

Destination λ -calculus

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1 Term and value syntax

tmv, x, y Term-level variable name
 $hdmv$ Hole or destination static name
 k Index for ranges

$hddyn, d$::= Hole or destination dynamic name
| \star Root namespace
| $d.1$ Subnamespace 1
| $d.2$ Subnamespace 2
| $d.3$ Subnamespace 3

$hdnm, h$::= Hole or destination name
| d Dynamic name
| $hdmv$ Static name

val, v ::= Term value
| $-h$ Hole
| $+h$ Destination
| $()$ Unit
| $\lambda x \mapsto t$ Linear function
| $lnl\ v$ Left variant for sum
| $lnr\ v$ Right variant for sum
| $\mathbin{\mathbb{J}}^m\ v$ Exponential
| (v_1, v_2) Product
| $\langle v_1, v_2 \rangle_\Delta$ Ampar

$term, t, u$::= Term
| v Term value
| x Variable
| $t \succ u$ Application
| $t ; u$ Pattern-match on unit
| $t \succ \text{case } \{ lnl\ x_1 \mapsto u_1, lnr\ x_2 \mapsto u_2 \}$ Pattern-match on sum

| $t \succ \text{case } (x_1, x_2) \mapsto u$ Pattern-match on product

| $t \succ \text{case } \mathbin{\mathbb{J}}^m\ x \mapsto u$ Pattern-match on exponential
| $t \succ \text{map } x \mapsto u$ Map over the left side of the ampar
| $\text{to}_{\times} t$ Wrap t into a trivial ampar
| $\text{from}_{\times} t$ Extract value from trivial ampar
| alloc_T Return a fresh "identity" ampar object
| $t \triangleleft ()$ Fill destination with unit
| $t \triangleleft lnl$ Fill destination with left variant
| $t \triangleleft lnr$ Fill destination with right variant
| $t \triangleleft (,)$ Fill destination with product constructor
| $t \triangleleft \mathbin{\mathbb{J}}^m$ Fill destination with exponential constructor
| $t \triangleleft \bullet u$ Fill destination with root of ampar u

2 Type system

type, T, U	$::=$	Type
	$\mathbf{1}$	Unit
	$\mathsf{T}_1 \oplus \mathsf{T}_2$	Sum
	$\mathsf{T}_1 \otimes \mathsf{T}_2$	Product
	$!^m \mathsf{T}$	Exponential
	$\mathsf{T}_1 \ltimes \mathsf{T}_2$	Ampar type (consuming T_1 yields T_2)
	$\mathsf{T}_1 \xrightarrow{m_1} \mathsf{T}_2$	Linear function
	$^m[\mathsf{T}]$	Destination
mode, m, n	$::=$	Mode (Semiring)
	pa	Pair of a multiplicity and age
	ω	Error case (incompatible types, multiplicities, or ages)
	$m_1 \cdot \dots \cdot m_k$	Semiring product
mul, p	$::=$	Multiplicity (first component of modality)
	1	Linear. Neutral element of the product
	ω	Non-linear. Absorbing for the product
	$p_1 \cdot \dots \cdot p_k$	Semiring product
age, a	$::=$	Age (second component of modality)
	ν	Born now. Neutral element of the product
	\uparrow	One scope older
	∞	Infinitely old / static. Absorbing for the product
	$a_1 \cdot \dots \cdot a_k$	Semiring product
ctx, Γ, Δ	$::=$	Typing context
	$\{b_1, \dots, b_k\}$	
	$^m\Gamma$	Multiply each binding by m
	$\Gamma_1 \uplus \Gamma_2$	
	$\Gamma_1 \cup \Gamma_2$	
	$-\Gamma$	
bndr, b	$::=$	Type assignment to either variable, destination or hole
	$x :_m \mathsf{T}$	Variable
	$+h :_m \xrightarrow{n}[\mathsf{T}]$	Destination (m is its own modality; n is the modality for values it accepts)
	$-h :^n \mathsf{T}$	Hole (n is the modality for values it accepts, it doesn't have a modality on its own)

