17-18 Individual	**1**	1966	**2**	19	**3**	718	**4**	99999	**5**	$\frac{1}{2}$
	6	2.5	**7**	$2\sqrt{5}$	**8**	$2(\sqrt{3}-1)$	**9**	$\frac{61}{8}=7.625$	**10**	54
	11	13	**12**	16π	**13**	2000	**14**	$-\frac{2018}{2019}$	**15**	289578289
17-18 Group	**1**	$\frac{4}{3}$	**2**	3999766	**3**	10	**4**	275	**5**	–15
	6	5	**7**	$\frac{9}{16}$	**8**	2380	**9**	2475	**10**	$\frac{900}{7}=128\frac{4}{7}$

Individual Events

I1 若 a 及 b 均為實數，求 $a^2 + b^2 + 12a - 8b + 2018$ 的最小值。

If a and b are real numbers, find the minimum value of $a^2 + b^2 + 12a - 8b + 2018$.

Reference: 1999 HG7, 2001 HI3, 2012 HG5

$a^2 + b^2 + 12a - 8b + 2018$ $= a^2 + 12a + 36 + b^2 - 8b + 16 + 1966$ $= (a + 6)^2 + (b - 4)^2 + 1966 \geq 1966$ 最小值為 1966。	$a^2 + b^2 + 12a - 8b + 2018$ $= a^2 + 12a + 36 + b^2 - 8b + 16 + 1966$ $= (a + 6)^2 + (b - 4)^2 + 1966 \geq 1966$ The minimum value is 1966 .

I2 設 a 及 k 均為常數。若 $(6x^3 + ax^2 + 7x - 3)\div(2x^2 + kx - 1)$ 的商和餘式分別為 $3x + 5$ 及 $-5x + 2$，求 a 的值。

Let a and k be constants. If the quotient and the remainder of $(6x^3 + ax^2 + 7x - 3)\div(2x^2 + kx - 1)$ are $3x + 5$ and $-5x + 2$ respectively, find the value of a .

利用除法定理：被除數 = 除數×商 + 餘數 $6x^3 + ax^2 + 7x - 3 = (2x^2 + kx - 1)\times(3x + 5) - 5x + 2$ 比較兩邊的 x 的係數：$7 = 5k - 3 - 5$ $\Rightarrow k = 3$ 比較兩邊的 x^2 的係數：$a = 2\times5 + 3k$ 代 $k = 3$ 入右式中： $a = 10 + 9 = 19$	Using division algorithms: Dividend = divisor×quotient + remainder $6x^3 + ax^2 + 7x - 3 = (2x^2 + kx - 1)\times(3x + 5) - 5x + 2$ Compare coefficients of x on both sides: $7 = 5k - 3 - 5 \Rightarrow k = 3$ Compare coefficients of x^2 on both sides: $a = 2\times5 + 3k$ Sub. $k = 3$ into the right side: $a = 10 + 9 = 19$

I3 在編制某雜誌中每頁的頁碼時，總共用去了 2,046 個數字，問該雜誌總共有多少頁？(假設該雜誌第一頁的頁碼是 1。)

In numbering the pages of a magazine, 2046 digits were used. How many pages are there in the magazine? (Assume the page number of the magazine starts from 1 .)

由第一頁至第九頁： 共 9 個數字 由第十頁至第九十九頁： 共$(99 - 9)\times2 = 180$ 個數字 由第一百頁至第九百九十九頁： 共$(999 - 99)\times3 = 2700$ 個數字 $9 + 180 < 2046 < 9 + 180 + 2700$ 假設該雜誌共有 x 頁，其中 $100 < x < 999$。 $9 + 180 + (x - 99)\times3 = 2046$ $x = 718$，該雜誌共有 718 頁。	Page 1 to 9: 9 digits Page 10 to 99: $(99 - 9)\times2 = 180$ digits Page 100 to 999: $(999 - 99)\times3 = 2700$ digits $9 + 180 < 2046 < 9 + 180 + 2700$ Suppose there are x pages in the magazine, where $100 < x < 999$. $9 + 180 + (x - 99)\times3 = 2046$ $\Rightarrow x = 718$, there 718 pages in the magazine.

I4 解 $\log\left(1+\frac{1}{1}\right)+\log\left(1+\frac{1}{2}\right)+\log\left(1+\frac{1}{3}\right)+\cdots+\log\left(1+\frac{1}{n}\right)=5$。

Solve $\log\left(1+\frac{1}{1}\right)+\log\left(1+\frac{1}{2}\right)+\log\left(1+\frac{1}{3}\right)+\cdots+\log\left(1+\frac{1}{n}\right)=5$.

$$\log 2+\log\frac{3}{2}+\log\frac{4}{3}+\cdots+\log\frac{n+1}{n}=5$$

$$\log\left(2\times\frac{3}{2}\times\frac{4}{3}\times\cdots\times\frac{n+1}{n}\right)=5$$

$n + 1 = 10^5 = 100000$

$n = 99999$

I5 已知 $\dfrac{1-2^{-\frac{1}{x}}}{2^{-\frac{1}{x}}-2^{-\frac{2}{x}}}=4$。求 x 的值。

Given that $\dfrac{1-2^{-\frac{1}{x}}}{2^{-\frac{1}{x}}-2^{-\frac{2}{x}}}=4$. Find the value of x . (**Reference 2018 FG2.1**)

$$\frac{\left(1-\frac{1}{2^{\frac{1}{x}}}\right)}{\left(\frac{1}{2^{\frac{1}{x}}}-\frac{1}{2^{\frac{2}{x}}}\right)}\cdot\frac{2^{\frac{2}{x}}}{2^{\frac{2}{x}}}=4$$

$$\frac{2^{\frac{1}{x}}\left(2^{\frac{1}{x}}-1\right)}{2^{\frac{1}{x}}-1}=4$$

$$2^{\frac{1}{x}}=2^2$$

$$x=\frac{1}{2}$$

I6 若 x 為有理數，求 x 的值滿足聯立方程 $\begin{cases} y=2x^2-11x+15 \\ y=2x^3-17x^2+16x+35 \end{cases}$。

If x is a rational number, find the value of x satisfying the simultaneous equations

$$\begin{cases} y = 2x^2 - 11x + 15 \\ y = 2x^3 - 17x^2 + 16x + 35 \end{cases}.$$

$2x^2 - 11x + 15 = (x - 3)(2x - 5)$

$2(3)^3 - 17(3)^2 + 16(3) + 35 = 54 - 153 + 48 + 35 = -86$

$2(2.5)^3 - 17(2.5)^2 + 16(2.5) + 35 = \frac{125}{4} - \frac{425}{4} + 40 + 35 = 0$

$2x^3 - 17x^2 + 16x + 35 = 2x^2 - 11x + 15$

$(2x - 5)(x^2 - 6x - 7) - (2x - 5)(x + 3) = 0$

$(2x - 5)(x^2 - 6x - 7 - x - 3) = 0$

$(2x - 5)(x^2 - 7x - 10) = 0$

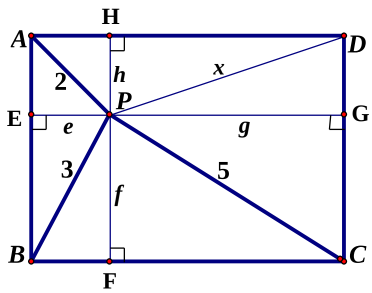
$x = 2.5$ or $x = \frac{7 \pm \sqrt{89}}{2}$ (無理根，捨去 irrational roots, rejected)

$$\begin{array}{r|l} & x^2 - 6x - 7 \\ \hline 2x-5 & 2x^3 - 17x^2 + 16x + 35 \\ & 2x^3 - 5x^2 \\ \hline & -12x^2 + 16x \\ & -12x^2 + 30x \\ \hline & -14x + 35 \\ & -14x + 35 \\ \hline \end{array}$$

I7 如圖一所示，P 為長方形 $ABCD$ 內的一點，使得 $PA = 2$，$PB = 3$ 及 $PC = 5$。求 PD 的長度。

As shown in Figure 1, P is a point inside a rectangle $ABCD$ such that $PA = 2$, $PB = 3$ and $PC = 5$. Find the length of PD .

