

Destination λ -calculus

Thomas BAGREL

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1 Term and value syntax

var, x, y Term-level variable name
 k Index for ranges

hdn, h	$::=$ $ \quad h + h'$ $ \quad h[H \pm h']$ $ \quad \max(H)$	Hole or destination name (\mathbb{N}) M M Shift by h' if $h \in H$ M Maximum of a set of holes
hdns, H	$::=$ $ \quad \{h_1, \dots, h_k\}$ $ \quad H_1 \cup H_2$ $ \quad H \pm h'$ $ \quad \text{hnames}(\Gamma)$ $ \quad \text{hnames}(C)$	Set of hole names M Union of sets M Shift all names from H by h' . M Hole names of a context (requires $\text{ctx_NoVar}(\Gamma)$) M Hole names of an evaluation context
term, t, u	$::=$ $ \quad v$ $ \quad x$ $ \quad t \succ t'$ $ \quad t ; u$ $ \quad t \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}$ $ \quad t \succ \text{case}_m (x_1, x_2) \mapsto u$ $ \quad t \succ \text{case}_m E^n x \mapsto u$ $ \quad t \succ \text{map } x \mapsto t'$ $ \quad \text{to}_\times u$ $ \quad \text{from}_\times t$ $ \quad t \triangleleft ()$ $ \quad t \triangleleft \text{Inl}$ $ \quad t \triangleleft \text{Inr}$ $ \quad t \triangleleft E^m$ $ \quad t \triangleleft (,)$ $ \quad t \triangleleft (\lambda x_m \mapsto u)$ $ \quad t \triangleleft \bullet t'$ $ \quad t[x := v]$	Term Value Variable Application Pattern-match on unit Pattern-match on sum Pattern-match on product Pattern-match on exponential Map over the right side of ampar t Wrap u into a trivial ampar Extract value from trivial ampar Fill destination with unit Fill destination with left variant Fill destination with right variant Fill destination with exponential constructor Fill destination with product constructor Fill destination with function Fill destination with root of ampar t' M
val, v	$::=$ $ \quad -h$ $ \quad +h$ $ \quad ()$ $ \quad \lambda^v x_m \mapsto u$ $ \quad \text{Inl } v$ $ \quad \text{Inr } v$ $ \quad E^m v$ $ \quad (v_1, v_2)$ $ \quad H \langle v_2, v_1 \rangle$ $ \quad v[H \pm h']$	Term value Hole Destination Unit Lambda abstraction Left variant for sum Right variant for sum Exponential Product Ampar M Shift hole names inside v by h' if they belong to H .

ectx, c	::=	$\square \succ t'$ $v \succ \square$ $\square ; u$ $\square \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}$ $\square \succ \text{case}_m (x_1, x_2) \mapsto u$ $\square \succ \text{case}_m E^n x \mapsto u$ $\square \succ \text{map } x \mapsto t'$ $\text{to}_\times \square$ $\text{from}_\times \square$ $\square \triangleleft ()$ $\square \triangleleft \text{Inl}$ $\square \triangleleft \text{Inr}$ $\square \triangleleft E^m$ $\square \triangleleft ()$ $\square \triangleleft (\lambda x_m \mapsto u)$ $\square \triangleleft \bullet t'$ $v \triangleleft \bullet \square$ $\overset{\text{op}}{\text{H}} \langle v_2, \square$	<p>Evaluation context component</p> <p>Application</p> <p>Application</p> <p>Pattern-match on unit</p> <p>Pattern-match on sum</p> <p>Pattern-match on product</p> <p>Pattern-match on exponential</p> <p>Map over the right side of ampar</p> <p>Wrap into a trivial ampar</p> <p>Extract value from trivial ampar</p> <p>Fill destination with unit</p> <p>Fill destination with left variant</p> <p>Fill destination with right variant</p> <p>Fill destination with exponential constructor</p> <p>Fill destination with product constructor</p> <p>Fill destination with function</p> <p>Fill destination with root of ampar</p> <p>Fill destination with root of ampar</p> <p>Open ampar. Only new addition to term shapes</p>
ectxs, C	::=	\square $C \circ c$ $C[\overset{\text{h}}{\text{h}} :=_{\text{H}} v]$	<p>Evaluation context stack</p> <p>Represent the empty stack / "identity" evaluation context</p> <p>Push c on top of C</p> <p>M Fill h in C with value v (that may contain holes)</p>

