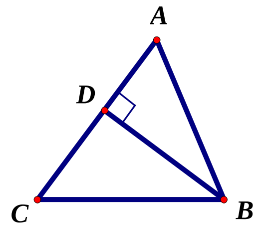
07-08 Individual	**1**	1.8	**2**	96	**3**	64	**4**	$\frac{12}{61}$	**5**	300
	6	27	**7**	3	**8**	$\frac{2007}{1004}$	**9**	2	**10**	–3

07-08 Group	**1**	1	**2**	34891	**3**	$\frac{8}{9}$	**4**	48	**5**	$\frac{1}{2}$
	6	1	**7**	$\frac{25}{12}$	**8**	3	**9**	6	**10**	$\frac{4\sqrt{41}}{5}$

Individual Events

I1 In Figure 1, ABC is a triangle, AB = 13 cm, BC = 14 cm and AC = 15 cm. D is a point on AC such that $BD \perp AC$. If CD is longer than AD by X cm, find the value of X.



$X = CD - AD = BC \cos C - AB \cos A$

$$X = 14\cdot\frac{15^2+14^2-13^2}{2\cdot 15\cdot 14} - 13\cdot\frac{15^2+13^2-14^2}{2\cdot 15\cdot 13} = 2\cdot\frac{14^2-13^2}{30}$$

$$X = \frac{27}{15} = \frac{9}{5}$$

Method 2

$BD^2 = 14^2 - CD^2 = 13^2 - AD^2$

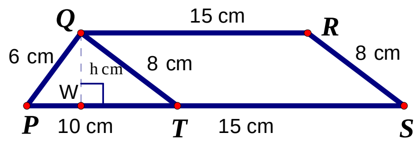
$14^2 - 13^2 = CD^2 - AD^2$

$(14 + 13)(14 - 13) = (CD + AD)(CD - AD)$

$27 = 15(CD - AD) \Rightarrow X = \frac{27}{15} = \frac{9}{5}$

I2 Given that a trapezium $PQRS$ with dimensions PQ = 6 cm, QR = 15 cm, RS = 8 cm and SP = 25 cm, also QR // PS. If the area of $PQRS$ is Y cm^2, find the value of Y.

Let the height of the trapezium be h cm (= QW).



From Q, draw QT // RS, which intersect PS at T.

$QRST$ is a //-gram (2 pairs of //-lines)

TS = 15 cm, QT = 8 cm (opp. sides, //-gram)

PT = (25 – 15) cm = 10 cm

$QP^2 + QT^2 = PT^2$

$\angle PQT = 90°$ (Converse, Pyth. Theorem)

$\frac{1}{2}6\times 8 =$ area of $\Delta PQT = \frac{1}{2}10\times h$, h = 4.8

Area of the trapezium = $Y = \frac{15+25}{2}\times 4.8 = 96$

I3 Given that x_0 and y_0 are positive integers satisfying the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{15}$. If $35 < y_0 < 50$ and $x_0 + y_0 = z_0$, find the value of z_0.

Reference: 2009 HG3

$15(x + y) = xy$

$xy - 15xy - 15y + 225 = 225$

$(x - 15)(y - 15) = 225$

$(x - 15, y - 15)$ = (1, 225), (3, 75), (5, 45), (9, 25), (15, 15), (25, 9), (45, 5), (75, 3), (225, 1)

$\because 35 < y_0 < 50$, $y_0 - 15 = 25$, $x_0 - 15 = 9$

$y_0 = 40$ and $x_0 = 24$; $z_0 = 24 + 40 = 64$

I4 Let a, b, c and d be real numbers. If $\frac{a}{b}=\frac{1}{2}$, $\frac{b}{c}=\frac{3}{2}$, $\frac{c}{d}=\frac{4}{5}$ and $\frac{ac}{b^2+d^2}=e$, find the value of e.

