

# Destination $\lambda$ -calculus

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## 1 Term and value syntax

$\text{var}, x, y$  Term-level variable name  
 $k$  Index for ranges

$\text{hvar}, h$	$::=$ $  \quad h + h'$ $  \quad h[H \pm h']$ $  \quad \max(H)$	Hole or destination name ( $\mathbb{N}$ ) M M Shift by $h'$ if $h \in H$ M Maximum of a set of holes
$\text{hvars}, H$	$::=$ $  \quad \{h_1, \dots, h_k\}$ $  \quad H_1 \cup H_2$ $  \quad H \pm h'$ $  \quad \text{hvars}(\Gamma)$ $  \quad \text{hvars}(C)$	Set of hole names M Union of sets M Shift all names from $H$ by $h'$ . M Hole names of a context (requires $\text{ctx\_NoVar}(\Gamma)$ ) M Hole names of an evaluation context
term, $t, u$	$::=$ $  \quad v$ $  \quad x$ $  \quad t \succ t'$ $  \quad t ; u$ $  \quad t \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}$ $  \quad t \succ \text{case}_m (x_1, x_2) \mapsto u$ $  \quad t \succ \text{case}_m E_n x \mapsto u$ $  \quad t \succ \text{map } x \mapsto t'$ $  \quad \text{to}_\times u$ $  \quad \text{from}_\times t$ $  \quad t \triangleleft ()$ $  \quad t \triangleleft \text{Inl}$ $  \quad t \triangleleft \text{Inr}$ $  \quad t \triangleleft E_m$ $  \quad t \triangleleft (,)$ $  \quad t \triangleleft (\lambda x_m \mapsto u)$ $  \quad t \triangleleft \bullet t'$ $  \quad t[x := v]$	Term Value Variable Application Pattern-match on unit Pattern-match on sum Pattern-match on product Pattern-match on exponential Map over the right side of ampar $t$ Wrap $u$ into a trivial ampar Convert ampar with no dest remaining into pair Fill destination with unit Fill destination with left variant Fill destination with right variant Fill destination with exponential constructor Fill destination with product constructor Fill destination with function Fill destination with root of ampar $t'$ M
val, $v$	$::=$ $  \quad -h$ $  \quad +h$ $  \quad ()$ $  \quad \lambda^v x_m \mapsto u$ $  \quad \text{Inl } v$ $  \quad \text{Inr } v$ $  \quad E_m v$ $  \quad (v_1, v_2)$ $  \quad H \langle v_2 \circ v_1 \rangle$ $  \quad v[H \pm h']$	Term value Hole Destination Unit Lambda abstraction Left variant for sum Right variant for sum Exponential Product Ampar M Shift hole names inside $v$ by $h'$ if they belong to $H$ .

ectx, c	::=	$\square \succ t'$ $v \succ \square$ $\square ; u$ $\square \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}$ $\square \succ \text{case}_m (x_1, x_2) \mapsto u$ $\square \succ \text{case}_m E_n x \mapsto u$ $\square \succ \text{map } x \mapsto t'$ $\text{to}_\times \square$ $\text{from}_\times \square$ $\square \triangleleft ()$ $\square \triangleleft \text{Inl}$ $\square \triangleleft \text{Inr}$ $\square \triangleleft E_m$ $\square \triangleleft ()$ $\square \triangleleft (\lambda x. x \mapsto u)$ $\square \triangleleft \bullet t'$ $v \triangleleft \bullet \square$ $\overset{\text{op}}{\text{H}} \langle v_2, \square$	Evaluation context component Application Application Pattern-match on unit Pattern-match on sum Pattern-match on product Pattern-match on exponential Map over the right side of ampar Wrap into a trivial ampar Convert ampar with no dest remaining into pair Fill destination with unit Fill destination with left variant Fill destination with right variant Fill destination with exponential constructor Fill destination with product constructor Fill destination with function Fill destination with root of ampar Fill destination with root of ampar Open ampar. <b>Only new addition to term shapes</b>
ectxs, C	::=	$\square$ $C \circ c$ $C[\overset{\text{h}}{\text{h}} := \overset{\text{H}}{\text{H}} v]$	Evaluation context stack Represent the empty stack / "identity" evaluation context Push c on top of C M Fill <b>h</b> in C with value v (that may contain holes)

