 0 \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{4.3.7}$$

[2]This 'divide and conquer' strategy is a useful tool we will repeatedly apply: see for example, the proof of Proposition 4.3.2, Lemma 4.4.2 (i), and Lemma 4.4.4.

Furthermore, $|\partial_j U \chi_{\{|u_\varepsilon|\geq k\}\cap\{|U|<k\}}|^{p_j} \leq |\partial_j U|^{p_j} \in L^1(\Omega)$. Hence, by the Dominated Convergence Theorem, we get

$$\int_\Omega |\partial_j U \chi_{\{|u_\varepsilon|\geq k\}\cap\{|U|<k\}}|^{p_j}\, dx \to 0 \text{ as } \varepsilon \to 0. \tag{4.3.8}$$

That is, $\partial_j U \chi_{\{|u_\varepsilon|\geq k\}\cap\{|U|<k\}}$ converges to 0 in $L^{p_j}(\Omega)$. Now, the same calculation as in Lemma 4.2.1 showing well-definedness of $\hat{A}_j$ and $\hat{\Phi}_\varepsilon$ shows that both

$$\{A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon))\}_\varepsilon \quad \text{and} \quad \{A_j(x, u_\varepsilon, \nabla T_k(U))\}_\varepsilon \tag{4.3.9}$$

are bounded uniformly in $L^{p_j'}(\Omega)$, so viewing (4.3.9) as sequences in the sense of Remark 4.3.1, they must, up to subsequence, converge weakly in $L^{p_j'}(\Omega)$. We conclude that

$$\int_\Omega D_{\varepsilon,k}\chi_{\{|u_\varepsilon|\geq k\}}\, dx \to 0 \text{ as } \varepsilon \to 0,$$

since the integrand is the product of a weakly convergent sequence in $L^{p_j'}(\Omega)$ and a strongly convergent sequence in $L^{p_j}(\Omega)$ converging to 0 (see Proposition 1.2.1 (vii)). This concludes Step 1. We now consider the term $\int_\Omega D_{\varepsilon,k}\chi_{\{|u|<k\}}\, dx$ in (4.3.5). By combining the convergence from Step 1 and Proposition A.2.1, we can write

$$\limsup_{\varepsilon\to0}\int_\Omega D_{\varepsilon,k}\, dx = \lim_{\varepsilon\to0}\int_\Omega D_{\varepsilon,k}\chi_{\{|u_\varepsilon|\geq k\}}\, dx + \limsup_{\varepsilon\to0}\int_\Omega D_{\varepsilon,k}\chi_{\{|u_\varepsilon|<k\}}\, dx = \limsup_{\varepsilon\to0}\int_\Omega D_{\varepsilon,k}\chi_{\{|u_\varepsilon|<k\}}\, dx.$$

Hence, it suffices to show the following to prove Lemma 4.3.2:

$$\limsup_{\varepsilon\to0}\int_\Omega D_{\varepsilon,k}\chi_{\{u_\varepsilon<k\}}\, dx \leq 0, \tag{4.3.10}$$

which is the aim of *Step 2*. For ease of notation, write $z_{\varepsilon,k} := T_k(u_\varepsilon) - T_k(U) \in W_0^{1,\vec{p}}(\Omega)$ and define, for some $\lambda \in \mathbb{R}$, the function $\varphi_\lambda : \mathbb{R} \to \mathbb{R}$ by $\varphi_\lambda(t) = t\exp(\lambda t^2)$ for $t \in \mathbb{R}$. Notice that $\varphi_\lambda'(t) = (1 + 2\lambda t^2)\exp(\lambda t^2)$, which implies that

$$\varphi_\lambda'(t) - \frac{\zeta(k)}{\nu_0}|\varphi_\lambda(t)| = \exp(\lambda t^2)\left(1 - \frac{\zeta(k)}{\nu_0}|t| + 2\lambda t^2\right). \tag{4.3.11}$$

We can bound (4.3.11) from below by $1/2$ with an appropriately chosen λ. Indeed, since $\exp(\lambda t^2) \geq 1$, it suffices to find λ such that

$$1 - \frac{\zeta(k)}{\nu_0}|t| + 2\lambda t^2 > \frac{1}{2} \implies 2\lambda t^2 - \frac{\zeta(k)}{\nu_0}|t| + \frac{1}{2} > 0 \text{ for all } t \in \mathbb{R}.$$

By noting that the discriminant $\Delta = \frac{\zeta^2(k)}{4\nu_0^2} - \lambda$ is invariant with respect to the sign of t, it suffices to pick $\lambda = \lambda(k)$ large so that $4\nu_0^2\lambda > \zeta^2(k)$ to guarantee this inequality. Now, for $v \in W_0^{1,\vec{p}}(\Omega)$, define

$$\mathcal{E}_{\varepsilon,k}(v) = \sum_{j=1}^N \int_\Omega A_j(x, u_\varepsilon, \nabla v)(\partial_j z_{\varepsilon,k})\left(\varphi_\lambda'(z_{\varepsilon,k}) - \frac{\zeta(k)}{\nu_0}|\varphi_\lambda(z_{\varepsilon,k})|\right)\chi_{\{|u_\varepsilon|<k\}}\, dx. \tag{4.3.12}$$

Then we can expand $\mathcal{E}_{\varepsilon,k}(T_k(u_\varepsilon)) - \mathcal{E}_{\varepsilon,k}(T_k(U))$ to get

$$\begin{aligned} &\mathcal{E}_{\varepsilon,k}(T_k(u_\varepsilon)) - \mathcal{E}_{\varepsilon,k}(T_k(U)) \\ &= \int_\Omega D_{\varepsilon,k}\left(\varphi_\lambda'(z_{\varepsilon,k}) - \frac{\zeta(k)}{\nu_0}|\varphi_\lambda(z_{\varepsilon,k})|\right)\chi_{\{|u_\varepsilon|<k\}}\, dx > \frac{1}{2}\int_\Omega D_{\varepsilon,k}\chi_{\{|u_\varepsilon|<k\}}\, dx \geq 0 \end{aligned} \tag{4.3.13}$$

where the last inequality also uses the fact that $D_{\varepsilon,k} \geq 0$ a.e. in Ω. To finish the proof of Lemma 4.3.2, in light of (4.3.10) and Proposition A.2.1, it suffices to show the following:

Proposition 4.3.2. For fixed positive integer $k > 0$, we have the following:

$$\lim_{\varepsilon\to 0} \mathcal{E}_{\varepsilon,k}(T_k(U)) = 0 \quad \text{and} \quad \limsup_{\varepsilon\to 0} \mathcal{E}_{\varepsilon,k}(T_k(u_\varepsilon)) \leq 0. \tag{4.3.14}$$

Proof. For the inequality in (4.3.14), the idea is to bound the expression $|\mathcal{E}_{\varepsilon,k}(T_k(U))|$, which we expand into

$$\left|\sum_{j=1}^{N} \int_\Omega A_j(x, u_\varepsilon, \nabla T_k(U))(\partial_j z_{\varepsilon,k}) \left(\varphi'_\lambda(z_{\varepsilon,k}) - \frac{\zeta(k)}{\nu_0}|\varphi_\lambda(z_{\varepsilon,k})|\right) \chi_{\{|u_\varepsilon|<k\}}\, dx\right|, \tag{4.3.15}$$

from above with a function which converges to 0 as $\varepsilon \to 0$ using weak convergence properties of $\partial_j z_{\varepsilon,k}$ and the boundedness of other terms. Indeed, since φ_λ and φ'_λ are continuous functions on $\mathbb{R}$ and we have $|z_{\varepsilon,k}| \leq 2k$ by the triangle inequality, there exists a constant $C_k > 0$ such that

$$\left|\varphi'_\lambda(z_{\varepsilon,k}) - \frac{\zeta(k)}{\nu_0}|\varphi_\lambda(z_{\varepsilon,k})|\right| \leq C_k.$$

Furthermore, the calculations of Lemma 4.2.1 (which are a consequence of the growth conditions) shows that on the set $\{|u_\varepsilon| < k\}$, for all $1 \leq j \leq N$ we can bound $A_j(x, u_\varepsilon, \nabla T_k(U))$ by a function $F_j \in L^{p'_j}(\Omega)$ for all $1 \leq j \leq N$ by Proposition 1.3.1. It remains to consider the behaviour of $\partial_j z_{\varepsilon,k}$ on the set $\{|u_\varepsilon| < k\}$. We notice that

$$\begin{aligned} \partial_j z_{\varepsilon,k} = \chi_{\{|u_\varepsilon|<k\}}\partial_j z_{\varepsilon,k} + \chi_{\{|u_\varepsilon|\geq k\}}\partial_j z_{\varepsilon,k} &= \chi_{\{|u_\varepsilon|<k\}}\partial_j z_{\varepsilon,k} - \chi_{\{|u_\varepsilon|\geq k\}\cap\{|U|<k\}}\partial_j U \\ \implies \chi_{\{|u_\varepsilon|<k\}}\partial_j z_{\varepsilon,k} &= \partial_j z_{\varepsilon,k} + \chi_{\{|u_\varepsilon|\geq k\}\cap\{U<k\}}\partial_j U. \end{aligned}$$

Recall from (4.3.8) that $\partial_j U \chi_{\{|u_\varepsilon|\geq k\}\cap\{|U|<k\}}$ strongly converges to 0 in $L^{p_j}(\Omega)$. Furthermore, from the convergence $z_{\varepsilon,k} \rightharpoonup 0$ weakly in $W_0^{1,\vec{p}}(\Omega)$, we get that $\partial_j z_{\varepsilon,k} \rightharpoonup 0$ weakly in $L^{p_j}(\Omega)$. To see this, note that for all $f \in L^{p'_j}(\Omega)$, there exists (by density) a sequence $f_n \in C_c^\infty(\Omega)$ converging to f in $L^{p'_j}(\Omega)$. It follows then, for fixed $\varepsilon > 0$ and $k \geq 1$, that we get

$$\begin{aligned} \left|\int_\Omega f(\partial_j z_{\varepsilon,k})\, dx\right| &\leq \int_\Omega |f - f_n||\partial_j z_{\varepsilon,k}|\, dx + \left|\int_\Omega f_n(\partial_j z_{\varepsilon,k})\, dx\right| \\ &\leq \|f - f_n\|_{L^{p'_j}(\Omega)}\|\partial_j z_{\varepsilon,k}\|_{L^{p_j}(\Omega)}\, dx + \left|\int_\Omega (\partial_j f_n) z_{\varepsilon,k}\, dx\right|. \end{aligned}$$

This can be made arbitrarily small as long as $n \in \mathbb{N}$ is picked large enough. Indeed, the first term in the upper bound vanishes since $f_n \to f$ in $L^{p'_j}(\Omega)$ and $z_{\varepsilon,k}$ is bounded in $W_0^{1,\vec{p}}(\Omega)$, and the second integral term vanishes since $z_{\varepsilon,k} \rightharpoonup 0$ weakly in $W_0^{1,\vec{p}}(\Omega)$: the map $z \mapsto \int_\Omega (\partial_j u_n) z\, dx$ is a well-defined, linear functional on $W_0^{1,\vec{p}}(\Omega)$, since $\partial_j f_n \in C_c^\infty(\Omega)$ for all $n \in \mathbb{N}$. We conclude that $\partial_j z_{\varepsilon,k} \rightharpoonup 0$ weakly in $L^{p_j}(\Omega)$. Since $F_j \in L^{p'_j}(\Omega)$, we obtain

$$|\mathcal{E}_{\varepsilon,k}(T_k(U))| \leq C_k \sum_{j=1}^{N} \int_\Omega F_j \chi_{\{|u_\varepsilon|<k\}}(\partial_j z_{\varepsilon,k})\, dx \to 0 \text{ as } \varepsilon \to 0$$

which is what we wanted for the first convergence in (4.3.14). □

For the second convergence of (4.3.14): by definition of $\mathcal{E}_{\varepsilon,k}$, we can replace all instances of $T_k(u_\varepsilon)$ with u_ε, since we are always working on the set $\{|u_\varepsilon| < k\}$. Now, by the continuity of φ_λ

and the boundedness of $z_{\varepsilon,k}$ a.e. on Ω, we have $\varphi_\lambda(z_{\varepsilon,k}) \in L^\infty(\Omega)$. By the anisotropic chain rule (Proposition 3.2.1), since $\varphi_\lambda \in C^1(\mathbb{R})$, we have $\varphi_\lambda(z_{\varepsilon,k}) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. By Lemma 4.3.1, we know that $z_{\varepsilon,k} \rightharpoonup 0$ weakly in $W_0^{1,\vec{p}}(\Omega)$ and $z_{\varepsilon,k} \to 0$ a.e. in Ω as $\varepsilon \to 0$. It follows then that, up to subsequence,

