Destination λ-calculus

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1 Term and value syntax

termvar, x, y, d	Term-level variable
holevar, h	Hole

term_value, v	::=		Term value
	|	$\langle v_1 \,;\, \overline{v_2} \rangle_H$	Ampar
	|	$@h$	Destination
	|	$()$	Unit
	|	$\mathsf{Inl}\, v$	Left variant for sum
	|	$\mathsf{Inr}\, v$	Right variant for sum
	|	$(v_1\, ,\, v_2)$	Product
	|	$\mathcal{D}^{m}\, v$	Exponential
	|	$\lambda x\, .\, t$	Linear function

$\overline{\text{extended\_value}}$, $\overline{v}$	::=		Pseudo-value that may contain holes
	|	v	Term value
	|	h	Hole
	|	$\mathsf{Inl}\, \overline{v}$	Left variant with val or hole
	|	$\mathsf{Inr}\, \overline{v}$	Right variant with val or hole
	|	$(\overline{v_1}\, ,\, \overline{v_2})$	Product with val or hole
	|	$\mathcal{D}^{m}\, \overline{v}$	Exponential with val or hole

term, t, u	::=		Term
	|	v	Term value
	|	x	Variable
	|	$t \succ u$	Application
	|	$t \succ \mathbf{case}\ () \mapsto u$	Pattern-match on unit
	|	$t \succ \mathbf{case}\ \{\, \mathsf{Inl}\, x_1 \mapsto u_1\, ,\ \mathsf{Inr}\, x_2 \mapsto u_2 \,\}$	Pattern-match on sum
	|	$t \succ \mathbf{case}\ (x_1\, ,\, x_2) \mapsto u$	Pattern-match on product
	|	$t \succ \mathbf{case}\ \mathcal{D}^{m}\, x \mapsto u$	Pattern-match on exponential
	|	$t \succ \mathbf{mapL}\ x \mapsto u$	Map over the left side of the ampar
	|	$\mathbf{to}_{\ltimes}\, t$	Wrap t into a trivial ampar
	|	$\mathbf{from}_{\ltimes}\, t$	Extract value from trivial ampar
	|	$\mathbf{alloc}_{A}$	Return a fresh "identity" ampar object
	|	$t \triangleleft ()$	Fill destination with unit
	|	$t \triangleleft \mathsf{Inl}$	Fill destination with left variant
	|	$t \triangleleft \mathsf{Inr}$	Fill destination with right variant
	|	$t \triangleleft (,)$	Fill destination with product constructor
	|	$t \triangleleft \mathcal{D}^{m}$	Fill destination with exponential constructor
	|	$t \triangleleft\!\bullet\, u$	Fill destination with root of ampar u

2 Type system

$\mathsf{type},\ \mathsf{A},\ \mathsf{B}$	$::=$		Type
	$\mid$	$\mathbf{1}$	Unit
	$\mid$	$\mathsf{A}_1 \oplus \mathsf{A}_2$	Sum
	$\mid$	$\mathsf{A}_1 \otimes \mathsf{A}_2$	Product
	$\mid$	$!^{m}\ \mathsf{A}$	Exponential
	$\mid$	$\mathsf{A}_1 \rtimes \mathsf{A}_2$	Ampar type (consuming A_1 yields A_2)
	$\mid$	$\mathsf{A}_1\ {}_{m}\!\!\to \mathsf{A}_2$	Linear function
	$\mid$	${}^{m}\lfloor \mathsf{A} \rfloor$	Destination
multiplicity, m, n	$::=$		Multiplicity (Semiring with product $\cdot$)
	$\mid$	ν	Born now. Identity of the product
	$\mid$	$\uparrow$	One scope older
	$\mid$	∞	Infinitely old / static. Absorbing for product
	$\mid$	$m_1 \cdot m_2$	Semiring product
typing_context, Δ	$::=$		Typing context
	$\mid$	Γ	
	$\mid$	H	
	$\mid$	$\Gamma \sqcup \mathrm{H}$	
	$\mid$	$m \cdot \Delta$	Increase age of bindings by m
pos_context, Γ	$::=$		Positive typing context
	$\mid$	$\{\mathsf{pos\_assign}*\}$	
	$\mid$	$\Gamma_1 \sqcup \Gamma_2$	
	$\mid$	$\Gamma^{@}$	Positive context restricted to destinations only
	$\mid$	$m \cdot \Gamma$	Increase age of bindings by m
pos_assign	$::=$		Positive type assignment
	$\mid$	$\mathsf{x} :_{m} \mathsf{A}$	Variable
	$\mid$	$@\mathsf{h} :_{m} {}^{n}\lfloor \mathsf{A} \rfloor$	Destination (m is its own age; n is the age of values it accepts)
neg_context, H	$::=$		Negative typing context
	$\mid$	$\{\mathsf{neg\_assign}*\}$	
	$\mid$	$\mathrm{H}_1 \sqcup \mathrm{H}_2$	
	$\mid$	$@^{-1}\Gamma$	Inverse the sign of the context
	$\mid$	$m \cdot \mathrm{H}$	Increase age of bindings by m
neg_assign	$::=$		Negative type assignment
	$\mid$	$\mathsf{h} :^{n} \mathsf{A}$	Hole (n is the age of values it accepts, its own age is undefined)

