Destination λ-calculus

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1 Term and value syntax

var, x, y	Term-level variable name
k	Index for ranges

hdn, h	::=			Hole or destination name ($\mathbb{N}$)
	|	$h + h'$	M	Sum
	|	$\max(H)$	M	Maximum of a set of holes
hdns, H	::=			Set of hole names
	|	$\{h_1, .., h_k\}$		
	|	$H_1 \cup H_2$	M	Union of sets
	|	$H \pm h$	M	Increase all names from H by h.
	|	$\mathrm{hnames}(\Gamma)$	M	Hole names of a context (requires ctx_NoVar(Γ))
	|	$\mathrm{hnames}(C)$	M	Hole names of an evaluation context
term, t, u	::=			Term
	|	v		Value
	|	x		Variable
	|	$t \succ u$		Application
	|	$t \,;\, u$		Pattern-match on unit
	|	$t \succ \mathbf{case}_m \{ \mathsf{Inl}\, x_1 \mapsto u_1 ,\ \mathsf{Inr}\, x_2 \mapsto u_2 \}$	bind x_1 in u_1 bind x_2 in u_2	Pattern-match on sum
	|	$t \succ \mathbf{case}_m (x_1 , x_2) \mapsto u$	bind x_1 in u bind x_2 in u	Pattern-match on product
	|	$t \succ \mathbf{case}_m \mathrm{E}^n\, x \mapsto u$	bind x in u	Pattern-match on exponential
	|	$t \succ \mathbf{map}\ x \mapsto u$	bind x in u	Map over the right side of ampar t
	|	$\mathbf{to}_{\ltimes}\ t$		Wrap t into a trivial ampar
	|	$\mathbf{from}_{\ltimes}\ t$		Extract value from trivial ampar
	|	**alloc**		Return a fresh "identity" ampar object
	|	$t \triangleleft ()$		Fill destination with unit
	|	$t \triangleleft (\lambda x\ {}_m \mapsto u)$	bind x in u	Fill destination with function
	|	$t \triangleleft \mathsf{Inl}$		Fill destination with left variant
	|	$t \triangleleft \mathsf{Inr}$		Fill destination with right variant
	|	$t \triangleleft (,)$		Fill destination with product constructor
	|	$t \triangleleft \mathrm{E}^m$		Fill destination with exponential constructor
	|	$t \triangleleft\!\bullet\, u$		Fill destination with root of ampar u
	|	$t[x \coloneqq v]$	M	
val, v	::=			Term value
	|	$-h$		Hole
	|	$+h$		Destination
	|	$()$		Unit
	|	$\lambda^{v} x\ {}_m \mapsto t$	bind x in t	Lambda abstraction
	|	$\mathsf{Inl}\ v$		Left variant for sum
	|	$\mathsf{Inr}\ v$		Right variant for sum
	|	$\mathrm{E}^m\ v$		Exponential
	|	(v_1 , v_2)		Product
	|	${}_H\langle v_1 \,;\, v_2 \rangle$		Ampar
	|	$v \pm h$	M	Rename hole names inside v by shifting them by h

eterm, j	::=			Pseudo-term
	|	$C[t]$		
ectx, C	::=			Evaluation context
	|	$\square$		Identity
	|	$C \succ u$		Application
	|	$v \succ C$		Application
	|	$C \; ; \; u$		Pattern-match on unit
	|	$C \succ \mathbf{case}_m \{ \mathsf{Inl}\, x_1 \mapsto u_1 \,,\ \mathsf{Inr}\, x_2 \mapsto u_2 \}$	bind x_1 in u_1 bind x_2 in u_2	Pattern-match on sum
	|	$C \succ \mathbf{case}_m\, (x_1 \,,\, x_2) \mapsto u$	bind x_1 in u bind x_2 in u	Pattern-match on product
	|	$C \succ \mathbf{case}_m\, \mathrm{E}^n\, x \mapsto u$	bind x in u	Pattern-match on exponential
	|	$C \succ \mathbf{map}\; x \mapsto u$	bind x in u	Map over the right side of ampar
	|	$\mathbf{to}_{\ltimes}\, C$		Wrap into a trivial ampar
	|	$\mathbf{from}_{\ltimes}\, C$		Extract value from trivial ampar
	|	$C \triangleleft ()$		Fill destination with unit
	|	$C \triangleleft (\lambda x\, {}_m \mapsto u)$	bind x in u	Fill destination with function
	|	$C \triangleleft \mathsf{Inl}$		Fill destination with left variant
	|	$C \triangleleft \mathsf{Inr}$		Fill destination with right variant
	|	$C \triangleleft (,)$		Fill destination with product constructor
	|	$C \triangleleft \mathrm{E}^m$		Fill destination with exponential constructor
	|	$C \triangleleft\!\bullet\, u$		Fill destination with root of ampar
	|	$v \triangleleft\!\bullet\, C$		Fill destination with root of ampar
	|	${}^{\mathrm{op}}_{H}\langle v_1 \,⨾\, C$		Open ampar. Only new addition to term shapes
	|	$C \circ C'$	M	Compose evaluation contexts
	|	$C[h \coloneqq_H v]$	M	Fill h with value v (that may contain holes)

