2016 Heat Individual (Sample Paper) answer

I1	237	**I2**	201499	**I3**	90	**I4**	30°	**I5**	7
I6	11	**I7**	23	**I8**	10	**I9**	$\frac{293}{34}$ $(=8\frac{21}{34})$	**I10**	1016064
I11	4	**I12**	2012	**I13**	730639	**I14**	13	**I15**	81.64

I1 An integer x minus 12 is the square of an integer. x plus 19 is the square of another integer. Find the value of x.

$x - 12 = n^2$ ········ (1); $x + 19 = m^2$ ········(2), where m, n are integers.

(2) – (1): $(m + n)(m - n) = 31$

$\because$ 31 is a prime number

$\therefore$ $m + n = 31$ and $m - n = 1$

$m = 16$, $n = 15$

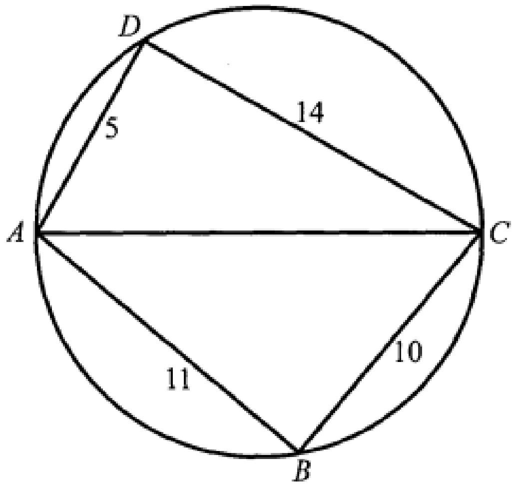
$x = 15^2 + 12 = 237$

I2 Given that $\left(10^{2015}\right)^{-10^2} = 0.\underbrace{000\cdots0}_{n\text{ times}}1$. Find the value of n.

$10^{-201500} = 0.\underbrace{000\cdots0}_{n\text{ times}}1$

$n = 201500 - 1 = 201499$

I3 As shown in Figure 2, $ABCD$ is a cyclic quadrilateral, where $AD = 5$, $DC = 14$, $BC = 10$ and $AB = 11$. Find the area of quadrilateral $ABCD$.



Reference: 2002 HI6

$AC^2 = 10^2 + 11^2 - 2\times11\times10 \cos \angle B$ ········(1)

$AC^2 = 5^2 + 14^2 - 2\times5\times14 \cos \angle D$ ········ (2)

(1) = (2): $221 - 220 \cos\angle B = 221 - 140 \cos\angle D$ …(3)

$\angle B + \angle D = 180°$ (opp. $\angle$s, cyclic quad.)

$\therefore \cos \angle D = -\cos \angle B$

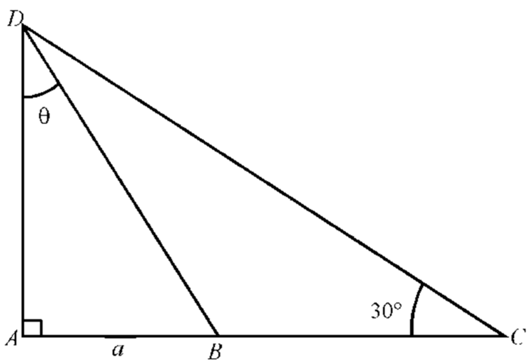
(3): $(220 + 140) \cos \angle B = 0 \Rightarrow \angle B = 90° = \angle D$

Area of the cyclic quadrilateral

= area of ΔABC + area of ΔACD

$= \frac{1}{2}\cdot11\cdot10 + \frac{1}{2}\cdot5\cdot14 = 90$

I4 Figure 1 shows a right-angled triangle ACD where B is a point on AC and $BC = 2AB$. Given that $AB = a$ and $\angle ACD = 30°$, find the value of θ.



In ΔABD, $AD = \frac{a}{\tan\theta}$

In ΔACD, $AC = \frac{AD}{\tan 30^\circ} = \frac{\sqrt{3}a}{\tan\theta}$

However, $AC = AB + BC = a + 2a = 3a$

$\therefore\ \frac{\sqrt{3}a}{\tan\theta} = 3a$

$\tan\theta = \frac{\sqrt{3}}{3}$

$\Rightarrow \theta = 30°$

I5 A school issues 4 types of raffle tickets with face values \$10, \$15, \$25 and \$40. Class A uses several one-hundred dollar notes to buy 30 raffle tickets, including 5 tickets each for two of the types and 10 tickets each for the other two types. How many one-hundred dollars notes Class A use to buy the raffle tickets?

100 is an even number, the face values \$15 and \$25 are odd numbers. Only 5 tickets of \$15 and 5 tickets of \$25 can make a sum of even numbers.

$10(10) + 15(5) + 25(5) + 40(10) = 700 \Rightarrow$ Class A uses 7 \$100 notes.

I6 Find the remainder when 2^{2011} is divided by 13.

$2^6 = 64 = 13\times5 - 1 \equiv -1 \bmod 13$; $2^{12} \equiv 1 \bmod 13$

$2011 = 12\times167 + 7$

$2^{2011} = 2^{12\times167+7} = (2^{12})^{167}\times2^7 \equiv 2^7 \equiv 2^6\times2 \equiv -1\times2 \equiv -2 \equiv 11 \bmod 13$

I7 Find the number of places of the number $2^{20}\times25^{12}$. (**Reference: 1982 FG10.1, 1992 HI17**)

$2^{20}\times25^{12} = 2^{20}\times5^{24} = 10^{20}\times5^4 = 625\times10^{20}$

The number of places = 23

I8 *A*, *B* and *C* pass a ball among themselves. *A* is the first one to pass the ball to other one. In how many ways will the ball be passed back to *A* after 5 passes?

Construct the following table:

Number of passes	1	2	3	4	5
A	0	1+1=2	1+1=2	3+3=6	5+5=10
B	1	0+1=1	1+2=3	3+2=5	5+6=11
C	1	0+1=1	1+2=3	3+2=5	5+6=11

There will be 10 ways for the ball to pass back to *A*.

I9 Given that a and b are distinct prime numbers, $a^2 - 19a + m = 0$ and $b^2 - 19b + m = 0$. Find the value of $\frac{a}{b}+\frac{b}{a}$. **Reference: 1996 HG8, 1996FG7.1, 2001 FG4.4, 2005 FG1.2, 2012 HI6**

a and b are prime distinct roots of $x^2 - 19x + m = 0$

$a + b$ = sum of roots = 19 (odd)

$\because$ a and b are prime number and all prime number except 2, are odd.

$\therefore$ $a = 2$, $b = 17$ (or $a = 17$, $b = 2$)

$\frac{a}{b}+\frac{b}{a}=\frac{17}{2}+\frac{2}{17}=\frac{293}{34}\left(=8\frac{21}{34}\right)$

I10 It is given that $a_1, a_2, \dots, a_n, \dots$ is a sequence of positive real numbers such that $a_1 = 1$ and $a_{n+1} = a_n + \sqrt{a_n} + \frac{1}{4}$. Find the value of a_{2015}.

