Linearized Einstein equations

FJ

1 Introduction

Maxwell equations, the equations describing the behaviour of the electromagnetic field, come with rather startling implications. One of the implications becomes aparent, when considering the Maxwell equations for vacuum (setting charge and current to 0) and substituting from one equation into another, one can produce the well-known *wave equation*, meaning that disturbances in the electromagnetic field are able to propagate though otherwise empty space as a wave. The wave equation also predicts the speed at which this wave should propagate and which turns out to be the speed of light, directly hinting to the fact, that light itself is an electromagnetic wave.

Now as is the case with Maxwell equations, the Einstein equations (1) can be manipulated to produce a wave equation for a small perturbation $h_{\mu\nu}$ in a nearly flat spacetime given by metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, showing that disturbances in a metric tensor can propagate as a waves too.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1}$$

The Einstein equations quantify a relationship between a metric tensor $g_{\mu\nu}$ (Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are both functions of $g_{\mu\nu}$ and it's derivatives), which quantifies how distances in space and time should be measured, and the energy-momentum tensor $T_{\mu\nu}$ containing the energy and momentum contents of space. In other words, it's according to this equation (1) that matter tells spacetime how to curve and spacetime tells matter how to move.

2 Gravitational waves

Gravitational waves are essentially vibrations in this small deviation $h_{\mu\nu}$ from a otherwise flat spacetime, as is given by the relationship

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{2}$$

We also assume that the preturbation as well as it's rate of change are small compared to the components of $\eta_{\mu\nu}$

$$|h_{\mu\nu}| \ll 1, |h_{\mu\nu,\sigma}| \ll 1. \tag{3}$$

This let's us ignore any terms which would be proportional the the perturbation squared as well as the product of the perturbation and it's derivative.

To deduce the equation for gravitational waves, we are going to need to calculate:

- the inverse metric $g^{\mu\nu}$,
- the connection coefficients $\Gamma^{\sigma}_{\ \mu\nu}$,
- the Riemann tensor $R^{\rho}_{\ \sigma\mu\nu}$,
- the Ricci tensor $R_{\mu\nu}$ and
- the Ricci scalar R.

2.1 The inverse metric $g^{\mu\nu}$

Let us assume that the inverse metric will be given by

$$g^{\mu\nu} = \eta^{\mu\nu} + k^{\mu\nu}, \tag{4}$$

with $k^{\mu\nu}$ being some small ($|k^{\mu\nu}| \ll 1$) but arbitrary different perturbation generally not inverse to the $h_{\mu\nu}$. We can calculate the $k^{\mu\nu}$ in the following way.

$$\begin{gathered}
g_{\mu\sigma} g^{\sigma\nu} = \delta_{\mu}^{\ \nu} \\
(\eta_{\mu\sigma} + h_{\mu\sigma})(\eta^{\sigma\nu} + k^{\sigma\nu}) = \delta_{\mu}^{\ \nu} \\
\eta_{\mu\sigma}\eta^{\sigma\nu} + \eta_{\mu\sigma}k^{\sigma\nu} + h_{\mu\sigma}\eta^{\sigma\nu} + h_{\mu\sigma}k^{\sigma\nu} = \delta_{\mu}^{\ \nu} \\
\delta_{\mu}^{\ \nu} + \eta_{\mu\sigma}k^{\sigma\nu} + h_{\mu\sigma}\eta^{\sigma\nu} + h_{\mu\sigma}k^{\sigma\nu} = \delta_{\mu}^{\ \nu}
\end{gathered}$$

We can subtract the $\delta_{\mu}^{\ \nu}$ from both sides, the $h_{\mu\sigma}k^{\sigma\nu}$ term we can ignore, for it is a product of two small terms.

$$\eta_{\mu\sigma}k^{\sigma\nu} + h_{\mu\sigma}\eta^{\sigma\nu} = 0 \tag{5}$$

Now we can use $\eta_{\mu\sigma}$ to raise and lower the indicies, normally we should use the entire metric $g_{\mu\nu}$ to do so, but since that would lead to

$$k^{\mu}_{\ \nu} = g_{\nu\sigma}k^{\mu\sigma} = (\eta_{\nu\sigma} + h_{\nu\sigma})k^{\mu\sigma} \approx \eta_{\nu\sigma}k^{\mu\sigma},$$

because the term $h_{\nu\sigma}k^{\mu\sigma}$ is second order in small quantities we can ignore it and essentially use only the $\eta_{\nu\sigma}$ to lower the index anyway. This lets us rewrite the equation 5 in a following way.

$$k_{\mu}^{\ \nu} = -h_{\mu}^{\ \nu}$$

$$k_{\mu}^{\ \nu}\eta^{\mu\rho} = -h_{\mu}^{\ \nu}\eta^{\mu\rho}$$

$$k^{\ \rho\nu} = -h^{\rho\nu}$$

We can see that the perturbation $k^{\mu\nu}$ in the inverse metric $g^{\mu\nu}$ is equal to $-h^{\mu\nu}$, therefore we get

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}$$

$$\boxed{g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}}$$

while $h^{\mu\nu}$ is not inverse to $h_{\mu\nu}$ but is just equal to $h_{\mu\nu}$ with both indices raised by the Minkowski metric $h^{\mu\nu} = h_{\rho\sigma}\eta^{\rho\mu}\eta^{\sigma\nu}$.

