Destination λ-calculus

Thomas Bagrel

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1 Term and value syntax

var, x, y	Term-level variable name		
k	Index for ranges		

hvar, h	::=		Hole or destination name ($\mathbb{N}$)
	$\mathtt{h}+\mathtt{h}'$	M	
	$\mathtt{h}[\mathtt{H}\pm\mathtt{h}']$	M	Shift by $\mathtt{h}'$ if $\mathtt{h} \in \mathtt{H}$
	$\mathtt{max}(\mathtt{H})$	M	Maximum of a set of holes

hvars, H	::=		Set of hole names
	$\{\mathtt{h}_1, .., \mathtt{h}_k\}$		
	$\mathtt{H}_1 \cup \mathtt{H}_2$	M	Union of sets
	$\mathtt{H}\pm\mathtt{h}'$	M	Shift all names from H by $\mathtt{h}'$.
	$\mathtt{hvars}(\Gamma)$	M	Hole names of a context (requires $\mathtt{ctx\_NoVar}(\Gamma)$)
	$\mathtt{hvars}(\mathsf{C})$	M	Hole names of an evaluation context

term, t, u	::=		Term
	v		Value
	x		Variable
	$\mathsf{t} \succ \mathsf{t}'$		Application
	$\mathsf{t} \,;\, \mathsf{u}$		Pattern-match on unit
	$\mathsf{t} \succ \mathsf{case}_m \{ \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \}$		Pattern-match on sum
	$\mathsf{t} \succ \mathsf{case}_m (\mathsf{x}_1 \,,\, \mathsf{x}_2) \mapsto \mathsf{u}$		Pattern-match on product
	$\mathsf{t} \succ \mathsf{case}_m \mathrm{E}^n\, \mathsf{x} \mapsto \mathsf{u}$		Pattern-match on exponential
	$\mathsf{t} \succ \mathsf{map}\ \mathsf{x} \mapsto \mathsf{t}'$		Map over the right side of ampar t
	$\mathsf{to}_{\ltimes}\ \mathsf{u}$		Wrap u into a trivial ampar
	$\mathsf{from}_{\ltimes}\ \mathsf{t}$		Extract value from trivial ampar
	$\mathsf{t} \lhd ()$		Fill destination with unit
	$\mathsf{t} \lhd \mathsf{Inl}$		Fill destination with left variant
	$\mathsf{t} \lhd \mathsf{Inr}$		Fill destination with right variant
	$\mathsf{t} \lhd \mathrm{E}^m$		Fill destination with exponential constructor
	$\mathsf{t} \lhd (,)$		Fill destination with product constructor
	$\mathsf{t} \lhd (\lambda \mathsf{x}_m \mapsto \mathsf{u})$		Fill destination with function
	$\mathsf{t} \lhd_\bullet \mathsf{t}'$		Fill destination with root of ampar t′
	$\mathsf{t}[\mathsf{x} := \mathsf{v}]$	M	

val, v	::=		Term value
	$-\mathtt{h}$		Hole
	$+\mathtt{h}$		Destination
	$()$		Unit
	$\lambda^{\mathsf{v}} \mathsf{x}_m \mapsto \mathsf{u}$		Lambda abstraction
	$\mathsf{Inl}\, \mathsf{v}$		Left variant for sum
	$\mathsf{Inr}\, \mathsf{v}$		Right variant for sum
	$\mathrm{E}^m\, \mathsf{v}$		Exponential
	$(\mathsf{v}_1 \,,\, \mathsf{v}_2)$		Product
	$_{\mathtt{H}}\langle \mathsf{v}_2 \,,\, \mathsf{v}_1 \rangle$		Ampar
	$\mathsf{v}[\mathtt{H}\pm\mathtt{h}']$	M	Shift hole names inside v by $\mathtt{h}'$ if they belong to H.

ectx, c	::=			Evaluation context component
	|	$\square \succ t'$		Application
	|	$v \succ \square$		Application
	|	$\square \,;\, u$		Pattern-match on unit
	|	$\square \succ \mathsf{case}_m \{ \mathsf{Inl}\, x_1 \mapsto u_1 \,,\ \mathsf{Inr}\, x_2 \mapsto u_2 \}$		Pattern-match on sum
	|	$\square \succ \mathsf{case}_m (x_1 \,, x_2) \mapsto u$		Pattern-match on product
	|	$\square \succ \mathsf{case}_m \, \mathrm{E}^n \, x \mapsto u$		Pattern-match on exponential
	|	$\square \succ \mathsf{map}\ x \mapsto t'$		Map over the right side of ampar
	|	$\mathsf{to}_{\ltimes}\ \square$		Wrap into a trivial ampar
	|	$\mathsf{from}_{\ltimes}\ \square$		Extract value from trivial ampar
	|	$\square \triangleleft ()$		Fill destination with unit
	|	$\square \triangleleft \mathsf{Inl}$		Fill destination with left variant
	|	$\square \triangleleft \mathsf{Inr}$		Fill destination with right variant
	|	$\square \triangleleft \mathrm{E}^m$		Fill destination with exponential constructor
	|	$\square \triangleleft (,)$		Fill destination with product constructor
	|	$\square \triangleleft (\lambda x \,_m \mapsto u)$		Fill destination with function
	|	$\square \triangleleft_\bullet t'$		Fill destination with root of ampar
	|	$v \triangleleft_\bullet \square$		Fill destination with root of ampar
	|	$\overset{\mathrm{op}}{\mathtt{H}}\langle v_2 \,, \square$		Open ampar. Only new addition to term shapes
ectxs, C	::=			Evaluation context stack
	|	$\square$		Represent the empty stack / "identity" evaluation context
	|	$C \circ c$		Push c on top of C
	|	$C[\mathtt{h} :=_{\mathtt{H}} v]$	M	Fill h in C with value v (that may contain holes)