$a_2 = 2 + \frac{1}{4} = \frac{9}{4}$

$a_3 = \frac{9}{4} + \frac{3}{2} + \frac{1}{4} = \frac{16}{4}$

Claim: $a_n = \frac{(n+1)^2}{4}$ for $n \ge 1$

Pf: By M.I. $n = 1, 2, 3$, proved already.

Suppose $a_k = \frac{(k+1)^2}{4}$ for some positive integer k.

$a_{k+1} = a_k + \sqrt{a_k} + \frac{1}{4} = \frac{(k+1)^2}{4} + \frac{k+1}{2} + \frac{1}{4} = \frac{(k+1)^2 + 2(k+1) + 1}{4} = \frac{(k+1+1)^2}{4}$

By M.I., the statement is true for $n \ge 1$

$a_{2015} = \frac{2016^2}{4} = 1008^2 = 1016064$

I11 If the quadratic equation $(k^2 - 4)x^2 - (14k + 4)x + 48 = 0$ has two distinct positive integral roots, find the value(s) of k.

Clearly $k^2 - 4 \neq 0$; otherwise, the equation cannot have two real roots.

Let the roots be α, β.

$\Delta = (14k + 4)^2 - 4(48)(k^2 - 4) = 2^2[(7k + 2)^2 - 48k^2 + 192] = 2^2(k^2 + 28k + 196) = [2(k + 14)]^2$

$$\alpha = \frac{14k+4+\sqrt{[2(k+14)]^2}}{2(k^2-4)} = \frac{7k+2+k+14}{k^2-4} = \frac{8k+16}{k^2-4} = \frac{8}{k-2},\ \beta = \frac{6k-12}{k^2-4} = \frac{6}{k+2}.$$

For positive integral roots, $k - 2$ is a positive factor of 8 and $k + 2$ is a positive factor of 6.

$k - 2 = 1, 2, 4, 8$ and $k + 2 = 1, 2, 3, 6$

$k = 3, 4, 6, 10$ and $k = -1, 0, 1, 4$

$\therefore k = 4$ only

Method 2 provided by Mr. Jimmy Pang from Po Leung Kuk Lee Shing Pik College

The quadratic equation can be factorised as: $[(k - 2)x - 8][(k + 2)x - 6] = 0$

$\because k \neq 2$ and $k \neq -2$ $\therefore x = \dfrac{8}{k-2}$ or $\dfrac{6}{k+2}$

By similar argument as before, for positive integral root, $k = 4$ only.

I12 Given that $y = (x + 1)(x + 2)(x + 3)(x + 4) + 2013$, find the minimum value of y.

Reference 1993HG5, 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3

$y = (x + 1)(x + 4)(x + 2)(x + 3) + 2013 = (x^2 + 5x + 4)(x^2 + 5x + 6) + 2013$

$= (x^2 + 5x)^2 + 10(x^2 + 5x) + 24 + 2013 = (x^2 + 5x)^2 + 10(x^2 + 5x) + 25 + 2012$

$= (x^2 + 5x + 5)^2 + 2012 \geq 2012$

The minimum value of y is 2012.

I13 How many pairs of distinct integers between 1 and 2015 inclusively have their products as multiple of 5?

Multiples of 5 are 5, 10, 15, 20, 25, 30, … , 2015. Number = 403

Numbers which are not multiples of 5 = 2015 – 403 = 1612

Let the first number be x, the second number be y.

Number of pairs = No. of ways of choosing any two numbers from 1 to 2015 – no. of ways of choosing such that both x, y are not multiples of 5.

$$= C_2^{2015} - C_2^{1612} = \frac{2015\times2014}{2} - \frac{1612\times1611}{2} = 403\times\left(\frac{5\times2014}{2} - \frac{4\times1611}{2}\right)$$

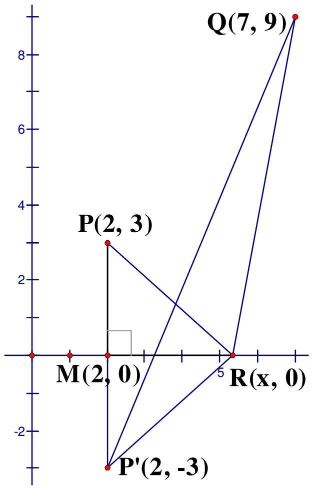
$$= 403\times(5\times1007 - 2\times1611) = 403\times(5035 - 3222) = 403\times1813 = 730639$$

I14 Let x be a real number. Find the minimum value of $\sqrt{x^2-4x+13}+\sqrt{x^2-14x+130}$.

Reference 2010 FG4.2

Consider the following problem:

Let $P(2, 3)$ and $Q(7, 9)$ be two points. $R(x, 0)$ is a variable point on x-axis. To find the minimum sum of distances $PR + RQ$.



Let y = sum of distances $= \sqrt{(x-2)^2+9} + \sqrt{(x-7)^2+81}$

If we reflect $P(2, 3)$ along x-axis to $P'(2, -3)$, $M(2, 0)$ is the foot of perpendicular,

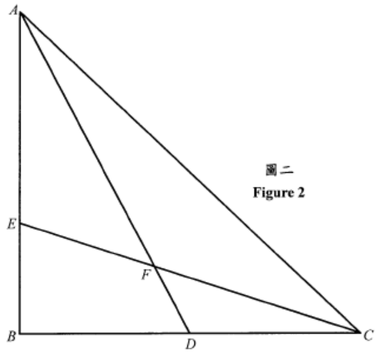
then $\Delta PMR \cong \Delta P'MR$ (S.A.S.)

$y = PR + RQ = P'R + RQ \geq P'Q$ (triangle inequality)

$y \geq \sqrt{(7-2)^2+(9+3)^2} = 13$

The minimum value of $\sqrt{x^2-4x+13}+\sqrt{x^2-14x+130}$ is 13.

I15 In figure 2, $AE = 14$, $EB = 7$, $AC = 29$ and $BD = DC = 10$. Find the value of BF^2.