2 Type system

type, T, U	::=			Type
	|	1		Unit
	|	$T_1 \oplus T_2$		Sum
	|	$T_1 \otimes T_2$		Product
	|	$!^m\, T$		Exponential
	|	$T_1 \ltimes T_2$		Ampar type (consuming T_2 yields T_1)
	|	$T_1\ {}_{m_1}\!\!\to T_2$		Function
	|	$\lfloor T \rfloor^m$		Destination
mode, *m*, *n*	::=			Mode (Semiring)
	|	pa		Pair of a multiplicity and age
	|	☠		Error case (incompatible types, multiplicities, or ages)
	|	$m_1 \cdot \ldots \cdot m_k$	M	Semiring product
mul, *p*	::=			Multiplicity (first component of modality)
	|	1		Linear. Neutral element of the product
	|	ω		Non-linear. Absorbing for the product
	|	$p_1 . \ldots . p_k$	M	Semiring product
age, *a*	::=			Age (second component of modality)
	|	ν		Born now. Neutral element of the product
	|	$\uparrow$		One scope older
	|	∞		Infinitely old / static. Absorbing for the product
	|	$a_1 \cdot \ldots \cdot a_k$	M	Semiring product
bndr, b	::=			Type assignment to either variable, destination or hole
	|	$x :_m T$		Variable
	|	$+h :_m \lfloor T \rfloor^n$		Destination (m is its own modality; n is the modality for values it accepts)
	|	$-h : T^n$		Hole (n is the modality for values it accepts, it doesn't have a modality on its own)

$\mathsf{ctx},\ \Gamma,\ \Delta$	::=			Typing context
	|	$\{\mathsf{b}_1, \ldots, \mathsf{b}_{\mathsf{k}}\}$		List of bindings
	|	$m \cdot \Gamma$	M	Multiply each binding by m
	|	$\Gamma_1 \uplus \Gamma_2$	M	Sum contexts Γ_1 and Γ_2. Duplicates/incompatible elements will give bindings with mod
	|	$-\Gamma$	M	Transforms every dest binding into a hole binding (requires `ctx_DestOnly` Γ)

$\boxed{\Gamma \Vdash \mathsf{v} : \mathsf{T}}$ *(Typing of values (raw))*

$$\textsc{TyR-val-H}\quad \frac{}{\{-h : \mathsf{T}^{1\nu}\} \Vdash -h : \mathsf{T}}$$

$$\textsc{TyR-val-D}\quad \frac{\texttt{ctx\_Compatible}\ \Gamma\ \ {+h} :_{1\nu} \lfloor \mathsf{T} \rfloor^{n}}{\Gamma \Vdash +h : \lfloor \mathsf{T} \rfloor^{n}}$$

$$\textsc{TyR-val-U}\quad \frac{}{\{\,\} \Vdash () : \mathsf{1}}$$

$$\textsc{TyR-val-F}\quad \frac{\texttt{ctx\_DestOnly}\ \Gamma \qquad \Gamma \uplus \{x :_m \mathsf{T}_1\} \vdash \mathsf{t} : \mathsf{T}_2}{\Gamma \Vdash \lambda^{\mathsf{v}} x_{\,m} \mapsto \mathsf{t} : \mathsf{T}_1 \ {}_m\!\!\to \mathsf{T}_2}$$

$$\textsc{TyR-val-L}\quad \frac{\Gamma \Vdash \mathsf{v} : \mathsf{T}_1}{\Gamma \Vdash \mathsf{Inl}\, \mathsf{v} : \mathsf{T}_1 \oplus \mathsf{T}_2}$$

$$\textsc{TyR-val-R}\quad \frac{\Gamma \Vdash \mathsf{v} : \mathsf{T}_2}{\Gamma \Vdash \mathsf{Inr}\, \mathsf{v} : \mathsf{T}_1 \oplus \mathsf{T}_2}$$

$$\textsc{TyR-val-P}\quad \frac{\Gamma_1 \Vdash \mathsf{v}_1 : \mathsf{T}_1 \qquad \Gamma_2 \Vdash \mathsf{v}_2 : \mathsf{T}_2}{\Gamma_1 \uplus \Gamma_2 \Vdash (\mathsf{v}_1\,,\, \mathsf{v}_2) : \mathsf{T}_1 \otimes \mathsf{T}_2}$$

$$\textsc{TyR-val-E}\quad \frac{\Gamma \Vdash \mathsf{v} : \mathsf{T}}{n\cdot\Gamma \Vdash \mathrm{E}^{n}\, \mathsf{v} : !^{n}\, \mathsf{T}}$$

$$\textsc{TyR-val-A}\quad \frac{\begin{array}{c}\texttt{ctx\_DestOnly}\ \Gamma_2 \uplus \Gamma_3 \\ \texttt{ctx\_DestOnly}\ \Gamma_1 \\ \Gamma_1 \uplus (-\Gamma_3) \Vdash \mathsf{v}_1 : \mathsf{T}_1 \\ \Gamma_2 \uplus \Gamma_3 \Vdash \mathsf{v}_2 : \mathsf{T}_2\end{array}}{\Gamma_1 \uplus \Gamma_2 \Vdash {}_{\mathtt{hnames}(-\Gamma_3)}\langle \mathsf{v}_1 \,ˎ\, \mathsf{v}_2 \rangle : \mathsf{T}_1 \ltimes \mathsf{T}_2}$$