$$\boxed{\Gamma \Vdash t : \mathbf{T}}$$

(Typing of terms (raw))

$$\begin{array}{c}
\text{TYR-TERM-H} \quad \frac{}{\{-\mathbf{h} : {}^{\textcolor{teal}{1}\nu} \mathbf{T}\} \Vdash -\mathbf{h} : \mathbf{T}} \quad
\text{TYR-TERM-D} \quad \frac{}{\{+\mathbf{h} : {}^{\textcolor{teal}{1}\nu} {}^n[\mathbf{T}]\} \Vdash +\mathbf{h} : {}^n[\mathbf{T}]} \quad
\text{TYR-TERM-U} \quad \frac{}{\{\} \Vdash () : \mathbf{1}} \quad
\text{TYR-TERM-L} \quad \frac{\Gamma \Vdash v : \mathbf{T}_1}{\Gamma \Vdash \text{Inl } v : \mathbf{T}_1 \oplus \mathbf{T}_2} \quad
\text{TYR-TERM-R} \quad \frac{\Gamma \Vdash v : \mathbf{T}_2}{\Gamma \Vdash \text{Inr } v : \mathbf{T}_1 \oplus \mathbf{T}_2} \\
\\
\text{TYR-TERM-P} \quad \frac{\Gamma_1 \Vdash v_1 : \mathbf{T}_1 \quad \Gamma_2 \Vdash v_2 : \mathbf{T}_2}{\Gamma_1 \uplus \Gamma_2 \Vdash (v_1, v_2) : \mathbf{T}_1 \otimes \mathbf{T}_2} \quad
\text{TYR-TERM-E} \quad \frac{\Gamma \Vdash v : \mathbf{T}}{m \cdot \Gamma \Vdash \mathbin{\text{\textit{D}}}^m v : !^m \mathbf{T}} \quad
\text{TYR-TERM-A} \quad \frac{\Gamma_1 \Vdash v_1 : \mathbf{T}_1 \quad \Gamma_2 \Vdash v_2 : \mathbf{T}_2 \quad \text{ctx_DestOnly } \Gamma_1 \quad \text{ctx_SubsetEq } - \Gamma_1 \Gamma_2}{\Gamma_1 \uplus \Gamma_2 \Vdash \langle v_1, v_2 \rangle_{-\Gamma_1} : \mathbf{T}_1 \times \mathbf{T}_2} \quad
\text{TYR-TERM-F} \quad \frac{\Gamma \uplus \{\mathbf{x} : {}^m \mathbf{T}_1\} \Vdash t : \mathbf{T}_2}{\Gamma \Vdash \lambda \mathbf{x} \mapsto t : \mathbf{T}_1 \xrightarrow{m} \mathbf{T}_2} \\
\\
\text{TYR-TERM-VAR} \quad \frac{\text{ctx_Compatible } \Gamma \quad \{\mathbf{x} : {}^{\textcolor{teal}{1}\nu} \mathbf{T}\}}{\Gamma \Vdash \mathbf{x} : \mathbf{T}} \quad
\text{TYR-TERM-APP} \quad \frac{\Gamma_1 \Vdash t : \mathbf{T}_1 \quad \Gamma_2 \Vdash u : \mathbf{T}_1 \xrightarrow{m} \mathbf{T}_2}{m \cdot \Gamma_1 \uplus \Gamma_2 \Vdash t \succ u : \mathbf{T}_2} \quad
\text{TYR-TERM-PATU} \quad \frac{\Gamma_1 \Vdash t : \mathbf{1} \quad \Gamma_2 \Vdash u : \mathbf{U}}{\Gamma_1 \uplus \Gamma_2 \Vdash t ; u : \mathbf{U}} \\
\\
\text{TYR-TERM-PATS} \quad \frac{\Gamma_1 \Vdash t : \mathbf{T}_1 \oplus \mathbf{T}_2 \quad \Gamma_2 \uplus \{\mathbf{x}_1 : {}^m \mathbf{T}_1\} \Vdash u_1 : \mathbf{U} \quad \Gamma_2 \uplus \{\mathbf{x}_2 : {}^m \mathbf{T}_2\} \Vdash u_2 : \mathbf{U}}{m \cdot \Gamma_1 \uplus \Gamma_2 \Vdash t \succ \text{case } \{\text{Inl } \mathbf{x}_1 \mapsto u_1, \text{Inr } \mathbf{x}_2 \mapsto u_2\} : \mathbf{U}} \quad
