

# Kolmogorov's Theorem

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Our goal is to understand the following theorem.

### Kolmogorov's Theorem

Let  $\rho, \gamma > 0$  be given, and let  $h(\mathbf{q}, \mathbf{p}) = h_0(\mathbf{p}) + h_1(\mathbf{q}, \mathbf{p})$  be a Hamiltonian, with  $h_0, h_1 \in \mathcal{A}_\rho$  and  $\|h\|_\rho \leq 1$ . Suppose the Taylor polynomial of  $h_0$  is

$$h_0(\mathbf{p}) = a + \omega \mathbf{p} + \frac{1}{2} \mathbf{p} \cdot C \mathbf{p} + o(|\mathbf{p}|^2),$$

with  $\omega \in \Omega_\gamma$  and  $C$  is symmetric and invertible. Then for any  $\rho_* \leq \rho$ , there exists  $\epsilon > 0$ , which depends on  $C$  and  $\gamma$ , but not on the remainder term in  $o(|\mathbf{p}|^2)$ , such that if  $\|h_1\|_\rho \leq \epsilon$ , there exists a symplectic mapping  $\Phi : A_{\rho_*} \rightarrow A_\rho$  such that if we set  $(\mathbf{q}, \mathbf{p}) = \Phi(\mathbf{Q}, \mathbf{P})$  and  $H = h \circ \Phi$ , we have

$$H(\mathbf{Q}, \mathbf{P}) = A + \omega \mathbf{P} + R(\mathbf{Q}, \mathbf{P}),$$

with  $R(\mathbf{Q}, \mathbf{P}) \in O(|\mathbf{P}|^2)$ .

# Kolmogorov's Theorem

The **KAM** theory is named after  
Andrei **K**olmogorov,  
Vladimir **A**rnold,  
Jürgen **M**oser.

Kolmogorov gave an original proof in the 1950s but never published it. Arnold gave a proof of the theorem in 1963 and Moser published a different but related result in 1962.

# A Motivating Example

The solar system when planets have zero mass.

A system with  $n$  masses  $m_i$  and positions  $\mathbf{x}_i$  satisfies Newton's

**F = ma** law:

$$m_i \ddot{\mathbf{x}}_i = \sum_{j \neq i} G m_i m_j \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|^3}.$$

We take the sun to be given by index 0 and the planets to be given by the indices  $1, \dots, n$ .

# A Motivating Example

Take the sun to be at the center of the solar system ( $\mathbf{x}_0 = 0$ ) and setting the masses of the planets to zero ( $m_i = 0$  for  $i \neq 0$ ).

We obtain  $\mathbf{x}_0'' = 0$  and for  $i \neq 0$

$$\mathbf{x}_i'' = -Gm_0 \frac{\mathbf{x}_i}{|\mathbf{x}_i|^3}.$$

# Hamiltonian Mechanics

Named for William Rowan Hamilton (1805-1865).  
A reformulation of classical Newtonian mechanics.  
Relies on a symplectic structure.

# Hamiltonian Mechanics

Let  $(X, \sigma)$  be a symplectic manifold. That is,  $X$  is a differentiable manifold and  $\sigma$  is a nowhere vanishing 2-form such that  $d\sigma = 0$ . A function  $H$  on a  $X$  has a *symplectic gradient* denoted  $\nabla_\sigma H$ . This gradient is the unique vector field such that for any vector field  $\xi$ , we have

$$\sigma(\xi, \nabla_\sigma H) = dH(\xi).$$

We will consider a *Hamiltonian differential equation* that makes use of this symplectic gradient:

$$\dot{\mathbf{x}} = (\nabla_\sigma H)(\mathbf{x}).$$

# Hamiltonian Mechanics

Let's compare the symplectic gradient and the gradient  $\nabla f$  of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  from multivariable calculus.

The gradient is the unique vector field such that

$$df(\xi) = \langle \xi, \nabla f \rangle.$$

We could also consider the “gradient differential equation”

$$\dot{\mathbf{x}} = \nabla f(\mathbf{x}).$$

Since  $f$  increases along its solutions, they can never return to their starting point.

This prohibits recurrence, whereas the Hamiltonian equations allow and almost impose recurrence (see the Poincaré recurrence theorem).



