Destination λ-calculus

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1 Term and value syntax

var, x, y	Term-level variable name
k	Index for ranges

$\mathtt{hvar}$, $\mathtt{h}$	$::=$			Hole or destination name ($\mathbb{N}$)
	$\mid$	$\mathtt{h}+\mathtt{h}'$	M	
	$\mid$	$\mathtt{h}[\mathtt{H}\pm\mathtt{h}']$	M	Shift by $\mathtt{h}'$ if $\mathtt{h} \in \mathtt{H}$
	$\mid$	$\mathtt{max}(\mathtt{H})$	M	Maximum of a set of holes
$\mathtt{hvars}$, $\mathtt{H}$	$::=$			Set of hole names
	$\mid$	$\{\mathtt{h}_1, .., \mathtt{h}_{\mathsf{k}}\}$		
	$\mid$	$\mathtt{H}_1 \cup \mathtt{H}_2$	M	Union of sets
	$\mid$	$\mathtt{H}\pm\mathtt{h}'$	M	Shift all names from $\mathtt{H}$ by $\mathtt{h}'$.
	$\mid$	$\mathtt{hvars}(\Gamma)$	M	Hole names of a context (requires $\mathtt{ctx\_NoVar}(\Gamma)$)
	$\mid$	$\mathtt{hvars}(\mathsf{C})$	M	Hole names of an evaluation context
term, t, u	$::=$			Term
	$\mid$	v		Value
	$\mid$	x		Variable
	$\mid$	$\mathsf{t} \succ \mathsf{t}'$		Application
	$\mid$	$\mathsf{t} \,;\, \mathsf{u}$		Pattern-match on unit
	$\mid$	$\mathsf{t} \succ \mathbf{case}_{m}\, \{\, \mathsf{Inl}\, \mathsf{x}_1 \mapsto \mathsf{u}_1\,,\ \mathsf{Inr}\, \mathsf{x}_2 \mapsto \mathsf{u}_2 \,\}$		Pattern-match on sum
	$\mid$	$\mathsf{t} \succ \mathbf{case}_{m}\, (\mathsf{x}_1\,,\, \mathsf{x}_2) \mapsto \mathsf{u}$		Pattern-match on product
	$\mid$	$\mathsf{t} \succ \mathbf{case}_{m}\, \mathrm{E}_{n}\, \mathsf{x} \mapsto \mathsf{u}$		Pattern-match on exponential
	$\mid$	$\mathsf{t} \succ \mathbf{map}\ \mathsf{x} \mapsto \mathsf{t}'$		Map over the right side of ampar t
	$\mid$	$\mathbf{to}_{\ltimes}\ \mathsf{u}$		Wrap u into a trivial ampar
	$\mid$	$\mathbf{from}_{\ltimes}\ \mathsf{t}$		Convert ampar with no dest remaining into pair
	$\mid$	$\mathsf{t} \triangleleft ()$		Fill destination with unit
	$\mid$	$\mathsf{t} \triangleleft \mathsf{Inl}$		Fill destination with left variant
	$\mid$	$\mathsf{t} \triangleleft \mathsf{Inr}$		Fill destination with right variant
	$\mid$	$\mathsf{t} \triangleleft \mathrm{E}_{m}$		Fill destination with exponential constructor
	$\mid$	$\mathsf{t} \triangleleft (,)$		Fill destination with product constructor
	$\mid$	$\mathsf{t} \triangleleft (\lambda \mathsf{x}\ {}_{m}\!\mapsto\ \mathsf{u})$		Fill destination with function
	$\mid$	$\mathsf{t} \triangleleft\!\bullet\, \mathsf{t}'$		Fill destination with root of ampar t'
	$\mid$	$\mathsf{t}[\mathsf{x} := \mathsf{v}]$	M	
val, v	$::=$			Term value
	$\mid$	$-\mathtt{h}$		Hole
	$\mid$	$+\mathtt{h}$		Destination
	$\mid$	$()$		Unit
	$\mid$	$\lambda^{\mathsf{v}} \mathsf{x}\ {}_{m}\!\mapsto\ \mathsf{u}$		Lambda abstraction
	$\mid$	$\mathsf{Inl}\, \mathsf{v}$		Left variant for sum
	$\mid$	$\mathsf{Inr}\, \mathsf{v}$		Right variant for sum
	$\mid$	$\mathrm{E}_{m}\, \mathsf{v}$		Exponential
	$\mid$	$(\mathsf{v}_1\,,\, \mathsf{v}_2)$		Product
	$\mid$	${}_{\mathtt{H}}\langle \mathsf{v}_2 \,,\, \mathsf{v}_1 \rangle$		Ampar
	$\mid$	$\mathsf{v}[\mathtt{H}\pm\mathtt{h}']$	M	Shift hole names inside v by $\mathtt{h}'$ if they belong to $\mathtt{H}$.