2.2 The connection coefficients $\Gamma^{\sigma}_{\ \mu\nu}$

In any general metric, the connection coefficients are calculated as

$$\Gamma^{\sigma}_{\ \mu\nu} = \frac{1}{2}g^{\sigma\rho}(-g_{\mu\nu,\rho} + g_{\rho\mu,\nu} + g_{\nu\rho,\mu}).$$

Given our metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, any derivatives of type $\eta_{\mu\nu,\rho}$ vanish since $\eta_{\mu\nu}$ is contant and the connection coefficient simplifies to

$$\Gamma^{\sigma}_{\ \mu\nu} = \frac{1}{2}g^{\sigma\rho}(-h_{\mu\nu,\rho} + h_{\rho\mu,\nu} + h_{\nu\rho,\mu}).$$

Now if we substitute $(\eta^{\sigma\rho} - h^{\sigma\rho})$ for $g^{\sigma\rho}$ and multiply the brackets, we can leave out all the terms proportional to product of $h^{\sigma\rho}$ and it's derivative, leaving us with only 3 terms inside the bracket. Namely those that were multiplied by $\eta^{\sigma\rho}$. The connection coefficients take the following form.

$$\boxed{\Gamma^{\sigma}_{\ \mu\nu} = \frac{1}{2}\eta^{\sigma\rho}(-h_{\mu\nu,\rho} + h_{\rho\mu,\nu} + h_{\nu\rho,\mu})} \tag{6}$$

2.3 Riemann tensor $R^{\rho}{}_{\sigma\mu\nu}$

Now that we have found the connection coefficients, we can use them to calculate the Riemann tensor as follows.

$$R^{\rho}{}_{\sigma\mu\nu} = -\Gamma^{\rho}{}_{\sigma\mu,\nu} + \Gamma^{\rho}{}_{\sigma\nu,\mu} - \Gamma^{\lambda}{}_{\sigma\mu}\Gamma^{\rho}{}_{\nu\lambda} + \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\rho}{}_{\mu\lambda}$$

The last two terms, being products of two connection coefficient, yield 9 terms each but all of them proportional to the product of two $h_{\mu\nu}$ derivatives and can therefore be ignored. This fact lets us simplify the Riemann tensor to a much simpler form.

$$R^{\rho}{}_{\sigma\mu\nu} = -\Gamma^{\rho}{}_{\sigma\mu,\nu} + \Gamma^{\rho}{}_{\sigma\nu,\mu}$$

$$\Gamma^{\rho}{}_{\sigma\mu,\nu} = \frac{1}{2}[\eta^{\rho\alpha}(-h_{\sigma\mu,\alpha} + h_{\alpha\sigma,\mu} + h_{\mu\alpha,\sigma})]_{,\nu}$$

$$\Gamma^{\rho}{}_{\sigma\mu,\nu} = \frac{1}{2}\eta^{\rho\alpha}(-h_{\sigma\mu,\alpha\nu} + h_{\alpha\sigma,\mu\nu} + h_{\mu\alpha,\sigma\nu})$$

$$R^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}(h_{\sigma\mu,\alpha\nu} - h_{\alpha\sigma,\mu\nu} - h_{\mu\alpha,\sigma\nu} - h_{\sigma\nu,\alpha\mu} + h_{\alpha\sigma,\nu\mu} + h_{\nu\alpha,\sigma\mu})$$

We are of course taking derivatives of the products $\eta^{\rho\alpha}h_{\sigma\mu,\alpha}$, but $\eta^{\rho\alpha}$ is just a matrix of constants, so we're just left with the $h_{\sigma\mu,\alpha}$ like derivatives. Now since the order of partial derivatives does not matter, the terms $h_{\alpha\sigma,\mu\nu}$ and $h_{\alpha\sigma,\nu\mu}$ are identical and cancel out.

$$\boxed{R^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}(h_{\sigma\mu,\alpha\nu} - h_{\mu\alpha,\sigma\nu} - h_{\sigma\nu,\alpha\mu} + h_{\nu\alpha,\sigma\mu})} \tag{7}$$

2.4 Ricci tensor $R_{\sigma\nu}$

The Ricci tensor is calculated by contracting the Riemann tensor in first and third index $R_{\sigma\nu} = R^{\mu}{}_{\sigma\mu\nu}$.

$$R^{\mu}{}_{\sigma\mu\nu} = \frac{1}{2}\eta^{\mu\alpha}(h_{\sigma\mu,\alpha\nu} - h_{\mu\alpha,\sigma\nu} - h_{\sigma\nu,\alpha\mu} + h_{\nu\alpha,\sigma\mu})$$

$$R^{\mu}{}_{\sigma\mu\nu} = \frac{1}{2}(h_{\sigma}{}^{\alpha}{}_{,\alpha\nu} - h_{\mu\alpha,\sigma\nu} - h_{\sigma\nu,\alpha\mu} + h_{\nu}{}^{\mu}{}_{,\sigma\mu})$$