# Hamiltonian Mechanics

For a simple example of a Hamiltonian system, consider  $X = \mathbb{R}^{2n}$  with coordinates  $(q_1, \dots, q_n, p_1, \dots, p_n)$ . Take  $\sigma = \sum_i dp_i \wedge dq_i$ .

Our Hamiltonian differential equation is

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i},\end{aligned}$$

where these are called the *Hamiltonian equations of motion*.

# Hamiltonian Mechanics

How does the Solar System fit this model? Consider the case of a single body of zero mass. (It is sufficient to consider the single body system, since the planets having zero mass means that they do not effect each other). For  $\mathbf{x} \in \mathbb{R}^2$  the equation  $\ddot{\mathbf{x}} = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$  is

Hamilton's equation for the manifold  $X = \mathbb{R}^2 \times \mathbb{R}^2$  with points  $(\mathbf{q}, \mathbf{p})$ , standard symplectic form

$$\sigma = dp_1 \wedge dq_1 + dp_2 \wedge dq_2,$$

and Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\sqrt{q_1^2 + q_2^2}}.$$

# Hamiltonian Mechanics

After some computation, we can obtain

$$\nabla_{\sigma} H = \left( \mathbf{p}, -\frac{1}{(q_1^2 + q_2^2)^{\frac{3}{2}}} \mathbf{q} \right).$$

The Hamiltonian Differential Equation  $\dot{\mathbf{x}} = \nabla_{\sigma} H(\mathbf{x})$  is

$$\begin{aligned} q_1' &= p_1 & p_1' &= -\frac{q_1}{(q_1^2 + q_2^2)^{\frac{3}{2}}}, \\ q_2' &= p_2 & p_2' &= -\frac{q_2}{(q_1^2 + q_2^2)^{\frac{3}{2}}}. \end{aligned}$$

From here we can recover  $\ddot{\mathbf{x}} = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$ .

# Hamiltonian Mechanics

The vector field  $\nabla_{\sigma} f$  has a flow  $\phi_f^t$  such that

$$\phi_f^t \text{ preserves } f: f \circ \phi_f^t = f,$$

$$\phi_f^t \text{ preserves } \sigma: (\phi_f^t)^* \sigma = \sigma.$$

Our proof of Kolmogorov's Theorem will involve the construction of a symplectic diffeomorphism  $\Phi$  that is a composition of Hamiltonian flows.

# Hamiltonian Mechanics

We will need to construct Taylor polynomials of functions of the form  $t \mapsto g \circ \phi_f^t$ .

To do this we use the Poisson bracket.

## Poisson Bracket

*Let  $f$  and  $g$  be two functions on  $X$ . The Poisson bracket, denoted  $\{f, g\}$ , is defined by*

$$\{f, g\} = \sigma(\nabla_\sigma f, \nabla_\sigma g) = df(\nabla_\sigma g) = -dg(\nabla_\sigma f).$$

# Hamiltonian Mechanics

The gradient of the Poisson bracket corresponds to the Lie bracket:

$$\nabla_{\sigma}\{f, g\} = [\nabla_{\sigma}f, \nabla_{\sigma}g].$$

Proving this equality is a computation that is not difficult, but rather tedious.

Two function  $f$  and  $g$  commute if their Poisson bracket is zero:  $\{f, g\} = 0$ .

This implies that their flows commute:

$$\phi_f(s) \circ \phi_g(t) = \phi_g(t) \circ \phi_f(s).$$

# Hamiltonian Mechanics

In the example involving the Hamiltonian equations of motion, we can write a Poisson bracket as

$$\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

The Poisson bracket will allow us to write Taylor polynomials:

$$f \circ \phi_g^t = f + t\{f, g\} + \frac{t^2}{2}\{\{f, g\}, g\} + \frac{t^3}{6}\{\{\{f, g\}, g\}, g\} + \cdots.$$

# Hamiltonian Mechanics

Denote the torus by  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . A *totally integrable system* is a symplectic manifold  $X = \mathbb{T}^n \times \mathbb{R}^n$  with variables  $(\mathbf{q} \in \mathbb{T}^n, \mathbf{p} \in \mathbb{R}^n)$ , symplectic form  $\sigma = \sum_i dp_i \wedge dq_i$ , and Hamiltonian function  $H(\mathbf{p})$  that depends only on  $\mathbf{p}$ .