$\boxed{\mathrm{H}_1 = \mathrm{H}_2}$ *($@^{-1}$: "Inverse sign of context" operation)*

AtApp-Empty
$$\frac{}{@^{-1}\emptyset = \emptyset}$$

AtApp-Rec
$$\frac{}{@^{-1}(\{@\mathsf{h} :_{m} {}^{n}\lfloor \mathsf{A} \rfloor\} \sqcup \Gamma) = \{\mathsf{h} :^{m \cdot n} \mathsf{A}\} \sqcup @^{-1}\Gamma}$$

$\boxed{\Delta \Vdash \mathsf{e}}$ *(Typing of effects (require both positive and negative contexts))*

TyEff-NoEff
$$\frac{}{\emptyset \sqcup \emptyset \Vdash \varepsilon}$$

TyEff-Single
$$\frac{\Gamma \sqcup \mathrm{H} \Vdash \overline{\mathsf{v}} : \mathsf{A} \qquad \mathsf{h} \notin \mathsf{names}(\Gamma)}{m \cdot ((n \cdot \uparrow) \cdot \Gamma \sqcup \{@\mathsf{h} :_{\nu} {}^{n}\lfloor \mathsf{A} \rfloor\} \sqcup n \cdot \mathrm{H}) \Vdash \mathsf{h} := \overline{\mathsf{v}}}$$

TyEff-Union
$$\frac{\begin{array}{c}\Gamma_1 \sqcup \mathrm{H}_1 \sqcup @^{-1}\Gamma_{22} \Vdash \mathsf{e}_1 \\ \Gamma_{21} \sqcup \Gamma_{22} \sqcup \mathrm{H}_2 \Vdash \mathsf{e}_2 \\ \mathsf{names}(\Gamma_1 \sqcup \mathrm{H}_1) \cap \mathsf{names}(\Gamma_{21} \sqcup \mathrm{H}_2) = \emptyset\end{array}}{\Gamma_1 \sqcup \Gamma_{21} \sqcup \mathrm{H}_1 \sqcup \mathrm{H}_2 \Vdash \mathsf{e}_1 \cdot \mathsf{e}_2}$$

$\boxed{\Gamma \vdash \mathsf{v} \mid \mathsf{e} : \mathsf{A}}$ *(Typing of commands (only a positive context is needed))*

TyCmd-Cmd
$$\frac{\begin{array}{c}\Gamma_{11} \sqcup \Gamma_{12} \vdash \mathsf{v} : \mathsf{A} \\ \Gamma_2 \sqcup @^{-1}\Gamma_{12} \Vdash \mathsf{e} \\ \mathsf{names}(\Gamma_{11}) \cap \mathsf{names}(\Gamma_2) = \emptyset\end{array}}{\Gamma_{11} \sqcup \Gamma_2 \vdash \mathsf{v} \mid \mathsf{e} : \mathsf{A}}$$

$\boxed{\Delta \Vdash \overline{v} : A}$ *(Typing of extended values (require both positive and negative contexts))*