## 2 Type system

type, T, U	::=	$1$ $T_1 \oplus T_2$ $T_1 \otimes T_2$ $!_m T$ $U \ltimes T$ $T \xrightarrow{m} U$ $[T]^m$	Type Unit Sum Product Exponential Ampar type (consuming T yields U) Function Destination
mode, m, n	::=	$pa$ $\omega$ $m_1 \cdot \dots \cdot m_k$	Mode (Semiring) Pair of a multiplicity and age Error case (incompatible types, multiplicities, or ages) M Semiring product
mul, p	::=	$1$ $\omega$ $p_1 \cdot \dots \cdot p_k$	Multiplicity (first component of modality) Linear. Neutral element of the product Non-linear. Absorbing for the product M Semiring product
age, a	::=	$\nu$ $\uparrow$ $\infty$ $a_1 \cdot \dots \cdot a_k$	Age (second component of modality) Born now. Neutral element of the product One scope older Infinitely old / static. Absorbing for the product M Semiring product
ctx, $\Gamma, \Delta, \Pi$	::=	$x :_m T$ $+h :_m [T]^n$ $-h : T^n$ $m \cdot \Gamma$ $\Gamma_1 + \Gamma_2$ $\Gamma_1, \Gamma_2$ $-\Gamma$	Typing context   M Multiply each binding by m M Sum contexts $\Gamma_1$ and $\Gamma_2$ . Duplicate keys with incompatible values will be tagged with m M Disjoint sum/union of contexts $\Gamma_1$ and $\Gamma_2$ . M Transforms dest bindings into a hole bindings (requires ctx_DestOnly $\Gamma$ and ctx_L

	$-^i \Gamma$	M	Transforms hole bindings into dest bindings with left mode $1\nu$ (requires <code>ctx_HoleOnly</code> $\Gamma$ )
	$\Gamma[\mathbf{H} \pm \mathbf{h}']$	M	Shift hole/dest names by $\mathbf{h}'$ if they belong to $\mathbf{H}$

$\Gamma \Vdash v : \mathbf{T}$

(Typing of values (raw))

$\frac{\text{TYR-VAL-H}}{-\mathbf{h} : \mathbf{T}^{1\nu} \Vdash -\mathbf{h} : \mathbf{T}}$	$\frac{\text{TYR-VAL-D}}{+\mathbf{h} :_{1\nu} [\mathbf{T}]^n \Vdash +\mathbf{h} : [\mathbf{T}]^n}$	$\frac{\text{TYR-VAL-U}}{\Vdash () : \mathbf{1}}$	$\frac{\text{TYR-VAL-F}}{\Delta \Vdash \lambda^v \mathbf{x} :_m \mathbf{T} \mapsto u : \mathbf{T} \xrightarrow{m} \mathbf{U}}$	$\frac{\text{TYR-VAL-L}}{\Gamma \Vdash \text{Inl } v_1 : \mathbf{T}_1 \oplus \mathbf{T}_2}$
$\frac{\text{TYR-VAL-R}}{\Gamma \Vdash v_2 : \mathbf{T}_2 \quad \Gamma \Vdash \text{Inr } v_2 : \mathbf{T}_1 \oplus \mathbf{T}_2}$	$\frac{\text{TYR-VAL-P}}{\Gamma_1 \Vdash v_1 : \mathbf{T}_1 \quad \Gamma_2 \Vdash v_2 : \mathbf{T}_2 \quad \Gamma_1 + \Gamma_2 \Vdash (v_1, v_2) : \mathbf{T}_1 \otimes \mathbf{T}_2}$	$\frac{\text{TYR-VAL-E}}{n \cdot \Gamma \Vdash \mathbf{E}_n v' : !_n \mathbf{T}}$	$\frac{\text{TYR-VAL-A} \quad \text{LinOnly } \Delta_3 \quad \text{FinAgeOnly } \Delta_3 \quad \mathcal{I}\uparrow \cdot \Delta_1, \Delta_3 \Vdash v_1 : \mathbf{T} \quad \Delta_2, (-\Delta_3) \Vdash v_2 : \mathbf{U}}{\Delta_1, \Delta_2 \Vdash \text{hvars}(-\Delta_3) \langle v_2, v_1 \rangle : \mathbf{U} \ltimes \mathbf{T}}$	

$\Pi \vdash t : \mathbf{T}$

(Typing of terms)