$\boxed{\Gamma \vdash \mathsf{t} : \mathsf{T}}$ *(Typing of terms)*

$$\textsc{Ty-term-Val}\quad \frac{\texttt{ctx\_DestOnly}\ \Gamma \qquad \Gamma \Vdash \mathsf{v} : \mathsf{T}}{\Gamma \vdash \mathsf{v} : \mathsf{T}}$$

$$\textsc{Ty-term-Var}\quad \frac{\texttt{ctx\_Compatible}\ \Gamma\ \ x :_{1\nu} \mathsf{T}}{\Gamma \vdash x : \mathsf{T}}$$

$$\textsc{Ty-term-App}\quad \frac{\Gamma_1 \vdash \mathsf{t} : \mathsf{T}_1 \qquad \Gamma_2 \vdash \mathsf{u} : \mathsf{T}_1 \ {}_m\!\!\to \mathsf{T}_2}{m\cdot\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t} \succ \mathsf{u} : \mathsf{T}_2}$$

$$\textsc{Ty-term-PatU}\quad \frac{\Gamma_1 \vdash \mathsf{t} : \mathsf{1} \qquad \Gamma_2 \vdash \mathsf{u} : \mathsf{U}}{\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t}\ ;\ \mathsf{u} : \mathsf{U}}$$

$$\textsc{Ty-term-PatS}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{x_1 :_m \mathsf{T}_1\} \\ \texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{x_2 :_m \mathsf{T}_2\} \\ \Gamma_1 \vdash \mathsf{t} : \mathsf{T}_1 \oplus \mathsf{T}_2 \\ \Gamma_2 \uplus \{x_1 :_m \mathsf{T}_1\} \vdash \mathsf{u}_1 : \mathsf{U} \\ \Gamma_2 \uplus \{x_2 :_m \mathsf{T}_2\} \vdash \mathsf{u}_2 : \mathsf{U}\end{array}}{m\cdot\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t} \succ \mathbf{case}_m \{\, \mathsf{Inl}\, x_1 \mapsto \mathsf{u}_1\,,\ \mathsf{Inr}\, x_2 \mapsto \mathsf{u}_2 \,\} : \mathsf{U}}$$

$$\textsc{Ty-term-PatP}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{x_1 :_m \mathsf{T}_1\} \\ \texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{x_2 :_m \mathsf{T}_2\} \\ \texttt{ctx\_Disjoint}\ \ \{x_1 :_m \mathsf{T}_1\}\ \ \{x_2 :_m \mathsf{T}_2\} \\ \Gamma_1 \vdash \mathsf{t} : \mathsf{T}_1 \otimes \mathsf{T}_2 \\ \Gamma_2 \uplus \{x_1 :_m \mathsf{T}_1, x_2 :_m \mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{m\cdot\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t} \succ \mathbf{case}_m\, (x_1\,,\, x_2) \mapsto \mathsf{u} : \mathsf{U}}$$

$$\textsc{Ty-term-PatE}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{x :_{m\cdot n} \mathsf{T}\} \\ \Gamma_1 \vdash \mathsf{t} : !^{n}\, \mathsf{T} \\ \Gamma_2 \uplus \{x :_{m\cdot n} \mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{m\cdot\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t} \succ \mathbf{case}_m\, \mathrm{E}^{n}\, x \mapsto \mathsf{u} : \mathsf{U}}$$

$$\textsc{Ty-term-Map}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{x :_{1\nu} \mathsf{T}_2\} \\ \Gamma_1 \vdash \mathsf{t} : \mathsf{T}_1 \ltimes \mathsf{T}_2 \\ 1{\uparrow}\cdot\Gamma_2 \uplus \{x :_{1\nu} \mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Gamma_1 \uplus \Gamma_2 \vdash \mathsf{t} \succ \mathbf{map}\ x \mapsto \mathsf{u} : \mathsf{T}_1 \ltimes \mathsf{U}}$$

$$\textsc{Ty-term-ToA}\quad \frac{\Gamma \vdash \mathsf{t} : \mathsf{T}}{\Gamma \vdash \mathbf{to}_{\ltimes}\, \mathsf{t} : \mathsf{T} \ltimes \mathsf{1}}$$

