Destination λ-calculus

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April 18, 2024

1 Term and value syntax

var, x, y	Term-level variable name		
k	Index for ranges		

hvar, h	::=		Hole or destination name ($\mathbb{N}$)
	$\mathtt{h}+\mathtt{h}'$	M	
	$\mathtt{h}[\mathtt{H}\dot{+}\mathtt{h}']$	M	Shift by $\mathtt{h}'$ if $\mathtt{h} \in \mathtt{H}$
	$\mathtt{max}(\mathtt{H})$	M	Maximum of a set of holes

hvars, H	::=		Set of hole names
	$\{\mathtt{h}_1, .., \mathtt{h}_k\}$		
	$\mathtt{H}_1 \cup \mathtt{H}_2$	M	Union of sets
	$\mathtt{H}\dot{+}\mathtt{h}'$	M	Shift all names from H by $\mathtt{h}'$.
	$\mathtt{hvars}(\Gamma)$	M	Hole names of a context (requires $\mathtt{ctx\_NoVar}(\Gamma)$)
	$\mathtt{hvars}(\mathsf{C})$	M	Hole names of an evaluation context

term, t, u	::=		Term
	v		Value
	x		Variable
	$\mathsf{t} \succ \mathsf{t}'$		Application
	$\mathsf{t} \,;\, \mathsf{u}$		Pattern-match on unit
	$\mathsf{t} \succ \mathsf{case}_m \{ \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \}$		Pattern-match on sum
	$\mathsf{t} \succ \mathsf{case}_m (\mathsf{x}_1 \,,\, \mathsf{x}_2) \mapsto \mathsf{u}$		Pattern-match on product
	$\mathsf{t} \succ \mathsf{case}_m \mathrm{E}^n \mathsf{x} \mapsto \mathsf{u}$		Pattern-match on exponential
	$\mathsf{t} \succ \mathsf{map}\ \mathsf{x} \mapsto \mathsf{t}'$		Map over the right side of ampar t
	$\mathsf{to}_{\ltimes}\ \mathsf{u}$		Wrap u into a trivial ampar
	$\mathsf{from}_{\ltimes}\ \mathsf{t}$		Extract value from trivial ampar
	$\mathsf{t} \lhd ()$		Fill destination with unit
	$\mathsf{t} \lhd \mathsf{Inl}$		Fill destination with left variant
	$\mathsf{t} \lhd \mathsf{Inr}$		Fill destination with right variant
	$\mathsf{t} \lhd \mathrm{E}^m$		Fill destination with exponential constructor
	$\mathsf{t} \lhd (,)$		Fill destination with product constructor
	$\mathsf{t} \lhd (\lambda \mathsf{x}_m \mapsto \mathsf{u})$		Fill destination with function
	$\mathsf{t} \lhd_\bullet \mathsf{t}'$		Fill destination with root of ampar t′
	$\mathsf{t}[\mathsf{x} := \mathsf{v}]$	M	

val, v	::=		Term value
	$-\mathtt{h}$		Hole
	$+\mathtt{h}$		Destination
	$()$		Unit
	$\lambda^{\mathsf{v}} \mathsf{x}_m \mapsto \mathsf{u}$		Lambda abstraction
	$\mathsf{Inl}\, \mathsf{v}$		Left variant for sum
	$\mathsf{Inr}\, \mathsf{v}$		Right variant for sum
	$\mathrm{E}^m\, \mathsf{v}$		Exponential
	$(\mathsf{v}_1 \,,\, \mathsf{v}_2)$		Product
	$_{\mathtt{H}}\langle \mathsf{v}_2 \,,\, \mathsf{v}_1 \rangle$		Ampar
	$\mathsf{v}[\mathtt{H}\dot{+}\mathtt{h}']$	M	Shift hole names inside v by $\mathtt{h}'$ if they belong to H.

ectx, c	::=			Evaluation context component
	|	$\square \succ t'$		Application
	|	$v \succ \square$		Application
	|	$\square \,;\, u$		Pattern-match on unit
	|	$\square \succ \mathsf{case}_m \{ \mathsf{Inl}\, x_1 \mapsto u_1 \,,\ \mathsf{Inr}\, x_2 \mapsto u_2 \}$		Pattern-match on sum
	|	$\square \succ \mathsf{case}_m (x_1 \,, x_2) \mapsto u$		Pattern-match on product
	|	$\square \succ \mathsf{case}_m \, \mathrm{E}^n \, x \mapsto u$		Pattern-match on exponential
	|	$\square \succ \mathsf{map}\ x \mapsto t'$		Map over the right side of ampar
	|	$\mathsf{to}_{\ltimes}\ \square$		Wrap into a trivial ampar
	|	$\mathsf{from}_{\ltimes}\ \square$		Extract value from trivial ampar
	|	$\square \triangleleft ()$		Fill destination with unit
	|	$\square \triangleleft \mathsf{Inl}$		Fill destination with left variant
	|	$\square \triangleleft \mathsf{Inr}$		Fill destination with right variant
	|	$\square \triangleleft \mathrm{E}^m$		Fill destination with exponential constructor
	|	$\square \triangleleft (,)$		Fill destination with product constructor
	|	$\square \triangleleft (\lambda x_{\,m} \mapsto u)$		Fill destination with function
	|	$\square \triangleleft_\bullet t'$		Fill destination with root of ampar
	|	$v \triangleleft_\bullet \square$		Fill destination with root of ampar
	|	${}^{\mathrm{op}}_{\mathtt{H}}\langle v_2 \, , \square$		Open ampar. Only new addition to term shapes
ectxs, C	::=			Evaluation context stack
	|	$\square$		Represent the empty stack / "identity" evaluation context
	|	$C \circ c$		Push c on top of C
	|	$C[\mathtt{h} :=_{\mathtt{H}} v]$	M	Fill h in C with value v (that may contain holes)