$\frac{\text{TY-TERM-VAL} \quad \text{DisposableOnly } \Pi \quad \Delta \Vdash v : \mathbf{T}}{\Pi, \Delta \vdash v : \mathbf{T}}$	$\frac{\text{TY-TERM-VAR} \quad \text{DisposableOnly } \Pi \quad 1\nu < : \quad m}{\Pi, \mathbf{x} :_m \mathbf{T} \vdash \mathbf{x} : \mathbf{T}}$	$\frac{\text{TY-TERM-APP} \quad \Pi_1 \vdash t : \mathbf{T} \quad \Pi_2 \vdash t' : \mathbf{T} \xrightarrow{m} \mathbf{U}}{m \cdot \Pi_1 + \Pi_2 \vdash t \succ t' : \mathbf{U}}$		
$\frac{\text{TY-TERM-PATU} \quad \Pi_1 \vdash t : \mathbf{1} \quad \Pi_2 \vdash u : \mathbf{U}}{\Pi_1 + \Pi_2 \vdash t ; u : \mathbf{U}}$	$\frac{\text{TY-TERM-PATS} \quad \begin{array}{l} \Pi_1 \vdash t : \mathbf{T}_1 \oplus \mathbf{T}_2 \\ \Pi_2, \mathbf{x}_1 :_m \mathbf{T}_1 \vdash u_1 : \mathbf{U} \\ \Pi_2, \mathbf{x}_2 :_m \mathbf{T}_2 \vdash u_2 : \mathbf{U} \end{array}}{m \cdot \Pi_1 + \Pi_2 \vdash t \succ \text{case}_m \{ \text{Inl } \mathbf{x}_1 \mapsto u_1, \text{Inr } \mathbf{x}_2 \mapsto u_2 \} : \mathbf{U}}$			
$\frac{\text{TY-TERM-PATP} \quad \begin{array}{l} \Pi_1 \vdash t : \mathbf{T}_1 \otimes \mathbf{T}_2 \\ \Pi_2, \mathbf{x}_1 :_m \mathbf{T}_1, \mathbf{x}_2 :_m \mathbf{T}_2 \vdash u : \mathbf{U} \end{array}}{m \cdot \Pi_1 + \Pi_2 \vdash t \succ \text{case}_m (\mathbf{x}_1, \mathbf{x}_2) \mapsto u : \mathbf{U}}$	$\frac{\text{TY-TERM-PATE} \quad \Pi_1 \vdash t : !_n \mathbf{T} \quad \Pi_2, \mathbf{x} :_{m \cdot n} \mathbf{T} \vdash u : \mathbf{U}}{m \cdot \Pi_1 + \Pi_2 \vdash t \succ \text{case}_m \mathbf{E}_n \mathbf{x} \mapsto u : \mathbf{U}}$	$\frac{\text{TY-TERM-MAP} \quad \begin{array}{l} \Pi_1 \vdash t : \mathbf{U} \ltimes \mathbf{T} \\ \mathcal{I}\uparrow \cdot \Pi_2, \mathbf{x} :_{1\nu} \mathbf{T} \vdash t' : \mathbf{T}' \end{array}}{\Pi_1 + \Pi_2 \vdash t \succ \text{map } \mathbf{x} \mapsto t' : \mathbf{U} \ltimes \mathbf{T}'}$		
$\frac{\text{TY-TERM-TOA} \quad \Pi \vdash u : \mathbf{U}}{\Pi \vdash \text{to}_\ltimes u : \mathbf{U} \ltimes \mathbf{1}}$	$\frac{\text{TY-TERM-FROMA} \quad \Pi \vdash t : \mathbf{U} \ltimes (!_{\omega\nu} \mathbf{T})}{\Pi \vdash \text{from}_\ltimes t : \mathbf{U} \otimes (!_{\omega\nu} \mathbf{T})}$	$\frac{\text{TY-TERM-FILLU} \quad \Pi \vdash t : [\mathbf{1}]^n}{\Pi \vdash t \triangleleft () : \mathbf{1}}$	$\frac{\text{TY-TERM-FILLL} \quad \Pi \vdash t : [\mathbf{T}_1 \oplus \mathbf{T}_2]^n}{\Pi \vdash t \triangleleft \text{Inl} : [\mathbf{T}_1]^n}$	$\frac{\text{TY-TERM-FILLR} \quad \Pi \vdash t : [\mathbf{T}_1 \oplus \mathbf{T}_2]^n}{\Pi \vdash t \triangleleft \text{Inr} : [\mathbf{T}_2]^n}$
$\frac{\text{TY-TERM-FILLP} \quad \Pi \vdash t : [\mathbf{T}_1 \otimes \mathbf{T}_2]^n}{\Pi \vdash t \triangleleft (,) : [\mathbf{T}_1]^n \otimes [\mathbf{T}_2]^n}$	$\frac{\text{TY-TERM-FILLE} \quad \Pi \vdash t : [!_{n'} \mathbf{T}]^n}{\Pi \vdash t \triangleleft \mathbf{E}_{n'} : [\mathbf{T}]^{n' \cdot n}}$	$\frac{\text{TY-TERM-FILLF} \quad \begin{array}{l} \Pi_1 \vdash t : [\mathbf{T} \xrightarrow{m} \mathbf{U}]^n \\ \Pi_2, \mathbf{x} :_m \mathbf{T} \vdash u : \mathbf{U} \end{array}}{\Pi_1 + (\mathcal{I}\uparrow \cdot n) \cdot \Pi_2 \vdash t \triangleleft (\lambda \mathbf{x} :_m \mapsto u) : \mathbf{1}}$		
	$\frac{\text{TY-TERM-FILLC} \quad \Pi_1 \vdash t : [\mathbf{U}]^n \quad \Pi_2 \vdash t' : \mathbf{U} \ltimes \mathbf{T}}{\Pi_1 + (\mathcal{I}\uparrow \cdot n) \cdot \Pi_2 \vdash t \triangleleft \bullet t' : \mathbf{T}}$			