Reference: 2005 HI5

$AB = 14 + 7 = 21$, $BC = 10 + 10 = 20$

$AB^2 + BC^2 = 21^2 + 20^2 = 841 = 29^2 = AC^2$

$\therefore \angle ABC = 90°$ (converse, Pythagoras' theorem)

Let $BF = a$, $\angle CBF = \theta$, $\angle ABF = 90° - \theta$

Area of ΔBEF + area of ΔBCF = area of ΔBCE

$$\frac{1}{2}\cdot 20\times a\sin\theta+\frac{1}{2}\cdot a\times 7\cos\theta=\frac{20\times 7}{2}$$

$20a \sin \theta + 7a \cos \theta = 140$ …… (1)

Area of ΔBDF + area of ΔABF = area of ΔABD

$$\frac{1}{2}\cdot 21\times a\cos\theta+\frac{1}{2}\cdot a\times 10\sin\theta=\frac{10\times 21}{2}$$

$21a \cos \theta + 10a \sin \theta = 210$ …… (2)

$2(2) - (1)$: $35\, a \cos \theta = 280$

$a \cos \theta = 8$ …… (3)

$3(1) - (2)$: $50\, a \sin \theta = 210$

$a \sin \theta = \frac{21}{5}$ …… (4)

$(3)^2 + (4)^2$: $BF^2 = a^2 = 8^2 + \left(\frac{21}{5}\right)^2 = \frac{2041}{25}$ $(= 81\frac{16}{25} = 81.64)$

15-16 Individual	**1**	8	**2**	–1007	**3**	45	**4**	12	**5**	5985
	6	14	**7**	200	**8**	46	**9**	14	**10**	32
	11	$\frac{1}{2016}$	**12**	$\frac{17}{3}$	**13**	522	**14**	20	**15**	–1
15-16 Group	**1**	$\frac{1}{2}$	**2**	75 cm^2	**3**	$*\frac{63}{2^{2011}}$ see the remark	**4**	6	**5**	$\sqrt{7}$
	6	672	**7**	72	**8**	386946	**9**	$\frac{281}{13}=21\frac{8}{13}$	**10**	4062241

Individual Events

I1 計算 $0.125^{2016}\times(2^{2017})^3$ 的值。

Find the value of $0.125^{2016}\times(2^{2017})^3$.

$0.125^{2016}\times(2^{2017})^3=\left(\frac{1}{8}\right)^{2016}\times\left(2^3\right)^{2017}\quad=\left(\frac{1}{8}\times 8\right)^{2016}\times 8=8$

I2 已知方程 $\begin{cases}x_1+x_2=x_2+x_3=x_3+x_4=\cdots=x_{2014}+x_{2015}=x_{2015}+x_{2016}=1\\x_1+x_2+x_3+\cdots+x_{2015}+x_{2016}=x_{2016}\end{cases}$，求 x_1 的值。

Given the equations $\begin{cases}x_1+x_2=x_2+x_3=x_3+x_4=\cdots=x_{2014}+x_{2015}=x_{2015}+x_{2016}=1\\x_1+x_2+x_3+\cdots+x_{2015}+x_{2016}=x_{2016}\end{cases}$,

find the value of x_1.

$x_1+x_2=x_2+x_3\Rightarrow x_1=x_3$

$x_2+x_3=x_3+x_4\Rightarrow x_2=x_4$

$x_3+x_4=x_4+x_5\Rightarrow x_3=x_5$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

Inductively, we can prove that $x_1=x_3=\cdots\cdots\cdots=x_{2015}$; $x_2=x_4=\cdots\cdots\cdots=x_{2016}$

Let $a=x_1+x_3+\cdots\cdots\cdots+x_{2015}=1008x_1$; $b=x_2+x_4+\cdots\cdots\cdots+x_{2016}=1008x_2$.

Sub. the above results into equation (2): $1008(x_1+x_2)=x_{2016}=x_2$

$1008=x_2$

$x_1+x_2=1\Rightarrow x_1=1-1008=-1007$

I3 有多少個 x 使得$\sqrt{2016-\sqrt{x}}$ 為整數？

How many x are there so that $\sqrt{2016-\sqrt{x}}$ is an integer?

Reference: 2018 FG4.1, 2019 FG2.1

$45=\sqrt{2025}>\sqrt{2016-\sqrt{x}}$

$\sqrt{2016-\sqrt{x}}=0, 1, 2, \cdots$ or 44.

There are 45 different x to make $\sqrt{2016-\sqrt{x}}$ an integer.

I4 若 x、y 為整數，有多少對 x、y 且滿足 $(x+1)^2+(y-2)^2=50$？

If x, y are integers, how many pairs of x, y are there which satisfy the equation $(x+1)^2+(y-2)^2=50$?

The integral solutions to $a^2+b^2=50$ are $(a, b)=(\pm5, \pm5), (\pm7, \pm1)$ or $(\pm1, \pm7)$.

$\therefore$ The number of pairs of integral solutions are $2\times2\times3=12$.

I5 63 個連續整數的和是 2016，求緊接該 63 個連續整數後的 63 個連續整數的和。
The sum of 63 consecutive integers is 2016, find the sum of the next 63 consecutive integers.
The sum of next 63 consecutive integers = 2016 + 63×63 = 5985

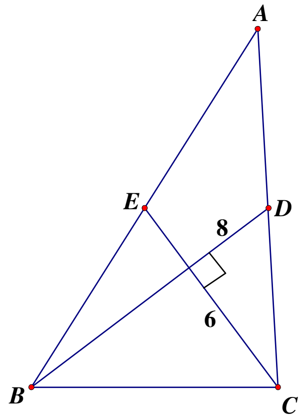
I6 已知 8 個整數的平均數、中位數、分佈域及唯一眾數均為 8。若 A 為該 8 個整數中的最大數，求 A 的最大值。
Given that the mean, median, range and the only mode of 8 integers are also 8. If A is the largest integer among those 8 integers, find the maximum value of A.
Suppose the 8 integers, arranged in ascending order, are $a \le b \le c \le d \le e \le f \le g \le A$.
$\frac{a+b+c+d+e+f+g+A}{8} = 8 \Rightarrow a+b+c+d+e+f+g+A = 64$ …… (1)
$\frac{d+e}{2} = 8 \Rightarrow d+e = 16$ …… (2)
$a = A - 8$ …… (3)
Sub. (2) and (3) into (1): $A - 8 + b + c + 16 + f + g + A = 64 \Rightarrow 2A + b + c + f + g = 56$ ⋯ (4)
$\because$ Median = 8 $\Rightarrow d \le 8 \le e$
$\because$ Mode = 8 $\Rightarrow d = e = 8$
In order to maximize A and satisfy equation (4), b, c, f, g must be as small as possible.
$f = g = 8$, $b = A - 8$, $c = A - 8$; sub. these assumptions into (4):
$2A + 2A - 16 + 8 + 8 = 56$
$\Rightarrow$ The maximum value of A = 14.

I7 在整數 1 至 500 之間出現了多少個數字「2」？
How many ‘2’s are there in the numbers between 1 to 500?
1 to 9, ‘2’ appears once. 10 to 99, ‘2’ appears in 12, 20, 21, 22, ⋯ , 29, 32, ⋯ , 92: 19 times.
100 to 199, ‘2’ appears 20 times, 200 to 299, ‘2’ appears 120 times,
300 to 399, ‘2’ appears 20 times, 400 to 499, ‘2’ appears 20 times.
‘2’ appears 200 times.

