Destination λ-calculus

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1 Term and value syntax

var, x, y	Term-level variable name
k	Index for ranges

$\mathtt{hdn}$, $\mathtt{h}$	::=			Hole or destination name ($\mathbb{N}$)
	|	$\mathtt{h}+\mathtt{h}'$	M	Sum
	|	$\mathtt{max}(\mathtt{H})$	M	Maximum of a set of holes
$\mathtt{hdns}$, $\mathtt{H}$	::=			Set of hole names
	|	$\{\mathtt{h}_1, .., \mathtt{h}_{\mathsf{k}}\}$		
	|	$\mathtt{H}_1 \cup \mathtt{H}_2$	M	Union of sets
	|	$\mathtt{H} \pm \mathtt{h}$	M	Increase all names from $\mathtt{H}$ by $\mathtt{h}$.
	|	$\mathtt{hnames}(\Gamma)$	M	Hole names of a context (requires $\mathtt{ctx\_NoVar}(\Gamma)$)
	|	$\mathtt{hnames}(\mathsf{C})$	M	Hole names of an evaluation context
term, t, u	::=			Term
	|	v		Value
	|	x		Variable
	|	$\mathsf{t} \succ \mathsf{u}$		Application
	|	$\mathsf{t}\ ;\ \mathsf{u}$		Pattern-match on unit
	|	$\mathsf{t} \succ \mathbf{case}_{m}\ \{\mathsf{Inl}\,\mathsf{x}_1 \mapsto \mathsf{u}_1\ ,\ \mathsf{Inr}\,\mathsf{x}_2 \mapsto \mathsf{u}_2\}$	bind x_1 in u_1 bind x_2 in u_2	Pattern-match on sum
	|	$\mathsf{t} \succ \mathbf{case}_{m}\ (\mathsf{x}_1\ ,\ \mathsf{x}_2) \mapsto \mathsf{u}$	bind x_1 in u bind x_2 in u	Pattern-match on product
	|	$\mathsf{t} \succ \mathbf{case}_{m}\ \mathrm{E}^{n}\,\mathsf{x} \mapsto \mathsf{u}$	bind x in u	Pattern-match on exponential
	|	$\mathsf{t} \succ \mathbf{map}\ \mathsf{x} \mapsto \mathsf{u}$	bind x in u	Map over the right side of ampar t
	|	$\mathbf{to}_{\ltimes}\ \mathsf{t}$		Wrap t into a trivial ampar
	|	$\mathbf{from}_{\ltimes}\ \mathsf{t}$		Extract value from trivial ampar
	|	$\mathbf{alloc}$		Return a fresh "identity" ampar object
	|	$\mathsf{t} \triangleleft ()$		Fill destination with unit
	|	$\mathsf{t} \triangleleft (\lambda \mathsf{x}\ {}_{m} \mapsto \mathsf{u})$	bind x in u	Fill destination with function
	|	$\mathsf{t} \triangleleft \mathsf{Inl}$		Fill destination with left variant
	|	$\mathsf{t} \triangleleft \mathsf{Inr}$		Fill destination with right variant
	|	$\mathsf{t} \triangleleft (,)$		Fill destination with product constructor
	|	$\mathsf{t} \triangleleft \mathrm{E}^{m}$		Fill destination with exponential constructor
	|	$\mathsf{t} \triangleleft\!\bullet\ \mathsf{u}$		Fill destination with root of ampar u
	|	$\mathsf{t}[\mathsf{x} := \mathsf{v}]$	M	
val, v	::=			Term value
	|	$-\mathtt{h}$		Hole
	|	$+\mathtt{h}$		Destination
	|	$()$		Unit
	|	$\lambda^{\mathsf{v}} \mathsf{x}\ {}_{m} \mapsto \mathsf{t}$	bind x in t	Lambda abstraction
	|	$\mathsf{Inl}\,\mathsf{v}$		Left variant for sum
	|	$\mathsf{Inr}\,\mathsf{v}$		Right variant for sum
	|	$\mathrm{E}^{m}\,\mathsf{v}$		Exponential
	|	$(\mathsf{v}_1\ ,\ \mathsf{v}_2)$		Product
	|	${}_{\mathtt{H}}\langle \mathsf{v}_1\ ,\ \mathsf{v}_2 \rangle$		Ampar
	|	$\mathsf{v} \pm \mathtt{h}$	M	Rename hole names inside v by shifting them by $\mathtt{h}$

eterm, j	::=			Pseudo-term
	|	C[t]		

