Math 340 Student Guide

Fall 2024

How to use this guide

This is the student guide to Math 320. It has information about the course content organized by textbook chapter. For each chapter, you will find learning goals, suggested practice problems from the book, and an overview of the main concepts.

General Information about Math 340

Math 340 is Elementary Matrix and Linear Algebra. It is a linear algebra course that focuses on the computational aspects of the subject. We have other introductory linear algebra courses (Math 341, Math 375) which focus on developing proof-writing skills. Additionally, we have a combined introductory differential equations/linear algebra course (Math 320).

We will still require you to prove/verify claims, but in a way that's more systematic than a theory-based course. For example, we will ask you to prove/disprove sets are subspaces, prove/disprove functions are linear transformations, and we will often require you to explain/justify your claims on exams. You should plan to practice problems from the textbook in addition to the online homework to prepare for quizzes and exams.

Key: P=Practice Problems from book (not graded) R=Reading

Chapter 1 (§1.1-§1.3)

R: §1.1-§1.3
P: §1.1: 1,7, 8, 10, 14 §1.2: 1-5, 8-12, 22 §1.3: 1-7, 11, 12

Learning Goals:

- Be able to solve linear systems using the method of elimination
- Be able to set up an augmented matrix for a linear system of equations AND be able to interpret an augmented matrix as a linear system of equations
- Be able to perform elementary row operations to a matrix
- Know how to find a REF or the RREF form for a matrix
- Be able to solve linear system using the method of Gaussian Elimination
- Be able to find the rank of a matrix and know what it tells you about the number of solutions for the corresponding linear system
- Know how to build the basic solutions for a homogeneous system
- Know how to determine the number of basic solutions of a homogeneous system using the rank

Student Notes: Linear Algebra is well-named; it focuses on algebraic equations which are linear. The highly structured nature of these equations allows us to develop rigorous methods of solving them as well as prove theorems about their structure. We will see in this course how this theory can be applied to stranger looking mathematical systems. Your book also provides many examples of applications of linear algebra to other areas: chemistry, differential equations, economics, and more.

Most students have solved systems of linear equations in past courses with one of two methods; either substitution or elimination. In §1.1, we will see how we can represent the method of elimination in a more streamlined way to solve systems efficiently.

In §1.2, we will rigorously define this process in terms of matrices, and begin discussing the structure of the matrices and how it relates to the solution set of the system.

We will finish Chapter 1 by focusing on linear systems with a special structure; the constant part of each equation will be set to zero (these systems are called *homogeneous*). Since these systems always have at least one solution (where each unknown is set to zero, called the trivial solution), these systems are always consistent. The question then becomes: when will the system have infinitely many solutions, and when will it have only the trivial solution? If the system has infinitely many solutions, is there a “finite way” to store these solutions?

Chapter 2 (§2.1-§2.4, §2.6)

R: §2.1-§2.4, §2.6
P: §2.1: 1-4, 12-17, 19 §2.2: 1-4, 9, 10, 14, 18 §2.3: 1, 3, 4, 6-9, 27, 29, 30 §2.4: 1(b has an error), 2, 4, 5, 9, 19, 21 §2.6: 1, 3, 5, 7, 14, 15, 20

Learning Goals:

- Be able to pick out an element of a matrix given its indices
- Know what it means for two matrices to be equal vs. row equivalent
- Know how to perform matrix operations: sum/difference, scalar multiplication, transpose, matrix-vector product (both as a linear combination and using dot product), Know how to perform matrix multiplication (both as repeated matrix-vector multiplication and using dot product)
- Know the basic properties of matrix operations
- Know which sizes of matrices can be multiplied together, and what the resulting product matrix's size will be
- Know the properties of matrix multiplication and how they interact with previous operations
- Be familiar with properties of inverses of matrices: uniqueness, inverse of a product, inverse of inverse, inverse of transpose
- Know how to use matrix inverses to solve linear systems of equations
- Be able to compute inverse matrices directly using row reduction
- Know the equivalent properties to determine if a matrix is invertible (see box on page 87 of your textbook, i.e. "Box of Facts")
- Know how to find the inverse of a matrix transformation
- Be able to compute the image of a vector under a matrix transformation
- Be able to determine if a vector is in the range/image of a matrix transformation
- Find the matrix representation of a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

Student Notes:
Chapter 2 discusses the algebra of matrices. We can do a lot of arithmetic operations with matrices in an analogous way to real numbers; we can add them, subtract them, and multiply them. Pay attention to which arithmetic properties from real numbers carry over to matrix operations, and which do not.