\text{TYR-TERM-PATP} \quad \frac{\Gamma_1 \Vdash t : \mathbf{T}_1 \otimes \mathbf{T}_2 \quad \Gamma_2 \uplus \{\mathbf{x}_1 : {}^m \mathbf{T}_1, \mathbf{x}_2 : {}^m \mathbf{T}_2\} \Vdash u : \mathbf{U}}{m \cdot \Gamma_1 \uplus \Gamma_2 \Vdash t \succ \text{case } (\mathbf{x}_1, \mathbf{x}_2) \mapsto u : \mathbf{U}} \\
\\
\text{TYR-TERM-PATE} \quad \frac{\Gamma_1 \Vdash t : !^n \mathbf{T} \quad \Gamma_2 \uplus \{\mathbf{x} : {}^{m \cdot n} \mathbf{T}\} \Vdash u : \mathbf{U}}{m \cdot \Gamma_1 \uplus \Gamma_2 \Vdash t \succ \text{case } \mathbin{\text{\textit{D}}}^n \mathbf{x} \mapsto u : \mathbf{U}} \quad
\text{TYR-TERM-MAP} \quad \frac{\Gamma_1 \Vdash t : \mathbf{T}_1 \times \mathbf{T}_2 \quad \textcolor{teal}{I}\Uparrow \Gamma_2 \uplus \{\mathbf{x} : {}^{\textcolor{teal}{1}\nu} \mathbf{T}_1\} \Vdash u : \mathbf{U}}{\Gamma_1 \uplus \Gamma_2 \Vdash t \succ \text{map } \mathbf{x} \mapsto u : \mathbf{U} \times \mathbf{T}_2} \quad
\text{TYR-TERM-FILLC} \quad \frac{\Gamma_1 \Vdash t : {}^n[\mathbf{T}_2] \quad \Gamma_2 \Vdash u : \mathbf{T}_1 \times \mathbf{T}_2}{\Gamma_1 \uplus (\textcolor{teal}{I}\Uparrow \cdot n) \cdot \Gamma_2 \Vdash t \blacktriangleleft u : \mathbf{T}_1} \\
\\
\text{TYR-TERM-FILLU} \quad \frac{}{\Gamma \Vdash t : {}^n[\mathbf{1}]} \quad
\text{TYR-TERM-FILLL} \quad \frac{}{\Gamma \Vdash t : {}^n[\mathbf{T}_1 \oplus \mathbf{T}_2]} \quad
\text{TYR-TERM-FILLR} \quad \frac{}{\Gamma \Vdash t : {}^n[\mathbf{T}_1 \otimes \mathbf{T}_2]} \quad
\text{TYR-TERM-FILLP} \quad \frac{}{\Gamma \Vdash t : {}^n[\mathbf{T}_1 \times \mathbf{T}_2]} \quad
\text{TYR-TERM-FILLE} \quad \frac{}{\Gamma \Vdash t : {}^m[!^n \mathbf{T}]} \\
\\
\text{TYR-TERM-FILLU} \quad \frac{}{\Gamma \Vdash t \triangleleft () : \mathbf{1}} \quad
\text{TYR-TERM-FILLL} \quad \frac{}{\Gamma \Vdash t \triangleleft \text{Inl} : {}^n[\mathbf{T}_1]} \quad
\text{TYR-TERM-FILLR} \quad \frac{}{\Gamma \Vdash t \triangleleft \text{Inr} : {}^n[\mathbf{T}_2]} \quad
\text{TYR-TERM-FILLP} \quad \frac{}{\Gamma \Vdash t \triangleleft (,) : {}^n[\mathbf{T}_1] \otimes {}^n[\mathbf{T}_2]} \quad
\text{TYR-TERM-FILLE} \quad \frac{}{\Gamma \Vdash t \triangleleft \mathbin{\text{\textit{D}}}^n : {}^{m \cdot n}[\mathbf{T}]} \\
\\
\text{TYR-TERM-ALLOC} \quad \frac{}{\{\} \Vdash \text{alloc}_{\mathbf{T}} : {}^{\textcolor{teal}{1}\nu}[\mathbf{T}] \times \mathbf{T}} \quad
\text{TYR-TERM-TOA} \quad \frac{}{\Gamma \Vdash \text{to}_{\times} t : \mathbf{1} \times \mathbf{T}} \quad
\text{TYR-TERM-FROMA} \quad \frac{}{\Gamma \Vdash \text{from}_{\times} t : \mathbf{T}}
\end{array}$$

