The Kolmogorov Theorem

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Abstract

This paper gives a proof of the Kolmogorov Theorem on the conservation of invariant tori. We follow the approach given by Hubbard and Ilyashenko in . Their proof is similar to the one given by Bennettin, Galgani, Giorgilli, and Strelcyn in , which itself resembles Kolmogorov's original argument.

1 Introduction

But before moving on, let's take a look at our main goal:

Theorem 1.1 (The Kolmogorov Theorem)**.** *Let $\rho, \gamma > 0$ be given, and let $h(\boldsymbol{q}, \boldsymbol{p}) = h_0(\boldsymbol{p}) + h_1(\boldsymbol{q}, \boldsymbol{p})$ be a Hamiltonian, with $h_0, h_1 \in \mathcal{A}_\rho$ and $\|h\|_\rho \leq 1$. Suppose the Taylor polynomial of h_0 is*

$$h_0(\boldsymbol{p}) = a + \omega \boldsymbol{p} + \frac{1}{2}\boldsymbol{p} \cdot C\boldsymbol{p} + o(|\boldsymbol{p}|^2),$$

with $\omega \in \Omega_\gamma$ and C is symmetric and invertible. Then for any $\rho_* \leq \rho$, there exists $\epsilon > 0$, which depends on C and γ, but not on the remainder term in $o(|\boldsymbol{p}|^2)$, such that if $\|h_1\|_\rho \leq \epsilon$, there exists a symplectic mapping $\Phi : A_{\rho_*} \to A_\rho$ such that if we set $(\boldsymbol{q}, \boldsymbol{p}) = \Phi(\boldsymbol{Q}, \boldsymbol{P})$ and $H = h \circ \Phi$, we have

$$H(\boldsymbol{Q}, \boldsymbol{P}) = A + \omega \boldsymbol{P} + R(\boldsymbol{Q}, \boldsymbol{P}), \tag{1}$$

with $R(\boldsymbol{Q}, \boldsymbol{P}) \in O(|\boldsymbol{P}|^2)$.

We will return to this statement after building up some intuition for it.

2 A Motivating Example

To start understanding the theorem, let's consider an example to which we would like to apply it. While staring up at the night sky, one might be driven to wonder, "Why do the planets orbit around the sun and just not fly off by themselves in their own directions?"

3 Hamiltonian Mechanics

4 Irrationality

In our statement of Kolmogorov's Theorem, we included the hypothesis that $\omega \in \Omega_\gamma$. We now define this notation and begin to explain its importance. The set Ω_γ consists of vectors that are "sufficiently irrational," a notion that we need to make more precise.

Let's first consider the definition of an irrational number. If a real number θ is irrational, then for all pairs if integers p and q, with q positive, we have the following

$$\left|\theta - \frac{p}{q}\right| \neq 0.$$

This equation tells us simply that there does not exist and rational number $\frac{p}{q}$ that equals our irrational number θ. Here, the "not equal to zero" part of the equation will be stressed, as an expression similar to the one on the left hand side will later appear as the denominator of a fraction (see section 7). As dividing by zero can be rather troublesome, we wish to avoid it. This condition of irrationality is the tool we use to do so: if θ is irrational, then the left side will not be zero, so we can divide by it without any problems.

Our condition of "sufficiently irrational" will mean that $\left|\theta - \frac{p}{q}\right|$ is "sufficiently nonzero," or since we are using an absolute value, "sufficiently big." However, as we learn in our introductory courses in real analysis, the rationals are dense in the reals, and every real number, specifically every irrational number θ, may be approximated arbitrarily closely by the rationals. More precisely, given any real $\epsilon > 0$, there exists a rational number $\frac{p}{q}$ such that $\left|\theta - \frac{p}{q}\right| < \epsilon$. Thus trying to coerce $\left|\theta - \frac{p}{q}\right|$ to be big is quite impossible.

Unsatisfied with our answer, let's instead consider a different question. Instead of wanting $\left|\theta - \frac{p}{q}\right|$ to be "big,", we ask that it is small *only if the denominator is big.* This is the beginning of the theory of Diophantine approximation.

The numbers that we seek will satisfy the following definition.

Definition 4.1 (Diophantine Number of Exponent d)**.** A number θ is *Diophantine of exponent d* if there exists a constant $\gamma > 0$ such that for all coprime integers p and q we have

$$\left|\theta - \frac{p}{q}\right| > \frac{\gamma}{|q|^d}.$$

From this definition we see that it is a stronger requirement for a number to be Diophantine of a smaller exponent. For all irrational numbers θ there exist arbitrarily large q and p prime to q such that

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

We see that no number is Diophantine of any exponent smaller than 2. And the number that are Diophantine of exponent exactly 2 are precisely the numbers whose continued fractions have bounded entries. These numbers form a set of measure zero.

But what about exponents greater than 2, or rather, of the form $2+\epsilon$ for $\epsilon > 0$? In the sense of Lebesgue measure, these numbers are quite abundant, as for any $\epsilon > 0$ they form a set of full measure.

Proposition 4.2 (Diophantine Numbers with Full Measure)**.** *For all $\epsilon > 0$, the set of Diophantine numbers of exponent $2+\epsilon$ is of full measure.*

Proof. We consider numbers in $\mathbb{R}/\mathbb{Z}$. Given any positive integer q, there are at most q elements of $\mathbb{Q}/\mathbb{Z}$ that, in reduced form, have denominator q. Hence for any constant γ, we consider the set

$$\left\{\theta \in \mathbb{R}/\mathbb{Z} : \left|\theta - \frac{p}{q}\right| < \frac{\gamma}{|q|^{2+\epsilon}}\right\}.$$

The length of this set is at most $\frac{2\gamma}{q^{1+\epsilon}}$. Summing over all q, we see that the set of numbers θ for with there exists q such that

$$\left|\theta - \frac{p}{q}\right| < \frac{\gamma}{2^{2+\epsilon}}$$

has length strictly less than

$$2\gamma \sum_{q=1}^{\infty} \frac{1}{q^{1+\epsilon}}.$$

Then we take the intersection over all these sets as $\gamma \to 0$, and we note that this intersection has measure 0. But this set is the complement of the set of Diophantine numbers of exponent $2+\epsilon$. Hence the claim holds. QED

5 Analytic Functions

As this is a paper for a complex analysis class, we begin this section by noting that the complex analysis is located here.

Consider the function $f(x) = x^3 - x + \frac{\sqrt{2}}{2}$.

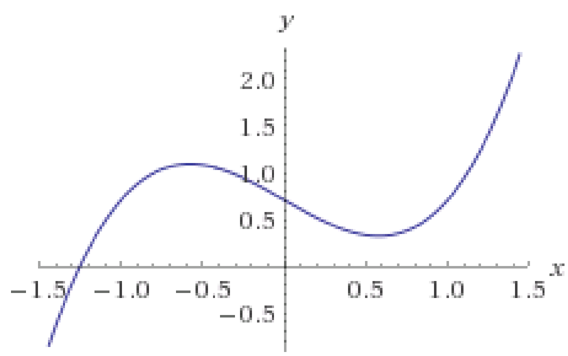

Figure 1: $f(x) = x^3 - x + \frac{\sqrt{2}}{2}$

6 Main Idea of the Proof

7 Diophantine Differential Equations

8 Solving Equations