Method 1

$b=2a$; $\frac{a}{b}\times\frac{b}{c}=\frac{1}{2}\times\frac{3}{2}\Rightarrow c=\frac{4a}{3}$; $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{1}{2}\times\frac{3}{2}\times\frac{4}{5}\Rightarrow d=\frac{5a}{3}$

$$e=\frac{ac}{b^2+d^2}=\frac{a\times\frac{4a}{3}}{(2a)^2+\left(\frac{5a}{3}\right)^2}=\frac{\frac{4}{3}}{4+\frac{25}{9}}=\frac{4}{3}\times\frac{9}{61}=\frac{12}{61}$$

Method 2 $a:b=1:2$, $b:c=3:2$

$a:b:c$

$1:2$

$\underline{\quad 3:2}$

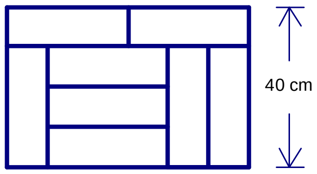
$3:6:4$

$\therefore a:b:c:d=3:6:4:5$

Let $a=3k$, $b=6k$, $c=4k$, $d=5k$

$$e=\frac{ac}{b^2+d^2}=\frac{(3k)(4k)}{(6k)^2+(5k)^2}=\frac{12}{61}$$

I5 In Figure 2, the large rectangle is formed by eight identical small rectangles. Given that the length of the shorter side of the large rectangle is 40 cm and the area of the small rectangle is A cm^2, find the value of A.



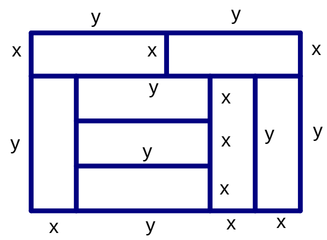
Let the length of a small rectangle be y cm;

the width of a small rectangle be x cm.

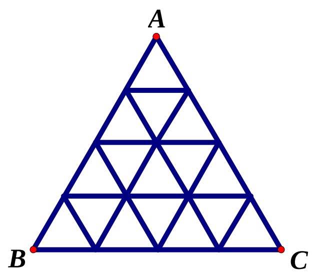
Then $x+y=40$, $y=3x$

$x=10$, $y=30$

$A=xy=300$



I6 In Figure 3, ΔABC is an equilateral triangle. It is formed by several identical equilateral triangles. If there are altogether N equilateral triangles in the figure, find the value of N.



Number of equilateral triangles of side 1 is 16.
Number of equilateral triangles of side 2 is 7.
Number of equilateral triangles of side 3 is 3.
Number of equilateral triangles of side 4 is 1.
Total number of equilateral triangles is 27.

I7 Let r be the larger real root of the equation $\frac{4}{y+1}+\frac{5}{y-5}=-\frac{3}{2}$. Find the value of r.

$8(y-5)+10(y+1)=-3(y^2-4y-5)$

$3y^2+6y-45=0$

$y^2+2y-15=0$

$(y+5)(y-3)=0$

$y=-5$ or 3; $r=3$

I8 Let x be a rational number and $w = \left|x+\frac{2007}{2008}\right| + \left|x-\frac{2007}{2008}\right|$. Find the smallest possible value of w.

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 FI1.3, 2010 HG6, 2011 FGS.1, 2012 FG2.3

Method 1

If $x < -\frac{2007}{2008}$, $w = -x - \frac{2007}{2008} - x + \frac{2007}{2008} = -2x$, $\therefore w = -2x > \frac{2007}{1004}$

If $-\frac{2007}{2008} \le x < \frac{2007}{2008}$, $w = x + \frac{2007}{2008} - x + \frac{2007}{2008} = \frac{2007}{1004}$, $\therefore w = \frac{2007}{1004}$

If $\frac{2007}{2008} \le x$, $w = x + \frac{2007}{2008} + x - \frac{2007}{2008} = 2x$, $\therefore w = 2x \ge \frac{2007}{1004}$

The smallest possible value of w is $\frac{2007}{1004}$.

Method 2 Using the triangle inequality: $|a| + |b| \ge |a + b|$

$$w = \left|x+\frac{2007}{2008}\right| + \left|x-\frac{2007}{2008}\right| \ge \left|x+\frac{2007}{2008}+\frac{2007}{2008}-x\right| = \frac{2007}{1004}$$

I9 Let m and n be a positive integers. Given that the number 2 appears m times and the number 4 appears n times in the expansion $\left(\left(\left((2)^2\right)^2\right)^{\cdot^{\cdot}}\right)^2 = \left(\left(\left((4)^4\right)^4\right)^{\cdot^{\cdot}}\right)^4$. If $k = \frac{m}{n}$, find the value of k.