2 Type system

type, **T**, **U**	::=			Type
	|	$\mathbf{1}$		Unit
	|	$\mathbf{T}_1 \oplus \mathbf{T}_2$		Sum
	|	$\mathbf{T}_1 \otimes \mathbf{T}_2$		Product
	|	$!^m \mathbf{T}$		Exponential
	|	$\mathbf{U} \ltimes \mathbf{T}$		Ampar type (consuming **T** yields **U**)
	|	$\mathbf{T} \,{}_m\!\!\rightarrow \mathbf{U}$		Function
	|	$\lfloor \mathbf{T} \rfloor^m$		Destination
mode, *m*, *n*	::=			Mode (Semiring)
	|	pa		Pair of a multiplicity and age
	|	$☠$		Error case (incompatible types, multiplicities, or ages)
	|	$m_1 \cdot \ldots \cdot m_k$	M	Semiring product
mul, *p*	::=			Multiplicity (first component of modality)
	|	1		Linear. Neutral element of the product
	|	ω		Non-linear. Absorbing for the product
	|	$p_1 \cdot \ldots \cdot p_k$	M	Semiring product
age, *a*	::=			Age (second component of modality)
	|	ν		Born now. Neutral element of the product
	|	$\uparrow$		One scope older
	|	∞		Infinitely old / static. Absorbing for the product
	|	$a_1 \cdot \ldots \cdot a_k$	M	Semiring product
ctx, Γ, Δ, Π	::=			Typing context
	|	$\{x :{}_m \mathbf{T}\}$		
	|	$\{+\mathtt{h} :{}_m \lfloor \mathbf{T} \rfloor^n\}$		
	|	$\{-\mathtt{h} : \mathbf{T}^n\}$		
	|	$m \cdot \Gamma$	M	Multiply each binding by m
	|	$\Gamma_1 + \Gamma_2$	M	Sum contexts Γ_1 and Γ_2. Duplicate keys with incompatible values will be tagged
	|	Γ_1, Γ_2	M	Disjoint sum/union of contexts Γ_1 and Γ_2.
	|	$-\Gamma$	M	Transforms dest bindings into a hole bindings (requires `ctx_DestOnly` Γ and `ctx`

	$-^{-1}\Gamma$	M	Transforms hole bindings into dest bindings with left mode 1_ν (requires `ctx_HoleOnly` Γ)
	$\Gamma[\mathtt{H}\pm\mathtt{h}']$	M	Shift hole/dest names by $\mathtt{h}'$ if they belong to $\mathtt{H}$

$\boxed{\Gamma \Vdash \mathsf{v} : \mathsf{T}}$ *(Typing of values (raw))*

TyR-val-H
$$\frac{}{\{-\mathtt{h} : \mathsf{T}^{1_\nu}\} \Vdash -\mathtt{h} : \mathsf{T}}$$

TyR-val-D
$$\frac{}{\{+\mathtt{h} : {}_{1_\nu}\lfloor \mathsf{T} \rfloor^n\} \Vdash +\mathtt{h} : \lfloor \mathsf{T} \rfloor^n}$$

TyR-val-U
$$\frac{}{\{\} \Vdash () : \mathbf{1}}$$

TyR-val-F
$$\frac{\Delta + \{\mathsf{x} : {}_m\mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}}{\Delta \Vdash \lambda^{\mathsf{v}} \mathsf{x}\, {}_m \mapsto \mathsf{u} : \mathsf{T} \,{}_m\!\!\to \mathsf{U}}$$

TyR-val-L
$$\frac{\Gamma \Vdash \mathsf{v}_1 : \mathsf{T}_1}{\Gamma \Vdash \mathsf{Inl}\, \mathsf{v}_1 : \mathsf{T}_1 \oplus \mathsf{T}_2}$$

TyR-val-R
$$\frac{\Gamma \Vdash \mathsf{v}_2 : \mathsf{T}_2}{\Gamma \Vdash \mathsf{Inr}\, \mathsf{v}_2 : \mathsf{T}_1 \oplus \mathsf{T}_2}$$

TyR-val-P
$$\frac{\Gamma_1 \Vdash \mathsf{v}_1 : \mathsf{T}_1 \qquad \Gamma_2 \Vdash \mathsf{v}_2 : \mathsf{T}_2}{\Gamma_1 + \Gamma_2 \Vdash (\mathsf{v}_1 , \mathsf{v}_2) : \mathsf{T}_1 \otimes \mathsf{T}_2}$$