$\Delta \dashv C : \mathbf{T} \twoheadrightarrow \mathbf{U}_0$

(Typing of evaluation contexts)

$\frac{\text{TY-ECTXS-ID}}{\dashv \square : \mathbf{U}_0 \twoheadrightarrow \mathbf{U}_0}$	$\frac{\text{TY-ECTXS-APPFOC1} \quad m \cdot \Delta_1, \Delta_2 \dashv C : \mathbf{U} \twoheadrightarrow \mathbf{U}_0 \quad \Delta_2 \vdash t' : \mathbf{T} \xrightarrow{m} \mathbf{U}}{\Delta_1 \dashv C \circ (\square \succ t') : \mathbf{T} \twoheadrightarrow \mathbf{U}_0}$	$\frac{\text{TY-ECTXS-APPFOC2} \quad m \cdot \Delta_1, \Delta_2 \dashv C : \mathbf{U} \twoheadrightarrow \mathbf{U}_0 \quad \Delta_1 \vdash v : \mathbf{T}}{\Delta_2 \dashv C \circ (v \succ \square) : (\mathbf{T} \xrightarrow{m} \mathbf{U}) \twoheadrightarrow \mathbf{U}_0}$
$\frac{\text{TY-ECTXS-PATUFOC} \quad \Delta_1, \Delta_2 \dashv C : \mathbf{U} \twoheadrightarrow \mathbf{U}_0 \quad \Delta_2 \vdash u : \mathbf{U}}{\Delta_1 \dashv C \circ (\square ; u) : \mathbf{1} \twoheadrightarrow \mathbf{U}_0}$	$\frac{\text{TY-ECTXS-PATSFOC} \quad m \cdot \Delta_1, \Delta_2 \dashv C : \mathbf{U} \twoheadrightarrow \mathbf{U}_0 \quad \Delta_2, \mathbf{x}_1 :_m \mathbf{T}_1 \vdash u_1 : \mathbf{U} \quad \Delta_2, \mathbf{x}_2 :_m \mathbf{T}_2 \vdash u_2 : \mathbf{U}}{\Delta_1 \dashv C \circ (\square \succ \text{case}_m \{ \text{Inl } \mathbf{x}_1 \mapsto u_1, \text{Inr } \mathbf{x}_2 \mapsto u_2 \}) : (\mathbf{T}_1 \oplus \mathbf{T}_2) \twoheadrightarrow \mathbf{U}_0}$	
$\frac{\text{TY-ECTXS-PATPFOC} \quad m \cdot \Delta_1, \Delta_2 \dashv C : \mathbf{U} \twoheadrightarrow \mathbf{U}_0 \quad \Delta_2, \mathbf{x}_1 :_m \mathbf{T}_1, \mathbf{x}_2 :_m \mathbf{T}_2 \vdash u : \mathbf{U}}{\Delta_1 \dashv C \circ (\square \succ \text{case}_m (\mathbf{x}_1, \mathbf{x}_2) \mapsto u) : (\mathbf{T}_1 \otimes \mathbf{T}_2) \twoheadrightarrow \mathbf{U}_0}$	$\frac{\text{TY-ECTXS-PATEFOC} \quad m \cdot \Delta_1, \Delta_2 \dashv C : \mathbf{U} \twoheadrightarrow \mathbf{U}_0 \quad \Delta_2, \mathbf{x} :_{m \cdot m'} \mathbf{T} \vdash u : \mathbf{U}}{\Delta_1 \dashv C \circ (\square \succ \text{case}_m \mathbf{E}_{m'} \mathbf{x} \mapsto u) : !_m \mathbf{T} \twoheadrightarrow \mathbf{U}_0}$	

TY-ECTXS-MAPFOC

$$\frac{\Delta_1, \Delta_2 \vdash C : \mathbf{U} \ltimes \mathbf{T}' \rightarrow \mathbf{U}_0 \quad \mathcal{I}\uparrow \cdot \Delta_2, \mathbf{x} :_{\mathcal{I}\downarrow} \mathbf{T} \vdash \mathbf{t}' : \mathbf{T}'}{\Delta_1 \vdash C \circ (\Box \succ \mathbf{map} \mathbf{x} \mapsto \mathbf{t}') : (\mathbf{U} \ltimes \mathbf{T}) \rightarrow \mathbf{U}_0}$$