ectx, C	::=			Evaluation context
	|	$\square$		Identity
	|	$C \succ u$		Application
	|	$v \succ C$		Application
	|	$C \; ; \; u$		Pattern-match on unit
	|	$C \succ \mathbf{case}_{m}\ \{ \mathsf{Inl}\, x_1 \mapsto u_1\ ,\ \mathsf{Inr}\, x_2 \mapsto u_2 \}$	bind x_1 in u_1 bind x_2 in u_2	Pattern-match on sum
	|	$C \succ \mathbf{case}_{m}\ (x_1\ ,\ x_2) \mapsto u$	bind x_1 in u bind x_2 in u	Pattern-match on product
	|	$C \succ \mathbf{case}_{m}\ \mathrm{E}^{n}\, x \mapsto u$	bind x in u	Pattern-match on exponential
	|	$C \succ \mathbf{map}\ x \mapsto u$	bind x in u	Map over the right side of ampar
	|	$\mathbf{to}_{\ltimes}\ C$		Wrap into a trivial ampar
	|	$\mathbf{from}_{\ltimes}\ C$		Extract value from trivial ampar
	|	$C \triangleleft ()$		Fill destination with unit
	|	$C \triangleleft (\lambda x\ {}_{m} \mapsto u)$	bind x in u	Fill destination with function
	|	$C \triangleleft \mathsf{Inl}$		Fill destination with left variant
	|	$C \triangleleft \mathsf{Inr}$		Fill destination with right variant
	|	$C \triangleleft (,)$		Fill destination with product constructor
	|	$C \triangleleft \mathrm{E}^{m}$		Fill destination with exponential constructor
	|	$C \triangleleft\!\bullet\ u$		Fill destination with root of ampar
	|	$v \triangleleft\!\bullet\ C$		Fill destination with root of ampar
	|	${}^{\mathrm{op}}_{\mathrm{H}}\langle v_1\ ⨾\ C$		Open ampar. Only new addition to term shapes
	|	$C \circ C'$	M	Compose evaluation contexts
	|	$C[h :=_{\mathrm{H}} v]$	M	Fill h with value v (that may contain holes)

2 Type system

type, **T**, **U**	::=			Type
	|	$\mathbf{1}$		Unit
	|	$T_1 \oplus T_2$		Sum
	|	$T_1 \otimes T_2$		Product
	|	$!^{m}\, T$		Exponential
	|	$T_1 \ltimes T_2$		Ampar type (consuming T_2 yields T_1)
	|	$T_1\ {}_{m_1}\!\!\to T_2$		Function
	|	$\lfloor T \rfloor^{m}$		Destination

mode, m, n	::=			Mode (Semiring)
	|	pa		Pair of a multiplicity and age
	|	☠		Error case (incompatible types, multiplicities, or ages)
	|	$m_1 \cdot \ldots \cdot m_k$	M	Semiring product

mul, p	::=			Multiplicity (first component of modality)
	|	1		Linear. Neutral element of the product
	|	ω		Non-linear. Absorbing for the product
	|	$p_1 . \ldots . p_k$	M	Semiring product

age, a	::=			Age (second component of modality)
	|	ν		Born now. Neutral element of the product
	|	$\uparrow$		One scope older
	|	∞		Infinitely old / static. Absorbing for the product
	|	$a_1 \cdot \ldots \cdot a_k$	M	Semiring product

bndr, b	::=			Type assignment to either variable, destination or hole
	|	$x :_{m} T$		Variable
	|	$+h :_{m} \lfloor T \rfloor^{n}$		Destination (m is its own modality; n is the modality for values it accepts)
	|	$-h : T^{n}$		Hole (n is the modality for values it accepts, it doesn't have a modality on its own)

ctx, Γ, Δ	::=			Typing context
	|	$\{\mathsf{b}_1, .., \mathsf{b}_k\}$		List of bindings
	|	$m{\cdot}\Gamma$	M	Multiply each binding by m
	|	$\Gamma_1 \uplus \Gamma_2$	M	Sum contexts Γ_1 and Γ_2. Duplicates/incompatible elements will give bindings with mo
	|	$-\Gamma$	M	Transforms every dest binding into a hole binding (requires `ctx_DestOnly` Γ)