# Hamiltonian Mechanics

The Hamiltonian equations of motion can be integrated.

The solution with initial value  $(\mathbf{q}_0, \mathbf{p}_0)$  is

$$\mathbf{q}(t) = \mathbf{q}_0 + t \frac{\partial H}{\partial \mathbf{p}} = \mathbf{q}_0 + t\omega(\mathbf{p}_0),$$

$$\mathbf{p}(t) = \mathbf{p}_0.$$

In particular, each coordinate  $p_1, \dots, p_n$  is conserved, and the trajectory is a linear motion on the torus  $\mathbb{T}^n \times \{\mathbf{p}_0\}$ .

# Irrationality

We want to make formal the notion of “sufficiently irrational.”  
(Later we will see that this will help us to avoid dividing by zero).

A number  $\theta$  is irrational if for all pairs  $p, q \in \mathbb{Z}$ , we have

$$\left| \theta - \frac{p}{q} \right| \neq 0.$$

So for  $\theta$  to be “very” irrational, we want  $\left| \theta - \frac{p}{q} \right|$  to be “very” different from 0, or “very big.”

But that is not quite possible, as we can always approximate an irrational number by rational numbers.

# Irrationality

Instead we want to consider what happens with small divisors.

We want  $\left| \theta - \frac{p}{q} \right|$  to be “small” *only if* the denominator is “big.”

# Irrationality

We start with *diophantine conditions*.

## Diophantine of Exponent $d$

A number  $\theta$  is *Diophantine of exponent  $d$*  if there exists a constant  $\gamma > 0$  such that for all coprime integers  $p$  and  $q$  we have

$$\left| \theta - \frac{p}{q} \right| > \frac{\gamma}{|q|^d}.$$

# Irrationality

Note that it is a stronger condition to be Diophantine of a smaller exponent.

## Diophantine Numbers with Full Measure

*For all  $\epsilon > 0$ , the set of Diophantine numbers of exponent  $2 + \epsilon$  is of full measure.*

# Irrationality

## Diophantine Numbers with Full Measure.

Consider numbers in  $\mathbb{R}/\mathbb{Z}$ .

For all positive integers  $q$ , there are at most  $q$  elements of  $\mathbb{Q}/\mathbb{Z}$  with denominator  $q$  in reduced form. Thus for all  $\gamma > 0$ ,

$$\left| \left\{ \theta \in \mathbb{R}/\mathbb{Z} : \left| \theta - \frac{p}{q} \right| < \frac{\gamma}{|q|^{2+\epsilon}} \right\} \right| < 2\gamma \sum_{q=1}^{\infty} \frac{1}{q^{1+\epsilon}}.$$

The intersection of these sets as  $\gamma \rightarrow 0$  has measure 0, and this intersection is the complement of the set of Diophantine numbers of exponent  $2 + \epsilon$ . QED

# Irrationality

We generalize “sufficiently irrational” from numbers to vectors.

## Diophantine Vector in $\mathbb{R}^n$

Let  $\omega = (\omega_1, \dots, \omega_n)$ . We say  $\omega$  is Diophantine if there exists  $\gamma > 0$  such that for all vectors with integer coefficients  $(k_1, \dots, k_n)$ , we have

$$|k_1\omega_1 + \dots + k_n\omega_n| \geq \frac{\gamma}{(k_1^2 + \dots + k_n^2)^{\frac{n}{2}}}.$$

Let  $\Omega_\gamma^n$  be the subset of such  $\omega \in \mathbb{R}^n$ .

We could also rewrite this condition as a dot product

$$|\mathbf{k} \cdot \omega| \leq \frac{\gamma}{|\mathbf{k}|^n}.$$

In our example of the solar system, the  $\omega_i$  represent the frequencies of the planets' orbits.

# Irrationality

How common are the vectors  $\omega$ ? Actually very common, as if we pick components  $\omega_i$  at random, we will almost surely select such a vector.

## Diophantine Vectors are of Full Measure

*The union  $\Omega = \bigcup_{\gamma > 0} \Omega_\gamma$  is of full measure.*



# Irrationality

## Diophantine Vectors are of Full Measure.

Consider  $S_{\mathbf{k},\gamma}$ , the region around the hyperplane orthogonal to  $\mathbf{k}$  of thickness  $\frac{2\gamma}{|\mathbf{k}|^{n+1}}$ .