2 Type system

type, T, U	::=	1 $T_1 \oplus T_2$ $T_1 \otimes T_2$ $!^m T$ $U \ltimes T$ $T \xrightarrow{m} U$ $[T]^m$	<p>Type</p> <p>Unit</p> <p>Sum</p> <p>Product</p> <p>Exponential</p> <p>Ampar type (consuming T yields U)</p> <p>Function</p> <p>Destination</p>
mode, m, n	::=	pa ω $m_1 \cdot \dots \cdot m_k$	<p>Mode (Semiring)</p> <p>Pair of a multiplicity and age</p> <p>Error case (incompatible types, multiplicities, or ages)</p> <p>M Semiring product</p>
mul, p	::=	1 ω $p_1 \cdot \dots \cdot p_k$	<p>Multiplicity (first component of modality)</p> <p>Linear. Neutral element of the product</p> <p>Non-linear. Absorbing for the product</p> <p>M Semiring product</p>
age, a	::=	ν \uparrow ∞ $a_1 \cdot \dots \cdot a_k$	<p>Age (second component of modality)</p> <p>Born now. Neutral element of the product</p> <p>One scope older</p> <p>Infinitely old / static. Absorbing for the product</p> <p>M Semiring product</p>
ctx, Γ, Δ, Π	::=	$\{x :_m T\}$ $\{+h :_m [T]^n\}$ $\{-h : T^n\}$ $m \cdot \Gamma$ $\Gamma_1 \uplus \Gamma_2$ $-\Gamma$ $-^{-1}\Gamma$	<p>Typing context</p> <p>M Multiply each binding by m</p> <p>M Sum contexts Γ_1 and Γ_2. Duplicate keys with incompatible values will be tagged</p> <p>M Transforms dest bindings into a hole bindings (requires ctx_DestOnly Γ and ctx_Hol</p> <p>M Transforms hole bindings into dest bindings with left mode 1ν (requires ctx_Hol</p>

| $\Gamma[\mathbf{H} \pm \mathbf{h}']$ M Shift hole/dest names by \mathbf{h}' if they belong to \mathbf{H}

$\boxed{\Gamma \Vdash v : \mathbf{T}}$

(Typing of values (raw))

$$\begin{array}{c}
\text{TYR-VAL-H} \quad \frac{\{-\mathbf{h} : \mathbf{T}^{\text{lv}}\} \Vdash -\mathbf{h} : \mathbf{T}}{} \quad \text{TYR-VAL-D} \quad \frac{\{+\mathbf{h} : \mathbf{T}^{\text{lv}}[\mathbf{T}]^n\} \Vdash +\mathbf{h} : [\mathbf{T}]^n}{} \quad \text{TYR-VAL-U} \quad \frac{\{\} \Vdash () : \mathbf{1}}{} \\
\text{TYR-VAL-L} \quad \frac{\Gamma \Vdash v_1 : \mathbf{T}_1}{\Gamma \Vdash \text{Inl } v_1 : \mathbf{T}_1 \oplus \mathbf{T}_2} \quad \text{TYR-VAL-R} \quad \frac{\Gamma \Vdash v_2 : \mathbf{T}_2}{\Gamma \Vdash \text{Inr } v_2 : \mathbf{T}_1 \oplus \mathbf{T}_2} \quad \text{TYR-VAL-P} \quad \frac{\Gamma_1 \Vdash v_1 : \mathbf{T}_1 \quad \Gamma_2 \Vdash v_2 : \mathbf{T}_2}{\Gamma_1 \uplus \Gamma_2 \Vdash (v_1, v_2) : \mathbf{T}_1 \otimes \mathbf{T}_2} \\
\text{TYR-VAL-E} \quad \frac{\Gamma \Vdash v' : \mathbf{T} \quad \text{mode_IsValid } n}{n \cdot \Gamma \Vdash \mathbf{E}^n v' : !^n \mathbf{T}} \\
\text{TYR-VAL-A} \quad \frac{\begin{array}{l} \text{ctx_DestOnly } \Delta_1 \\ \text{ctx_DestOnly } \Delta_2 \\ \text{ctx_DestOnly } \Delta_3 \\ \text{ctx_LinOnly } \Delta_3 \\ \text{ctx_FinAgeOnly } \Delta_3 \\ \text{ctx_ValidOnly } \Delta_3 \\ \text{ctx_Disjoint } \Delta_1 \Delta_2 \\ \text{ctx_Disjoint } \Delta_1 \Delta_3 \\ \text{ctx_Disjoint } \Delta_2 \Delta_3 \\ \text{!} \uparrow \cdot \Delta_1 \uplus \Delta_3 \Vdash v_1 : \mathbf{T} \\ \Delta_2 \uplus (-\Delta_3) \Vdash v_2 : \mathbf{U} \end{array}}{\Delta_1 \uplus \Delta_2 \Vdash \text{hnames}(-\Delta_3) \langle v_2, v_1 \rangle : \mathbf{U} \ltimes \mathbf{T}}
\end{array}$$

$\boxed{\Pi \vdash t : \mathbf{T}}$

(Typing of terms)

