01-02 Individual	1	$\frac{5}{42}$	**2**	180	**3**	8	**4**	93324	**5**	7.5
	6	120	**7**	$\frac{2}{3}$	**8**	3	**9**	4.5	**10**	23

01-02 Group	**1**	360	**2**	221	**3**	18	**4**	43	**5**	7
	6	65	**7**	$\frac{9}{20}$	**8**	48	**9**	28	**10**	8

Individual Events

I1 There are 9 cards, numbered from 1 to 9, in a bag. If 3 cards are drawn together at random, find the probability that all are odd. (Express your answer in the simplest fraction.)

P(all are odd) $=\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}=\frac{5}{42}$

I2 Given $a^3 = 150b$ and a, b are positive integers, find the least value of b.

Reference: 2000 FG5.1

$150 = 2\times3\times5^2$

For the least value of b, $a^3 = 2^3\times3^3\times5^3$

$b = 2^2\times3^2\times5 = 180$

I3 Suppose $\cos 15° = \frac{\sqrt{a}+\sqrt{b}}{4}$ and a, b are natural numbers. If $a + b = y$, find the value of y.

$$\begin{aligned}\cos 15° &= \cos (60° - 45°)\\ &= \cos 60° \cos 45° + \sin 60° \sin 45°\\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

$y = a + b = 2 + 6 = 8$

I4 Each of the digits 2, 3, 4, 5 can be used once and once only in writing a four-digit number. Find the sum of all such numbers.

Reference: 1994 FG8.4

A number starting with ‘2’ may be 2345, 2354, 2435, 2453, 2534, 2543. So, there are 6 numbers starting with ‘2’. Similarly, there are 6 numbers starting with ‘3’, 6 numbers starting with ‘4’, 6 numbers starting with ‘5’.

The sum of all possible thousands-digits are: $(6\times2 + 6\times3 + 6\times4 + 6\times5)\times1000 = 84000$

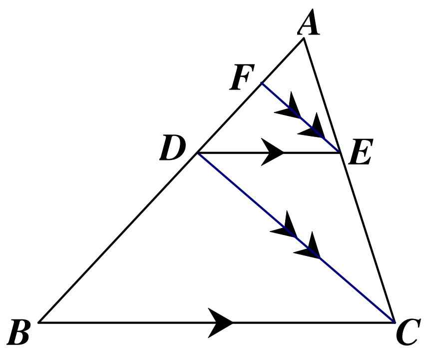
Similarly, the sum of all possible hundreds-digits is: $(6\times2 + 6\times3 + 6\times4 + 6\times5)\times100 = 8400$,

the sum of all possible tens-digits are: $(6\times2 + 6\times3 + 6\times4 + 6\times5)\times10 = 840$,

the sum of all possible units-digits are: $(6\times2 + 6\times3 + 6\times4 + 6\times5)\times1 = 84$.

The sum of all possible numbers are: $84000 + 8400 + 840 + 84 = 93324$

I5 In ΔABC, $DE \,//\, BC$, $FE \,//\, DC$, $AF = 2$, $FD = 3$ and $DB = X$. Find the value of X.

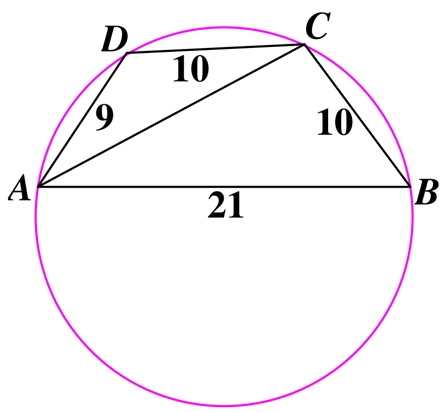


$AE : EC = 2 : 3$ (theorem of eq. ratio)

$AD : DB = 2 : 3$ (theorem of eq. ratio)

$DB = (2+3)\times\frac{3}{2} = 7.5$

I6 If the lengths of the sides of a cyclic quadrilateral are 9, 10, 10 and 21 respectively, find the area of the cyclic quadrilateral. (**Reference: 2014 HI5**)



Let $AB = 21$, $BC = 10 = CD$, $DA = 9$, join AC.

