

# Destination $\lambda$ -calculus

Thomas BAGREL

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## 1 Term and value syntax

$tmv, x, y$  Term-level variable name  
 $hdmv$  Hole or destination static name  
 $k$  Index for ranges

$hddyn, d$  ::= Hole or destination dynamic name  
|  $\star$  Root namespace  
|  $d.1$  Subnamespace 1  
|  $d.2$  Subnamespace 2  
|  $d.3$  Subnamespace 3

$hdnm, h$  ::= Hole or destination name  
|  $d$  Dynamic name  
|  $hdmv$  Static name

$val, v$  ::= Term value  
|  $-h$  Hole  
|  $+h$  Destination  
|  $()$  Unit  
|  $\lambda x \mapsto t$  Lambda abstraction  
|  $lnl\ v$  Left variant for sum  
|  $lnr\ v$  Right variant for sum  
|  $\mathbin{\mathbb{J}}^m v$  Exponential  
|  $(v_1, v_2)$  Product  
|  $\langle v_1, v_2 \rangle_\Delta$  Ampar

$term, t, u$  ::= Term  
|  $v$  Value  
|  $x$  Variable  
|  $t \succ u$  Application  
|  $t ; u$  Pattern-match on unit  
|  $t \succ \text{case } \{ lnl\ x_1 \mapsto u_1, lnr\ x_2 \mapsto u_2 \}$  Pattern-match on sum  
  
|  $t \succ \text{case } (x_1, x_2) \mapsto u$  Pattern-match on product  
  
|  $t \succ \text{case } \mathbin{\mathbb{J}}^m x \mapsto u$  Pattern-match on exponential  
|  $t \succ \text{map } x \mapsto u$  Map over the right side of the ampar  
|  $\text{to}_\times t$  Wrap  $t$  into a trivial ampar  
|  $\text{from}_\times t$  Extract value from trivial ampar  
|  $\text{alloc}_T$  Return a fresh "identity" ampar object  
|  $t \triangleleft ()$  Fill destination with unit  
|  $t \triangleleft lnl$  Fill destination with left variant  
|  $t \triangleleft lnr$  Fill destination with right variant  
|  $t \triangleleft (,)$  Fill destination with product constructor  
|  $t \triangleleft \mathbin{\mathbb{J}}^m$  Fill destination with exponential constructor  
|  $t \triangleleft \bullet u$  Fill destination with root of ampar  $u$

## 2 Type system

<b>type, <math>\mathsf{T}</math>, <math>\mathsf{U}</math></b>	$::=$	Type
	$\mathbf{1}$	Unit
	$\mathsf{T}_1 \oplus \mathsf{T}_2$	Sum
	$\mathsf{T}_1 \otimes \mathsf{T}_2$	Product
	$!^m \mathsf{T}$	Exponential
	$\mathsf{T}_1 \ltimes \mathsf{T}_2$	Ampar type (consuming $\mathsf{T}_2$ yields $\mathsf{T}_1$ )
	$\mathsf{T}_1 \xrightarrow{m_1} \mathsf{T}_2$	Function
	$\llbracket \mathsf{T} \rrbracket^m$	Destination
<b>mode, <math>m</math>, <math>n</math></b>	$::=$	Mode (Semiring)
	$pa$	Pair of a multiplicity and age
	$\omega$	Error case (incompatible types, multiplicities, or ages)
	$m_1 \cdot \dots \cdot m_k$	Semiring product
<b>mul, <math>p</math></b>	$::=$	Multiplicity (first component of modality)
	$1$	Linear. Neutral element of the product
	$\omega$	Non-linear. Absorbing for the product
	$p_1 \cdot \dots \cdot p_k$	Semiring product
<b>age, <math>a</math></b>	$::=$	Age (second component of modality)
	$\nu$	Born now. Neutral element of the product
	$\uparrow$	One scope older
	$\infty$	Infinitely old / static. Absorbing for the product
	$a_1 \cdot \dots \cdot a_k$	Semiring product
<b>ctx, <math>\Gamma</math>, <math>\Delta</math></b>	$::=$	Typing context
	$\{b_1, \dots, b_k\}$	List of bindings
	$m \cdot \Gamma$	Multiply each binding by $m$
	$\Gamma_1 \uplus \Gamma_2$	Sum contexts $\Gamma_1$ and $\Gamma_2$ . Duplicates/incompatible elements will give bindings with modality $\omega$
	$\Gamma_1 \overset{-}{\uplus} \overset{+}{\Gamma_2}$	Sum contexts, but allow linear holes from $\Gamma_1$ to be compensated by linear dests from $\Gamma_2$
	$-\Gamma$	Transforms every hole binding into a dest binding (requires <code>ctx_DestOnly</code> $\Gamma$ )
<b>bndr, <math>b</math></b>	$::=$	Type assignment to either variable, destination or hole
	$x :_m \mathsf{T}$	Variable
	$+h :_m \llbracket \mathsf{T} \rrbracket^n$	Destination ( $m$ is its own modality; $n$ is the modality for values it accepts)
	$-h :^n \mathsf{T}$	Hole ( $n$ is the modality for values it accepts, it doesn't have a modality on its own)

