Kolmogorov's Theorem

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Theorem 0.1 (Kolmogorov's Theorem)**.** *Let $\rho, \gamma > 0$ be given, and let $h(\boldsymbol{q}, \boldsymbol{p}) = h_0(\boldsymbol{p}) + h_1(\boldsymbol{q}, \boldsymbol{p})$ be a Hamiltonian, with $h_0, h_1 \in \mathcal{A}_\rho$ and $\|h\|_\rho \le 1$. Suppose the Taylor polynomial of h_0 is*

$$h_0(\boldsymbol{p}) = a + \omega \boldsymbol{p} + \frac{1}{2}\boldsymbol{p} \cdot C\boldsymbol{p} + o(|\boldsymbol{p}|^2),$$

with $\omega \in \Omega_\gamma$ and C is symmetric and invertible. Then for any $\rho_* \le \rho$, there exists $\epsilon > 0$, which depends on C and γ, but not on the remainder term in $o(|\boldsymbol{p}|^2)$, such that if $\|h_1\|_\rho \le \epsilon$, there exists a symplectic mapping $\Phi : A_{\rho_*} \to A_\rho$ such that if we set $(\boldsymbol{q}, \boldsymbol{p}) = \Phi(\boldsymbol{Q}, \boldsymbol{P})$ and $H = h \circ \Phi$, we have

$$H(\boldsymbol{Q}, \boldsymbol{P}) = A + \omega \boldsymbol{P} + R(\boldsymbol{Q}, \boldsymbol{P}),$$

with $R(\boldsymbol{Q}, \boldsymbol{P}) \in O(|\boldsymbol{P}|^2)$.

In particular the motion

$$\begin{aligned} \boldsymbol{Q}(t) &= \boldsymbol{Q}(0) + t\omega_0, \\ \boldsymbol{P}(t) &= 0, \end{aligned}$$

is a solution of the Hamiltonian equation that is conjugate to the linear flow with direction ω_0, so that the invariant torus $\boldsymbol{p} = 0$ is preserved by the perturbation.

If (X, σ) is a symplectic manifold, then any function H on X has a symplectic gradient $\nabla_\sigma H$ defined as the unique vector field such that for any vector field ξ, it holds that $\sigma(\xi, \nabla_\sigma H) = dH(\xi)$. The Hamiltonian differential equation is $\dot{x} = (\nabla_\sigma H)(x)$.

$\nabla_\sigma f$ has a flow ϕ_f^t such that $f \circ \phi_f^t = f$ and $(\phi_f^t)^* \sigma = \sigma$.

Definition 0.2 (Poisson Bracket)**.** The *Poisson Bracket* of X is defined by

$$\{f, g\} = \sigma(\nabla_\sigma g, \nabla_\sigma f) = df(\nabla_\sigma g) = -dg(\nabla_\sigma f).$$

Functions f and g *commute* if $\{f, g\} = 0$. This condition implies that their flows commute:

$$\phi_f(s) \circ \phi_g(t) = \phi_g(t) \circ \phi_f(s).$$

The symplectic gradient of the Poisson bracket is the Lie bracket:

$$\nabla_\sigma \{f, g\} = [\nabla_\sigma f, \nabla_\sigma g].$$

Example 0.3 (Hamiltonian Equations of Motion)**.** Take $X = \mathbb{R}^{2n}$ with coordinates $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $\sigma = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$. Then the Hamiltonian differential equation becomes the *Hamiltonian Equations of Motion*:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

We can integrate these equations. The solution with initial conditions $(\boldsymbol{q}_0, \boldsymbol{p}_0)$ is

$$\begin{aligned} \boldsymbol{q}(t) &= \boldsymbol{q}_0 + t\frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{p}_0) =: \boldsymbol{q}_0 + t\omega(\boldsymbol{p}_0), \\ \boldsymbol{p}(t) &= \boldsymbol{p}_0. \end{aligned}$$

Each coordinate $p_1, \ldots p_n$ is conserved, and the motion is a linear motion on the torus $\mathbb{T}^n \times \{\boldsymbol{p}_0\}$. $\triangle$