2 Type system

type, T, U	::=			Type
	|	$\mathbf{1}$		Unit
	|	$\mathbf{T}_1 \oplus \mathbf{T}_2$		Sum
	|	$\mathbf{T}_1 \otimes \mathbf{T}_2$		Product
	|	$!^m \mathbf{T}$		Exponential
	|	$\mathbf{U} \ltimes \mathbf{T}$		Ampar type (consuming T yields U)
	|	$\mathbf{T} \,_m\!\!\rightarrow \mathbf{U}$		Function
	|	$\lfloor \mathbf{T} \rfloor^m$		Destination
mode, m, n	::=			Mode (Semiring)
	|	pa		Pair of a multiplicity and age
	|	$☠$		Error case (incompatible types, multiplicities, or ages)
	|	$m_1 \cdot \ldots \cdot m_k$	M	Semiring product
mul, p	::=			Multiplicity (first component of modality)
	|	1		Linear. Neutral element of the product
	|	ω		Non-linear. Absorbing for the product
	|	$p_1 \cdot \ldots \cdot p_k$	M	Semiring product
age, a	::=			Age (second component of modality)
	|	ν		Born now. Neutral element of the product
	|	$\uparrow$		One scope older
	|	∞		Infinitely old / static. Absorbing for the product
	|	$a_1 \cdot \ldots \cdot a_k$	M	Semiring product
ctx, Γ, Δ, Π	::=			Typing context
	|	$\{x :\, _m \mathbf{T}\}$		
	|	$\{+\mathtt{h} :\, _m \lfloor \mathbf{T} \rfloor^n\}$		
	|	$\{-\mathtt{h} : \mathbf{T}^n\}$		
	|	$m \cdot \Gamma$	M	Multiply each binding by m
	|	$\Gamma_1 \uplus \Gamma_2$	M	Sum contexts Γ_1 and Γ_2. Duplicate keys with incompatible values will be tagged
	|	$-\Gamma$	M	Transforms dest bindings into a hole bindings (requires `ctx_DestOnly` Γ and `ctx`
	|	$-^{-1}\Gamma$	M	Transforms hole bindings into dest bindings with left mode 1ν (requires `ctx_Hol`

| $\Gamma[\mathtt{H}\pm\mathtt{h}']$ M Shift hole/dest names by $\mathtt{h}'$ if they belong to $\mathtt{H}$

$\boxed{\Gamma \Vdash \mathsf{v} : \mathsf{T}}$ *(Typing of values (raw))*

TyR-val-H
$$\frac{}{\{-\mathtt{h} : \mathsf{T}^{\mathit{1}\nu}\} \Vdash -\mathtt{h} : \mathsf{T}}$$

TyR-val-D
$$\frac{}{\{+\mathtt{h} : {}_{\mathit{1}\nu}\lfloor \mathsf{T} \rfloor^{n}\} \Vdash +\mathtt{h} : \lfloor \mathsf{T} \rfloor^{n}}$$

TyR-val-U
$$\frac{}{\{\} \Vdash () : \mathsf{1}}$$

TyR-val-F
$$\frac{\begin{array}{c}\mathtt{mode\_IsValid}\ m \\ \mathtt{ctx\_DestOnly}\ \Delta \\ \Delta \uplus \{\mathsf{x} : {}_{m}\mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Delta \Vdash \lambda^{\mathsf{v}} \mathsf{x}\, {}_{m} \mapsto \mathsf{u} : \mathsf{T} \,{}_{m}\!\!\rightarrow \mathsf{U}}$$

TyR-val-L
$$\frac{\Gamma \Vdash \mathsf{v}_1 : \mathsf{T}_1}{\Gamma \Vdash \mathsf{Inl}\, \mathsf{v}_1 : \mathsf{T}_1 \oplus \mathsf{T}_2}$$

TyR-val-R
$$\frac{\Gamma \Vdash \mathsf{v}_2 : \mathsf{T}_2}{\Gamma \Vdash \mathsf{Inr}\, \mathsf{v}_2 : \mathsf{T}_1 \oplus \mathsf{T}_2}$$

TyR-val-P
$$\frac{\Gamma_1 \Vdash \mathsf{v}_1 : \mathsf{T}_1 \qquad \Gamma_2 \Vdash \mathsf{v}_2 : \mathsf{T}_2}{\Gamma_1 \uplus \Gamma_2 \Vdash (\mathsf{v}_1 , \mathsf{v}_2) : \mathsf{T}_1 \otimes \mathsf{T}_2}$$

TyR-val-E
$$\frac{\Gamma \Vdash \mathsf{v}' : \mathsf{T} \qquad \mathtt{mode\_IsValid}\ n}{n{\cdot}\Gamma \Vdash \mathrm{E}^{n}\, \mathsf{v}' : !^{n}\, \mathsf{T}}$$

TyR-val-A
$$\frac{\begin{array}{c}\mathtt{ctx\_DestOnly}\ \Delta_1 \\ \mathtt{ctx\_DestOnly}\ \Delta_2 \\ \mathtt{ctx\_DestOnly}\ \Delta_3 \\ \mathtt{ctx\_LinOnly}\ \Delta_3 \\ \mathtt{ctx\_FinAgeOnly}\ \Delta_3 \\ \mathtt{ctx\_ValidOnly}\ \Delta_3 \\ \Delta_1\ \#\ \Delta_2 \qquad \Delta_1\ \#\ \Delta_3 \\ \Delta_2\ \#\ \Delta_3 \qquad \mathit{1}\!\uparrow{\cdot}\Delta_1 \uplus \Delta_3 \Vdash \mathsf{v}_1 : \mathsf{T} \\ \Delta_2 \uplus (-\Delta_3) \Vdash \mathsf{v}_2 : \mathsf{U}\end{array}}{\Delta_1 \uplus \Delta_2 \Vdash {}_{\mathtt{hvars}(-\Delta_3)}\langle \mathsf{v}_2 \, , \, \mathsf{v}_1 \rangle : \mathsf{U} \ltimes \mathsf{T}}$$

$\boxed{\Pi \vdash \mathsf{t} : \mathsf{T}}$ *(Typing of terms)*

Ty-term-Val
$$\frac{\begin{array}{c}\mathtt{ctx\_DestOnly}\ \Delta \\ \mathtt{ctx\_DisposableOnly}\ \Pi \\ \Delta \Vdash \mathsf{v} : \mathsf{T}\end{array}}{\Pi \uplus \Delta \vdash \mathsf{v} : \mathsf{T}}$$

Ty-term-Var
$$\frac{\begin{array}{c}\mathtt{ctx\_DisposableOnly}\ \Pi \\ \Pi\ \#\ \{\mathsf{x} : {}_{m}\mathsf{T}\} \\ \mathtt{mode\_IsSubtype}\ m\ \mathit{1}\nu\end{array}}{\Pi \uplus \{\mathsf{x} : {}_{m}\mathsf{T}\} \vdash \mathsf{x} : \mathsf{T}}$$