$\boxed{\Gamma \Vdash \mathsf{v} : \mathsf{T}}$ *(Typing of values (raw))*

$$\textsc{TyR-val-H}\quad \frac{}{\{-\mathsf{h} : \mathsf{T}^{\mathit{1}\nu}\} \Vdash -\mathsf{h} : \mathsf{T}}$$

$$\textsc{TyR-val-D}\quad \frac{\texttt{ctx\_Compatible}\ \Gamma\ \ +\mathsf{h} :_{\mathit{1}\nu} \lfloor \mathsf{T} \rfloor^{n}}{\Gamma \Vdash +\mathsf{h} : \lfloor \mathsf{T} \rfloor^{n}}$$

$$\textsc{TyR-val-U}\quad \frac{}{\{\,\} \Vdash () : \mathbf{1}}$$

$$\textsc{TyR-val-F}\quad \frac{\begin{array}{c}\texttt{ctx\_DestOnly}\ \Gamma\\ \Gamma \uplus \{\mathsf{x} :_{m} \mathsf{T}_1\} \vdash \mathsf{t} : \mathsf{T}_2\end{array}}{\Gamma \Vdash \lambda^{\mathsf{v}} \mathsf{x}\,_{m} \mapsto \mathsf{t} : \mathsf{T}_1\ _{m}\!\!\to \mathsf{T}_2}$$

$$\textsc{TyR-val-L}\quad \frac{\Gamma \Vdash \mathsf{v} : \mathsf{T}_1}{\Gamma \Vdash \mathsf{Inl}\,\mathsf{v} : \mathsf{T}_1 \oplus \mathsf{T}_2}$$

$$\textsc{TyR-val-R}\quad \frac{\Gamma \Vdash \mathsf{v} : \mathsf{T}_2}{\Gamma \Vdash \mathsf{Inr}\,\mathsf{v} : \mathsf{T}_1 \oplus \mathsf{T}_2}$$

$$\textsc{TyR-val-P}\quad \frac{\Gamma_1 \Vdash \mathsf{v}_1 : \mathsf{T}_1 \qquad \Gamma_2 \Vdash \mathsf{v}_2 : \mathsf{T}_2}{\Gamma_1 \uplus \Gamma_2 \Vdash (\mathsf{v}_1\,,\,\mathsf{v}_2) : \mathsf{T}_1 \otimes \mathsf{T}_2}$$

$$\textsc{TyR-val-E}\quad \frac{\Gamma \Vdash \mathsf{v} : \mathsf{T}}{n{\cdot}\Gamma \Vdash \mathrm{E}^{n}\,\mathsf{v} : !^{n}\,\mathsf{T}}$$

$$\textsc{TyR-val-A}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_1\ \Gamma_2\\ \texttt{ctx\_DestOnly}\ \Gamma_2 \uplus \Gamma_3\\ \texttt{ctx\_DestOnly}\ \Gamma_1\\ \Gamma_1 \uplus (-\Gamma_3) \Vdash \mathsf{v}_1 : \mathsf{T}_1\\ \Gamma_2 \uplus \Gamma_3 \Vdash \mathsf{v}_2 : \mathsf{T}_2\end{array}}{\Gamma_1 \uplus \Gamma_2 \Vdash {}_{\texttt{hnames}(-\Gamma_3)}\langle \mathsf{v}_1\,,\,\mathsf{v}_2\rangle : \mathsf{T}_1 \ltimes \mathsf{T}_2}$$

$\boxed{\Gamma \vdash \mathsf{t} : \mathsf{T}}$ *(Typing of terms)*

$$\textsc{Ty-term-Val}\quad \frac{\texttt{ctx\_NoHole}\ \Gamma \qquad \Gamma \Vdash \mathsf{v} : \mathsf{T}}{\Gamma \vdash \mathsf{v} : \mathsf{T}}$$

$$\textsc{Ty-term-Var}\quad \frac{\texttt{ctx\_Compatible}\ \Gamma\ \ \mathsf{x} :_{\mathit{1}\nu} \mathsf{T}}{\Gamma \vdash \mathsf{x} : \mathsf{T}}$$

$$\textsc{Ty-term-App}\quad \frac{\Gamma_1 \vdash \mathsf{t} : \mathsf{T}_1 \qquad \Gamma_2 \vdash \mathsf{u} : \mathsf{T}_1\ _{m}\!\!\to \mathsf{T}_2}{m{\cdot}\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t} \succ \mathsf{u} : \mathsf{T}_2}$$

$$\textsc{Ty-term-PatU}\quad \frac{\Gamma_1 \vdash \mathsf{t} : \mathbf{1} \qquad \Gamma_2 \vdash \mathsf{u} : \mathsf{U}}{\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t}\ ;\ \mathsf{u} : \mathsf{U}}$$