The part within the unit cube  $Q$  has measure at most  $\frac{M\gamma}{|\mathbf{k}|^{n+1}}$ , with  $M$  the constant giving the max  $(n-1)$ -dim measure of a hyperplane intersected with  $Q$ .

The sum  $\sum_{\mathbf{k} \in \mathbb{Z}^n - \{0\}} \frac{1}{|\mathbf{k}|^{n+1}}$  is finite, so the volume of  $\bigcup_{\mathbf{k} \in \mathbb{Z}^n - \{0\}} S_{\mathbf{k},\gamma} \cap Q$  is at most a constant times  $\gamma$ .

The intersection of these sets has measure 0, and  $\Omega$  is the complement, similar to the previous proof.

QED

# Torus

We view the solar system as an  $n$  dimensional torus.

Give the planets initial positions  $\mathbf{a} = (a_1, \dots, a_n)$ .

Then the motion  $t \mapsto \mathbf{a} + t\omega = (a_1 + t\omega_1, \dots, a_n + t\omega_n)$  is called a *linear flow on  $(\mathbb{R}/\mathbb{Z})^n$  in the direction  $\omega$* .

The trajectory is dense on the torus if and only if the trajectory  $\omega$  is irrational.

A motion  $\mathbf{x}(t)$  of a perturbed system is “the same” as the motion of an unperturbed system  $\mathbf{x}_1(t)$  dense on a torus  $T_1$  means that  $\mathbf{x}(t)$  is dense on a torus  $T$  and that there exists a homeomorphism  $\Phi : T \rightarrow T_1$  such that  $\Phi(\mathbf{x}(t)) = \mathbf{x}_1(t)$ .

# Torus

## KAM for the Solar System

*Let  $\mathbf{x}_1(t)$  be a motion of the zero-masses system for which the frequency vector  $\omega$  is Diophantine. (Then  $\mathbf{x}_1(t)$  is dense on the corresponding torus  $T_1$ ).*

*Then there exists  $\epsilon > 0$  such that, if the planets have masses  $m_i < \epsilon$ , then there exists a trajectory  $\mathbf{x}(t)$  of the system that is dense on a torus  $T$ , and there exists a homeomorphism  $\Phi : T \rightarrow T_1$  such that  $\Phi(\mathbf{x}(t)) = \mathbf{x}_1(t)$ .*

*The probability of being on such a trajectory goes to 1 as  $\epsilon$  goes to 0.*

# Another Look at the Theorem

## Kolmogorov's Theorem

Let  $\rho, \gamma > 0$  be given, and let  $h(\mathbf{q}, \mathbf{p}) = h_0(\mathbf{p}) + h_1(\mathbf{q}, \mathbf{p})$  be a Hamiltonian, with  $h_0, h_1 \in \mathcal{A}_\rho$  and  $\|h\|_\rho \leq 1$ . Suppose the Taylor polynomial of  $h_0$  is

$$h_0(\mathbf{p}) = a + \omega \mathbf{p} + \frac{1}{2} \mathbf{p} \cdot C \mathbf{p} + o(|\mathbf{p}|^2),$$

with  $\omega \in \Omega_\gamma$  and  $C$  is symmetric and invertible. Then for any  $\rho_* \leq \rho$ , there exists  $\epsilon > 0$ , which depends on  $C$  and  $\gamma$ , but not on the remainder term in  $o(|\mathbf{p}|^2)$ , such that if  $\|h_1\|_\rho \leq \epsilon$ , there exists a symplectic mapping  $\Phi : A_{\rho_*} \rightarrow A_\rho$  such that if we set  $(\mathbf{q}, \mathbf{p}) = \Phi(\mathbf{Q}, \mathbf{P})$  and  $H = h \circ \Phi$ , we have

$$H(\mathbf{Q}, \mathbf{P}) = A + \omega \mathbf{P} + R(\mathbf{Q}, \mathbf{P}),$$

with  $R(\mathbf{Q}, \mathbf{P}) \in O(|\mathbf{P}|^2)$ .