I8 某數的 16 進制位是 1140。而同一數字的 a 進制位是 240，求 a。
A number in base 16 is 1140. The same number in base a is 240, what is a?
$1140_{16} = 16^3 + 16^2 + 4\times16 = 4416_{10} = 240_a = 2a^2 + 4a$
$a^2 + 2a - 2208 = 0$
$(a - 46)(a + 48) = 0$
$a = 46$ or -48 (rejected)

I9 P 點的極座標為 $(6, 240°)$。若 P 向右平移 16 單位，求 P 的像與極點之間的距離。
The polar coordinates of P are $(6, 240°)$. If P is translated to the right by 16 units, find the distance between its image and the pole. **Reference: 2019 HI2**
Before translation, the rectangular coordinates of P is $(6 \cos 240°, 6 \sin 240°) = (-3, -3\sqrt{3})$.
After translation, the rectangular coordinates of P is $(13, -3\sqrt{3})$.
The distance from the pole is $\sqrt{13^2 + \left(-3\sqrt{3}\right)^2}$ = 14 units

I10 如圖一，在 ΔABC 中，BD 和 CE 分別是 AC 和 AB 兩邊上的中綫，且 $BD \perp CE$。已知 $BD = 8$，$CE = 6$，求ΔABC 的面積。
As shown in Figure 1, BD and CE are the medians of the sides AC and AB of ΔABC respectively, and $BD \perp CE$. Given that $BD = 8$, $CE = 6$, find the area of ΔABC.



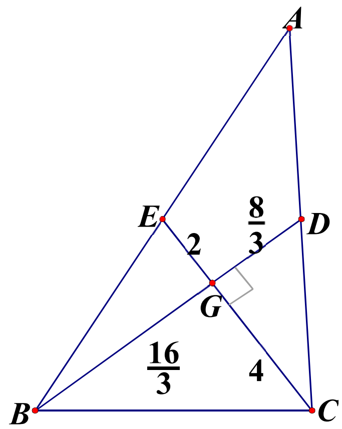
圖一
Figure 1

Suppose BD and CE intersect at the centriod G.
Then G divides each median in the ratio 1 : 2.

$CG = 4$, $GE = 2$; $BG = \frac{16}{3}$, $GD = \frac{8}{3}$.

$S_{\Delta BCE} = \frac{1}{2} \cdot 6 \cdot \frac{16}{3} = 16$ sq. units

$S_{\Delta ABC} = 2\ S_{\Delta BCE} = 32$ sq. units



I11 已知方程 $100[\log(63x)][\log(32x)] + 1 = 0$ 有兩個相異的實數根 α 及 β，求 $\alpha\beta$ 的值。
It is known that the equation $100[\log(63x)][\log(32x)] + 1 = 0$ has two distinct real roots α and β. Find the value of $\alpha\beta$.

$100[\log(63x)][\log(32x)] + 1 = 0 \Rightarrow 100(\log 63 + \log x)(\log 32 + \log x) + 1 = 0$

$100 (\log x)^2 + 100(\log 32 + \log 63) \log x + (100 \log 32 \log 63 + 1) = 0$

This is a quadratic equation in $\log x$. The two distinct real roots are $\log \alpha$ and $\log \beta$.

$\log \alpha\beta = \log \alpha + \log \beta =$ sum of roots

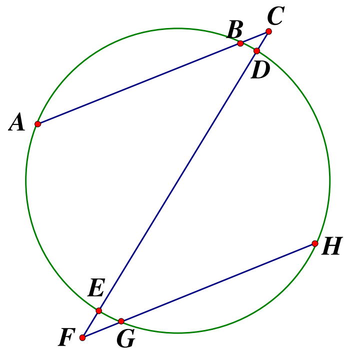
$$= -\frac{100(\log 32 + \log 63)}{100}$$

$$= \log \frac{1}{32 \times 63}$$

$$\Rightarrow \alpha\beta = \frac{1}{2016}$$

I12 如圖二所示，ABC，$CDEF$ 及 FGH 皆為直綫，且 ABC // FGH。$AB = 42$，$GH = 40$，$EF = 6$ 及 $FG = 8$。已知 ABC 與 FGH 之間的距離為 41，求 BC。

As shown in Figure 2, ABC, $CDEF$ and FGH are straight lines, ABC // FGH, $AB = 42$, $GH = 40$, $EF = 6$ and $FG = 8$. Given that the distance between ABC and FGH is 41, find BC.



圖二 Figure 2

Let the mid-point of AB be M.

Draw the perpendicular bisector MN of AB cutting GH at N.

$AM = MB = 21$ and $AB \perp MN$.

$\angle HNM = \angle AMN = 90°$ (alt. $\angle$s, AB // GH)

MN must pass through the centre O of the circle.

$GN = NH = 20$ ($\perp$ from centre bisect chord)

Let $ON = x$, then $OM = 41 - x$. Join OA, OH. Let the radius be r.

$21^2 + (41 - x)^2 = r^2 \cdots\cdots (1)$ (Pythagoras' theorem on ΔAMO)

$20^2 + x^2 = r^2 \cdots\cdots (2)$ (Pythagoras' theorem on ΔHNO)

(1) = (2): $441 + 1681 - 82x + x^2 = 400 + x^2$

$x = 21$

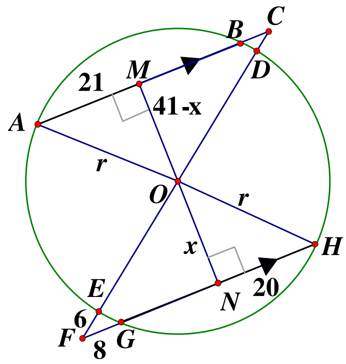
Sub. $x = 21$ into (2): $r^2 = 20^2 + 21^2 \Rightarrow r = 29$

$FG \times FH = FE \times FD$ (intersecting chords theorem)

$8 \times 48 = 6 \times (6 + ED)$

$ED = 58 = 2r$ = diameter of the circle

$\therefore$ O is the mid-point of ED.



It is easy to show that $\Delta OMC \sim \Delta ONF$ (equiangular)

$$\frac{MC}{OM} = \frac{NF}{ON}$$ (corr. sides, ~Δs)

$$\frac{21 + BC}{41 - 21} = \frac{8 + 20}{21}$$

$$BC = \frac{17}{3}$$

I13 設 A、B 和 C 為三個數字。利用這三個數字組成的三位數有以下性質：

(a) ACB 可以被 3 整除；

(b) BAC 可以被 4 整除；

(c) BCA 可以被 5 整除；及

(d) CBA 的因數數目為單數。

求三位數 ABC。

Let A, B and C be three digits. The number formed by these three digits has the following properties:

(a) ACB is divisible by 3;

(b) BAC is divisible by 4;

(c) BCA is divisible by 5;

(d) CBA has an odd number of factors.

Find the 3-digit number ABC.

From (a), $A + B + C = 3m$ $\cdots\cdots$ (1), where m is a positive integer.