ectx, c	::=			Evaluation context component
	|	$\square \succ t'$		Application
	|	$v \succ \square$		Application
	|	$\square \; ; \; u$		Pattern-match on unit
	|	$\square \succ \mathsf{case}_{m} \{ \mathsf{Inl}\, x_1 \mapsto u_1 \,,\; \mathsf{Inr}\, x_2 \mapsto u_2 \}$		Pattern-match on sum
	|	$\square \succ \mathsf{case}_{m} (x_1 \,,\, x_2) \mapsto u$		Pattern-match on product
	|	$\square \succ \mathsf{case}_{m}\, \mathrm{E}_{n}\, x \mapsto u$		Pattern-match on exponential
	|	$\square \succ \mathsf{map}\; x \mapsto t'$		Map over the right side of ampar
	|	$\mathsf{to}_{\ltimes}\; \square$		Wrap into a trivial ampar
	|	$\mathsf{from}_{\ltimes}\; \square$		Convert ampar with no dest remaining into pair
	|	$\square \triangleleft ()$		Fill destination with unit
	|	$\square \triangleleft \mathsf{Inl}$		Fill destination with left variant
	|	$\square \triangleleft \mathsf{Inr}$		Fill destination with right variant
	|	$\square \triangleleft \mathrm{E}_{m}$		Fill destination with exponential constructor
	|	$\square \triangleleft (,)$		Fill destination with product constructor
	|	$\square \triangleleft (\lambda x \,{}_{m}\!\mapsto\; u)$		Fill destination with function
	|	$\square \triangleleft\!\bullet\, t'$		Fill destination with root of ampar
	|	$v \triangleleft\!\bullet\, \square$		Fill destination with root of ampar
	|	${}^{\mathrm{op}}_{\mathrm{H}}\langle v_2 \,⨟\, \square$		Open ampar. Only new addition to term shapes
ectxs, C	::=			Evaluation context stack
	|	$\square$		Represent the empty stack / "identity" evaluation context
	|	$C \circ c$		Push c on top of C
	|	$C[h :=_{\mathrm{H}} v]$	M	Fill h in C with value v (that may contain holes)

2 Type system

type, **T**, **U**	::=			Type
	|	$\mathbf{1}$		Unit
	|	$\mathsf{T}_1 \oplus \mathsf{T}_2$		Sum
	|	$\mathsf{T}_1 \otimes \mathsf{T}_2$		Product
	|	$!_{m}\, \mathsf{T}$		Exponential
	|	$\mathsf{U} \ltimes \mathsf{T}$		Ampar type (consuming **T** yields **U**)
	|	$\mathsf{T} \;{}_{m}\!\!\to \mathsf{U}$		Function
	|	$\lfloor \mathsf{T} \rfloor^{m}$		Destination
mode, *m*, *n*	::=			Mode (Semiring)
	|	pa		Pair of a multiplicity and age
	|	☠		Error case (incompatible types, multiplicities, or ages)
	|	$m_1 \cdot \ldots \cdot m_k$	M	Semiring product
mul, *p*	::=			Multiplicity (first component of modality)
	|	1		Linear. Neutral element of the product
	|	ω		Non-linear. Absorbing for the product
	|	$p_1 . \ldots . p_k$	M	Semiring product
age, *a*	::=			Age (second component of modality)
	|	ν		Born now. Neutral element of the product
	|	$\uparrow$		One scope older
	|	∞		Infinitely old / static. Absorbing for the product
	|	$a_1 \cdot \ldots \cdot a_k$	M	Semiring product
ctx, Γ, Δ, Π	::=			Typing context
	|	$x :_{m} \mathsf{T}$		
	|	$+h :_{m} \lfloor \mathsf{T} \rfloor^{n}$		
	|	$-h : \mathsf{T}^{n}$		
	|	$m \cdot \Gamma$	M	Multiply each binding by m
	|	$\Gamma_1 + \Gamma_2$	M	Sum contexts Γ_1 and Γ_2. Duplicate keys with incompatible values will be tagged wi
	|	$\Gamma_1,\; \Gamma_2$	M	Disjoint sum/union of contexts Γ_1 and Γ_2.
	|	$-\Gamma$	M	Transforms dest bindings into a hole bindings (requires `ctx_DestOnly` Γ and `ctx_L`

$\mid\ {-}^{-1}\Gamma$	M	Transforms hole bindings into dest bindings with left mode 1ν (requires `ctx_HoleOnly` Γ)
$\mid\ \Gamma[\mathtt{H}{\pm}\mathtt{h}']$	M	Shift hole/dest names by $\mathtt{h}'$ if they belong to $\mathtt{H}$