$$\textsc{Ty-term-PatS}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{\mathsf{x}_1 :_{m} \mathsf{T}_1\}\\ \texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{\mathsf{x}_2 :_{m} \mathsf{T}_2\}\\ \Gamma_1 \vdash \mathsf{t} : \mathsf{T}_1 \oplus \mathsf{T}_2\\ \Gamma_2 \uplus \{\mathsf{x}_1 :_{m} \mathsf{T}_1\} \vdash \mathsf{u}_1 : \mathsf{U}\\ \Gamma_2 \uplus \{\mathsf{x}_2 :_{m} \mathsf{T}_2\} \vdash \mathsf{u}_2 : \mathsf{U}\end{array}}{m{\cdot}\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t} \succ \mathbf{case}_{m}\,\{\,\mathsf{Inl}\,\mathsf{x}_1 \mapsto \mathsf{u}_1\,,\ \mathsf{Inr}\,\mathsf{x}_2 \mapsto \mathsf{u}_2\,\} : \mathsf{U}}$$

$$\textsc{Ty-term-PatP}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{\mathsf{x}_1 :_{m} \mathsf{T}_1\}\\ \texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{\mathsf{x}_2 :_{m} \mathsf{T}_2\}\\ \texttt{ctx\_Disjoint}\ \ \{\mathsf{x}_1 :_{m} \mathsf{T}_1\}\ \ \{\mathsf{x}_2 :_{m} \mathsf{T}_2\}\\ \Gamma_1 \vdash \mathsf{t} : \mathsf{T}_1 \otimes \mathsf{T}_2\\ \Gamma_2 \uplus \{\mathsf{x}_1 :_{m} \mathsf{T}_1, \mathsf{x}_2 :_{m} \mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{m{\cdot}\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t} \succ \mathbf{case}_{m}\,(\mathsf{x}_1\,,\,\mathsf{x}_2) \mapsto \mathsf{u} : \mathsf{U}}$$

$$\textsc{Ty-term-PatE}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{\mathsf{x} :_{m \cdot n} \mathsf{T}\}\\ \Gamma_1 \vdash \mathsf{t} : !^{n}\,\mathsf{T}\\ \Gamma_2 \uplus \{\mathsf{x} :_{m\cdot n} \mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{m{\cdot}\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t} \succ \mathbf{case}_{m}\,\mathrm{E}^{n}\,\mathsf{x} \mapsto \mathsf{u} : \mathsf{U}}$$

$$\textsc{Ty-term-Map}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{\mathsf{x} :_{\mathit{1}\nu} \mathsf{T}_2\}\\ \Gamma_1 \vdash \mathsf{t} : \mathsf{T}_1 \ltimes \mathsf{T}_2\\ \mathit{1}{\uparrow}{\cdot}\Gamma_2 \uplus \{\mathsf{x} :_{\mathit{1}\nu} \mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t} \succ \mathbf{map}\ \mathsf{x} \mapsto \mathsf{u} : \mathsf{T}_1 \ltimes \mathsf{U}}$$

$$\textsc{Ty-term-ToA}\quad \frac{\Gamma \vdash \mathsf{t} : \mathsf{T}}{\Gamma \vdash \mathbf{to}_{\ltimes}\,\mathsf{t} : \mathsf{T} \ltimes \mathbf{1}}$$

$$\textsc{Ty-term-FromA}\quad \frac{\Gamma \vdash \mathsf{t} : \mathsf{T} \ltimes \mathbf{1}}{\Gamma \vdash \mathbf{from}_{\ltimes}\,\mathsf{t} : \mathsf{T}}$$

$$\textsc{Ty-term-Alloc}\quad \frac{}{\{\,\} \vdash \mathbf{alloc} : \mathsf{T} \ltimes \lfloor \mathsf{T} \rfloor^{\mathit{1}\nu}}$$

$$\textsc{Ty-term-FillU}\quad \frac{\Gamma \vdash \mathsf{t} : \lfloor \mathbf{1} \rfloor^{n}}{\Gamma \vdash \mathsf{t} \triangleleft () : \mathbf{1}}$$

$$\textsc{Ty-term-FillF}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{\mathsf{x} :_{m} \mathsf{T}_1\}\\ \Gamma_1 \vdash \mathsf{t} : \lfloor \mathsf{T}_1\ _{m}\!\!\to \mathsf{T}_2 \rfloor^{n}\\ \Gamma_2 \uplus \{\mathsf{x} :_{m} \mathsf{T}_1\} \vdash \mathsf{u} : \mathsf{T}_2\end{array}}{\Gamma_1 \uplus (\mathit{1}{\uparrow}{\cdot}n){\cdot}\Gamma_2 \vdash \mathsf{t} \triangleleft (\lambda \mathsf{x}\,_{m} \mapsto \mathsf{u}) : \mathbf{1}}$$