From (b), $10A + C = 4n$ $\cdots\cdots$ (2), where n is a positive integer.

From (c), $A = 0$ or 5 $\cdots\cdots$ (3)

If $A = 0$, then ACB is not a three digit number. $\therefore$ rejected

Sub. $A = 5$ into (2), $C = 2$ or 6

From (d), CBA has an odd number of factors $\Rightarrow$ CBA is a perfect square $\cdots\cdots$ (4)

Sub. $A = 5$, $C = 6$ into (1): $B = 1, 4$ or 7

$CBA = 615, 645$ or 675, all these numbers are not perfect square, rejected.

Sub. $A = 5$, $C = 2$ into (1): $B = 2, 5$ or 8

$CBA = 225, 255$ or 285

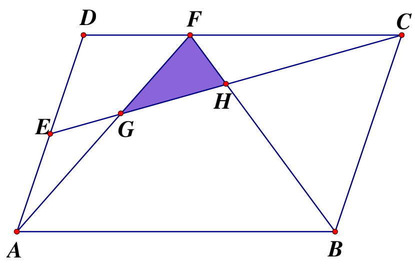
Of these numbers, only 225 is a perfect square

$\therefore A = 5, B = 2, C = 2$

$ABC = 522$

I14 在圖三中，$ABCD$ 為一平行四邊形，E 為 AD 上的中點及 F 為 DC 上的點且滿足 $DF : FC = 1 : 2$。FA 及 FB 分別相交 EC 於 G 及 H，求 $\dfrac{ABCD\text{的面積}}{\Delta FGH\text{的面積}}$ 的值。

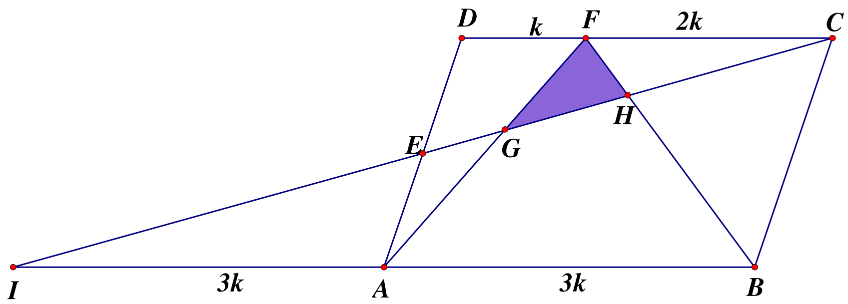
As shown in Figure 3, $ABCD$ is a parallelogram. E is the mid-point of AD and F is a point on DC such that $DF : FC = 1 : 2$. FA and FB intersect EC at G and H respectively. Find the value of $\dfrac{\text{Area of } ABCD}{\text{Area of } \Delta FGH}$.



圖三

Figure 3

Reference: 1998 HG5, 2019 HI11



Produce CE to meet BA produced at I. Let $DF = k$, $CF = 2k$.

$AB = 3k$ (opp. sides //-gram)

$\Delta CDE \cong \Delta IAE$ ($DE = EA$, given, A.A.S.)

$IA = DC = 3k$ (corr. sides, $\cong \Delta$s)

$\Delta CFG \sim \Delta IAE$ (equiangular)

$CG : GI = CF : IA = FG : GA = 2 : 3 \cdots\cdots (1)$ (corr. sides, ~Δs)

$\Delta CFH \sim \Delta IBH$ (equiangular)

$CH : HI = CF : IB = FH : HB = 2k : 6k = 1 : 3 \cdots\cdots (2)$ (corr. sides, ~Δs)

Let $IC = 20m$. By (1), $CG = 8m$, $GI = 12m$.

By (2), $CH = 5m$, $HI = 15k$

$\therefore\ GH = CG - CH = 8m - 5m = 3m$

$CH : HG = 5m : 3m = 5 : 3$

Let $S_{\Delta FGH} = 3p$, then $S_{\Delta CFH} = 5p$ (ΔFGH and ΔCFH have the same height)

$\Rightarrow S_{\Delta CFG} = 3p + 5p = 8p$

$S_{\Delta CAG} = \dfrac{3}{2} \times 8p = 12p$ (ΔCFG and ΔCAG have the same height & by (1))

$\Rightarrow S_{\Delta CAF} = 8p + 12p = 20p$

$S_{\Delta DAF} = \dfrac{1}{2} \times 20p = 10p$ (ΔDAF and ΔCAF have the same height)

$\Rightarrow S_{\Delta CAD} = 10p + 20p = 30p$

$\Delta ACD \cong \Delta CAB$ (A.S.A.)

$\Rightarrow S_{\Delta CAB} = 30p$

$\Rightarrow S_{ABCD} = 30p + 30p = 60p$

$$\frac{\text{Area of } ABCD}{\text{Area of } \Delta FGH} = \frac{60p}{3p} = 20$$

I15 已知數列$\{a_n\}$，其中 $a_{n+2} = a_{n+1} - a_n$。若 $a_2 = -1$ 及 $a_3 = 1$，求 a_{2016} 的值。

Given a sequence $\{a_n\}$, where $a_{n+2} = a_{n+1} - a_n$. If $a_2 = -1$ and $a_3 = 1$, find the value of a_{2016}.

$a_4 = a_3 - a_2 = 1 - (-1) = 2$

$a_5 = a_4 - a_3 = 2 - 1 = 1$

$a_6 = a_5 - a_4 = 1 - 2 = -1$

$a_7 = a_6 - a_5 = -1 - 1 = -2$

$a_8 = a_7 - a_6 = -2 - (-1) = -1 = a_2$

$a_9 = a_8 - a_7 = -1 - (-2) = 1 = a_3$

$a_{10} = a_9 - a_8 = 1 - (-1) = 2 = a_4$

$\therefore$ The sequence repeats the cycle $(-1, 1, 2, 1, -1, -2)$ for every 6 terms.

$2016 = 6\times336$

$a_{2016} = a_{2010} = \cdots\cdots = a_6 = -1$

Group Events

G1 最初甲瓶裝有 1 公升酒精，乙瓶是空的。

第 1 次將甲瓶全部的酒精倒入乙瓶中，第 2 次將乙瓶酒精的 $\frac{1}{2}$ 倒回甲瓶，

第 3 次將甲瓶酒精的 $\frac{1}{3}$ 倒回乙瓶，第 4 次將乙瓶酒精的 $\frac{1}{4}$ 倒回甲瓶，……。

第 2016 次後，甲瓶還有多少公升酒精？

At the beginning, there was 1 litre of alcohol in bottle *A* and bottle *B* is an empty bottle. First, pour all alcohol from bottle *A* to bottle *B*; second, pour $\frac{1}{2}$ of the alcohol from bottle *B* back to bottle *A*; third, pour $\frac{1}{3}$ of the alcohol from bottle *A* to bottle *B*; fourth, pour $\frac{1}{4}$ of the alcohol from bottle *B* back to bottle *A*, $\cdots$. After the 2016th pouring, how much alcohol was left in bottle *A*?