(1) $\varphi_\lambda(z_{\varepsilon,k}) \to 0$ a.e. in Ω as $\varepsilon \to 0$, by the continuity of φ_λ, and

(2) $\varphi_\lambda(z_{\varepsilon,k}) \rightharpoonup 0$ weakly in $W_0^{1,\vec{p}}(\Omega)$ as $\varepsilon \to 0$. To see this, we first note that $(\varphi_\lambda(z_{\varepsilon,k}))_\varepsilon$ is bounded in $W_0^{1,\vec{p}}(\Omega)$. Indeed, $\partial_j \varphi_\lambda(z_{\varepsilon,k}) = (\partial_j z_{\varepsilon,k})\varphi'(z_{\varepsilon,k})$ by the anisotropic chain rule. Then $\varphi'(z_{\varepsilon,k})$ is in $L^\infty(\Omega)$, since φ' is continuous and $z_{\varepsilon,k} \in L^\infty(\Omega)$. Furthermore, $\partial_j z_{\varepsilon,k} = \partial_j T_k(u_\varepsilon) - \partial_j T_k(U)$ is bounded in $L^{p_j}(\Omega)$ since $(u_\varepsilon)_\varepsilon$ is bounded in $W_0^{1,\vec{p}}(\Omega)$, from which it follows that the sequence $(\varphi_\lambda(z_{\varepsilon,k}))_\varepsilon$ is bounded in $W_0^{1,\vec{p}}(\Omega)$. Then up to subsequence, $\varphi_\lambda(z_{\varepsilon,k})$ converges weakly in $W_0^{1,\vec{p}}(\Omega)$, the limit of which must be 0 by Proposition 1.3.2 and the a.e. convergence of $z_{\varepsilon,k}$ from (1).

Thus, both (1) and (2) hold. Since u_ε is a weak solution of the equation and $\varphi_\lambda(z_{\varepsilon,k}) \in W_0^{1,\vec{p}}(\Omega)$, we can substitute for v in (4.2.3) to get

$$\sum_{j=1}^N \int_\Omega \hat{A}_j(u_\varepsilon)\partial_j \varphi_\lambda(z_{\varepsilon,k})\,dx + \int_\Omega \Phi_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_{\varepsilon,k})\,dx = \langle \mathfrak{B}u_\varepsilon, \varphi_\lambda(z_{\varepsilon,k})\rangle \tag{4.3.16}$$

To simplify this equation and form an inequality, we split up the second term on the left-hand side:

$$\int_\Omega \Phi_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_{\varepsilon,k})\,dx = \int_\Omega \Phi_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_{\varepsilon,k})\chi_{\{|u_\varepsilon|<k\}}\,dx + \underbrace{\int_\Omega \Phi_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_{\varepsilon,k})\chi_{\{|u_\varepsilon|\geq k\}}\,dx}_{(B)}$$

and show that the term (B) is a non-negative quantity. Indeed, $\operatorname{sgn}(\Phi_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)) = \operatorname{sgn}(u_\varepsilon)$ by the sign condition on Φ (and hence Φ_ε). Then

$$\begin{aligned}\operatorname{sgn}\left(\Phi_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_{\varepsilon,k})\chi_{\{|u_\varepsilon|\geq k\}}\right) &= \operatorname{sgn}\left(u_\varepsilon z_{\varepsilon,k}\exp(\lambda(z_{\varepsilon,k})^2)\chi_{\{|u_\varepsilon|\geq k\}}\right)\\ &= \operatorname{sgn}(u_\varepsilon z_{\varepsilon,k}) \quad \text{on } \{|u_\varepsilon|\geq k\}\\ &= \operatorname{sgn}(u_\varepsilon(T_k(u_\varepsilon)-T_k(U)))\geq 0 \quad \text{on } \{|u_\varepsilon|\geq k\}.\end{aligned}$$

This last inequality follows by seeing that $|u_\varepsilon| \geq k$ implies that $T_k(U) \geq T_k(u_\varepsilon)$ if $u_\varepsilon < 0$ and $T_k(U) \leq T_k(u_\varepsilon)$ if $u_\varepsilon > 0$, whence we see all signs cancel, so expression B is non-negative. We can remove (B) from (4.3.16) to obtain the inequality

$$\underbrace{\sum_{j=1}^N \int_\Omega \hat{A}_j(u_\varepsilon)(\partial_j \varphi_\lambda(z_{\varepsilon,k}))\,dx}_{(\alpha)} + \underbrace{\int_\Omega \Phi_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_{\varepsilon,k})\chi_{\{|u_\varepsilon|<k\}}\,dx}_{(\beta)} \leq \langle \mathfrak{B}u_\varepsilon, \varphi_\lambda(z_{\varepsilon,k})\rangle. \tag{4.3.17}$$

To make the expressions more palatable, we use the notation

$$\begin{aligned} X_k(\varepsilon) &:= \frac{\zeta(k)}{\nu_0}\sum_{j=1}^N \int_\Omega \left(\hat{A}_j(u_\varepsilon)\partial_j(T_k(U)) + c(x)\right)|\varphi_\lambda(z_{\varepsilon,k})|\chi_{\{|u_\varepsilon|<k\}}\,dx,\\ Y_k(\varepsilon) &:= \sum_{j=1}^N \int_\Omega \hat{A}_j(u_\varepsilon)(\partial_j U)\varphi'_\lambda(z_{\varepsilon,k})\chi_{\{|U|<k\}\cap\{|u_\varepsilon|\geq k\}}\,dx.\end{aligned} \tag{4.3.18}$$

Now, recall that we have $\partial_j \varphi_\lambda(z_{\varepsilon,k}) = \partial_j z_{\varepsilon,k} \varphi'_\lambda(z_{\varepsilon,k})$, so we can rewrite (α) as

$$\begin{aligned}\int_\Omega \hat{A}_j(u_\varepsilon)(\partial_j \varphi_\lambda(z_{\varepsilon,k}))\,dx &= \int_\Omega \hat{A}_j(u_\varepsilon)(\partial_j z_{\varepsilon,k})\varphi'_\lambda(z_{\varepsilon,k})\chi_{\{|u_\varepsilon|<k\}}\,dx + \int_\Omega \hat{A}_j(u_\varepsilon)(\partial_j z_{\varepsilon,k})\varphi'_\lambda(z_{\varepsilon,k})\chi_{\{|u_\varepsilon|\geq k\}}\,dx \\ &= \int_\Omega \hat{A}_j(u_\varepsilon)(\partial_j z_{\varepsilon,k})\varphi'_\lambda(z_{\varepsilon,k})\chi_{\{|u_\varepsilon|<k\}}\,dx - \int_\Omega \hat{A}_j(u_\varepsilon)(\partial_j U)\varphi'_\lambda(z_{\varepsilon,k})\chi_{\{|u_\varepsilon|\geq k\}\cap\{|U|<k\}}\,dx \\ &= \int_\Omega \hat{A}_j(u_\varepsilon)(\partial_j z_{\varepsilon,k})\varphi'_\lambda(z_{\varepsilon,k})\chi_{\{|u_\varepsilon|<k\}}\,dx - Y_k(\varepsilon).\end{aligned}$$

Turning our attention to (β), we can use the growth condition (4.1.3) for Φ followed by the coercivity condition (4.1.2) to see that

$$|\hat{\Phi}_\varepsilon(u_\varepsilon)|\chi_{\{|u_\varepsilon|<k\}} \leq \zeta(k)\left(\frac{1}{\nu_0}\sum_{j=1}^N \hat{A}_j(u_\varepsilon)\partial_j u_\varepsilon + c(x)\right)\chi_{\{|u_\varepsilon|<k\}}. \tag{4.3.19}$$

Since we have the formula

$$\partial_j z_{\varepsilon,k} = \partial_j(T_k(u_\varepsilon) - T_k(U)) = \partial_j u_\varepsilon - \partial_j T_k(U) \quad \text{on} \quad \{|u_\varepsilon| < k\},$$

we can replace all instances of $\partial_j u_\varepsilon$ with $\partial_j z_{\varepsilon,k} + \partial_j T_k(U)$. Recalling the form of (β) in (4.3.17), we multiply both sides of (4.3.19) by $|\varphi_\lambda(z_{\varepsilon,k})|$, and integrate, to obtain

$$\begin{aligned}\int_\Omega |\Phi_\varepsilon(u_\varepsilon)||\varphi_\lambda(z_{\varepsilon,k})|\,dx &\leq \zeta(k)\int_\Omega\left(\frac{1}{\nu_0}\sum_{j=1}^N \hat{A}_j(u_\varepsilon)(\partial_j z_{\varepsilon,k} + \partial_j T_k(U)) + c(x)\right)|\varphi_\lambda(z_{\varepsilon,k})|\chi_{\{|u_\varepsilon|<k\}}\,dx \\ &= \zeta(k)\int_\Omega\left(\frac{1}{\nu_0}\sum_{j=1}^N \hat{A}_j(u_\varepsilon)\partial_j z_{\varepsilon,k}\right)|\varphi_\lambda(z_{\varepsilon,k})|\chi_{\{|u_\varepsilon|<k\}} + X_k(\varepsilon).\end{aligned}$$

Using $\int_\Omega a\,dx \geq -\int_\Omega |a|\,dx$ for integrable a, we find a lower bound on (4.3.17)-(β):

$$\begin{aligned}\int_\Omega \Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(z_{\varepsilon,k})\chi_{\{|u_\varepsilon|<k\}}\,dx &\geq -\int_\Omega |\Phi_\varepsilon(u_\varepsilon)||\varphi_\lambda(z_{\varepsilon,k})|\,dx \\ &\geq -\zeta(k)\int_\Omega\left(\frac{1}{\nu_0}\sum_{j=1}^N \hat{A}_j(u_\varepsilon)\partial_j z_{\varepsilon,k}\right)|\varphi_\lambda(z_{\varepsilon,k})|\chi_{\{|u_\varepsilon|<k\}}\,dx - X_k(\varepsilon).\end{aligned}$$

Then by combining the bounds on (4.3.17)-(α) and (β), we find that

$$\mathcal{E}_{\varepsilon,k}(u_\varepsilon) \leq X_k(\varepsilon) + Y_k(\varepsilon) + \langle \mathfrak{B}u_\varepsilon, \varphi_\lambda(z_{\varepsilon,k})\rangle. \tag{4.3.20}$$

To finish the proof of Proposition 4.3.2, it suffices to show that $X_k(\varepsilon), Y_k(\varepsilon)$ and $\langle \mathfrak{B}u_\varepsilon, \varphi_\lambda(z_{\varepsilon,k})\rangle$ all converge to 0 (as sequences of real numbers) as $\varepsilon \to 0$.

For $\langle \mathfrak{B}u_\varepsilon, \varphi_\lambda(z_{\varepsilon,k})\rangle$, we recall that $\mathfrak{B}$ satisfies the property (P_2) and that $\varphi(z_{\varepsilon,k}) \rightharpoonup 0$ weakly in $W_0^{1,\vec{p}}(\Omega)$ (see 2) to conclude that this term converges to 0.

