

IVP Analytic vs. Numerical Solution

June 29, 2020

1 Problem

Compare the analytic and numerical solution of η of the following shallow water problem:

$$\begin{aligned}\eta &= e^{-(x-3.5)^2} \\ u &= 0 \\ h &= x \\ m &= \infty\end{aligned}$$

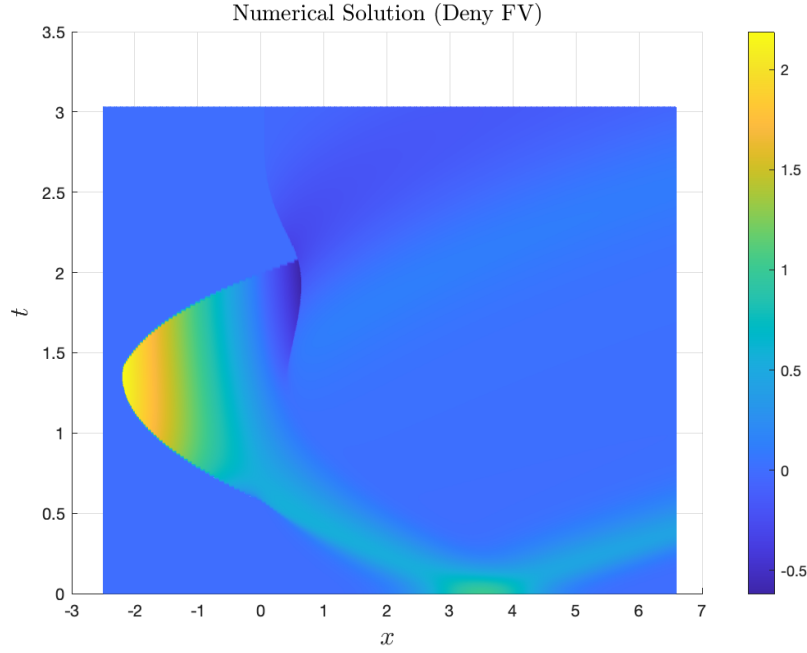
In other words, a Gaussian initial wave with no initial velocity, and a plane-inclined shape (y^∞). This reduces to a 1-1 SWE. We can reproduce this with a different slope and initial conditions easily.

2 Setup

Statistical comparison was done on an equally spaced grid of 1000 points in time on $[0,3]$ and at 1000 points in x on $[-2.5, 6.5]$

2.1 Numerical

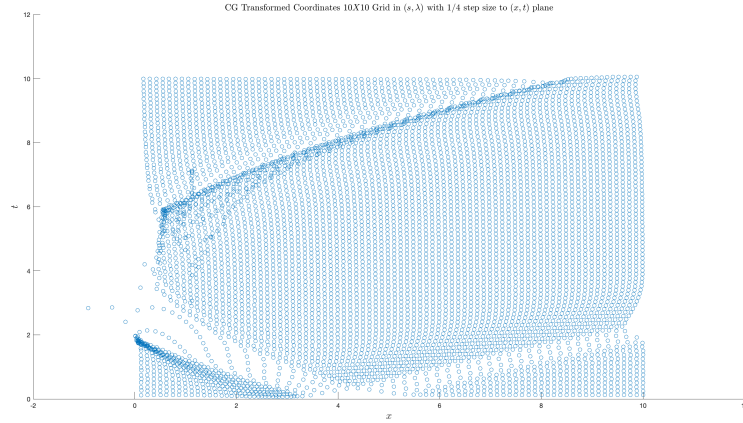
I set Deny's Catalina 1 "runwave.m" with the initial conditions. The following displays eta in the (x, t) plane



2.2 Analytic

Chebfun was used to calculate the Hankel transform solution to the CG transform on a grid in (s, λ) then CG transform to (x, t)

The following figure shows the a grid in (s, λ) transformed to (x, t)



Note the distinct non-linear nature caused by the $-u^2$ of η
The analytical solution of η was computed using formulas in Nicolsky (2018)

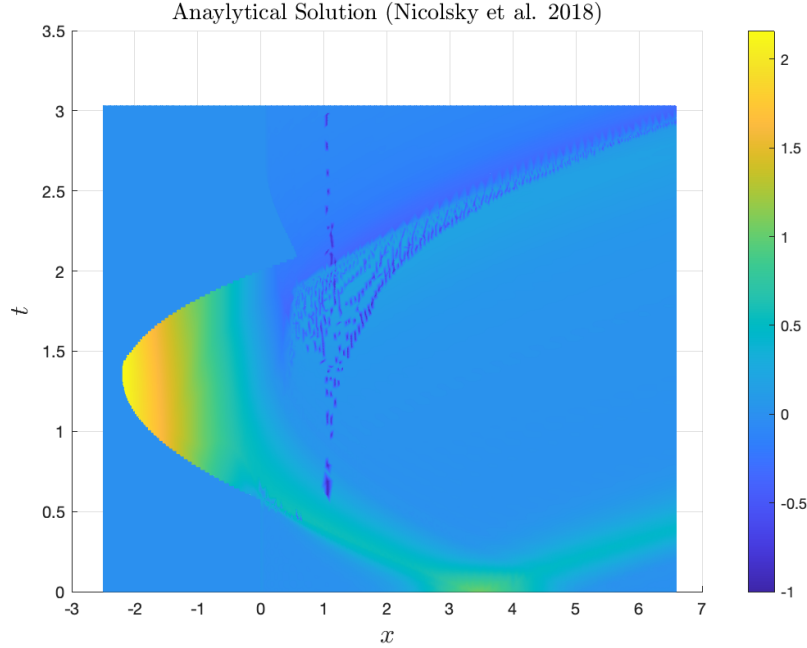
$$\begin{aligned}\psi(s, \lambda) &= \int_0^\infty (a(k)\cos(\beta k\lambda) + b(k)\sin(\beta k\lambda))J_0(2k\sqrt{s})dk \\ \varphi(s, \lambda) &= s^{-1/2} \int_0^\infty (a(k)\sin(\beta k\lambda) + b(k)\cos(\beta k\lambda))J_1(2k\sqrt{s})dk\end{aligned}$$