$$\begin{array}{c}
\text{TY-TERM-VAL} \quad \frac{\begin{array}{l} \text{ctx_DestOnly } \Delta \\ \text{ctx_DisposableOnly } \Pi \\ \Delta \Vdash v : \mathbf{T} \end{array}}{\Pi \uplus \Delta \vdash v : \mathbf{T}} \quad \text{TY-TERM-VAR} \quad \frac{\begin{array}{l} \text{ctx_DisposableOnly } \Pi \\ \text{ctx_Disjoint } \Pi \{x : \mathbf{T}\} \\ \text{mode_IsSubtype } m \text{ !} \nu \end{array}}{\Pi \uplus \{x : \mathbf{T}\} \vdash x : \mathbf{T}} \quad \text{TY-TERM-APP} \quad \frac{\Pi_1 \vdash t : \mathbf{T} \quad \Pi_2 \vdash t' : \mathbf{T} \rightarrow \mathbf{U}}{m \cdot \Pi_1 \uplus \Pi_2 \vdash t \succ t' : \mathbf{U}} \\
\text{TY-TERM-PATS} \quad \frac{\begin{array}{l} \text{mode_IsValid } m \\ \text{ctx_Disjoint } \Pi_2 \{x_1 : \mathbf{T}_1\} \\ \text{ctx_Disjoint } \Pi_2 \{x_2 : \mathbf{T}_2\} \\ \Pi_1 \vdash t : \mathbf{T}_1 \oplus \mathbf{T}_2 \\ \Pi_2 \uplus \{x_1 : \mathbf{T}_1\} \vdash u_1 : \mathbf{U} \\ \Pi_2 \uplus \{x_2 : \mathbf{T}_2\} \vdash u_2 : \mathbf{U} \end{array}}{m \cdot \Pi_1 \uplus \Pi_2 \vdash t \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \} : \mathbf{U}} \\
\text{TY-TERM-PATU} \quad \frac{\Pi_1 \vdash t : \mathbf{1} \quad \Pi_2 \vdash u : \mathbf{U}}{\Pi_1 \uplus \Pi_2 \vdash t ; u : \mathbf{U}} \\
\text{TY-TERM-PATP} \quad \frac{\begin{array}{l} \text{mode_IsValid } m \\ \text{ctx_Disjoint } \Pi_2 \{x_1 : \mathbf{T}_1\} \\ \text{ctx_Disjoint } \Pi_2 \{x_2 : \mathbf{T}_2\} \\ \text{ctx_Disjoint } \{x_1 : \mathbf{T}_1\} \{x_2 : \mathbf{T}_2\} \\ \Pi_1 \vdash t : \mathbf{T}_1 \otimes \mathbf{T}_2 \\ \Pi_2 \uplus \{x_1 : \mathbf{T}_1\} \uplus \{x_2 : \mathbf{T}_2\} \vdash u : \mathbf{U} \end{array}}{m \cdot \Pi_1 \uplus \Pi_2 \vdash t \succ \text{case}_m (x_1, x_2) \mapsto u : \mathbf{U}} \\
\text{TY-TERM-PATE} \quad \frac{\begin{array}{l} \text{mode_IsValid } m \\ \text{mode_IsValid } n \\ \text{ctx_Disjoint } \Pi_2 \{x : \mathbf{T}\} \\ \Pi_1 \vdash t : !^n \mathbf{T} \\ \Pi_2 \uplus \{x : \mathbf{T}\} \vdash u : \mathbf{U} \end{array}}{m \cdot \Pi_1 \uplus \Pi_2 \vdash t \succ \text{case}_m \mathbf{E}^n x \mapsto u : \mathbf{U}} \\
\text{TY-TERM-MAP} \quad \frac{\begin{array}{l} \text{ctx_Disjoint } \Pi_2 \{x : \mathbf{T}\} \\ \Pi_1 \vdash t : \mathbf{U} \ltimes \mathbf{T} \\ \text{!} \uparrow \cdot \Pi_2 \uplus \{x : \mathbf{T}\} \vdash t' : \mathbf{T}' \end{array}}{\Pi_1 \uplus \Pi_2 \vdash t \succ \text{map } x \mapsto t' : \mathbf{U} \ltimes \mathbf{T}'} \\
\text{TY-TERM-TOA} \quad \frac{\Pi \vdash u : \mathbf{U}}{\Pi \vdash \text{to}_\times u : \mathbf{U} \ltimes \mathbf{1}} \quad \text{TY-TERM-FROMA} \quad \frac{\Pi \vdash t : \mathbf{U} \ltimes \mathbf{1}}{\Pi \vdash \text{from}_\times t : \mathbf{U}} \quad \text{TY-TERM-FILLU} \quad \frac{\Pi \vdash t : [\mathbf{1}]^n}{\Pi \vdash t \triangleleft () : \mathbf{1}} \\
\text{TY-TERM-FILLL} \quad \frac{\Pi \vdash t : [\mathbf{T}_1 \oplus \mathbf{T}_2]^n}{\Pi \vdash t \triangleleft \text{Inl} : [\mathbf{T}_1]^n} \quad \text{TY-TERM-FILLR} \quad \frac{\Pi \vdash t : [\mathbf{T}_1 \oplus \mathbf{T}_2]^n}{\Pi \vdash t \triangleleft \text{Inr} : [\mathbf{T}_2]^n} \\
\text{TY-TERM-FILLF} \quad \frac{\begin{array}{l} \text{mode_IsValid } m \\ \text{mode_IsValid } n \\ \text{ctx_Disjoint } \Pi_2 \{x : \mathbf{T}\} \\ \Pi_1 \vdash t : [\mathbf{T} \rightarrow \mathbf{U}]^n \\ \Pi_2 \uplus \{x : \mathbf{T}\} \vdash u : \mathbf{U} \end{array}}{\Pi_1 \uplus (\text{!} \uparrow \cdot n) \cdot \Pi_2 \vdash t \triangleleft (\lambda x. m \mapsto u) : \mathbf{1}} \\
\text{TY-TERM-FILLE} \quad \frac{\begin{array}{l} \text{mode_IsValid } n \\ \Pi \vdash t : [!^{n'} \mathbf{T}]^n \end{array}}{\Pi \vdash t \triangleleft \mathbf{E}^{n'} : [\mathbf{T}]^{n' \cdot n}} \\
\text{TY-TERM-FILLP} \quad \frac{\Pi \vdash t : [\mathbf{T}_1 \otimes \mathbf{T}_2]^n}{\Pi \vdash t \triangleleft (,) : [\mathbf{T}_1]^n \otimes [\mathbf{T}_2]^n}
\end{array}$$

