

# Introduction to Matroids

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## 1 Introduction

In this course we've seen several examples of algorithms that are "Greedy." A Greedy algorithm is an algorithm that makes a "locally best" decision. For example, in Dykstra's algorithm we use a priority queue to keep track of nodes on the frontier and pop them off using the priority of shortest path distance. And in Kruskal's algorithm for finding minimum weight spanning trees, at each step we select the minimum weight edge that does not form a cycle.

We can generalize the idea of "locally make a best decision" by using matroids. The word *matroid* should make you think of the word *matrix*, which you use in linear algebra and multivariable or vector calculus. We'll begin by reviewing the concepts of linear independence, then we'll discuss analogous concepts in graph theory. Then we will relate these concepts by generalizing them and giving the definition of a matroid. And finally we will use matroids to give a proof of the correctness of Kruskal's algorithm.

## 2 Independence in Linear Algebra

As you take Linear Algebra and Multivariable Calculus you will gain lots of experience working with vectors and matrices. In these notes we'll denote vectors using boldface letters at the end of the alphabet, such as  $\vec{u}$  and  $\vec{v}$ , and matrices using capital letters at the beginning of the alphabet, such as  $A$  and  $B$ . We'll denote by  $\vec{0}$  the vector of all 0's, and we'll also write vectors as columns and matrices as rectangles containing numbers:

$$\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 5 \\ -2 & 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{bmatrix}.$$

We say that vectors are elements of a set called a Vector Space, and we also have the ability to add vectors and to multiply them by scalars. In multivariable calculus the most common example of vector spaces are  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , where vectors consist of tuples of 2 and 3 numbers, respectively. We can add vectors and multiply them by scalars as follows:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}, \quad a \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \end{bmatrix}.$$

A linear combination of vectors is a sum of scalar multiples of vectors. For example

$$2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}.$$

We could say that the 3rd vector depends on the first two vectors, since it is a linear combination of them.

A set of vectors<sup>1</sup>  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  is *linearly independent* if the only scalars  $a_1, a_2, \dots, a_n$  that satisfy

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n = \vec{0}$$

are  $a_1 = a_2 = \dots = a_n = 0$ . This means there is no way to write one of the vectors as a combination of the others, unless we make all the coefficients 0. That is, none of the vectors in the set depend on the others. For example, the standard basis vectors in  $\mathbb{R}^3$  are linearly independent,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

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<sup>1</sup> We can also treat the columns of matrices as vectors and define a notion of linear independence on them, although we should be careful that a column may be repeated, whereas a vector in a set is not repeated.

as none of them can be written as a linear combination of the others. The following set of vectors is linearly dependent:

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}, \quad 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 4+2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

There is a limit to the number of vectors that we can add to a set while maintaining independence. For example, in  $\mathbb{R}^3$ , any set of 4 vectors is linearly dependent. An independent set to which we can't add any more vectors is called a *maximal* independent set, with the word maximal meaning we have included as many vectors as possible.<sup>1</sup>

### 3 Independence in Graphs

### 4 Definition of Matroids

Now that we have some intuition about the concept of “independence,” let's give a definition of matroids.<sup>2</sup>

**Definition 4.1** (Matroid). A *matroid*  $M$  is a pair  $(E, \mathcal{I})$  consisting of a finite set  $E$  and a collection  $\mathcal{I}$  of subsets of  $E$ , called the *independent sets*, satisfying the following properties:

1. The collection is nonempty. That is,  $\mathcal{I} \neq \emptyset$ .
2. All subsets of an independent set are independent. That is, if  $A \in \mathcal{I}$  and  $B \subseteq A$ , then  $B \in \mathcal{I}$ .
3. If  $A$  and  $B$  are independent sets and  $A$  is larger than  $B$ , then we can take an element that is in  $A$  but not in  $B$  and add it to  $B$  to construct an independent set. That is, if  $A, B \in \mathcal{I}$  with  $|A| > |B|$ , then there exists  $x \in A \setminus B$  such that  $B \cup \{x\} \in \mathcal{I}$ . This property is known as the *independent set exchange property*—we can “exchange” an element from one set to another.

### 5 Kruskal's Algorithm

<sup>1</sup> A maximal linearly independent set of vectors is called a *basis*.

<sup>2</sup> We could define matroids in many other ways. Matroids are referred to as *cryptomorphic*, which means they have many equivalent but ostensibly unrelated definitions.