TyValExt-Hole
$$\frac{}{\emptyset \uplus \{h :^{\nu} A\} \Vdash h : A}$$

TyValExt-Dest
$$\frac{}{\{@h :_{\nu} {}^{n}\lfloor A \rfloor\} \uplus \emptyset \Vdash @h : {}^{n}\lfloor A \rfloor}$$

TyValExt-Unit
$$\frac{}{\emptyset \uplus \emptyset \Vdash () : 1}$$

TyValExt-Inl
$$\frac{\Gamma \uplus H \Vdash \overline{v} : A_1}{\Gamma \uplus H \Vdash \mathsf{Inl}\,\overline{v} : A_1 \oplus A_2}$$

TyValExt-Inr
$$\frac{\Gamma \uplus H \Vdash \overline{v} : A_2}{\Gamma \uplus H \Vdash \mathsf{Inr}\,\overline{v} : A_1 \oplus A_2}$$

TyValExt-Prod
$$\frac{\Gamma_1 \uplus H_1 \Vdash \overline{v_1} : A_1 \qquad \Gamma_2 \uplus H_2 \Vdash \overline{v_2} : A_2 \qquad \mathbf{names}(\Gamma_1 \uplus H_1) \cap \mathbf{names}(\Gamma_2 \uplus H_2) = \emptyset}{\Gamma_1 \sqcup \Gamma_2 \uplus H_1 \sqcup H_2 \Vdash (\overline{v_1}, \overline{v_2}) : A_1 \otimes A_2}$$

TyValExt-Exp
$$\frac{\Gamma \uplus H \Vdash \overline{v} : A}{m\cdot\Gamma \uplus m\cdot H \Vdash \big)^{m}\,\overline{v} : !^{m} A}$$

TyValExt-Ampar
$$\frac{\Gamma_1 \uplus \emptyset \Vdash v_1 : A_1 \qquad \Gamma_2 \uplus @^{-1}\Gamma_1 \Vdash \overline{v_2} : A_2}{\Gamma_2 \uplus \emptyset \Vdash \langle v_1 , \overline{v_2} \rangle_H : A_1 \rtimes A_2}$$

TyValExt-Lambda
$$\frac{\Gamma \sqcup \{x :_m A_1\} \vdash t : A_2}{\Gamma \uplus \emptyset \Vdash \lambda x\,.\,t : A_1 {}_{m}\!\!\to A_2}$$

$\boxed{\Gamma \vdash t : A}$ *(Typing of terms (only a positive context is needed))*

TyTerm-Val
$$\frac{\Gamma \uplus \emptyset \Vdash v : A}{\Gamma \vdash v : A}$$

TyTerm-VarNow
$$\frac{}{\{x :_{\nu} A\} \vdash x : A}$$

TyTerm-VarInf
$$\frac{}{\{x :_{\infty} A\} \vdash x : A}$$

TyTerm-App
$$\frac{\Gamma_1 \vdash t : A_1 \qquad \Gamma_2 \vdash u : A_1 {}_{m}\!\!\to A_2 \qquad \mathbf{names}(\Gamma_1) \cap \mathbf{names}(\Gamma_2) = \emptyset}{m\cdot\Gamma_1 \sqcup \Gamma_2 \vdash t \succ u : A_2}$$

TyTerm-PatUnit
$$\frac{\Gamma_1 \vdash t : 1 \qquad \Gamma_2 \vdash u : B \qquad \mathbf{names}(\Gamma_1) \cap \mathbf{names}(\Gamma_2) = \emptyset}{\Gamma_1 \sqcup \Gamma_2 \vdash t \succ \mathbf{case}\ () \mapsto u : B}$$

TyTerm-PatSum
$$\frac{\Gamma_1 \vdash t : A_1 \oplus A_2 \qquad \Gamma_2 \sqcup \{x_1 :_m A_1\} \vdash u_1 : B \qquad \Gamma_2 \sqcup \{x_2 :_m A_2\} \vdash u_2 : B \qquad \mathbf{names}(\Gamma_1) \cap \mathbf{names}(\Gamma_2) = \emptyset}{m\cdot\Gamma_1 \sqcup \Gamma_2 \vdash t \succ \mathbf{case}\,\{\mathsf{Inl}\,x_1 \mapsto u_1 ,\ \mathsf{Inr}\,x_2 \mapsto u_2\} : B}$$