$$\text{TY-TERM-FILLC} \quad \frac{\text{mode_IsValid } n \quad \Pi_1 \vdash t : [\mathbf{U}]^n \quad \Pi_2 \vdash t' : \mathbf{U} \ltimes \mathbf{T}}{\Pi_1 \uplus (\mathcal{I} \uparrow \cdot n) \cdot \Pi_2 \vdash t \triangleleft \bullet t' : \mathbf{T}}$$

$$\boxed{\Delta \vdash C : \mathbf{T} \multimap \mathbf{U}_0}$$

(Typing of evaluation contexts)

$\text{TY-ECTXS-ID} \quad \frac{\{\} \vdash \square : \mathbf{U}_0 \multimap \mathbf{U}_0}{\Delta_1 \vdash C \circ (\square \succ t') : \mathbf{T} \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-APPFOC1} \quad \frac{\begin{array}{l} \text{ctx_Disjoint } \Delta_1 \Delta_2 \\ \text{ctx_DestOnly } \Delta_1 \\ \text{ctx_DestOnly } \Delta_2 \\ \text{mode_IsValid } m \\ \text{ctx_ValidOnly } \Delta_2 \\ m \cdot \Delta_1 \uplus \Delta_2 \vdash C : \mathbf{U} \multimap \mathbf{U}_0 \\ \Delta_2 \vdash t' : \mathbf{T} \xrightarrow{m} \mathbf{U} \end{array}}{\Delta_1 \vdash C \circ (\square \succ t') : \mathbf{T} \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-APPFOC2} \quad \frac{\begin{array}{l} \text{ctx_Disjoint } \Delta_1 \Delta_2 \\ \text{ctx_DestOnly } \Delta_1 \\ \text{ctx_DestOnly } \Delta_2 \\ \text{mode_IsValid } m \\ \text{ctx_ValidOnly } \Delta_1 \\ m \cdot \Delta_1 \uplus \Delta_2 \vdash C : \mathbf{U} \multimap \mathbf{U}_0 \\ \Delta_1 \vdash v : \mathbf{T} \end{array}}{\Delta_2 \vdash C \circ (v \succ \square) : (\mathbf{T} \xrightarrow{m} \mathbf{U}) \multimap \mathbf{U}_0}$
$\text{TY-ECTXS-PATSFoc} \quad \frac{\begin{array}{l} \text{ctx_Disjoint } \Delta_1 \Delta_2 \\ \text{ctx_DestOnly } \Delta_1 \\ \text{ctx_DestOnly } \Delta_2 \\ \text{mode_IsValid } m \\ \text{ctx_ValidOnly } \Delta_2 \\ m \cdot \Delta_1 \uplus \Delta_2 \vdash C : \mathbf{U} \multimap \mathbf{U}_0 \\ \Delta_2 \uplus \{x_1 : m \mathbf{T}_1\} \vdash u_1 : \mathbf{U} \\ \Delta_2 \uplus \{x_2 : m \mathbf{T}_2\} \vdash u_2 : \mathbf{U} \end{array}}{\Delta_1 \vdash C \circ (\square \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}) : (\mathbf{T}_1 \oplus \mathbf{T}_2) \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-PATPFoc} \quad \frac{\begin{array}{l} \text{ctx_Disjoint } \Delta_1 \Delta_2 \\ \text{ctx_DestOnly } \Delta_1 \\ \text{ctx_DestOnly } \Delta_2 \\ \text{ctx_Disjoint } \{x_1 : m \mathbf{T}_1\} \{x_2 : m \mathbf{T}_2\} \\ \text{mode_IsValid } m \\ \text{ctx_ValidOnly } \Delta_2 \\ m \cdot \Delta_1 \uplus \Delta_2 \vdash C : \mathbf{U} \multimap \mathbf{U}_0 \\ \Delta_2 \uplus \{x_1 : m \mathbf{T}_1\} \uplus \{x_2 : m \mathbf{T}_2\} \vdash u : \mathbf{U} \end{array}}{\Delta_1 \vdash C \circ (\square \succ \text{case}_m (x_1, x_2) \mapsto u) : (\mathbf{T}_1 \otimes \mathbf{T}_2) \multimap \mathbf{U}_0}$	
$\text{TY-ECTXS-PATEFoc} \quad \frac{\begin{array}{l} \text{ctx_Disjoint } \Delta_1 \Delta_2 \\ \text{ctx_DestOnly } \Delta_1 \\ \text{ctx_DestOnly } \Delta_2 \\ \text{mode_IsValid } m \\ \text{mode_IsValid } m' \\ \text{ctx_ValidOnly } \Delta_2 \\ m \cdot \Delta_1 \uplus \Delta_2 \vdash C : \mathbf{U} \multimap \mathbf{U}_0 \\ \Delta_2 \uplus \{x : m \cdot m' \mathbf{T}\} \vdash u : \mathbf{U} \end{array}}{\Delta_1 \vdash C \circ (\square \succ \text{case}_m \text{E}^{m'} x \mapsto u) : !