No. of times	Amount of alcohol in A	Amount of alcohol in B
1	0	1
2	$1-\frac{1}{2}=\frac{1}{2}$	$\frac{1}{2}$
3	$\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$	$1-\frac{1}{3}=\frac{2}{3}$
4	$1-\frac{1}{2}=\frac{1}{2}$	$\frac{2}{3}\times\frac{3}{4}=\frac{1}{2}$
5	$\frac{1}{2}\times\frac{4}{5}=\frac{2}{5}$	$1-\frac{2}{5}=\frac{3}{5}$
6	$1-\frac{1}{2}=\frac{1}{2}$	$\frac{3}{5}\times\frac{5}{6}=\frac{1}{2}$

Let the amount of alcohol in *A* and *B* be a_n and b_n after *n* trails.

Claim: For $n > 1$, $a_{2n} = b_{2n} = \frac{1}{2}$, $a_{2n-1} = \frac{n-1}{2n-1}$, $b_{2n-1} = \frac{n}{2n-1}$.

Proof: Mathematical induction on *n*. $n = 2, 3$, proved by the above table.

Suppose $a_{2k} = b_{2k} = \frac{1}{2}$ for some positive integer $k > 1$.

$a_{2k+1} = \frac{1}{2}\times\frac{2k}{2k+1} = \frac{k}{2k+1} = \frac{(k+1)-1}{2(k+1)-1}$; $b_{2k+1} = 1 - \frac{k}{2k+1} = \frac{k+1}{2k+1} = \frac{(k+1)}{2(k+1)-1}$

Suppose $a_{2k-1} = \frac{k-1}{2k-1}$, $b_{2k-1} = \frac{k}{2k-1}$ for some positive integer $k > 1$.

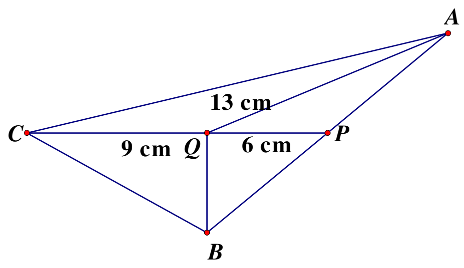
$b_{2k} = \frac{k}{2k-1}\times\frac{2k-1}{2k} = \frac{1}{2}$; $b_{2k} = 1 - \frac{1}{2} = \frac{1}{2}$

By the principal of mathematical induction, the claim is true for all positive integer $n>1$.

$a_{2016} = a_{2(1008)} = \frac{1}{2}$

G2 圖一顯示ΔABC，P 為 AB 的中點及 Q 是 CP 上的一點。已知 $BQ \perp CP$，$PQ = 6$ cm、$CQ = 9$ cm 及 $AQ = 13$ cm。求ΔABC 的面積。

Figure 1 shows ΔABC, P is the mid-point of AB and Q is a point on CP. It is known that $BQ \perp CP$, $PQ = 6$ cm, $CQ = 9$ cm and $AQ = 13$ cm. Find the area of ΔABC.



Produce QP to D so that $PD = QP = 6$ cm

$AP = PB$ (given that P is the mid-point of AB)

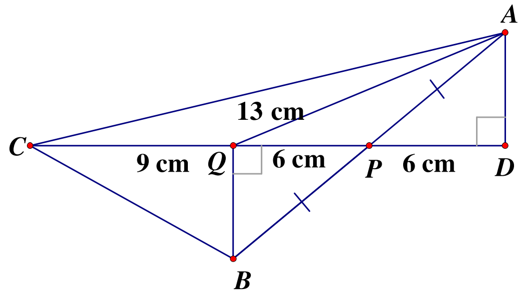
$\angle APD = \angle BPQ$ (vert. opp. $\angle$s)

$\therefore\ \Delta APD \cong \Delta BPQ$ (S.A.S.)

$\angle ADP = \angle BQP = 90°$ (corr. $\angle$s, $\cong \Delta$s)

$AD = \sqrt{13^2 - 12^2}$ cm $= 5$ cm (Pythagoras' theorem)

$QB = AD = 5$ cm (corr. sides, $\cong \Delta$s)



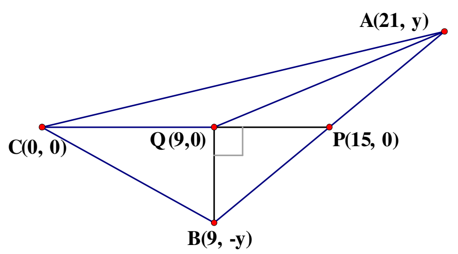
$S_{\Delta BCP} = \frac{1}{2} \cdot 15 \cdot 5 \text{ cm}^2 = \frac{75}{2} \text{ cm}^2$

$S_{\Delta ACP} = S_{\Delta BCP} = \frac{75}{2} \text{ cm}^2$ (They have the same base and the same height)

$S_{\Delta ABC} = 2 \times \frac{75}{2} \text{ cm}^2 = 75 \text{ cm}^2$

Method 2 (Provided by Mr. Mak Hugo Wai Leung)

Using coordinate geometry method, we denote C as the origin, then $Q = (9, 0)$, $P = (15, 0)$. We may let $B = (9, -y)$, where $y > 0$. Since P is the midpoint of A and B, the coordinates $A = (21, y)$.



Now $AQ = 13$ gives $\sqrt{(y-0)^2 + (21-9)^2} = 13$,

solving yields $y = 5$ (since $y > 0$).

Therefore, $A = (21, 5)$, $B = (9, -5)$, $C = (0, 0)$, and the area of triangle ABC is given by:

$$\frac{1}{2}\begin{vmatrix} 0 & 0 \\ 9 & -5 \\ 21 & 5 \\ 0 & 0 \end{vmatrix} \text{cm}^2 = \frac{1}{2}(45 + 105)\text{ cm}^2 = 75 \text{ cm}^2$$

G3 考慮數列 $a_1, a_2, a_3, \cdots$ 。定義 $S_n = a_1 + a_2 + \cdots + a_n$ 其中 n 為任何正整數。

若 $S_n = 2 - a_n - \frac{1}{2^{n-1}}$ ，求 a_{2016} 的值。

Consider a sequence of numbers $a_1, a_2, a_3, \cdots$. Define $S_n = a_1 + a_2 + \cdots + a_n$ for any positive integer n. Find the value of a_{2016} if $S_n = 2 - a_n - \frac{1}{2^{n-1}}$.

Claim: $a_n = \frac{n}{2^n}$

Prove by induction. $S_1 = a_1 = 2 - a_1 - 1 \Rightarrow a_1 = \frac{1}{2}$

$S_2 = a_1 + a_2 = 2 - a_2 - \frac{1}{2} \Rightarrow \frac{1}{2} + 2a_2 = 2 - \frac{1}{2} \Rightarrow a_2 = \frac{1}{2} = \frac{2}{4}$

Suppose $a_m = \frac{m}{2^m}$ is true for $m = 1, 2, \cdots , k$, where k is a positive integer.