$AC^2 = 21^2 + 10^2 - 2\times 21\times 10 \cos B \quad \cdots\cdots\cdots (1)$

$AC^2 = 9^2 + 10^2 - 2\times 9\times 10 \cos D \quad \cdots\cdots\cdots (2)$

(1) = (2): $541 - 420 \cos B = 181 - 180 \cos D$ ....(3)

$B + D = 180°$ (opp. $\angle$s, cyclic quad.)

$\therefore \cos D = -\cos B$

(3): $(420 + 180) \cos B = 541 - 181$

$\Rightarrow \cos B = \frac{3}{5}$

$\sin B = \sin D = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}$

Area of the cyclic quadrilateral = area of ΔABC + area of ΔACD

$$= \frac{1}{2}\cdot 21\cdot 10\cdot\frac{4}{5} + \frac{1}{2}\cdot 9\cdot 10\cdot\frac{4}{5}$$

$$= \frac{1}{2}\cdot 30\cdot 10\cdot\frac{4}{5}$$

$$= 120$$

I7 If $\frac{(a-b)(c-d)}{(b-c)(d-a)} = 3$, find the value of $\frac{(a-c)(b-d)}{(a-b)(c-d)}$.

Reference: 2006 HG6, 2009 FI3.3, 2014 HG7

$\frac{(a-b)(c-d)}{(b-c)(d-a)} = 3 \cdots\cdots\cdots (1)$

$\Rightarrow ac - bc - ad + bd = 3bd - 3cd - 3ab + 3ac$

$3ab - bc - ad + 3cd = 2ac + 2bd$

$\Rightarrow ab - bc - ad + cd = 2ac - 2ab - 2cd + 2bd \cdots\cdots\cdots (2)$

$$\frac{(a-c)(b-d)}{(a-b)(c-d)} = \frac{ab-bc-ad+cd}{ac-bc-ad+bd} = \frac{2(ac-ab-cd+bd)}{ac-bc-ad+bd} \quad \text{by (2)}$$

$$= \frac{2(b-c)(d-a)}{(a-b)(c-d)} = \frac{2}{3} \quad \text{by (1)}$$

I8 When the expression $x^3 + kx^2 + 3$ is divided by $x + 3$, the remainder is 2 less than when divided by $(x + 1)$. Find the value of k.

Let $f(x) = x^3 + kx^2 + 3$

$f(-1) - f(-3) = 2$

$\Rightarrow -1 + k + 3 - (-27 + 9k + 3) = 2$

$\Rightarrow k = 3$

I9 Given that the perimeter of a sector of a circle is 18. When the radius is r, the area of the sector attains the maximum value, find the value of r.

Let θ be the angle (in radians) subtended at centre, A be the area of the sector.

$$\begin{cases} 2r + r\theta = 18 \\ A = \frac{1}{2}r^2\theta \end{cases}$$

$$\Rightarrow \begin{cases} \theta = \frac{18-2r}{r} \\ A = \frac{1}{2}r^2\theta \end{cases}$$

$$\Rightarrow A = \frac{1}{2}r^2 \cdot \frac{18-2r}{r}$$

$= r(9 - r)$

$= -(r - 4.5)^2 + 4.5^2$

When the area is a maximum, $r = 4.5$

I10 Given $f\left(x+\frac{1}{x}\right) = x^2 + \frac{1}{x^2}$, find the value of f(5).

Reference: 2002 FG2.2

$$f\left(x+\frac{1}{x}\right) = x^2 + \frac{1}{x^2} + 2 - 2 = \left(x+\frac{1}{x}\right)^2 - 2$$

$\Rightarrow f(x) = x^2 - 2$

$\Rightarrow f(5) = 23$

Group Events

G1 A bag of sweets is distributed to three persons *A*, *B* and *C*. The numbers of sweets obtained by *A*, *B* and *C* are in the ratios of 5 : 4 : 3 respectively. If the sweets are re-distributed to *A*, *B*, *C* according to the ratios 7 : 6 : 5 respectively, then one of them would get 40 more sweets than his original number. Find the original number of sweets obtained by this person.