$\boxed{\Gamma \Vdash v : T}$ *(Typing of values (raw))*

$$\textsc{TyR-val-H}\quad \frac{}{-\mathtt{h} : T^{1\nu} \Vdash -\mathtt{h} : T} \qquad \textsc{TyR-val-D}\quad \frac{}{+\mathtt{h} :_{1\nu} \lfloor T \rfloor^{n} \Vdash +\mathtt{h} : \lfloor T \rfloor^{n}} \qquad \textsc{TyR-val-U}\quad \frac{}{\Vdash () : 1}$$

$$\textsc{TyR-val-F}\quad \frac{\Delta + x :_m T \vdash u : U}{\Delta \Vdash \lambda^{v} x \,{}_{m}{\mapsto}\, u : T \,{}_{m}{\to}\, U} \qquad \textsc{TyR-val-L}\quad \frac{\Gamma \Vdash v_1 : T_1}{\Gamma \Vdash \mathsf{Inl}\, v_1 : T_1 \oplus T_2}$$

$$\textsc{TyR-val-R}\quad \frac{\Gamma \Vdash v_2 : T_2}{\Gamma \Vdash \mathsf{Inr}\, v_2 : T_1 \oplus T_2} \qquad \textsc{TyR-val-P}\quad \frac{\Gamma_1 \Vdash v_1 : T_1 \qquad \Gamma_2 \Vdash v_2 : T_2}{\Gamma_1 + \Gamma_2 \Vdash (v_1 , v_2) : T_1 \otimes T_2} \qquad \textsc{TyR-val-E}\quad \frac{\Gamma \Vdash v' : T}{n\cdot\Gamma \Vdash \mathrm{E}_n\, v' : !_n T}$$

$$\textsc{TyR-val-A}\quad \frac{\mathtt{LinOnly}\ \Delta_3 \qquad \mathtt{FinAgeOnly}\ \Delta_3 \qquad 1{\uparrow}\cdot\Delta_1, \ \Delta_3 \Vdash v_1 : T \qquad \Delta_2, \ (-\Delta_3) \Vdash v_2 : U}{\Delta_1, \ \Delta_2 \Vdash_{\mathtt{hvars}(-\Delta_3)} \langle v_2 \,;\, v_1 \rangle : U \ltimes T}$$

$\boxed{\Pi \vdash t : T}$ *(Typing of terms)*

$$\textsc{Ty-term-Val}\quad \frac{\mathtt{DisposableOnly}\ \Pi \qquad \Delta \Vdash v : T}{\Pi, \ \Delta \vdash v : T} \qquad \textsc{Ty-term-Var}\quad \frac{\mathtt{DisposableOnly}\ \Pi \qquad 1\nu \ <: \ m}{\Pi, \ x :_m T \vdash x : T}$$

$$\textsc{Ty-term-App}\quad \frac{\Pi_1 \vdash t : T \qquad \Pi_2 \vdash t' : T \,{}_{m}{\to}\, U}{m\cdot\Pi_1 + \Pi_2 \vdash t \succ t' : U}$$

$$\textsc{Ty-term-PatU}\quad \frac{\Pi_1 \vdash t : 1 \qquad \Pi_2 \vdash u : U}{\Pi_1 + \Pi_2 \vdash t \,;\, u : U}$$

$$\textsc{Ty-term-PatS}\quad \frac{\Pi_1 \vdash t : T_1 \oplus T_2 \qquad \Pi_2, \ x_1 :_m T_1 \vdash u_1 : U \qquad \Pi_2, \ x_2 :_m T_2 \vdash u_2 : U}{m\cdot\Pi_1 + \Pi_2 \vdash t \succ \mathsf{case}_m \{ \mathsf{Inl}\, x_1 \mapsto u_1 , \ \mathsf{Inr}\, x_2 \mapsto u_2 \} : U}$$

$$\textsc{Ty-term-PatP}\quad \frac{\Pi_1 \vdash t : T_1 \otimes T_2 \qquad \Pi_2, \ x_1 :_m T_1, \ x_2 :_m T_2 \vdash u : U}{m\cdot\Pi_1 + \Pi_2 \vdash t \succ \mathsf{case}_m\, (x_1 , x_2) \mapsto u : U}$$

$$\textsc{Ty-term-PatE}\quad \frac{\Pi_1 \vdash t : !_n T \qquad \Pi_2, \ x :_{m\cdot n} T \vdash u : U}{m\cdot\Pi_1 + \Pi_2 \vdash t \succ \mathsf{case}_m\, \mathrm{E}_n\, x \mapsto u : U}$$

$$\textsc{Ty-term-Map}\quad \frac{\Pi_1 \vdash t : U \ltimes T \qquad 1{\uparrow}\cdot\Pi_2, \ x :_{1\nu} T \vdash t' : T'}{\Pi_1 + \Pi_2 \vdash t \succ \mathsf{map}\ x \mapsto t' : U \ltimes T'}$$

$$\textsc{Ty-term-ToA}\quad \frac{\Pi \vdash u : U}{\Pi \vdash \mathsf{to}_{\ltimes}\, u : U \ltimes 1} \qquad \textsc{Ty-term-FromA}\quad \frac{\Pi \vdash t : U \ltimes (!_{\omega\nu} T)}{\Pi \vdash \mathsf{from}_{\ltimes}\, t : U \otimes (!_{\omega\nu} T)} \qquad \textsc{Ty-term-FillU}\quad \frac{\Pi \vdash t : \lfloor 1 \rfloor^{n}}{\Pi \vdash t \triangleleft () : 1}$$