You may have seen matrix multiplication in the past. We will define matrix multiplication carefully by first setting up matrix-vector multiplication in §2.2, then define matrix multiplication in §2.3. You might be wondering why we define multiplication differently if you end up getting the same final answer. The reason is that we will be using multiplication in many different contexts, and to justify that our methods are correct, it may be advantageous to use one definition of multiplication versus the other. You should practice multiplying matrices using both methods to become comfortable with thinking about matrix multiplication using either notion, depending on the context.

In §2.4, we will learn how to find the inverse of a matrix. You can think about matrix inverses as being the analogue to reciprocals for real numbers; however, while 0 is the only real number without a reciprocal, there are many nonzero matrices that do not have an inverse (for example, $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$). One of the big learning goals is to know (and understand!) the conditions for invertibility of a matrix.

We skip §2.5, but you're welcome to read it to learn more about the theory of inverses.

We will end Chapter 2 by learning about matrix transformations of vectors. Matrix transformations have a lot of real life applications, such as data analysis and visual art (see your Canvas course main image for an example!).

Chapter 3 (§3.1-§3.3)

R: §3.1, §3.2 (until "Adjugates"), §3.3 (until "Linear Dynamical Systems")
P: §3.1: 1afh, 3, 5, 7, 8, 9, 13 §3.2: 2-7, 10, 12, 13, 20bcdeijk, 21, 26 §3.3: 1, 3, 8, 9, 12, 18, 20, 21

Learning Goals:

- Be able to compute determinants of small (2×2 and 3×3) matrices using cofactor expansion
- Be familiar with the properties of determinants
- Be able to compute determinants of larger matrices using properties of determinants
- Be able to compute determinants of larger matrices using cofactor expansion through specific choice of row/column
- Know the relationship between determinants and inverses
- Know how to verify if a vector is an eigenvector for a matrix
- Be able to compute the eigenvectors/eigenvalues (real eigenvalues only) for a matrix using the characteristic equation
- Be able to find a set of basic eigenvectors for an eigenvalue of a matrix
- Be able to determine if a matrix is diagonalizable
- Be able to diagonalize a matrix (i.e. find P and D)

Student Notes:

In Chapter 3, we will learn about the determinant, which is a number associated with a matrix. Among other things, the determinant will tell you if a matrix is invertible.

There are several ways to calculate the determinant; we will compute the determinant using *cofactor expansion.* Calculating the determinant for 3x3 matrices will probably look familiar to you; in Calculus 2 and 3, you use this method to calculate the cross product of a pair of vectors. Note that the determinant will give you a number instead of a vector. While calculating the determinant, note that the choice of row/column does not affect the value of the determinant. Keep this in mind when we're proving the numerous properties of the determinant.

We will end Chapter 3 by seeing one application of determinants; finding the eigenvalues and eigenvectors of a matrix. Thinking about a matrix as a linear transformation, the eigenvectors are the specific vectors where the image of the vector is merely scaled. While this problem may seem complicated, we will see that it really boils down to solving a system of homogeneous equations!

The book goes into more detail about applications of determinants than we will cover in lecture. You're welcome to read all of the sections if you're interested, but otherwise you can stop reading §3.2 at the subsection "Adjugates", and you can stop reading §3.3 at the subsection "Linear Dynamical Systems."

