Kolmogorov's Theorem

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Our goal is to understand the following theorem.

Kolmogorov's Theorem

Let $\rho, \gamma > 0$ be given, and let $h(\mathbf{q}, \mathbf{p}) = h_0(\mathbf{p}) + h_1(\mathbf{q}, \mathbf{p})$ be a Hamiltonian, with $h_0, h_1 \in \mathcal{A}_\rho$ and $\|h\|_\rho \leq 1$. Suppose the Taylor polynomial of h_0 is

$$h_0(\mathbf{p}) = a + \omega \mathbf{p} + \frac{1}{2}\mathbf{p} \cdot C\mathbf{p} + o(|\mathbf{p}|^2),$$

with $\omega \in \Omega_\gamma$ and C is symmetric and invertible. Then for any $\rho_* \leq \rho$, there exists $\epsilon > 0$, which depends on C and γ, but not on the remainder term in $o(|\mathbf{p}|^2)$, such that if $\|h_1\|_\rho \leq \epsilon$, there exists a symplectic mapping $\Phi : A_{\rho_*} \to A_\rho$ such that if we set $(\mathbf{q}, \mathbf{p}) = \Phi(\mathbf{Q}, \mathbf{P})$ and $H = h \circ \Phi$, we have

$$H(\mathbf{Q}, \mathbf{P}) = A + \omega \mathbf{P} + R(\mathbf{Q}, \mathbf{P}),$$

with $R(\mathbf{Q}, \mathbf{P}) \in O(|\mathbf{P}|^2)$.

Kolmogorov's Theorem

The **KAM** theory is named after

Andreï **K**olmogorov,

Vladimir **A**rnold,

Jürgen **M**oser.

Kolmogorov gave an original proof in the 1950s but never published it. Arnold gave a proof of the theorem in 1963 and Moser published a different but related result in 1962.

A Motivating Example

The solar system when planets have zero mass.

Hamiltonian Mechanics

Named for William Rowan Hamilton (1805-1865).
A reformulation of classical Newtonian mechanics.
Relies on a symplectic structure.

Hamiltonian Mechanics

Let (X, σ) be a symplectic manifold. That is, X is a differentiable manifold and σ is a nowhere vanishing 2-form such that $d\sigma = 0$. A function H on a X has a *symplectic gradient* denoted $\nabla_\sigma H$. This gradient is the unique vector field such that for any vector field ξ, we have

$$\sigma(\xi, \nabla_\sigma H) = dH(\xi).$$

We will consider a *Hamiltonian differential equation* that makes use of this symplectic gradient:

$$\dot{\mathbf{x}} = (\nabla_\sigma H)(\mathbf{x}).$$

Hamiltonian Mechanics

For a simple example of a Hamiltonian system, consider $X = \mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. Take $\sigma = \sum_i dp_i \wedge dq_i$.

Our Hamiltonian differential equation is

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i},$$

where these are called the *Hamiltonian equations of motion*.

Hamiltonian Mechanics

How does the Solar System fit this model? Consider the case of a single body of zero mass. (It is sufficient to consider the single body system, since the planets having zero mass means that they do not effect each other). For $\mathbf{x} \in \mathbb{R}^2$ the equation $\ddot{\mathbf{x}} = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$ is Hamilton's equation for the manifold $X = \mathbb{R}^2 \times \mathbb{R}^2$ with points $(\mathbf{q}, \mathbf{p})$, standard symplectic form

$$\sigma = dp_1 \wedge dq_1 + dp_2 \wedge dq_2,$$

and Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\sqrt{q_1^2 + q_2^2}}.$$

Hamiltonian Mechanics

After some computation, we can obtain

$$\nabla_\sigma H = \left(\mathbf{p}, -\frac{1}{(q_1^2 + q_2^2)^{\frac{3}{2}}} \mathbf{q} \right).$$

The Hamiltonian Differential Equation $\dot{\mathbf{x}} = \nabla_\sigma H(\mathbf{x})$ is

$$q_1' = p_1 \qquad p_1' = -\frac{q_1}{(q_1^2 + q_2^2)^{\frac{3}{2}}},$$

$$q_2' = p_2 \qquad p_2' = -\frac{q_2}{(q_1^2 + q_2^2)^{\frac{3}{2}}}.$$

From here we can recover $\ddot{\mathbf{x}} = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$.

Hamiltonian Mechanics

The vector field $\nabla_\sigma f$ has a flow ϕ_f^t such that

ϕ_f^t preserves f: $f \circ \phi_f^t = f$,

ϕ_f^t preserves σ: $(\phi_f^t)^* \sigma = \sigma$.

Our proof of Kolmogorov's Theorem will involve the construction of a symplectic diffeomorphism Φ that is a composition of Hamiltonian flows.

Hamiltonian Mechanics

We will need to construct Taylor polynomials of functions of the form $t \mapsto g \circ \phi_f^t$.
To do this we use the Poisson bracket.