TyR-val-E
$$\frac{\Gamma \Vdash \mathsf{v}' : \mathsf{T}}{n{\cdot}\Gamma \Vdash \mathrm{E}^n \mathsf{v}' : !^n \mathsf{T}}$$

TyR-val-A
$$\frac{\mathtt{LinOnly}\ \Delta_3 \qquad \mathtt{FinAgeOnly}\ \Delta_3 \qquad 1\!\uparrow{\cdot}\Delta_1, \Delta_3 \Vdash \mathsf{v}_1 : \mathsf{T} \qquad \Delta_2, (-\Delta_3) \Vdash \mathsf{v}_2 : \mathsf{U}}{\Delta_1, \Delta_2 \Vdash {}_{\mathtt{hvars}(-\Delta_3)}\langle \mathsf{v}_2 , \mathsf{v}_1 \rangle : \mathsf{U} \ltimes \mathsf{T}}$$

$\boxed{\Pi \vdash \mathsf{t} : \mathsf{T}}$ *(Typing of terms)*

Ty-term-Val
$$\frac{\mathtt{DisposableOnly}\ \Pi \qquad \Delta \Vdash \mathsf{v} : \mathsf{T}}{\Pi, \Delta \vdash \mathsf{v} : \mathsf{T}}$$

Ty-term-Var
$$\frac{\mathtt{DisposableOnly}\ \Pi \qquad m \mathrel{<:} 1_\nu}{\Pi, \{\mathsf{x} : {}_m\mathsf{T}\} \vdash \mathsf{x} : \mathsf{T}}$$

Ty-term-App
$$\frac{\Pi_1 \vdash \mathsf{t} : \mathsf{T} \qquad \Pi_2 \vdash \mathsf{t}' : \mathsf{T} \,{}_m\!\!\to \mathsf{U}}{m{\cdot}\Pi_1 + \Pi_2 \vdash \mathsf{t} \succ \mathsf{t}' : \mathsf{U}}$$

Ty-term-PatU
$$\frac{\Pi_1 \vdash \mathsf{t} : \mathbf{1} \qquad \Pi_2 \vdash \mathsf{u} : \mathsf{U}}{\Pi_1 + \Pi_2 \vdash \mathsf{t} \,;\, \mathsf{u} : \mathsf{U}}$$

Ty-term-PatS
$$\frac{\begin{array}{c}\Pi_1 \vdash \mathsf{t} : \mathsf{T}_1 \oplus \mathsf{T}_2 \\ \Pi_2, \{\mathsf{x}_1 : {}_m\mathsf{T}_1\} \vdash \mathsf{u}_1 : \mathsf{U} \\ \Pi_2, \{\mathsf{x}_2 : {}_m\mathsf{T}_2\} \vdash \mathsf{u}_2 : \mathsf{U}\end{array}}{m{\cdot}\Pi_1 + \Pi_2 \vdash \mathsf{t} \succ \mathsf{case}_m \{\, \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \,\} : \mathsf{U}}$$

Ty-term-PatP
$$\frac{\begin{array}{c}\Pi_1 \vdash \mathsf{t} : \mathsf{T}_1 \otimes \mathsf{T}_2 \\ \Pi_2, \{\mathsf{x}_1 : {}_m\mathsf{T}_1\}, \{\mathsf{x}_2 : {}_m\mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{m{\cdot}\Pi_1 + \Pi_2 \vdash \mathsf{t} \succ \mathsf{case}_m\, (\mathsf{x}_1 , \mathsf{x}_2) \mapsto \mathsf{u} : \mathsf{U}}$$

Ty-term-PatE
$$\frac{\Pi_1 \vdash \mathsf{t} : !^n \mathsf{T} \qquad \Pi_2, \{\mathsf{x} : {}_{m\cdot n}\mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}}{m{\cdot}\Pi_1 + \Pi_2 \vdash \mathsf{t} \succ \mathsf{case}_m\, \mathrm{E}^n\, \mathsf{x} \mapsto \mathsf{u} : \mathsf{U}}$$

Ty-term-Map
$$\frac{\Pi_1 \vdash \mathsf{t} : \mathsf{U} \ltimes \mathsf{T} \qquad 1\!\uparrow{\cdot}\Pi_2, \{\mathsf{x} : {}_{1_\nu}\mathsf{T}\} \vdash \mathsf{t}' : \mathsf{T}'}{\Pi_1 + \Pi_2 \vdash \mathsf{t} \succ \mathsf{map}\ \mathsf{x} \mapsto \mathsf{t}' : \mathsf{U} \ltimes \mathsf{T}'}$$

Ty-term-ToA
$$\frac{\Pi \vdash \mathsf{u} : \mathsf{U}}{\Pi \vdash \mathsf{to}_\ltimes\, \mathsf{u} : \mathsf{U} \ltimes \mathbf{1}}$$

Ty-term-FromA
$$\frac{\Pi \vdash \mathsf{t} : \mathsf{U} \ltimes \mathbf{1}}{\Pi \vdash \mathsf{from}_\ltimes\, \mathsf{t} : \mathsf{U}}$$

Ty-term-FillU
$$\frac{\Pi \vdash \mathsf{t} : \lfloor \mathbf{1} \rfloor^n}{\Pi \vdash \mathsf{t} \triangleleft () : \mathbf{1}}$$