$$\textsc{Ty-term-FillL}\quad \frac{\Gamma \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^{n}}{\Gamma \vdash \mathsf{t} \triangleleft \mathsf{Inl} : \lfloor \mathsf{T}_1 \rfloor^{n}}$$

$$\textsc{Ty-term-FillR}\quad \frac{\Gamma \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^{n}}{\Gamma \vdash \mathsf{t} \triangleleft \mathsf{Inr} : \lfloor \mathsf{T}_2 \rfloor^{n}}$$

$$\textsc{Ty-term-FillP}\quad \frac{\Gamma \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \otimes \mathsf{T}_2 \rfloor^{n}}{\Gamma \vdash \mathsf{t} \triangleleft (,) : \lfloor \mathsf{T}_1 \rfloor^{n} \otimes \lfloor \mathsf{T}_2 \rfloor^{n}}$$

$$\textsc{Ty-term-FillE}\quad \frac{\Gamma \vdash \mathsf{t} : \lfloor !^{n'}\,\mathsf{T} \rfloor^{n}}{\Gamma \vdash \mathsf{t} \triangleleft \mathrm{E}^{n'} : \lfloor \mathsf{T} \rfloor^{n' \cdot n}}$$

$$\textsc{Ty-term-FillC}\quad \frac{\Gamma_1 \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \rfloor^{n} \qquad \Gamma_2 \vdash \mathsf{u} : \mathsf{T}_1 \ltimes \mathsf{T}_2}{\Gamma_1 \uplus (\mathit{1}{\uparrow}{\cdot}n){\cdot}\Gamma_2 \vdash \mathsf{t} \triangleleft\!\bullet\, \mathsf{u} : \mathsf{T}_2}$$

$\boxed{\Gamma \Vdash \mathsf{C} : \mathsf{T}_1 \rightarrowtail \mathsf{T}_2}$ *(Typing of evaluation contexts)*

$$\textsc{TyR-ectx-Id}\quad \frac{}{\{\,\} \Vdash \square : \mathsf{T} \rightarrowtail \mathsf{T}}$$

$$\textsc{TyR-ectx-App1}\quad \frac{\begin{array}{c}\Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail \mathsf{T}_1\\ \Gamma_2 \vdash \mathsf{u} : \mathsf{T}_1\ _{m}\!\!\to \mathsf{T}_2\end{array}}{m{\cdot}\Gamma_1 \uplus \Gamma_2 \Vdash \mathsf{C} \succ \mathsf{u} : \mathsf{T}_0 \rightarrowtail \mathsf{T}_2}$$

$$\textsc{TyR-ectx-App2}\quad \frac{\begin{array}{c}\Gamma_1 \vdash \mathsf{v} : \mathsf{T}_1\\ \Gamma_2 \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail (\mathsf{T}_1\ _{m}\!\!\to \mathsf{T}_2)\end{array}}{m{\cdot}\Gamma_1 \uplus \Gamma_2 \Vdash \mathsf{C} \succ \mathsf{u} : \mathsf{T}_0 \rightarrowtail \mathsf{T}_2}$$

$$\textsc{TyR-ectx-PatU}\quad \frac{\Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail \mathbf{1} \qquad \Gamma_2 \vdash \mathsf{u} : \mathsf{U}}{\Gamma_1 \uplus \Gamma_2 \Vdash \mathsf{C}\ ;\ \mathsf{u} : \mathsf{T}_0 \rightarrowtail \mathsf{U}}$$

TyR-ectx-PatS

$$\frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \{x_1 :_m \mathsf{T}_1\} \\ \texttt{ctx\_Disjoint}\ \Gamma_2\ \{x_2 :_m \mathsf{T}_2\} \\ \Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail (\mathsf{T}_1 \oplus \mathsf{T}_2) \\ \Gamma_2 \uplus \{x_1 :_m \mathsf{T}_1\} \vdash \mathsf{u}_1 : \mathsf{U} \\ \Gamma_2 \uplus \{x_2 :_m \mathsf{T}_2\} \vdash \mathsf{u}_2 : \mathsf{U}\end{array}}{m{\cdot}\Gamma_1 \uplus \Gamma_2 \Vdash \mathsf{C} \succ \mathbf{case}_m \{ \mathsf{Inl}\, x_1 \mapsto \mathsf{u}_1 ,\ \mathsf{Inr}\, x_2 \mapsto \mathsf{u}_2 \} : \mathsf{T}_0 \rightarrowtail \mathsf{U}}$$