$5 + 4 + 3 = 12$; $7 + 6 + 5 = 18$; $\frac{3}{12}=\frac{9}{36}$, $\frac{5}{18}=\frac{10}{36}$; *C* would get more.

Let the original number of sweets be x.

$x\left(\frac{10}{36}-\frac{9}{36}\right)=40$

$x = 1440$

C originally obtained $1440\times\frac{3}{12} = 360$ sweets.

G2 Given that a, b, c are three consecutive odd numbers and $b^3 = 3375$, find the value of ac.

$b^3 = 3375 = 3^3\times5^3$

$\Rightarrow b = 15$

$a = b - 2$

$c = b + 2$

$ac = (b - 2)(b + 2)$

$= b^2 - 4$

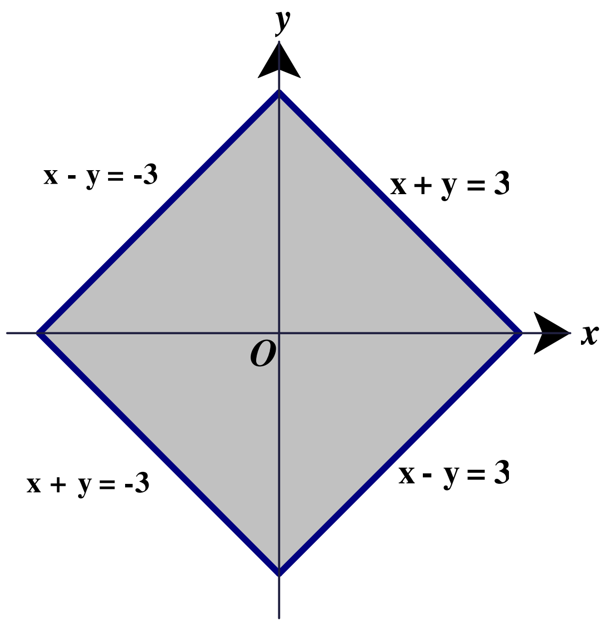
$= 225 - 4 = 221$

G3 Let p be the area of the polygon formed by the inequality $|x| + |y| \le 3$ in the Cartesian plane. Find the value of p.

The graph is shown on the right.

The polygon formed by the inequality is the shaded region.

p = shaded area $= 4\times\frac{1}{2}\times3^2 = 18$



G4 Find the remainder of $7^{2003} \div 100$. **(Reference: 1989 HG6)**

The question is equivalent to find the last 2 digits of 7^{2003}.

$7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$; the last 2 digits repeats for every multiples of 4.

$7^{2003} = 7^{4\times500+3}$, the last 2 digits is 43.

G5 If real numbers x, y satisfy the equation $x^2 + y^2 + 3xy = 35$, find the maximum value of xy.

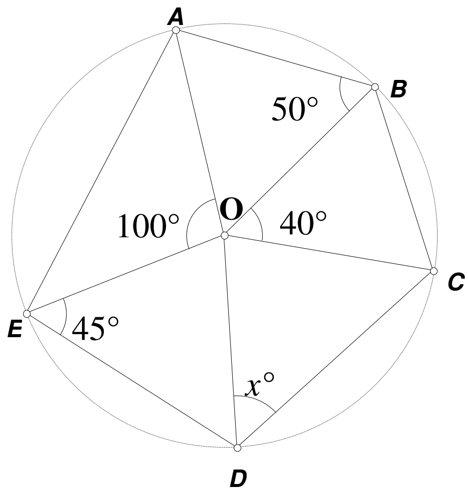
$35 = x^2 - 2xy + y^2 + 5xy = (x - y)^2 + 5xy \ge 5xy$

$\Rightarrow 7 \ge xy$, equality holds when $x = y$.

The maximum value of $xy = 7$.