$$\boxed{\Gamma \vdash t : \mathbf{T}}$$

(Typing of terms (valid ones only))

$$\begin{array}{c}
\text{TY-TERM-T} \\
\frac{\Gamma \Vdash t : \mathbf{T} \quad \text{ctx_Valid } \Gamma \quad \text{ctx_NoHole } \Gamma}{\Gamma \vdash t : \mathbf{T}}
\end{array}$$

3 Effects and big-step semantics

$eff, e ::=$ Effect

ε

has

$e_1 \gg \dots \gg e_k$ Chain effects

$\boxed{\Gamma \Vdash e}$ (Typing of effects (raw))

TYR-EFF-N

$$\frac{\{\} \Vdash \varepsilon}{\text{TYR-EFF-N}}$$

TYR-EFF-A

$$\frac{\Gamma \uplus \Delta \Vdash v : \mathbf{T} \quad \text{ctx_DestOnly } \Gamma \quad \text{ctx_HoleOnly } \Delta}{(\uparrow \cdot m \cdot n) \cdot \Gamma \uplus \{+\mathbf{h} : \textcolor{red}{m} \textcolor{blue}{n}[\mathbf{T}]\} \uplus (m \cdot n) \cdot \Delta \Vdash \mathbf{h} := v}$$

TYR-EFF-C

$$\frac{\Gamma_1 \Vdash e_1 \quad \Gamma_2 \Vdash e_2}{\Gamma_1 \uplus \Gamma_2 \Vdash e_1 \gg e_2}$$

$\boxed{\Gamma \vdash e}$ (Typing of effects (valid ones only))

TY-EFF-T

$$\frac{\Gamma \Vdash e \quad \text{ctx_Valid } \Gamma}{\Gamma \vdash e}$$

$\boxed{\Gamma \vdash v \diamond e : \mathbf{T}}$ (Typing of commands (valid ones only))

TY-CMD-C

$$\frac{\Gamma_1 \vdash v : \mathbf{T} \quad \Gamma_2 \vdash e \quad \text{ctx_DestOnly } \Gamma_1 \uplus \Gamma_2}{\Gamma_1 \uplus \Gamma_2 \vdash v \diamond e : \mathbf{T}}$$

$\boxed{v_1 \Gamma_1 \mid e_1 \Downarrow v_2 \Gamma_2 \mid e_2}$ (Big-step evaluation of effects on values (with potential holes))

SEM-EFF-N

$$\frac{}{v_1 \Gamma_1 \mid \varepsilon \Downarrow v_1 \Gamma_1 \mid \varepsilon}$$