TY-ECTXS-TOAFOC

$$\frac{\Delta \vdash C : (\mathbf{U} \ltimes \mathbf{1}) \rightarrow \mathbf{U}_0}{\Delta \vdash C \circ (\mathbf{to}_{\ltimes} \Box) : \mathbf{U} \rightarrow \mathbf{U}_0}$$

TY-ECTXS-FROMAFOC

$$\frac{\Delta \vdash C : (\mathbf{U} \otimes (!_{\omega} \mathbf{T})) \rightarrow \mathbf{U}_0}{\Delta \vdash C \circ (\mathbf{from}_{\ltimes} \Box) : (\mathbf{U} \ltimes (!_{\omega} \mathbf{T})) \rightarrow \mathbf{U}_0}$$

TY-ECTXS-FILLUFOC

$$\frac{\Delta \vdash C : \mathbf{1} \rightarrow \mathbf{U}_0}{\Delta \vdash C \circ (\Box \triangleleft ()) : [\mathbf{1}]^n \rightarrow \mathbf{U}_0}$$

TY-ECTXS-FILLLFOC

$$\frac{\Delta \vdash C : [\mathbf{T}_1]^n \rightarrow \mathbf{U}_0}{\Delta \vdash C \circ (\Box \triangleleft \mathbf{Inl}) : [\mathbf{T}_1 \oplus \mathbf{T}_2]^n \rightarrow \mathbf{U}_0}$$

TY-ECTXS-FILLRFOC

$$\frac{\Delta \vdash C : [\mathbf{T}_2]^n \rightarrow \mathbf{U}_0}{\Delta \vdash C \circ (\Box \triangleleft \mathbf{Inr}) : [\mathbf{T}_1 \oplus \mathbf{T}_2]^n \rightarrow \mathbf{U}_0}$$

TY-ECTXS-FILLPFOC

$$\frac{\Delta \vdash C : ([\mathbf{T}_1]^n \otimes [\mathbf{T}_2]^n) \rightarrow \mathbf{U}_0}{\Delta \vdash C \circ (\Box \triangleleft (,)) : [\mathbf{T}_1 \otimes \mathbf{T}_2]^n \rightarrow \mathbf{U}_0}$$

TY-ECTXS-FILLEFOC

$$\frac{\Delta \vdash C : [\mathbf{T}]^{m \cdot n} \rightarrow \mathbf{U}_0}{\Delta \vdash C \circ (\Box \triangleleft \mathbf{E}_m) : [!_m \mathbf{T}]^n \rightarrow \mathbf{U}_0}$$

TY-ECTXS-FILLFFOC

$$\frac{\Delta_1, (\mathcal{I}\uparrow \cdot n) \cdot \Delta_2 \vdash C : \mathbf{1} \rightarrow \mathbf{U}_0 \quad \Delta_2, \mathbf{x} :_m \mathbf{T} \vdash \mathbf{u} : \mathbf{U}}{\Delta_1 \vdash C \circ (\Box \triangleleft (\lambda \mathbf{x}_m \mapsto \mathbf{u})) : [\mathbf{T}_m \rightarrow \mathbf{U}]^n \rightarrow \mathbf{U}_0}$$

TY-ECTXS-FILLCFOC1

$$\frac{\Delta_1, (\mathcal{I}\uparrow \cdot n) \cdot \Delta_2 \vdash C : \mathbf{T} \rightarrow \mathbf{U}_0 \quad \Delta_2 \vdash \mathbf{t}' : \mathbf{U} \ltimes \mathbf{T}}{\Delta_1 \vdash C \circ (\Box \triangleleft \bullet \mathbf{t}') : [\mathbf{U}]^n \rightarrow \mathbf{U}_0}$$

TY-ECTXS-FILLCFOC2

$$\frac{\Delta_1, (\mathcal{I}\uparrow \cdot n) \cdot \Delta_2 \vdash C : \mathbf{T} \rightarrow \mathbf{U}_0 \quad \Delta_1 \vdash \mathbf{v} : [\mathbf{U}]^n}{\Delta_2 \vdash C \circ (\mathbf{v} \triangleleft \bullet \Box) : \mathbf{U} \ltimes \mathbf{T} \rightarrow \mathbf{U}_0}$$