TyTerm-PatProd
$$\frac{\Gamma_1 \vdash t : A_1 \otimes A_2 \qquad \Gamma_2 \sqcup \{x_1 :_m A_1, x_2 :_m A_2\} \vdash u : B \qquad \mathbf{names}(\Gamma_1) \cap \mathbf{names}(\Gamma_2) = \emptyset}{m\cdot\Gamma_1 \sqcup \Gamma_2 \vdash t \succ \mathbf{case}\ (x_1 , x_2) \mapsto u : B}$$

TyTerm-PatExp
$$\frac{\Gamma_1 \vdash t : !^{m'} A \qquad \Gamma_2 \sqcup \{x :_{m\cdot m'} A_1\} \vdash u : B \qquad \mathbf{names}(\Gamma_1) \cap \mathbf{names}(\Gamma_2) = \emptyset}{m\cdot\Gamma_1 \sqcup \Gamma_2 \vdash t \succ \mathbf{case}\ \big)^{m'}\,x \mapsto u : B}$$

TyTerm-MapAmpar
$$\frac{\Gamma_1 \vdash t : A_1 \rtimes A_2 \qquad \uparrow\cdot\Gamma_2 \sqcup \{x :_{\nu} A_1\} \vdash u : B \qquad \mathbf{names}(\Gamma_1) \cap \mathbf{names}(\Gamma_2) = \emptyset}{\Gamma_1 \sqcup \Gamma_2 \vdash t \succ \mathbf{mapL}\ x \mapsto u : B \rtimes A_2}$$

TyTerm-FillComp
$$\frac{\Gamma_1 \vdash t : {}^{n}\lfloor A_2 \rfloor \qquad \Gamma_2 \vdash u : A_1 \rtimes A_2 \qquad \mathbf{names}(\Gamma_1) \cap \mathbf{names}(\Gamma_2) = \emptyset}{\Gamma_1 \sqcup (n\cdot\uparrow)\cdot\Gamma_2 \vdash t \triangleleft\!\bullet\, u : A_1}$$

TyTerm-FillUnit
$$\frac{\Gamma \vdash t : {}^{n}\lfloor 1 \rfloor}{\Gamma \vdash t \triangleleft () : 1}$$

TyTerm-FillInl
$$\frac{\Gamma \vdash t : {}^{n}\lfloor A_1 \oplus A_2 \rfloor}{\Gamma \vdash t \triangleleft \mathsf{Inl} : {}^{n}\lfloor A_1 \rfloor}$$

TyTerm-FillInr
$$\frac{\Gamma \vdash t : {}^{n}\lfloor A_1 \oplus A_2 \rfloor}{\Gamma \vdash t \triangleleft \mathsf{Inr} : {}^{n}\lfloor A_2 \rfloor}$$

TyTerm-FillProd
$$\frac{\Gamma \vdash t : {}^{n}\lfloor A_1 \otimes A_2 \rfloor}{\Gamma \vdash t \triangleleft (,) : {}^{n}\lfloor A_1 \rfloor \otimes {}^{n}\lfloor A_2 \rfloor}$$

TyTerm-FillExp
$$\frac{\Gamma \vdash t : {}^{n}\lfloor !^{n'} A \rfloor}{\Gamma \vdash t \triangleleft \big)^{n'} : {}^{n\cdot n'}\lfloor A \rfloor}$$

TyTerm-Alloc
$$\frac{}{\emptyset \vdash \mathbf{alloc}_A : {}^{\nu}\lfloor A \rfloor \rtimes A}$$

TyTerm-ToAmpar
$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathbf{to}_{\rtimes}\ t : 1 \rtimes A}$$

TyTerm-FromAmpar
$$\frac{\Gamma \vdash t : 1 \rtimes A}{\Gamma \vdash \mathbf{from}_{\rtimes}\ t : A}$$