Reference: 1994 FG10.1-2, 2001 FG2.2, 2003 FI3.4

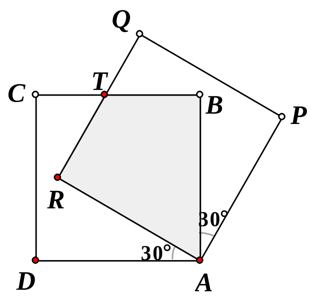


圖一 Figure 1

假設分別由 P 至 AB、BC、CD 及 DA 之垂足為 E、F、G 及 H。設 $PD = x$、$PE = e$、$PF = f$、$PG = g$、$PH = h$。 那麼，由畢氏定理可得知： $e^2 + h^2 = 2^2$ …… (1) $e^2 + f^2 = 3^2$ …… (2) $f^2 + g^2 = 5^2$ …… (3) $g^2 + h^2 = x^2$ …… (4) (1) + (3) – (2) – (4): $0 = 4 + 29 - 9 - x^2$ $PD = x = 2\sqrt{5}$	Let E, F, G, H be the feet of perpendiculars drawn from P onto AB, BC, CD and DA respectively. Let $PD = x$, $PE = e$, $PF = f$, $PG = g$, $PH = h$. Then by Pythagoras' theorem, $e^2 + h^2 = 2^2$ …… (1) $e^2 + f^2 = 3^2$ …… (2) $f^2 + g^2 = 5^2$ …… (3) $g^2 + h^2 = x^2$ …… (4) (1) + (3) – (2) – (4): $0 = 4 + 29 - 9 - x^2$ $PD = x = 2\sqrt{5}$

I8 如圖二所示，兩個邊長為 x cm 的正方形於一角重疊。若兩個正方形的非重疊部分與重疊部分面積的比是 $a:1$，求 a 的值。

As shown in Figure 2, two squares with side x cm coincides at one corner. If the ratio of the non-overlapping area to the overlapping area of the two squares is a : 1, find the value of a .



圖二 Figure 2

將兩個正方形的角重新命名為 $ABCD$ 及 $APQR$ 如圖所示。 假設 BC 與 QR 相交於 T。假設正方形的每邊邊長為 x 。 $\angle B = \angle R = 90°$ (正方形的性質) $\angle BAR = 60°$	Label the corners of the two squares as $ABCD$ and $APQR$ as shown. Suppose BC intersects QR at T . Let the length of side of each square be x . $\angle B = \angle R = 90°$ (property of a square) $\angle BAR = 60°$ $\Delta ABT \cong \Delta ART$ (R.H.S.)

$\Delta ABT \cong \Delta ART$ (R.H.S.)

$\therefore \angle BAT = \angle RAT = 30°$ (全等三角形的對應角)

$BT = RT = x \tan 30° = \dfrac{x}{\sqrt{3}}$

$ABTR$ 的面積$= 2\times\dfrac{1}{2}x\cdot\dfrac{x}{\sqrt{3}} = \dfrac{x^2}{\sqrt{3}}$

非重疊部分的面積$= 2\left(x^2 - \dfrac{x^2}{\sqrt{3}}\right) = 2x^2\left(\dfrac{\sqrt{3}-1}{\sqrt{3}}\right)$

非重疊部分：重疊部分面積的比

$= 2x^2\left(\dfrac{\sqrt{3}-1}{\sqrt{3}}\right) : \dfrac{x^2}{\sqrt{3}} = 2\left(\sqrt{3}-1\right):1$

$a = 2\left(\sqrt{3}-1\right)$

$\therefore \angle BAT = \angle RAT = 30°$ (corr. $\angle$s $\cong \Delta$s)

$BT = RT = x \tan 30° = \dfrac{x}{\sqrt{3}}$

Area of $ABTR = 2\times\dfrac{1}{2}x\cdot\dfrac{x}{\sqrt{3}} = \dfrac{x^2}{\sqrt{3}}$

Area of the unshaded part $= 2\left(x^2 - \dfrac{x^2}{\sqrt{3}}\right) = 2x^2\left(\dfrac{\sqrt{3}-1}{\sqrt{3}}\right)$

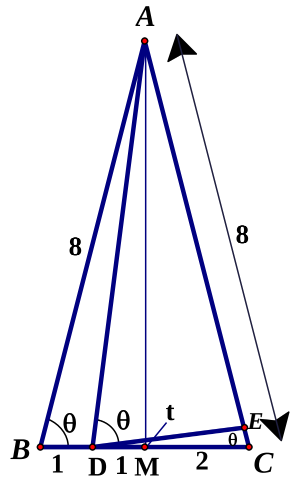
The non-overlapping area : the overlapping area of the two squares

$= 2x^2\left(\dfrac{\sqrt{3}-1}{\sqrt{3}}\right) : \dfrac{x^2}{\sqrt{3}} = 2\left(\sqrt{3}-1\right):1$

$a = 2\left(\sqrt{3}-1\right)$

I9 如圖三所示，ABC 是一個等腰三角形，其中 $AB = AC = 8$ 及 $BC = 4$。D 及 E 分別為 BC 及 AC 上的點使得 $BD = 1$ 及 $\angle ABC = \angle ADE$。求 AE 的值。

As shown in Figure 3, ABC is an isosceles triangle with $AB = AC = 8$ and $BC = 4$. D and E are points lying on BC and AC respectively such that $BD = 1$ and $\angle ABC = \angle ADE$. Find the length of AE.



圖三 Figure 3

假設 M 為 BC 的中點。

$BM = MC = 2$

$\Delta ABM \cong \Delta ACM$ (S.S.S.)

$\angle AMB = \angle AMC = 90°$ (全等三角形的對應角)

Let $\angle ABM = \theta = \angle ACM$ (等腰三角形底角)

$\angle ADE = \theta = \angle ACB$

在 ΔABM 中， $\cos\theta = \dfrac{2}{8} = \dfrac{1}{4}$

於ΔABD 應用餘弦定理：

$AD^2 = 1^2 + 8^2 - 2\times1\times8\cos\theta$

$AD = \sqrt{61}$

設 $\angle CDE = t$

$\angle AED = t + \theta = \angle ADC$ (ΔCDE 的外角)

$\Delta ADE \sim \Delta ACD$ (等角)

$\dfrac{AE}{AD} = \dfrac{AD}{AC}$ (相似三角形的對應邊)

$\dfrac{AE}{\sqrt{61}} = \dfrac{\sqrt{61}}{8}$

Let M be the mid-point of BC.

$BM = MC = 2$

$\Delta ABM \cong \Delta ACM$ (S.S.S.)

$\angle AMB = \angle AMC = 90°$ (corr. $\angle$s $\cong \Delta$'s)

Let $\angle ABM = \theta = \angle ACM$ (base $\angle$s isos. Δ)

$\angle ADE = \theta = \angle ACB$

In ΔABM, $\cos\theta = \dfrac{2}{8} = \dfrac{1}{4}$

Apply cosine formula on ΔABD:

$AD^2 = 1^2 + 8^2 - 2\times1\times8\cos\theta$

$AD = \sqrt{61}$

Let $\angle CDE = t$

$\angle AED = t + \theta = \angle ADC$ (ext. $\angle$ of ΔCDE)

$\Delta ADE \sim \Delta ACD$ (equiangular)

$\dfrac{AE}{AD} = \dfrac{AD}{AC}$ (corr. sides, ~Δ's)

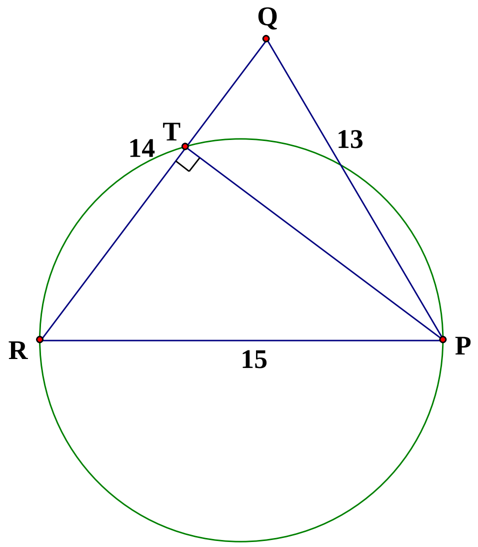
$\dfrac{AE}{\sqrt{61}} = \dfrac{\sqrt{61}}{8}$

$AE = \dfrac{61}{8} = 7.625$

$AE=\frac{61}{8}=7.625$	

I10 PQR 是一個三角形，其中 $PQ=13$、$QR=14$ 及 $PR=15$。以 PR 為直徑繪畫出圓 C，C 相交 QR 於點 T。求 ΔPTR 的面積。

PQR is a triangle with $PQ = 13$, $QR = 14$ and $PR = 15$. The circle C is drawn with diameter PR. C intersects QR at a point T. Find the area of ΔPTR.