$$\textsc{Ty-term-FromA}\quad \frac{\Gamma \vdash \mathsf{t} : \mathsf{T} \ltimes \mathsf{1}}{\Gamma \vdash \mathbf{from}_{\ltimes}\, \mathsf{t} : \mathsf{T}}$$

$$\textsc{Ty-term-Alloc}\quad \frac{}{\{\,\} \vdash \mathbf{alloc} : \mathsf{T} \ltimes \lfloor \mathsf{T} \rfloor^{1\nu}}$$

$$\textsc{Ty-term-FillU}\quad \frac{\Gamma \vdash \mathsf{t} : \lfloor \mathsf{1} \rfloor^{n}}{\Gamma \vdash \mathsf{t} \triangleleft () : \mathsf{1}}$$

$$\textsc{Ty-term-FillF}\quad \frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \ \{x :_m \mathsf{T}_1\} \\ \Gamma_1 \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \ {}_m\!\!\to \mathsf{T}_2 \rfloor^{n} \\ \Gamma_2 \uplus \{x :_m \mathsf{T}_1\} \vdash \mathsf{u} : \mathsf{T}_2\end{array}}{\Gamma_1 \uplus (1{\uparrow}\cdot n)\cdot\Gamma_2 \vdash \mathsf{t} \triangleleft (\lambda x_{\,m} \mapsto \mathsf{u}) : \mathsf{1}}$$

$$\textsc{Ty-term-FillL}\quad \frac{\Gamma \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^{n}}{\Gamma \vdash \mathsf{t} \triangleleft \mathsf{Inl} : \lfloor \mathsf{T}_1 \rfloor^{n}}$$

$$\textsc{Ty-term-FillR}\quad \frac{\Gamma \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \oplus \mathsf{T}_2 \rfloor^{n}}{\Gamma \vdash \mathsf{t} \triangleleft \mathsf{Inr} : \lfloor \mathsf{T}_2 \rfloor^{n}}$$

$$\textsc{Ty-term-FillP}\quad \frac{\Gamma \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \otimes \mathsf{T}_2 \rfloor^{n}}{\Gamma \vdash \mathsf{t} \triangleleft (,) : \lfloor \mathsf{T}_1 \rfloor^{n} \otimes \lfloor \mathsf{T}_2 \rfloor^{n}}$$

$$\textsc{Ty-term-FillE}\quad \frac{\Gamma \vdash \mathsf{t} : \lfloor !^{n'}\, \mathsf{T} \rfloor^{n}}{\Gamma \vdash \mathsf{t} \triangleleft \mathrm{E}^{n'} : \lfloor \mathsf{T} \rfloor^{n'\cdot n}}$$

$$\textsc{Ty-term-FillC}\quad \frac{\Gamma_1 \vdash \mathsf{t} : \lfloor \mathsf{T}_1 \rfloor^{n} \qquad \Gamma_2 \vdash \mathsf{u} : \mathsf{T}_1 \ltimes \mathsf{T}_2}{\Gamma_1 \uplus (1{\uparrow}\cdot n)\cdot\Gamma_2 \vdash \mathsf{t} \triangleleft\!\bullet\, \mathsf{u} : \mathsf{T}_2}$$

$\boxed{\Gamma \Vdash \mathsf{C} : \mathsf{T}_1\, {}_m\!\!\rightarrowtail \mathsf{T}_2}$ *(Typing of evaluation contexts)*

$$\textsc{TyR-ectx-Id}\quad \frac{}{\{\,\} \Vdash \square : \mathsf{U}_0\ {}_{1\nu}\!\!\rightarrowtail \mathsf{U}_0}$$

$$\textsc{TyR-ectx-App1}\quad \frac{\begin{array}{c}\texttt{ctx\_DestOnly}\ \Gamma_0 \\ -(n\cdot\Gamma_0) \uplus \Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_2\, {}_n\!\!\rightarrowtail \mathsf{U}_0 \\ \Gamma_0 \uplus \Gamma_2 \vdash \mathsf{u} : \mathsf{T}_1 \ {}_m\!\!\to \mathsf{T}_2\end{array}}{\Gamma_1 \uplus n\cdot\Gamma_2 \Vdash \mathsf{C} \circ (\square \succ \mathsf{u}) : \mathsf{T}_1\, {}_{(m\cdot n)}\!\!\rightarrowtail \mathsf{U}_0}$$

$$\textsc{TyR-ectx-App2}\quad \frac{\begin{array}{c}\texttt{ctx\_DestOnly}\ \Gamma_0 \\ -((m\cdot n)\cdot\Gamma_0) \uplus \Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_2\, {}_n\!\!\rightarrowtail \mathsf{U}_0 \\ \Gamma_0 \uplus \Gamma_2 \vdash \mathsf{v} : \mathsf{T}_1\end{array}}{\Gamma_1 \uplus (m\cdot n)\cdot\Gamma_2 \Vdash \mathsf{C} \circ (\mathsf{v} \succ \square) : (\mathsf{T}_1 \ {}_m\!\!\to \mathsf{T}_2)\, {}_n\!\!\rightarrowtail \mathsf{U}_0}$$