Ty-term-App
$$\frac{\begin{array}{c}\mathtt{mode\_IsValid}\ m \\ \Pi_1 \vdash \mathsf{t} : \mathsf{T} \qquad \Pi_2 \vdash \mathsf{t}' : \mathsf{T} \,{}_{m}\!\!\rightarrow \mathsf{U}\end{array}}{m{\cdot}\Pi_1 \uplus \Pi_2 \vdash \mathsf{t} \succ \mathsf{t}' : \mathsf{U}}$$

Ty-term-PatU
$$\frac{\Pi_1 \vdash \mathsf{t} : \mathsf{1} \qquad \Pi_2 \vdash \mathsf{u} : \mathsf{U}}{\Pi_1 \uplus \Pi_2 \vdash \mathsf{t} \,;\, \mathsf{u} : \mathsf{U}}$$

Ty-term-PatS
$$\frac{\begin{array}{c}\mathtt{mode\_IsValid}\ m \\ \Pi_2\ \#\ \{\mathsf{x}_1 : {}_{m}\mathsf{T}_1\} \\ \Pi_2\ \#\ \{\mathsf{x}_2 : {}_{m}\mathsf{T}_2\} \\ \Pi_1 \vdash \mathsf{t} : \mathsf{T}_1 \oplus \mathsf{T}_2 \\ \Pi_2 \uplus \{\mathsf{x}_1 : {}_{m}\mathsf{T}_1\} \vdash \mathsf{u}_1 : \mathsf{U} \\ \Pi_2 \uplus \{\mathsf{x}_2 : {}_{m}\mathsf{T}_2\} \vdash \mathsf{u}_2 : \mathsf{U}\end{array}}{m{\cdot}\Pi_1 \uplus \Pi_2 \vdash \mathsf{t} \succ \mathsf{case}_{m}\, \{\, \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \,\} : \mathsf{U}}$$

Ty-term-PatP
$$\frac{\begin{array}{c}\mathtt{mode\_IsValid}\ m \\ \Pi_2\ \#\ \{\mathsf{x}_1 : {}_{m}\mathsf{T}_1\} \\ \Pi_2\ \#\ \{\mathsf{x}_2 : {}_{m}\mathsf{T}_2\} \\ \{\mathsf{x}_1 : {}_{m}\mathsf{T}_1\}\ \#\ \{\mathsf{x}_2 : {}_{m}\mathsf{T}_2\} \\ \Pi_1 \vdash \mathsf{t} : \mathsf{T}_1 \otimes \mathsf{T}_2 \\ \Pi_2 \uplus \{\mathsf{x}_1 : {}_{m}\mathsf{T}_1\} \uplus \{\mathsf{x}_2 : {}_{m}\mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{m{\cdot}\Pi_1 \uplus \Pi_2 \vdash \mathsf{t} \succ \mathsf{case}_{m}\, (\mathsf{x}_1 \, , \, \mathsf{x}_2) \mapsto \mathsf{u} : \mathsf{U}}$$

Ty-term-PatE
$$\frac{\begin{array}{c}\mathtt{mode\_IsValid}\ m \\ \mathtt{mode\_IsValid}\ n \\ \Pi_2\ \#\ \{\mathsf{x} : {}_{m{\cdot}n}\mathsf{T}\} \qquad \Pi_1 \vdash \mathsf{t} : !^{n}\, \mathsf{T} \\ \Pi_2 \uplus \{\mathsf{x} : {}_{m{\cdot}n}\mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{m{\cdot}\Pi_1 \uplus \Pi_2 \vdash \mathsf{t} \succ \mathsf{case}_{m}\, \mathrm{E}^{n}\, \mathsf{x} \mapsto \mathsf{u} : \mathsf{U}}$$

Ty-term-Map
$$\frac{\begin{array}{c}\Pi_2\ \#\ \{\mathsf{x} : {}_{\mathit{1}\nu}\mathsf{T}\} \qquad \Pi_1 \vdash \mathsf{t} : \mathsf{U} \ltimes \mathsf{T} \\ \mathit{1}\!\uparrow{\cdot}\Pi_2 \uplus \{\mathsf{x} : {}_{\mathit{1}\nu}\mathsf{T}\} \vdash \mathsf{t}' : \mathsf{T}'\end{array}}{\Pi_1 \uplus \Pi_2 \vdash \mathsf{t} \succ \mathsf{map}\ \mathsf{x} \mapsto \mathsf{t}' : \mathsf{U} \ltimes \mathsf{T}'}$$

Ty-term-ToA
$$\frac{\Pi \vdash \mathsf{u} : \mathsf{U}}{\Pi \vdash \mathsf{to}_{\ltimes}\, \mathsf{u} : \mathsf{U} \ltimes \mathsf{1}}$$

Ty-term-FromA
$$\frac{\Pi \vdash \mathsf{t} : \mathsf{U} \ltimes \mathsf{1}}{\Pi \vdash \mathsf{from}_{\ltimes}\, \mathsf{t} : \mathsf{U}}$$

Ty-term-FillU
$$\frac{\Pi \vdash \mathsf{t} : \lfloor \mathsf{1} \rfloor^{n}}{\Pi \vdash \mathsf{t} \triangleleft () : \mathsf{1}}$$

Ty-term-FillL
$$\frac{\Pi \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^{n}}{\Pi \vdash \mathsf{t} \triangleleft \mathsf{Inl} : \lfloor \mathsf{T}_1 \rfloor^{n}}$$

Ty-term-FillR
$$\frac{\Pi \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^{n}}{\Pi \vdash \mathsf{t} \triangleleft \mathsf{Inr} : \lfloor \mathsf{T}_2 \rfloor^{n}}$$

Ty-term-FillP
$$\frac{\Pi \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \otimes \mathsf{T}_2 \rfloor^{n}}{\Pi \vdash \mathsf{t} \triangleleft (,) : \lfloor \mathsf{T}_1 \rfloor^{n} \otimes \lfloor \mathsf{T}_2 \rfloor^{n}}$$

Ty-term-FillE
$$\frac{\begin{array}{c}\mathtt{mode\_IsValid}\ n \\ \Pi \vdash \mathsf{t} : \lfloor !^{n'}\, \mathsf{T} \rfloor^{n}\end{array}}{\Pi \vdash \mathsf{t} \triangleleft \mathrm{E}^{n'} : \lfloor \mathsf{T} \rfloor^{n' \cdot n}}$$