$$\textsc{Ty-term-FillL}\quad \frac{\Pi \vdash t : \lfloor T_1 \oplus T_2 \rfloor^{n}}{\Pi \vdash t \triangleleft \mathsf{Inl} : \lfloor T_1 \rfloor^{n}} \qquad \textsc{Ty-term-FillR}\quad \frac{\Pi \vdash t : \lfloor T_1 \oplus T_2 \rfloor^{n}}{\Pi \vdash t \triangleleft \mathsf{Inr} : \lfloor T_2 \rfloor^{n}}$$

$$\textsc{Ty-term-FillP}\quad \frac{\Pi \vdash t : \lfloor T_1 \otimes T_2 \rfloor^{n}}{\Pi \vdash t \triangleleft (,) : \lfloor T_1 \rfloor^{n} \otimes \lfloor T_2 \rfloor^{n}} \qquad \textsc{Ty-term-FillE}\quad \frac{\Pi \vdash t : \lfloor !_{n'} T \rfloor^{n}}{\Pi \vdash t \triangleleft \mathrm{E}_{n'} : \lfloor T \rfloor^{n'\cdot n}}$$

$$\textsc{Ty-term-FillF}\quad \frac{\Pi_1 \vdash t : \lfloor T \,{}_{m}{\to}\, U \rfloor^{n} \qquad \Pi_2, \ x :_m T \vdash u : U}{\Pi_1 + (1{\uparrow}\cdot n)\cdot\Pi_2 \vdash t \triangleleft (\lambda x \,{}_{m}{\mapsto}\, u) : 1}$$

$$\textsc{Ty-term-FillC}\quad \frac{\Pi_1 \vdash t : \lfloor U \rfloor^{n} \qquad \Pi_2 \vdash t' : U \ltimes T}{\Pi_1 + (1{\uparrow}\cdot n)\cdot\Pi_2 \vdash t \triangleleft\!\bullet\, t' : T}$$

$\boxed{\Delta \dashv C : T \rightarrowtail U_0}$ *(Typing of evaluation contexts)*

$$\textsc{Ty-ectxs-Id}\quad \frac{}{\dashv \square : U_0 \rightarrowtail U_0}$$

$$\textsc{Ty-ectxs-AppFoc1}\quad \frac{m\cdot\Delta_1, \ \Delta_2 \dashv C : U \rightarrowtail U_0 \qquad \Delta_2 \vdash t' : T \,{}_{m}{\to}\, U}{\Delta_1 \dashv C \circ (\square \succ t') : T \rightarrowtail U_0}$$

$$\textsc{Ty-ectxs-AppFoc2}\quad \frac{m\cdot\Delta_1, \ \Delta_2 \dashv C : U \rightarrowtail U_0 \qquad \Delta_1 \vdash v : T}{\Delta_2 \dashv C \circ (v \succ \square) : (T \,{}_{m}{\to}\, U) \rightarrowtail U_0}$$

$$\textsc{Ty-ectxs-PatUFoc}\quad \frac{\Delta_1, \ \Delta_2 \dashv C : U \rightarrowtail U_0 \qquad \Delta_2 \vdash u : U}{\Delta_1 \dashv C \circ (\square \,;\, u) : 1 \rightarrowtail U_0}$$

$$\textsc{Ty-ectxs-PatSFoc}\quad \frac{m\cdot\Delta_1, \ \Delta_2 \dashv C : U \rightarrowtail U_0 \qquad \Delta_2, \ x_1 :_m T_1 \vdash u_1 : U \qquad \Delta_2, \ x_2 :_m T_2 \vdash u_2 : U}{\Delta_1 \dashv C \circ (\square \succ \mathsf{case}_m \{ \mathsf{Inl}\, x_1 \mapsto u_1 , \ \mathsf{Inr}\, x_2 \mapsto u_2 \}) : (T_1 \oplus T_2) \rightarrowtail U_0}$$

$$\textsc{Ty-ectxs-PatPFoc}\quad \frac{m\cdot\Delta_1, \ \Delta_2 \dashv C : U \rightarrowtail U_0 \qquad \Delta_2, \ x_1 :_m T_1, \ x_2 :_m T_2 \vdash u : U}{\Delta_1 \dashv C \circ (\square \succ \mathsf{case}_m\, (x_1 , x_2) \mapsto u) : (T_1 \otimes T_2) \rightarrowtail U_0}$$

$$\textsc{Ty-ectxs-PatEFoc}\quad \frac{m\cdot\Delta_1, \ \Delta_2 \dashv C : U \rightarrowtail U_0 \qquad \Delta_2, \ x :_{m\cdot m'} T \vdash u : U}{\Delta_1 \dashv C \circ (\square \succ \mathsf{case}_m\, \mathrm{E}_{m'}\, x \mapsto u) : !_{m'} T \rightarrowtail U_0}$$