$\angle PTR=90°$ (半圓上的圓周角) $\cos\angle PTR=\frac{14^2+15^2-13^2}{2\times14\times15}=\frac{3}{5}$ $\sin\angle PTR=\frac{4}{5}$ $RT=PR\cos\angle PTR=15\times\frac{3}{5}=9$ ΔPRT 的面積$=\frac{1}{2}RP\cdot RT\sin\angle PRT$ $=\frac{1}{2}\times15\times9\times\frac{4}{5}$ $=54$	$\angle PTR=90°$ ($\angle$ in semi-circle) $\cos\angle PTR=\frac{14^2+15^2-13^2}{2\times14\times15}=\frac{3}{5}$ $\sin\angle PTR=\frac{4}{5}$ $RT=PR\cos\angle PTR=15\times\frac{3}{5}=9$ Area of $\Delta PRT=\frac{1}{2}RP\cdot RT\sin\angle PRT$ $=\frac{1}{2}\times15\times9\times\frac{4}{5}$ $=54$

I11 求 $3^x+5+\frac{36}{3^x+4}$ 的最小值。

Find the minimum value of $3^x+5+\frac{36}{3^x+4}$.

設 $y=3^x+4$，則此該表達式變成： $y+\frac{36}{y}+1\ \ge\ 2\sqrt{y\times\frac{36}{y}}+1$ (A.M. ≥ G.M.) $=13$ 等式成立當 $y=\frac{36}{y}$; 即 $y=6$ $3^x+4=6$ $\Rightarrow x=\log 2\div\log 3$ $\therefore$ 最小值為 13。	Let $y=3^x+4$, then the expression becomes: $y+\frac{36}{y}+1\ \ge\ 2\sqrt{y\times\frac{36}{y}}+1$ (A.M. ≥ G.M.) $=13$ Equality holds when $y=\frac{36}{y}$; 即 i.e. $y=6$ $3^x+4=6$ $\Rightarrow x=\log 2\div\log 3$ $\therefore$ The minimum value is 13 .

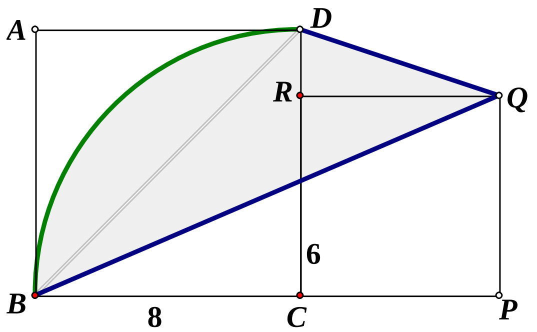
The following method is suggested by Mr. Ma Shing, a secondary school teacher:

Let $y=3^x+4$ and $T=y+\frac{36}{y}+1\ \ge 0$, then the equation becomes: $yT=y^2+y+36$

$\Rightarrow y^2 + (1 - T)y + 36 = 0$, a quadratic equation in y. For real values of y, $\Delta = (1 - T)^2 - 4(36) \ge 0$
$\Rightarrow T - 1 \ge 12$ or $T - 1 \le -12$ (rejected) $\Rightarrow T \ge 13$.
The minimum value is 13 .

I12 如圖四所示，$ABCD$ 及 $PQRC$ 為 兩個連接的正方形。以 C 為圓心及 CB 為半徑繪畫出弧 BD 。已知 $BC = 8$ 及 $RC = 6$ 。 求弧 BD 及綫段 DQ 與 BQ 所圍成的區域的面積。

As shown in Figure 4, two squares $ABCD$ and $PQRC$ are joined together. An arc BD is drawn with centre C and radius CB. Given that $BC = 8$ and $RC = 6$. Find the area of the region bounded by the arc BD, line segments DQ and BQ .



圖四 Figure 4

Reference: 2000 FI4.2, 2004 HI9, 2005 HG7

陰影面積 $=$弓形 $BD + S_{\Delta BRD} + S_{\Delta QRD} + S_{\Delta BRQ}$ $=\frac{1}{4}\pi\cdot 8^2-\frac{1}{2}\cdot 8^2+\frac{1}{2}(8-6)\cdot 8+\frac{1}{2}(8-6)\cdot 6+\frac{1}{2}\cdot 6\times 6$ $= 16\pi - 32 + 8 + 6 + 18 = 16\pi$ **方法二** 連接 BD 及 CQ。 $\angle CBD = \angle PCQ = 45°$ (正方形的性質) $\therefore BD \parallel CQ$ (對應角相等) $S_{\Delta BDQ} = S_{\Delta BDC}$ (兩三角形同底同高) 陰影面積 $=$弓形 $BD + S_{\Delta BDQ}$ $=$弓形 $BD + S_{\Delta BDC}$ $=$ 扇形 $BDC = 16\pi$	Shaded area $=$ segment $BD + S_{\Delta BRD} + S_{\Delta QRD} + S_{\Delta BRQ}$ $=\frac{1}{4}\pi\cdot 8^2-\frac{1}{2}\cdot 8^2+\frac{1}{2}(8-6)\cdot 8+\frac{1}{2}(8-6)\cdot 6+\frac{1}{2}\cdot 6\times 6$ $= 16\pi - 32 + 8 + 6 + 18 = 16\pi$ **Method 2** Join BD and CQ. $\angle CBD = \angle PCQ = 45°$ (property of a square) $\therefore BD \parallel CQ$ (corr. $\angle$s eq.) $S_{\Delta BDQ} = S_{\Delta BDC}$ (same bases and same heights) Shaded area $=$ segment $BD + S_{\Delta BDQ}$ $=$ segment $BD + S_{\Delta BDC}$ $=$ sector $BDC = 16\pi$

I13 一個四位數可以透過把它的所有數字加起來，變成另一個數。例如：1234 可以變成 10，因為 $1+2+3+4=10$。究竟從 1998 至 4998 (包括此兩個數)有多少個四位數，經上述變換後不可以被 3 整除？

A 4-digit number can be transformed into another number by adding its digits. For example, 1234 is transformed into 10 as $1 + 2 + 3 + 4 = 10$. How many transformed numbers from 1998 to 4998 inclusive are **NOT** divisible by 3?

已給一正整數，易證該數能被 3 整除的充分及必要條件是該數的數位之和是 3 的倍數。 我們只須數一數由 1998 至 4998 之間的 3 的倍數。 $1998 = 3\times 666$, $4998 = 3\times 1666$ $\therefore$ 3 的倍數數量是 $1666 - 666 + 1 = 1001$ 不能被 3 整除的數目有 $4998 - 1998 + 1 - 1001 = 2000$	It is easy to show that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3. It suffices to count the number of multiples of 3 from 1998 to 4998. $1998 = 3\times 666$, $4998 = 3\times 3666$ $\therefore$ The number of multiples of 3 is $1666 - 666 + 1 = 1001$ Number of integers which are **NOT** divisible by 3 is $4998 - 1998 + 1 - 1001 = 2000$

I14 對任意實數 x ($x \neq 1$)，定義函數 $f(x)=\frac{x}{1-x}$ 及 $f\circ f(x) = f(f(x))$。

求 $2017\underbrace{f\circ f\circ f\circ\cdots\circ f}_{2018個f}(2018)$ 的值。

For any real number x ($x \neq 1$), define a function $f(x)=\frac{x}{1-x}$ and $f\circ f(x) = f(f(x))$.

Find the value of $2017\underbrace{f\circ f\circ f\circ\cdots\circ f}_{2018\text{ copies of f}}(2018)$.