Chapter 5 (§5.1-§5.2, §5.4-§5.5)

R: §5.1-§5.2, §5.4-§5.5
P: §5.1: 1-5, 7-9, 12, 15, 16, 17, 20 §5.2: 1, 2, 4, 6, 7, 10, §5.4: 1-3, 7 §5.5: 1-4, 6-8, 10

Learning Goals:

- Be able to verify that a subset of a vector space is or is NOT a subspace
- Find a spanning set for a subspace
- Be able to verify if a vector is in the span of other vectors
- Find a spanning set for the nullspace of a matrix
- Find a spanning set for the image space of matrix
- Be able to determine if a set of vectors is linearly independent via a system of equations
- Be able to determine if a set of vectors is linearly independent via the determinant of a matrix or demonstrating a linear combination
- Know how span/linear independence connects to the invertibility criteria of matrices
- Know how to show a set is or is NOT a basis for a subspace
- Know how to find the dimension of a subspace
- Know how rank connects to dimension of row spaces/column spaces
- Know how to find a basis for the column space of a matrix (pick up the pivots!)
- Understand the difference between similarity and row equivalence
- Know how to use similarity to find shared properties of similar matrices
- Be able to find the algebraic multiplicity and geometric multiplicity of an eigenvalue
- Be able to determine if a matrix is diagonalizable

Student Notes: We're going to now start looking at how vectors can interact with each other in a more general sense. We'll start by investigating when a subset of n-dimensional space can "act like" a smaller dimensional space sitting inside the bigger space. Such sets are called *subspaces*. Visually, subspaces of 3-space will look like lines and planes running through the origin. While it's helpful to think about the geometric visualization of subspaces, it's important to understand their algebraic properties, as well.

We will ask you to prove that sets are (or are NOT) subspaces. A subset is a subspace if it satisfies three criteria; (1) the zero vector is in the subset, (2) for every pair of vectors in the subset, their sum is also in the subset, (3) for every vector in the subset and every scalar (real number), the scalar multiple of the vector is also in the subset. Notice that the criteria says "for every": this means that showing that the sum of a specific pair of vectors is in the set is NOT sufficient; this only shows that criterion (2) holds FOR THAT PAIR. To justify property (2) holds, you need to use variables to run through every possible element of that subset. The same method holds for property (3): you need to use variables to run through every possible element of the subset and every possible scalar. To show a subset is NOT a subspace, you want to show that the "for every" condition does not hold. This means you should use explicit examples (explicit means with actual numbers, not variables). For property (2), that means you should pick an explicit pair of vectors that are in the subset, but their sum is NOT in the subset. For property (3), you should pick an explicit vector that is in the subset, and an explicit scalar, and show that the scalar multiple is NOT in the subset. You will see examples in lecture and discussion of subsets which are (or are not) subspaces; pay attention to the proofs to see how to correctly justify your answers.

All nontrivial subspaces have an infinite number of vectors; however, we will see how to describe a subspace with a finite number of vectors. In other words, we will learn how to build every vector in a subspace through a linear combination of a specific fixed set of vectors. This set of “generating vectors” is called the span of the subspace, and we’ll start learning methods for finding such sets.

How do we know when a spanning set is “minimal”? You can think about this problem as having vectors which are “unique” enough that the linear combination has a unique set of scalars. Such sets of vectors are called linearly independent, and we’ll learn how to test for this (hint: we’re going to use some of our old tricks!)

In §5.5, we return to the concept of eigenvalues/eigenvectors (which we first saw in §3.3), and learn more about diagonalizable matrices. Diagonalizable matrices can be associated in a structured way to diagonal matrices, which helps us compute properties of these matrices efficiently. However, not all matrices are diagonalizable! We’ll learn a condition for diagonalizability that will utilize our new concepts; specifically, linear independence.

Chapter 6 (§6.1-§6.4)

R: §6.1-§6.4
P: §6.1: 1-4, 6, 7 §6.2: 1, 2, 3abcdg (replace $F[0, 1]$ with P_2) 4-7, 9, 11, 13, 21 §6.3: 1, 2a-d, 4-7, 12, 18, 23 §6.4: 1, 2, 5, 7, 8, 11

Learning Goals:

- If given one of the 10 vector space properties, be able to verify if the property holds/does not hold for a potential vector space
- Be able to come up with a counterexample to show a set/operation is NOT a vector space or violates a vector space property
- Be able to verify that a subset of a vector space is or is NOT a subspace
- Be able to determine if a vector is in the span of a set of vectors
- Be able to determine if a set spans a vector space
- Be able to determine if a set is linearly independent/linearly dependent (via systems of equations, the determinant of a matrix, or demonstrating a linear combination)
- Know how to show a set is or is NOT a basis for a vector space
- Know how to find a basis from a set of vectors
- Be able to find the dimension of a vector space/subspace
- Know the dimensions of the common finite-dimensional vector spaces: $\mathbb{R}^n$, $\mathbb{M}_{mn}$, P_d
- Know how to use the dimension of a vector space to justify if a set is a basis

Student Notes: Chapter 6 will be our first big step towards abstraction (don't panic!) There are many nice properties about vectors and vector arithmetic in $\mathbb{R}^n$ that apply to other things that have addition and scalar multiplication: for example, matrices and polynomials. Instead of proving that these properties are true separately for each of these sets, mathematicians unite them under a common name, then prove these properties ONCE for the common name.