For $X_k(\varepsilon)$, the idea is to prove that both $\hat{A}_j(u_\varepsilon)\partial_j T_k(U)|\varphi_\lambda(z_{\varepsilon,k})|\chi_{\{|u_\varepsilon|<k\}}$ and $c|\varphi_\lambda(z_{\varepsilon,k})|\chi_{\{|u_\varepsilon|<k\}}$ converge to 0 in $L^1(\Omega)$. Since $c|\varphi_\lambda(z_{\varepsilon,k})|\chi_{\{|u_\varepsilon|<k\}}$ is pointwise convergent to 0 and is bounded from above a.e. in Ω by $c|\varphi_\lambda(z_{\varepsilon,k})| \in L^1(\Omega)$, we conclude by Dominated Convergence Theorem that

$$c|\varphi_\lambda(z_{\varepsilon,k})|\chi_{\{|u_\varepsilon|<k\}} \to 0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0.$$

For the second term in $X_k(\varepsilon)$, since $\hat{A}_j(u_\varepsilon)$ is bounded in $L^{p_j'}(\Omega)$ for all $1 \leq j \leq N$, up to subsequence $\hat{A}_j(u_\varepsilon)$ converges weakly in $L^{p_j'}(\Omega)$ for all $1 \leq j \leq N$. On this subsequence, $\sum_{j=1}^N \hat{A}_j(u_\varepsilon)\partial_j U$ converges strongly in $L^1(\Omega)$, since $U \in W_0^{1,\vec{p}}(\Omega)$ implies $\partial_j U \in L^{p_j}(\Omega)$ for all $1 \leq j \leq N$. Hence, by Proposition 1.3.1, there exists a non-negative $F \in L^1(\Omega)$ such that up to subsequence, we have $\left|\sum_{j=1}^N \hat{A}_j(u_\varepsilon)\partial_j U\right| \leq F$ a.e. in Ω. We obtain

$$\sum_{j=1}^N \hat{A}_j(u_\varepsilon)\partial_j T_k(U)\left|\varphi_\lambda(z_{\varepsilon,k})\right|\chi_{\{|u_\varepsilon|<k\}} \to 0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0$$

by taking the pointwise a.e. upper bound $F|\varphi_\lambda(z_{\varepsilon,k})|$ along with the a.e. convergence of $|\varphi_\lambda(z_{\varepsilon,k})|$ to 0 and applying the Dominated Convergence Theorem. This shows exactly that $X_k(\varepsilon) \to 0$ as $\varepsilon \to 0$, as we wanted.

Showing that $Y_k(\varepsilon) \to 0$ follows the same idea as the proof for $X_k(\varepsilon)$. In this case, we recall the a.e. convergence of $\chi_{\{|u_\varepsilon|\geq k\}\cap\{|U|<k\}}$ (see (4.3.7)). Then we can take a dominating function $F\varphi_\lambda'(z_{\varepsilon,k})$ (This is in $L^1(\Omega)$ since $\varphi'(z_{\varepsilon,k})$ is bounded and F is in $L^1(\Omega)$) to see that $Y_k(\varepsilon) \to 0$ as $\varepsilon \to 0$, which, combined with the convergence of $X_k(\varepsilon)$, shows that $\limsup_{\varepsilon\to 0} \mathcal{E}_{\varepsilon,k}(u_\varepsilon) \leq 0$. This finishes the proof of Proposition 4.3.2 and hence Lemma 4.3.2. ☐

Lemma 4.3.3. Let $k \geq 1$ be a fixed integer, and suppose up to a subsequence of $\{u_\varepsilon\}_\varepsilon$, we have $u_\varepsilon \to U$ a.e. in Ω and $D_{\varepsilon,k} \to 0$ in $L^1(\Omega)$ as $\varepsilon \to 0$. Then up to subsequence, we have the following convergences:

$$\nabla T_k(u_\varepsilon) \to \nabla T_k(U) \text{ a.e. in } \Omega \quad \text{and} \quad T_k(u_\varepsilon) \to T_k(U) \text{ in } W_0^{1,\vec{p}}(\Omega). \tag{4.3.21}$$

Proof. The a.e. convergence in (4.3.21) is exactly the result of applying Proposition 4.3.1 to the sequence of weak solutions $(u_\varepsilon)_\varepsilon$, recalling from Lemma 4.3.1 that $u_\varepsilon \rightharpoonup U$ weakly in $W_0^{1,\vec{p}}(\Omega)$ along with the convergence $D_{\varepsilon,k} \to 0$ in $L^1(\Omega)$ from Lemma 4.3.2. It remains to show the strong convergence $T_k(u_\varepsilon) \to T_k(U)$ in $W_0^{1,\vec{p}}(\Omega)$. For $v, w \in W_0^{1,\vec{p}}(\Omega)$, define

$$H_{\varepsilon,k}(v, w) := \sum_{j=1}^N A_j(x, u_\varepsilon, \nabla T_k(v))\partial_j T_k(w). \tag{4.3.22}$$

Since we fix $k \geq 1$, we write H_ε to denote $H_{\varepsilon,k}$. In Step 1, we prove a technical convergence result that will be required for our proof of the convergence $T_k(u_\varepsilon) \to T_k(U)$ in $W_0^{1,\vec{p}}(\Omega)$. Specifically, we claim that

$$H_\varepsilon(u_\varepsilon, U) \text{ and } H_\varepsilon(U, u_\varepsilon) \text{ converge to } \sum_{j=1}^N A_j(x, U, \nabla T_k(U))\partial_j T_k(U) \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.3.23}$$

Step 1. We make the observation that $(A_j(x, u_\varepsilon, T_k(u_\varepsilon))_\varepsilon$ is bounded in $L^{p_j'}(\Omega)$, using the growth conditions and that $(u_\varepsilon)_\varepsilon$ is bounded in $W_0^{1,\vec{p}}(\Omega)$ from Lemma 4.3.1. Since $L^{p_j'}(\Omega)$ is reflexive, it follows that up to subsequence we have $\{A_j(x, u_\varepsilon, \nabla T_k(u_\varepsilon))\}_\varepsilon$ converges weakly in $L^{p_j'}(\Omega)$. The weak limit must be $A_j(x, U, \nabla T_k(U))$, by applying Proposition 1.3.2 and the a.e. convergence

$T_k(u_\varepsilon) \to T_k(U)$ from Step 2 along with the continuity of A_j in the last two variables. Hence, we have the convergence

$$H_\varepsilon(u_\varepsilon, U) \to A_j(x, U, \nabla T_k(U))\partial_j(T_k(U)) \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0,$$

which is exactly the convergence of $H_\varepsilon(u_\varepsilon, U)$ that we wanted.

Next, we show that $H_\varepsilon(U, u_\varepsilon) \to \sum_{j=1}^N A_j(x, U, \nabla T_k(U))\partial_j T_k(U)$ in $L^1(\Omega)$ as $\varepsilon \to 0$ using Vitali's Theorem, which requires us to show a.e. convergence and uniform integrability of $H_\varepsilon(u_\varepsilon, U)$ (recall also Remark 1.4.2). Using the convergences $u_\varepsilon \to u$ a.e. and $\nabla T_k(u_\varepsilon) \to \nabla T_k(U)$ a.e. in Ω and that A_j is a Carathéodory function, we have the convergence

$$A_j(x, u_\varepsilon, \nabla T_k(U))\partial_j T_k(u_\varepsilon) \to A_j(x, U, \nabla T_k(U))\partial_j T_k(U) \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0 \tag{4.3.24}$$

for all $1 \le j \le N$, so we only need the uniform integrability. Recall from Theorem 3.2.2 that $\partial_j T_k(u_\varepsilon) = \chi_{\{|u_\varepsilon|<k\}}\partial_j u_\varepsilon$. It follows that for any measurable $E \subseteq \Omega$,

$$\begin{aligned}
\int_E |A_j(x, u_\varepsilon, \nabla T_k(U))\partial_j T_k(u_\varepsilon)|\,dx &= \int_E \chi_{\{|u_\varepsilon|<k\}}|A_j(x, u_\varepsilon, \nabla T_k(U))\partial_j u_\varepsilon|\,dx \\
&\le \int_{E\cap\chi_{\{|u_\varepsilon|<k\}}} |A_j(x, u_\varepsilon, \nabla T_k(U))||\partial_j u_\varepsilon|\,dx \\
&\le \left(\int_{E\cap\chi_{\{|u_\varepsilon|<k\}}} |A_j(x, u_\varepsilon, \nabla T_k(U))|^{p_j'}\,dx\right)^{1/p_j'} \|\partial_j u_\varepsilon\|_{L^{p_j}(E\cap\chi_{\{|u_\varepsilon|<k\}})} \\
&\le C\left(\int_{E\cap\chi_{\{|u_\varepsilon|<k\}}} |A_j(x, u_\varepsilon, \nabla T_k(U))|^{p_j'}\,dx\right)^{1/p_j'},
\end{aligned}$$

where we use the fact that $\{u_\varepsilon\}_\varepsilon$ is uniformly bounded in $W_0^{1,\vec{p}}(E \cap \chi_{\{|u_\varepsilon|<k\}})$ from Lemma 4.3.1. This upper bound estimate shows that it's sufficient to prove $\{\chi_{\{|u_\varepsilon|<k\}}|A_j(x, u_\varepsilon, \nabla T_k(U))|^{p_j'}\}_\varepsilon$ is uniformly integrable on Ω. The growth condition on A_j gives, for every $E \subseteq \Omega$ measurable,

$$\int_E \chi_{\{|u_\varepsilon|<k\}}|A_j(x, u_\varepsilon, \nabla T_k(U))|^{p_j'}\,dx \le C\int_{E\cap\{|u_\varepsilon|<k\}} \left((\eta_j(x))^{p_j'} + |u_\varepsilon|^{p^*} + \left(\sum_{l=1}^N |\partial_l T_k(U)|^{p_l}\right)\right) dx$$

for a positive $C > 0$. The integrand is the sum of integrable functions (note that u_ε is bounded a.e. on $E \cap \{|u_\varepsilon| < k\}$), and by Remark 1.4.1, we conclude that $\{\chi_{\{|u_\varepsilon|<k\}}|A_j(x, u_\varepsilon, \nabla T_k(U))|^{p_j'}\}_\varepsilon$ is uniformly integrable on Ω. It follows that

$$\{A_j(x, u_\varepsilon, \nabla T_k(U))\partial_j T_k(u_\varepsilon)\}_\varepsilon \text{ is uniformly integrable over } \Omega \tag{4.3.25}$$

for all $1 \le j \le N$. Hence, we can apply Vitali's Theorem (and Remark 1.4.2) to conclude the convergence

$$H_\varepsilon(U, u_\varepsilon) \to \sum_{j=1}^N A_j(x, U, \nabla T_k(U))\partial_j T_k(U) \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0$$

which finishes the proof of Step 1.

Step 2. In Step 2, we show that $T_k(u_\varepsilon) \to T_k(U)$ strongly in $W_0^{1,\vec{p}}(\Omega)$ as $\varepsilon \to 0$ with the help of the convergence results in Step 1. That is, we want to show that $\partial_j T_k(u_\varepsilon) \to \partial_j T_k(u_\varepsilon)$ in $L^{p_j}(\Omega)$ for all $1 \leq j \leq N$. By Remark 1.4.2, it is sufficient to show that $\{|\partial_j T_k(u_\varepsilon)|^{p_j}\}_\varepsilon$ is uniformly integrable over Ω for all $1 \leq j \leq N$, but in fact, we can prove a stronger condition:

$$\left\{ \sum_{j=1}^{N} |\partial_j T_k(u_\varepsilon)|^{p_j} \right\}_\varepsilon \text{ is uniformly integrable over } \Omega. \tag{4.3.26}$$

Now, we know $D_{\varepsilon,k} \to 0$ in $L^1(\Omega)$ by Lemma 4.3.2 and we also have the inequality

$$D_{\varepsilon,k}(x) \geq \nu_0 \sum_{j=1}^{N} |\partial_j T_k(u_\varepsilon)|^{p_j} - |H_\varepsilon(u_\varepsilon, U)| - |H_\varepsilon(U, u_\varepsilon)| \quad \text{a.e. in } \Omega \tag{4.3.27}$$

from Lemma 4.3.2 and (4.2.6) (replacing ∇u_ε with $\nabla T_k(u_\varepsilon)$ and expanding appropriately). Integrating a rearrangement of (4.3.27) over an arbitrary measurable set $E \subseteq \Omega$ gives

$$\int_E \sum_{j=1}^{N} |\partial_j T_k(u_\varepsilon)|^{p_j} \, dx \leq \frac{1}{\nu_0} \int_E D_{\varepsilon,k} + |H_\varepsilon(u_\varepsilon, U)| + |H_\varepsilon(U, u_\varepsilon)| \, dx.$$

Since we know that $D_{\varepsilon,k}$, $H_\varepsilon(u_\varepsilon, U)$ and $H_\varepsilon(U, u_\varepsilon)$ all converge in $L^1(\Omega)$ as $\varepsilon \to 0$ (the former by Lemma 4.3.2 and the latter two in Step 1), we conclude by Corollary 1.4.1 and Remark 1.4.1 that $\{\sum_{j=1}^{N} |\partial_j T_k(u_\varepsilon)|^{p_j}\}_\varepsilon$ is uniformly integrable over ε in Ω, as we wanted in (4.3.26). This finishes the proof of Step 2, and the proof of Lemma 4.3.3.