where

$$\begin{aligned}a(k) &= 2k \int_0^\infty \psi(s*, 0)J_0(2k\sqrt{s*})ds* \\ b(k) &= 2k \int_0^\infty \varphi(s*, 0)s^{*1/2}J_1(2k\sqrt{s*})ds*\end{aligned}$$

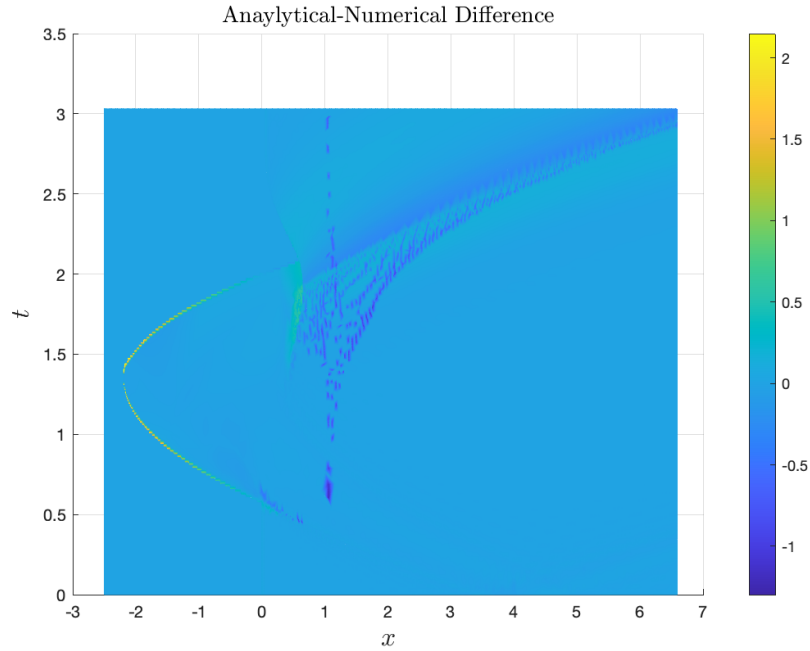
and the projection of φ and ψ onto $\lambda = 0$ were computed via a first order taylor expansion. Note that these eqauntions require $\eta'_0(x) > -1$. See Nicolsky for derivation.

$$\begin{aligned}\Phi(s, \lambda) &= \begin{pmatrix} \varphi(s, \lambda) \\ \psi(s, \lambda) \end{pmatrix} \\ \Phi_0(x) &= \begin{pmatrix} u_0(x) \\ \eta_0(x) + u_0^2(x)/2 \end{pmatrix} \\ \Phi_1 &= \Phi_0 + u_0(u'_0AD^{-1}B\Phi_0 - B\Phi_0 - AD^{-1}\Phi'_0)\end{aligned}$$

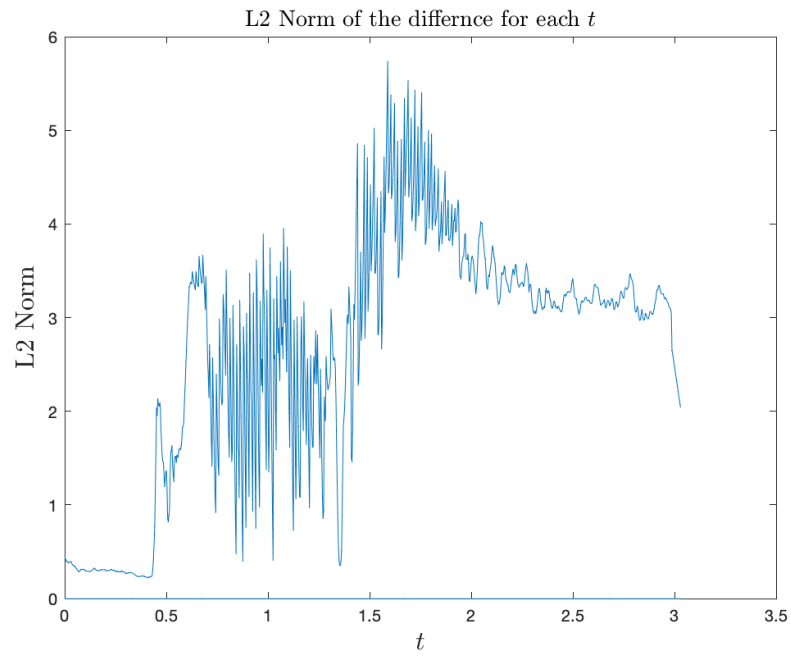


3 Statistical Analysis

This is the difference between the two ie. numerical - analytical



The following is the L2 norm at each value of t . The difference increases in a sporadic fashion at the beginning and end of run-up. The primary explanation for this is problems with the computation of the analytic solution.



4 Further Problems

1. Analytic solution stability.
2. Comparison of the speed wasn't completed.
3. Different initial conditions.
4. NOAA analytic solution.