$$\boxed{\Gamma \Vdash t : \mathbf{T}}$$

(Typing of terms (raw))

$$\begin{array}{c}
\text{TYR-TERM-H} \quad \frac{}{\{-\mathbf{h} : {}^{\textcolor{teal}{I}\nu} \mathbf{T}\} \Vdash -\mathbf{h} : \mathbf{T}} \quad
\text{TYR-TERM-D} \quad \frac{}{\{+\mathbf{h} : {}^{\textcolor{teal}{I}\nu} [\mathbf{T}]^n\} \Vdash +\mathbf{h} : [\mathbf{T}]^n} \quad
\text{TYR-TERM-U} \quad \frac{}{\{\} \Vdash () : \mathbf{1}} \quad
\text{TYR-TERM-L} \quad \frac{\Gamma \Vdash v : \mathbf{T}_1}{\Gamma \Vdash \text{Inl } v : \mathbf{T}_1 \oplus \mathbf{T}_2} \quad
\text{TYR-TERM-R} \quad \frac{\Gamma \Vdash v : \mathbf{T}_2}{\Gamma \Vdash \text{Inr } v : \mathbf{T}_1 \oplus \mathbf{T}_2} \\
\\
\text{TYR-TERM-P} \quad \frac{\Gamma_1 \Vdash v_1 : \mathbf{T}_1 \quad \Gamma_2 \Vdash v_2 : \mathbf{T}_2}{\Gamma_1 \uplus \Gamma_2 \Vdash (v_1, v_2) : \mathbf{T}_1 \otimes \mathbf{T}_2} \quad
\text{TYR-TERM-E} \quad \frac{\Gamma \Vdash v : \mathbf{T}}{m \cdot \Gamma \Vdash \mathbin{\text{\textcircled{\textcolor{teal}{I}}}}^m v : !^m \mathbf{T}} \quad
\text{TYR-TERM-A} \quad \frac{\Gamma_1 \Vdash v_1 : \mathbf{T}_1 \quad \Gamma_2 \Vdash v_2 : \mathbf{T}_2 \quad \text{ctx\_DestOnly } \Gamma_2 \quad \text{ctx\_SubsetEq } - \Gamma_2 \Gamma_1}{\Gamma_1 \text{ } ^{-} \uplus \textcolor{teal}{I} \Gamma_2 \Vdash \langle v_1, v_2 \rangle_{-\Gamma_1} : \mathbf{T}_1 \ltimes \mathbf{T}_2} \quad
\text{TYR-TERM-F} \quad \frac{\Gamma \uplus \{\mathbf{x} : {}^{\textcolor{teal}{I}m} \mathbf{T}_1\} \Vdash t : \mathbf{T}_2}{\Gamma \Vdash \lambda \mathbf{x} \mapsto t : \mathbf{T}_1 \xrightarrow{m} \mathbf{T}_2} \\
\\
\text{TYR-TERM-VAR} \quad \frac{\text{ctx\_Compatible } \Gamma \quad \{\mathbf{x} : {}^{\textcolor{teal}{I}\nu} \mathbf{T}\}}{\Gamma \Vdash \mathbf{x} : \mathbf{T}} \quad
\text{TYR-TERM-APP} \quad \frac{\Gamma_1 \Vdash t : \mathbf{T}_1 \quad \Gamma_2 \Vdash u : \mathbf{T}_1 \xrightarrow{m} \mathbf{T}_2}{m \cdot \Gamma_1 \uplus \Gamma_2 \Vdash t \succ u : \mathbf{T}_2} \quad
\text{TYR-TERM-PATU} \quad \frac{\Gamma_1 \Vdash t : \mathbf{1} \quad \Gamma_2 \Vdash u : \mathbf{U}}{\Gamma_1 \uplus \Gamma_2 \Vdash t ; u : \mathbf{U}} \\
\\
\text{TYR-TERM-PATS} \quad \frac{\Gamma_1 \Vdash t : \mathbf{T}_1 \oplus \mathbf{T}_2 \quad \Gamma_2 \uplus \{\mathbf{x}_1 : {}^{\textcolor{teal}{I}m} \mathbf{T}_1\} \Vdash u_1 : \mathbf{U} \quad \Gamma_2 \uplus \{\mathbf{x}_2 : {}^{\textcolor{teal}{I}m} \mathbf{T}_2\} \Vdash u_2 : \mathbf{U}}{m \cdot \Gamma_1 \uplus \Gamma_2 \Vdash t \succ \text{case } \{\text{Inl } \mathbf{x}_1 \mapsto u_1, \text{Inr } \mathbf{x}_2 \mapsto u_2\} : \mathbf{U}} \quad