^{m'} \mathbf{T} \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-MAPFoc} \quad \frac{\begin{array}{l} \text{ctx_Disjoint } \Delta_1 \Delta_2 \\ \text{ctx_DestOnly } \Delta_1 \\ \text{ctx_DestOnly } \Delta_2 \\ \text{ctx_ValidOnly } \Delta_2 \\ \Delta_1 \uplus \Delta_2 \vdash C : \mathbf{U} \ltimes \mathbf{T}' \multimap \mathbf{U}_0 \\ \mathcal{I} \uparrow \cdot \Delta_2 \uplus \{x : \mathcal{I} \nu \mathbf{T}\} \vdash t' : \mathbf{T}' \end{array}}{\Delta_1 \vdash C \circ (\square \succ \text{map } x \mapsto t') : (\mathbf{U} \ltimes \mathbf{T}) \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-TOAFoc} \quad \frac{\Delta \vdash C : (\mathbf{U} \ltimes \mathbf{1}) \multimap \mathbf{U}_0}{\Delta \vdash C \circ (\text{to}_\ltimes \square) : \mathbf{U} \multimap \mathbf{U}_0}$
$\text{TY-ECTXS-FROMAFoc} \quad \frac{\Delta \vdash C : \mathbf{U} \multimap \mathbf{U}_0}{\Delta \vdash C \circ (\text{from}_\ltimes \square) : (\mathbf{U} \ltimes \mathbf{1}) \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-FILLUFoc} \quad \frac{\Delta \vdash C : \mathbf{1} \multimap \mathbf{U}_0}{\Delta \vdash C \circ (\square \triangleleft ()) : [\mathbf{1}]^n \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-FILLLFoc} \quad \frac{\Delta \vdash C : [\mathbf{T}_1]^n \multimap \mathbf{U}_0}{\Delta \vdash C \circ (\square \triangleleft \text{Inl}) : [\mathbf{T}_1 \oplus \mathbf{T}_2]^n \multimap \mathbf{U}_0}$
$\text{TY-ECTXS-FILLRFoc} \quad \frac{\Delta \vdash C : [\mathbf{T}_2]^n \multimap \mathbf{U}_0}{\Delta \vdash C \circ (\square \triangleleft \text{Inr}) : [\mathbf{T}_1 \oplus \mathbf{T}_2]^n \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-FILLPFoc} \quad \frac{\Delta \vdash C : ([\mathbf{T}_1]^n \otimes [\mathbf{T}_2]^n) \multimap \mathbf{U}_0}{\Delta \vdash C \circ (\square \triangleleft (,)) : [\mathbf{T}_1 \otimes \mathbf{T}_2]^n \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-FILLEFoc} \quad \frac{\begin{array}{l} \text{mode_IsValid } m \\ \Delta \vdash C : [\mathbf{T}]^{m \cdot n} \multimap \mathbf{U}_0 \end{array}}{\Delta \vdash C \circ (\square \triangleleft \text{E}^m) : [!^m \mathbf{T}]^n \multimap \mathbf{U}_0}$
$\text{TY-ECTXS-FILLFFoc} \quad \frac{\begin{array}{l} \text{ctx_Disjoint } \Delta_1 \Delta_2 \\ \text{ctx_DestOnly } \Delta_1 \\ \text{ctx_DestOnly } \Delta_2 \\ \text{ctx_ValidOnly } \Delta_2 \\ \text{mode_IsValid } m \\ \text{mode_IsValid } n \\ \Delta_1 \uplus (\mathcal{I} \uparrow \cdot n) \cdot \Delta_2 \vdash C : \mathbf{1} \multimap \mathbf{U}_0 \\ \Delta_2 \uplus \{x : m \mathbf{T}\} \vdash u : \mathbf{U} \end{array}}{\Delta_1 \vdash C \circ (\square \triangleleft (\lambda x_m \mapsto u)) : [\mathbf{T} \xrightarrow{m} \mathbf{U}]^n \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-FILLCFoc1} \quad \frac{\begin{array}{l} \text{ctx_Disjoint } \Delta_1 \Delta_2 \\ \text{ctx_DestOnly } \Delta_1 \\ \text{ctx_DestOnly } \Delta_2 \\ \text{ctx_ValidOnly } \Delta_2 \\ \text{mode_IsValid } n \\ \Delta_1 \uplus (\mathcal{I} \uparrow \cdot n) \cdot \Delta_2 \vdash C : \mathbf{T} \multimap \mathbf{U}_0 \\ \Delta_2 \vdash t' : \mathbf{U} \ltimes \mathbf{T} \end{array}}{\Delta_1 \vdash C \circ (\square \triangleleft \bullet t') : [\mathbf{U}]^n \multimap \mathbf{U}_0}$	$\text{TY-ECTXS-FILLCFoc2} \quad \frac{\begin{array}{l} \text{ctx_Disjoint } \Delta_1 \Delta_2 \\ \text{ctx_DestOnly } \Delta_1 \\ \text{ctx_DestOnly } \Delta_2 \\ \text{ctx_ValidOnly } \Delta_1 \\ \text{mode_IsValid } n \\ \Delta_1 \uplus (\mathcal{I} \uparrow \cdot n) \cdot \Delta_2 \vdash C : \mathbf{T} \multimap \mathbf{U}_0 \\ \Delta_1 \vdash v : [\mathbf{U}]^n \end{array}}{\Delta_2 \vdash C \circ (v \triangleleft \bullet) : \mathbf{U} \ltimes \mathbf{T} \multimap \mathbf{U}_0}$