SEM-EFF-S

$$\frac{\text{ctx_HdnmNotMem } \mathbf{h} \Gamma_1 \quad v_1 \Gamma_1 \mid e_1 \Downarrow v_2 \Gamma_2 \mid e_2}{v_1 \Gamma_1 \mid \mathbf{h} := v' \gg e_1 \Downarrow v_2 \Gamma_2 \mid \mathbf{h} := v' \gg e_2}$$

SEM-EFF-F

$$\frac{\Gamma_0 \Vdash v_0 : \mathbf{T} \quad \text{ctx_Valid } \Gamma_0 \quad v_1[\mathbf{h} := v_0] (\Gamma_1 \uplus \textcolor{blue}{n} \cdot \Gamma_0) \mid e_1 \Downarrow v_2 \Gamma_2 \mid e_2}{v_1 \Gamma_1 \uplus \{-\mathbf{h} : \textcolor{blue}{n} \mathbf{T}\} \mid \mathbf{h} := v_0 \gg e_1 \Downarrow v_2 \Gamma_2 \mid e_2}$$

$\boxed{t \Downarrow_d v \diamond e}$ (Big-step evaluation into commands)

SEM-TERM-V

$$\frac{}{v \Downarrow_d v \diamond \varepsilon}$$

SEM-TERM-APP

$$\frac{t_1 \Downarrow_{d.1} v_1 \diamond e_1 \quad t_2 \Downarrow_{d.2} \lambda \mathbf{x} \mapsto u \diamond e_2 \quad u[\mathbf{x} := v_1] \Downarrow_{d.3} v_3 \diamond e_3}{t_1 \succ t_2 \Downarrow_d v_3 \diamond e_1 \gg e_2 \gg e_3}$$

SEM-TERM-PATU

$$\frac{t_1 \Downarrow_{d.1} () \diamond e_1 \quad t_2 \Downarrow_{d.2} v_2 \diamond e_2}{t_1 ; t_2 \Downarrow_d v_2 \diamond e_1 \gg e_2}$$

SEM-TERM-PATL

$$\frac{t \Downarrow_{d.1} \text{Inl } v_1 \diamond e_1 \quad u_1[\mathbf{x}_1 := v_1] \Downarrow_{d.2} v_2 \diamond e_2}{t \succ \text{case } \{\text{Inl } \mathbf{x}_1 \mapsto u_1, \text{Inr } \mathbf{x}_2 \mapsto u_2\} \Downarrow_d v_2 \diamond e_1 \gg e_2}$$

SEM-TERM-PATR

$$\frac{t \Downarrow_{d.1} \text{Inr } v_1 \diamond e_1 \quad u_2[\mathbf{x}_2 := v_1] \Downarrow_{d.2} v_2 \diamond e_2}{t \succ \text{case } \{\text{Inl } \mathbf{x}_1 \mapsto u_1, \text{Inr } \mathbf{x}_2 \mapsto u_2\} \Downarrow_d v_2 \diamond e_1 \gg e_2}$$

SEM-TERM-PATP

$$\frac{t \Downarrow_{d.1} (v_1, v_2) \diamond e_1 \quad u[\mathbf{x}_1 := v_1][\mathbf{x}_2 := v_2] \Downarrow_{d.2} v_2 \diamond e_2}{t \succ \text{case } (\mathbf{x}_1, \mathbf{x}_2) \mapsto u \Downarrow_d v_2 \diamond e_1 \gg e_2}$$

SEM-TERM-MAP

$$\frac{t \Downarrow_{d.1} \langle v_1, v_2 \rangle_{\Delta} \diamond e_1 \quad u[\mathbf{x} := v_1] \Downarrow_{d.2} v_3 \diamond e_2 \quad v_2 \Delta \mid e_2 \Downarrow_d v_4 \Delta' \mid e_3}{t \succ \text{map } \mathbf{x} \mapsto u \Downarrow_d \langle v_3, v_4 \rangle_{\Delta'} \diamond e_1 \gg e_3}$$