This common name is called a *vector space*. To be a vector space, a (nonempty) set needs to have addition and scalar multiplication defined, and these operations need to satisfy 10 properties. Again, don't panic! For exams, you won't need to memorize all 10 properties of a vector space, but you will need to know the definition of subspace (and the subspace test). If we want you to prove a vector space axiom for a given operation, we'll give you the general axiom. Again, just like with subspaces in $\mathbb{R}^n$, to prove something is true "for all" or "for every", you need to use variables.

We will define span and linear independence for general vector spaces. When doing computational problems, think about how your computations compare with those for $\mathbb{R}^n$. You should be able to quickly come up with a basis for the "common" vector spaces: $\mathbb{R}^n$, $\mathbb{M}_{mn}$, P_d, and know the dimensions for each. We will learn how knowing the dimension of the vector space will help you determine if a set is a basis to find more interesting-looking bases.

Chapter 7 (§7.1-§7.3)

R: §7.1-§7.3
P: §7.1: 1a-b,d-k, 2-5, 17 §7.2: 1, 2, 6, 10, 11, 14, 15 §7.3: 1, 2, 4, 22, 23, 26

Learning Goals:

- Be able to show a function between vector spaces is or is NOT a linear transformation
- Know how to determine if a linear transformation is one-to-one and/or onto
- Know how to find the kernel and/or range/image of a matrix transformation (as well as a basis for the kernel/range)
- Know how to find vectors in the kernel/range of a general linear transformation
- Know how the dimension of vector spaces for a linear transformation interacts with the problem of being one-to-one/onto
- Be able to determine if a linear transformation is an isomorphism
- Know how to determine if two finite-dimensional vector spaces are isomorphic

Student Notes: In Chapter 7, we will again take a step towards abstraction. We will generalize our notion of linear transformations to other vector spaces. One of the big goals of Chapter 7 will be to rigorously bring together all of our vector spaces into a common structure. We define the notions of one-to-one and onto for linear transformations along with the kernel and range of a linear transformation. If $L: V \to W$ is a linear transformation, then the kernel of Lis a subspace of the vector space V, and it is all the vectors in V which are sent to the zero vector in W. The range of a linear transformation is a subspace of the vector space W, and it is all the vectors in W which are the image (via L) of a vector from V.

You've probably seen one-to-one and onto before in other math courses (for example, in precalculus, many students learn that one-to-one functions satisfy the "horizontal line test."). One of the cool things about linear transformations is that the problem of being one-to-one is completely decided by the kernel; if the kernel has only the zero vector, the linear transformation is one-to-one; otherwise, it's not.

Since the kernel and the range are subspaces, they each have a dimension, and we will see that these dimensions have a beautiful relationship, called the Dimension Theorem. We can use this theorem to calculate unknown dimensions and see what kinds of linear transformations are possible between different vector spaces.

We end this chapter by rigorously creating a matching between vector spaces with the same dimension using linear transformations, called an *isomorphism*. This demonstrates that the structure of vector spaces of the same dimension are the same. This construction allows us to move between different representations of vector spaces depending on the context. However, it's still important to declare your isomorphism before moving to a different vector space. We will see more details about this later.