Ty-term-FillF
$$\frac{\begin{array}{c}\mathtt{mode\_IsValid}\ m \\ \mathtt{mode\_IsValid}\ n \\ \Pi_2\ \#\ \{\mathsf{x} : {}_{m}\mathsf{T}\} \\ \Pi_1 \vdash \mathsf{t} : \lfloor \mathsf{T} \,{}_{m}\!\!\rightarrow \mathsf{U} \rfloor^{n} \\ \Pi_2 \uplus \{\mathsf{x} : {}_{m}\mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Pi_1 \uplus (\mathit{1}\!\uparrow{\cdot}n){\cdot}\Pi_2 \vdash \mathsf{t} \triangleleft (\lambda \mathsf{x}\, {}_{m} \mapsto \mathsf{u}) : \mathsf{1}}$$

Ty-term-FillC
$$\frac{\begin{array}{c}\mathtt{mode\_IsValid}\ n \\ \Pi_1 \vdash \mathsf{t} : \lfloor \mathsf{U} \rfloor^{n} \qquad \Pi_2 \vdash \mathsf{t}' : \mathsf{U} \ltimes \mathsf{T}\end{array}}{\Pi_1 \uplus (\mathit{1}\!\uparrow{\cdot}n){\cdot}\Pi_2 \vdash \mathsf{t} \triangleleft\!\bullet \mathsf{t}' : \mathsf{T}}$$

$\boxed{\Delta \dashv \mathsf{C} : \mathsf{T} \rightarrowtail \mathsf{U}_0}$ *(Typing of evaluation contexts)*

Ty-ectxs-Id
$$\frac{}{\{\} \dashv \square : \mathsf{U}_0 \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-AppFoc1
$$\frac{\begin{array}{c}\Delta_1 \mathbin{\#} \Delta_2 \quad \mathtt{ctx\_DestOnly}\ \Delta_1 \\ \mathtt{ctx\_DestOnly}\ \Delta_2 \\ \mathtt{mode\_IsValid}\ m \\ \mathtt{ctx\_ValidOnly}\ \Delta_2 \\ m \cdot \Delta_1 \uplus \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \\ \Delta_2 \vdash \mathsf{t}' : \mathsf{T} \mathrel{}_m\!\!\rightarrow \mathsf{U}\end{array}}{\Delta_1 \dashv \mathsf{C} \circ (\square \succ \mathsf{t}') : \mathsf{T} \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-AppFoc2
$$\frac{\begin{array}{c}\Delta_1 \mathbin{\#} \Delta_2 \quad \mathtt{ctx\_DestOnly}\ \Delta_1 \\ \mathtt{ctx\_DestOnly}\ \Delta_2 \\ \mathtt{mode\_IsValid}\ m \\ \mathtt{ctx\_ValidOnly}\ \Delta_1 \\ m \cdot \Delta_1 \uplus \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \\ \Delta_1 \vdash \mathsf{v} : \mathsf{T}\end{array}}{\Delta_2 \dashv \mathsf{C} \circ (\mathsf{v} \succ \square) : (\mathsf{T} \mathrel{}_m\!\!\rightarrow \mathsf{U}) \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-PatUFoc
$$\frac{\begin{array}{c}\Delta_1 \mathbin{\#} \Delta_2 \quad \mathtt{ctx\_DestOnly}\ \Delta_1 \\ \mathtt{ctx\_DestOnly}\ \Delta_2 \\ \mathtt{ctx\_ValidOnly}\ \Delta_2 \\ \Delta_1 \uplus \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \\ \Delta_2 \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Delta_1 \dashv \mathsf{C} \circ (\square\,;\mathsf{u}) : \mathbf{1} \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-PatSFoc
$$\frac{\begin{array}{c}\Delta_1 \mathbin{\#} \Delta_2 \quad \mathtt{ctx\_DestOnly}\ \Delta_1 \\ \mathtt{ctx\_DestOnly}\ \Delta_2 \\ \mathtt{mode\_IsValid}\ m \\ \mathtt{ctx\_ValidOnly}\ \Delta_2 \\ m \cdot \Delta_1 \uplus \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \\ \Delta_2 \uplus \{\mathsf{x}_1 : {}_m\mathsf{T}_1\} \vdash \mathsf{u}_1 : \mathsf{U} \\ \Delta_2 \uplus \{\mathsf{x}_2 : {}_m\mathsf{T}_2\} \vdash \mathsf{u}_2 : \mathsf{U}\end{array}}{\Delta_1 \dashv \mathsf{C} \circ (\square \succ \mathsf{case}_m\,\{\,\mathsf{Inl}\,\mathsf{x}_1 \mapsto \mathsf{u}_1\,,\ \mathsf{Inr}\,\mathsf{x}_2 \mapsto \mathsf{u}_2\,\}) : (\mathsf{T}_1 \oplus \mathsf{T}_2) \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-PatPFoc
$$\frac{\begin{array}{c}\Delta_1 \mathbin{\#} \Delta_2 \quad \mathtt{ctx\_DestOnly}\ \Delta_1 \\ \mathtt{ctx\_DestOnly}\ \Delta_2 \\ \{\mathsf{x}_1 : {}_m\mathsf{T}_1\} \mathbin{\#} \{\mathsf{x}_2 : {}_m\mathsf{T}_2\} \\ \mathtt{mode\_IsValid}\ m \\ \mathtt{ctx\_ValidOnly}\ \Delta_2 \\ m \cdot \Delta_1 \uplus \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \\ \Delta_2 \uplus \{\mathsf{x}_1 : {}_m\mathsf{T}_1\} \uplus \{\mathsf{x}_2 : {}_m\mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Delta_1 \dashv \mathsf{C} \circ (\square \succ \mathsf{case}_m\,(\mathsf{x}_1\,,\,\mathsf{x}_2) \mapsto \mathsf{u}) : (\mathsf{T}_1 \otimes \mathsf{T}_2) \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-PatEFoc
$$\frac{\begin{array}{c}\Delta_1 \mathbin{\#} \Delta_2 \quad \mathtt{ctx\_DestOnly}\ \Delta_1 \\ \mathtt{ctx\_DestOnly}\ \Delta_2 \\ \mathtt{mode\_IsValid}\ m \\ \mathtt{mode\_IsValid}\ m' \\ \mathtt{ctx\_ValidOnly}\ \Delta_2 \\ m \cdot \Delta_1 \uplus \Delta_2 \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0 \\ \Delta_2 \uplus \{\mathsf{x} : {}_{m \cdot m'}\mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Delta_1 \dashv \mathsf{C} \circ (\square \succ \mathsf{case}_m\,\mathrm{E}^{m'}\,\mathsf{x} \mapsto \mathsf{u}) : !^{m'}\,\mathsf{T} \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-MapFoc
$$\frac{\begin{array}{c}\Delta_1 \mathbin{\#} \Delta_2 \quad \mathtt{ctx\_DestOnly}\ \Delta_1 \\ \mathtt{ctx\_DestOnly}\ \Delta_2 \\ \mathtt{ctx\_ValidOnly}\ \Delta_2 \\ \Delta_1 \uplus \Delta_2 \dashv \mathsf{C} : \mathsf{U} \ltimes \mathsf{T}' \rightarrowtail \mathsf{U}_0 \\ \mathit{1}\!\uparrow \cdot \Delta_2 \uplus \{\mathsf{x} : {}_{1\nu}\mathsf{T}\} \vdash \mathsf{t}' : \mathsf{T}'\end{array}}{\Delta_1 \dashv \mathsf{C} \circ (\square \succ \mathsf{map}\ \mathsf{x} \mapsto \mathsf{t}') : (\mathsf{U} \ltimes \mathsf{T}) \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-ToAFoc
$$\frac{\Delta \dashv \mathsf{C} : (\mathsf{U} \ltimes 1) \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\mathsf{to}_{\ltimes}\,\square) : \mathsf{U} \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FromAFoc
$$\frac{\Delta \dashv \mathsf{C} : \mathsf{U} \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\mathsf{from}_{\ltimes}\,\square) : (\mathsf{U} \ltimes 1) \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillUFoc
$$\frac{\Delta \dashv \mathsf{C} : \mathbf{1} \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\square \triangleleft ()) : \lfloor \mathbf{1} \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillLFoc
$$\frac{\Delta \dashv \mathsf{C} : \lfloor \mathsf{T}_1 \rfloor^n \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\square \triangleleft \mathsf{Inl}) : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillRFoc
$$\frac{\Delta \dashv \mathsf{C} : \lfloor \mathsf{T}_2 \rfloor^n \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\square \triangleleft \mathsf{Inr}) : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillFFoc
$$\frac{\begin{array}{c}\Delta_1 \mathbin{\#} \Delta_2 \quad \mathtt{ctx\_DestOnly}\ \Delta_1 \\ \mathtt{ctx\_DestOnly}\ \Delta_2 \\ \mathtt{ctx\_ValidOnly}\ \Delta_2 \\ \mathtt{mode\_IsValid}\ m \\ \mathtt{mode\_IsValid}\ n \\ \Delta_1 \uplus (\mathit{1}\!\uparrow \cdot n) \cdot \Delta_2 \dashv \mathsf{C} : \mathbf{1} \rightarrowtail \mathsf{U}_0 \\ \Delta_2 \uplus \{\mathsf{x} : {}_m\mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Delta_1 \dashv \mathsf{C} \circ (\square \triangleleft (\lambda \mathsf{x}_m \mapsto \mathsf{u})) : \lfloor \mathsf{T} \mathrel{}_m\!\!\rightarrow \mathsf{U} \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillPFoc
$$\frac{\Delta \dashv \mathsf{C} : (\lfloor \mathsf{T}_1 \rfloor^n \otimes \lfloor \mathsf{T}_2 \rfloor^n) \rightarrowtail \mathsf{U}_0}{\Delta \dashv \mathsf{C} \circ (\square \triangleleft (,)) : \lfloor \mathsf{T}_1 \otimes \mathsf{T}_2 \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillEFoc
$$\frac{\begin{array}{c}\mathtt{mode\_IsValid}\ m \\ \Delta \dashv \mathsf{C} : \lfloor \mathsf{T} \rfloor^{m \cdot n} \rightarrowtail \mathsf{U}_0\end{array}}{\Delta \dashv \mathsf{C} \circ (\square \triangleleft \mathrm{E}^m) : \lfloor !^m\,\mathsf{T} \rfloor^n \rightarrowtail \mathsf{U}_0}$$