\text{TYR-TERM-PATP} \quad \frac{\Gamma_1 \Vdash t : \mathbf{T}_1 \otimes \mathbf{T}_2 \quad \Gamma_2 \uplus \{\mathbf{x}_1 : {}^{\textcolor{teal}{I}m} \mathbf{T}_1, \mathbf{x}_2 : {}^{\textcolor{teal}{I}m} \mathbf{T}_2\} \Vdash u : \mathbf{U}}{m \cdot \Gamma_1 \uplus \Gamma_2 \Vdash t \succ \text{case } (\mathbf{x}_1, \mathbf{x}_2) \mapsto u : \mathbf{U}} \\
\\
\text{TYR-TERM-PATE} \quad \frac{\Gamma_1 \Vdash t : !^n \mathbf{T} \quad \Gamma_2 \uplus \{\mathbf{x} : {}^{\textcolor{teal}{I}m \cdot n} \mathbf{T}\} \Vdash u : \mathbf{U}}{m \cdot \Gamma_1 \uplus \Gamma_2 \Vdash t \succ \text{case } \mathbin{\text{\textcircled{\textcolor{teal}{I}}}}^n \mathbf{x} \mapsto u : \mathbf{U}} \quad
\text{TYR-TERM-MAP} \quad \frac{\Gamma_1 \Vdash t : \mathbf{T}_1 \ltimes \mathbf{T}_2 \quad \textcolor{teal}{I} \uparrow \Gamma_2 \uplus \{\mathbf{x} : {}^{\textcolor{teal}{I}\nu} \mathbf{T}_2\} \Vdash u : \mathbf{U}}{\Gamma_1 \uplus \Gamma_2 \Vdash t \succ \text{map } \mathbf{x} \mapsto u : \mathbf{T}_1 \ltimes \mathbf{U}} \quad
\text{TYR-TERM-FILLC} \quad \frac{\Gamma_1 \Vdash t : [\mathbf{T}_1]^n \quad \Gamma_2 \Vdash u : \mathbf{T}_1 \ltimes \mathbf{T}_2}{\Gamma_1 \uplus (\textcolor{teal}{I} \uparrow \cdot n) \cdot \Gamma_2 \Vdash t \blacktriangleleft u : \mathbf{T}_2} \\
\\
\text{TYR-TERM-FILLU} \quad \frac{\Gamma \Vdash t : [\mathbf{1}]^n}{\Gamma \Vdash t \triangleleft () : \mathbf{1}} \quad
\text{TYR-TERM-FILLL} \quad \frac{\Gamma \Vdash t : [\mathbf{T}_1 \oplus \mathbf{T}_2]^n}{\Gamma \Vdash t \triangleleft \text{Inl} : [\mathbf{T}_1]^n} \quad
\text{TYR-TERM-FILLR} \quad \frac{\Gamma \Vdash t : [\mathbf{T}_1 \oplus \mathbf{T}_2]^n}{\Gamma \Vdash t \triangleleft \text{Inr} : [\mathbf{T}_2]^n} \quad
\text{TYR-TERM-FILLP} \quad \frac{\Gamma \Vdash t : [\mathbf{T}_1 \otimes \mathbf{T}_2]^n}{\Gamma \Vdash t \triangleleft (,) : [\mathbf{T}_1]^n \otimes [\mathbf{T}_2]^n} \quad
\text{TYR-TERM-FILLE} \quad \frac{\Gamma \Vdash t : [!^n \mathbf{T}]^m}{\Gamma \Vdash t \triangleleft \mathbin{\text{\textcircled{\textcolor{teal}{I}}}}^n : [\mathbf{T}]^{m \cdot n}} \\
\\
\text{TYR-TERM-ALLOC} \quad \frac{}{\{\} \Vdash \text{alloc}_{\mathbf{T}} : \mathbf{T} \ltimes [\mathbf{T}]^{\textcolor{teal}{I}\nu}} \quad
\text{TYR-TERM-TOA} \quad \frac{\Gamma \Vdash t : \mathbf{T}}{\Gamma \Vdash \text{to}_{\ltimes} t : \mathbf{T} \ltimes \mathbf{1}} \quad
\text{TYR-TERM-FROMA} \quad \frac{\Gamma \Vdash t : \mathbf{T} \ltimes \mathbf{1}}{\Gamma \Vdash \text{from}_{\ltimes} t : \mathbf{T}}
\end{array}$$