# Analytic Functions

We will be solving a system of equations iteratively, using an analog of Newton's method.

Newton's method for solving an equation  $f(\mathbf{x}) = 0$  consists of choosing a starting guess  $\mathbf{x}_0$  and then defining

$$\mathbf{x}_{i+1} = \mathbf{x}_i - [Df(\mathbf{x}_i)]^{-1}f(\mathbf{x}_i).$$

But we might pick a “bad” initial guess for which Newton's method does not converge to a solution.

For example, taking  $f(x) = x^3 - x + \frac{\sqrt{2}}{2}$  and the initial guess  $x_0 = 0$ , Newton's method will oscillate between 0 and  $\frac{\sqrt{2}}{2}$ .

# Analytic Functions

In order to guarantee convergence, we use Kantorovich's theorem. This theorem requires a bound on the second derivative. We want an analogous condition for our problem.

# Analytic Functions

We need a way to measure “size,” that is, we need to choose a norm for our functions.


Let  $X \subseteq \mathbb{C}^k$  be compact.<sup>1</sup>

Let the caligraphic letter  $\mathcal{X}$  be the Banach algebra of continuous functions on  $X$  that are analytic in the interior and have the sup norm

$$\|f\|_X = \sum_{\mathbf{x} \in X} |f(\mathbf{x})|.$$

We use the absolute value as the standard Euclidean norm on  $\mathbb{C}^n$ .

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<sup>1</sup>Or as Alex might like to say, let  $X$  be a compact. 

# Analytic Functions

We consider three regions:

$$B_\rho = \{\mathbf{p} \in \mathbb{C} : |\mathbf{p}| \leq \rho\},$$

$$C_\rho = \{\mathbf{q} \in \mathbb{C}^n / \mathbb{Z}^n : |\operatorname{Im}(\mathbf{q})| \leq \rho\},$$

$$A_\rho = C_\rho \times B_\rho = \{(\mathbf{q}, \mathbf{p}) \in \mathbb{C}^n / \mathbb{Z}^n \times \mathbb{C}^n : |\mathbf{p}| \leq \rho, |\operatorname{Im}(\mathbf{q})| \leq \rho\}.$$

Denote by  $\mathcal{B}_\rho$ ,  $\mathcal{C}_\rho$ , and  $\mathcal{A}_\rho$  the corresponding Banach algebras.

We can expand elements of  $\mathcal{B}_\rho$  as power series, and elements of  $\mathcal{C}_\rho$  as Fourier series:

$$f(\mathbf{z}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} f_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{z}}.$$



# Analytic Functions

We can bound derivatives of analytic functions on balls in terms of the values of the function itself.

## Cauchy's Inequalities on Balls

*If  $f \in \mathcal{B}_\rho$ , then*

$$\|Df\|_{\rho-\delta} \leq \frac{1}{\delta} \|f\|_\rho,$$

$$\|D^2f\|_{\rho-\delta} \leq \frac{4}{\delta^2} \|f\|_\rho.$$

*As a corollary, the case  $\delta = \rho$  bounds derivatives at the center of balls.*

$$|Df(0)| \leq \frac{1}{\rho} \|f\|_\rho,$$

$$|D^2f(0)| \leq \frac{4}{\rho^2} \|f\|_\rho.$$

# Analytic Functions

## Cauchy's Inequalities on Balls.

Take  $\mathbf{z} \in B_{\rho-\delta}$  and  $\mathbf{u} \in \mathbb{C}^n$ . Since  $B_\delta(\mathbf{z}) \subseteq B_\rho$ , the function  $g : t \mapsto f(\mathbf{z} + t\delta\mathbf{u})$  is defined on the unit disc, and the normal Cauchy inequality implies that

$$\delta |(Df(\mathbf{z}))\mathbf{u}| = |g'(0)| \leq \|g\|_1 \leq \|f\|_\rho.$$

Applying the argument twice gives the result for the second derivative:

$$|D^2f(\mathbf{z})(\mathbf{u}, \mathbf{v})| \leq \frac{2}{\delta} \|Df(\mathbf{z})(\mathbf{u})\|_{\rho-\frac{\delta}{2}} |\mathbf{v}| \leq \frac{4}{\delta^2} \|f\|_\rho |\mathbf{u}||\mathbf{v}|.$$

QED

# Smell of the Proof

Recall the final equation in Kolmogorov's theorem:

$$H(\mathbf{Q}, \mathbf{P}) = A + \omega \mathbf{P} + R(\mathbf{Q}, \mathbf{P}).$$

This is an equation for a diffeomorphism  $\Phi$ . We solve for this diffeomorphism. To do so we would like to use Newton's method, but using it is not quite sufficient. However we can still do something with a similar flavor.