$2^{\left(2^{m-1}\right)} = 4^{\left(4^{n-1}\right)}$

$2^{\left(2^{m-1}\right)} = 2^{2\left(2^{2n-2}\right)}$

$2^{m-1} = 2^{2n-1} \Rightarrow m = 2n$

$k = 2$

I10 Find the value of $\log_2(\sin^2 45°) + \log_2(\cos^2 60°) + \log_2(\tan^2 45°)$.

$\log_2 \frac{1}{2} + \log_2 \frac{1}{4} + \log_2 1 = -1 - 2 + 0 = -3$

Group Events

G1 Given that the decimal part of $5+\sqrt{11}$ is A and the decimal part of $5-\sqrt{11}$ is B.
Let $C = A + B$, find the value of C.

$3<\sqrt{11}<4$; $5+\sqrt{11}=5+3+A=8+A$

$5-\sqrt{11}=5-(3+A)=2-A=1+(1-A)$

$B = 1 - A$; $A + B = 1$

$C = 1$

G2 A total number of x candies, x is a positive integer, can be evenly distributed to 851 people as well as 943 people. Find the least possible value of x.

$851 = 23\times37$; $943 = 23\times41$

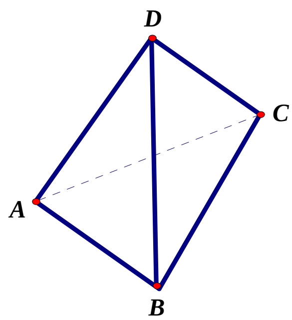
$x = 23\times37\times m = 23\times41\times n$, where m, n are positive integers.

$37m = 41n$

$\because$ 37, 41 are relatively prime, the minimum $m = 41$

The least possible value of $x = 23\times37\times41 = 851\times41 = 34891$

G3 In Figure 1, $ABCD$ is a regular tetrahedron with side length of 2 cm.
If the volume of the tetrahedron is $\sqrt{R}$ cm^3, find the value of R.



Let M be the mid point of BC. ($BM = MC = 1$ cm)

$\Delta ABM \cong \Delta ACM$ (SSS)

$AM=\sqrt{3}$ cm (Pythagoras Theorem)

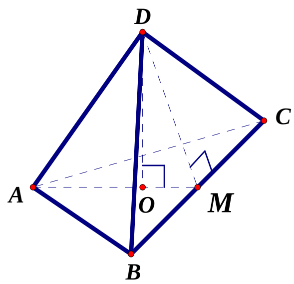
O is the centriod of ΔABC. $AO=\dfrac{2\sqrt{3}}{3}$ cm

DO = height of the tetrahedron $=\sqrt{AD^2-AO^2}=\sqrt{\dfrac{8}{3}}$ cm

Volume $=\sqrt{R}\text{ cm}^3=\dfrac{1}{3}\cdot\dfrac{1}{2}\cdot 2^2\sin 60^\circ\sqrt{\dfrac{8}{3}}\text{ cm}^3$

$\sqrt{R}=\dfrac{1}{3}\sqrt{8}$

$R=\dfrac{8}{9}$



G4 Given that x is a positive integer and $x < 60$. If x has exactly 10 positive factors, find the value of x.

Method 1

Note that except for the perfect square numbers (say 25), all positive integers have even numbers of factors. For a number $x < 60$ which has 10 positive factors, x will be divisible by as many numbers < 8 as possible. One possible choice would be 48. The positive factors are 1, 2, 3, 4, 6, 8, 12, 16, 24 and 48.

Method 2 Express N as unique prime factorization: $p_1^{r_1}p_2^{r_2}\cdots p_n^{r_n}$, then the number of factors is $(1 + r_1)(1 + r_2) \ldots (1 + r_n) = 10 = 2\times5 \Rightarrow n = 2$, $1 + r_1 = 2$, $1 + r_2 = 5 \Rightarrow r_1 = 1$, $r_2 = 4$

For prime numbers 2, 3, 5, …, $2^4\times3 = 48 < 60$, other combinations will exceed 60.