TyR-ectx-PatP

$$\frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \{x_1 :_m \mathsf{T}_1\} \\ \texttt{ctx\_Disjoint}\ \Gamma_2\ \{x_2 :_m \mathsf{T}_2\} \\ \texttt{ctx\_Disjoint}\ \{x_1 :_m \mathsf{T}_1\}\ \{x_2 :_m \mathsf{T}_2\} \\ \Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail (\mathsf{T}_1 \otimes \mathsf{T}_2) \\ \Gamma_2 \uplus \{x_1 :_m \mathsf{T}_1, x_2 :_m \mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{m{\cdot}\Gamma_1 \uplus \Gamma_2 \Vdash \mathsf{C} \succ \mathbf{case}_m\, (x_1 , x_2) \mapsto \mathsf{u} : \mathsf{T}_0 \rightarrowtail \mathsf{U}}$$

TyR-ectx-PatE

$$\frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \{x :_{m\cdot n} \mathsf{T}\} \\ \Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail !^n\, \mathsf{T} \\ \Gamma_2 \uplus \{x :_{m\cdot n} \mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{m{\cdot}\Gamma_1 \uplus \Gamma_2 \Vdash \mathsf{C} \succ \mathbf{case}_m\, \mathrm{E}^n\, x \mapsto \mathsf{u} : \mathsf{T}_0 \rightarrowtail \mathsf{U}}$$

TyR-ectx-Map

$$\frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \{x :_{1\nu} \mathsf{T}_2\} \\ \Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail (\mathsf{T}_1 \ltimes \mathsf{T}_2) \\ 1{\uparrow}{\cdot}\Gamma_2 \uplus \{x :_{1\nu} \mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Gamma_1 \uplus \Gamma_2 \Vdash \mathsf{C} \succ \mathbf{map}\ x \mapsto \mathsf{u} : \mathsf{T}_0 \rightarrowtail (\mathsf{T}_1 \ltimes \mathsf{U})}$$

TyR-ectx-ToA

$$\frac{\Gamma \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail \mathsf{T}}{\Gamma \Vdash \mathbf{to}_{\ltimes}\, \mathsf{C} : \mathsf{T}_0 \rightarrowtail (\mathsf{T} \ltimes \mathbf{1})}$$

TyR-ectx-FromA

$$\frac{\Gamma \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail (\mathsf{T} \ltimes \mathbf{1})}{\Gamma \Vdash \mathbf{from}_{\ltimes}\, \mathsf{C} : \mathsf{T}_0 \rightarrowtail \mathsf{T}}$$

TyR-ectx-FillU

$$\frac{\Gamma \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail \lfloor \mathbf{1} \rfloor^n}{\Gamma \Vdash \mathsf{C} \lhd () : \mathsf{T}_0 \rightarrowtail \mathbf{1}}$$

TyR-ectx-FillF

$$\frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \{x :_m \mathsf{T}_1\} \\ \Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail \lfloor \mathsf{T}_1\ {}_m{\to}\ \mathsf{T}_2 \rfloor^n \\ \Gamma_2 \uplus \{x :_m \mathsf{T}_1\} \vdash \mathsf{u} : \mathsf{T}_2\end{array}}{\Gamma_1 \uplus (1{\uparrow}{\cdot}n){\cdot}\Gamma_2 \Vdash \mathsf{C} \lhd (\lambda x\ {}_m \mapsto \mathsf{u}) : \mathsf{T}_0 \rightarrowtail \mathbf{1}}$$

TyR-ectx-FillL

$$\frac{\Gamma \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^n}{\Gamma \Vdash \mathsf{C} \lhd \mathsf{Inl} : \mathsf{T}_0 \rightarrowtail \lfloor \mathsf{T}_1 \rfloor^n}$$

TyR-ectx-FillR

$$\frac{\Gamma \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^n}{\Gamma \Vdash \mathsf{C} \lhd \mathsf{Inr} : \mathsf{T}_0 \rightarrowtail \lfloor \mathsf{T}_2 \rfloor^n}$$

TyR-ectx-FillP

$$\frac{\Gamma \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail \lfloor \mathsf{T}_1 \otimes \mathsf{T}_2 \rfloor^n}{\Gamma \Vdash \mathsf{C} \lhd (,) : \mathsf{T}_0 \rightarrowtail \lfloor \mathsf{T}_1 \rfloor^n \otimes \lfloor \mathsf{T}_2 \rfloor^n}$$