3 Effects and big-step semantics

effect, e	::=		Effect
	|	ε	No effect
	|	$h := \overline{v}$	
	|	$e_1 \cdot e_2$	

$\boxed{\mathit{eff\_app}_1 = \mathit{eff\_app}_2}$ *(**apply**: how effects are applied locally or winded up (we assume effect lists are ε-terminated))*

EffApp-NoEff
$$\frac{}{\mathbf{apply}\,(\varepsilon, \overline{v}_{\mathrm{H}}) = \varepsilon, \overline{v}_{\mathrm{H}}}$$

EffApp-WindUp
$$\frac{h \notin \mathbf{names}\,(\mathrm{H})}{\mathbf{apply}\,(h := \overline{v_2} \cdot e, \overline{v_1}_{\mathrm{H}}) = h := \overline{v_2} \mathbin{\hat{\cdot}} \mathbf{apply}\,(e, \overline{v_1}_{\mathrm{H}})}$$

EffApp-Fill
$$\frac{\_ \sqcup \mathrm{H}' \Vdash \overline{v_2} : \mathsf{A} \qquad \mathbf{names}(\mathrm{H} \sqcup \{h :^{n} \mathsf{A}\}) \cap \mathbf{names}(\mathrm{H}') = \emptyset}{\mathbf{apply}\,(h := \overline{v_2} \cdot e, \overline{v_1}_{\mathrm{H} \sqcup \{h :^{n} \mathsf{A}\}}) = \mathbf{apply}\,(e, \overline{v_1}[h := \overline{v_2}]_{\mathrm{H} \sqcup n \cdot \mathrm{H}'})}$$

$\boxed{t \Downarrow v \mid e}$ *(Big-step evaluation into commands)*

BigStep-Val
$$\frac{}{v \Downarrow v \mid \varepsilon}$$

BigStep-App
$$\frac{t_1 \Downarrow v_1 \mid e_1 \qquad t_2 \Downarrow \lambda x \,.\, u \mid e_2 \qquad u[x := v_1] \Downarrow v_3 \mid e_3}{t_1 \succ t_2 \Downarrow v_3 \mid e_1 \cdot e_2 \cdot e_3}$$

BigStep-PatUnit
$$\frac{t_1 \Downarrow () \mid e_1 \qquad t_2 \Downarrow v_2 \mid e_2}{t_1 \succ \mathbf{case}\,() \mapsto t_2 \Downarrow v_2 \mid e_1 \cdot e_2}$$

BigStep-PatInl
$$\frac{t \Downarrow \mathbf{Inl}\, v_1 \mid e_1 \qquad u_1[x_1 := v_1] \Downarrow v_2 \mid e_2}{t \succ \mathbf{case}\,\{\mathbf{Inl}\, x_1 \mapsto u_1 , \mathbf{Inr}\, x_2 \mapsto u_2\} \Downarrow v_2 \mid e_1 \cdot e_2}$$

BigStep-PatInr
$$\frac{t \Downarrow \mathbf{Inr}\, v_1 \mid e_1 \qquad u_2[x_2 := v_1] \Downarrow v_2 \mid e_2}{t \succ \mathbf{case}\,\{\mathbf{Inl}\, x_1 \mapsto u_1 , \mathbf{Inr}\, x_2 \mapsto u_2\} \Downarrow v_2 \mid e_1 \cdot e_2}$$

BigStep-PatProd
$$\frac{t \Downarrow (v_1 , v_2) \mid e_1 \qquad u[x_1 := v_1, x_2 := v_2] \Downarrow v_2 \mid e_2}{t \succ \mathbf{case}\,(x_1 , x_2) \mapsto u \Downarrow v_2 \mid e_1 \cdot e_2}$$

BigStep-MapAmpar
$$\frac{t \Downarrow \langle v_1 , \overline{v_2} \rangle_{\mathrm{H}} \mid e_1 \qquad u[x := v_1] \Downarrow v_3 \mid e_2 \qquad e_3, \overline{v_4}_{\mathrm{H}'} = \mathbf{apply}\,(e_2, \overline{v_2}_{\mathrm{H}})}{t \succ \mathbf{mapL}\, x \mapsto u \Downarrow \langle v_3 , \overline{v_4} \rangle_{\mathrm{H}'} \mid e_1 \cdot e_3}$$