G5 Given that $90^\circ < \theta < 180^\circ$ and $\sin\theta=\dfrac{\sqrt{3}}{2}$. If $A = \cos(180^\circ - \theta)$, find the value of A.

$\theta = 120^\circ$, $A=\cos 60^\circ=\dfrac{1}{2}$

G6 Let x be a positive real number. Find the minimum value of $x^{2008} - x^{1004} + \frac{1}{x^{1004}}$.

Let $t = x^{1004}$, then $t^2 = x^{2008}$

$$x^{2008} - x^{1004} + \frac{1}{x^{1004}} = t^2 - t + \frac{1}{t} = t^2 - 1 - \frac{t^2-1}{t} + 1$$

$$= (t^2-1)\left(1-\frac{1}{t}\right) + 1 = \frac{(t^2-1)(t-1)}{t} + 1 = \frac{(t-1)^2(t+1)}{t} + 1$$

Clearly $t = x^{1004} > 0$, $(t-1)^2(t+1) \ge 0$, $\frac{(t-1)^2(t+1)}{t} + 1 \ge 0 + 1 = 1$, equality holds when $t = 1$.

$\therefore$ When $x = 1$, the minimum value of $x^{2008} - x^{1004} + \frac{1}{x^{1004}}$ is 1.

G7 Let x and y be real numbers satisfying $\begin{cases} \left(x-\frac{1}{3}\right)^3 + 2008\left(x-\frac{1}{3}\right) = -5 \\ \left(y-\frac{7}{4}\right)^3 + 2008\left(y-\frac{7}{4}\right) = 5 \end{cases}$. If $z = x + y$, find the value of z.

Let $a = x - \frac{1}{3}$, $b = y - \frac{7}{4}$. Add up the two equations: $a^3 + b^3 + 2008(a + b) = 0$

$(a + b)(a^2 - ab + b^2) + 2008(a + b) = 0$

$(a + b)(a^2 - ab + b^2 + 2008) = 0$

$a + b = 0$ or $a^2 - ab + b^2 + 2008 = 0$

But $a^2 - ab + b^2 + 2008 = \left(a - \frac{b}{2}\right)^2 + \frac{b^2}{4} + 2008 \ge 2008 \ne 0$

$\therefore a + b = x - \frac{1}{3} + y - \frac{7}{4} = 0$; $z = x + y = \frac{25}{12}$

G8 Let R be the remainder of $1\times3\times5\times7\times9\times11\times13\times15\times17\times19$ divided by 4. Find the value of R.

Note that if N and m are positive integers, $0 \le m \le 99$ and $x = 100N + m$, then the remainder when x divided by 4 is the same as that when m is divided by 4.

Product $=1\times(4-1)\times(4+1)\times(4\times2-1)\times(4\times2+1)\times(4\times3-1)\times(4\times3+1)\times(4\times4-1)\times(4\times4+1)\times(4\times5-1)$

$=1\times(4+1)\times(4\times2+1)\times(4\times3+1)\times(4\times4+1)\times(4-1)\times(4\times2-1)\times(4\times3-1)\times(4\times4-1)\times(4\times5-1)$

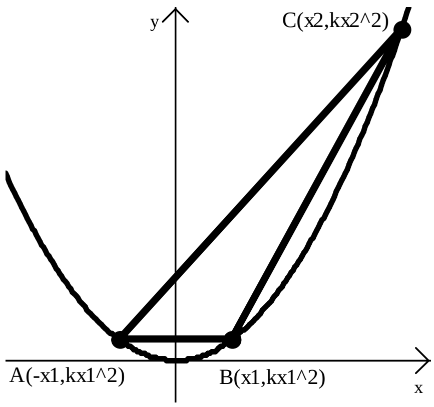
$= (4a + 1)(4b - 1)$, where a, and b are integers.

$= 16ab + 4(b - a) - 1$, the remainder is 3.