TyR-ectx-PatU

$$\frac{\texttt{ctx\_DestOnly}\ \Gamma_0 \quad -(n{\cdot}\Gamma_0)\uplus\Gamma_1 \Vdash \mathsf{C} : \mathsf{U}\,{}_{n}{\rightarrowtail}\mathsf{U}_0 \quad \Gamma_0\uplus\Gamma_2 \vdash \mathsf{u} : \mathsf{U}}{\Gamma_1\uplus n{\cdot}\Gamma_2 \Vdash \mathsf{C}\circ(\square\ ;\ \mathsf{u}) : \mathsf{1}\,{}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-PatS

$$\frac{\begin{array}{c}\texttt{ctx\_DestOnly}\ \Gamma_0\\ \texttt{ctx\_Disjoint}\ \Gamma_0\uplus\Gamma_2\ \{x_1 :_m \mathsf{T}_1\}\\ \texttt{ctx\_Disjoint}\ \Gamma_0\uplus\Gamma_2\ \{x_2 :_m \mathsf{T}_2\}\\ -(n{\cdot}\Gamma_0)\uplus\Gamma_1 \Vdash \mathsf{C} : \mathsf{U}\,{}_{n}{\rightarrowtail}\mathsf{U}_0\\ \Gamma_0\uplus\Gamma_2\uplus\{x_1 :_m \mathsf{T}_1\} \vdash \mathsf{u}_1 : \mathsf{U}\\ \Gamma_0\uplus\Gamma_2\uplus\{x_2 :_m \mathsf{T}_2\} \vdash \mathsf{u}_2 : \mathsf{U}\end{array}}{\Gamma_1\uplus n{\cdot}\Gamma_2 \Vdash \mathsf{C}\circ(\square \succ \mathbf{case}_m\,\{\,\mathsf{Inl}\,x_1 \mapsto \mathsf{u}_1\ ,\ \mathsf{Inr}\,x_2 \mapsto \mathsf{u}_2\,\}) : (\mathsf{T}_1{\oplus}\mathsf{T}_2)\,{}_{(m\cdot n)}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-PatP

$$\frac{\begin{array}{c}\texttt{ctx\_DestOnly}\ \Gamma_0\\ \texttt{ctx\_Disjoint}\ \Gamma_0\uplus\Gamma_2\ \{x_1 :_m \mathsf{T}_1\}\\ \texttt{ctx\_Disjoint}\ \Gamma_0\uplus\Gamma_2\ \{x_2 :_m \mathsf{T}_2\}\\ \texttt{ctx\_Disjoint}\ \{x_1 :_m \mathsf{T}_1\}\ \{x_2 :_m \mathsf{T}_2\}\\ -(n{\cdot}\Gamma_0)\uplus\Gamma_1 \Vdash \mathsf{C} : \mathsf{U}\,{}_{n}{\rightarrowtail}\mathsf{U}_0\\ \Gamma_0\uplus\Gamma_2\uplus\{x_1 :_m \mathsf{T}_1, x_2 :_m \mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Gamma_1\uplus n{\cdot}\Gamma_2 \Vdash \mathsf{C}\circ(\square \succ \mathbf{case}_m\,(x_1\ ,\ x_2)\mapsto \mathsf{u}) : (\mathsf{T}_1{\otimes}\mathsf{T}_2)\,{}_{(m\cdot n)}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-PatE

$$\frac{\begin{array}{c}\texttt{ctx\_DestOnly}\ \Gamma_0\\ \texttt{ctx\_Disjoint}\ \Gamma_0\uplus\Gamma_2\ \{x :_{m\cdot m'} \mathsf{T}\}\\ -(n{\cdot}\Gamma_0)\uplus\Gamma_1 \Vdash \mathsf{C} : \mathsf{U}\,{}_{n}{\rightarrowtail}\mathsf{U}_0\\ \Gamma_0\uplus\Gamma_2\uplus\{x :_{m\cdot m'} \mathsf{T}\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Gamma_1\uplus n{\cdot}\Gamma_2 \Vdash \mathsf{C}\circ(\square \succ \mathbf{case}_m\,\mathrm{E}^{m'}\,x \mapsto \mathsf{u}) : !^{m'}\,\mathsf{T}\,{}_{(m\cdot n)}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-Map

$$\frac{\begin{array}{c}\texttt{ctx\_DestOnly}\ \Gamma_0\\ \texttt{ctx\_Disjoint}\ 1{\uparrow}{\cdot}(\Gamma_0\uplus\Gamma_2)\ \{x :_{1\nu} \mathsf{T}_2\}\\ -(n{\cdot}\Gamma_0)\uplus\Gamma_1 \Vdash \mathsf{C} : \mathsf{U}\,{}_{n}{\rightarrowtail}\mathsf{U}_0\\ 1{\uparrow}{\cdot}(\Gamma_0\uplus\Gamma_2)\uplus\{x :_{1\nu} \mathsf{T}_2\} \vdash \mathsf{u} : \mathsf{U}\end{array}}{\Gamma_1\uplus n{\cdot}\Gamma_2 \Vdash \mathsf{C}\circ(\square \succ \mathbf{map}\ x\mapsto \mathsf{u}) : (\mathsf{T}_1 \ltimes \mathsf{T}_2)\,{}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-ToA