Chapter 9 (§9.1-§9.3)

R: §9.1-§9.3 (see below)
P: §9.1: 1-4, 8, 9, 15, 16a, 17, 18 §9.2: 1, 2, 3, 4, 6, 7, 8, 9 §9.3: 1

Learning Goals:
- Be able to give the coordinate vector with respect to an ordered basis
- Know how to find the matrix representing a linear transformation $L: V \to W$ using ordered bases for vector spaces V and W
- Know how to verify if a linear transformation is invertible using matrices
- Know how to compute the change matrix given two ordered bases for a vector space, and how to use it to find the coordinate vector
- Know how to change the representation of a linear operator from one basis to another, both directly and using the Similarity Theorem (pg. 507)
- Be able to find the eigenvalues/eigenvectors of a linear operator
- Be able to find a diagonal matrix representation of a linear operator (if possible)

Student Notes: In Chapter 9, we'll learn how to build a matrix representation for a general linear transformation $L: V \to W$. This construction has obvious advantages; by moving to a matrix, we can typically compute linear transformations more easily, and we can utilize many of our known results of matrix transformations.

However, we need to set up this construction carefully. The choice of bases for our vector spaces V and W affects the final matrix. You may be wondering why we don't always pick the same/easiest basis to work with. We will see the advantages of picking a "non-standard" basis later on.

In §9.2, we focus on linear transformations which are *linear operators* (they are functions between the exact same vector space). Our matrix representation will then be square, and we can use the same ordered basis for the beginning and end. Our choice of basis will affect our matrix representation. However, there are properties of these matrices that will be unchanged depending on the choice of basis. We will learn how to move between different representations using transition matrices.

The main goal of §9.3 is to learn how to find eigenvalues and eigenvectors of a linear operator. We can utilize the eigenvectors to help build a "nicer" matrix representation for the linear operator.Your book goes deeper into the theory of this topic than we're planning to do for this course. You're encouraged to read the whole section for §9.3, but please focus on the following parts: pages 514-516 (before Theorem 9.3.1) and pages 518-520 (starting at Eigenvalues, and stopping at Direct Sums).

Chapter 10 (§5.3, §10.1-§10.2)

R: §5.3, §10.1-§10.2 (stop after Example 10.2.4)
P: §5.3: 2, 5, 6, 8 §10.1: 1, 3, 4, 7, 13ab, 16, 17, 22, 26, 29a §10.2: 1, 2a, 5

Learning Goals:

- Be able to compute length of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Be able to compute the standard inner product/dot product of vectors
- Be familiar with the basic properties of the dot product (including the test for orthogonality
- Know how to verify if a function is an inner product
- Be able to compute the length of a vector and find a unit vector given an inner product
- Be able to find the distance between two vectors using an inner product
- Be familiar with the Cauchy–Bunyakovsky–Schwarz (CBS) Inequality and its implications
- Know how to check if two vectors are orthogonal given an inner product
- Know how to find the coefficients of a linear combination using an orthogonal/orthonormal basis
- Be able to build an orthonormal basis given an orthogonal basis
- Be able to build an orthogonal basis given any basis using the Gram-Schmidt algorithm (note: not on final exam)

Student Notes: The last part of this course is to extend our notions of length, distance, and angle to abstract vector spaces. We will start by reviewing the standard notion of length in 2-space and 3-space along with properties of the dot product in §5.3. We'll then take our last leap into abstraction by generalizing the idea of the dot product to an inner product. The "nice" properties of the dot product will carry over to the inner product, allowing us to extend the notion of length, distance, and orthogonality to inner products.

We will ask you to prove/disprove functions are inner products. Just like before, when proving a property that is stated as "for all" or "for every," you need to use variables. If you are showing a function is NOT an inner product, you should choose explicit vectors which "break" one of the inner product criteria.

Also note that the Cauchy–Bunyakovsky–Schwarz (CBS) Inequality is also often called just the Cauchy-Schwarz Inequality.

We finish the course by looking at specific bases of vector spaces which have a special geometric construction. For a given inner product, we look at orthogonal bases which are pairwise orthogonal to each other, and orthonormal bases, which are orthogonal bases where every vector is unit length.

These bases are useful due to the fact that the coordinates/scalars of a given vector can be computed directly using the inner product (instead of solving a linear system of equations).

Not every basis is orthonormal nor orthogonal; however, given an orthogonal basis, it’s easy to find an orthonormal basis (just divide each vector by its length). To construct an orthogonal basis from a given starting basis, we can use the Gram-Schmidt method. We will learn how to implement this method for small bases. Note that the Gram-Schmidt method can be used for any inner product space, not just the standard inner product. Also note that while Gram-Schmidt is not on the final, you will have homework assigned on it.