G6 In figure 1, points A, B, C, D, E are on a circle with centre at O.

Given $\angle DEO = 45°$, $\angle AOE = 100°$, $\angle ABO = 50°$, $\angle BOC = 40°$, and $\angle ODC = x°$, find the value of x.



$\angle AOB = 180° - 2\times 50° = 80°$ ($\angle$s sum of Δ)

$\angle DOE = 180° - 2\times 45° = 90°$ ($\angle$s sum of Δ)

$\angle COD = 360° - 40° - 80° - 100° - 90°$ ($\angle$s at a pt.)

$= 50°$

$x° = (180° - 50°) \div 2 = 65°$ ($\angle$s sum of Δ)

$x = 65$

G7 20 balls are put into 2 bags with 10 balls in each bag. The balls in each bag are labeled numbers 1 to 10, all balls in one bag are white and all balls in the other bag are black. If one ball is drawn from each of two bags, find the probability that the number of the white ball is greater than that of the black ball.

Reference: 1989 HI11

Let the number shown on the white ball drawn be x, and the number shown on the black ball drawn be y. To find $P(x > y)$.

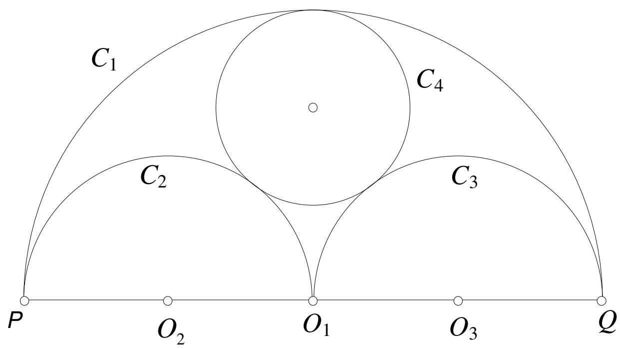
By symmetry, $P(x > y) = P(x < y)$

Further, $P(x > y) + P(x > y) + P(x = y) = 1$

$2P(x > y) + 10\times\frac{1}{10}\times\frac{1}{10} = 1$

$P(x > y) = \frac{9}{20}$

G8 In figure 2, PQ, PO_1, O_1Q are diameters of semi-circles C_1, C_2, C_3 with centres at O_1, O_2, O_3 respectively, and the circle C_4 touches C_1, C_2, and C_3. If $PQ = 24$, find the area of circle C_4. (Take $\pi = 3$).



$O_1O_2 = O_1O_3 = 6$

Let the centre of C_4 be O_4, the radius $= r$.

$O_4O_1 \perp PQ$

$O_3O_4 = r + 6$; $O_1O_4 = 12 - r$

$(12 - r)^2 + 6^2 = (r + 6)^2$ (Pythagoras' Theorem on $\Delta O_1O_3O_4$)

$144 - 24r + r^2 + 36 = r^2 + 12r + 36$

$\Rightarrow r = 4$

Area of $C_4 = \pi(4^2) = 48$

G9 Given that a and b are positive integers satisfying the equation $ab - a - b = 12$, find the value of ab.

Reference: 1987 FG10.4, 2012 FI4.2

$ab - a - b + 1 = 13$

$\Rightarrow (a - 1)(b - 1) = 13$

$\Rightarrow a - 1 = 13, b - 1 = 1$

$\Rightarrow a = 14, b = 2$

$ab = 28$

G10 Given that $\angle A$ is a right angle in triangle ABC, $\sin^2 C - \cos^2 C = \frac{1}{4}$, $AB = \sqrt{40}$ and $BC = x$, find the value of x.

$\sin^2 C - \cos^2 C = \frac{1}{4}$

$\Rightarrow \sin^2 C - (1 - \sin^2 C) = \frac{1}{4}$

$\Rightarrow \sin C = \sqrt{\frac{5}{8}}$

$B + C = 90°$

$\Rightarrow \cos B = \sin C = \sqrt{\frac{5}{8}} = \frac{\sqrt{40}}{x}$

$x = 8$