Reference: 1997 HG2

$f\circ f(x)=f(f(x))=f\left(\frac{x}{1-x}\right)=\frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}=\frac{x}{1-x-x}=\frac{x}{1-2x}$ $f\circ f\circ f(x)=f\left(\frac{x}{1-2x}\right)=\frac{\frac{x}{1-x}}{1-\frac{2x}{1-x}}=\frac{x}{1-x-2x}=\frac{x}{1-3x}$ 聲明： $\underbrace{f\circ f\circ f\circ\cdots\circ f}_{n\text{ copies of f}}(x)=\frac{x}{1-nx}$ 證明：利用數學歸納法，$n=1$、2、3，上文已證。 假設 $\underbrace{f\circ f\circ f\circ\cdots\circ f}_{k\text{ 個 f}}(x)=\frac{x}{1-kx}$ 成立。 $\underbrace{f\circ f\circ f\circ\cdots\circ f}_{k+1\text{ 個 f}}(x)=f\left(\frac{x}{1-kx}\right)=\frac{\frac{x}{1-x}}{1-\frac{kx}{1-x}}=\frac{x}{1-(k+1)x}$ 如果 $n=k$ 等式成立，$n=k+1$ 時等式依然成立。 由數學歸納法的原則，對於所有正整數，該等式成立。 $2017\underbrace{f\circ f\circ f\circ\cdots\circ f}_{2018個f}(2018)$ $=2017\times\frac{2018}{1-2018\times2018}$ $=2017\times\frac{2018}{(1-2018)\times(1+2018)}$ $=-\frac{2018}{2019}$	$f\circ f(x)=f(f(x))=f\left(\frac{x}{1-x}\right)=\frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}=\frac{x}{1-x-x}=\frac{x}{1-2x}$ $f\circ f\circ f(x)=f\left(\frac{x}{1-2x}\right)=\frac{\frac{x}{1-x}}{1-\frac{2x}{1-x}}=\frac{x}{1-x-2x}=\frac{x}{1-3x}$ Claim: $\underbrace{f\circ f\circ f\circ\cdots\circ f}_{n\text{ copies of f}}(x)=\frac{x}{1-nx}$ Proof: By M.I.. $n = 1, 2, 3$, proved above Suppose $\underbrace{f\circ f\circ f\circ\cdots\circ f}_{k\text{ copies of f}}(x)=\frac{x}{1-kx}$ $\underbrace{f\circ f\circ f\circ\cdots\circ f}_{k+1\text{ copies of f}}(x)=f\left(\frac{x}{1-kx}\right)=\frac{\frac{x}{1-x}}{1-\frac{kx}{1-x}}=\frac{x}{1-(k+1)x}$ If it is true for $n=k$, then it is also true for $n=k+1$ By the principle of mathematical induction, the formula is true for all positive integer n. $2017\underbrace{f\circ f\circ f\circ\cdots\circ f}_{2018個f}(2018)$ $=2017\times\frac{2018}{1-2018\times2018}$ $=2017\times\frac{2018}{(1-2018)\times(1+2018)}$ $=-\frac{2018}{2019}$

I15 設 $N^2=\overline{abcdefabc}$ 為一個 9 位整數，其中 N 是 4 個相異質數的積及 a、b、c、d、e、f 均為非零數字且滿足 $\overline{def}=2\times\overline{abc}$。求 N^2 的最小值。

Let $N^2=\overline{abcdefabc}$ be a nine-digit positive integer, where N is the product of four distinct primes and a, b, c, d, e, f are non-zero digits that satisfy $\overline{def}=2\times\overline{abc}$. Find the least value of N^2.

$N^2=\overline{abcdefabc}=1000000\,\overline{abc}+1000\,\overline{def}+\overline{abc}$ $=1000001\,\overline{abc}+2000\,\overline{abc}$ $=1002001\,\overline{abc}$ $=1001^2\times\overline{abc}=(7\times11\times13)^2\times\overline{abc}$ $\therefore\ \overline{abc}=$ 一個三位數不等於 7、11 或 13 的質數的平方 $\overline{abc}$ 的最小值 $=17^2=289$ N^2 的最小值 $=1002001\times289=289578289$	$N^2=\overline{abcdefabc}=1000000\,\overline{abc}+1000\,\overline{def}+\overline{abc}$ $=1000001\,\overline{abc}+2000\,\overline{abc}$ $=1002001\,\overline{abc}$ $=1001^2\times\overline{abc}=(7\times11\times13)^2\times\overline{abc}$ $\therefore\ \overline{abc}=$ a 3-digit number which is the square of a prime number different from 7, 11 and 13 The least possible $\overline{abc}=17^2=289$ The least possible value of N^2

	$= 1002001 \times 289 = 289578289$

Group Events

G1 設 f(x) 為二次多項式，其中 $f(1)=\frac{1}{2}$，$f(2)=\frac{1}{6}$，$f(3)=\frac{1}{12}$。求 f(6) 的值。

Let f(x) be a polynomial of degree 2, where $f(1)=\frac{1}{2}$, $f(2)=\frac{1}{6}$, $f(3)=\frac{1}{12}$. Find the value of f(6) .

Reference 2003 FG4.2

設 $f(x)=ax^2+bx+c$, 則 $f(1)=a+b+c=\frac{1}{2}\quad\cdots(1)$ $f(2)=4a+2b+c=\frac{1}{6}\quad\cdots(2)$ $f(3)=9a+3b+c=\frac{1}{12}\quad\cdots(3)$ $(2)-(1)$: $3a+b=-\frac{1}{3}\quad\cdots(4)$ $(3)-(2)$: $5a+b=-\frac{1}{12}\quad\cdots(5)$ $(5)-(4)$: $2a=\frac{1}{4}\Rightarrow a=\frac{1}{8}$ 代 $a=\frac{1}{8}$ 入 (4): $\frac{3}{8}+b=-\frac{1}{3}\Rightarrow b=-\frac{17}{24}$ 代 $a=\frac{1}{8}$、$b=-\frac{17}{24}$ 入(1): $\frac{1}{8}-\frac{17}{24}+c=\frac{1}{2}\Rightarrow c=\frac{13}{12}$ $f(x)=\frac{1}{8}x^2-\frac{17}{24}x+\frac{13}{12}$ $f(6)=\frac{1}{8}\times 6^2-\frac{17}{24}\times 6+\frac{13}{12}$ $=\frac{9}{2}-\frac{17}{4}+\frac{13}{12}=\frac{54}{12}-\frac{51}{12}+\frac{13}{12}=\frac{4}{3}$	Let $f(x)=ax^2+bx+c$, then $f(1)=a+b+c=\frac{1}{2}\quad\cdots(1)$ $f(2)=4a+2b+c=\frac{1}{6}\quad\cdots(2)$ $f(3)=9a+3b+c=\frac{1}{12}\quad\cdots(3)$ $(2)-(1)$: $3a+b=-\frac{1}{3}\quad\cdots(4)$ $(3)-(2)$: $5a+b=-\frac{1}{12}\quad\cdots(5)$ $(5)-(4)$: $2a=\frac{1}{4}\Rightarrow a=\frac{1}{8}$ Sub. $a=\frac{1}{8}$ into (4): $\frac{3}{8}+b=-\frac{1}{3}\Rightarrow b=-\frac{17}{24}$ Sub. $a=\frac{1}{8}, b=-\frac{17}{24}$ into (1): $\frac{1}{8}-\frac{17}{24}+c=\frac{1}{2}\Rightarrow c=\frac{13}{12}$ $f(x)=\frac{1}{8}x^2-\frac{17}{24}x+\frac{13}{12}$ $f(6)=\frac{1}{8}\times 6^2-\frac{17}{24}\times 6+\frac{13}{12}$ $=\frac{9}{2}-\frac{17}{4}+\frac{13}{12}=\frac{54}{12}-\frac{51}{12}+\frac{13}{12}=\frac{4}{3}$
方法二 設 $F(x)=x(x+1)f(x)-1$， 這是一多項式，其冪次為 4。 $F(1)=2f(1)-1=0\Rightarrow(x-1)$ 為 F(x) 的因式 $F(2)=6f(2)-1=0\Rightarrow(x-2)$ 為 F(x) 的因式 $F(3)=12f(3)-1=0\Rightarrow(x-3)$為 F($x$) 的因式 $F(x)=(x-1)(x-2)(x-3)(ax+b)=x(x+1)f(x)-1$ $F(0)=(-1)(-2)(-3)(0+b)=0(0+1)f(0)-1$ $\Rightarrow b=\frac{1}{6}$ $F(-1)=(-2)(-3)(-4)(-a+b)=-1\times 0\times f(-1)-1$ $-24\left(-a+\frac{1}{6}\right)=-1\Rightarrow a=\frac{1}{8}$ $F(6)=(6-1)(6-2)(6-3)(\frac{1}{8}\cdot 6+\frac{1}{6})=6\times 7f(6)-1$ $5\times 4\times 3\times\left(\frac{3}{4}+\frac{1}{6}\right)=42f(6)-1$ $f(6)=\frac{4}{3}$	**Method 2** Let $F(x)=x(x+1)f(x)-1$ This is a polynomial of degree 4 $F(1)=2f(1)-1=0\Rightarrow(x-1)$ is a factor of F(x) $F(2)=6f(2)-1=0\Rightarrow(x-2)$ is a factor of F(x) $F(3)=12f(3)-1=0\Rightarrow(x-3)$ is a factor of F(x) $F(x)=(x-1)(x-2)(x-3)(ax+b)=x(x+1)f(x)-1$ $F(0)=(-1)(-2)(-3)(0+b)=0(0+1)f(0)-1$ $\Rightarrow b=\frac{1}{6}$ $F(-1)=(-2)(-3)(-4)(-a+b)=-1\times 0\times f(-1)-1$ $-24\left(-a+\frac{1}{6}\right)=-1\Rightarrow a=\frac{1}{8}$ $F(6)=(6-1)(6-2)(6-3)(\frac{1}{8}\cdot 6+\frac{1}{6})=6\times 7f(6)-1$ $5\times 4\times 3\times\left(\frac{3}{4}+\frac{1}{6}\right)=42f(6)-1$ $f(6)=\frac{4}{3}$