$$\begin{array}{c}
\text{TY-ECTXS-AOPENFOC} \\
\text{ctx_Disjoint } \Delta_1 \ \Delta_2 \\
\text{ctx_Disjoint } \Delta_1 \ \Delta_3 \\
\text{hdns_Disjoint } \text{hnames}(\text{C}) \ \text{hnames}(-\Delta_3) \\
\text{ctx_DestOnly } \Delta_1 \\
\text{ctx_DestOnly } \Delta_2 \\
\text{ctx_DestOnly } \Delta_3 \\
\text{ctx_LinOnly } \Delta_3 \\
\text{ctx_FinAgeOnly } \Delta_3 \\
\text{ctx_ValidOnly } \Delta_3 \\
\Delta_1 \uplus \Delta_2 \vdash \text{C} : (\mathbf{U} \ltimes \mathbf{T}') \multimap \mathbf{U}_0 \\
\Delta_2 \uplus -\Delta_3 \Vdash v_2 : \mathbf{U} \\
\hline
\textcolor{teal}{t} \uparrow \Delta_1 \uplus \Delta_3 \vdash \text{C} \circ (\overset{\text{op}}{\text{hnames}}(-\Delta_3)) \langle v_2, \square \rangle : \mathbf{T}' \multimap \mathbf{U}_0
\end{array}$$

$$\vdash \text{C}[t] : \mathbf{T}$$

(Typing of extended terms (pair of evaluation context and term))

$$\begin{array}{c}
\text{TY-ETERM-CLOSEDTERM} \\
\text{ctx_ValidOnly } \Delta \\
\text{ctx_DestOnly } \Delta \\
\Delta \vdash \text{C} : \mathbf{T} \multimap \mathbf{U}_0 \quad \Delta \vdash t : \mathbf{T} \\
\hline
\vdash \text{C}[t] : \mathbf{U}_0
\end{array}$$