Ty-term-FillL
$$\frac{\Pi \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^n}{\Pi \vdash \mathsf{t} \triangleleft \mathsf{Inl} : \lfloor \mathsf{T}_1 \rfloor^n}$$

Ty-term-FillR
$$\frac{\Pi \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^n}{\Pi \vdash \mathsf{t} \triangleleft \mathsf{Inr} : \lfloor \mathsf{T}_2 \rfloor^n}$$

Ty-term-FillP
$$\frac{\Pi \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \otimes \mathsf{T}_2 \rfloor^n}{\Pi \vdash \mathsf{t} \triangleleft (,) : \lfloor \mathsf{T}_1 \rfloor^n \otimes \lfloor \mathsf{T}_2 \rfloor^n}$$

Ty-term-FillE
$$\frac{\Pi \vdash \mathsf{t} : \lfloor !^{n'}\, \mathsf{T} \rfloor^n}{\Pi \vdash \mathsf{t} \triangleleft \mathrm{E}^{n'} : \lfloor \mathsf{T} \rfloor^{n' \cdot n}}$$

Ty-term-FillF
$$\frac{\Pi_1 \vdash \mathsf{t} : \lfloor \mathsf{T} \,{}_m\!\!\to \mathsf{U} \rfloor^n \qquad \Pi_2, \{\mathsf{x} : {}_m\mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}}{\Pi_1 + (1\!\uparrow{\cdot}n){\cdot}\Pi_2 \vdash \mathsf{t} \triangleleft (\lambda \mathsf{x}\, {}_m \mapsto \mathsf{u}) : \mathbf{1}}$$

Ty-term-FillC
$$\frac{\Pi_1 \vdash \mathsf{t} : \lfloor \mathsf{U} \rfloor^n \qquad \Pi_2 \vdash \mathsf{t}' : \mathsf{U} \ltimes \mathsf{T}}{\Pi_1 + (1\!\uparrow{\cdot}n){\cdot}\Pi_2 \vdash \mathsf{t} \triangleleft\!\bullet \mathsf{t}' : \mathsf{T}}$$

$\boxed{\Delta \dashv \mathsf{C} : \mathsf{T} \rightarrowtail \mathsf{U}_0}$ *(Typing of evaluation contexts)*