$$\textsc{Ty-ectxs-MapFoc}\quad \frac{\Delta_1,\ \Delta_2 \dashv C : \mathsf{U} \ltimes \mathsf{T}' \rightarrowtail \mathsf{U}_0 \qquad 1{\uparrow}{\cdot}\Delta_2,\ x :_{1\nu} \mathsf{T} \vdash t' : \mathsf{T}'}{\Delta_1 \dashv C \circ (\square \succ \mathbf{map}\ x \mapsto t') : (\mathsf{U} \ltimes \mathsf{T}) \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-ToAFoc}\quad \frac{\Delta \dashv C : (\mathsf{U} \ltimes \mathsf{1}) \rightarrowtail \mathsf{U}_0}{\Delta \dashv C \circ (\mathbf{to}_{\ltimes}\ \square) : \mathsf{U} \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-FromAFoc}\quad \frac{\Delta \dashv C : (\mathsf{U} \otimes (!_{\omega\nu}\, \mathsf{T})) \rightarrowtail \mathsf{U}_0}{\Delta \dashv C \circ (\mathbf{from}_{\ltimes}\ \square) : (\mathsf{U} \ltimes (!_{\omega\nu}\, \mathsf{T})) \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-FillUFoc}\quad \frac{\Delta \dashv C : \mathsf{1} \rightarrowtail \mathsf{U}_0}{\Delta \dashv C \circ (\square \triangleleft ()) : \lfloor \mathsf{1} \rfloor^{n} \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-FillLFoc}\quad \frac{\Delta \dashv C : \lfloor \mathsf{T}_1 \rfloor^{n} \rightarrowtail \mathsf{U}_0}{\Delta \dashv C \circ (\square \triangleleft \mathsf{Inl}) : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^{n} \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-FillRFoc}\quad \frac{\Delta \dashv C : \lfloor \mathsf{T}_2 \rfloor^{n} \rightarrowtail \mathsf{U}_0}{\Delta \dashv C \circ (\square \triangleleft \mathsf{Inr}) : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^{n} \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-FillPFoc}\quad \frac{\Delta \dashv C : (\lfloor \mathsf{T}_1 \rfloor^{n} \otimes \lfloor \mathsf{T}_2 \rfloor^{n}) \rightarrowtail \mathsf{U}_0}{\Delta \dashv C \circ (\square \triangleleft (,)) : \lfloor \mathsf{T}_1 \otimes \mathsf{T}_2 \rfloor^{n} \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-FillEFoc}\quad \frac{\Delta \dashv C : \lfloor \mathsf{T} \rfloor^{m \cdot n} \rightarrowtail \mathsf{U}_0}{\Delta \dashv C \circ (\square \triangleleft \mathrm{E}_m) : \lfloor !_m\, \mathsf{T} \rfloor^{n} \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-FillFFoc}\quad \frac{\Delta_1,\ (1{\uparrow}{\cdot}n){\cdot}\Delta_2 \dashv C : \mathsf{1} \rightarrowtail \mathsf{U}_0 \qquad \Delta_2,\ x :_m \mathsf{T} \vdash u : \mathsf{U}}{\Delta_1 \dashv C \circ (\square \triangleleft (\lambda x\ {}_m\!\mapsto u)) : \lfloor \mathsf{T}\ {}_m\!\to \mathsf{U} \rfloor^{n} \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-FillCFoc1}\quad \frac{\Delta_1,\ (1{\uparrow}{\cdot}n){\cdot}\Delta_2 \dashv C : \mathsf{T} \rightarrowtail \mathsf{U}_0 \qquad \Delta_2 \vdash t' : \mathsf{U} \ltimes \mathsf{T}}{\Delta_1 \dashv C \circ (\square \triangleleft\!\cdot\, t') : \lfloor \mathsf{U} \rfloor^{n} \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-FillCFoc2}\quad \frac{\Delta_1,\ (1{\uparrow}{\cdot}n){\cdot}\Delta_2 \dashv C : \mathsf{T} \rightarrowtail \mathsf{U}_0 \qquad \Delta_1 \vdash v : \lfloor \mathsf{U} \rfloor^{n}}{\Delta_2 \dashv C \circ (v \triangleleft\!\bullet\, \square) : \mathsf{U} \ltimes \mathsf{T} \rightarrowtail \mathsf{U}_0}$$

$$\textsc{Ty-ectxs-AOpenFoc}\quad \frac{\begin{array}{c} \mathtt{hvars}(C)\ \mathtt{\#\#}\ \mathtt{hvars}(-\Delta_3) \\ \mathtt{LinOnly}\ \Delta_3 \qquad \mathtt{FinAgeOnly}\ \Delta_3 \\ \Delta_1,\ \Delta_2 \dashv C : (\mathsf{U} \ltimes \mathsf{T}') \rightarrowtail \mathsf{U}_0 \\ \Delta_2,\ -\Delta_3 \Vdash v_2 : \mathsf{U} \end{array}}{1{\uparrow}{\cdot}\Delta_1,\ \Delta_3 \dashv C \circ ({}^{\mathrm{op}}_{\mathtt{hvars}(-\Delta_3)}\langle v_2 \mathbin{,} \square) : \mathsf{T}' \rightarrowtail \mathsf{U}_0}$$