TY-ECTXS-AOPENFOC

$$\frac{\text{hvars}(C) \text{ \#\# } \text{hvars}(-\Delta_3) \quad \text{LinOnly } \Delta_3 \quad \text{FinAgeOnly } \Delta_3 \quad \Delta_1, \Delta_2 \vdash C : (\mathbf{U} \ltimes \mathbf{T}') \rightarrow \mathbf{U}_0 \quad \Delta_2, -\Delta_3 \Vdash \mathbf{v}_2 : \mathbf{U}}{\mathcal{I}\uparrow \cdot \Delta_1, \Delta_3 \vdash C \circ (\overset{\text{op}}{\text{hvars}}(-\Delta_3) \langle \mathbf{v}_2, \Box \rangle) : \mathbf{T}' \rightarrow \mathbf{U}_0}$$

$$\boxed{\vdash C[\mathbf{t}] : \mathbf{T}}$$

(Typing of extended terms (pair of evaluation context and term))

TY-ETERM-CLOSEDETERM

$$\frac{\Delta \vdash C : \mathbf{T} \rightarrow \mathbf{U}_0 \quad \Delta \vdash \mathbf{t} : \mathbf{T}}{\vdash C[\mathbf{t}] : \mathbf{U}_0}$$

### 3 Small-step semantics

$C[t] \longrightarrow C'[t']$			<i>(Small-step evaluation of terms using evaluation contexts)</i>		
SEM-ETERM-APPFOC1 <b>NotVal</b> $t$	SEM-ETERM-APPUNFOC1	SEM-ETERM-APPFOC2 <b>NotVal</b> $t'$			
$\overline{C[t \succ t'] \longrightarrow (C \circ (\Box \succ t'))[t]}$	$\overline{(C \circ (\Box \succ t'))[v] \longrightarrow C[v \succ t']}$	$\overline{C[v \succ t'] \longrightarrow (C \circ (v \succ \Box))[t']}$			
SEM-ETERM-APPUNFOC2	SEM-ETERM-APPRED	SEM-ETERM-PATUFOC <b>NotVal</b> $t$			
$\overline{(C \circ (v \succ \Box))[v'] \longrightarrow C[v \succ v']}$	$\overline{C[v \succ (\lambda^v x_m \mapsto u)] \longrightarrow C[u[x := v]]}$	$\overline{C[t; u] \longrightarrow (C \circ (\Box; u))[t]}$			
SEM-ETERM-PATUUNFOC		SEM-ETERM-PATURED			
$\overline{(C \circ (\Box; u))[v] \longrightarrow C[v; u]}$		$\overline{C[(\cdot); u] \longrightarrow C[u]}$			
<b>NotVal</b> $t$					
$\overline{C[t \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}] \longrightarrow (C \circ (\Box \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}))[t]}$					
SEM-ETERM-PATSUNFOC					
$\overline{(C \circ (\Box \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}))[v] \longrightarrow C[v \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}]}$					
SEM-ETERM-PATLRED					
$\overline{C[(\text{Inl } v_1) \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}] \longrightarrow C[u_1[x_1 := v_1]]}$					
SEM-ETERM-PATRRRED					
$\overline{C[(\text{Inr } v_2) \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}] \longrightarrow C[u_2[x_2 := v_2]]}$					
SEM-ETERM-PATPFOC					
<b>NotVal</b> $t$					
$\overline{C[t \succ \text{case}_m (x_1, x_2) \mapsto u] \longrightarrow (C \circ (\Box \succ \text{case}_m (x_1, x_2) \mapsto u))[t]}$					
SEM-ETERM-PATPUNFOC					
$\overline{(C \circ (\Box \succ \text{case}_m (x_1, x_2) \mapsto u))[v] \longrightarrow C[v \succ \text{case}_m (x_1, x_2) \mapsto u]}$					
SEM-ETERM-PATPREd			SEM-ETERM-PATEFOC		
$\overline{C[(v_1, v_2) \succ \text{case}_m (x_1, x_2) \mapsto u] \longrightarrow C[u[x_1 := v_1][x_2 := v_2]]}$			<b>NotVal</b> $t$		
$\overline{C[(v_1, v_2) \succ \text{case}_m (x_1, x_2) \mapsto u] \longrightarrow C[u[x_1 := v_1][x_2 := v_2]]}$			$\overline{C[t \succ \text{case}_m E_n x \mapsto u] \longrightarrow (C \circ (\Box \succ \text{case}_m E_n x \mapsto u))[t]}$		
SEM-ETERM-PATEUNFOC			SEM-ETERM-PATERED		
$\overline{(C \circ (\Box \succ \text{case}_m E_n x \mapsto u))[v] \longrightarrow C[v \succ \text{case}_m E_n x \mapsto u]}$			$\overline{C[E_n v' \succ \text{case}_m E_n x \mapsto u] \longrightarrow C[u[x := v']]}$		
SEM-ETERM-MAPFOC			SEM-ETERM-MAPUNFOC		
<b>NotVal</b> $t$					
$\overline{C[t \succ \text{map } x \mapsto t'] \longrightarrow (C \circ (\Box \succ \text{map } x \mapsto t'))[t]}$			$\overline{(C \circ (\Box \succ \text{map } x \mapsto t'))[v] \longrightarrow C[v \succ \text{map } x \mapsto t']}$		
SEM-ETERM-MAPREDAOPENFOC			SEM-ETERM-AOPENUNFOC		
$\overline{C[\text{h} \langle v_2, v_1 \rangle \succ \text{map } x \mapsto t'] \longrightarrow (C \circ (\overset{\text{op}}{\text{H}} \langle v_2, \text{h} \rangle, \Box))[t' [x := v_1 [\text{h}]]]}$			$\overline{(C \circ \overset{\text{op}}{\text{H}} \langle v_2, \Box \rangle)[v_1] \longrightarrow C[\text{h} \langle v_2, v_1 \rangle]}$		
$\text{h}' = \max(\text{hvars}(C)) + 1$					
SEM-ETERM-ToAFOC			SEM-ETERM-ToAUNFOC		
<b>NotVal</b> $u$					
$\overline{C[\text{to}_x u] \longrightarrow (C \circ (\text{to}_x \Box))[u]}$			$\overline{(C \circ (\text{to}_x \Box))[v_2] \longrightarrow C[\text{to}_x v_2]}$		
SEM-ETERM-ToARED					
$\overline{C[\text{to}_x v_2] \longrightarrow C[\{\cdot\} \langle v_2, () \rangle]}$					
SEM-ETERM-FROMAFOC			SEM-ETERM-FROMAUNFOC		
<b>NotVal</b> $t$					
$\overline{C[\text{from}_x t] \longrightarrow (C \circ (\text{from}_x \Box))[t]}$			$\overline{(C \circ (\text{from}_x \Box))[v] \longrightarrow C[\text{from}_x v]}$		
SEM-ETERM-FROMARED					
$\overline{C[\text{from}_x \{ \cdot \} \langle v_2, E_{\omega} v_1 \rangle] \longrightarrow C[(v_2, E_{\omega} v_1)]}$					
SEM-ETERM-FILLUFOC			SEM-ETERM-FILLUUNFOC		
<b>NotVal</b> $t$					
$\overline{C[t \triangleleft ()] \longrightarrow (C \circ (\Box \triangleleft ()))[t]}$			$\overline{(C \circ (\Box \triangleleft ()))[v] \longrightarrow C[v \triangleleft ()]}$		
SEM-ETERM-FILLURED					
$\overline{C[+\text{h} \triangleleft ()] \longrightarrow C[\text{h} := \{ \cdot \} ()][()]}$					