We obtain  $\Phi$  as a limit of  $\Phi_i$ , where

$$\Phi_i = \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1.$$

Here,  $\phi_i$  is the Hamiltonian flow for a Hamiltonian function  $g_i$ . This  $g_i$  is the unknown for which we solve.

# Smell of the Proof

At the  $i$ th step we have a Hamiltonian  $\tilde{h} = \Phi_i^* h$ .

We expand  $\tilde{h}$  up to order 2 in  $\mathbf{p}$ , and the coefficients will be Fourier series in  $\mathbf{q}$ .

We write  $\tilde{h} = \tilde{h}_0 + \tilde{h}_1$ :

$\tilde{h}_1$  has the terms constant or linear in  $\mathbf{p}$ , except the constant in  $\mathbf{q}$ ,

$\tilde{h}_0$  is everything else.

We would like to eliminate  $\tilde{h}_1$ , but we won't be able to do so by solving a linear equation.

Instead we will solve a linear equation for a function  $g$  such that  $\phi_g^* \tilde{h}$  is “better” than  $\tilde{h}$ , in some reasonable sense of “better.”

# Smell of the Proof

We expand  $\phi_g^* \tilde{h}$  to first order in  $g$ :

$$\begin{aligned}\phi_g^* \tilde{h} &= \tilde{h} + \{g, \tilde{h}\} + o(|g|) \\ &= \tilde{h}_0 + \tilde{h}_1 + \{g, \tilde{h}_0\} + \{g, \tilde{h}_1\} + o(|g|).\end{aligned}$$

We want to eliminate the terms, other than the term constant in  $\mathbf{q}$ , that are not  $O(|\mathbf{p}|)^2$ .

Applying the standard Newton's method would necessitate solving the equation

$$\tilde{h}_1 + \{g, \tilde{h}_0\} + \{g, \tilde{h}_1\} \in o(|\mathbf{p}|).$$

But we will do something a bit different.

# Smell of the Proof

We can assume that anything we want is small as long as the choice is justified by the resulting inequalities.

So suppose that  $\{g, \tilde{h}_1\}$  is of order 2 since  $g$  and  $\tilde{h}_1$  are both small. Then the linear equation that we need to solve is

$$\tilde{h}_1 + \{g, \tilde{h}_0\} \in o(|\mathbf{p}|).$$

This is a *Diophantine differential equation*, and our ability to solve it will rely on the Diophantine vectors that we saw previously.

# Diophantine Differential Equations

Let  $g \in \mathcal{C}_\rho$ . We are going to be solving linear equations of the form

$$Df(\omega) = \sum_{i=1}^n \omega_i \frac{\partial f}{\partial q_i} = g,$$

with  $f \in \mathcal{C}_{\rho'}$  for some  $\rho' < \rho$ .

Write the functions  $f$  and  $g$  as Fourier Series:

$$f(\mathbf{q}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} f_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{q}}, \quad g(\mathbf{q}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} g_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{q}}.$$

The solution is given by

$$f_{\mathbf{k}} = \frac{1}{2\pi i (\mathbf{k} \cdot \omega)} g_{\mathbf{k}}.$$

We need  $g_0$  to be zero. Then  $f_0$  is arbitrary, and otherwise the series for  $f$  is unique.

# Diophantine Differential Equations

The convergence properties of  $f$  depend on the Diophantine properties of  $\omega$ .

We cannot divide by zero when computing the Fourier coefficients  $\hat{f}_k$ .

Even though we have convergence for  $\rho$ , we might not have boundedness. So we need some  $\rho' < \rho$  where we have boundedness, but large enough so that the limit is nonempty.



# Diophantine Differential Equations

We will use the following tool for choosing  $\rho'$ .