Ty-ectxs-Id
$$\frac{}{\{\} \dashv \Box : \mathsf{U}_0 \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-AppFoc1
$$\frac{m{\cdot}\Delta_1, \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \qquad \Delta_2 \vdash \mathsf{t}' : \mathsf{T} \,{}_m\!\!\to \mathsf{U}}{\Delta_1 \dashv \mathsf{C} \circ (\Box \succ \mathsf{t}') : \mathsf{T} \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-AppFoc2
$$\frac{m{\cdot}\Delta_1, \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \qquad \Delta_1 \vdash \mathsf{v} : \mathsf{T}}{\Delta_2 \dashv \mathsf{C} \circ (\mathsf{v} \succ \Box) : (\mathsf{T} \,{}_m\!\!\to \mathsf{U}) \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-PatUFoc
$$\frac{\Delta_1, \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \qquad \Delta_2 \vdash \mathsf{u} : \mathsf{U}}{\Delta_1 \dashv \mathsf{C} \circ (\Box \,;\, \mathsf{u}) : \mathbf{1} \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-PatSFoc
$$\frac{\begin{array}{c}m{\cdot}\Delta_1, \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \\ \Delta_2, \{\mathsf{x}_1 : {}_m\mathsf{T}_1\} \vdash \mathsf{u}_1 : \mathsf{U} \\ \Delta_2, \{\mathsf{x}_2 : {}_m\mathsf{T}_2\} \vdash \mathsf{u}_2 : \mathsf{U}\end{array}}{\Delta_1 \dashv \mathsf{C} \circ (\Box \succ \mathsf{case}_m \{\, \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \,\}) : (\mathsf{T}_1 \oplus \mathsf{T}_2) \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-PatPFoc
$$\frac{\begin{array}{c}m{\cdot}\Delta_1, \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \\ \Delta_2, \{\mathsf{x}_1 : {}_m\mathsf{T}_1\}, \{\mathsf{x}_2 : {}_m\mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Delta_1 \dashv \mathsf{C} \circ (\Box \succ \mathsf{case}_m\, (\mathsf{x}_1 , \mathsf{x}_2) \mapsto \mathsf{u}) : (\mathsf{T}_1 \otimes \mathsf{T}_2) \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-PatEFoc
$$\frac{m{\cdot}\Delta_1, \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \qquad \Delta_2, \{\mathsf{x} : {}_{m\cdot m'}\mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}}{\Delta_1 \dashv \mathsf{C} \circ (\Box \succ \mathsf{case}_m\, \mathrm{E}^{m'}\, \mathsf{x} \mapsto \mathsf{u}) : !^{m'}\, \mathsf{T} \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-MapFoc
$$\frac{\Delta_1, \Delta_2 \dashv \mathsf{C} : \mathsf{U} \ltimes \mathsf{T}' \rightarrowtail \mathsf{U}_0 \quad \mathit{1}\uparrow\cdot\Delta_2, \{\mathsf{x} : {}_{\mathit{1}_\nu}\mathsf{T}\} \vdash \mathsf{t}' : \mathsf{T}'}{\Delta_1 \dashv \mathsf{C} \circ (\Box \succ \mathbf{map}\ \mathsf{x} \mapsto \mathsf{t}') : (\mathsf{U} \ltimes \mathsf{T}) \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-ToAFoc
$$\frac{\Delta \dashv \mathsf{C} : (\mathsf{U} \ltimes \mathbf{1}) \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\mathbf{to}_\ltimes\ \Box) : \mathsf{U} \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FromAFoc
$$\frac{\Delta \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\mathbf{from}_\ltimes\ \Box) : (\mathsf{U} \ltimes \mathbf{1}) \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillUFoc
$$\frac{\Delta \dashv \mathsf{C} : \mathbf{1} \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\Box \triangleleft ()) : \lfloor \mathbf{1} \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillLFoc
$$\frac{\Delta \dashv \mathsf{C} : \lfloor \mathsf{T}_1 \rfloor^n \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\Box \triangleleft \mathsf{Inl}) : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillRFoc
$$\frac{\Delta \dashv \mathsf{C} : \lfloor \mathsf{T}_2 \rfloor^n \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\Box \triangleleft \mathsf{Inr}) : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillPFoc
$$\frac{\Delta \dashv \mathsf{C} : (\lfloor \mathsf{T}_1 \rfloor^n \otimes \lfloor \mathsf{T}_2 \rfloor^n) \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\Box \triangleleft (,)) : \lfloor \mathsf{T}_1 \otimes \mathsf{T}_2 \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillEFoc
$$\frac{\Delta \dashv \mathsf{C} : \lfloor \mathsf{T} \rfloor^{m \cdot n} \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\Box \triangleleft \mathrm{E}^m) : \lfloor !^m\, \mathsf{T} \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillFFoc
$$\frac{\Delta_1, (\mathit{1}\uparrow\cdot n)\cdot\Delta_2 \dashv \mathsf{C} : \mathbf{1} \rightarrowtail \mathsf{U}_0 \quad \Delta_2, \{\mathsf{x} : {}_m\mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}}{\Delta_1 \dashv \mathsf{C} \circ (\Box \triangleleft (\lambda \mathsf{x}_m \mapsto \mathsf{u})) : \lfloor \mathsf{T} \xrightarrow{m} \mathsf{U} \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillCFoc1
$$\frac{\Delta_1, (\mathit{1}\uparrow\cdot n)\cdot\Delta_2 \dashv \mathsf{C} : \mathsf{T} \rightarrowtail \mathsf{U}_0 \quad \Delta_2 \vdash \mathsf{t}' : \mathsf{U} \ltimes \mathsf{T}}{\Delta_1 \dashv \mathsf{C} \circ (\Box \triangleleft_\bullet \mathsf{t}') : \lfloor \mathsf{U} \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillCFoc2
$$\frac{\Delta_1, (\mathit{1}\uparrow\cdot n)\cdot\Delta_2 \dashv \mathsf{C} : \mathsf{T} \rightarrowtail \mathsf{U}_0 \quad \Delta_1 \vdash \mathsf{v} : \lfloor \mathsf{U} \rfloor^n}{\Delta_2 \dashv \mathsf{C} \circ (\mathsf{v} \triangleleft_\bullet \Box) : \mathsf{U} \ltimes \mathsf{T} \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-AOpenFoc
$$\frac{\begin{array}{c}\mathtt{hvars}(\mathsf{C})\ \#\#\ \mathtt{hvars}(-\Delta_3) \\ \mathtt{LinOnly}\ \Delta_3 \qquad \mathtt{FinAgeOnly}\ \Delta_3 \\ \Delta_1, \Delta_2 \dashv \mathsf{C} : (\mathsf{U} \ltimes \mathsf{T}') \rightarrowtail \mathsf{U}_0 \\ \Delta_2, -\Delta_3 \Vdash \mathsf{v}_2 : \mathsf{U}\end{array}}{\mathit{1}\uparrow\cdot\Delta_1, \Delta_3 \dashv \mathsf{C} \circ ({}^{\mathrm{op}}_{\mathtt{hvars}(-\Delta_3)}\langle \mathsf{v}_2 \,, \Box) : \mathsf{T}' \rightarrowtail \mathsf{U}_0}$$

$\boxed{\vdash \mathsf{C}[\mathsf{t}] : \mathsf{T}}$ *(Typing of extended terms (pair of evaluation context and term))*

Ty-eterm-ClosedEterm
$$\frac{\Delta \dashv \mathsf{C} : \mathsf{T} \rightarrowtail \mathsf{U}_0 \qquad \Delta \vdash \mathsf{t} : \mathsf{T}}{\vdash \mathsf{C}[\mathsf{t}] : \mathsf{U}_0}$$

3 Small-step semantics

$\boxed{C[t] \longrightarrow C'[t']}$ *(Small-step evaluation of terms using evaluation contexts)*

Sem-eterm-AppFoc1
$$\frac{\texttt{NotVal}\ t}{C[t \succ t'] \longrightarrow (C \circ (\square \succ t'))[t]}$$