Method 2 $1\times3\times5\times7\times9\times11\times13\times15\times17\times19 \equiv 1\cdot(-1)\times1\times(-1)\times1\times(-1)\times1\times(-1)\times1\times(-1) \equiv 3 \bmod 4$

G9 Given that k, x_1 and x_2 are positive integers with $x_1 < x_2$. Let A, B and C be three points on the curve $y = kx^2$, with x-coordinates being $-x_1$, x_1 and x_2 respectively. If the area of ΔABC is 15 square units, find the sum of all possible values of k.

$\because k > 0$, the parabola opens upwards with following shape:



Area of $\Delta ABC = \frac{1}{2} AB \times \text{height}$

$\frac{1}{2}(x_1 + x_1)\times(kx_2^2 - kx_1^2) = 15$

$kx_1(x_2^2 - x_1^2) = 15$

Possible $k = 1, 3, 5, 15$

When $k = 1$, $x_1 = 1$, $x_2 = 4$

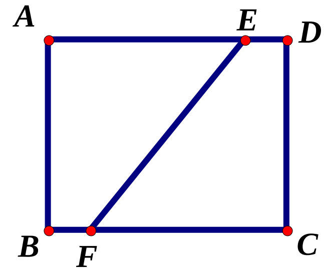
When $k = 3$, no solution

When $k = 5$, $x_1 = 1$, $x_2 = 2$

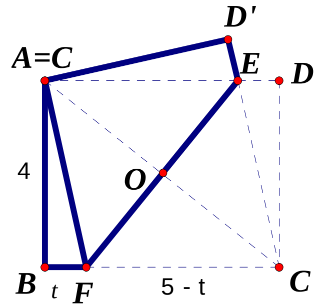
When $k = 15$, no solution

Sum of all possible values of $k = 1 + 5 = 6$

G10 In Figure 2, $ABCD$ is rectangular piece of paper with $AB = 4$ cm and $BC = 5$ cm. Fold the paper by putting point C onto A to create a crease EF. If $EF = r$ cm, find the value of r.



Suppose D will be folded to a position D'.



$AE = CE$, $AF = CF$ (by the property of folding)
$EF = EF$ (common sides)
$\Delta EFA \cong \Delta EFC$ (S.S.S.)
$\therefore AE = CE$, $AF = CF$ (corr. sides $\cong \Delta$'s)
$\angle AEF = \angle CEF$, $\angle AFE = \angle CFE$ (corr. $\angle$s $\cong \Delta$'s)
$\angle AEF = \angle CFE$ (alt. $\angle$s, AD // BC)
$\therefore \angle AEF = \angle AFE$, $\angle CEF = \angle CFE$
$AE = CE = CF = AF$ (sides opp. equal $\angle$s)
$\therefore AECF$ is a rhombus (4 sides equal)
Let O be the intersection of AC and EF.
In ΔABC, $AC^2 = 4^2 + 5^2$ (Pythagoras' Theorem)
$AC = \sqrt{41}$; $AO = \dfrac{\sqrt{41}}{2}$ (diagonal of a rhombus)
Let $BF = DE = t$
$AE = 5 - t = EC = CF = AF$
In ΔABF, $4^2 + t^2 = (5 - t)^2 = 25 - 10t + t^2$ (Pythagoras' Theorem)
$t = \dfrac{9}{10}$, $AF = 5 - \dfrac{9}{10} = \dfrac{41}{10}$
In ΔAOF, $OF^2 = AF^2 - AO^2$ (Pythagoras' Theorem)
$OF = \sqrt{(4.1)^2 - \dfrac{41}{4}} = \dfrac{2\sqrt{41}}{5}$
$r = EF = 2OF = \dfrac{4\sqrt{41}}{5}$ (diagonal of a rhombus)

Method 2 As before, $AECF$ is a rhombus. AC and EF bisect each other at O. $AO = \dfrac{\sqrt{41}}{2}$.

$\angle EAO = \angle ACB$ (alt. $\angle$s AD // BC)
$\angle AOE = 90° = \angle ABC$ (property of rhombus)
$\tan \angle EAO = \tan \angle ACB$

$$\frac{EO}{\frac{\sqrt{41}}{2}} = \frac{4}{5}$$

$$EO = \frac{2\sqrt{41}}{5}$$

$$EF = 2EO = \frac{4\sqrt{41}}{5}$$