## Proposition

*If  $g \in \mathcal{C}_\rho$  and  $\omega \in \Omega_\gamma$ , then for all  $\delta \in (0, \rho)$  we have*

$$\|f\|_{\rho-\delta} \leq \frac{\kappa_n}{\gamma \delta^{2n}} \|g\|_\rho,$$

$$\|Df\|_{\rho-\delta} \leq \frac{\kappa_n}{\gamma \delta^{2n+1}} \|g\|_\rho,$$

*where  $\kappa_n$  is a constant that depends only on  $n$ .*

Next we will prove this proposition.

# Diophantine Differential Equations

For  $\mathbf{y} \in \mathbb{R}^n$  with  $|\mathbf{y}| \leq \rho$  (in particular  $\mathbf{y} = \rho \frac{\mathbf{k}}{|\mathbf{k}|}$ ), the function  $\mathbf{q} \mapsto g(\mathbf{q} - i\mathbf{y})$  is continuous and periodic in  $\mathbf{q}$  of period 1.

This function can be written

$$g(\mathbf{q} - i\mathbf{y}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} g_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot (\mathbf{q} - i\mathbf{y})} = \sum_{\mathbf{k} \in \mathbb{Z}^n} \left( g_{\mathbf{k}} e^{2\pi \mathbf{k} \cdot \mathbf{y}} \right) e^{2\pi i \mathbf{k} \cdot \mathbf{q}}.$$

By Parseval's theorem, we have

$$\|g\|_{\rho}^2 \geq \int_{\mathbb{T}^n} |g(\mathbf{q} - i\mathbf{y})|^2 |d^n \mathbf{q}| = \sum_{\mathbf{k} \in \mathbb{Z}^n} |g_{\mathbf{k}}|^2 e^{4\pi \mathbf{k} \cdot \mathbf{y}}.$$

Since the series has positive numbers, we have

$$\|g\|_{\rho}^2 \geq |g_{\mathbf{k}}|^2 e^{4\pi \rho |\mathbf{k}|}.$$

# Diophantine Differential Equations

Using the Diophantine-ness of  $\omega$  and the Fourier coefficients  $f_{\mathbf{k}}$ , we have

$$|f_{\mathbf{k}}| \leq \frac{1}{2\pi\gamma} \|g\|_{\rho} |\mathbf{k}|^n e^{-2\pi|\mathbf{k}|\rho}.$$

Next we split the proposition's proof into two parts: the  $f$  case and the  $Df$  case.

# Diophantine Differential Equations

For  $|\mathbf{q}| \leq \rho - \delta$ , we can write

$$\begin{aligned} \left| f_{\mathbf{k}} e^{2\pi i(\mathbf{k} \cdot \mathbf{q})} \right| &\leq \sum_{\mathbf{k} \in \mathbb{Z}^n} \frac{|\mathbf{k}|^n}{2\pi\gamma} \|g\|_{\rho} e^{-2\pi\rho|\mathbf{k}|(\rho-\delta)} \\ &= \frac{\|g\|_{\rho}}{2\pi\gamma} \sum_{\mathbf{k} \in \mathbb{Z}^n} |\mathbf{k}|^n e^{-2\pi|\mathbf{k}|\delta}. \end{aligned}$$

We then rewrite the sum on the right:

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} |\mathbf{k}|^n e^{-2\pi|\mathbf{k}|\delta} = \frac{1}{(2\pi\delta)^{2n}} \left( (2\pi\delta)^n \sum_{\mathbf{k}' \in 2\pi\delta\mathbb{Z}^n} |\mathbf{k}'|^n e^{-|\mathbf{k}'|} \right).$$

# Diophantine Differential Equations

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} |\mathbf{k}|^n e^{-2\pi |\mathbf{k}| \delta} = \frac{1}{(2\pi \delta)^{2n}} \left( (2\pi \delta)^n \sum_{\mathbf{k}' \in 2\pi \delta \mathbb{Z}^n} |\mathbf{k}'|^n e^{-|\mathbf{k}'|} \right).$$

As  $\delta \rightarrow 0$ , the expression inside the parentheses approaches

$$\int_{\mathbb{R}^n} |\mathbf{x}|^n e^{-|\mathbf{x}|} d^n \mathbf{x}.$$

Since the integral is convergent there exists  $\kappa'_n$  such that for  $\delta \leq 1$ , it holds that

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} |\mathbf{k}|^n e^{-2\pi |\mathbf{k}| \delta} \leq \frac{\kappa'_n}{(2\pi \delta)^{2n}}.$$

This is the “f” part of the proof.