Sem-eterm-AppUnfoc1
$$\frac{}{(C \circ (\square \succ t'))[v] \longrightarrow C[v \succ t']}$$

Sem-eterm-AppFoc2
$$\frac{\texttt{NotVal}\ t'}{C[v \succ t'] \longrightarrow (C \circ (v \succ \square))[t']}$$

Sem-eterm-AppUnfoc2
$$\frac{}{(C \circ (v \succ \square))[v'] \longrightarrow C[v \succ v']}$$

Sem-eterm-AppRed
$$\frac{}{C[v \succ (\lambda^{v} x_{\,m} \mapsto u)] \longrightarrow C[u[x := v]]}$$

Sem-eterm-PatUFoc
$$\frac{\texttt{NotVal}\ t}{C[t\ ;\ u] \longrightarrow (C \circ (\square\ ;\ u))[t]}$$

Sem-eterm-PatUUnfoc
$$\frac{}{(C \circ (\square\ ;\ u))[v] \longrightarrow C[v\ ;\ u]}$$

Sem-eterm-PatURed
$$\frac{}{C[()\ ;\ u] \longrightarrow C[u]}$$

Sem-eterm-PatSFoc
$$\frac{\texttt{NotVal}\ t}{C[t \succ \mathsf{case}_m\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}] \longrightarrow (C \circ (\square \succ \mathsf{case}_m\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}))[t]}$$

Sem-eterm-PatSUnfoc
$$\frac{}{(C \circ (\square \succ \mathsf{case}_m\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}))[v] \longrightarrow C[v \succ \mathsf{case}_m\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}]}$$

Sem-eterm-PatLRed
$$\frac{}{C[(\mathsf{Inl}\,v_1) \succ \mathsf{case}_m\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}] \longrightarrow C[u_1[x_1 := v_1]]}$$

Sem-eterm-PatRRed
$$\frac{}{C[(\mathsf{Inr}\,v_2) \succ \mathsf{case}_m\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}] \longrightarrow C[u_2[x_2 := v_2]]}$$

Sem-eterm-PatPFoc
$$\frac{\texttt{NotVal}\ t}{C[t \succ \mathsf{case}_m\,(x_1\,,\,x_2) \mapsto u] \longrightarrow (C \circ (\square \succ \mathsf{case}_m\,(x_1\,,\,x_2) \mapsto u))[t]}$$

Sem-eterm-PatPUnfoc
$$\frac{}{(C \circ (\square \succ \mathsf{case}_m\,(x_1\,,\,x_2) \mapsto u))[v] \longrightarrow C[v \succ \mathsf{case}_m\,(x_1\,,\,x_2) \mapsto u]}$$

Sem-eterm-PatPRed
$$\frac{}{C[(v_1\,,\,v_2) \succ \mathsf{case}_m\,(x_1\,,\,x_2) \mapsto u] \longrightarrow C[u[x_1 := v_1][x_2 := v_2]]}$$

Sem-eterm-PatEFoc
$$\frac{\texttt{NotVal}\ t}{C[t \succ \mathsf{case}_m\,\mathrm{E}^{n}\,x \mapsto u] \longrightarrow (C \circ (\square \succ \mathsf{case}_m\,\mathrm{E}^{n}\,x \mapsto u))[t]}$$

Sem-eterm-PatEUnfoc
$$\frac{}{(C \circ (\square \succ \mathsf{case}_m\,\mathrm{E}^{n}\,x \mapsto u))[v] \longrightarrow C[v \succ \mathsf{case}_m\,\mathrm{E}^{n}\,x \mapsto u]}$$

Sem-eterm-PatERed
$$\frac{}{C[\mathrm{E}^{n}\,v' \succ \mathsf{case}_m\,\mathrm{E}^{n}\,x \mapsto u] \longrightarrow C[u[x := v']]}$$

Sem-eterm-MapFoc
$$\frac{\texttt{NotVal}\ t}{C[t \succ \mathsf{map}\ x \mapsto t'] \longrightarrow (C \circ (\square \succ \mathsf{map}\ x \mapsto t'))[t]}$$

Sem-eterm-MapUnfoc
$$\frac{}{(C \circ (\square \succ \mathsf{map}\ x \mapsto t'))[v] \longrightarrow C[v \succ \mathsf{map}\ x \mapsto t']}$$

Sem-eterm-MapRedAOpenFoc
$$\frac{h' = \mathtt{max(hvars(C))+1}}{C[{}_{\mathrm{H}}\langle v_2\,,\,v_1\rangle \succ \mathsf{map}\ x \mapsto t'] \longrightarrow (C \circ ({}^{\mathrm{op}}_{\mathrm{H}\pm h'}\langle v_2[\mathrm{H}\pm h']\,,\,\square\rangle))[t'[x := v_1[\mathrm{H}\pm h']]]}$$

Sem-eterm-AOpenUnfoc
$$\frac{}{(C \circ {}^{\mathrm{op}}_{\mathrm{H}}\langle v_2\,,\,\square\rangle)[v_1] \longrightarrow C[{}_{\mathrm{H}}\langle v_2\,,\,v_1\rangle]}$$