$\frac{\text{SEM-ETERM-FILLFOC} \quad \text{NotVal } t}{C[t \triangleleft \text{Inl}] \longrightarrow (C \circ (\Box \triangleleft \text{Inl}))[t]}$	$\frac{\text{SEM-ETERM-FILLUNFOC}}{(C \circ (\Box \triangleleft \text{Inl}))[v] \longrightarrow C[v \triangleleft \text{Inl}]}$	$\frac{\text{SEM-ETERM-FILLRED} \quad h' = \max(\text{hvars}(C) \cup \{h\}) + 1}{C[+h \triangleleft \text{Inl}] \longrightarrow C[h :=_{\{h'+1\}} \text{Inl} - (h'+1)][+(h'+1)]}$
$\frac{\text{SEM-ETERM-FILLRFOC} \quad \text{NotVal } t}{C[t \triangleleft \text{Inr}] \longrightarrow (C \circ (\Box \triangleleft \text{Inr}))[t]}$	$\frac{\text{SEM-ETERM-FILLRUNFOC}}{(C \circ (\Box \triangleleft \text{Inr}))[v] \longrightarrow C[v \triangleleft \text{Inr}]}$	
$\frac{\text{SEM-ETERM-FILLRRED} \quad h' = \max(\text{hvars}(C) \cup \{h\}) + 1}{C[+h \triangleleft \text{Inr}] \longrightarrow C[h :=_{\{h'+1\}} \text{Inr} - (h'+1)][+(h'+1)]}$	$\frac{\text{SEM-ETERM-FILLEFOC} \quad \text{NotVal } t}{C[t \triangleleft E_m] \longrightarrow (C \circ (\Box \triangleleft E_m))[t]}$	
$\frac{\text{SEM-ETERM-FILLEUNFOC}}{(C \circ (\Box \triangleleft E_m))[v] \longrightarrow C[v \triangleleft E_m]}$	$\frac{\text{SEM-ETERM-FILLERED} \quad h' = \max(\text{hvars}(C) \cup \{h\}) + 1}{C[+h \triangleleft E_m] \longrightarrow C[h :=_{\{h'+1\}} E_m - (h'+1)][+(h'+1)]}$	
$\frac{\text{SEM-ETERM-FILLPFOC} \quad \text{NotVal } t}{C[t \triangleleft (,)] \longrightarrow (C \circ (\Box \triangleleft (,)))[t]}$	$\frac{\text{SEM-ETERM-FILLPUNFOC}}{(C \circ (\Box \triangleleft (,)))[v] \longrightarrow C[v \triangleleft (,)]}$	
$\frac{\text{SEM-ETERM-FILLPRE} \quad h' = \max(\text{hvars}(C) \cup \{h\}) + 1}{C[+h \triangleleft (,)] \longrightarrow C[h :=_{\{h'+1, h'+2\}} (- (h'+1), - (h'+2))][+(h'+1), +(h'+2)]}$		
$\frac{\text{SEM-ETERM-FILLFFOC} \quad \text{NotVal } t}{C[t \triangleleft (\lambda x_m \mapsto u)] \longrightarrow (C \circ (\Box \triangleleft (\lambda x_m \mapsto u)))[t]}$	$\frac{\text{SEM-ETERM-FILLFUNFOC}}{(C \circ (\Box \triangleleft (\lambda x_m \mapsto u)))[v] \longrightarrow C[v \triangleleft (\lambda x_m \mapsto u)]}$	
$\frac{\text{SEM-ETERM-FILLFRED}}{C[+h \triangleleft (\lambda x_m \mapsto u)] \longrightarrow C[h :=_{\{ \}} \lambda^v x_m \mapsto u][()]}$	$\frac{\text{SEM-ETERM-FILLCFOC1} \quad \text{NotVal } t}{C[t \triangleleft \bullet t'] \longrightarrow (C \circ (\Box \triangleleft \bullet t'))[t]}$	$\frac{\text{SEM-ETERM-FILLCUNFOC1}}{(C \circ (\Box \triangleleft \bullet t'))[v] \longrightarrow C[v \triangleleft \bullet t']}$
$\frac{\text{SEM-ETERM-FILLCFOC2} \quad \text{NotVal } t'}{C[v \triangleleft \bullet t'] \longrightarrow (C \circ (v \triangleleft \bullet \Box))[t']}$	$\frac{\text{SEM-ETERM-FILLCUNFOC2}}{(C \circ (v \triangleleft \bullet \Box))[v'] \longrightarrow C[v \triangleleft \bullet v']}$	$\frac{\text{SEM-ETERM-FILLCRE} \quad h' = \max(\text{hvars}(C) \cup \{h\}) + 1}{C[+h \triangleleft \bullet_h (v_2, v_1)] \longrightarrow C[h :=_{(H \pm h')} v_2 [H \pm h']][v_1 [H \pm h']}]}$