G2 求 $\sqrt{2018\times2012\times1988\times1982+8100}$。

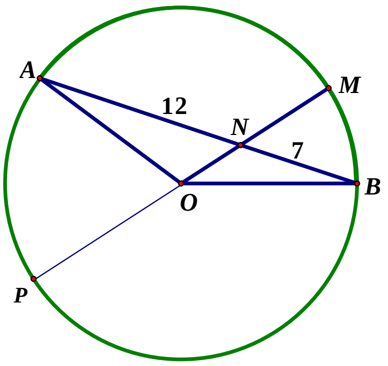
Evaluate $\sqrt{2018\times2012\times1988\times1982+8100}$.

Reference: 2000 FG3.1, 2004 FG3.1, 2012 FI2.3, 2013 HI5

設 $a = 2000$， 則 $\sqrt{2018\times2012\times1988\times1982+8100}$ $=\sqrt{(a+18)\times(a+12)\times(a-12)\times(a-18)+8100}$ $=\sqrt{(a^2-324)\times(a^2-144)+8100}$ $=\sqrt{a^4-468a^2+18^2\times12^2+90^2}$ $=\sqrt{a^4-468a^2+18^2\times(12^2+5^2)}$ $=\sqrt{a^4-468a^2+18^2\times13^2}$ $=\sqrt{a^4-468a^2+234^2}$ $=\sqrt{(a^2-234)^2}$ $= 2000^2 - 234$ $= 4000000 - 234$ $= 3999766$	Let $a = 2000$, then $\sqrt{2018\times2012\times1988\times1982+8100}$ $=\sqrt{(a+18)\times(a+12)\times(a-12)\times(a-18)+8100}$ $=\sqrt{(a^2-324)\times(a^2-144)+8100}$ $=\sqrt{a^4-468a^2+18^2\times12^2+90^2}$ $=\sqrt{a^4-468a^2+18^2\times(12^2+5^2)}$ $=\sqrt{a^4-468a^2+18^2\times13^2}$ $=\sqrt{a^4-468a^2+234^2}$ $=\sqrt{(a^2-234)^2}$ $= 2000^2 - 234$ $= 4000000 - 234$ $= 3999766$

G3 如圖一所示，OAB 是一個以 O 為圓心的扇形。N 為半徑 OM 與 AB 的交點。已知 $AN = 12$，$BN = 7$ 及 $3ON = 2MN$。求 OM 的長度。

As shown in Figure 1, OAB is a sector with centre O. N is the intersecting point of radius OM and AB. Given that $AN = 12$, $BN = 7$ and $3ON = 2MN$. Find the length of OM.



圖一 Figure 1

將扇形畫至圓形，延長 MO 並交圓形於 P。 設 $ON = 2k$、$MN = 3k$。半徑$= 5k$ 利用相交弦定理， $PN\times NM = AN\times NB$ $7k\times3k = 12\times7$ $k = 2$ $OM = 5k = 10$	Complete the circle. Produce MO to meet the circle again at P. Let $ON = 2k$, $MN = 3k$. The radius $= 5k$ $PN = PO + ON = 5k + 2k = 7k$ By intersecting chords theorem, $PN\times NM = AN\times NB$ $7k\times3k = 12\times7$ $k = 2$ $OM = 5k = 10$

G4 對任意非零實數 x，函數 $f(x)$ 有以下特性： $2f(x)+f\left(\frac{1}{x}\right)=11x+4$。設 S 為所有滿足於 $f(x)=2018$ 的根之和。求 S 之值。

For any non-zero real number x, the function $f(x)$ has the following property: $2f(x)+f\left(\frac{1}{x}\right)=11x+4$. Let S be the sum of all roots satisfying the equation $f(x) = 2018$. Find the value of S. **Reference: 2019 HG5**

$2f(x)+f\left(\frac{1}{x}\right)=11x+4 \quad \cdots (1)$ $2f\left(\frac{1}{x}\right)+f(x)=\frac{11}{x}+4 \quad \cdots (2)$ 2(1) – (2): $3f(x)=22x-\frac{11}{x}+4$ $\Rightarrow f(x)=\frac{1}{3}\left(22x-\frac{11}{x}+4\right)$ $f(x) = 2018 \Rightarrow \frac{1}{3}\left(22x-\frac{11}{x}+4\right)=2018$ $22x-\frac{11}{x}+4=6054$ $22x^2 - 6050x - 11 = 0$ $2x^2 - 550x - 1 = 0$ $S=$ 兩根之和 $=-\frac{b}{a}=275$	$2f(x)+f\left(\frac{1}{x}\right)=11x+4 \quad \cdots (1)$ $2f\left(\frac{1}{x}\right)+f(x)=\frac{11}{x}+4 \quad \cdots (2)$ 2(1) – (2): $3f(x)=22x-\frac{11}{x}+4$ $\Rightarrow f(x)=\frac{1}{3}\left(22x-\frac{11}{x}+4\right)$ $f(x) = 2018 \Rightarrow \frac{1}{3}\left(22x-\frac{11}{x}+4\right)=2018$ $22x-\frac{11}{x}+4=6054$ $22x^2 - 6050x - 11 = 0$ $2x^2 - 550x - 1 = 0$ $S=$ sum of roots $=-\frac{b}{a}=275$

G5 求可滿足下列方程組的 x 的值：$\begin{cases} x^2+9x-10y-220=0 & \cdots(1) \\ y^2-5x+6y-166=0 & \cdots(2) \\ xy=195 & \cdots(3) \end{cases}$.

Find the value of x that satisfy the following system of equations: $\begin{cases} x^2+9x-10y-220=0 \\ y^2-5x+6y-166=0 \\ xy=195 \end{cases}$.