□

Although Lemma 4.3.3 shows that we can obtain a.e. convergence for the truncations $\nabla T_k(u_\varepsilon)$ to $\nabla T_k(U)$ for any fixed integer $k \geq 1$, we would like this to hold for the total sequence $(\nabla u_\varepsilon)_\varepsilon$. The subject of the next result proves that we are able to do so - however, the argument we use cannot remove the dependence on k of the convergences of both $T_k(\nabla u_\varepsilon)$ and $T_k(u_\varepsilon)$.

Corollary 4.3.1. Up to subsequence, for every integer $k \geq 1$ we have

$$\nabla u_\varepsilon \to \nabla U \text{ a.e. in } \Omega \quad \text{and} \quad T_k(u_\varepsilon) \to T_k(U) \text{ in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0. \tag{4.3.28}$$

Proof. We use a diagonal argument. Lemma 4.3.3 says that for every fixed $k \in \mathbb{N}$, up to a subsequence of $\{u_\varepsilon\}_\varepsilon$,

$$\nabla T_k(u_\varepsilon) \to \nabla T_k(U) \text{ a.e. in } \Omega \quad \text{and} \quad T_k(u_\varepsilon) \to T_k(U) \text{ in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0. \tag{4.3.29}$$

For $k = 1$, the lemma gives us a sequence $\{u_{\varepsilon_n^{(1)}}\}_{n\in\mathbb{N}}$ with $\varepsilon_n^{(1)} \to 0$ as $n \to \infty$ such that

$$\nabla T_1(u_{\varepsilon_n^{(1)}}) \to \nabla T_1(U) \text{ a.e in } \Omega \quad \text{and} \quad T_k(u_{\varepsilon_n^{(1)}}) \to T_k(U) \text{ in } W_0^{1,\vec{p}}(\Omega).$$

Since $\{\varepsilon_n^{(1)}\}_{n\in\mathbb{N}}$ is a sequence that converges to 0, we can appeal to Lemma 4.3.3 again to get a subsequence $\{u_{\varepsilon_n^{(2)}}\}_{n\in\mathbb{N}}$ of $\{u_{\varepsilon_n^{(1)}}\}_{n\in\mathbb{N}}$ satisfying the convergences in (4.3.29) with $k = 2$. Continuing inductively, we obtain a countable list of subsequences $\{u_{\varepsilon_n^{(j)}}\}_{n\in\mathbb{N}}$, indexed in j:

$$\begin{array}{ccccccc}
u_{\varepsilon_1^{(1)}} & u_{\varepsilon_2^{(1)}} & u_{\varepsilon_3^{(1)}} & u_{\varepsilon_4^{(1)}} & \ldots & u_{\varepsilon_n^{(1)}} & \ldots \\
u_{\varepsilon_1^{(2)}} & u_{\varepsilon_2^{(2)}} & u_{\varepsilon_3^{(2)}} & u_{\varepsilon_4^{(2)}} & \ldots & u_{\varepsilon_n^{(2)}} & \ldots \\
u_{\varepsilon_1^{(3)}} & u_{\varepsilon_2^{(3)}} & u_{\varepsilon_3^{(3)}} & u_{\varepsilon_4^{(3)}} & \ldots & u_{\varepsilon_n^{(3)}} & \ldots \\
\vdots & \vdots & \vdots & \vdots & & \vdots &
\end{array}$$

Now take the diagonal sequence $\{u_{\varepsilon_j^{(j)}}\}_{j\in\mathbb{N}}$. We claim that this subsequence satisfies the convergence requirements we need. Notice that by construction, $\{u_{\varepsilon_j^{(j)}}\}_{j\geq k}$ is a subsequence of $\{u_{\varepsilon_j^{(l)}}\}_{j\in\mathbb{N}}$ for $1 \leq l \leq k-1$ with $\varepsilon_j^{(j)} \to 0$ as $j \to \infty$, so $\nabla T_l(u_{\varepsilon_j^{(j)}}) \to \nabla T_l(U)$ a.e. in Ω and $T_l(u_{\varepsilon_j^{(j)}}) \to T_l(U)$ as $j \to \infty$ for all l in this range. Now, since u_ε converges to U a.e. in Ω as $\varepsilon \to 0$, where U is defined almost everywhere on Ω, there exists $M \geq 0$ such that $|u_\varepsilon(x)| \leq M$ for a.e. $x \in \Omega$. Consider the subsequence $\{u_{\varepsilon_j^{(j)}}\}_{j>\lceil M\rceil}$. Then

$$
\begin{aligned}
&\nabla u_{\varepsilon_j^{(j)}} = \nabla T_{\lceil M\rceil}(u_{\varepsilon_j^{(j)}}) \to \nabla T_{\lceil M\rceil}(U) = \nabla U \text{ a.e in } \Omega,\\
&T_k(u_{\varepsilon_j^{(j)}}) \to T_k(U) \text{ in } W_0^{1,\vec{p}}(\Omega), \quad \text{for } 1 \leq k \leq \lceil M\rceil.
\end{aligned}
\tag{4.3.30}
$$

Since the convergences in (4.3.30) hold if we replace $\lceil M\rceil$ with an arbitrarily large constant, we conclude that (4.3.28) holds, as required. □

4.3.2. Obtaining Solutions to Approximate Problem in the Limit. The original aim in introducing the bounded approximate PDE is to obtain a sequence of solutions which converge to a solution of the original PDE (4.1.1) with $f = 0$. The results of Lemma 4.3.1 and Corollary 4.3.1 give us useful properties of the convergence of $(u_\varepsilon)_\varepsilon$ that we can now leverage. In particular, our work suggests that U is the appropriate candidate to be a weak solution of (4.1.1) with $f = 0$. This requires U to satisfy

$$
\begin{cases}
\sum_{j=1}^N \int_\Omega A_j(x, U, \nabla U)\partial_j v\, dx + \int_\Omega \Phi(x, U, \nabla U) v\, dx = \langle \mathfrak{B}U, v\rangle,\\
\Phi(x, U, \nabla U) \in L^1(\Omega)
\end{cases}
\tag{4.3.31}
$$

for all $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. This suggests that our first step in showing (4.3.31) is the following lemma:

Lemma 4.3.4. We have $\hat{\Phi}(U) \in L^1(\Omega)$, and

$$
\hat{\Phi}_\varepsilon(u_\varepsilon) \to \hat{\Phi}(U) \text{ strongly in } L^1(\Omega) \text{ as } \varepsilon \to 0.
\tag{4.3.32}
$$

Proof. We first show that $\hat{\Phi}(U) \in L^1(\Omega)$, which will follow from a two-step process where we use Fatou's Lemma to show that $\hat{\Phi}(U)U \in L^1(\Omega)$, then use this fact to show $\hat{\Phi}(U) \in L^1(\Omega)$. We recall from Lemma 4.3.1 and Corollary 4.3.1 the following convergences:

$$
u_\varepsilon \to U \text{ a.e. in } \Omega \quad \text{and} \quad \nabla u_\varepsilon \to \nabla U \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0.
$$

Since $\hat{\Phi}$ is a Carathéodory function and hence continuous in its last two variables, the convergences of u_ε and ∇u_ε imply we have $\hat{\Phi}_\varepsilon(u_\varepsilon)u_\varepsilon \to \hat{\Phi}(U)U$ a.e. in Ω as $\varepsilon \to 0$. Furthermore, we know that $\{\hat{\Phi}_\varepsilon(u_\varepsilon)u_\varepsilon\}_\varepsilon$ is non-negative by the sign condition and is uniformly bounded in $L^1(\Omega)$ over ε by Lemma 4.3.1. Hence, by Fatou's Lemma, we have

$$
\int_\Omega \hat{\Phi}(U)U\, dx = \int_\Omega \liminf_{\varepsilon\to 0} \hat{\Phi}_\varepsilon(u_\varepsilon)u_\varepsilon\, dx \leq \liminf_{\varepsilon\to 0} \int_\Omega \hat{\Phi}_\varepsilon(u_\varepsilon)u_\varepsilon\, dx \leq C_1
$$

with $C_1 > 0$ a uniform upper bound. This shows that $\hat{\Phi}(U)U \in L^1(\Omega)$. To show that $\hat{\Phi}(U) \in L^1(\Omega)$, we would like to use the growth condition (4.1.3), but since this only works when U is bounded (due to the increasing ζ term), we use again the 'divide and conquer' strategy. For any $M > 0$, we have

$$
\int_\Omega \hat{\Phi}(U)\, dx = \int_\Omega \hat{\Phi}(U)\chi_{\{|U|\leq M\}}\, dx + \int_\Omega \hat{\Phi}(U)\chi_{\{|U|>M\}}\, dx.
$$

It follows then that

$$\hat{\Phi}(U)\chi_{\{|U|\leq M\}} \leq \zeta(|M|)\left(\sum_{j=1}^{N}|\partial_j U|^{p_j} + c\right) \in L^1(\Omega),$$
$$\hat{\Phi}(U)\chi_{\{|U|>M\}} \leq \frac{1}{M}\hat{\Phi}(U)U \in L^1(\Omega).$$

Hence, $\hat{\Phi}(U) \in L^1(\Omega)$.

To prove the convergence in (4.3.32), we employ Vitali's Theorem. We already know that $\hat{\Phi}_\varepsilon(u_\varepsilon)$ converges to $\hat{\Phi}(U)$ a.e. in Ω and that $\hat{\Phi}(U) \in L^1(\Omega)$, so it remains to show that $\{\hat{\Phi}_\varepsilon(u_\varepsilon)\}_\varepsilon$ is uniformly integrable. Fix some $\eta > 0$ arbitrary and let $E \subseteq \Omega$ be a measurable set. We wish to show that there exists $\delta > 0$ such that whenever $|E| < \delta$,

$$\int_E \hat{\Phi}_\varepsilon(u_\varepsilon)\,dx < \eta. \tag{4.3.33}$$

'Divide and conquer' comes into play again. Indeed, let $M > 0$ be an arbitrary real constant. We have

$$\int_E \hat{\Phi}_\varepsilon(u_\varepsilon)\,dx = \underbrace{\int_E \hat{\Phi}_\varepsilon(u_\varepsilon)\chi_{\{|u_\varepsilon|\leq M\}}\,dx}_{(A)} + \underbrace{\int_E \hat{\Phi}_\varepsilon(u_\varepsilon)\chi_{\{|u_\varepsilon|>M\}}\,dx}_{(B)}.$$

The growth condition (4.1.3) bounds the integrand of (A) from above:

$$\hat{\Phi}_\varepsilon(u_\varepsilon)\chi_{\{u_\varepsilon\}\leq M} \leq \zeta(M)\left(\sum_{j=1}^{N}|\partial_j(T_M u_\varepsilon)|^{p_j} + c\right).$$

Since both terms in the right-hand side are integrable, we use Proposition (1.4.1) to get the existence of $\delta_1 > 0$ such that whenever $\mathrm{meas}(E) < \delta$, we have $\int_E \hat{\Phi}_\varepsilon(u_\varepsilon)\,dx \leq \eta/2$. We now consider integral (B) - in light of the uniform bound on $\|\hat{\Phi}_\varepsilon(u_\varepsilon)\|_{L^1(\Omega)}$ from Lemma 4.3.1, we have

$$(B) \leq \int_E \frac{\hat{\Phi}(u_\varepsilon)u_\varepsilon}{M}\,dx \leq \frac{C}{M}.$$

Since M was arbitrary, taking $M > \frac{\eta}{2C}$ gives $(B) < \frac{\eta}{2}$. Combining the bounds on (A) and (B), we obtain (4.3.33). By Vitali's Theorem, the convergence (4.3.32) follows, and we are done. □

Lemma 4.3.5. The following convergences hold for all $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$:

$$\sum_{j=1}^{N}\int_\Omega A_j(x,u_\varepsilon,\nabla u_\varepsilon)\partial_j v\,dx \to \sum_{j=1}^{N}\int_\Omega A_j(x,U,\nabla U)\partial_j v\,dx, \tag{4.3.34}$$
$$\langle \mathfrak{B}u_\varepsilon, v\rangle \to \langle \mathfrak{B}U, v\rangle$$