$\boxed{\vdash C[t] : \mathsf{T}}$ *(Typing of extended terms (pair of evaluation context and term))*

$$\textsc{Ty-eterm-ClosedEterm}\quad \frac{\Delta \dashv C : \mathsf{T} \rightarrowtail \mathsf{U}_0 \qquad \Delta \vdash t : \mathsf{T}}{\vdash C[t] : \mathsf{U}_0}$$

3 Small-step semantics

$\boxed{C[t] \longrightarrow C'[t']}$ *(Small-step evaluation of terms using evaluation contexts)*

$$\textsc{Sem-eterm-AppFoc1}\quad \frac{\mathtt{NotVal}\ t}{C[t \succ t'] \longrightarrow (C \circ (\square \succ t'))[t]}$$

$$\textsc{Sem-eterm-AppUnfoc1}\quad \frac{}{(C \circ (\square \succ t'))[v] \longrightarrow C[v \succ t']}$$

$$\textsc{Sem-eterm-AppFoc2}\quad \frac{\mathtt{NotVal}\ t'}{C[v \succ t'] \longrightarrow (C \circ (v \succ \square))[t']}$$

$$\textsc{Sem-eterm-AppUnfoc2}\quad \frac{}{(C \circ (v \succ \square))[v'] \longrightarrow C[v \succ v']}$$

$$\textsc{Sem-eterm-AppRed}\quad \frac{}{C[v \succ (\lambda^{v} x\ {}_{m}\!\mapsto u)] \longrightarrow C[u[x := v]]}$$

$$\textsc{Sem-eterm-PatUFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t\ ;\ u] \longrightarrow (C \circ (\square\ ;\ u))[t]}$$

$$\textsc{Sem-eterm-PatUUnfoc}\quad \frac{}{(C \circ (\square\ ;\ u))[v] \longrightarrow C[v\ ;\ u]}$$

$$\textsc{Sem-eterm-PatURed}\quad \frac{}{C[()\ ;\ u] \longrightarrow C[u]}$$

$$\textsc{Sem-eterm-PatSFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t \succ \mathbf{case}_{m}\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}] \longrightarrow (C \circ (\square \succ \mathbf{case}_{m}\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}))[t]}$$

$$\textsc{Sem-eterm-PatSUnfoc}\quad \frac{}{(C \circ (\square \succ \mathbf{case}_{m}\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}))[v] \longrightarrow C[v \succ \mathbf{case}_{m}\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}]}$$

$$\textsc{Sem-eterm-PatLRed}\quad \frac{}{C[(\mathsf{Inl}\,v_1) \succ \mathbf{case}_{m}\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}] \longrightarrow C[u_1[x_1 := v_1]]}$$

$$\textsc{Sem-eterm-PatRRed}\quad \frac{}{C[(\mathsf{Inr}\,v_2) \succ \mathbf{case}_{m}\,\{\,\mathsf{Inl}\,x_1 \mapsto u_1\,,\ \mathsf{Inr}\,x_2 \mapsto u_2\,\}] \longrightarrow C[u_2[x_2 := v_2]]}$$

$$\textsc{Sem-eterm-PatPFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t \succ \mathbf{case}_{m}\,(x_1\,,\,x_2) \mapsto u] \longrightarrow (C \circ (\square \succ \mathbf{case}_{m}\,(x_1\,,\,x_2) \mapsto u))[t]}$$

$$\textsc{Sem-eterm-PatPUnfoc}\quad \frac{}{(C \circ (\square \succ \mathbf{case}_{m}\,(x_1\,,\,x_2) \mapsto u))[v] \longrightarrow C[v \succ \mathbf{case}_{m}\,(x_1\,,\,x_2) \mapsto u]}$$

$$\textsc{Sem-eterm-PatPRed}\quad \frac{}{C[(v_1\,,\,v_2) \succ \mathbf{case}_{m}\,(x_1\,,\,x_2) \mapsto u] \longrightarrow C[u[x_1 := v_1][x_2 := v_2]]}$$

$$\textsc{Sem-eterm-PatEFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t \succ \mathbf{case}_{m}\,\mathrm{E}_{n}\,x \mapsto u] \longrightarrow (C \circ (\square \succ \mathbf{case}_{m}\,\mathrm{E}_{n}\,x \mapsto u))[t]}$$

$$\textsc{Sem-eterm-PatEUnfoc}\quad \frac{}{(C \circ (\square \succ \mathbf{case}_{m}\,\mathrm{E}_{n}\,x \mapsto u))[v] \longrightarrow C[v \succ \mathbf{case}_{m}\,\mathrm{E}_{n}\,x \mapsto u]}$$

$$\textsc{Sem-eterm-PatERed}\quad \frac{}{C[\mathrm{E}_{n}\,v' \succ \mathbf{case}_{m}\,\mathrm{E}_{n}\,x \mapsto u] \longrightarrow C[u[x := v']]}$$

$$\textsc{Sem-eterm-MapFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t \succ \mathbf{map}\ x \mapsto t'] \longrightarrow (C \circ (\square \succ \mathbf{map}\ x \mapsto t'))[t]}$$

$$\textsc{Sem-eterm-MapUnfoc}\quad \frac{}{(C \circ (\square \succ \mathbf{map}\ x \mapsto t'))[v] \longrightarrow C[v \succ \mathbf{map}\ x \mapsto t']}$$