(1)+(2)–2(3): $x^2-2xy+y^2+4x-4y-386+390=0$ $(x-y)^2+4(x-y)+4=0 \Rightarrow (x-y+2)^2=0$ $x = y - 2 \cdots\cdots (4)$ 代(4)入(3)：$(y-2)y = 195$ $y^2 - 2y - 195 = 0$ $(y - 15)(y + 13) = 0$ $y = 15$ 或 -13 當 $y = 15$，$x = 13$；當 $y = -13$，$x = -15$ 稽核：代 $x = 13$、$y = 15$ 入(1)： 左式$= 13^2 +9(13) -10(15) -220 = -84$ (捨去) 代 $x = -15$、$y = -13$ 入 (1)： 左式$=(-15)^2+9(-15)-10(-13)-220= 0$ =右式 代 $x = -15$、$y = -13$ 入 (2)： 左式$= (-13)^2 -5(-15)+6(-13)-166 = 0$ =右式 $\therefore x = -15$	(1)+(2)–2(3): $x^2-2xy+y^2+4x-4y-386+390= 0$ $(x-y)^2+4(x-y)+4=0 \Rightarrow (x-y+2)^2=0$ $x = y - 2 \cdots\cdots (4)$ Sub. (4) into (3): $(y-2)y = 195$ $y^2 - 2y - 195 = 0$ $(y - 15)(y + 13) = 0$ $y = 15$ or -13 When $y = 15$, $x = 13$; when $y = -13$, $x = -15$ Check: Sub. $x = 13$, $y = 15$ into (1): L.H.S.$=13^2+9(13)-10(15)-220= -84$ (rejected) Sub. $x = -15$, $y = -13$ into (1): L.H.S.$=(-15)^2+9(-15)-10(-13)-220=0$= R.H.S. Sub. $x = -15$, $y = -13$ into (2): L.H.S.$=(-13)^2-5(-15)+6(-13)-166= 0$ = R.H.S. $\therefore x = -15$

G6 已知 $n^4 + 104 = 3^m$，其中 n、m 為正整數。求 n 的最小值。

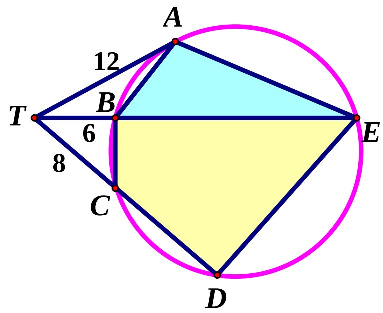
Given that $n^4 + 104 = 3^m$, where n, m are positive integers. Find the least value of n.

Reference: 2013 HI4

$3^3 = 27$，$3^4 = 81$，$3^5 = 243$，$3^6 = 729$ $243 - 104 = 139 \neq n^4$，$729 - 104 = 625 = 5^4$ n 的最小值為 5。	$3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = 729$ $243 - 104 = 139 \neq n^4$, $729 - 104 = 625 = 5^4$ The least value of n is 5 .

G7 如圖二所示，A、B、C、D 及 E 為圓上的點。T 是該圓外的一點。TA 是該圓在點 A 的切綫， TBE 及 TCD 為直綫。已知 TBE 是 $\angle ATD$ 的角平分綫、$TA = 12$、$TB = 6$ 及 $TC = 8$。求 ΔABE 與四邊形 $BCDE$ 的面積比。

As shown in Figure 2, A, B, C, D and E are points on the circle. T is a point outside the circle such that TA is a tangent to the circle at A and TBE and TCD are straight lines. It is given that TBE is the angle bisector of $\angle ATD$, $TA = 12$, $TB = 6$ and $TC = 8$. Find the ratio of the area of ΔABE to the area of quadrilateral $BCDE$.



圖二 Figure 2

Reference 2015 HI10

利用相交弦定理， $TB\times TE = TA^2$ $6\times TE = 12^2$ $TE = 24$ $BE = 24 - 6 = 18$ 設 $\angle ATB = \theta = \angle CTB$ $\dfrac{S_{\Delta ABT}}{S_{\Delta CTB}} = \dfrac{\frac{1}{2}TA\cdot TB\sin\theta}{\frac{1}{2}TC\cdot TB\sin\theta} = \dfrac{12}{8} = \dfrac{3}{2}$ …… (1) 考慮 ΔABT 及 ΔABE 它們同高不同底 $\dfrac{S_{\Delta ABE}}{S_{\Delta ABT}} = \dfrac{BE}{TB} = \dfrac{18}{6} = 3$ …… (2) 考慮 ΔCTB 及 ΔETD $\angle BTC = \angle DTE$ (公共角) $\angle TBC = \angle TDE$ (圓內接四邊形外角) $\angle TCB = \angle TED$ (圓內接四邊形外角) $\therefore\ \Delta CTB \sim \Delta ETD$ (等角) $\dfrac{S_{\Delta ETD}}{S_{\Delta CTB}} = \left(\dfrac{TE}{TC}\right)^2 = \left(\dfrac{24}{8}\right)^2 = 9$ $\Rightarrow \dfrac{S_{BCDE}}{S_{\Delta CTB}} = 9 - 1 = 8$ $\Rightarrow \dfrac{S_{\Delta CTB}}{S_{BCDE}} = \dfrac{1}{8}$ …… (3) (1)×(2)×(3): $\dfrac{\Delta ABE\text{ 的面積}}{BCDE\text{ 的面積}} = \dfrac{S_{\Delta ABT}}{S_{\Delta CTB}}\times\dfrac{S_{\Delta ABE}}{S_{\Delta ABT}}\times\dfrac{S_{\Delta CTB}}{S_{BCDE}}$ $= \dfrac{3}{2}\times 3\times\dfrac{1}{8} = \dfrac{9}{16}$	By intersecting chords theorem, $TB\times TE = TA^2$ $6\times TE = 12^2$ $TE = 24$ $BE = 24 - 6 = 18$ Let $\angle ATB = \theta = \angle CTB$ $\dfrac{S_{\Delta ABT}}{S_{\Delta CTB}} = \dfrac{\frac{1}{2}TA\cdot TB\sin\theta}{\frac{1}{2}TC\cdot TB\sin\theta} = \dfrac{12}{8} = \dfrac{3}{2}$ …… (1) Consider ΔABT and ΔABE They have the same height but different bases. $\dfrac{S_{\Delta ABE}}{S_{\Delta ABT}} = \dfrac{BE}{TB} = \dfrac{18}{6} = 3$ …… (2) Consider ΔCTB and ΔETD $\angle BTC = \angle DTE$ (common ∠s) $\angle TBC = \angle TDE$ (ext. ∠, cyclic quad.) $\angle TCB = \angle TED$ (ext. ∠, cyclic quad.) $\therefore\ \Delta CTB \sim \Delta ETD$ (equiangular) $\dfrac{S_{\Delta ETD}}{S_{\Delta CTB}} = \left(\dfrac{TE}{TC}\right)^2 = \left(\dfrac{24}{8}\right)^2 = 9$ $\Rightarrow \dfrac{S_{BCDE}}{S_{\Delta CTB}} = 9 - 1 = 8$ $\Rightarrow \dfrac{S_{\Delta CTB}}{S_{BCDE}} = \dfrac{1}{8}$ …… (3) (1)×(2)×(3): $\dfrac{\text{area of }\Delta ABE}{\text{area of }BCDE} = \dfrac{S_{\Delta ABT}}{S_{\Delta CTB}}\times\dfrac{S_{\Delta ABE}}{S_{\Delta ABT}}\times\dfrac{S_{\Delta CTB}}{S_{BCDE}}$ $= \dfrac{3}{2}\times 3\times\dfrac{1}{8} = \dfrac{9}{16}$

G8 已知 a、b、c、d、e、f、g 及 h 為正整數，使得 $a>b>c>d>e>f>g>h$ 及 $a+h=b+g=c+f=d+e=35$，問有多少組可行答案 $\{a, b, c, d, e, f, g, h\}$ 存在？

Given that a, b, c, d, e, f, g and h are positive integers such that $a>b>c>d>e>f>g>h$ and $a+h=b+g=c+f=d+e=35$. How many possible solution sets of $\{a, b, c, d, e, f, g, h\}$ exist?