The Poisson bracket allows us to write Taylor Series as

$$f \circ \phi_g^t = f + t\{f, g\} + \frac{t^2}{2}\{\{f, g\}, g\} + \frac{t^3}{3!}\{\{\{f, g\}, g\}, g\} + \cdots .$$

Definition 0.4 (Diophantine Number of Exponent d)**.** A number θ is *Diophantine of exponent d* if there exists a constant $\gamma > 0$ such that for all coprime integers p and q we have

$$\left|\theta - \frac{p}{q}\right| > \frac{\gamma}{|q|^d}.$$

Definition 0.5 (Diophantine Vector)**.** Let $\omega = (\omega_1, \dots, \omega_n)$. We say ω is Diophantine if there exists $\gamma > 0$ such that for all vectors with integer coefficients $(k_1, \dots, k_n)$, we have

$$|k_1\omega_1 + \cdots + k_n\omega_n| \geq \frac{\gamma}{(k_1^2 + \cdots + k_n^2)^{\frac{n}{2}}}.$$

Let Ω_γ^n be the subset of such $\omega \in \mathbb{R}^n$.

If an analytic function f is bounded on an open set U and if V is relatively compact in U, then we can bound the second derivatives of f on V in terms of $\sup_U |f|$. The important domains are

$$\begin{aligned}
B_\rho &= \{\boldsymbol{p} \in \mathbb{C} : |\boldsymbol{p}| \leq \rho\}, \\
C_\rho &= \{\boldsymbol{q} \in \mathbb{C}^n/\mathbb{Z}^n : |\operatorname{Im}(\boldsymbol{q})| \leq \rho\}, \\
A_\rho &= C_\rho \times B_\rho = \{(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^n/\mathbb{Z}^n \times \mathbb{C}^n : |\boldsymbol{p}| \leq \rho, |\operatorname{Im}(\boldsymbol{q})| \leq \rho\}.
\end{aligned}$$

Denote by $\mathcal{B}_\rho$, $\mathcal{C}_\rho$, and $\mathcal{A}_\rho$ the corresponding Banach algebras of functions continuous on these compact sets and analytic on the interiors, with sup-norm $\|f\|_\rho$.

Definition 0.6 (Banach Algebra)**.** Let k be $\mathbb{R}$ or $\mathbb{C}$. A *normed algebra* over k is an algebra $\mathcal{A}$ over k with a sub-multiplicative norm $\|.\|$, that is, for all $x, y \in \mathcal{A}$, we have $\|xy\| \leq \|x\| \, \|y\|$, If $\mathcal{A}$ is a Banach space, then it is called a *Banach algebra.*

Elements of $\mathcal{B}_\rho$ can be expanded as Power series. Elements of $\mathcal{C}_\rho$ can be expanded as Fourier series $f(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^{2n}} f_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i}\, \boldsymbol{k} \cdot \boldsymbol{z}}$.

Let $g \in \mathcal{C}_\rho$. We are going to be solving linear equations of the form

$$\mathrm{D}f(\omega) = \sum_{i=1}^n \omega_i \frac{\partial f}{\partial q_i} = g,$$

with $f \in \mathcal{C}_{\rho'}$ for some $\rho' < \rho$.

Write the functions f and g as Fourier Series:

$$f(\boldsymbol{q}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^n} f_{\boldsymbol{k}} e^{2\pi i\, \boldsymbol{k} \cdot \boldsymbol{q}}, \qquad g(\boldsymbol{q}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^n} g_{\boldsymbol{k}} e^{2\pi i\, \boldsymbol{k} \cdot \boldsymbol{q}}.$$

The solution is given by

$$f_{\boldsymbol{k}} = \frac{1}{2\pi i\, (\boldsymbol{k} \cdot \omega)} g_{\boldsymbol{k}}.$$

We need to solve the equations

$$\begin{aligned}
DX(\boldsymbol{q})(\omega) &= -A(\boldsymbol{q}) \\
DY(\boldsymbol{q})(\omega) &= -B(\boldsymbol{q}) - \big(\lambda + DX(\boldsymbol{q})\big)C(\boldsymbol{q})
\end{aligned}$$

for X and Y_i, $i = 1, \dots, n$.