$$\boxed{\Gamma \vdash t : \mathbf{T}}$$

(Typing of terms (valid ones only))

$$\begin{array}{c}
\text{TY-TERM-T} \\
\frac{\Gamma \Vdash t : \mathbf{T} \quad \text{ctx\_Valid } \Gamma \quad \text{ctx\_NoHole } \Gamma}{\Gamma \vdash t : \mathbf{T}}
\end{array}$$

### 3 Effects and big-step semantics

$eff, e$	$::=$	Effect
	$\varepsilon$	No-op effect
	$f$	Single hole assignment
	$e_1 \gg \dots \gg e_k$	Chain effects
$hf, f$	$::=$	Hole filling
	$\mathbf{h} := v$	Fill $\mathbf{h}$ with value $v$ (that may contain holes)

$\boxed{\Gamma \Vdash e}$  (Typing of effects (raw))

TYR-EFF-N	TYR-EFF-A	TYR-EFF-C
$\frac{}{\{\} \Vdash \varepsilon}$	$\frac{\Gamma \uplus \Delta \Vdash v : \mathbf{T} \quad \text{ctx\_DestOnly } \Gamma}{\text{ctx\_HoleOnly } \Delta}$	$\frac{\Gamma_1 \Vdash e_1 \quad \Gamma_2 \Vdash e_2}{\Gamma_1 \text{ } ^-\uplus^+ \Gamma_2 \Vdash e_1 \gg e_2}$
	$(I \uparrow \cdot m \cdot n) \cdot \Gamma \uplus \{+\mathbf{h} :_m [\mathbf{T}]^n\} \uplus (m \cdot n) \cdot \Delta \Vdash \mathbf{h} := v$	

$\boxed{\Gamma \vdash e}$  (Typing of effects (valid ones only))

TY-EFF-T
$\frac{\Gamma \Vdash e \quad \text{ctx\_Valid } \Gamma}{\Gamma \vdash e}$

$\boxed{\Gamma \vdash v \diamond e : \mathbf{T}}$  (Typing of commands (valid ones only))

TY-CMD-C
$\frac{\Gamma_1 \vdash e \quad \Gamma_2 \vdash v : \mathbf{T} \quad \text{ctx\_DestOnly } \Gamma_1 \text{ } ^-\uplus^+ \Gamma_2}{\Gamma_1 \text{ } ^-\uplus^+ \Gamma_2 \vdash v \diamond e : \mathbf{T}}$