$$\frac{\Gamma \Vdash \mathsf{C} : (\mathsf{T} \ltimes \mathsf{1})\,{}_{n}{\rightarrowtail}\mathsf{U}_0}{\Gamma \Vdash \mathsf{C}\circ(\mathbf{to}_{\ltimes}\ \square) : \mathsf{T}\,{}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-FromA

$$\frac{\Gamma \Vdash \mathsf{C} : \mathsf{T}\,{}_{n}{\rightarrowtail}\mathsf{U}_0}{\Gamma \Vdash \mathsf{C}\circ(\mathbf{from}_{\ltimes}\ \square) : (\mathsf{T} \ltimes \mathsf{1})\,{}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-FillU

$$\frac{\Gamma \Vdash \mathsf{C} : \mathsf{1}\,{}_{n}{\rightarrowtail}\mathsf{U}_0}{\Gamma \Vdash \mathsf{C}\circ(\square \triangleleft ()) : \lfloor \mathsf{1}\rfloor^{m'}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-FillF

$$\frac{\begin{array}{c}\texttt{ctx\_Disjoint}\ \Gamma_2\ \{x :_m \mathsf{T}_1\}\\ \Gamma_1 \Vdash \mathsf{C} : \mathsf{1}\,{}_{n}{\rightarrowtail}\mathsf{U}_0\\ \Gamma_2\uplus\{x :_m \mathsf{T}_1\} \vdash \mathsf{u} : \mathsf{T}_2\end{array}}{\Gamma_1\uplus(1{\uparrow}{\cdot}m'{\cdot}n){\cdot}\Gamma_2 \Vdash \mathsf{C}\circ(\square \triangleleft (\lambda x\,{}_m \mapsto \mathsf{u})) : \lfloor \mathsf{T}_1\,{}_m{\to}\,\mathsf{T}_2\rfloor^{m'}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-FillL

$$\frac{\Gamma \Vdash \mathsf{C} : \lfloor \mathsf{T}_1\rfloor^{m'}_{n}{\rightarrowtail}\mathsf{U}_0}{\Gamma \Vdash \mathsf{C}\circ(\square \triangleleft \mathsf{Inl}) : \lfloor \mathsf{T}_1{\oplus}\mathsf{T}_2\rfloor^{m'}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-FillR

$$\frac{\Gamma \Vdash \mathsf{C} : \lfloor \mathsf{T}_2\rfloor^{m'}_{n}{\rightarrowtail}\mathsf{U}_0}{\Gamma \Vdash \mathsf{C}\circ(\square \triangleleft \mathsf{Inr}) : \lfloor \mathsf{T}_1{\oplus}\mathsf{T}_2\rfloor^{m'}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-FillP

$$\frac{\Gamma \Vdash \mathsf{C} : (\lfloor \mathsf{T}_1\rfloor^{m'}{\otimes}\lfloor \mathsf{T}_2\rfloor^{m'})\,{}_{n}{\rightarrowtail}\mathsf{U}_0}{\Gamma \Vdash \mathsf{C}\circ(\square \triangleleft (,)) : \lfloor \mathsf{T}_1{\otimes}\mathsf{T}_2\rfloor^{m'}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-FillE

$$\frac{\Gamma \Vdash \mathsf{C} : \lfloor \mathsf{T}\rfloor^{m\cdot m'}_{n}{\rightarrowtail}\mathsf{U}_0}{\Gamma \Vdash \mathsf{C}\circ(\square \triangleleft \mathrm{E}^{m}) : \lfloor !^{m}\,\mathsf{T}\rfloor^{m'}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-FillC1

$$\frac{\Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_2\,{}_{n}{\rightarrowtail}\mathsf{U}_0 \quad \Gamma_2 \vdash \mathsf{u} : \mathsf{T}_1 \ltimes \mathsf{T}_2}{\Gamma_1\uplus(1{\uparrow}{\cdot}m'{\cdot}n){\cdot}\Gamma_2 \Vdash \mathsf{C}\circ(\square \triangleleft\!\bullet\, \mathsf{u}) : \lfloor \mathsf{T}_1\rfloor^{m'}_{n}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-FillC2

$$\frac{\begin{array}{c}\texttt{ctx\_DestOnly}\ \Gamma_0\\ -(n{\cdot}\Gamma_0)\uplus\Gamma_1 \Vdash \mathsf{C} : \mathsf{T}_2\,{}_{n}{\rightarrowtail}\mathsf{U}_0\\ \Gamma_0\uplus\Gamma_2 \vdash \mathsf{v} : \lfloor \mathsf{T}_1\rfloor^{m'}\end{array}}{\Gamma_1\uplus n{\cdot}\Gamma_2 \Vdash \mathsf{C}\circ(\mathsf{v} \triangleleft\!\bullet\, \square) : \mathsf{T}_1 \ltimes \mathsf{T}_2\,{}_{(m'\cdot n)}{\rightarrowtail}\mathsf{U}_0}$$