Ty-ectxs-FillCFoc1

$$\frac{\Delta_1 \mathbin{\#} \Delta_2 \quad \mathtt{ctx\_DestOnly}\ \Delta_1 \quad \mathtt{ctx\_DestOnly}\ \Delta_2 \quad \mathtt{ctx\_ValidOnly}\ \Delta_2 \quad \mathtt{mode\_IsValid}\ n \quad \Delta_1 \uplus (\mathit{1}\!\uparrow\cdot n)\cdot\Delta_2 \dashv \mathsf{C} : \mathbf{T} \rightarrowtail \mathbf{U}_0 \quad \Delta_2 \vdash \mathsf{t}' : \mathbf{U} \ltimes \mathbf{T}}{\Delta_1 \dashv \mathsf{C} \circ (\square \triangleleft\bullet\, \mathsf{t}') : \lfloor \mathbf{U} \rfloor^n \rightarrowtail \mathbf{U}_0}$$

Ty-ectxs-FillCFoc2

$$\frac{\Delta_1 \mathbin{\#} \Delta_2 \quad \mathtt{ctx\_DestOnly}\ \Delta_1 \quad \mathtt{ctx\_DestOnly}\ \Delta_2 \quad \mathtt{ctx\_ValidOnly}\ \Delta_1 \quad \mathtt{mode\_IsValid}\ n \quad \Delta_1 \uplus (\mathit{1}\!\uparrow\cdot n)\cdot\Delta_2 \dashv \mathsf{C} : \mathbf{T} \rightarrowtail \mathbf{U}_0 \quad \Delta_1 \vdash \mathsf{v} : \lfloor \mathbf{U} \rfloor^n}{\Delta_2 \dashv \mathsf{C} \circ (\mathsf{v} \triangleleft\bullet\, \square) : \mathbf{U} \ltimes \mathbf{T} \rightarrowtail \mathbf{U}_0}$$