BigStep-Alloc
$$\frac{\mathbf{fresh}\, h}{\mathbf{alloc}_{\mathsf{A}} \Downarrow \langle @h , h \rangle_{\{h :^{\nu} \mathsf{A}\}} \mid \varepsilon}$$

BigStep-ToAmpar
$$\frac{t \Downarrow v \mid e}{\mathbf{to}_{\rtimes}\, t \Downarrow \langle () , v \rangle_{\emptyset} \mid e}$$

BigStep-FromAmpar
$$\frac{t \Downarrow \langle () , v \rangle_{\emptyset} \mid e}{\mathbf{from}_{\rtimes}\, t \Downarrow v \mid e}$$

BigStep-FillUnit
$$\frac{t \Downarrow @h \mid e}{t \triangleleft () \Downarrow () \mid e \cdot h := ()}$$

BigStep-FillInl
$$\frac{t \Downarrow @h \mid e \qquad \mathbf{fresh}\, h'}{t \triangleleft \mathbf{Inl} \Downarrow @h' \mid e \cdot h := \mathbf{Inl}\, h'}$$

BigStep-FillInr
$$\frac{t \Downarrow @h \mid e}{t \triangleleft \mathbf{Inr} \Downarrow @h' \mid e \cdot h := \mathbf{Inr}\, h'}$$

BigStep-FillProd
$$\frac{t \Downarrow @h \mid e \qquad \mathbf{fresh}\, h_1 \qquad \mathbf{fresh}\, h_2}{t \triangleleft (,) \Downarrow (@h_1 , @h_2) \mid e \cdot h := (h_1 , h_2)}$$

BigStep-FillComp
$$\frac{t \Downarrow @h \mid e_1 \qquad u \Downarrow \langle v_1 , \overline{v_2} \rangle_{\mathrm{H}} \mid e_2}{t \triangleleft\!\bullet u \Downarrow v_1 \mid e_1 \cdot e_2 \cdot h := \overline{v_2}}$$

4 Type safety

Theorem 1 (Type safety)**.** *If* $\Gamma^{@} \vdash \mathsf{t} : \mathbf{A}$ *then* $\mathsf{t} \Downarrow \mathsf{v} \mid \mathsf{e}$ *and* $\Gamma^{@} \vdash \mathsf{v} \mid \mathsf{e} : \mathbf{A}$.

Proof. By induction on the typing derivation.

- TyTerm_Val: (0) $\Gamma^{@} \vdash \mathsf{v} : \mathbf{A}$
  (0) gives (1) $\mathsf{v} \Downarrow \mathsf{v} \mid \varepsilon$ immediately. From TyEff_NoEff and TyCmd_Cmd we conclude (2) $\Gamma^{@} \vdash \mathsf{v} \mid \mathsf{e} : \mathbf{A}$.