3 Small-step semantics

$C[t] \longrightarrow C'[t']$			<i>(Small-step evaluation of terms using evaluation contexts)</i>		
SEM-ETERM-APPFOC1 $\text{term_NotVal } t$	SEM-ETERM-APPUNFOC1	SEM-ETERM-APPFOC2 $\text{term_NotVal } t'$			
$\overline{C[t \succ t'] \longrightarrow (C \circ (\Box \succ t'))[t]}$	$\overline{(C \circ (\Box \succ t'))[v] \longrightarrow C[v \succ t']}$	$\overline{C[v \succ t'] \longrightarrow (C \circ (v \succ \Box))[t']}$			
SEM-ETERM-APPUNFOC2	SEM-ETERM-APPRED	SEM-ETERM-PATUFOC $\text{term_NotVal } t$			
$\overline{(C \circ (v \succ \Box))[v'] \longrightarrow C[v \succ v']}$	$\overline{C[v \succ (\lambda^v x_m \mapsto u)] \longrightarrow C[u[x := v]]}$	$\overline{C[t ; u] \longrightarrow (C \circ (\Box ; u))[t]}$			
SEM-ETERM-PATUUNFOC		SEM-ETERM-PATURED			
$\overline{(C \circ (\Box ; u))[v] \longrightarrow C[v ; u]}$		$\overline{C[(\cdot) ; u] \longrightarrow C[u]}$			
SEM-ETERM-PATSFOC					
$\text{term_NotVal } t$					
$\overline{C[t \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}] \longrightarrow (C \circ (\Box \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}))[t]}$					
SEM-ETERM-PATSUNFOC					
$\overline{(C \circ (\Box \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}))[v] \longrightarrow C[v \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}]}$					
SEM-ETERM-PATLRED					
$\overline{C[(\text{Inl } v_1) \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}] \longrightarrow C[u_1[x_1 := v_1]]}$					
SEM-ETERM-PATRRED					
$\overline{C[(\text{Inr } v_2) \succ \text{case}_m \{ \text{Inl } x_1 \mapsto u_1, \text{Inr } x_2 \mapsto u_2 \}] \longrightarrow C[u_2[x_2 := v_2]]}$					
SEM-ETERM-PATPFOC					
$\text{term_NotVal } t$					
$\overline{C[t \succ \text{case}_m (x_1, x_2) \mapsto u] \longrightarrow (C \circ (\Box \succ \text{case}_m (x_1, x_2) \mapsto u))[t]}$					
SEM-ETERM-PATPUNFOC					
$\overline{(C \circ (\Box \succ \text{case}_m (x_1, x_2) \mapsto u))[v] \longrightarrow C[v \succ \text{case}_m (x_1, x_2) \mapsto u]}$					
SEM-ETERM-PATPREd			SEM-ETERM-PATEFOC		
$\overline{C[(v_1, v_2) \succ \text{case}_m (x_1, x_2) \mapsto u] \longrightarrow C[u[x_1 := v_1][x_2 := v_2]]}$			$\text{term_NotVal } t$		
$\overline{C[(v_1, v_2) \succ \text{case}_m (x_1, x_2) \mapsto u] \longrightarrow C[u[x_1 := v_1][x_2 := v_2]]}$			$\overline{C[t \succ \text{case}_m E^n x \mapsto u] \longrightarrow (C \circ (\Box \succ \text{case}_m E^n x \mapsto u))[t]}$		
SEM-ETERM-PATEUNFOC			SEM-ETERM-PATERED		
$\overline{(C \circ (\Box \succ \text{case}_m E^n x \mapsto u))[v] \longrightarrow C[v \succ \text{case}_m E^n x \mapsto u]}$			$\overline{C[E^n v' \succ \text{case}_m E^n x \mapsto u] \longrightarrow C[u[x := v']]}$		
SEM-ETERM-MAPFOC			SEM-ETERM-MAPUNFOC		
$\text{term_NotVal } t$			$\text{term_NotVal } t$		
$\overline{C[t \succ \text{map } x \mapsto t'] \longrightarrow (C \circ (\Box \succ \text{map } x \mapsto t'))[t]}$			$\overline{(C \circ (\Box \succ \text{map } x \mapsto t'))[v] \longrightarrow C[v \succ \text{map } x \mapsto t']}$		
SEM-ETERM-MAPREDAOPENFOC			SEM-ETERM-AOPENUNFOC		
$h' = \max(hnames(C)) + 1$			$h' = \max(hnames(C)) + 1$		
$\overline{C[\langle v_2, v_1 \rangle \succ \text{map } x \mapsto t'] \longrightarrow (C \circ (\overset{\text{op}}{H \pm h} \langle v_2, \Box \rangle, \Box))[t' [x := v_1 [H \pm h']]]}$			$\overline{(C \circ \overset{\text{op}}{H} \langle v_2, \Box \rangle)[v_1] \longrightarrow C[\langle v_2, v_1 \rangle]}$		
SEM-ETERM-TOAFOC			SEM-ETERM-TOAUNFOC		
$\text{term_NotVal } u$			$\text{term_NotVal } u$		
$\overline{C[\text{to}_x u] \longrightarrow (C \circ (\text{to}_x \Box))[u]}$			$\overline{(C \circ (\text{to}_x \Box))[v_2] \longrightarrow C[\text{to}_x v_2]}$		
SEM-ETERM-TOARED			SEM-ETERM-TOARED		
$\overline{C[\text{to}_x v_2] \longrightarrow C[\{\cdot\} \langle v_2, (\cdot) \rangle]}$			$\overline{C[\text{to}_x v_2] \longrightarrow C[\{\cdot\} \langle v_2, (\cdot) \rangle]}$		
SEM-ETERM-FROMAFOC			SEM-ETERM-FROMAUNFOC		
$\text{term_NotVal } t$			$\text{term_NotVal } t$		
$\overline{C[\text{from}_x t] \longrightarrow (C \circ (\text{from}_x \Box))[t]}$			$\overline{(C \circ (\text{from}_x \Box))[v] \longrightarrow C[\text{from}_x v]}$		
SEM-ETERM-FROMARED			SEM-ETERM-FROMARED		
$\overline{C[\text{from}_x \{\cdot\} \langle v_2, (\cdot) \rangle] \longrightarrow C[v_2]}$			$\overline{C[\text{from}_x \{\cdot\} \langle v_2, (\cdot) \rangle] \longrightarrow C[v_2]}$		
SEM-ETERM-FILLUFOC			SEM-ETERM-FILLUUNFOC		
$\text{term_NotVal } t$			$\text{term_NotVal } t$		
$\overline{C[t \triangleleft (\cdot)] \longrightarrow (C \circ (\Box \triangleleft (\cdot)))[t]}$			$\overline{(C \circ (\Box \triangleleft (\cdot)))[v] \longrightarrow C[v \triangleleft (\cdot)]}$		
SEM-ETERM-FILLURED			SEM-ETERM-FILLURED		
$\overline{C[+h \triangleleft (\cdot)] \longrightarrow C[h := \{\cdot\} (\cdot)]}$			$\overline{C[+h \triangleleft (\cdot)] \longrightarrow C[h := \{\cdot\} (\cdot)]}$		