Proof. Let $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. The convergence of $\langle \mathfrak{B}u_\varepsilon, v\rangle$ follows immediately from the weak convergence $u_\varepsilon \rightharpoonup U$ in $W_0^{1,\vec{p}}(\Omega)$ and assumption (P_2) on $\mathfrak{B}$. For the first convergence, the pointwise convergence $u_\varepsilon \to U$ and $\nabla u_\varepsilon \to \nabla U$ a.e. in Ω along with A_j being Carathéodory implies we have $A_j(x,u_\varepsilon,\nabla u_\varepsilon) \to A_j(x,U,\nabla U)$ a.e. in Ω for all $1 \leq j \leq N$. The growth condition

(4.1.3) along with Lemma 4.3.1 shows that $\{A_j(x, u_\varepsilon, \nabla u_\varepsilon)\}_\varepsilon$ is uniformly bounded in $L^{p'_j}(\Omega)$, so up to subsequence, $A_j(x, u_\varepsilon, \nabla u_\varepsilon)$ converges weakly in $L^{p'_j}(\Omega)$, and this limit must coincide with $A_j(x, U, \nabla U)$ for all $1 \le j \le N$ by Proposition 1.3.2. It follows then that

$$\int_\Omega A_j(x, u_\varepsilon, \nabla u_\varepsilon)\, dx \to \int_\Omega A_j(x, U, \nabla U)\partial_j v\, dx \quad \text{for all } 1 \le j \le N,$$

since $\partial_j v \in L^{p_j}(\Omega)$, which is exactly what we wanted for (4.3.34). □

We can now conclude that U is a weak solution of the PDE 4.1.1 with $f = 0$ satisfying (4.3.31). Indeed, we have already proven that $\Phi(x, U, \nabla U) \in L^1(\Omega)$ in Lemma 4.3.4; now, for all $\varepsilon > 0$, the function $u_\varepsilon \in W_0^{1,\vec{p}}(\Omega)$ satisfies

$$\sum_{j=1}^N \int_\Omega A_j(x, u_\varepsilon, \nabla u_\varepsilon)\partial_j v\, dx + \int_\Omega \Phi_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) v\, dx - \langle \mathfrak{B} u_\varepsilon, v\rangle = 0 \tag{4.3.35}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$; by taking the limit as $\varepsilon \to 0$, using the convergences in Lemma 4.3.4 and Lemma 4.3.5 with $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, we conclude that $U \in W_0^{1,\vec{p}}(\Omega)$ satisfies

$$\sum_{j=1}^N \int_\Omega \hat{A}_j(U)\partial_j v\, dx + \int_\Omega \hat{\Phi}(U) v\, dx = \langle Bu, v\rangle \quad \text{for all } v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega),$$

whence we obtain that U satisfies (4.3.31). This finishes the first approximation process.

4.4. Anisotropic Equation with Non-Zero L^1-data

We now consider the original PDE (4.1.1) in its totality. Recall in the previous section that we have obtained a weak solution U to the bounded approximate problem (4.2.2) when $f = 0$. In order to generate a weak solution of the original problem (4.1.1) with non-zero input data $f \in L^1(\Omega)$, the basic idea is the same as the first approximation: introduce a sequence of approximate PDE for which the problem of finding weak solutions is possible, by approximating f with L^∞ functions f_ε. After obtaining a sequence of approximate solutions, we aim to show that, up to subsequence, we can extract a solution of the original PDE (4.1.1) using a priori estimates and convergence arguments. For $\varepsilon > 0$, define

$$f_\varepsilon(x) := \frac{f(x)}{1 + \varepsilon|f(x)|} \quad \text{for a.e. } x \in \Omega.$$

We record the following useful facts:

$$|f_\varepsilon| \le \min\left\{\frac{1}{\varepsilon}, |f|\right\} \text{ a.e. in } \Omega, \quad f_\varepsilon \to f \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega \text{ as } \varepsilon \to 0.$$

We are now considering the approximate PDE

$$\begin{cases} \mathcal{A}u - \mathfrak{B}u + \hat{\Phi}(u) = f_\varepsilon, \\ u \in W_0^{1,\vec{p}}(\Omega), \quad \hat{\Phi}(u) \in L^1(\Omega). \end{cases} \tag{4.4.1}$$

We assume that (4.1.2), (4.1.3), (4.1.4) hold, with $\mathfrak{B}$ satisfying properties (P_1) and (P_2) and $\hat{\Phi}(u) \in L^1(\Omega)$. Recall as well that since the right-hand side of (4.4.1) is non-zero, we must also assume the lower bound on Φ in (4.1.5) holds; as we will see, this is what allows us to find a priori estimates on the set of weak solutions to the PDE (4.4.1). A weak solution of (4.4.1) is a function $u \in W_0^{1,\vec{p}}(\Omega)$ satisfying the integral equation

$$\sum_{j=1}^{N} \int_\Omega \hat{A}_j(u)\partial_j v\, dx + \int_\Omega \hat{\Phi}(u) v\, dx = \int_\Omega f_\varepsilon v\, dx + \langle Bu, v\rangle \text{ for all } v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega). \tag{4.4.2}$$

Again, the test function v must be bounded since we are testing it with $\hat{\Phi} \in L^1(\Omega)$. In fact, our work from the *previous* approximation gives us the following lemma:

Lemma 4.4.1. There exists a set of weak solutions $\{U_\varepsilon\}_\varepsilon$ of (4.4.1), indexed by ε, satisfying (4.4.2).

Proof. Define the operator $\mathfrak{B}_\varepsilon : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ by

$$\langle \mathfrak{B}_\varepsilon u, v\rangle = \langle \mathfrak{B} u, v\rangle + \int_\Omega f_\varepsilon v\, dx \quad \text{for all } u, v \in W_0^{1,\vec{p}}(\Omega).$$

It suffices to show that $\mathfrak{B}_\varepsilon$ satisfies (P_1) and (P_2), whence we derive the existence of a weak solution U_ε satisfying (4.4.2) by applying Lemma 4.3.5 to the approximate PDE with $\mathfrak{B}u + f_\varepsilon$ replaced by $\mathfrak{B}_\varepsilon$. Now, the continuity property (P_2) of $\mathfrak{B}_\varepsilon$ follows easily by using that $\mathfrak{B}$ satisfies (P_2) and that the functional F_ε defined on $W_0^{1,\vec{p}}(\Omega)$ by $F_\varepsilon(v) = \int_\Omega f_\varepsilon v\, dx$ is in $W^{-1,\vec{p}'}(\Omega)$. To show that $\mathfrak{B}_\varepsilon$ satisfies (P_1), we have the following:

$$\begin{aligned}
|\langle \mathfrak{B}_\varepsilon u, v\rangle| &\le |\langle \mathfrak{B} u, v\rangle| + \left|\int_\Omega f_\varepsilon v\, dx\right| \\
&\le \mathfrak{C}\left(1 + \|u\|_{W_0^{1,\vec{p}}(\Omega)}\right)\left(\mathfrak{a}_0 \|v\|_{W_0^{1,\vec{p}}(\Omega)} + \|v\|_{L^s(\Omega)}\right) + C_1 \|f_\varepsilon\|_{L^\infty(\Omega)} \|v\|_{L^1(\Omega)} \\
&\le \mathfrak{K}\left(1 + \|u\|_{W_0^{1,\vec{p}}(\Omega)}\right)\left(\mathfrak{a}_0 \|v\|_{W_0^{1,\vec{p}}(\Omega)} + \|v\|_{L^s(\Omega)}\right) \quad (*)
\end{aligned}$$

for constants $\mathfrak{K} > 0$, $s \in [1, p^*)$, $\mathfrak{a}_0 \ge 0$, $\mathfrak{b} \in (0, p_1 - 1)$ if $\mathfrak{a}_0 > 0$ and $\mathfrak{b} \in (0, p_1/p')$ if $\mathfrak{a}_0 = 0$. To obtain the inequality $(*)$, in the case $\mathfrak{a}_0 = 0$, we use the continuous embedding $L^1(\Omega) \overset{\text{cont.}}{\hookrightarrow} L^s(\Omega)$ on $\|v\|_{L^1(\Omega)}$, and if $\mathfrak{a}_0 > 0$, then we apply the embedding $W_0^{1,\vec{p}}(\Omega) \overset{\text{cont.}}{\hookrightarrow} L^1(\Omega)$. This concludes the proof. $\square$

Remark 4.4.1. The proof of Lemma 4.4.1 shows that the properties (P_1) and (P_2) hold when the operator $\mathfrak{B}$ is perturbed by an element in the dual space $W^{-1,\vec{p}}(\Omega)$ (in our case, this is the addition of the linear functional $\phi : W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}, v \mapsto \int_\Omega f_\varepsilon v\, dx$). Indeed, this can be seen in Example 4.1.1, where $\mathfrak{B}$ as a constant operator trivially satisfies (P_1) and (P_2), and (P_1) and (P_2) are invariant under sums.

Even as the a priori estimates of Lemma 4.3.1 provided a foundation for the proofs of Lemmas 4.3.2 and 4.3.3, we require a priori estimates for our new approximate problem. In fact, near-analogues of Lemma 4.3.1 hold:

Lemma 4.4.2. (i) There exists a uniform upper bound $K > 0$ such that for all $\varepsilon > 0$,

$$\|U_\varepsilon\|_{W_0^{1,\vec{p}}(\Omega)} + \int_\Omega |\hat{\Phi}(U_\varepsilon)|\, dx \le K.$$

(ii) There exists a function $U_0 \in W_0^{1,\vec{p}}(\Omega)$ such that, up to subsequence, we have the following convergences:

$$U_\varepsilon \to U_0 \text{ a.e. in } \Omega \quad \text{and} \quad U_\varepsilon \rightharpoonup U_0 \text{ weakly in } W_0^{1,\vec{p}}(\Omega).$$

Proof. The proof of (i) differs from the proof of Lemma 4.3.1, with the major culprits being the lack of boundedness of $\hat{\Phi}(U_\varepsilon)$ and the introduction of f_ε changing several bounds. We take $\tau > 0$ as the constant from (4.1.5). Taking $T_\tau(U_\varepsilon) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ as a test function in (4.4.2), we have

$$\begin{aligned}
&\sum_{j=1}^N \int_\Omega \hat{A}_j(U_\varepsilon)\partial_j T_\tau(U_\varepsilon)\,dx + \tau\int_\Omega |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|\geq\tau\}}\,dx\\
&= \sum_{j=1}^N \int_\Omega \hat{A}_j(U_\varepsilon)\partial_j U_\varepsilon \chi_{\{|U_\varepsilon|<\tau\}}\,dx + \tau\int_\Omega |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|\geq\tau\}}\,dx\\
&\leq \sum_{j=1}^N \int_\Omega \hat{A}_j(U_\varepsilon)\partial_j U_\varepsilon\,dx + \int_\Omega \hat{\Phi}(U_\varepsilon)T_\tau(U_\varepsilon)\,dx\\
&= \langle \mathfrak{B}U_\varepsilon, T_\tau(U_\varepsilon)\rangle + \int_\Omega f_\varepsilon T_\tau(U_\varepsilon)\,dx \leq |\langle \mathfrak{B}U_\varepsilon, T_\tau(U_\varepsilon)\rangle| + \tau\|f\|_{L^1(\Omega)}.
\end{aligned} \tag{4.4.3}$$

By the coercivity condition and the lower bound (4.1.5), we immediately get a lower bound for $\sum_{j=1}^N \int_\Omega \hat{A}_j(U_\varepsilon)\partial_j T_\tau(U_\varepsilon)\chi_{\{|U_\varepsilon|<\tau\}}\,dx + \tau\int_\Omega |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|\geq\tau\}}\,dx$ of the form

$$\nu_0 \sum_{j=1}^N \int_\Omega |\partial_j U_\varepsilon|^{p_j}\chi_{\{|U_\varepsilon|<\tau\}}\,dx + \tau\gamma\int_\Omega |\partial_j U_\varepsilon|^{p_j}\chi_{\{|U_\varepsilon|\geq\tau\}}\,dx.$$