TyR-ectx-FillE

$$\frac{\Gamma \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail \lfloor !^{n'}\, \mathsf{T} \rfloor^n}{\Gamma \Vdash \mathsf{C} \lhd \mathrm{E}^{n'} : \mathsf{T}_0 \rightarrowtail \lfloor \mathsf{T} \rfloor^{n'\cdot n}}$$

TyR-ectx-FillC

$$\frac{\begin{array}{c}\Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_0 \rightarrowtail \lfloor \mathsf{T}_1 \rfloor^n \\ \Gamma_2 \vdash \mathsf{u} : \mathsf{T}_1 \ltimes \mathsf{T}_2\end{array}}{\Gamma_1 \uplus (1{\uparrow}{\cdot}n){\cdot}\Gamma_2 \Vdash \mathsf{C} \lhd\!\bullet\, \mathsf{u} : \mathsf{T}_0 \rightarrowtail \mathsf{T}_2}$$

$\boxed{\vdash \mathsf{j} : \mathsf{T}}$ *(Typing of extended terms (pair of evaluation context and term))*

Ty-eterm-ClosedEterm

$$\frac{-\Gamma \Vdash \mathsf{C} : \mathsf{T}_1 \rightarrowtail \mathsf{T}_2 \qquad \Gamma \vdash \mathsf{t} : \mathsf{T}_1}{\vdash \mathsf{C}[\mathsf{t}] : \mathsf{T}_2}$$

3 Small-step semantics

$\boxed{\mathsf{j} \longrightarrow \mathsf{j}'}$ *(Small-step evaluation of terms using evaluation contexts)*

$$\dfrac{}{\mathsf{C}[\mathsf{v} \succ (\lambda^{\mathsf{v}} \mathsf{x}_{\,m} \mapsto \mathsf{t})] \longrightarrow \mathsf{C}[\mathsf{t}[\mathsf{x} \coloneqq \mathsf{v}]]}\ \textsc{Sem-eterm-App}$$

$$\dfrac{}{\mathsf{C}[() \mathbin{;} \mathsf{t}_2] \longrightarrow \mathsf{C}[\mathsf{t}_2]}\ \textsc{Sem-eterm-PatU}$$

$$\dfrac{}{\mathsf{C}[(\mathsf{Inl}\,\mathsf{v}) \succ \mathbf{case}_m \{\, \mathsf{Inl}\,\mathsf{x}_1 \mapsto \mathsf{u}_1 ,\ \mathsf{Inr}\,\mathsf{x}_2 \mapsto \mathsf{u}_2 \,\}] \longrightarrow \mathsf{C}[\mathsf{u}_1[\mathsf{x} \coloneqq \mathsf{v}]]}\ \textsc{Sem-eterm-PatL}$$

$$\dfrac{}{\mathsf{C}[(\mathsf{Inr}\,\mathsf{v}) \succ \mathbf{case}_m \{\, \mathsf{Inl}\,\mathsf{x}_1 \mapsto \mathsf{u}_1 ,\ \mathsf{Inr}\,\mathsf{x}_2 \mapsto \mathsf{u}_2 \,\}] \longrightarrow \mathsf{C}[\mathsf{u}_2[\mathsf{x} \coloneqq \mathsf{v}]]}\ \textsc{Sem-eterm-PatR}$$

$$\dfrac{}{\mathsf{C}[(\mathsf{v}_1 , \mathsf{v}_2) \succ \mathbf{case}_m (\mathsf{x}_1 , \mathsf{x}_2) \mapsto \mathsf{u}] \longrightarrow \mathsf{C}[\mathsf{u}[\mathsf{x}_1 \coloneqq \mathsf{v}_1][\mathsf{x}_2 \coloneqq \mathsf{v}_2]]}\ \textsc{Sem-eterm-PatP}$$

$$\dfrac{}{\mathsf{C}[\mathrm{E}^{n}\,\mathsf{v} \succ \mathbf{case}_m \mathrm{E}^{n}\,\mathsf{x} \mapsto \mathsf{u}] \longrightarrow \mathsf{C}[\mathsf{u}[\mathsf{x} \coloneqq \mathsf{v}]]}\ \textsc{Sem-eterm-PatE}$$

$$\dfrac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}))}{\mathsf{C}[{}_{\mathtt{H}}\langle \mathsf{v}_1 , \mathsf{v}_2 \rangle \succ \mathbf{map}\ \mathsf{x} \mapsto \mathsf{u}] \longrightarrow (\mathsf{C} \circ ({}^{\mathrm{op}}_{\mathtt{H} \pm \mathtt{h}'}\langle \mathsf{v}_1 \pm \mathtt{h}' , \square \rangle))[\mathsf{u}[\mathsf{x} \coloneqq \mathsf{v}_2 \pm \mathtt{h}']]}\ \textsc{Sem-eterm-MapOpen}$$