Sem-eterm-ToAFoc
$$\frac{\texttt{NotVal}\ u}{C[\mathsf{to}_{\ltimes}\ u] \longrightarrow (C \circ (\mathsf{to}_{\ltimes}\ \square))[u]}$$

Sem-eterm-ToAUnfoc
$$\frac{}{(C \circ (\mathsf{to}_{\ltimes}\ \square))[v_2] \longrightarrow C[\mathsf{to}_{\ltimes}\ v_2]}$$

Sem-eterm-ToARed
$$\frac{}{C[\mathsf{to}_{\ltimes}\ v_2] \longrightarrow C[{}_{\{\,\}}\langle v_2\,,\,()\rangle]}$$

Sem-eterm-FromAFoc
$$\frac{\texttt{NotVal}\ t}{C[\mathsf{from}_{\ltimes}\ t] \longrightarrow (C \circ (\mathsf{from}_{\ltimes}\ \square))[t]}$$

Sem-eterm-FromAUnfoc
$$\frac{}{(C \circ (\mathsf{from}_{\ltimes}\ \square))[v] \longrightarrow C[\mathsf{from}_{\ltimes}\ v]}$$

Sem-eterm-FromARed
$$\frac{}{C[\mathsf{from}_{\ltimes}\ {}_{\{\,\}}\langle v_2\,,\,()\rangle] \longrightarrow C[v_2]}$$

Sem-eterm-FillUFoc
$$\frac{\texttt{NotVal}\ t}{C[t \triangleleft ()] \longrightarrow (C \circ (\square \triangleleft ()))[t]}$$

Sem-eterm-FillUUnfoc
$$\frac{}{(C \circ (\square \triangleleft ()))[v] \longrightarrow C[v \triangleleft ()]}$$

Sem-eterm-FillURed
$$\frac{}{C[+h \triangleleft ()] \longrightarrow C[h :=_{\{\,\}}\ ()][()]}$$

Sem-eterm-FillLFoc
$$\frac{\texttt{NotVal t}}{\mathsf{C}[\mathsf{t} \lhd \mathsf{Inl}] \longrightarrow (\mathsf{C} \circ (\Box \lhd \mathsf{Inl}))[\mathsf{t}]}$$

Sem-eterm-FillLUnfoc
$$\frac{}{(\mathsf{C} \circ (\Box \lhd \mathsf{Inl}))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \lhd \mathsf{Inl}]}$$

Sem-eterm-FillLRed
$$\frac{\mathsf{h'} = \mathsf{max}(\mathsf{hvars}(\mathsf{C}) \cup \{\mathsf{h}\}) + 1}{\mathsf{C}[+\mathsf{h} \lhd \mathsf{Inl}] \longrightarrow \mathsf{C}[\mathsf{h} :=_{\{\mathsf{h'}+1\}} \mathsf{Inl}\, -(\mathsf{h'}+1)][+(\mathsf{h'}+1)]}$$

Sem-eterm-FillRFoc
$$\frac{\texttt{NotVal t}}{\mathsf{C}[\mathsf{t} \lhd \mathsf{Inr}] \longrightarrow (\mathsf{C} \circ (\Box \lhd \mathsf{Inr}))[\mathsf{t}]}$$

Sem-eterm-FillRUnfoc
$$\frac{}{(\mathsf{C} \circ (\Box \lhd \mathsf{Inr}))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \lhd \mathsf{Inr}]}$$

Sem-eterm-FillRRed
$$\frac{\mathsf{h'} = \mathsf{max}(\mathsf{hvars}(\mathsf{C}) \cup \{\mathsf{h}\}) + 1}{\mathsf{C}[+\mathsf{h} \lhd \mathsf{Inr}] \longrightarrow \mathsf{C}[\mathsf{h} :=_{\{\mathsf{h'}+1\}} \mathsf{Inr}\, -(\mathsf{h'}+1)][+(\mathsf{h'}+1)]}$$

Sem-eterm-FillEFoc
$$\frac{\texttt{NotVal t}}{\mathsf{C}[\mathsf{t} \lhd \mathrm{E}^{m}] \longrightarrow (\mathsf{C} \circ (\Box \lhd \mathrm{E}^{m}))[\mathsf{t}]}$$

Sem-eterm-FillEUnfoc
$$\frac{}{(\mathsf{C} \circ (\Box \lhd \mathrm{E}^{m}))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \lhd \mathrm{E}^{m}]}$$

Sem-eterm-FillERed
$$\frac{\mathsf{h'} = \mathsf{max}(\mathsf{hvars}(\mathsf{C}) \cup \{\mathsf{h}\}) + 1}{\mathsf{C}[+\mathsf{h} \lhd \mathrm{E}^{m}] \longrightarrow \mathsf{C}[\mathsf{h} :=_{\{\mathsf{h'}+1\}} \mathrm{E}^{m}\, -(\mathsf{h'}+1)][+(\mathsf{h'}+1)]}$$

Sem-eterm-FillPFoc
$$\frac{\texttt{NotVal t}}{\mathsf{C}[\mathsf{t} \lhd (,)] \longrightarrow (\mathsf{C} \circ (\Box \lhd (,)))[\mathsf{t}]}$$