$S_{k+1} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots + \frac{k}{2^k} + a_{k+1} = 2 - a_{k+1} - \frac{1}{2^k}$ …… (1)

$2S_{k+1} = 1 + 1 + \frac{3}{4} + \frac{4}{8} + \cdots + \frac{k}{2^{k-1}} + 2a_{k+1} = 4 - 2a_{k+1} - \frac{1}{2^{k-1}}$ …… (2)

$2S_{k+1} - S_{k+1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{k-1}} - \frac{k}{2^k} + a_{k+1} = 2 - a_{k+1} - \frac{1}{2^k}$

$$\frac{1 - \frac{1}{2^k}}{1 - \frac{1}{2}} - \frac{k}{2^k} + 2a_{k+1} = 2 - \frac{1}{2^k} \Rightarrow 2 - \frac{2}{2^k} - \frac{k}{2^k} + 2a_{k+1} = 2 - \frac{1}{2^k} \Rightarrow a_{k+1} = \frac{k+1}{2^{k+1}}$$

By the principle of mathematical induction, the formula is true for all positive integer n.

$a_{2016} = \frac{2016}{2^{2016}} = \frac{32 \times 63}{32 \times 2^{2011}} = \frac{63}{2^{2011}}$

Remark: Original question

考慮數列 $a_1, a_2, a_3, \cdots$ 。定義 $S_n = a_1 + a_2 + \cdots + a_n$ 其中 n 為任何整數。

若 $S_n = 2 - a_n - \frac{1}{2^{n-1}}$ ，求 a_{2016} 的值。

Consider a sequence of numbers $a_1, a_2, a_3, \cdots$. Define $S_n = a_1 + a_2 + \cdots + a_n$ for any positive integer n. Find the value of a_{2016} if $S_n = 2 - a_n - \frac{1}{2^{n-1}}$.

The Chinese version is not the same as the English version. If n is **ANY** integer, S_n is undefined for negative values or zero of n.

G4 設 x 及 y 為正整數且滿足 $\log x + \log y = \log(2x - y) + 1$，求 (x, y) 的數量。

If x and y are positive integers that satisfy $\log x + \log y = \log(2x - y) + 1$, find the number of possible pairs of (x, y).

Reference: 2002 HG9, 2006 FI3.3, 2006 FG2.4, 2012 FI4.2

$\log(xy) = \log(2x - y) + \log 10 \Rightarrow xy = 10(2x - y)$

$20x - 10y - xy = 0$

$200 + 20x - y(10 + x) = 200 \Rightarrow (20 - y)(10 + x) = 200$

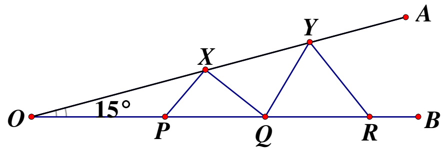
$10 + x$	$20 - y$	x	y
1	200	rejected	
2	100	rejected	
4	50	rejected	
5	40	rejected	
8	25	rejected	
10	20	rejected	
20	10	10	10
25	8	15	12
40	5	30	15
50	4	40	16
100	2	90	18
200	1	190	19

There are 6 pairs of (x, y) satisfying the equation.

G5 圖二中，$\angle AOB = 15°$。X、Y 是 OA 上的點，P、Q、R 是 OB 上的點使得 $OP = 1$ 及 $OR = 3$。若 $s = PX + XQ + QY + YR$，求 s 的最小值。

In Figure 2, $\angle AOB = 15°$. X, Y are points on OA, P, Q, R are points on OB such that $OP = 1$ and $OR = 3$.

If $s = PX + XQ + QY + YR$, find the least value of s.



圖二 Figure 2

Reference: 1999 HG9

Reflect O, P, Q, R, B along OA to give O, S, T, U, C.

Reflect O, X, Y, A along OC to give O, V, W, D.

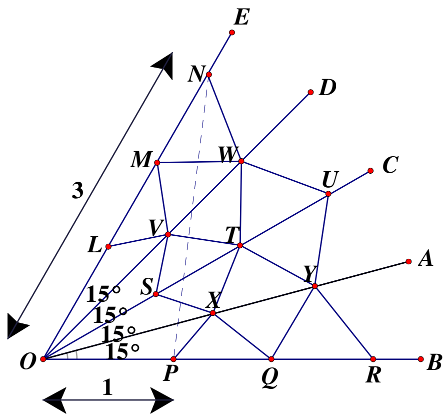
Reflect O, S, T, U, C along OD to give O, L, M, N, E.

By the definition of reflection,

$\angle AOC = \angle COD = \angle DOE = \angle AOB = 15°$, $\angle BOE = 60°$

$OS = OL = OP = 1$, $OT = OM = OQ$, $OU = ON = OR = 3$

$OV = OX$, $OW = OY$



$s = PX + XQ + QY + YR$

$\quad = PX + XT + TW + WN$

s is the least when P, X, T, W, N are collinear.

In this case, by cosine rule,

$s^2 = 1^2 + 3^2 - 2\times1\times3 \cos 60° = 7$

$s = \sqrt{7}$

G6 設 $y = px^2 + qx + r$ 為一二次函數。已知

(1) y 的對稱軸為 $x = 2016$。

(2) 該函數的圖像通過 x 軸於 A、B 兩點，其中 $AB = 4$ 單位。

(3) 該函數的圖像通過直綫 $y = -10$ 於 C、D 兩點，其中 $CD = 16$ 單位。

求 q 的值。

Let $y = px^2 + qx + r$ be a quadratic function. It is known that

(1) The axis of symmetry of y is $x = 2016$.

(2) The curve cuts the x-axis at two points A and B such that $AB = 4$ units.

(3) The curve cuts the line $y = -10$ at two points C and D such that $CD = 16$ units.

Find the value of q.