$d+e=35 \Rightarrow d>17.5>e$ $\Rightarrow a>b>c>d>17.5>e>f>g>h>0$ $\Rightarrow 17\geq e>f>g>h\geq 1$ 可行的組合 = 由 1 至 17 之中任意選取 4 個不同數字的方法。 選取方法的數量= $C_4^{17}=\frac{17\times 16\times 15\times 14}{1\times 2\times 3\times 4}=2380$	$d+e=35 \Rightarrow d>17.5>e$ $\Rightarrow a>b>c>d>17.5>e>f>g>h>0$ $\Rightarrow 17\geq e>f>g>h\geq 1$ It is equivalent to choose 4 distinct integers from 1 to 17 . No. of ways = $C_4^{17}=\frac{17\times 16\times 15\times 14}{1\times 2\times 3\times 4}=2380$

G9 求 $\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}\right)+\left(\frac{2}{3}+\frac{2}{4}+\cdots+\frac{2}{100}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{100}\right)+\cdots+\left(\frac{98}{99}+\frac{98}{100}\right)+\frac{99}{100}$ 的值。

Find the value of $\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}\right)+\left(\frac{2}{3}+\frac{2}{4}+\cdots+\frac{2}{100}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{100}\right)+\cdots+\left(\frac{98}{99}+\frac{98}{100}\right)+\frac{99}{100}$.

Reference 1995 HG3, 1996 FG9.4, 2004 HG1

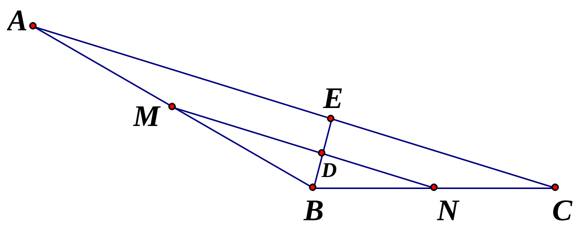
$$\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}\right)+\left(\frac{2}{3}+\frac{2}{4}+\cdots+\frac{2}{100}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{100}\right)+\cdots+\left(\frac{98}{99}+\frac{98}{100}\right)+\frac{99}{100}$$

$$=\frac{1}{2}+\frac{\frac{3\times 2}{2}}{3}+\frac{\frac{4\times 3}{2}}{4}+\frac{\frac{5\times 4}{2}}{5}+\cdots+\frac{\frac{100\times 99}{2}}{100}=\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\frac{4}{2}+\cdots+\frac{99}{2}=\frac{1}{2}(1+2+3+4+\cdots+99)$$

$$=\frac{1}{2}\times\frac{1}{2}\cdot 100\cdot 99=2475$$

G10 如圖三所示，ABC 是一個三角形，其中 $AB=40$、$BC=30$ 及 $\angle ABC=150°$。 M 及 N 分別為 AB 及 BC 的中點。$\angle ABC$ 的角平分綫分別相交 MN 及 AC 於 D 及 E。求 $AMDE$ 的面積。

As shown in Figure 3, ABC is a triangle with $AB=40$, $BC=30$ and $\angle ABC=150°$. M and N are the mid-points of AB and BC respectively. The angle bisector of $\angle ABC$ intersects MN and AC at D and E respectively. Find the area of quadrilateral $AMDE$.

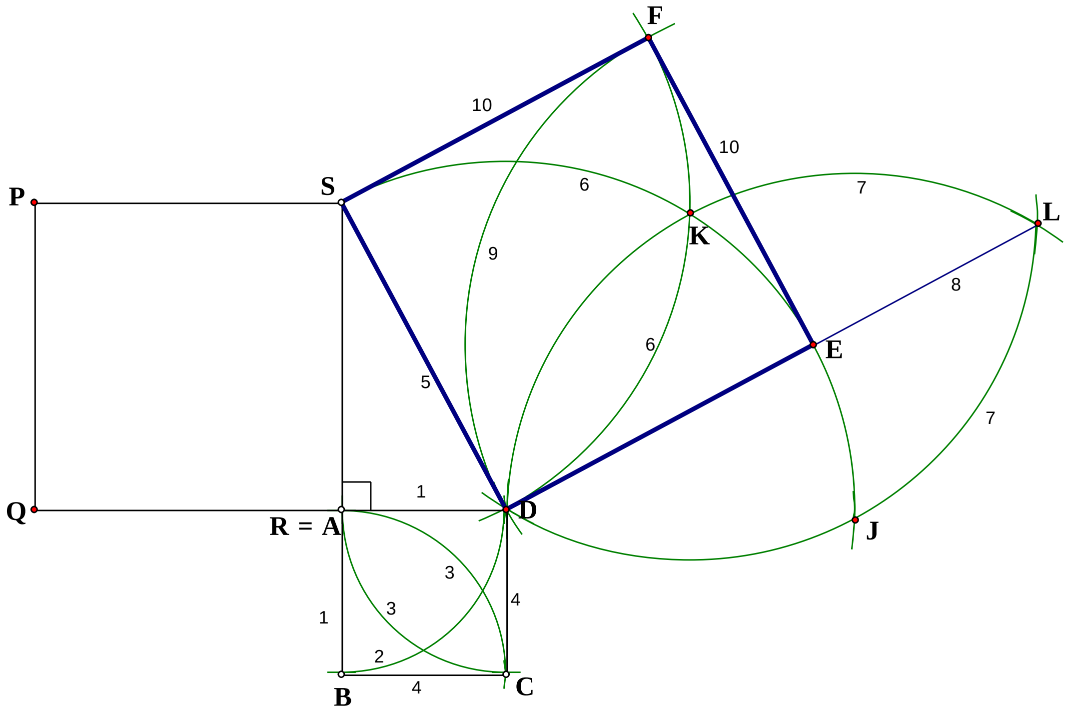


圖三 Figure 3

MN // AC (中點定理) Let $BD=x$, $BE=2x$ (截綫定理) $S_{\Delta BMN}=\frac{1}{2}\cdot 20\times 15\sin 150^\circ=75$ $S_{\Delta BAC}=\frac{1}{2}\cdot 40\times 30\sin 150^\circ=300$ $S_{\Delta BDN} : S_{\Delta BDM}=\frac{1}{2}\cdot 15x\sin 75^\circ : \frac{1}{2}\cdot 20x\sin 75^\circ=3:4$ $S_{\Delta BDM}=75\times\frac{4}{7}=\frac{300}{7}$ 同理，$S_{\Delta BEA}=300\times\frac{4}{7}=\frac{1200}{7}$ $S_{AMDE}=\frac{1200}{7}-\frac{300}{7}=\frac{900}{7}=128\frac{4}{7}$	MN // AC (mid-point theorem) Let $BD=x$, $BE=2x$ (intercept theorem) $S_{\Delta BMN}=\frac{1}{2}\cdot 20\times 15\sin 150^\circ=75$ $S_{\Delta BAC}=\frac{1}{2}\cdot 40\times 30\sin 150^\circ=300$ $S_{\Delta BDN} : S_{\Delta BDM}=\frac{1}{2}\cdot 15x\sin 75^\circ : \frac{1}{2}\cdot 20x\sin 75^\circ=3:4$ $S_{\Delta BDM}=75\times\frac{4}{7}=\frac{300}{7}$ Similarly, $S_{\Delta BEA}=300\times\frac{4}{7}=\frac{1200}{7}$ $S_{AMDE}=\frac{1200}{7}-\frac{300}{7}=\frac{900}{7}=128\frac{4}{7}$

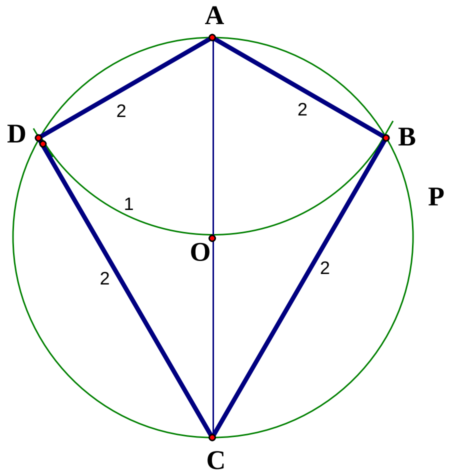
Geometrical Construction

1. 求作一個正方形使得其面積等於下圖的兩個正方形 $ABCD$ 及 $PQRS$ 的面積之和。
Construct a square whose area is equal to the sum of the areas of the squares $ABCD$ and $PQRS$ as shown below.
Reference: 2015 HC3