$\boxed{v_1 \Gamma_1 \mid e_1 \Downarrow v_2 \Gamma_2 \mid e_2}$  (Big-step evaluation of effects on values (with potential holes))

SEM-EFF-N	SEM-EFF-S	SEM-EFF-F
$\frac{}{v_1 \Gamma_1 \mid \varepsilon \Downarrow v_1 \Gamma_1 \mid \varepsilon}$	$\frac{\text{ctx\_HdnmNotMem } \mathbf{h} \Gamma_1 \quad v_1 \Gamma_1 \mid e_1 \Downarrow v_2 \Gamma_2 \mid e_2}{v_1 \Gamma_1 \mid \mathbf{h} := v' \gg e_1 \Downarrow v_2 \Gamma_2 \mid \mathbf{h} := v' \gg e_2}$	$\frac{\Gamma_0 \Vdash v_0 : \mathbf{T} \quad \text{ctx\_Valid } \Gamma_0 \quad v_1[\mathbf{h} := v_0] (\Gamma_1 \uplus n \cdot \Gamma_0) \mid e_1 \Downarrow v_2 \Gamma_2 \mid e_2}{v_1 \Gamma_1 \uplus \{-\mathbf{h} : n \cdot \mathbf{T}\} \mid \mathbf{h} := v_0 \gg e_1 \Downarrow v_2 \Gamma_2 \mid e_2}$

$\boxed{t \Downarrow_d v \diamond e}$  (Big-step evaluation into commands)

SEM-TERM-V	SEM-TERM-APP	SEM-TERM-PATU
$\frac{}{v \Downarrow_d v \diamond \varepsilon}$	$\frac{t_1 \Downarrow_{d.1} v_1 \diamond e_1 \quad t_2 \Downarrow_{d.2} \lambda \mathbf{x} \mapsto u \diamond e_2 \quad u[\mathbf{x} := v_1] \Downarrow_{d.3} v_3 \diamond e_3}{t_1 \succ t_2 \Downarrow_d v_3 \diamond e_1 \gg e_2 \gg e_3}$	$\frac{t_1 \Downarrow_{d.1} () \diamond e_1 \quad t_2 \Downarrow_{d.2} v_2 \diamond e_2}{t_1 ; t_2 \Downarrow_d v_2 \diamond e_1 \gg e_2}$

SEM-TERM-PATL	SEM-TERM-PATR
$\frac{t \Downarrow_{d.1} \text{Inl } v_1 \diamond e_1 \quad u_1[\mathbf{x}_1 := v_1] \Downarrow_{d.2} v_2 \diamond e_2}{t \succ \text{case } \{ \text{Inl } \mathbf{x}_1 \mapsto u_1, \text{Inr } \mathbf{x}_2 \mapsto u_2 \} \Downarrow_d v_2 \diamond e_1 \gg e_2}$	$\frac{t \Downarrow_{d.1} \text{Inr } v_1 \diamond e_1 \quad u_2[\mathbf{x}_2 := v_1] \Downarrow_{d.2} v_2 \diamond e_2}{t \succ \text{case } \{ \text{Inl } \mathbf{x}_1 \mapsto u_1, \text{Inr } \mathbf{x}_2 \mapsto u_2 \} \Downarrow_d v_2 \diamond e_1 \gg e_2}$

SEM-TERM-PATP	SEM-TERM-MAP	SEM-TERM-ALLOC
$\frac{t \Downarrow_{d.1} (v_1, v_2) \diamond e_1 \quad u[\mathbf{x}_1 := v_1][\mathbf{x}_2 := v_2] \Downarrow_{d.2} v_2 \diamond e_2}{t \succ \text{case } (\mathbf{x}_1, \mathbf{x}_2) \mapsto u \Downarrow_d v_2 \diamond e_1 \gg e_2}$	$\frac{t \Downarrow_{d.1} \langle v_1, v_2 \rangle_\Delta \diamond e_1 \quad u[\mathbf{x} := v_1] \Downarrow_{d.2} v_3 \diamond e_2 \quad v_2 \Delta \mid e_2 \Downarrow_d v_4 \Delta' \mid e_3}{t \succ \text{map } \mathbf{x} \mapsto u \Downarrow_d \langle v_3, v_4 \rangle_{\Delta'} \diamond e_1 \gg e_3}$	$\frac{}{\text{alloc } \mathbf{T} \Downarrow_d \langle +\mathbf{d}, -\mathbf{d} \rangle_{\{-\mathbf{d} : n \cdot \mathbf{T}\}} \diamond \varepsilon}$