## 4 Remarks on the Coq proofs

- Not particularly elegant. Max number of goals observed 232 (solved by a single call to the **congruence** tactic). When you have a computer, brute force is a viable strategy. (in particular, no semiring formalisation, it was quicker to do directly)
- Rules generated by ott, same as in the article (up to some notational difference). Contexts are not generated purely by syntax, and are interpreted in a semantic domain (finite functions).
- Reasoning on closed terms avoids almost all complications on binder manipulation. Makes proofs tractable.
- Finite functions: making a custom library was less headache than using existing libraries (including **MMap**). Existing libraries don't provide some of the tools that we needed, but the most important factor ended up being the need for a modicum of dependency between key and value. There wasn't really that out there. Backed by actual functions for simplicity; cost: equality is complicated.
- Most of the proofs done by author with very little prior experience to Coq.
- Did proofs in Coq because context manipulations are tricky.
- Context sum made total by adding an extra invalid *mode* (rather than an extra context). It seems to be much simpler this way.
- It might be a good idea to provide statistics on the number of lemmas and size of Coq codebase.
- (possibly) renaming as permutation, inspired by nominal sets, make more lemmas don't require a condition (but some lemmas that wouldn't in a straight renaming do in exchange).
- (possibly) methodology: assume a lot of lemmas, prove main theorem, prove assumptions, some wrong, fix. A number of wrong lemma initially assumed, but replacing them by correct variant was always easy to fix in proofs.
- Axioms that we use and why (in particular setoid equality not very natural with ott-generated typing rules).
- Talk about the use and benefits of Copilot.