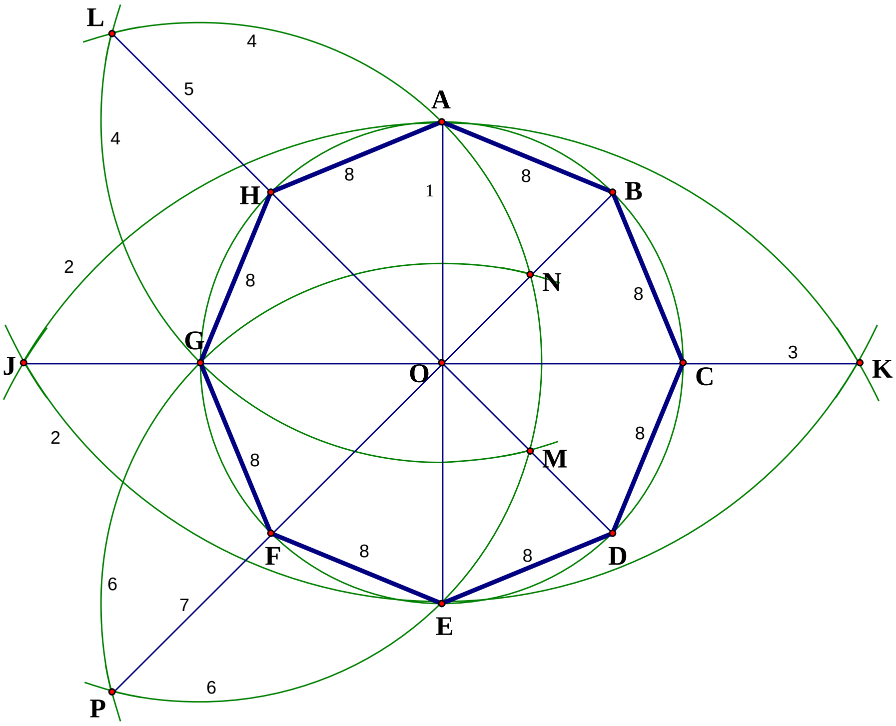
作圖方法如下：	Construction steps:
(1) 將較大的正方形 $PQRS$ 抄至上圖，延長 QR 及 SR。	(1) Copy the larger square $PQRS$ as shown, produce QR and SR.
(2) 以 R 為圓心，較小的正方形 $ABCD$ 的邊長 AD 為半徑作一弧，分別交 QR 及 SR 的延長綫於 D 及 B，重新命名 R 為 A。	(2) Use R as centre, the length of side AD of the smaller square $ABCD$ as radius to draw an arc, intersecting QR and SR produced at D and B respectively, rename R as A.
(3) 以 B 為圓心，BA 為半徑作一弧；以 D 為圓心，DA 為半徑作一弧。兩弧相交於 A 及 C。	(3) Use B as centre, BA as radius to draw an arc; use D as centre, DA as radius to draw an arc. The two arcs intersect at A and C.
(4) 連接 BC 及 DC，$ABCD$ 為較小的正方形，且 $RS \perp AD$。	(4) Join BC and DC, $ABCD$ is the smaller square and $RS \perp AD$.
(5) 連接 SD。	(5) Join SD.
(6) 以 D 為圓心，DS 為半徑作一弧；以 S 為圓心，SD 為半徑作一弧。兩弧相交於 K。	(6) Use D as centre, DS as radius to draw an arc; use S as centre, SD as radius to draw an arc. The two arcs intersect at K.
(7) 以 K 為圓心，KD 為半徑作一弧，交步驟(6)的弧於 D 及 J。以 J 為圓心，JD 為半徑作一弧，交剛才的弧於 D 及 L。	(7) Use K as centre, KD as radius to draw an arc, intersecting the arc in step (6) at D and J. Use J as centre, JD as radius to draw an arc. The two arcs intersect at D and L.
(8) 連接 DL，交步驟(6)的弧於 E。	(8) Join DL, intersecting the arc in step (6) at E.
(9) 以 E 為圓心，ED 為半徑作一弧，交步驟(6)的弧於 D 及 F。	(9) Use E as centre, ED as radius to draw an arc, intersecting the arc in step (6) at D and F.
(10) 連接 EF 及 SF。	(10) Join EF and SF.
$DEFS$ 便是所須的正方形，證明從略。	$DEFS$ is the required square, proof omitted.

2. 已知 AC 是一條通過一個以 O 作圓心的綫段，如下圖所示。求作一個鳶形 $ABCD$ 使得 $\angle BAD = 2\times\angle BCD$ 及 B、D 分別位於圓 APC 上。
Given that AC is a line segment passing through the centre O of a circle, as shown in the figure below. Construct a kite $ABCD$ such that $\angle BAD = 2\times\angle BCD$ and B, D lies on the circle APC.
Remark: There is a typing mistake in the Chinese old version: $\angle BAC = 2\times\angle BDC$.



作圖方法如下： (1) 以 A 為圓心，AO 為半徑作一弧，分別交圓於 B 及 D。 (2) 連接 AB、BC、CD 及 DA。 $ABCD$ 便是所須鳶形，作圖完畢。 證明如下： $AD = AO = OD$ (圓的半徑) ΔAOD 為等邊三角形。 同理，ΔAOB 亦為等邊三角形。 $\angle DAO = \angle BAO = 60°$ (等邊三角形的性質) $\angle BAD = 120°$ $\angle BCD = 60°$ (圓內接四邊形對角) $\therefore \angle BAD = 2\times\angle BCD$ 易證 $\Delta ABC \cong \Delta ADC$ (S.A.S.) $\therefore AB = AD$ 及 $BC = DC$ (全等三角形對應邊) $ABCD$ 是一個鳶形。	Construction steps: (1) Use A as centre, AO as radius to construct an arc, intersecting the circle at B and D respectively. (2) Join AB, BC, CD and DA. $ABCD$ is the required kite, construction complete. Proof: $AD = AO = OD$ (radii) ΔAOD is an equiangular triangle Similarly, ΔAOB is also an equiangular triangle. $\angle DAO = \angle BAO = 60°$ (Property of equilateral Δ) $\angle BAD = 120°$ $\angle BCD = 60°$ (opp. $\angle$s, cyclic quad.) $\therefore \angle BAD = 2\times\angle BCD$ It is easy to show that $\Delta ABC \cong \Delta ADC$ (S.A.S.) $\therefore AB = AD$ 及 $BC = DC$ (corr. sides, $\cong \Delta$'s) $ABCD$ is a kite

3. 作一個以 O 為圓心的圓上的正八邊形 $ABCDEFGH$。
Construct a regular octagon $ABCDEFGH$ on a circle with centre O .



作圖方法如下： (1) 作直徑 AOE。 (2) 以 A 為圓心，AE 為半徑作一弧；以 E 為圓心，EA 為半徑作一弧；兩弧相交於 J 及 K。 (3) 連接 JK，交圓於 G 及 C。 (4) 以 A 為圓心，AG 為半徑作一弧；以 G 為圓心，GA 為半徑作一弧；兩弧相交於 L 及 M。 (5) 連接並延長 LM，交圓於 H 及 D。 (6) 以 G 為圓心，GE 為半徑作一弧；以 E 為圓心，EG 為半徑作一弧；兩弧相交於 P 及 N。 (7) 連接並延長 PN，交圓於 F 及 B。 (8) 連接 AB、BC、CD、DE、EF、FG、GH 及 HA。 $ABCDEFGH$ 便是所須的正八邊形，證明從略。	Construction steps: (1) Construct a diameter AOE. (2) Use A as centre, AE as radius to draw an arc; use E as centre, EA as radius to draw another arc; the two arcs intersect at J and K. (3) Join JK, intersecting the circle at G and C. (4) Use A as centre, AG as radius to draw an arc; use G as centre, GA as radius to draw another arc; the two arcs intersect at L and M. (5) Join and produce LM, intersecting the circle at H and D. (6) Use G as centre, GE as radius to draw an arc; use E as centre, EG as radius to draw another arc; the two arcs intersect at P and N. (7) Join and produce PN, intersecting the circle at F and B. (8) Join AB, BC, CD, DE, EF, FG, GH and HA. $ABCDEFGH$ is the required regular octagon, proof omitted.