Ty-ectxs-AOpenFoc

$$\frac{\begin{array}{c}\Delta_1 \mathbin{\#} \Delta_2 \\ \Delta_1 \mathbin{\#} \Delta_3 \qquad \Delta_2 \mathbin{\#} \Delta_3 \\ \mathtt{hvars}(\mathsf{C}) \mathbin{\#\#} \mathtt{hvars}(-\Delta_3) \\ \mathtt{ctx\_DestOnly}\ \Delta_1 \\ \mathtt{ctx\_DestOnly}\ \Delta_2 \\ \mathtt{ctx\_DestOnly}\ \Delta_3 \\ \mathtt{ctx\_LinOnly}\ \Delta_3 \\ \mathtt{ctx\_FinAgeOnly}\ \Delta_3 \\ \mathtt{ctx\_ValidOnly}\ \Delta_3 \\ \Delta_1 \uplus \Delta_2 \dashv \mathsf{C} : (\mathbf{U} \ltimes \mathbf{T}') \rightarrowtail \mathbf{U}_0 \\ \Delta_2 \uplus -\Delta_3 \Vdash \mathsf{v}_2 : \mathbf{U}\end{array}}{\mathit{1}\!\uparrow\cdot\Delta_1 \uplus \Delta_3 \dashv \mathsf{C} \circ ({}^{\mathrm{op}}_{\mathtt{hvars}(-\Delta_3)}\langle \mathsf{v}_2 \mathbin{,} \square) : \mathbf{T}' \rightarrowtail \mathbf{U}_0}$$

$\boxed{\vdash \mathsf{C}[\mathsf{t}] : \mathbf{T}}$ *(Typing of extended terms (pair of evaluation context and term))*

Ty-eterm-ClosedEterm

$$\frac{\mathtt{ctx\_ValidOnly}\ \Delta \quad \mathtt{ctx\_DestOnly}\ \Delta \quad \Delta \dashv \mathsf{C} : \mathbf{T} \rightarrowtail \mathbf{U}_0 \quad \Delta \vdash \mathsf{t} : \mathbf{T}}{\vdash \mathsf{C}[\mathsf{t}] : \mathbf{U}_0}$$

3 Small-step semantics

$\boxed{\mathsf{C}[\mathsf{t}] \longrightarrow \mathsf{C}'[\mathsf{t}']}$ *(Small-step evaluation of terms using evaluation contexts)*

Sem-eterm-AppFoc1
$$\frac{\texttt{term\_NotVal t}}{\mathsf{C}[\mathsf{t} \succ \mathsf{t}'] \longrightarrow (\mathsf{C} \circ (\square \succ \mathsf{t}'))[\mathsf{t}]}$$

Sem-eterm-AppUnfoc1
$$\frac{}{(\mathsf{C} \circ (\square \succ \mathsf{t}'))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \succ \mathsf{t}']}$$

Sem-eterm-AppFoc2
$$\frac{\texttt{term\_NotVal t}'}{\mathsf{C}[\mathsf{v} \succ \mathsf{t}'] \longrightarrow (\mathsf{C} \circ (\mathsf{v} \succ \square))[\mathsf{t}']}$$

Sem-eterm-AppUnfoc2
$$\frac{}{(\mathsf{C} \circ (\mathsf{v} \succ \square))[\mathsf{v}'] \longrightarrow \mathsf{C}[\mathsf{v} \succ \mathsf{v}']}$$

Sem-eterm-AppRed
$$\frac{}{\mathsf{C}[\mathsf{v} \succ (\lambda^{\mathsf{v}} \mathsf{x}_m \mapsto \mathsf{u})] \longrightarrow \mathsf{C}[\mathsf{u}[\mathsf{x} := \mathsf{v}]]}$$

Sem-eterm-PatUFoc
$$\frac{\texttt{term\_NotVal t}}{\mathsf{C}[\mathsf{t} \,;\, \mathsf{u}] \longrightarrow (\mathsf{C} \circ (\square \,;\, \mathsf{u}))[\mathsf{t}]}$$

Sem-eterm-PatUUnfoc
$$\frac{}{(\mathsf{C} \circ (\square \,;\, \mathsf{u}))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \,;\, \mathsf{u}]}$$

Sem-eterm-PatURed
$$\frac{}{\mathsf{C}[() \,;\, \mathsf{u}] \longrightarrow \mathsf{C}[\mathsf{u}]}$$

Sem-eterm-PatSFoc
$$\frac{\texttt{term\_NotVal t}}{\mathsf{C}[\mathsf{t} \succ \mathsf{case}_m \{ \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \}] \longrightarrow (\mathsf{C} \circ (\square \succ \mathsf{case}_m \{ \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \}))[\mathsf{t}]}$$

Sem-eterm-PatSUnfoc
$$\frac{}{(\mathsf{C} \circ (\square \succ \mathsf{case}_m \{ \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \}))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \succ \mathsf{case}_m \{ \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \}]}$$

Sem-eterm-PatLRed
$$\frac{}{\mathsf{C}[(\mathsf{Inl}\, \mathsf{v}_1) \succ \mathsf{case}_m \{ \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \}] \longrightarrow \mathsf{C}[\mathsf{u}_1[\mathsf{x}_1 := \mathsf{v}_1]]}$$

Sem-eterm-PatRRed
$$\frac{}{\mathsf{C}[(\mathsf{Inr}\, \mathsf{v}_2) \succ \mathsf{case}_m \{ \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \}] \longrightarrow \mathsf{C}[\mathsf{u}_2[\mathsf{x}_2 := \mathsf{v}_2]]}$$

Sem-eterm-PatPFoc
$$\frac{\texttt{term\_NotVal t}}{\mathsf{C}[\mathsf{t} \succ \mathsf{case}_m (\mathsf{x}_1 \,,\, \mathsf{x}_2) \mapsto \mathsf{u}] \longrightarrow (\mathsf{C} \circ (\square \succ \mathsf{case}_m (\mathsf{x}_1 \,,\, \mathsf{x}_2) \mapsto \mathsf{u}))[\mathsf{t}]}$$

Sem-eterm-PatPUnfoc
$$\frac{}{(\mathsf{C} \circ (\square \succ \mathsf{case}_m (\mathsf{x}_1 \,,\, \mathsf{x}_2) \mapsto \mathsf{u}))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \succ \mathsf{case}_m (\mathsf{x}_1 \,,\, \mathsf{x}_2) \mapsto \mathsf{u}]}$$

Sem-eterm-PatPRed
$$\frac{}{\mathsf{C}[(\mathsf{v}_1 \,,\, \mathsf{v}_2) \succ \mathsf{case}_m (\mathsf{x}_1 \,,\, \mathsf{x}_2) \mapsto \mathsf{u}] \longrightarrow \mathsf{C}[\mathsf{u}[\mathsf{x}_1 := \mathsf{v}_1][\mathsf{x}_2 := \mathsf{v}_2]]}$$