Taking $k := \min\{\nu_0, \tau\gamma\} > 0$ and using (4.4.3), we then get the estimate

$$k\sum_{j=1}^N \int_\Omega |\partial_j U_\varepsilon|^{p_j}\,dx \leq |\langle \mathfrak{B}U_\varepsilon, T_\tau(U_\varepsilon)\rangle| + \tau\|f\|_{L^1(\Omega)},$$

so that the expression $k\sum_{j=1}^N \int_\Omega |\partial_j U_\varepsilon|^{p_j}\,dx - |\langle \mathfrak{B}U_\varepsilon, T_\tau(U_\varepsilon)\rangle|$ is bounded above by the constant $\tau\|f\|_{L^1(\Omega)}$. However, notice that for arbitrary $u \in W_0^{1,\vec{p}}(\Omega)$, using condition (P_1),

$$\begin{aligned}
&\frac{k\sum_{j=1}^N \int_\Omega |\partial_j u|^{p_j}\,dx - |\langle \mathfrak{B}u, T_\tau(u)\rangle|}{\|u\|_{W_0^{1,\vec{p}}(\Omega)}}\\
&\geq \frac{k\sum_{j=1}^N \|\partial_j u\|_{L^{p_j}(\Omega)}^{p_j} - \mathfrak{C}\left(1 + \|u\|_{W_0^{1,\vec{p}}(\Omega)}^{\mathfrak{b}}\right)\left(\mathfrak{a}_0\|u\|_{W_0^{1,\vec{p}}(\Omega)} + \|u\|_{L^s(\Omega)}\right)}{\|u\|_{W_0}^{1,\vec{p}(\Omega)}}
\end{aligned} \tag{4.4.4}$$

from which we derive that the left-hand side of (4.4.4) diverges as $\|u\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty$ by considering dominant terms with the largest exponent (Formally, this uses the Generalised Young's Inequality and considering the cases $\mathfrak{a}_0 = 0$ and $\mathfrak{a}_0 > 0$, see the proof of Lemma 4.2.4). Since we have already shown the left-hand side of (4.4.4) is bounded from above by a constant, this forces $\{U_\varepsilon\}_\varepsilon$ to be bounded uniformly by a constant $K_0 > 0$ in $W_0^{1,\vec{p}}(\Omega)$.

It now remains to show that we can bound $\hat{\Phi}(U_\varepsilon)$ in $L^1(\Omega)$. We use again the divide and conquer strategy. Indeed, since the coercivity condition (4.1.2) forces $\sum_{j=1}^N \int_\Omega \hat{A}_j(U_\varepsilon)\partial_j U_\varepsilon \, dx$ to be positive, (4.4.3) gives

$$\tau \int_\Omega |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|\geq\tau\}} \, dx \leq \int_\Omega \hat{\Phi}(U_\varepsilon)T_\tau(U_\varepsilon) \, dx \leq |\langle \mathfrak{B}U_\varepsilon, T_\tau(U_\varepsilon)\rangle| + \tau\|f\|_{L^1(\Omega)}.$$

Since $\{U_\varepsilon\}_\varepsilon$ is bounded by the constant K_0 in $W_0^{1,\vec{p}}(\Omega)$, the property (P_1) of $\mathfrak{B}$ forces $|\langle \mathfrak{B}U_\varepsilon, T_\tau(U_\varepsilon)\rangle|$ to be bounded above by a constant K_1 for all $\varepsilon > 0$, so that

$$\int_\Omega |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|\geq\tau\}} \, dx \leq \|f\|_{L^1(\Omega)} + \frac{K_1}{\tau}.$$

Furthermore, considering $|U_\varepsilon|\chi_{\{|U_\varepsilon|<\tau\}}$, we use the growth condition (4.1.3) to get

$$\int_\Omega |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|<\tau\}} \, dx \leq \zeta(\tau)\left(\sum_{j=1}^N \|\partial_j U_\varepsilon\|_{L^{p_j}(\Omega)}^{p_j} + \|c\|_{L^1(\Omega)}\right) \leq K_2$$

which is bounded by the constant $K_2 > 0$ since $c \in L^1(\Omega)$, $\{U_\varepsilon\}_\varepsilon$ is bounded in $W_0^{1,\vec{p}}(\Omega)$ from Lemma 4.4.2 (i), and $\zeta(\tau)$ is a constant. This leads to

$$\int_\Omega |\hat{\Phi}(U_\varepsilon)| \, dx = \int_\Omega |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|<\tau\}} \, dx + \int_\Omega |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|\geq\tau\}} \leq \|f\|_{L^1(\Omega)} + \frac{K_1}{\tau} + K_2.$$

Combining this upper bound with the upper bound on $\{U_\varepsilon\}_\varepsilon$, we conclude (i), as required.

Proof of (ii): The proof follows completely analogously with the proof of Lemma 4.3.1 (ii). □

Even though the proofs of Lemma 4.3.2 and Lemma 4.3.3 involve the approximation Φ_ε and not Φ, the only properties that were used of Φ_ε are the growth conditions (4.1.3) and sign condition that are also satisfied by Φ. Furthermore, the only properties of $\mathfrak{B}$ that are used are (P_1) and (P_2), which are also satisfied by $\mathfrak{B}_\varepsilon$. In addition, $\mathfrak{B}_\varepsilon$ 'absorbs' the additional term $\int_\Omega f_\varepsilon v \, dx$ in the weak solution definition (4.4.2). Hence, by replacing u_ε with U_ε, U with U_0 and $\langle \mathfrak{B}u, v\rangle + \int_\Omega f_\varepsilon v \, dx$ with $\langle \mathfrak{B}_\varepsilon u, v\rangle$, we can prove the equivalent versions of Lemma 4.3.2, Lemma 4.3.3 and Corollary 4.3.1, giving us the key result:

Corollary 4.4.1. Up to subsequence, for all integers $k \geq 1$ we have

$$\nabla U_\varepsilon \to \nabla U_0 \text{ a.e. in } \Omega \quad \text{and} \quad T_k(U_\varepsilon) \to T_k(U_0) \text{ strongly in } W_0^{1,\vec{p}}(\Omega) \text{ as } \varepsilon \to 0.$$

4.4.1. Obtaining Solutions in the Limit. The aim of this final section is to show that U_0 is indeed a solution of the original Dirichlet problem (4.1.1). That is, we want to show that the following integral equation holds:

$$\sum_{j=1}^N \int_\Omega \hat{A}_j(U_0)\partial_j v \, dx + \int_\Omega \hat{\Phi}(U_0)v \, dx = \langle \mathfrak{B}U_0, v\rangle + \int_\Omega f v \, dx \tag{4.4.5}$$

for all $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. Since U_ε is a solution of the approximate problem (4.4.1) for all $\varepsilon > 0$, we already have

$$\sum_{j=1}^N \int_\Omega \hat{A}_j(x, U_\varepsilon, \nabla U_\varepsilon)\partial_j v \, dx + \int_\Omega \hat{\Phi}(U_\varepsilon)v \, dx = \langle \mathfrak{B}U_\varepsilon, v\rangle + \int_\Omega f_\varepsilon v \, dx \tag{4.4.6}$$

for all $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, and it remains for us to take the limit as $\varepsilon \to 0$. We first deal with the convergence results that do not deviate much from our previous work.

Lemma 4.4.3. The following convergences hold as $\varepsilon \to 0$:

$$\begin{aligned} \sum_{j=1}^N \int_\Omega \hat{A}_j(U_\varepsilon)\partial_j v\, dx &\to \sum_{j=1}^N \int_\Omega \hat{A}_j(U_0)\partial_j v\, dx, \\ \langle \mathfrak{B}U_\varepsilon, v\rangle + \int_\Omega f_\varepsilon v\, dx &\to \langle \mathfrak{B}U_0, v\rangle + \int_\Omega f v\, dx. \end{aligned} \tag{4.4.7}$$

Proof. The proof of Lemma 4.3.5, replacing all u_ε with U_ε, gives us the convergences

$$\sum_{j=1}^N \int_\Omega \hat{A}_j(U_\varepsilon)\partial_j v\, dx \to \sum_{j=1}^N \int_\Omega \hat{A}_j(U_0)\partial_j v\, dx, \quad \text{and} \quad \langle \mathfrak{B}U_\varepsilon v\rangle \to \langle \mathfrak{B}U_0, v\rangle \text{ as } \varepsilon \to 0.$$

For the remaining term $\int_\Omega f_\varepsilon v\, dx$, we recall that by construction, $f_\varepsilon \to f$ in $L^1(\Omega)$. Since $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ by assumption, we obtain

$$\int_\Omega |f_\varepsilon v - f v|\, dx \le \|v\|_{L^\infty(\Omega)} \int_\Omega |f_\varepsilon - f|\, dx \to 0 \text{ as } \varepsilon \to 0.$$

This gives us the convergences in (4.4.7). □

Lemma 4.4.4. We have $\hat{\Phi}(U_0) \in L^1(\Omega)$ and the convergence

$$\hat{\Phi}(U_\varepsilon) \to \hat{\Phi}(U_0) \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.4.8}$$

Proof. We first show that $\hat{\Phi}(U_0) \in L^1(\Omega)$. From Corollary 4.4.1 we have, up to subsequence, the convergences $\nabla U_\varepsilon \to \nabla U_0$ and $U_\varepsilon \to U_0$ a.e. in Ω. Since Φ is Carathéodory, continuity in the last two variables imply we have $|\hat{\Phi}(U_\varepsilon)| \to |\hat{\Phi}(U_0)|$ a.e. in Ω. Fatou's Lemma then gives us

$$\int_\Omega |\hat{\Phi}(U_0)|\, dx = \int_\Omega \liminf_{\varepsilon\to 0} |\hat{\Phi}(U_\varepsilon)|\, dx \le \liminf_{\varepsilon\to 0} \int_\Omega |\hat{\Phi}(U_\varepsilon)|\, dx \le K$$

where K is the upper bound from Lemma 4.4.2 (i). This shows that $\hat{\Phi}(U_0) \in L^1(\Omega)$. It remains to show the convergence (4.4.8), for which we will use Vitali's Theorem. We already know $|\hat{\Phi}(U_\varepsilon)| \to |\hat{\Phi}(U_0)|$ a.e. in Ω, so it suffices to show that $\{|\hat{\Phi}(U_\varepsilon)|\}_\varepsilon$ is uniformly integrable, for which the divide and conquer strategy comes in handy. Let $M > 1$ be an arbitrary constant and E be an arbitrary measurable subset of Ω. We write $|\hat{\Phi}(U_\varepsilon)|$ as

$$|\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|\le M\}} + |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|> M\}}. \tag{4.4.9}$$

We can bound $|\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|\le M\}}$ using the growth condition on Φ in (4.1.3):

$$\begin{aligned} \int_E |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|\le M\}}\, dx &\le \int_E \chi_{\{|U_\varepsilon|\le M\}}\zeta(M)\left(\sum_{j=1}^N |\partial_j U_\varepsilon|^{p_j} + c\right) dx \\ &\le \int_E \zeta(M)\left(\sum_{j=1}^N |\partial_j T_M(U_\varepsilon)|^{p_j} + c\right) dx. \end{aligned} \tag{4.4.10}$$

We recall that $c \in L^1(\Omega)$ is a non-negative function by assumption and that $\partial_j T_M(U_\varepsilon) \to \partial_j T_M(U_0)$ in $L^{p_j}(\Omega)$ from Corollary 4.4.1. Applying Proposition 1.4.1 and Corollary 1.4.1, we see that $|\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|\le M\}}$ is uniformly integrable, i.e., the upper bound in (4.4.10) can be made uniformly small when meas(E) is small enough.