SEM-TERM-ALLOC

$$\frac{}{\text{alloc } \mathbf{T} \Downarrow_d \langle +\mathbf{d}, -\mathbf{d} \rangle_{\{-\mathbf{d} : \textcolor{blue}{n} \mathbf{T}\}} \diamond \varepsilon}$$

SEM-TERM-TOA

$$\frac{t \Downarrow_d v \diamond e}{\text{to}_{\mathbf{x}} t \Downarrow_d \langle (), v \rangle_{\{\}} \diamond e}$$

SEM-TERM-FROMA

$$\frac{t \Downarrow_d \langle (), v \rangle_{\{\}} \diamond e}{\text{from}_{\mathbf{x}} t \Downarrow_d v \diamond e}$$

SEM-TERM-FILLU

$$\frac{t \Downarrow_d +\mathbf{h} \diamond e}{t \triangleleft () \Downarrow_d () \diamond e \gg \mathbf{h} := ()}$$

SEM-TERM-FILLL

$$\frac{t \Downarrow_{d.1} +\mathbf{h} \diamond e}{t \triangleleft \text{Inl } \Downarrow_d +\mathbf{d}.2 \diamond e \gg \mathbf{h} := \text{Inl } -\mathbf{d}.2}$$

SEM-TERM-FILLR

$$\frac{t \Downarrow_{d.1} +\mathbf{h} \diamond e}{t \triangleleft \text{Inr } \Downarrow_d +\mathbf{d}.2 \diamond e \gg \mathbf{h} := \text{Inr } -\mathbf{d}.2}$$

SEM-TERM-FILLP

$$\frac{t \Downarrow_{d.1} +\mathbf{h} \diamond e}{t \triangleleft (,) \Downarrow_d (+\mathbf{d}.2, +\mathbf{d}.3) \diamond e \gg \mathbf{h} := (-\mathbf{d}.2, -\mathbf{d}.3)}$$

SEM-TERM-FILLC

$$\frac{t \Downarrow_{d.1} +\mathbf{h} \diamond e \quad u \Downarrow_{d.2} \langle v_1, v_2 \rangle_{\Delta} \diamond e_2}{t \triangleleft \bullet u \Downarrow_d v_1 \diamond e_1 \gg e_2 \gg \mathbf{h} := v_2}$$

4 Type safety

Theorem 1 (Type safety). *If $\text{ctx_DestOnly } \Gamma$ and $\Gamma \vdash t : \mathbf{T}$ then $t \Downarrow_{\mathbf{d}} v \diamond e$ and $\Gamma \vdash v \diamond e : \mathbf{T}$.*

Theorem 2 (Type safety for complete programs). *If $\{\} \vdash t : \mathbf{T}$ then $t \Downarrow_{\star} v \diamond \varepsilon$ and $\{\} \vdash v : \mathbf{T}$*

Proof. By induction on the typing derivation.

- **TYTERM_VAL:** (0) $\Gamma \vdash v : \mathbf{T}$
 (0) gives (1) $v \Downarrow_{\mathbf{d}} v \diamond \varepsilon$ immediately. From **TYEFF_NOEFF** and **TYCMD_CMD** we conclude (2) $\Gamma \vdash v \diamond \varepsilon : \mathbf{T}$.
- **TYTERM_APP:** (0) $m \cdot \Gamma_1 \uplus \Gamma_2 \vdash t \succ u : \mathbf{T}_2$
 We have
 (1) $\Gamma_1 \vdash t : \mathbf{T}_1$
 (2) $\Gamma_2 \vdash u : \mathbf{T}_1 \xrightarrow{m} \mathbf{T}_2$
 (3) $\text{ctx_Disjoint } \Gamma_1 \Gamma_2$
 Using recursion hypothesis on (1) we get (4) $t \Downarrow_{\mathbf{d},1} v_1 \diamond e_1$ where (5) $\Gamma_1 \vdash v_1 \diamond e_1 : \mathbf{T}_1$.
 Inverting **TYCMD_CMD** we get (5) $\Gamma_{11} \uplus \Gamma_{13} \vdash v_1 : \mathbf{T}_1$ and (6) $\Gamma_{12} \uplus -\Gamma_{13} \Vdash e_1$ where (7) $\Gamma_1 = \Gamma_{11} \uplus \Gamma_{12}$.
 Using recursion hypothesis on (2) we get (8) $u \Downarrow_{\mathbf{d},2} v_2 \diamond e_2$ where (9) $\Gamma_2 \vdash v_2 \diamond e_2 : \mathbf{T}_1 \xrightarrow{m} \mathbf{T}_2$.
 Inverting **TYCMD_CMD** we get (10) $\Gamma_{21} \uplus \Gamma_{23} \vdash v_2 : \mathbf{T}_1 \xrightarrow{m} \mathbf{T}_2$ and (11) $\Gamma_{22} \uplus -\Gamma_{23} \Vdash e_2$ where (12) $\Gamma_2 = \Gamma_{21} \uplus \Gamma_{22}$.
 Using Lemma ?? on (9) we get (13) $v_2 = \lambda x \mapsto t'$ and (14) $\Gamma_{21} \uplus \Gamma_{23} \uplus \{x : m \mathbf{T}_1\} \vdash t' : \mathbf{T}_2$.