Sem-eterm-PatEFoc
$$\frac{\texttt{term\_NotVal t}}{\mathsf{C}[\mathsf{t} \succ \mathsf{case}_m \mathrm{E}^n\, \mathsf{x} \mapsto \mathsf{u}] \longrightarrow (\mathsf{C} \circ (\square \succ \mathsf{case}_m \mathrm{E}^n\, \mathsf{x} \mapsto \mathsf{u}))[\mathsf{t}]}$$

Sem-eterm-PatEUnfoc
$$\frac{}{(\mathsf{C} \circ (\square \succ \mathsf{case}_m \mathrm{E}^n\, \mathsf{x} \mapsto \mathsf{u}))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \succ \mathsf{case}_m \mathrm{E}^n\, \mathsf{x} \mapsto \mathsf{u}]}$$

Sem-eterm-PatERed
$$\frac{}{\mathsf{C}[\mathrm{E}^n\, \mathsf{v}' \succ \mathsf{case}_m \mathrm{E}^n\, \mathsf{x} \mapsto \mathsf{u}] \longrightarrow \mathsf{C}[\mathsf{u}[\mathsf{x} := \mathsf{v}']]}$$

Sem-eterm-MapFoc
$$\frac{\texttt{term\_NotVal t}}{\mathsf{C}[\mathsf{t} \succ \mathsf{map}\, \mathsf{x} \mapsto \mathsf{t}'] \longrightarrow (\mathsf{C} \circ (\square \succ \mathsf{map}\, \mathsf{x} \mapsto \mathsf{t}'))[\mathsf{t}]}$$

Sem-eterm-MapUnfoc
$$\frac{}{(\mathsf{C} \circ (\square \succ \mathsf{map}\, \mathsf{x} \mapsto \mathsf{t}'))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \succ \mathsf{map}\, \mathsf{x} \mapsto \mathsf{t}']}$$

Sem-eterm-MapRedAOpenFoc
$$\frac{\mathsf{h}' = \mathtt{max(hvars(C))+1}}{\mathsf{C}[{}_{\mathrm{H}}\langle \mathsf{v}_2 \,,\, \mathsf{v}_1 \rangle \succ \mathsf{map}\, \mathsf{x} \mapsto \mathsf{t}'] \longrightarrow (\mathsf{C} \circ ({}^{\mathrm{op}}_{\mathrm{H} \pm \mathsf{h}'}\langle \mathsf{v}_2[\mathrm{H} \pm \mathsf{h}'] \,,\, \square))[\mathsf{t}'[\mathsf{x} := \mathsf{v}_1[\mathrm{H} \pm \mathsf{h}']]]}$$

Sem-eterm-AOpenUnfoc
$$\frac{}{(\mathsf{C} \circ {}^{\mathrm{op}}_{\mathrm{H}}\langle \mathsf{v}_2 \,,\, \square)[\mathsf{v}_1] \longrightarrow \mathsf{C}[{}_{\mathrm{H}}\langle \mathsf{v}_2 \,,\, \mathsf{v}_1 \rangle]}$$

Sem-eterm-ToAFoc
$$\frac{\texttt{term\_NotVal u}}{\mathsf{C}[\mathsf{to}_{\ltimes}\, \mathsf{u}] \longrightarrow (\mathsf{C} \circ (\mathsf{to}_{\ltimes}\, \square))[\mathsf{u}]}$$

Sem-eterm-ToAUnfoc
$$\frac{}{(\mathsf{C} \circ (\mathsf{to}_{\ltimes}\, \square))[\mathsf{v}_2] \longrightarrow \mathsf{C}[\mathsf{to}_{\ltimes}\, \mathsf{v}_2]}$$

Sem-eterm-ToARed
$$\frac{}{\mathsf{C}[\mathsf{to}_{\ltimes}\, \mathsf{v}_2] \longrightarrow \mathsf{C}[{}_{\{\}}\langle \mathsf{v}_2 \,,\, () \rangle]}$$

Sem-eterm-FromAFoc
$$\frac{\texttt{term\_NotVal t}}{\mathsf{C}[\mathsf{from}_{\ltimes}\, \mathsf{t}] \longrightarrow (\mathsf{C} \circ (\mathsf{from}_{\ltimes}\, \square))[\mathsf{t}]}$$

Sem-eterm-FromAUnfoc
$$\frac{}{(\mathsf{C} \circ (\mathsf{from}_{\ltimes}\, \square))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{from}_{\ltimes}\, \mathsf{v}]}$$

Sem-eterm-FromARed
$$\frac{}{\mathsf{C}[\mathsf{from}_{\ltimes}\, {}_{\{\}}\langle \mathsf{v}_2 \,,\, () \rangle] \longrightarrow \mathsf{C}[\mathsf{v}_2]}$$

Sem-eterm-FillUFoc
$$\frac{\texttt{term\_NotVal t}}{\mathsf{C}[\mathsf{t} \triangleleft ()] \longrightarrow (\mathsf{C} \circ (\square \triangleleft ()))[\mathsf{t}]}$$

Sem-eterm-FillUUnfoc
$$\frac{}{(\mathsf{C} \circ (\square \triangleleft ()))[\mathsf{v}] \longrightarrow \mathsf{C}[\mathsf{v} \triangleleft ()]}$$