We now consider $|\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|>M\}}$ in (4.4.9). The strategy here is to prove that by choosing $M > 1$ large enough independently of $\varepsilon > 0$, then we can make $\|\hat{\Phi}(U_\varepsilon)\chi_{\{|U_\varepsilon|>M\}}\|_{L^1(\Omega)}$ as small as desired; the choice of M must be independent of ε since we already have a restriction on meas(E) from bounding $\hat{\Phi}(U_\varepsilon)$ in $L^1(\Omega)$, which depends on the convergence $\partial_j T_M(U_\varepsilon) \to \partial_j T_M(U)$ on E and hence on M. Take as a test function $v = T_1(G_{M-1}(U_\varepsilon)) \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ in (4.4.6). Then we have

$$\begin{aligned}
\partial_j T_1(G_{M-1}(U_\varepsilon)) &= \chi_{\{|U_\varepsilon - T_{M-1}(U_\varepsilon)|\le 1\}}(\partial_j U_\varepsilon - \partial_j T_{M-1}(U_\varepsilon)) \\
&= \chi_{\{|U_\varepsilon - T_{M-1}(U_\varepsilon)|\le 1\}}(\partial_j U_\varepsilon - \chi_{\{|U_\varepsilon|<M-1\}}\partial_j U_\varepsilon) \\
&= \chi_{\{|U_\varepsilon - T_{M-1}(U_\varepsilon)|\le 1\}\cap\{|U_\varepsilon|>M-1\}}\partial_j U_\varepsilon \\
&= \chi_{\{M-1<|U_\varepsilon|<M\}}\partial_j U_\varepsilon.
\end{aligned}$$

By applying the coercivity condition for $\hat{A}_j$, for all $1 \le j \le N$ we obtain

$$\begin{aligned}
\sum_{j=1}^N \int_\Omega \hat{A}_j(U_\varepsilon)\partial_j T_1(G_{M-1}(U_\varepsilon))\,dx &= \sum_{j=1}^N \int_\Omega \chi_{\{M-1<|U_\varepsilon|<M\}}\hat{A}_j(U_\varepsilon)\partial_j U_\varepsilon\,dx \\
&\ge \nu_0 \int_\Omega \chi_{\{M-1<|U_\varepsilon|<M\}} \sum_{j=1}^N |\partial_j U_\varepsilon|^{p_j}\,dx \ge 0.
\end{aligned}$$

Since $\sum_{j=1}^N \int_\Omega \hat{A}_j(U_\varepsilon)\partial_j T_1(G_{M-1}(U_\varepsilon))\,dx$ is non-negative, we can ignore this term in the integral equation (4.4.2) with test function chosen to be $T_1(G_{M-1}(U_\varepsilon))$ to obtain the inequality

$$\int_\Omega \hat{\Phi}(U_\varepsilon)T_1(G_{M-1}(U_\varepsilon))\chi_{\{|U_\varepsilon|>M\}}\,dx \le \langle \mathfrak{B}U_\varepsilon, T_1(G_{M-1}(U_\varepsilon))\rangle + \int_\Omega f_\varepsilon T_1(G_{M-1}(U_\varepsilon))\,dx. \tag{4.4.11}$$

Now, from the sign condition, we know that $\operatorname{sgn}(\hat{\Phi}(U_\varepsilon)) = \operatorname{sgn}(U_\varepsilon)$. Since $tG_k(t) \ge 0$ for any $k > 0$ and $t \in \mathbb{R}$, we conclude that $\hat{\Phi}(U_\varepsilon)T_1(G_{M-1}(U_\varepsilon)) \ge 0$. We further notice that $\chi_{\{|U_\varepsilon|>M\}}G_{M-1}(U_\varepsilon) \ge 1$ if $U_\varepsilon > 0$ and $\chi_{\{|U_\varepsilon|>M\}}G_{M-1}(U_\varepsilon) \le -1$ if $U_\varepsilon < 0$. Hence, $\chi_{\{|U_\varepsilon|>M\}}|T_1(G_{M-1}(U_\varepsilon))| = 1$. Applying absolute values to (4.4.11) (the expressions are non-negative by the sign condition on Φ), we obtain

$$\int_\Omega |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|>M\}}\,dx \le |\langle \mathfrak{B}U_\varepsilon, T_1(G_{M-1}(U_\varepsilon))\rangle| + \int_\Omega |f_\varepsilon|\chi_{\{|U_\varepsilon|>M-1\}}\,dx \tag{4.4.12}$$

where we have also used that $|T_1(G_{M-1}(U_\varepsilon))| \le 1$ and is zero when $|U_\varepsilon| \le M-1$, in order to replace the term $\int_\Omega f_\varepsilon T_1(G_{M-1}(U_\varepsilon))\,dx$. Now, since up to subsequence $U_\varepsilon \rightharpoonup U_0$ weakly in $W_0^{1,\vec{p}}(\Omega)$ from Lemma 4.4.2 (ii), we have

$$T_1(G_{M-1}(U_\varepsilon)) = T_1(U_\varepsilon - T_{M-1}(U_\varepsilon)) \rightharpoonup T_1(U_0 - T_{M-1}(U_0)) = T_1(G_{M-1}(U_\varepsilon)) \text{ weakly in } W_0^{1,\vec{p}}(\Omega).$$

Using this weak convergence, property (P_2) of $\mathfrak{B}$ and the fact that $f_\varepsilon \to f$ in $L^1(\Omega)$, we take $\limsup_{\varepsilon\to 0}$ in (4.4.12) to get

$$\limsup_{\varepsilon\to 0}\int_\Omega |\hat{\Phi}(U_\varepsilon)|\chi_{\{|U_\varepsilon|>M\}}\,dx \le |\langle \mathfrak{B}U_0, T_1(G_{M-1}(U_0))\rangle| + \int_\Omega |f|\chi_{\{|U_0|>M-1\}}\,dx. \tag{4.4.13}$$

Here, we must recall that whenever we take the limit $\varepsilon \to 0$, we are actually taking a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$. By definition of lim sup, for any $\rho > 0$ there exist $\eta \in \mathbb{N}$ such that for all $j \geq \eta$, we can bound $\int_\Omega |\hat{\Phi}(U_{\varepsilon_j})|\chi_{\{|U_{\varepsilon_j}|>M\}}\,dx$ from above by

$$\rho + \limsup_{n\to\infty} \int_\Omega |\hat{\Phi}(U_{\varepsilon_n})| \leq \rho + |\langle \mathfrak{B}U_0, T_1(G_{M-1}(U_0))\rangle| + \int_\Omega |f|\chi_{\{|U_0|>M-1\}}\,dx. \tag{4.4.14}$$

Now, $f \in L^1(\Omega)$ and satisfies the uniform integrability condition by Proposition 1.4.1. Since U_0 is well-defined, we have $\operatorname{meas}(\{x \in \Omega \mid |U_0(x)| = \infty\}) = 0$. Hence, for some $M > 1$ large enough, we can force $\operatorname{meas}(\{|U_0| > M-1\})$ small enough so that

$$\int_\Omega |f|\chi_{\{|U_0|>M-1\}}\,dx = \int_{\Omega\cap\{\{|U_0|>M-1\}\}} |f|\,dx$$

is small, uniformly with respect to ε. It remains to show that $|\langle \mathfrak{B}U_0, T_1(G_{M-1}(U_0))\rangle|$ can be arbitrarily bounded by varying M, independently of E and ε. The inequality (4.2.22) gives us

$$\begin{aligned}
|\langle \mathfrak{B}U_0, T_1(G_{M-1}(U_0))\rangle| &\leq \mathfrak{C}_1\left(1 + \|U_0\|^{\mathfrak{b}}_{W_0^{1,\vec{p}}(\Omega)}\right)\|T_1(G_{M-1}(U_0))\|_{W_0^{1,\vec{p}}(\Omega)} \\
&= \mathfrak{C}_1\left(1 + \|U_0\|^{\mathfrak{b}}_{W_0^{1,\vec{p}}(\Omega)}\right)\sum_{j=1}^N \|\partial_j T_1(G_{M-1}(U_0))\|_{L^{p_j}(\Omega)} \\
&= \mathfrak{C}_1\left(1 + \|U_0\|^{\mathfrak{b}}_{W_0^{1,\vec{p}}(\Omega)}\right)\sum_{j=1}^N \left(\int_\Omega \chi_{\{M-1<|\partial_j U_0|<M\}}|\partial_j U_0|^{p_j}\,dx\right)^{1/p_j}.
\end{aligned}$$

Now, $\mathfrak{C}_1\left(1 + \|U_0\|^{\mathfrak{b}}_{W_0^{1,\vec{p}}(\Omega)}\right)$ is a fixed quantity. For each $1 \leq j \leq N$, we see that $|\partial_j U_0|^{p_j} \in L^1(\Omega)$ and hence is uniformly integrable. Since $|\partial_j U_0|^{p_j}$ is a well-defined function taking on finite values a.e. on Ω, we see that

$$\sum_{j=1}^N \left(\int_\Omega \chi_{\{M-1<|U_0|<M\}}|\partial_j U_0|^{p_j}\,dx\right)^{1/p_j} = \sum_{j=1}^N \left(\int_{\Omega\cap\{M-1<|U_0|<M\}} |\partial_j U_0|^{p_j}\,dx\right)^{1/p_j},$$

which can be made arbitrarily small by taking M large enough (indeed, $\operatorname{meas}(\{M-1 < |U_0| < M\})$ tends to 0 as $M \to \infty$). Hence, we see that for M large enough, independent of ε, $|\langle \mathfrak{B}U_0, T_1(G_{M-1}(U_0))\rangle|$ can be made arbitrarily small. It follows that we can bound $|\hat{\Phi}(U_{\varepsilon_n})|\chi_{\{|U_\varepsilon|>M\}}$ arbitrarily in $L^1(\Omega)$ with the inequality (4.4.14) by taking ρ small enough and fixing some M large for all $n > \eta$.

It remains to bound the remaining finite number of terms of $|\hat{\Phi}(U_{\varepsilon_n})|\chi_{\{|U_\varepsilon|>M\}}$ corresponding to $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\eta-1}$. To do so, we simply recall the inequality in (4.4.12) and note that for each $n = 1, 2, \ldots, \eta-1$, we can force $|\langle \mathfrak{B}U_{\varepsilon_n}\rangle, T_1(G_{M-1}(U_\varepsilon))|$ and $\int_\Omega |f_{\varepsilon_n}|\chi_{\{|U_{\varepsilon_n}>M-1|\}}$ arbitrarily small, using the same argument as we did above, by picking M large enough (concretely, one would perform the above argument for a fixed ε_n with $1 \leq n \leq \eta-1$, obtain a choice of M then redefine M to be the maximum over these finitely many values). This shows that $\{\hat{\Phi}(U_{\varepsilon_n})\chi_{\{|U_{\varepsilon_n}|>M\}}\}_{n\in\mathbb{N}}$ is uniformly integrable over Ω. Since we have already shown that $\hat{\Phi}(U_{\varepsilon_n})\chi_{\{|U_\varepsilon|\leq M\}}$ is uniformly integrable, we conclude that their sum $\hat{\Phi}(U_\varepsilon)$ is uniformly integrable over Ω. By Vitali's Theorem, we have the convergence of (4.4.8), as required. $\square$

Theorem 4.4.1. Suppose the assumptions (4.1.2), (4.1.3), (4.1.4) and (4.1.5) hold, with $\mathfrak{B} : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ satisfying properties (P_1) and (P_2) and $f \in L^1(\Omega)$ arbitrary. Then there exists a function $U_0 \in W_0^{1,\vec{p}}(\Omega)$ such that $\Phi(x, U_0, \nabla U_0) \in L^1(\Omega)$ and satisfies the anisotropic elliptic PDE

$$\begin{cases} \mathcal{A}u - \mathfrak{B}u + \Phi(x,u,\nabla u) = f & \text{in } \Omega, \\ u \in W_0^{1,\vec{p}}(\Omega), \quad \Phi(x,u,\nabla u) \in L^1(\Omega). \end{cases}$$

in a weak sense, i.e., in the sense of Definition 4.1.1.

Proof. We have already proved that $\Phi(U_0) \in L^1(\Omega)$ in Lemma 4.4.4. Since U_ε is a solution of the approximate PDE (4.4.1), it satisfies

$$\sum_{j=1}^{N} \int_\Omega \hat{A}_j(U_\varepsilon)\partial_j v\, dx + \int_\Omega \hat{\Phi}(U_\varepsilon) v\, dx = \langle \mathfrak{B}U_\varepsilon, v\rangle + \int_\Omega f_\varepsilon v\, dx \quad \text{for all } v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

By taking the limit as $\varepsilon \to 0$ and using the convergence results of Lemma 4.4.3 and Lemma 4.4.4, we conclude

$$\sum_{j=1}^{N} \int_\Omega \hat{A}_j(U_0)\partial_j v\, dx + \int_\Omega \hat{\Phi}(U_0) v\, dx = \langle \mathfrak{B}U_0, v\rangle + \int_\Omega f v\, dx \quad \text{for all } v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

That is, U_0 is a weak solution of the original anisotropic PDE (4.1.1) with arbitrary L^1 data f and lower order gradient dependent term Φ. This completes the proof, as well as the aim of this thesis. $\square$

Appendices

Chapter A

Remark and Relevant Results

In the appendix, we begin with an informal contextual remark on the methods that were applied in this thesis and the reasons for choosing particular methods of proof. We also include some results that are used in the text but are not included in the main body. Reasons for exclusion include non-relevance to other results or that it is taught in courses deemed to be assumed knowledge (e.g. Measure Theory, Functional Analysis, Variational Methods).

