Name: Solutions

Section 5.1

1. For each integer below (i) trace the standard algorithm (Algorithm 5.1.8 page 226) to determine if it is prime and (ii) find its prime factorization.

   (a) $n = 966$ (i) 2 divides 966 so it isn't prime
   (ii) $966 = 2\cdot 483 = 2\cdot 3\cdot 161 = 2\cdot 3\cdot 7\cdot 23$

   (i) 127 is not divisible by 2,3,5,7,11. Since $13 > \sqrt{127}$, 127 is prime.
   (b) $n = 127$ (ii) 127 (it is its own prime factorization)

2. For each pair of integers find (i) the greatest common divisor of the pair and (ii) the least common multiple of the pair.

   (a) $n = 30,\ m = 120$

   $\gcd(30,120) = 30$
   $\operatorname{lcm}(30,120) = 120$

   (b) $n = 104,\ m = 363$

   $n = 104 = 2^3\cdot 13$ $\gcd(104,363) = 1$
   $m = 3\cdot 11^2$ $\operatorname{lcm}(104,363) = 104\cdot 363 = 37{,}752$

   (c) $n = 72,\ m = 306$
   $n = 2^3\cdot 3^2$ $\gcd(72,306) = 2\cdot 3^2 = 18$
   $m = 2\cdot 3^2\cdot 17$ $\operatorname{lcm}(72,306) = 2^3\cdot 3^2\cdot 17 = 1224 = \frac{72\cdot 306}{18}$
   alternate view

   (d) $n = 2^2\cdot 3\cdot 5^4,\ m = 2^3\cdot 5^3\cdot 7$

   $\gcd(n,m) = 2^2\cdot 5^3 = 500$
   $\operatorname{lcm}(n,m) = 2^3\cdot 3\cdot 5^4\cdot 7 = 105{,}000$

3. For #2d, write n and m as products of the *same* set of prime factors.

   $n = 2^2\cdot 3\cdot 5^4\cdot \underline{7^0}$ add in p^0 when needed
   $m = 2^3\cdot \underline{3^0}\cdot 5^3\cdot 7$

4. Let $m = p_1^{a_1}p_2^{a_2}p_3^{a_3}\cdots p_n^{a_n}$ and $n = p_1^{b_1}p_2^{b_2}p_3^{b_3}\cdots p_n^{b_n}$ where $a_i,\ b_i \in \mathbb{Z}^{nonneg}$.

   (a) Is $p_1^{a_1}p_2^{a_2}p_3^{a_3}\cdots p_n^{a_n}$ necessarily the prime factorization of m? Explain.

   No.
   Since a_i and b_i could be zero, it is not necessarily the prime factorization of m.

   (b) Give formulas for the greatest common divisor and least common multiple of m and n.

   $$\gcd(m,n) = p_1^{\min(a_1,b_1)}\,p_2^{\min(a_2,b_2)}\cdots p_n^{\min(a_n,b_n)}$$

   $$\operatorname{lcm}(m,n) = p_1^{\max(a_1,b_1)}\,p_2^{\max(a_2,b_2)}\cdots p_n^{\max(a_n,b_n)}$$

5. Write a formal, direct proof of the following:

   Let n, c, and d be integers. If $dc \mid nc$, then $d \mid n$.

<u>Pf</u>: Let $n,c,d \in \mathbb{Z}$. If $dc \mid nc$, then $dc \neq 0$ and $\exists q \in \mathbb{Z}$ so that $dc \cdot q = nc$. Since $dc \neq 0$, we know $c \neq 0$. Thus we can divide the equation $dcq = nc$ by c to get $dq = n$. Since $q \in \mathbb{Z}$ and $d \neq 0$, we have shown $d \mid n$.

Section 5.2

1. When a number is represented in

- *decimal* form, digits are selected from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} and each position represents a power of 10

  So the expansion of the symbols: 8032 is $8 \times 10^3 + 0 \cdot 10^2 + 3 \cdot 10^1 + 2 \cdot 10^0$

- *binary* form, digits are selected from the set {0, 1} and each position represents a power of 2

  So the expansion of the symbols: 1101 is $1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$

- *hexadecimal* form, digits are selected from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F} and each position represents a power of 16

  So the expansion of the symbols: $20AF$ is $2 \cdot 16^3 + 0 \cdot 16^2 + 10 \cdot 16^1 + 15 \cdot 16^0$

2. Express the binary number 1101010 in decimal.

1101010 represents $1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^3 + 1 \cdot 2 = \underline{106}$ in decimal.

3. Express the decimal number 357 in binary.

Work

$2^8 = 256$

$357 - 2^8 = 101$

$2^6 = 64$

So $101 - 2^6 = 37$

Now $37 - 2^5 = 5$;

$5 - 2^2 = 1$.

Conclusion

$357 = 2^8 + 2^6 + 2^5 + 2^2 + 1$.

So in binary, 357 is

$\boxed{101100101}$

4. Express the hexadecimal number $A105$ in decimal.

A105 represents $10 \cdot 16^3 + 1 \cdot 16^2 + 0 \cdot 16 + 5 = \underline{41{,}221}$ in decimal.

5. Express the decimal number 10400 in hexadecimal.

Work

$10400 - 16^3 = 6304$

$6304 - 16^3 = 2208$

$2208 - 8 \cdot 16^2 = 160$

$160 - 10 \cdot 16 = 0$

Conclusion

$10{,}400 = 2 \cdot 16^3 + 8 \cdot 16^2 + 10 \cdot 16$

So, in hexadecimal, 10,400 is

$\boxed{28A}$

6. Assume you are given a decimal integer n, how many bits (digits) would you need to represent n in binary? (If you don't immediately know the answer, return to #3 and think about how you calculated it.)

In words: you need to find the largest power of 2 smaller than your n, say k. Then you need k+1 digits because digits start w/ exponent of 0 and you will need to reach exponent k.

In math:

$$\# \text{digits} = \lfloor \lg 2 \rfloor + 1$$

7. Without actually finding the binary representation, determine the number of bits needed to represent the decimal number 2,500,230.

$$\lfloor \lg (2500230) \rfloor = 21$$

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(b/c $2^{21} = 2{,}097{,}152$
and
$2^{22} = 4{,}194{,}304$)