TyR-ectx-AOpen

$$\frac{\begin{array}{c}\texttt{ctx\_DestOnly}\ \Gamma_0\\ \texttt{ctx\_DestOnly}\ \Gamma_3\\ \texttt{hdns\_Disjoint}\ \texttt{hnames}(\Gamma_3)\ \texttt{hnames}(\mathsf{C})\\ -(n{\cdot}\Gamma_0)\uplus\Gamma \Vdash \mathsf{C} : (\mathsf{T}_1 \ltimes \mathsf{U})\,{}_{n}{\rightarrowtail}\mathsf{U}_0\\ \Gamma_0\uplus\Gamma_1\uplus(-\Gamma_3) \Vdash \mathsf{v}_1 : \mathsf{T}_1\end{array}}{\Gamma\uplus n{\cdot}(\Gamma_1\uplus(-\Gamma_3)) \Vdash \mathsf{C}\circ({}^{\mathrm{op}}_{\texttt{hnames}(-\Gamma_3)}\langle \mathsf{v}_1\ ,\ \square) : \mathsf{U}\,{}_{n}{\rightarrowtail}\mathsf{U}_0}$$

$\boxed{\vdash \mathsf{j} : \mathsf{T}}$ *(Typing of extended terms (pair of evaluation context and term))*

Ty-eterm-ClosedEterm

$$\frac{-(n{\cdot}\Gamma) \Vdash \mathsf{C} : \mathsf{T}\,{}_{n}{\rightarrowtail}\mathsf{U}_0 \qquad \Gamma \vdash \mathsf{t} : \mathsf{T}}{\vdash \mathsf{C}[\mathsf{t}] : \mathsf{U}_0}$$

3 Small-step semantics

$\boxed{\mathsf{j} \longrightarrow \mathsf{j}'}$ *(Small-step evaluation of terms using evaluation contexts)*

$$\textsc{Sem-eterm-App}\quad \frac{}{\mathsf{C}[\mathsf{v} \succ (\lambda^{\mathsf{v}} \mathsf{x}_{\,m} \mapsto \mathsf{t})] \longrightarrow \mathsf{C}[\mathsf{t}[\mathsf{x} := \mathsf{v}]]}$$

$$\textsc{Sem-eterm-PatU}\quad \frac{}{\mathsf{C}[() \,;\, \mathsf{t}_2] \longrightarrow \mathsf{C}[\mathsf{t}_2]}$$

$$\textsc{Sem-eterm-PatL}\quad \frac{}{\mathsf{C}[(\mathsf{Inl}\,\mathsf{v}) \succ \mathbf{case}_m \{\, \mathsf{Inl}\,\mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\,\mathsf{x}_2 \mapsto \mathsf{u}_2 \,\}] \longrightarrow \mathsf{C}[\mathsf{u}_1[\mathsf{x} := \mathsf{v}]]}$$

$$\textsc{Sem-eterm-PatR}\quad \frac{}{\mathsf{C}[(\mathsf{Inr}\,\mathsf{v}) \succ \mathbf{case}_m \{\, \mathsf{Inl}\,\mathsf{x}_1 \mapsto \mathsf{u}_1 \,,\ \mathsf{Inr}\,\mathsf{x}_2 \mapsto \mathsf{u}_2 \,\}] \longrightarrow \mathsf{C}[\mathsf{u}_2[\mathsf{x} := \mathsf{v}]]}$$

$$\textsc{Sem-eterm-PatP}\quad \frac{}{\mathsf{C}[(\mathsf{v}_1 \,,\, \mathsf{v}_2) \succ \mathbf{case}_m (\mathsf{x}_1 \,,\, \mathsf{x}_2) \mapsto \mathsf{u}] \longrightarrow \mathsf{C}[\mathsf{u}[\mathsf{x}_1 := \mathsf{v}_1][\mathsf{x}_2 := \mathsf{v}_2]]}$$

$$\textsc{Sem-eterm-PatE}\quad \frac{}{\mathsf{C}[\mathrm{E}^{n}\,\mathsf{v} \succ \mathbf{case}_m\, \mathrm{E}^{n}\,\mathsf{x} \mapsto \mathsf{u}] \longrightarrow \mathsf{C}[\mathsf{u}[\mathsf{x} := \mathsf{v}]]}$$

$$\textsc{Sem-eterm-MapOpen}\quad \frac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}))}{\mathsf{C}[{}_{\mathtt{H}}\langle \mathsf{v}_1 \,,\, \mathsf{v}_2 \rangle \succ \mathbf{map}\ \mathsf{x} \mapsto \mathsf{u}] \longrightarrow (\mathsf{C} \circ ({}^{\mathrm{op}}_{\mathtt{H}\pm\mathtt{h}'}\langle \mathsf{v}_1 {\pm}\mathtt{h}' \,,\, \square \rangle))[\mathsf{u}[\mathsf{x} := \mathsf{v}_2 {\pm}\mathtt{h}']]}$$