Poisson Bracket

Let f and g be two functions on X. The Poisson bracket, denoted $\{f, g\}$, is defined by

$$\{f, g\} = \sigma(\nabla_\sigma f, \nabla_\sigma g) = df(\nabla_\sigma g) = -dg(\nabla_\sigma f).$$

Irrationality

We want to make formal the notion of “suffiently irrational.”
(Later we will see that this will help us to avoid dividing by zero).
A number θ is irrational if for all pairs $p, q \in \mathbb{Z}$, we have
$\left|\theta - \frac{p}{q}\right| \neq 0$.

So for θ to be “very” irrational, we want $\left|\theta - \frac{p}{q}\right|$ to be “very” different from 0, or “very big.”
But that is not quite possible, as we can always approximate an irrational number by rational numbers.

Irrationality

Instead we want to consider what happens with small divisors.

Irrationality

We start with *diophantine conditions*.

Diophantine of Exponent d

A number θ is diophantine of exponent d if there exists a constant $\gamma > 0$ such that for all coprime integers p and q we have

$$\left|\theta - \frac{p}{q}\right| > \frac{\gamma}{|q|^d}.$$

Irrationality

We generalize "sufficiently irrational" from numbers to vectors.

Diophantine Vector in $\mathbb{R}^n$

Let $\omega = (\omega_1, \ldots, \omega_n)$. We say ω is Diophantine if there exists $\gamma > 0$ such that for all vectors with integer coefficients $(k_1, \ldots, k_n)$, we have

$$|k_1\omega_1 + \cdots + k_n\omega_n| \geq \frac{\gamma}{(k_1^2 + \cdots + k_n^2)^{\frac{n}{2}}}.$$

Let Ω_γ^n be the subset of such $\omega \in \mathbb{R}^n$.

In our example of the solar system, the ω_i represent the frequencies of the planets' orbits.

Irrationality

How common are the vectors ω? Actually very common, as if we pick components ω_i at random, we will almost surely select such a vector.

Torus

Analytic Functions

We will be solving a system of equations iteratively, using an analog of Newton's method.
Recall that when using Newton's method, we need a way to bound the second derivatives.
We need a way to measure "size," that is, we need to choose a norm for our functions.

Analytic Functions

Let $X \subseteq \mathbb{C}^k$ be compact. [1]
Let the caligraphic letter $\mathcal{X}$ be the Banach algebra of continuous functions on X that are analytic in the interior and have the sup norm

$$\|f\|_X = \sum_{\mathbf{x} \in X} |f(\mathbf{x})|.$$

We use the absolute value as the standard Euclidean norm on $\mathbb{C}^n$.

[1] Or as Alex might like to say, let X be a compact.

Analytic Functions

We consider three regions:

$$B_\rho = \{\mathbf{p} \in \mathbb{C} : |\mathbf{p}| \le \rho\},$$
$$C_\rho = \{\mathbf{q} \in \mathbb{C}^n/\mathbb{Z}^n : |\operatorname{Im}(\mathbf{q})| \le \rho\},$$
$$A_\rho = C_\rho \times B_\rho = \{(\mathbf{q}, \mathbf{p}) \in \mathbb{C}^n/\mathbb{Z}^n \times \mathbb{C}^n : |\mathbf{p}| \le \rho, |\operatorname{Im}(\mathbf{q})| \le \rho\}.$$

Denote by $\mathcal{B}_\rho$, $\mathcal{C}_\rho$, and $\mathcal{A}_\rho$ the corresponding Banach algebras.

Smell of the Proof

The last equation in the theorem is an equation for a diffeomorphism Φ. We solve for this diffeomorphism. To do so we would like to use Newton's method, but using it is not quite sufficient. However we can still do something with a similar flavor. We obtain Φ as a limit of Φ_i, where

$$\Phi_i = \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1.$$

Here, ϕ_i is the Hamiltonian flow for a Hamiltonian function g_i. This g_i is the unknown for which we solve.

Diophantine Differential Equations

Let $g \in \mathcal{C}_\rho$. We are going to be solving linear equations of the form

$$\mathrm{D}f(\omega) = \sum_{i=1}^{n} \omega_i \frac{\partial f}{\partial q_i} = g,$$

with $f \in \mathcal{C}_{\rho'}$ for some $\rho' < \rho$.
Set

$$f(\mathbf{q}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} f_{\mathbf{k}} e^{2\pi i\, \mathbf{k}\cdot\mathbf{q}},$$

$$g(\mathbf{q}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} g_{\mathbf{k}} e^{2\pi i\, \mathbf{k}\cdot\mathbf{q}}.$$

The solution is

$$f_{\mathbf{k}} = \frac{1}{2\pi i\,(\mathbf{k}\cdot\omega)} g_{\mathbf{k}}.$$

We need g_0 to be zero. Then f_0 is arbitrary, and otherwise the series for f is unique.

Diophantine Differential Equations

The convergence properties of f depend on the Diophantine properties of ω.
We cannot divide by zero when computing the Fourier coefficients $f_{\mathbf{k}}$.
Even though we have convergence for ρ, we might not have boundedness. So we need some $\rho' < \rho$ where we have boundedness, but large enough so that the limit is nonempty.