$$\dfrac{}{(\mathsf{C} \circ {}^{\mathrm{op}}_{\mathtt{H}}\langle \mathsf{v}_1 , \square \rangle)[\mathsf{v}_2] \longrightarrow \mathsf{C}[{}_{\mathtt{H}}\langle \mathsf{v}_1 , \mathsf{v}_2 \rangle]}\ \textsc{Sem-eterm-MapClose}$$

$$\dfrac{}{\mathbf{alloc} \longrightarrow {}_{\{1\}}\langle +1 , -1 \rangle}\ \textsc{Sem-eterm-Alloc}$$

$$\dfrac{}{\mathsf{C}[\mathbf{to}_{\ltimes}\ \mathsf{v}] \longrightarrow \mathsf{C}[{}_{\{\,\}}\langle \mathsf{v} , () \rangle]}\ \textsc{Sem-eterm-ToA}$$

$$\dfrac{}{\mathsf{C}[\mathbf{from}_{\ltimes}\ {}_{\{\,\}}\langle \mathsf{v} , () \rangle] \longrightarrow \mathsf{v}}\ \textsc{Sem-eterm-FromA}$$

$$\dfrac{}{\mathsf{C}[+\mathtt{h} \triangleleft ()] \longrightarrow \mathsf{C}[\mathtt{h} \coloneqq_{\{\,\}} ()][()]}\ \textsc{Sem-eterm-FillU}$$

$$\dfrac{}{\mathsf{C}[+\mathtt{h} \triangleleft (\lambda \mathsf{x}_{\,m} \mapsto \mathsf{u})] \longrightarrow \mathsf{C}[\mathtt{h} \coloneqq_{\{\,\}} \lambda^{\mathsf{v}} \mathsf{x}_{\,m} \mapsto \mathsf{u}][()]}\ \textsc{Sem-eterm-FillF}$$

$$\dfrac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}) \cup \{\mathtt{h}\})}{\mathsf{C}[+\mathtt{h} \triangleleft \mathsf{Inl}] \longrightarrow \mathsf{C}[\mathtt{h} \coloneqq_{\{\mathtt{h}'+1\}} \mathsf{Inl}\ -(\mathtt{h}'+1)][+(\mathtt{h}'+1)]}\ \textsc{Sem-eterm-FillL}$$

$$\dfrac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}) \cup \{\mathtt{h}\})}{\mathsf{C}[+\mathtt{h} \triangleleft \mathsf{Inr}] \longrightarrow \mathsf{C}[\mathtt{h} \coloneqq_{\{\mathtt{h}'+1\}} \mathsf{Inr}\ -(\mathtt{h}'+1)][+(\mathtt{h}'+1)]}\ \textsc{Sem-eterm-FillR}$$

$$\dfrac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}) \cup \{\mathtt{h}\})}{\mathsf{C}[+\mathtt{h} \triangleleft \mathrm{E}^{m}] \longrightarrow \mathsf{C}[\mathtt{h} \coloneqq_{\{\mathtt{h}'+1\}} \mathrm{E}^{m}\ -(\mathtt{h}'+1)][+(\mathtt{h}'+1)]}\ \textsc{Sem-eterm-FillE}$$

$$\dfrac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}) \cup \{\mathtt{h}\})}{\mathsf{C}[+\mathtt{h} \triangleleft (,)] \longrightarrow \mathsf{C}[\mathtt{h} \coloneqq_{\{\mathtt{h}'+1, \mathtt{h}'+2\}} (-(\mathtt{h}'+1) , -(\mathtt{h}'+2))][(+(\mathtt{h}'+1) , +(\mathtt{h}'+2))]}\ \textsc{Sem-eterm-FillP}$$

$$\dfrac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}) \cup \{\mathtt{h}\})}{\mathsf{C}[+\mathtt{h} \triangleleft\!\bullet\ {}_{\mathtt{H}}\langle \mathsf{v}_1 , \mathsf{v}_2 \rangle] \longrightarrow \mathsf{C}[\mathtt{h} \coloneqq_{(\mathtt{H} \pm \mathtt{h}')} \mathsf{v}_1 \pm \mathtt{h}'][\mathsf{v}_2 \pm \mathtt{h}']}\ \textsc{Sem-eterm-FillC}$$