SEM-TERM-TOA	SEM-TERM-FROMA	SEM-TERM-FILLU	SEM-TERM-FILLL
$\frac{}{t \Downarrow_d v \diamond e}$	$\frac{}{t \Downarrow_d \langle (), v \rangle_{\{\}} \diamond e}$	$\frac{}{t \Downarrow_d +\mathbf{h} \diamond e}$	$\frac{}{t \Downarrow_{d.1} +\mathbf{h} \diamond e}$
$\text{to}_{\times} t \Downarrow_d \langle (), v \rangle_{\{\}} \diamond e$	$\text{from}_{\times} t \Downarrow_d v \diamond e$	$t \triangleleft () \Downarrow_d () \diamond e \gg \mathbf{h} := ()$	$t \triangleleft \text{Inl } \mathbf{d} \Downarrow_d +\mathbf{d}.2 \diamond e \gg \mathbf{h} := \text{Inl } -\mathbf{d}.2$

SEM-TERM-FILLR	SEM-TERM-FILLP	SEM-TERM-FILLC
$\frac{}{t \triangleleft \text{Inr } \mathbf{d} \Downarrow_d +\mathbf{d}.2 \diamond e \gg \mathbf{h} := \text{Inr } -\mathbf{d}.2}$	$\frac{}{t \triangleleft (,) \Downarrow_d (+\mathbf{d}.2, +\mathbf{d}.3) \diamond e \gg \mathbf{h} := (-\mathbf{d}.2, -\mathbf{d}.3)}$	$\frac{t \Downarrow_{d.1} +\mathbf{h} \diamond e_1 \quad u \Downarrow_{d.2} \langle v_1, v_2 \rangle_\Delta \diamond e_2}{t \triangleleft \bullet u \Downarrow_d v_1 \diamond e_1 \gg e_2 \gg \mathbf{h} := v_2}$

## 4 Type safety

**Theorem 1** (Type safety). *If  $\text{ctx\_DestOnly } \Gamma$  and  $\Gamma \vdash t : \mathbf{T}$  then  $t \Downarrow_{\mathbf{d}} v \diamond e$  and  $\Gamma \vdash v \diamond e : \mathbf{T}$ .*

**Theorem 2** (Type safety for complete programs). *If  $\{\} \vdash t : \mathbf{T}$  then  $t \Downarrow_{\mathbf{s}} v \diamond \varepsilon$  and  $\{\} \vdash v : \mathbf{T}$*

**Proof.** By induction on the typing derivation.

- **TYTERM\_VAL:** (0)  $\Gamma \vdash v : \mathbf{T}$   
(0) gives (1)  $v \Downarrow_{\mathbf{d}} v \diamond \varepsilon$  immediately. From **TYEFF\_NOEFF** and **TYCMD\_CMD** we conclude (2)  $\Gamma \vdash v \diamond e : \mathbf{T}$ .

- **TYTERM\_APP:** (0)  $m \cdot \Gamma_1 \uplus \Gamma_2 \vdash t \succ u : \mathbf{T}_2$

We have

- (1)  $\Gamma_1 \vdash t : \mathbf{T}_1$
- (2)  $\Gamma_2 \vdash u : \mathbf{T}_1 \xrightarrow{m} \mathbf{T}_2$
- (3) **ctx\_Disjoint**  $\Gamma_1 \Gamma_2$

Using recursion hypothesis on (1) we get (4)  $t \Downarrow_{\mathbf{d},1} v_1 \diamond e_1$  where (5)  $\Gamma_1 \vdash v_1 \diamond e_1 : \mathbf{T}_1$ .

Inverting **TYCMD\_CMD** we get (5)  $\Gamma_{11} \uplus \Gamma_{13} \vdash v_1 : \mathbf{T}_1$  and (6)  $\Gamma_{12} \uplus -\Gamma_{13} \Vdash e_1$  where (7)  $\Gamma_1 = \Gamma_{11} \uplus \Gamma_{12}$ .