Typing value part of the result

Using Lemma ?? on (14) and (5) we get (15) $m \cdot (\Gamma_{11} \uplus \Gamma_{13}) \uplus (\Gamma_{21} \uplus \Gamma_{23}) \vdash t'[x := v_1] : \mathbf{T}_2$.

Using recursion hypothesis on (15) we get (16) $t'[x := v_1] \Downarrow_{\mathbf{d},3} v_3 \diamond e_3$ where (17) $m \cdot (\Gamma_{11} \uplus \Gamma_{13}) \uplus (\Gamma_{21} \uplus \Gamma_{23}) \vdash v_3 \diamond e_3 : \mathbf{T}_2$.

Typing effect part of the result

We have

- (6) $\Gamma_{12} \uplus -\Gamma_{13} \Vdash e_1$
- (11) $\Gamma_{22} \uplus -\Gamma_{23} \Vdash e_2$

$\text{ctx_Disjoint } \Gamma_{12} \Gamma_{22}$ comes naturally from (3), (7) and (12).

We must show:

$\text{ctx_Disjoint } \Gamma_{12} \Gamma_{23}$: holes in e_2 (associated to u) are fresh so they cannot match a destination name from t as they don't exist yet when t is evaluated.

$\text{ctx_Disjoint } \Gamma_{22} \Gamma_{13}$: slightly harder. Holes in e_1 (associated to t) are fresh too, so I don't see a way for u to create a term that could mention them, but sequentially, at least, they exist during u evaluation. In fact, Γ_{22} might have intersection with Γ_{13} (see **TYEFF_UNION**) as long as they share the same modalities (it's even harder to prove I think).

$\text{ctx_Disjoint } \Gamma_{13} \Gamma_{23}$: freshness of holes in both effects, executed sequentially, should be enough.

Let say this is solved by Lemma 1, with no holes of e_1 negative context appearing as dests in e_2 positive context.

By **TYEFF_UNION** we get (18) $\Gamma_{12} \uplus \Gamma_{22} \uplus -\Gamma_{13} \uplus -\Gamma_{23} \Vdash e_1 \gg e_2$.

Inverting **TYCMD_CMD** on (17) we get (19) $m \cdot (\Gamma_{111} \uplus \Gamma_{131}) \uplus \Gamma_{211} \uplus \Gamma_{231} \uplus \Gamma_3 \vdash v_3 : \mathbf{T}_2$ and (20) $m \cdot (\Gamma_{112} \uplus \Gamma_{132}) \uplus \Gamma_{212} \uplus \Gamma_{232} \uplus \Gamma_3 \Vdash e_3$ where (21) $\Gamma_{k1} \uplus \Gamma_{k2} = \Gamma_k$