- TyTerm_App: (0) $m\cdot\Gamma_1^{@} \sqcup \Gamma_2^{@} \vdash \mathsf{t} \succ \mathsf{u} : \mathbf{A}_2$
  We have
  (1) $\Gamma_1^{@} \vdash \mathsf{t} : \mathbf{A}_1$
  (2) $\Gamma_2^{@} \vdash \mathsf{u} : \mathbf{A}_1 \,{}_{m}\!\!\rightarrow \mathbf{A}_2$
  (3) $\mathsf{names}(\Gamma_1^{@}) \cap \mathsf{names}(\Gamma_2^{@}) = \emptyset$
  Using recursion hypothesis on (1) we get (4) $\mathsf{t} \Downarrow \mathsf{v}_1 \mid \mathsf{e}_1$ where (5) $\Gamma_1^{@} \vdash \mathsf{v}_1 \mid \mathsf{e}_1 : \mathbf{A}_1$.
  Inverting TyCmd_Cmd we get (5) $\Gamma_{11}^{@} \sqcup \Gamma_{13}^{@} \vdash \mathsf{v}_1 : \mathbf{A}_1$ and (6) $\Gamma_{12} \sqcup @^{-1}(\Gamma_{13}^{@}) \Vdash \mathsf{e}_1$ where (7) $\Gamma_1^{@} = \Gamma_{11}^{@} \sqcup \Gamma_{12}^{@}$.
  Using recursion hypothesis on (2) we get (8) $\mathsf{u} \Downarrow \mathsf{v}_2 \mid \mathsf{e}_2$ where (9) $\Gamma_2^{@} \vdash \mathsf{v}_2 \mid \mathsf{e}_2 : \mathbf{A}_1 \,{}_{m}\!\!\rightarrow \mathbf{A}_2$.
  Inverting TyCmd_Cmd we get (10) $\Gamma_{21}^{@} \sqcup \Gamma_{23}^{@} \vdash \mathsf{v}_2 : \mathbf{A}_1 \,{}_{m}\!\!\rightarrow \mathbf{A}_2$ and (11) $\Gamma_{22} \sqcup @^{-1}(\Gamma_{23}^{@}) \Vdash \mathsf{e}_2$ where (12) $\Gamma_2^{@} = \Gamma_{21}^{@} \sqcup \Gamma_{22}^{@}$.
  Using Lemma **??** on (9) we get (13) $\mathsf{v}_2 = \lambda \mathsf{x} \,.\, \mathsf{t}'$ and (14) $\Gamma_{21}^{@} \sqcup \Gamma_{23}^{@} \sqcup \{\mathsf{x} :_m \mathbf{A}_1\} \vdash \mathsf{t}' : \mathbf{A}_2$.

  Typing value part of the result

  Using Lemma **??** on (14) and (5) we get (15) $m\cdot(\Gamma_{11}^{@} \sqcup \Gamma_{13}^{@}) \sqcup (\Gamma_{21}^{@} \sqcup \Gamma_{23}^{@}) \vdash \mathsf{t}'[\mathsf{x} \coloneqq \mathsf{v}_1] : \mathbf{A}_2$.
  Using recursion hypothesis on (15) we get (16) $\mathsf{t}'[\mathsf{x} \coloneqq \mathsf{v}_1] \Downarrow \mathsf{v}_3 \mid \mathsf{e}_3$ where (17) $m\cdot(\Gamma_{11}^{@} \sqcup \Gamma_{13}^{@}) \sqcup (\Gamma_{21}^{@} \sqcup \Gamma_{23}^{@}) \vdash \mathsf{v}_3 \mid \mathsf{e}_3 : \mathbf{A}_2$.

  Typing effect part of the result

  We have
  (6) $\Gamma_{12}^{@} \sqcup @^{-1}(\Gamma_{13}^{@}) \Vdash \mathsf{e}_1$
  (11) $\Gamma_{22}^{@} \sqcup @^{-1}(\Gamma_{23}^{@}) \Vdash \mathsf{e}_2$

  $\mathsf{names}(\Gamma_{12}^{@}) \cap \mathsf{names}(\Gamma_{22}^{@}) = \emptyset$ comes naturally from (3), (7) and (12).

  We must show:
  $\mathsf{names}(\Gamma_{12}^{@}) \cap \mathsf{names}(\Gamma_{23}^{@}) = \emptyset$: holes in e_2 (associated to u) are fresh so they cannot match a destination name from t as they don't exist yet when t is evaluated.
  $\mathsf{names}(\Gamma_{22}^{@}) \cap \mathsf{names}(\Gamma_{13}^{@}) = \emptyset$: slightly harder. Holes in e_1 (associated to t) are fresh too, so I don't see a way for u to create a term that could mention them, but sequentially, at least, they exist during u evaluation. In fact, Γ_{22} might have intersection with Γ_{13} (see TyEff_Union) as long as they share the same modalities (it's even harder to rove I think).
  $\mathsf{names}(\Gamma_{13}^{@}) \cap \mathsf{names}(\Gamma_{23}^{@}) = \emptyset$: freshness of holes in both effects, executed sequentially, should be enough.

Theorem 2 (Type safety for complete programs)**.** *If* $\emptyset \vdash \mathsf{t} : \mathbf{A}$ *then* $\mathsf{t} \Downarrow \mathsf{v} \mid \varepsilon$ *and* $\emptyset \vdash \mathsf{v} : \mathbf{A}$.