Using recursion hypothesis on (2) we get (8)  $u \Downarrow_{\mathbf{d},2} v_2 \diamond e_2$  where (9)  $\Gamma_2 \vdash v_2 \diamond e_2 : \mathbf{T}_1 \xrightarrow{m} \mathbf{T}_2$ .

Inverting **TYCMD\_CMD** we get (10)  $\Gamma_{21} \uplus \Gamma_{23} \vdash v_2 : \mathbf{T}_1 \xrightarrow{m} \mathbf{T}_2$  and (11)  $\Gamma_{22} \uplus -\Gamma_{23} \Vdash e_2$  where (12)  $\Gamma_2 = \Gamma_{21} \uplus \Gamma_{22}$ .

Using Lemma ?? on (9) we get (13)  $v_2 = \lambda x \mapsto t'$  and (14)  $\Gamma_{21} \uplus \Gamma_{23} \uplus \{x :_m \mathbf{T}_1\} \vdash t' : \mathbf{T}_2$ .

*Typing value part of the result*

Using Lemma ?? on (14) and (5) we get (15)  $m \cdot (\Gamma_{11} \uplus \Gamma_{13}) \uplus (\Gamma_{21} \uplus \Gamma_{23}) \vdash t'[x := v_1] : \mathbf{T}_2$ .

Using recursion hypothesis on (15) we get (16)  $t'[x := v_1] \Downarrow_{\mathbf{d},3} v_3 \diamond e_3$  where (17)  $m \cdot (\Gamma_{11} \uplus \Gamma_{13}) \uplus (\Gamma_{21} \uplus \Gamma_{23}) \vdash v_3 \diamond e_3 : \mathbf{T}_2$ .

*Typing effect part of the result*

We have

- (6)  $\Gamma_{12} \uplus -\Gamma_{13} \Vdash e_1$
- (11)  $\Gamma_{22} \uplus -\Gamma_{23} \Vdash e_2$

**ctx\_Disjoint**  $\Gamma_{12} \Gamma_{22}$  comes naturally from (3), (7) and (12).

We must show:

**ctx\_Disjoint**  $\Gamma_{12} \Gamma_{23}$ : holes in  $e_2$  (associated to  $u$ ) are fresh so they cannot match a destination name from  $t$  as they don't exist yet when  $t$  is evaluated.

**ctx\_Disjoint**  $\Gamma_{22} \Gamma_{13}$ : slightly harder. Holes in  $e_1$  (associated to  $t$ ) are fresh too, so I don't see a way for  $u$  to create a term that could mention them, but sequentially, at least, they exist during  $u$  evaluation. In fact,  $\Gamma_{22}$  might have intersection with  $\Gamma_{13}$  (see **TYEFF\_UNION**) as long as they share the same modalities (it's even harder to prove I think).

**ctx\_Disjoint**  $\Gamma_{13} \Gamma_{23}$ : freshness of holes in both effects, executed sequentially, should be enough.

Let say this is solved by Lemma 1, with no holes of  $e_1$  negative context appearing as dests in  $e_2$  positive context.

By **TYEFF\_UNION** we get (18)  $\Gamma_{12} \uplus \Gamma_{22} \uplus -\Gamma_{13} \uplus -\Gamma_{23} \Vdash e_1 \gg e_2$ .

Inverting **TYCMD\_CMD** on (17) we get (19)  $m \cdot (\Gamma_{111} \uplus \Gamma_{131}) \uplus \Gamma_{211} \uplus \Gamma_{231} \uplus \Gamma_3 \vdash v_3 : \mathbf{T}_2$  and (20)  $m \cdot (\Gamma_{112} \uplus \Gamma_{132}) \uplus \Gamma_{212} \uplus \Gamma_{232} \uplus \Gamma_3 \Vdash e_3$  where (21)  $\Gamma_{k1} \uplus \Gamma_{k2} = \Gamma_k$