We have

- (18) $\Gamma_{12} \uplus \Gamma_{22} \uplus -\Gamma_{13} \uplus -\Gamma_{23} \Vdash e_1 \gg e_2$
- (20) $m \cdot (\Gamma_{112} \uplus \Gamma_{132}) \uplus \Gamma_{212} \uplus \Gamma_{232} \uplus -\Gamma_3 \Vdash e_3$

Using (21) on (18) to decompose $-\Gamma_{23}$, we get (22) $\Gamma_{12} \uplus \Gamma_{22} \uplus -(\Gamma_{131} \uplus \Gamma_{231}) \uplus -(\Gamma_{132} \uplus \Gamma_{232}) \Vdash e_1 \gg e_2$

We want Γ_{132} from (22) to cancel $m \cdot \Gamma_{132}$ from (20), but the multiplicity doesn't match apparently.

Γ_{13} contains dests associated to holes that may have been created when evaluating t into $v_1 \diamond e_1$. If v_1 is used with delay (result of multiplying its context by m), then should we also delay the RHS of its associated effect? In other terms, if we have $\{+h : \iota \nu \text{ }^n[\mathbf{T}_1 \oplus \mathbf{T}_2]\} \vdash +h' \diamond h := \text{Inl } -h' : \text{ }^n[\mathbf{T}_1]$, and use h' with delay m (e.g stored inside another dest in the body of the function), should we also type the RHS of $h := \text{Inl } -h'$ with delay? I think so, if we want to keep the property that age of dests and age of the associated holes are the same. Which means a more refined substitution lemma.

$$\begin{aligned}
 & (+h_0 \triangleleft (,)) \succ \text{case } (x_1, x_2) \mapsto x_1 \triangleleft (\text{to}_x +h_1) ; x_2 \succ (\lambda x_2 \mapsto +h_3 \triangleleft (\text{to}_x x_2)) \\
 & +h_0 \triangleleft (,)_d \Downarrow (+d.2, +d.3) \diamond h_0 := (-d.2, -d.3) \\
 & (x_1 \triangleleft (\text{to}_x +h_1) ; x_2)[x_1 := +d.2][x_2 := +d.3]_{d'} \Downarrow +d.3 \diamond d.2 := +h_1 \\
 & (+h_0 \triangleleft (,)) \succ \text{case } (x_1, x_2) \mapsto x_1 \triangleleft (\text{to}_x +h_1) ; x_2 \Downarrow_{d''} +d.3 \diamond h_0 := (-d.2, -d.3) \gg d.2 := +h_1 \\
 & (+h_3 \triangleleft (\text{to}_x x_2))[x_2 := +d.3]_{d'''} \Downarrow () \diamond h_3 := +d.3 \\
 & t_{d''''} \Downarrow () \diamond h_0 := (-d.2, -d.3) \gg d.2 := +h_1 \gg h_3 := +d.3
 \end{aligned}$$

Lemma 1 (Freshness of holes). *Let t be a program with no pre-existing ampar sharing hole names.*

During the reduction of t , the only other place where the names of the holes on the RHS of an effect can appear is in the accompanying value of the command, as destinations.

Proof. Names of the holes on the RHS of a new effect:

- either are **fresh** (in all $\text{BIGSTEP_FILL}\langle Ctor \rangle$ rules), which means the only other place where those names are known and can show up is as destinations on the accompanying value of the command (Γ_{12} in TYCMD_CMD), but not in positive or negative contexts of the command given by the evaluation of a sibling subterm;
- or are those of pre-existing holes coming from the extended value v_2 of an ampar, when BIGSTEP_FILLCOMP is evaluated. Because they come from an ampar, they must be neutralized by this ampar, so the left value v_1 of the ampar is the only place where those names can show up, as destinations, if we disallow pre-existing ampar with shared hole names in the body of the initial program \cdot . And v_1 is exactly the accompanying value returned by the evaluation of BIGSTEP_FILLCOMP .

TODO: prove that this property is preserved by typing rules

□