A.1. Remark on Chapter 4.

Remark A.1.1. Figure 1 depicts a simplistic rendition of the problem-solving process that occurs in Chapter 4. We provide commentary on the figure and the thesis as a whole.

First, we note that it is natural to consider if the approximation steps for solving the PDE in Chapter 4 can be performed in a different order, i.e., treating the f term first and the Φ term second, or perhaps even simultaneously approximating both terms. The motivation for setting $f = 0$ and approximating Φ first is two-fold - we want to weaken the assumptions enough on the problem so that we can actually obtain a sequence of approximate weak solutions (this is Requirement 1), and we would like to guarantee "good properties" of these approximate solutions, such as having a priori estimates. This is Requirement 2. Indeed, in general, without treating Φ in some way first, it would not be possible to obtain approximate solutions, since a priori we do not even know $\Phi(x, u, \nabla u) \in L^1(\Omega)$ for $u \in W_0^{1,\vec{p}}(\Omega)$; any other method would fail Requirement 1. If we decide to approximate Φ first, the natural thought would be to consider if we could simultaneously approximate f with f_ε as well. Here, we can only give anecdotal evidence; the methods that were used can obtain a sequence of approximate solutions $(u_\varepsilon)_\varepsilon$ as we did in this thesis, but we fall afoul of Requirement 2 - we cannot obtain a priori estimates on $(u_\varepsilon)_\varepsilon$, which is the essential step allowing us to use weak convergence arguments in $W_0^{1,\vec{p}}(\Omega)$. In this way, we note that the approximation order employed in this thesis is, in a sense, natural, since all other paths fail at either Requirement 1 or Requirement 2.

We also point out some of the nuances and specific choices we make in the *convergence arguments*, where we let $\varepsilon \to 0$ in Figure 1 to obtain a limit which is a solution of the non-approximate PDE. Indeed, in the first approximation setting $f = 0$, we did not need to work with the truncations $T_k(u_\varepsilon)$ as we did starting in Proposition 4.3.1. Instead, we could have shown directly that, up to subsequence, $u_\varepsilon \to U$ strongly in $W_0^{1,\vec{p}}(\Omega)$, as shown in [3, Section 5.1]. It can be seen that the methods from the proof presented in [3] cannot be repeated in the second approximation, with the culprit being the much weaker assumptions in the second approximation (i.e., we have $f_\varepsilon \neq 0$ in the second approximated PDE and we do not have Φ is automatically bounded unlike Φ_ε)) - in particular, the estimates that we would like to have simply do not hold anymore in this weaker case. Although for most of the convergence arguments we only require the pointwise a.e. convergence

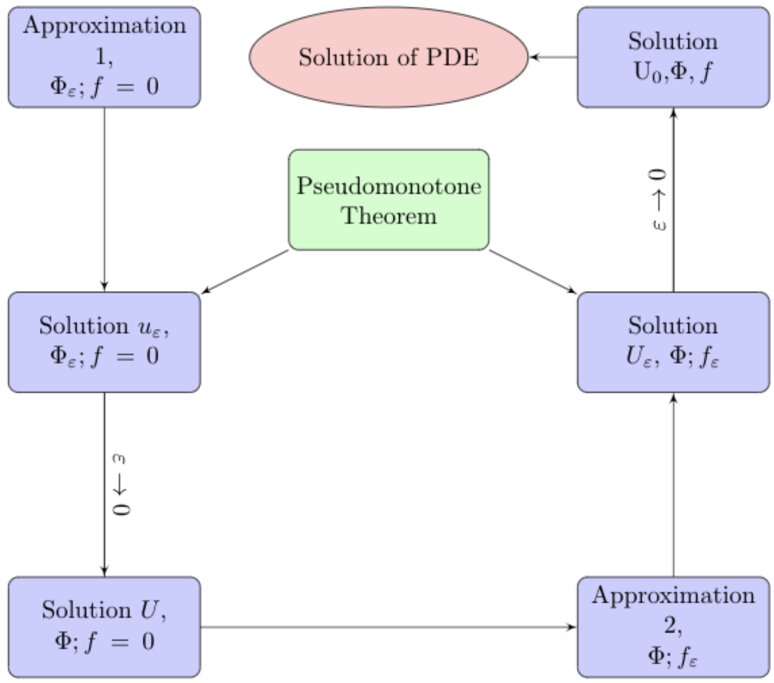

FIGURE 1. Diagram of approximation process for solving anisotropic elliptic PDE with arbitrary input data $f \in L^1(\Omega)$ and Φ.

$\nabla U_\varepsilon \to \nabla U_0$ on Ω, we point out in equation (4.4.10) the presence of $\nabla T_k(U_\varepsilon)$ and the requirement that $\nabla T_k(U_\varepsilon)$ converges strongly to $T_k(U)$ in $W_0^{1,\vec{p}}(\Omega)$ for any $k \geq 1$ to show uniform integrability and apply Vitali's Theorem. In general, it is not sufficient to just show that $\nabla u_\varepsilon \to \nabla U$ a.e. in Ω or that $\nabla U_\varepsilon \to \nabla U_0$ a.e. in Ω (in the second approximation).

Finally, the reason we opt to work with $\nabla T_k u_\varepsilon$ and $\nabla T_k u$ in Proposition 4.3.1 rather than ∇u_ε and ∇U, as we do in Proposition 4.2.2, is so that we can show $D_{\varepsilon,k} \to 0$ in $L^1(\Omega)$, which would not have been possible if we simply used the definition of D_n from Proposition 4.2.2 without the truncated derivatives.

A.2. Relevant Results

Theorem A.2.1 (Fatou's Lemma)**.** Let (Ω, σ, μ) be a measure space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions. Define

$$f(x) = \liminf_{n\to\infty} f_n(x)$$

for all $x \in \Omega$. Then

$$\int_\Omega f \, d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \, d\mu.$$

Theorem A.2.2 (Hölder's Inequality)**.** Let (Ω, μ) be a measure space and $p, p' \geq 1$ such that $1/p + 1/p' = 1$. Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$ and

$$\int_\Omega |fg|\, d\mu \leq \|f\|_p \|g\|_{p'}$$

Theorem A.2.3 (Monotone Convergence Theorem)**.** Suppose (Ω, σ, μ) is a measure space. Suppose we have a sequence of functions $(f_n)_{n\in\mathbb{N}}$ with $f_n : \Omega \to [0, \infty]$ measurable, satisfying

$$0 \leq f_n \leq f_{n+1} \text{ a.e. in } \Omega \text{ for all } n \in \mathbb{N}.$$

Then there exists a measurable function $f : \Omega \to [0, \infty]$ such that $f_n \to f$ a.e. in Ω and

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_X f\, d\mu.$$

Theorem A.2.4 (Dominated Convergence Theorem)**.** Let (Ω, σ, μ) be a measure space. Suppose we have a sequence of measurable functions $(f_n)_{n\in\mathbb{N}}$ with $f_n : \Omega \to \mathbb{R}$ and a measurable function $f : \Omega \to \mathbb{R}$ such that $f_n \to f$ a.e. in Ω. Furthermore, suppose that there exists a function $g \in L^1(\Omega)$ such that $|f_n| \leq g$ a.e. in Ω for all $n \in \mathbb{N}$. Then $f_n \to f$ in $L^1(\Omega)$ as $n \to \infty$.

Proposition A.2.1 (Limit Inferior and Superior of Sums)**.** Suppose $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}$ such that a_n converges to some limit $a \in \mathbb{R}$. Then

$$\liminf_{n\to\infty}(a_n + b_n) = a + \liminf_{n\to\infty} b_n \quad \text{and} \quad \limsup_{n\to\infty}(a_n + b_n) = a + \limsup_{n\to\infty} b_n.$$

Proposition A.2.2 (Sub-super additivity of lim inf and lim sup)**.** Suppose $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}$ such that $\liminf_{n\to\infty} a_n$ and $\liminf_{n\to\infty} b_n$ are finite. Then

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Proposition A.2.3 (Arithmetic-Geometric Mean Inequality)**.** Suppose $x_1, x_2, \dots, x_N$ are N non-negative real numbers. Then

$$\sqrt[n]{x_1 x_2 \dots x_N} \leq \frac{1}{N} \sum_{j=1}^{N} x_j.$$

Theorem A.2.5 (Green's Identity)**.** Let Ω be a bounded subset of $\mathbb{R}^N$ with boundary Γ of class C^1. Let $u \in C^2(\overline{\Omega})$ and $v \in C^1(\overline{\Omega})$. Then

$$\int_\Omega v\Delta u\, dx = \int_\Gamma v\frac{\partial u}{\partial n}\, d\sigma + \int_\Omega \nabla v \cdot \nabla u\, dx$$

where $\frac{\partial u}{\partial n}$ denotes the directional directive of u with respect to the outward normal vector n on the boundary Γ and $d\sigma$ is the measure on the boundary Γ. In particular, if either v or $\frac{\partial u}{\partial n}$ vanishes on Γ, we have the identity

$$\int_\Omega v\Delta u\, dx = \int_\Omega \nabla v \cdot \nabla u\, dx.$$

Proposition A.2.4 (Properties of Convolutions)**.** (See [4, Theorem 4.15 and Proposition 4.18, p. 104, 106]). Let $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$ with $1 \leq p \leq \infty$. For a.e. $x \in \mathbb{R}^N$, the function $y \mapsto f(x - y)g(y)$ is integrable on $\mathbb{R}^N$. The *convolution* of f and g, written as $f * g : \mathbb{R}^N \to \mathbb{R}$, is defined to be

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x - y)g(y)\, dy.$$

Furthermore, $f * g \in L^p(\mathbb{R}^N)$ with

$$\|f * g\|_{L^p(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)} \|g\|_{L^p(\mathbb{R}^N)}$$

and we can contain the support of $f * g$:

$$\operatorname{supp}(f * g) \subseteq \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}.$$

Proposition A.2.5 (Mollifiers)**.** (See [4, p. 108]). A sequence of mollifiers $(\rho_n)_{n\in\mathbb{N}}$ is any sequence of functions on $\mathbb{R}^N$ satisfying

$$\rho_n \geq 0, \quad \rho_n \in C_c^\infty(\mathbb{R}^N), \quad \operatorname{supp}\, \rho_n \subseteq \overline{B(0, 1/n)} \quad \text{and} \quad \int_{\mathbb{R}^N} \rho_n \, dx = 1.$$

Such a sequence can be generated by taking

$$\rho(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

and defining $\rho_n(x) := \frac{n^N \rho(nx)}{\int_{\mathbb{R}}^N \rho \, dx}$.

Theorem A.2.6 (Riesz Representation Theorem)**.** (See [4, Theorem 4.11 and Theorem 4.14, p. 97, 99]).

Let $1 < p < \infty$ and (Ω, σ, μ) be a measure space. Suppose $\phi \in (L^p(\Omega))^*$. Then there exists a unique $u \in L^{p'}(\Omega)$ such that

$$\langle \phi, f \rangle = \int_\Omega uf \, d\mu \quad \text{for all } f \in L^p(\Omega)$$

with

$$\|u\|_{p'} = \|\phi\|_{L^p(\Omega)^*}.$$

With the Riesz Representation Theorem, we canonically identify the space $L^{p'}(\Omega)$ and the dual space $(L^p(\Omega))^*$ by the isometry $L^{p'}(\Omega) \to (L^p(\Omega))^*, u \mapsto \varphi$, where φ maps $f \in L^p(\Omega)$ to $\int_\Omega uf \, d\mu$. Moreover, if (Ω, σ, μ) is σ-finite, then the Riesz Representation Theorem applies to the case $p = 1$, whence we obtain the canonical isomorphism between $L^1(\Omega)$ and $L^\infty(\Omega)$.

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