Sem-eterm-FillPUnfoc
$$\frac{}{(\mathsf{C} \circ (\Box \lhd (,)))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \lhd (,)]}$$

Sem-eterm-FillPRed
$$\frac{\mathsf{h'} = \mathsf{max}(\mathsf{hvars}(\mathsf{C}) \cup \{\mathsf{h}\}) + 1}{\mathsf{C}[+\mathsf{h} \lhd (,)] \longrightarrow \mathsf{C}[\mathsf{h} :=_{\{\mathsf{h'}+1, \mathsf{h'}+2\}} (-(\mathsf{h'}+1), -(\mathsf{h'}+2))][(+(\mathsf{h'}+1), +(\mathsf{h'}+2))]}$$

Sem-eterm-FillFFoc
$$\frac{\texttt{NotVal t}}{\mathsf{C}[\mathsf{t} \lhd (\lambda \mathsf{x}_{\,m} \mapsto \mathsf{u})] \longrightarrow (\mathsf{C} \circ (\Box \lhd (\lambda \mathsf{x}_{\,m} \mapsto \mathsf{u})))[\mathsf{t}]}$$

Sem-eterm-FillFUnfoc
$$\frac{}{(\mathsf{C} \circ (\Box \lhd (\lambda \mathsf{x}_{\,m} \mapsto \mathsf{u})))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \lhd (\lambda \mathsf{x}_{\,m} \mapsto \mathsf{u})]}$$

Sem-eterm-FillFRed
$$\frac{}{\mathsf{C}[+\mathsf{h} \lhd (\lambda \mathsf{x}_{\,m} \mapsto \mathsf{u})] \longrightarrow \mathsf{C}[\mathsf{h} :=_{\{\}} \lambda^{\mathsf{v}} \mathsf{x}_{\,m} \mapsto \mathsf{u}][()]}$$

Sem-eterm-FillCFoc1
$$\frac{\texttt{NotVal t}}{\mathsf{C}[\mathsf{t} \lhd_{\bullet} \mathsf{t'}] \longrightarrow (\mathsf{C} \circ (\Box \lhd_{\bullet} \mathsf{t'}))[\mathsf{t}]}$$

Sem-eterm-FillCUnfoc1
$$\frac{}{(\mathsf{C} \circ (\Box \lhd_{\bullet} \mathsf{t'}))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \lhd_{\bullet} \mathsf{t'}]}$$

Sem-eterm-FillCFoc2
$$\frac{\texttt{NotVal t'}}{\mathsf{C}[\mathsf{v} \lhd_{\bullet} \mathsf{t'}] \longrightarrow (\mathsf{C} \circ (\mathsf{v} \lhd_{\bullet} \Box))[\mathsf{t'}]}$$

Sem-eterm-FillCUnfoc2
$$\frac{}{(\mathsf{C} \circ (\mathsf{v} \lhd_{\bullet} \Box))[\mathsf{v'}] \longrightarrow \mathsf{C}[\mathsf{v} \lhd_{\bullet} \mathsf{v'}]}$$

Sem-eterm-FillCRed
$$\frac{\mathsf{h'} = \mathsf{max}(\mathsf{hvars}(\mathsf{C}) \cup \{\mathsf{h}\}) + 1}{\mathsf{C}[+\mathsf{h} \lhd_{\bullet \mathrm{H}} \langle \mathsf{v}_2 \,,\, \mathsf{v}_1 \rangle] \longrightarrow \mathsf{C}[\mathsf{h} :=_{(\mathrm{H} \pm \mathsf{h'})} \mathsf{v}_2[\mathrm{H} \pm \mathsf{h'}]][\mathsf{v}_1[\mathrm{H} \pm \mathsf{h'}]]}$$

4 Remarks on the Coq proofs

- Not particularly elegant. Max number of goals observed 232 (solved by a single call to the `congruence` tactic). When you have a computer, brute force is a viable strategy. (in particular, no semiring formalisation, it was quicker to do directly)
- Rules generated by ott, same as in the article (up to some notational difference). Contexts are not generated purely by syntax, and are interpreted in a semantic domain (finite functions).
- Reasoning on closed terms avoids almost all complications on binder manipulation. Makes proofs tractable.
- Finite functions: making a custom library was less headache than using existing libraries (including `MMap`). Existing libraries don't provide some of the tools that we needed, but the most important factor ended up being the need for a modicum of dependency between key and value. There wasn't really that out there. Backed by actual functions for simplicity; cost: equality is complicated.
- Most of the proofs done by author with very little prior experience to Coq.
- Did proofs in Coq because context manipulations are tricky.
- Context sum made total by adding an extra invalid *mode* (rather than an extra context). It seems to be much simpler this way.
- It might be a good idea to provide statistics on the number of lemmas and size of Coq codebase.
- (possibly) renaming as permutation, inspired by nominal sets, make more lemmas don't require a condition (but some lemmas that wouldn't in a straight renaming do in exchange).
- (possibly) methodology: assume a lot of lemmas, prove main theorem, prove assumptions, some wrong, fix. A number of wrong lemma initially assumed, but replacing them by correct variant was always easy to fix in proofs.
- Axioms that we use and why (in particular setoid equality not very natural with ott-generated typing rules).
- Talk about the use and benefits of Copilot.