NAME: Solutions

This quiz contains 4 problems worth 30 points. You may not use books, notes, or a calculator. You have 30 minutes to take the quiz.

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1. (6 points) Prove that for all integers m and n, if m is even and $m+n$ is even, then n is even.

Proof: Assume m and $m+n$ are both even. Thus, by definition of even, there exist integers k_1, k_2 so that

$$m = 2k_1 \quad \text{and} \quad m+n = 2k_2.$$

Now, $n = (m+n) - m = 2k_2 - 2k_1 = 2(k_2 - k_1)$, where $k_2 - k_1$ is an integer.

So by definition of even, n is even.

2. (7 points) Use Proof by Contrapositive to prove that for all real numbers x and y, if $x+2y \geq 3$, then $x \geq 1$ or $y \geq 1$.

[Recall the contrapositive of $p \to q$ is $\neg q \to \neg p$.]

So we will prove:

For all real numbers x and y, if $x<1$ and $y<1$, then $x+2y<3$.

Proof: Assume x and y are real numbers, and $x<1$ and $y<1$.

Then $x+2y < 1+2\cdot 1 = 3$. Thus, $x+2y<3$.

3. (8 points) Use Mathematical Induction to prove $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$.

$S(n)$ is the statement: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

basis step: $S(1)$: $\frac{1}{1\cdot 2} = \frac{1}{2} = \frac{1}{1+1}$, which is true.

inductive step: Assume $S(n)$: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ is true. We must show $S(n+1)$: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$ is true. Now,

$$LHS = \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}\right) + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \quad \text{by the inductive hypoth.}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)}$$

$$= \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2} = RHS.$$

4. (9 points) Use Mathematical Induction to prove $2n+1 \leq 2^n$ for $n = 3, 4, \cdots$.

Let $S(n)$ be the statement $2n+1 \leq 2^n$ for $n = 3, 4, \cdots$

basis step: $S(3)$ is $2\cdot 3+1 = 7 \leq 8 = 2^3$, which is true.

inductive step: Assume $S(n)$: $2n+1 \leq 2^n$ is true. We must show that $S(n+1)$: $2(n+1)+1 \leq 2^{n+1}$ is true. Now,

$$LHS = 2(n+1)+1 = (2n+1)+2 \leq 2^n + 2 \quad \text{by the inductive hypothesis}$$

$$\leq 2^n + 2^n \quad \text{because } 2 \leq 2^n \text{ for } n \geq 3.$$

$$= 2\cdot 2^n = 2^{n+1}.$$