We have

- (18)  $\Gamma_{12} \uplus \Gamma_{22} \uplus -\Gamma_{13} \uplus -\Gamma_{23} \Vdash e_1 \gg e_2$
- (20)  $m \cdot (\Gamma_{112} \uplus \Gamma_{132}) \uplus \Gamma_{212} \uplus \Gamma_{232} \uplus -\Gamma_3 \Vdash e_3$

Using (21) on (18) to decompose  $-\Gamma_{23}$ , we get (22)  $\Gamma_{12} \uplus \Gamma_{22} \uplus -(\Gamma_{131} \uplus \Gamma_{231}) \uplus -(\Gamma_{132} \uplus \Gamma_{232}) \Vdash e_1 \gg e_2$

We want  $\Gamma_{132}$  from (22) to cancel  $m \cdot \Gamma_{132}$  from (20), but the multiplicity doesn't match apparently.

$\Gamma_{13}$  contains dests associated to holes that may have been created when evaluating  $t$  into  $v_1 \diamond e_1$ . If  $v_1$  is used with delay (result of multiplying its context by  $m$ ), then should we also delay the RHS of its associated effect? In other terms, if we have  $\{+h :_{1\nu} [\mathbf{T}_1 \oplus \mathbf{T}_2]^n\} \vdash +h' \diamond h := \text{Inl } -h' : [\mathbf{T}_1]^n$ , and use  $h'$  with delay  $m$  (e.g stored inside another dest in the body of the function), should we also type the RHS of  $h := \text{Inl } -h'$  with delay? I think so, if we want to keep the property that age of dests and age of the associated holes are the same. Which means a more refined substitution lemma.

$$(+h_0 \triangleleft (,)) \succ \text{case } (x_1, x_2) \mapsto x_1 \triangleleft \bullet (\text{to}_\times +h_1) ; x_2 \succ (\lambda x_2 \mapsto +h_3 \triangleleft \bullet (\text{to}_\times x_2))$$

$$+h_0 \triangleleft (,) \Downarrow_{\mathbf{d}} (+d.2, +d.3) \diamond h_0 := (-d.2, -d.3)$$

$$(x_1 \triangleleft \bullet (\text{to}_\times +h_1) ; x_2)[x_1 := +d.2][x_2 := +d.3] \Downarrow_{\mathbf{d}'} +d.3 \diamond d.2 := +h_1$$

$$(+h_0 \triangleleft (,)) \succ \text{case } (x_1, x_2) \mapsto x_1 \triangleleft \bullet (\text{to}_\times +h_1) ; x_2 \Downarrow_{\mathbf{d}'} +d.3 \diamond h_0 := (-d.2, -d.3) \gg d.2 := +h_1$$

$$(+h_3 \triangleleft \bullet (\text{to}_\times x_2))[x_2 := +d.3] \Downarrow_{\mathbf{d}'''} () \diamond h_3 := +d.3$$

$$t \Downarrow_{\mathbf{d}''''} () \diamond h_0 := (-d.2, -d.3) \gg d.2 := +h_1 \gg h_3 := +d.3$$

**Lemma 1** (Freshness of holes). *Let  $t$  be a program with no pre-existing ampar sharing hole names.*

*During the reduction of  $t$ , the only other place where the names of the holes on the RHS of an effect can appear is in the accompanying value of the command, as destinations.*

*Proof.* Names of the holes on the RHS of a new effect:

- either are **fresh** (in all  $\text{BIGSTEP\_FILL}\langle Ctor \rangle$  rules), which means the only other place where those names are known and can show up is as destinations on the accompanying value of the command ( $\Gamma_{12}$  in  $\text{TYCMD\_CMD}$ ), but not in positive or negative contexts of the command given by the evaluation of a sibling subterm;
- or are those of pre-existing holes coming from the extended value  $v_2$  of an ampar, when  $\text{BIGSTEP\_FILLCOMP}$  is evaluated. Because they come from an ampar, they must be neutralized by this ampar, so the left value  $v_1$  of the ampar is the only place where those names can show up, as destinations, if we disallow pre-existing ampar with shared hole names in the body of the initial program  $\cdot$ . And  $v_1$  is exactly the accompanying value returned by the evaluation of  $\text{BIGSTEP\_FILLCOMP}$ .

TODO: prove that this property is preserved by typing rules

□