$\frac{\text{SEM-ETERM-FILLFOC} \quad \text{term_NotVal } t}{C[t \triangleleft \text{Inl}] \longrightarrow (C \circ (\Box \triangleleft \text{Inl}))[t]}$	$\frac{\text{SEM-ETERM-FILLUNFOC}}{(C \circ (\Box \triangleleft \text{Inl}))[v] \longrightarrow C[v \triangleleft \text{Inl}]}$	$\frac{\text{SEM-ETERM-FILLRED} \quad h' = \max(\text{hnames}(C) \cup \{h\}) + 1}{C[+h \triangleleft \text{Inl}] \longrightarrow C[h :=_{\{h'+1\}} \text{Inl} - (h'+1)][+(h'+1)]}$
$\frac{\text{SEM-ETERM-FILLRFOC} \quad \text{term_NotVal } t}{C[t \triangleleft \text{Inr}] \longrightarrow (C \circ (\Box \triangleleft \text{Inr}))[t]}$	$\frac{\text{SEM-ETERM-FILLRUNFOC}}{(C \circ (\Box \triangleleft \text{Inr}))[v] \longrightarrow C[v \triangleleft \text{Inr}]}$	
$\frac{\text{SEM-ETERM-FILLRRED} \quad h' = \max(\text{hnames}(C) \cup \{h\}) + 1}{C[+h \triangleleft \text{Inr}] \longrightarrow C[h :=_{\{h'+1\}} \text{Inr} - (h'+1)][+(h'+1)]}$	$\frac{\text{SEM-ETERM-FILLEFOC} \quad \text{term_NotVal } t}{C[t \triangleleft E^m] \longrightarrow (C \circ (\Box \triangleleft E^m))[t]}$	
$\frac{\text{SEM-ETERM-FILLEUNFOC}}{(C \circ (\Box \triangleleft E^m))[v] \longrightarrow C[v \triangleleft E^m]}$	$\frac{\text{SEM-ETERM-FILLERED} \quad h' = \max(\text{hnames}(C) \cup \{h\}) + 1}{C[+h \triangleleft E^m] \longrightarrow C[h :=_{\{h'+1\}} E^m - (h'+1)][+(h'+1)]}$	
$\frac{\text{SEM-ETERM-FILLPFOC} \quad \text{term_NotVal } t}{C[t \triangleleft (,)] \longrightarrow (C \circ (\Box \triangleleft (,)))[t]}$	$\frac{\text{SEM-ETERM-FILLPUNFOC}}{(C \circ (\Box \triangleleft (,)))[v] \longrightarrow C[v \triangleleft (,)]}$	
$\frac{\text{SEM-ETERM-FILLPRD} \quad h' = \max(\text{hnames}(C) \cup \{h\}) + 1}{C[+h \triangleleft (,)] \longrightarrow C[h :=_{\{h'+1, h'+2\}} (- (h'+1), - (h'+2))][+(h'+1), +(h'+2)]}$		
$\frac{\text{SEM-ETERM-FILLFFOC} \quad \text{term_NotVal } t}{C[t \triangleleft (\lambda x_m \mapsto u)] \longrightarrow (C \circ (\Box \triangleleft (\lambda x_m \mapsto u)))[t]}$	$\frac{\text{SEM-ETERM-FILLFUNFOC}}{(C \circ (\Box \triangleleft (\lambda x_m \mapsto u)))[v] \longrightarrow C[v \triangleleft (\lambda x_m \mapsto u)]}$	
$\frac{\text{SEM-ETERM-FILLFRD}}{C[+h \triangleleft (\lambda x_m \mapsto u)] \longrightarrow C[h :=_{\{ \}} \lambda^v x_m \mapsto u][()]} \quad \frac{\text{SEM-ETERM-FILLCFoc1} \quad \text{term_NotVal } t}{C[t \triangleleft \bullet t'] \longrightarrow (C \circ (\Box \triangleleft \bullet t'))[t]} \quad \frac{\text{SEM-ETERM-FILLCUNFOC1}}{(C \circ (\Box \triangleleft \bullet t'))[v] \longrightarrow C[v \triangleleft \bullet t']}$		
$\frac{\text{SEM-ETERM-FILLCFoc2} \quad \text{term_NotVal } t'}{C[v \triangleleft \bullet t'] \longrightarrow (C \circ (v \triangleleft \bullet \Box))[t']}$	$\frac{\text{SEM-ETERM-FILLCUNFOC2}}{(C \circ (v \triangleleft \bullet \Box))[v'] \longrightarrow C[v \triangleleft \bullet v']}$	$\frac{\text{SEM-ETERM-FILLCRD} \quad h' = \max(\text{hnames}(C) \cup \{h\}) + 1}{C[+h \triangleleft \bullet_H \langle v_2, v_1 \rangle] \longrightarrow C[h :=_{(H \pm h')} v_2 [H \pm h']][v_1 [H \pm h']}]}$

4 Remarks on the Coq proofs

- Not particularly elegant. Max number of goals observed 232 (solved by a single call to the `congruence` tactic). When you have a computer, brute force is a viable strategy. (in particular, no semiring formalisation, it was quicker to do directly)
- Rules generated by ott, same as in the article (up to some notational difference). Contexts are not generated purely by syntax, and are interpreted in a semantic domain (finite functions).
- Reasoning on closed terms avoids almost all complications on binder manipulation. Makes proofs tractable.
- Finite functions: making a custom library was less headache than using existing libraries (including `MMap`). Existing libraries don't provide some of the tools that we needed, but the most important factor ended up being the need for a modicum of dependency between key and value. There wasn't really that out there. Backed by actual functions for simplicity; cost: equality is complicated.
- Most of the proofs done by author with very little prior experience to Coq.
- Did proofs in Coq because context manipulations are tricky.
- A number of wrong lemma initially assumed, but replacing them by correct variant was always easy to fix in proofs.
- Context sum made total by adding an extra invalid *mode* (rather than an extra context). It seems to be much simpler this way.
- It might be a good idea to provide statistics on the number of lemmas and size of Coq codebase.