$$\textsc{Sem-eterm-MapRedAOpenFoc}\quad \frac{\mathtt{h'} = \mathtt{max(hvars(C))+1}}{C[{}_{\mathtt{H}}\langle v_2\,,\,v_1\rangle \succ \mathbf{map}\ x \mapsto t'] \longrightarrow (C \circ ({}^{\mathrm{op}}_{\mathtt{H}\pm\mathtt{h'}}\langle v_2[\mathtt{H}\pm\mathtt{h'}]\,,\,\square\rangle))[t'[x := v_1[\mathtt{H}\pm\mathtt{h'}]]]}$$

$$\textsc{Sem-eterm-AOpenUnfoc}\quad \frac{}{(C \circ {}^{\mathrm{op}}_{\mathtt{H}}\langle v_2\,,\,\square\rangle)[v_1] \longrightarrow C[{}_{\mathtt{H}}\langle v_2\,,\,v_1\rangle]}$$

$$\textsc{Sem-eterm-ToAFoc}\quad \frac{\mathtt{NotVal}\ u}{C[\mathbf{to}_{\ltimes}\ u] \longrightarrow (C \circ (\mathbf{to}_{\ltimes}\ \square))[u]}$$

$$\textsc{Sem-eterm-ToAUnfoc}\quad \frac{}{(C \circ (\mathbf{to}_{\ltimes}\ \square))[v_2] \longrightarrow C[\mathbf{to}_{\ltimes}\ v_2]}$$

$$\textsc{Sem-eterm-ToARed}\quad \frac{}{C[\mathbf{to}_{\ltimes}\ v_2] \longrightarrow C[{}_{\{\,\}}\langle v_2\,,\,()\rangle]}$$

$$\textsc{Sem-eterm-FromAFoc}\quad \frac{\mathtt{NotVal}\ t}{C[\mathbf{from}_{\ltimes}\ t] \longrightarrow (C \circ (\mathbf{from}_{\ltimes}\ \square))[t]}$$

$$\textsc{Sem-eterm-FromAUnfoc}\quad \frac{}{(C \circ (\mathbf{from}_{\ltimes}\ \square))[v] \longrightarrow C[\mathbf{from}_{\ltimes}\ v]}$$

$$\textsc{Sem-eterm-FromARed}\quad \frac{}{C[\mathbf{from}_{\ltimes}\ {}_{\{\,\}}\langle v_2\,,\,\mathrm{E}_{\omega\nu}\ v_1\rangle] \longrightarrow C[(v_2\,,\,\mathrm{E}_{\omega\nu}\ v_1)]}$$

$$\textsc{Sem-eterm-FillUFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t \triangleleft ()] \longrightarrow (C \circ (\square \triangleleft ()))[t]}$$

$$\textsc{Sem-eterm-FillUUnfoc}\quad \frac{}{(C \circ (\square \triangleleft ()))[v] \longrightarrow C[v \triangleleft ()]}$$

$$\textsc{Sem-eterm-FillURed}\quad \frac{}{C[+\mathtt{h} \triangleleft ()] \longrightarrow C[\mathtt{h} :=_{\{\,\}}\ ()][()]}$$

$$\textsc{Sem-eterm-FillLFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t \triangleleft \mathsf{Inl}] \longrightarrow (C \circ (\square \triangleleft \mathsf{Inl}))[t]}$$

$$\textsc{Sem-eterm-FillLUnfoc}\quad \frac{}{(C \circ (\square \triangleleft \mathsf{Inl}))[v] \longrightarrow C[v \triangleleft \mathsf{Inl}]}$$

$$\textsc{Sem-eterm-FillLRed}\quad \frac{h' = \max(\mathrm{hvars}(C) \cup \{h\})+1}{C[+h \triangleleft \mathsf{Inl}] \longrightarrow C[h \coloneqq_{\{h'+1\}} \mathsf{Inl}\ -(h'+1)][+(h'+1)]}$$

$$\textsc{Sem-eterm-FillRFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t \triangleleft \mathsf{Inr}] \longrightarrow (C \circ (\square \triangleleft \mathsf{Inr}))[t]}$$

$$\textsc{Sem-eterm-FillRUnfoc}\quad \frac{}{(C \circ (\square \triangleleft \mathsf{Inr}))[v] \longrightarrow C[v \triangleleft \mathsf{Inr}]}$$

$$\textsc{Sem-eterm-FillRRed}\quad \frac{h' = \max(\mathrm{hvars}(C) \cup \{h\})+1}{C[+h \triangleleft \mathsf{Inr}] \longrightarrow C[h \coloneqq_{\{h'+1\}} \mathsf{Inr}\ -(h'+1)][+(h'+1)]}$$

$$\textsc{Sem-eterm-FillEFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t \triangleleft \mathrm{E}_m] \longrightarrow (C \circ (\square \triangleleft \mathrm{E}_m))[t]}$$