$$\textsc{Sem-eterm-MapClose}\quad \frac{}{(\mathsf{C} \circ {}^{\mathrm{op}}_{\mathtt{H}}\langle \mathsf{v}_1 \,,\, \square \rangle)[\mathsf{v}_2] \longrightarrow \mathsf{C}[{}_{\mathtt{H}}\langle \mathsf{v}_1 \,,\, \mathsf{v}_2 \rangle]}$$

$$\textsc{Sem-eterm-Alloc}\quad \frac{}{\mathbf{alloc} \longrightarrow {}_{\{\mathtt{1}\}}\langle +\mathtt{1} \,,\, -\mathtt{1} \rangle}$$

$$\textsc{Sem-eterm-ToA}\quad \frac{}{\mathsf{C}[\mathbf{to}_{\ltimes}\ \mathsf{v}] \longrightarrow \mathsf{C}[{}_{\{\,\}}\langle \mathsf{v} \,,\, () \rangle]}$$

$$\textsc{Sem-eterm-FromA}\quad \frac{}{\mathsf{C}[\mathbf{from}_{\ltimes}\ {}_{\{\,\}}\langle \mathsf{v} \,,\, () \rangle] \longrightarrow \mathsf{v}}$$

$$\textsc{Sem-eterm-FillU}\quad \frac{}{\mathsf{C}[+\mathtt{h} \triangleleft ()] \longrightarrow \mathsf{C}[\mathtt{h} :=_{\{\,\}} ()][()]}$$

$$\textsc{Sem-eterm-FillF}\quad \frac{}{\mathsf{C}[+\mathtt{h} \triangleleft (\lambda \mathsf{x}_{\,m} \mapsto \mathsf{u})] \longrightarrow \mathsf{C}[\mathtt{h} :=_{\{\,\}} \lambda^{\mathsf{v}} \mathsf{x}_{\,m} \mapsto \mathsf{u}][()]}$$

$$\textsc{Sem-eterm-FillL}\quad \frac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}) \cup \{\mathtt{h}\})}{\mathsf{C}[+\mathtt{h} \triangleleft \mathsf{Inl}] \longrightarrow \mathsf{C}[\mathtt{h} :=_{\{\mathtt{h}'+\mathtt{1}\}} \mathsf{Inl}\ -(\mathtt{h}'+\mathtt{1})][+(\mathtt{h}'+\mathtt{1})]}$$

$$\textsc{Sem-eterm-FillR}\quad \frac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}) \cup \{\mathtt{h}\})}{\mathsf{C}[+\mathtt{h} \triangleleft \mathsf{Inr}] \longrightarrow \mathsf{C}[\mathtt{h} :=_{\{\mathtt{h}'+\mathtt{1}\}} \mathsf{Inr}\ -(\mathtt{h}'+\mathtt{1})][+(\mathtt{h}'+\mathtt{1})]}$$

$$\textsc{Sem-eterm-FillE}\quad \frac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}) \cup \{\mathtt{h}\})}{\mathsf{C}[+\mathtt{h} \triangleleft \mathrm{E}^{m}] \longrightarrow \mathsf{C}[\mathtt{h} :=_{\{\mathtt{h}'+\mathtt{1}\}} \mathrm{E}^{m}\ -(\mathtt{h}'+\mathtt{1})][+(\mathtt{h}'+\mathtt{1})]}$$

$$\textsc{Sem-eterm-FillP}\quad \frac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}) \cup \{\mathtt{h}\})}{\mathsf{C}[+\mathtt{h} \triangleleft (,)] \longrightarrow \mathsf{C}[\mathtt{h} :=_{\{\mathtt{h}'+\mathtt{1},\mathtt{h}'+\mathtt{2}\}} (-(\mathtt{h}'+\mathtt{1}) \,,\, -(\mathtt{h}'+\mathtt{2}))][(+(\mathtt{h}'+\mathtt{1}) \,,\, +(\mathtt{h}'+\mathtt{2}))]}$$

$$\textsc{Sem-eterm-FillC}\quad \frac{\mathtt{h}' \;=\; \mathtt{max}(\mathtt{hnames}(\mathsf{C}) \cup \{\mathtt{h}\})}{\mathsf{C}[+\mathtt{h} \triangleleft\!\bullet\, {}_{\mathtt{H}}\langle \mathsf{v}_1 \,,\, \mathsf{v}_2 \rangle] \longrightarrow \mathsf{C}[\mathtt{h} :=_{(\mathtt{H}\pm\mathtt{h}')} \mathsf{v}_1 {\pm}\mathtt{h}'][\mathsf{v}_2 {\pm}\mathtt{h}']}$$