Sem-eterm-FillURed
$$\frac{}{\mathsf{C}[+\mathsf{h} \triangleleft ()] \longrightarrow \mathsf{C}[\mathsf{h} :=_{\{\}}\, ()][()]}$$

$$\textsc{Sem-eterm-FillLFoc}\ \frac{\texttt{term\_NotVal}\ t}{C[t \triangleleft \mathsf{Inl}] \longrightarrow (C \circ (\Box \triangleleft \mathsf{Inl}))[t]} \qquad \textsc{Sem-eterm-FillLUnfoc}\ \frac{}{(C \circ (\Box \triangleleft \mathsf{Inl}))[v] \longrightarrow C[v \triangleleft \mathsf{Inl}]}$$

$$\textsc{Sem-eterm-FillLRed}\ \frac{h' = \max(\mathsf{hvars}(C) \cup \{h\}) + 1}{C[+h \triangleleft \mathsf{Inl}] \longrightarrow C[h :=_{\{h'+1\}} \mathsf{Inl}\ -(h'+1)][+(h'+1)]}$$

$$\textsc{Sem-eterm-FillRFoc}\ \frac{\texttt{term\_NotVal}\ t}{C[t \triangleleft \mathsf{Inr}] \longrightarrow (C \circ (\Box \triangleleft \mathsf{Inr}))[t]} \qquad \textsc{Sem-eterm-FillRUnfoc}\ \frac{}{(C \circ (\Box \triangleleft \mathsf{Inr}))[v] \longrightarrow C[v \triangleleft \mathsf{Inr}]}$$

$$\textsc{Sem-eterm-FillRRed}\ \frac{h' = \max(\mathsf{hvars}(C) \cup \{h\}) + 1}{C[+h \triangleleft \mathsf{Inr}] \longrightarrow C[h :=_{\{h'+1\}} \mathsf{Inr}\ -(h'+1)][+(h'+1)]}$$

$$\textsc{Sem-eterm-FillEFoc}\ \frac{\texttt{term\_NotVal}\ t}{C[t \triangleleft \mathrm{E}^m] \longrightarrow (C \circ (\Box \triangleleft \mathrm{E}^m))[t]} \qquad \textsc{Sem-eterm-FillEUnfoc}\ \frac{}{(C \circ (\Box \triangleleft \mathrm{E}^m))[v] \longrightarrow C[v \triangleleft \mathrm{E}^m]}$$

$$\textsc{Sem-eterm-FillERed}\ \frac{h' = \max(\mathsf{hvars}(C) \cup \{h\}) + 1}{C[+h \triangleleft \mathrm{E}^m] \longrightarrow C[h :=_{\{h'+1\}} \mathrm{E}^m\ -(h'+1)][+(h'+1)]}$$

$$\textsc{Sem-eterm-FillPFoc}\ \frac{\texttt{term\_NotVal}\ t}{C[t \triangleleft (,)] \longrightarrow (C \circ (\Box \triangleleft (,)))[t]} \qquad \textsc{Sem-eterm-FillPUnfoc}\ \frac{}{(C \circ (\Box \triangleleft (,)))[v] \longrightarrow C[v \triangleleft (,)]}$$

$$\textsc{Sem-eterm-FillPRed}\ \frac{h' = \max(\mathsf{hvars}(C) \cup \{h\}) + 1}{C[+h \triangleleft (,)] \longrightarrow C[h :=_{\{h'+1, h'+2\}} (-(h'+1), -(h'+2))][(+(h'+1), +(h'+2))]}$$

$$\textsc{Sem-eterm-FillFFoc}\ \frac{\texttt{term\_NotVal}\ t}{C[t \triangleleft (\lambda \mathsf{x}_m \mapsto u)] \longrightarrow (C \circ (\Box \triangleleft (\lambda \mathsf{x}_m \mapsto u)))[t]} \qquad \textsc{Sem-eterm-FillFUnfoc}\ \frac{}{(C \circ (\Box \triangleleft (\lambda \mathsf{x}_m \mapsto u)))[v] \longrightarrow C[v \triangleleft (\lambda \mathsf{x}_m \mapsto u)]}$$

$$\textsc{Sem-eterm-FillFRed}\ \frac{}{C[+h \triangleleft (\lambda \mathsf{x}_m \mapsto u)] \longrightarrow C[h :=_{\{\}} \lambda^{\mathsf{v}} \mathsf{x}_m \mapsto u][()]}$$

$$\textsc{Sem-eterm-FillCFoc1}\ \frac{\texttt{term\_NotVal}\ t}{C[t \triangleleft_\bullet t'] \longrightarrow (C \circ (\Box \triangleleft_\bullet t'))[t]} \qquad \textsc{Sem-eterm-FillCUnfoc1}\ \frac{}{(C \circ (\Box \triangleleft_\bullet t'))[v] \longrightarrow C[v \triangleleft_\bullet t']}$$

$$\textsc{Sem-eterm-FillCFoc2}\ \frac{\texttt{term\_NotVal}\ t'}{C[v \triangleleft_\bullet t'] \longrightarrow (C \circ (v \triangleleft_\bullet \Box))[t']} \qquad \textsc{Sem-eterm-FillCUnfoc2}\ \frac{}{(C \circ (v \triangleleft_\bullet \Box))[v'] \longrightarrow C[v \triangleleft_\bullet v']}$$

$$\textsc{Sem-eterm-FillCRed}\ \frac{h' = \max(\mathsf{hvars}(C) \cup \{h\}) + 1}{C[+h \triangleleft_\bullet {}_{\mathrm{H}}\langle v_2, v_1 \rangle] \longrightarrow C[h :=_{(\mathrm{H} \pm h')}\ v_2[\mathrm{H} \pm h']][v_1[\mathrm{H} \pm h']]}$$

4 Remarks on the Coq proofs

- Not particularly elegant. Max number of goals observed 232 (solved by a single call to the `congruence` tactic). When you have a computer, brute force is a viable strategy. (in particular, no semiring formalisation, it was quicker to do directly)
- Rules generated by ott, same as in the article (up to some notational difference). Contexts are not generated purely by syntax, and are interpreted in a semantic domain (finite functions).
- Reasoning on closed terms avoids almost all complications on binder manipulation. Makes proofs tractable.
- Finite functions: making a custom library was less headache than using existing libraries (including `MMap`). Existing libraries don't provide some of the tools that we needed, but the most important factor ended up being the need for a modicum of dependency between key and value. There wasn't really that out there. Backed by actual functions for simplicity; cost: equality is complicated.
- Most of the proofs done by author with very little prior experience to Coq.
- Did proofs in Coq because context manipulations are tricky.
- Context sum made total by adding an extra invalid *mode* (rather than an extra context). It seems to be much simpler this way.
- It might be a good idea to provide statistics on the number of lemmas and size of Coq codebase.
- (possibly) renaming as permutation, inspired by nominal sets, make more lemmas don't require a condition (but some lemmas that wouldn't in a straight renaming do in exchange).
- (possibly) methodology: assume a lot of lemmas, prove main theorem, prove assumptions, some wrong, fix. A number of wrong lemma initially assumed, but replacing them by correct variant was always easy to fix in proofs.
- Axioms that we use and why (in particular setoid equality not very natural with ott-generated typing rules).
- Talk about the use and benefits of Copilot.