# Diophantine Differential Equations

The “ $Df$ ” part of the proof is similar.

Use

$$Df(\mathbf{q})(\mathbf{u}) = 2\pi i \sum (\mathbf{k} \cdot \mathbf{u}) f_{\mathbf{k}} e^{2\pi i(\mathbf{k} \cdot \mathbf{q})}.$$

We obtain

$$|Df(\mathbf{q})(\mathbf{u})| \leq \frac{|\mathbf{u}| \|\mathbf{g}\|_{\rho} \kappa_n''}{\gamma (2\pi\gamma)^{n+1}}.$$

To complete the proof of the proposition, take

$$\kappa_n = \max \left\{ \frac{\kappa_n'}{(2\pi)^{2n}}, \frac{\kappa_n''}{(2\pi)^{2n+1}} \right\}.$$

# The Equations to Solve

Let's return to  $h_1 + \{g, h_0\} \in O(|\mathbf{p}|^2)$  (the Diophantine differential equation that we had before, but without the tildes and with  $o(|\mathbf{p}|)$  replaced by  $O(|\mathbf{p}|^2)$ , which is equivalent since our functions are analytic).

The unknown is  $g$ , which we take to be degree 1 in  $\mathbf{p}$ :

$$g = \lambda \mathbf{q} + X(\mathbf{q}) + \sum_{i=1}^n Y_i(\mathbf{q}) p_i$$

Note that only the linear terms of  $g$  contribute to the linear terms of  $\{g, h_0\}$ .

The unknowns are:  $\lambda$ ,  $X$ ,  $Y_i$ .

We can expand  $X$  and  $Y_i$  as Fourier series, since they are functions of  $\mathbf{q}$  only.

# The Equations to Solve

Let  $a \in \mathbb{R}$ ,  $\omega \in \Omega_\gamma$ ,  $R(\mathbf{q}, \mathbf{p}) \in O(|\mathbf{p}|^3)$ , and  $\bar{A} = 0$  be the average of  $A$  on the torus  $\mathbf{p} = 0$ , we can write

$$h_0(\mathbf{q}, \mathbf{p}) = a + \omega \mathbf{p} + \frac{1}{2} \mathbf{p} \cdot C(\mathbf{q}) \mathbf{p} + R(\mathbf{q}, \mathbf{p}),$$

$$h_1(\mathbf{q}, \mathbf{p}) = A(\mathbf{q}) + B(\mathbf{q}) \mathbf{p}.$$

We then have

$$\begin{aligned} & (h_1 + \{g, h_0\})(\mathbf{q}, \mathbf{p}) \\ &= \omega \cdot \lambda + A(\mathbf{q}) + DX(\mathbf{q})(\omega) \\ &+ \left( B(\mathbf{q}) + (\lambda + DX(\mathbf{q})) C(\mathbf{q}) + \omega DY(\mathbf{q}) \right) \cdot \mathbf{p} + O(|\mathbf{p}|^2). \end{aligned}$$



# The Equations to Solve

We need to solve the equations

$$DX(\mathbf{q})(\omega) = -A(\mathbf{q})$$

$$DY(\mathbf{q})(\omega) = -B(\mathbf{q}) - (\lambda + DX(\mathbf{q}))C(\mathbf{q})$$

for  $X$  and  $Y_i$ ,  $i = 1, \dots, n$ .

These are Diophantine differential equations.

Since we have the hypothesis  $\bar{A} = 0$ , we can solve the first equation and find  $X \in \mathcal{C}_{\rho'}$  for all  $\rho' < \rho$ .

We then substitute into the second equation.

# The Equations to Solve

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for  $X$  and  $Y_i$ ,  $i = 1, \dots, n$ .

We can then determine  $\lambda$  since the averages of the right-hand sides in the  $n$  are all zero.

Then we can solve the second equation and find  $Y_i \in \mathcal{C}_{\rho''}$  for all  $\rho'' < \rho'$ .

Iterating the process requires choosing  $\rho'$  and  $\rho''$  “carefully.”