$$\textsc{Sem-eterm-FillEUnfoc}\quad \frac{}{(C \circ (\square \triangleleft \mathrm{E}_m))[v] \longrightarrow C[v \triangleleft \mathrm{E}_m]}$$

$$\textsc{Sem-eterm-FillERed}\quad \frac{h' = \max(\mathrm{hvars}(C) \cup \{h\})+1}{C[+h \triangleleft \mathrm{E}_m] \longrightarrow C[h \coloneqq_{\{h'+1\}} \mathrm{E}_m\ -(h'+1)][+(h'+1)]}$$

$$\textsc{Sem-eterm-FillPFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t \triangleleft (,)] \longrightarrow (C \circ (\square \triangleleft (,)))[t]}$$

$$\textsc{Sem-eterm-FillPUnfoc}\quad \frac{}{(C \circ (\square \triangleleft (,)))[v] \longrightarrow C[v \triangleleft (,)]}$$

$$\textsc{Sem-eterm-FillPRed}\quad \frac{h' = \max(\mathrm{hvars}(C) \cup \{h\})+1}{C[+h \triangleleft (,)] \longrightarrow C[h \coloneqq_{\{h'+1,h'+2\}} (-(h'+1)\,,\ -(h'+2))][(+(h'+1)\,,\ +(h'+2))]}$$

$$\textsc{Sem-eterm-FillFFoc}\quad \frac{\mathtt{NotVal}\ t}{C[t \triangleleft (\lambda x\,{}_m\!\mapsto u)] \longrightarrow (C \circ (\square \triangleleft (\lambda x\,{}_m\!\mapsto u)))[t]}$$

$$\textsc{Sem-eterm-FillFUnfoc}\quad \frac{}{(C \circ (\square \triangleleft (\lambda x\,{}_m\!\mapsto u)))[v] \longrightarrow C[v \triangleleft (\lambda x\,{}_m\!\mapsto u)]}$$

$$\textsc{Sem-eterm-FillFRed}\quad \frac{}{C[+h \triangleleft (\lambda x\,{}_m\!\mapsto u)] \longrightarrow C[h \coloneqq_{\{\,\}} \lambda^{\mathsf{v}} x\,{}_m\!\mapsto u][()]}$$

$$\textsc{Sem-eterm-FillCFoc1}\quad \frac{\mathtt{NotVal}\ t}{C[t \triangleleft\!\bullet\, t'] \longrightarrow (C \circ (\square \triangleleft\!\bullet\, t'))[t]}$$

$$\textsc{Sem-eterm-FillCUnfoc1}\quad \frac{}{(C \circ (\square \triangleleft\!\bullet\, t'))[v] \longrightarrow C[v \triangleleft\!\bullet\, t']}$$

$$\textsc{Sem-eterm-FillCFoc2}\quad \frac{\mathtt{NotVal}\ t'}{C[v \triangleleft\!\bullet\, t'] \longrightarrow (C \circ (v \triangleleft\!\bullet\, \square))[t']}$$

$$\textsc{Sem-eterm-FillCUnfoc2}\quad \frac{}{(C \circ (v \triangleleft\!\bullet\, \square))[v'] \longrightarrow C[v \triangleleft\!\bullet\, v']}$$

$$\textsc{Sem-eterm-FillCRed}\quad \frac{h' = \max(\mathrm{hvars}(C) \cup \{h\})+1}{C[+h \triangleleft\!\bullet_{H} \langle v_2\,, v_1\rangle] \longrightarrow C[h \coloneqq_{(H \pm h')} v_2[H \pm h']][v_1[H \pm h']]}$$

4 Remarks on the Coq proofs

- Not particularly elegant. Max number of goals observed 232 (solved by a single call to the `congruence` tactic). When you have a computer, brute force is a viable strategy. (in particular, no semiring formalisation, it was quicker to do directly)
- Rules generated by ott, same as in the article (up to some notational difference). Contexts are not generated purely by syntax, and are interpreted in a semantic domain (finite functions).
- Reasoning on closed terms avoids almost all complications on binder manipulation. Makes proofs tractable.
- Finite functions: making a custom library was less headache than using existing libraries (including `MMap`). Existing libraries don't provide some of the tools that we needed, but the most important factor ended up being the need for a modicum of dependency between key and value. There wasn't really that out there. Backed by actual functions for simplicity; cost: equality is complicated.
- Most of the proofs done by author with very little prior experience to Coq.
- Did proofs in Coq because context manipulations are tricky.
- Context sum made total by adding an extra invalid *mode* (rather than an extra context). It seems to be much simpler this way.
- It might be a good idea to provide statistics on the number of lemmas and size of Coq codebase.
- (possibly) renaming as permutation, inspired by nominal sets, make more lemmas don't require a condition (but some lemmas that wouldn't in a straight renaming do in exchange).
- (possibly) methodology: assume a lot of lemmas, prove main theorem, prove assumptions, some wrong, fix. A number of wrong lemma initially assumed, but replacing them by correct variant was always easy to fix in proofs.
- Axioms that we use and why (in particular setoid equality not very natural with ott-generated typing rules).
- Talk about the use and benefits of Copilot.