$y = p(x - 2016)^2 + k$

Let α, β be the roots of $y = p(x - 2016)^2 + k = 0$

$p(x^2 - 4032x + 2016^2) + k = 0$

$px^2 - 4032px + 2016^2p + k = 0$

$\alpha + \beta = 4032$, $\alpha\beta = \dfrac{2016^2 p + k}{p} = 2016^2 + \dfrac{k}{p}$

$|\alpha - \beta| = 4 \Rightarrow (\alpha - \beta)^2 = 16$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 16$

$\Rightarrow (4032)^2 - 4(2016^2 + \dfrac{k}{p}) = 16$

$k = -4p$

$y = p(x - 2016)^2 - 4p = px^2 - 4032px + (2016^2 - 4)p$

Let r, s be the roots of $px^2 - 4032px + (2016^2 - 4)p = -10$

$r + s = 4032$, $rs = \dfrac{2014\times 2018p + 10}{p} = 2014\times 2018 + \dfrac{10}{p}$

$|r - s| = 16 \Rightarrow (r + s)^2 - 4rs = 256$

$4032^2 - 4(2016^2 - 4 + \dfrac{10}{p}) = 256$

$16 - \dfrac{40}{p} = 256$

$\dfrac{40}{p} = -240 \Rightarrow p = -\dfrac{1}{6}$

$q = -4032p = (-4032)\times\left(-\dfrac{1}{6}\right) = 672$

Method 2

$y = p(x - 2016)^2 + k$

Let α, β be the roots of $y = p(x - 2016)^2 + k = 0$

$\alpha = 2016 - 2 = 2014$, $\beta = 2016 + 2 = 2018$

$p(2018 - 2016)^2 + k = 0 \Rightarrow 4p + k = 0 \cdots (1)$

Let r, s be the roots of $p(x - 2016)^2 + k = -10$

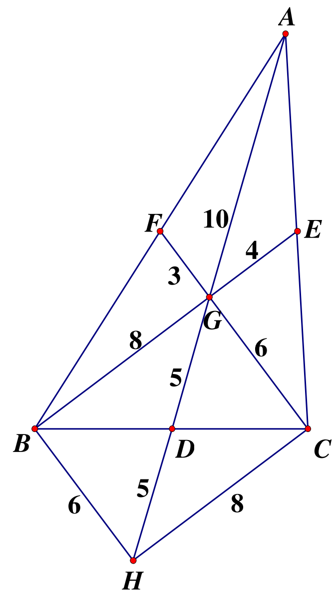
$r = 2016 - 8 = 2008$, $s = 2016 + 8 = 2024$

$p(2024 - 2016)^2 + k = 0 \Rightarrow 64p + k = -10 \cdots (2)$

Solving (1), (2) gives $p = -\dfrac{1}{6}$, $k = \dfrac{2}{3}$, $q = -4032p = (-4032)\times\left(-\dfrac{1}{6}\right) = 672$

G7 設三角形三條中綫的長度為 9、12 及 15。求該三角形的面積。

The lengths of the three medians of a triangle are 9, 12 and 15. Find the area of the triangle.



Let the triangle be ABC, with medians $AD = 15$, $BE = 12$, $CF = 9$.

The centriod G divides each median in the ratio 1 : 2.

$\therefore AG = 10$, $GD = 5$, $BG = 8$, $GE = 4$, $CG = 6$, $GF = 3$.

Produce GD to H so that $GD = DH = 5$.

Join BH, HC. By the definition of median, $BD = DC$.

$\therefore BHCG$ is a parallelogram. (diagonals bisect each other)

$CH = 8$, $BH = 6$ (opp. sides of //-gram)

In ΔBGH, $BG^2 + BH^2 = 6^2 + 8^2 = 36 + 64 = 100 = 10^2 = GH^2$

$\therefore \angle GBH = 90°$ (converse, Pythagoras' theorem)

$S_{\Delta BGH} = \frac{1}{2} BH \cdot BG = \frac{1}{2} \cdot (6\times 8) = 24$

$S_{\Delta BDG} = S_{\Delta BHG} = \frac{24}{2} = 12$ (equal base, same height)

$S_{\Delta CDG} = S_{\Delta BDG} = 12$ (equal base, same height)

$\Rightarrow S_{\Delta BCG} = 12 + 12 = 24$

$S_{\Delta BGF} = \frac{3}{6} S_{\Delta BCG} = \frac{24}{2} = 12$ (different bases, same height)

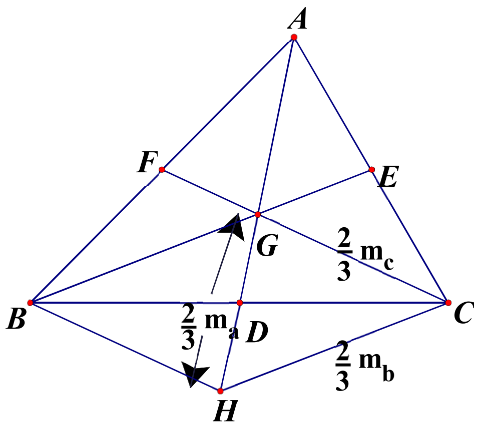
$\Rightarrow S_{\Delta BCF} = 12 + 24 = 36$

$S_{\Delta ACF} = S_{\Delta BCF} = 36$ (equal base, same height)

$S_{\Delta ABC} = 36 + 36 = 72$

Method 2 (Inspired by Mr. Mak Hugo Wai Leung)

Claim: Area of triangle $= \frac{4}{3}\sqrt{m(m-m_a)(m-m_b)(m-m_c)}$ …… (*), where m_a, m_b and m_c are the lengths of the 3 medians from vertices A, B and C respectively, and $m = \frac{m_a + m_b + m_c}{2}$.



The centriod G divides each median in the ratio 1 : 2.

$\therefore AG = \frac{2}{3}m_a$, $BG = \frac{2}{3}m_b$, $CG = \frac{2}{3}m_c$.

Produce GD to H so that $GD = DH = \frac{1}{3}m_a$.

Join BH, HC. By the definition of median, $BD = DC$.

$BHCG$ is a parallelogram (diagonals bisect each other)

$HC = BG$, $BH = CG$ (opp. sides of //-gram)

ΔCGH is similar to a larger triangle whose sides are m_a, m_b, m_c. By Heron' formula,

$S_{\Delta CGH} = \left(\frac{2}{3}\right)^2 \sqrt{m(m-m_a)(m-m_b)(m-m_c)} = S_{\Delta BGH}$

$S_{\Delta BCG} = S_{\Delta BCH} = \frac{1}{2} S_{BGCH} = \frac{4}{9}\sqrt{m(m-m_a)(m-m_b)(m-m_c)}$

$S_{\Delta ACG} = S_{\Delta ABG} = \frac{4}{9}\sqrt{m(m-m_a)(m-m_b)(m-m_c)}$

$\therefore S_{\Delta ABC} = 3\times\frac{4}{9}\sqrt{m(m-m_a)(m-m_b)(m-m_c)} = \frac{4}{3}\sqrt{m(m-m_a)(m-m_b)(m-m_c)}$

$m = \frac{1}{2}(9+12+15) = 18$, $m - m_a = 18 - 9 = 9$, $m - m_b = 18 - 12 = 6$, $m - m_c = 18 - 15 = 3$

$S_{\Delta ABC} = \frac{4}{3}\sqrt{18(9)(6)(3)} = 72$

G8 若某正整數的二進位表示有以下特質：

(1) 有 11 個位，

(2) 有六個位是 1，有五個位是零，

則稱該數為「好數」。

(例如：2016 是一個「好數」，因為 $2016 = 11111100000_2$。)

求所有「好數」的和。

If the binary representation of a positive integer has the following properties:

(1) the number of digits = 11,

(2) the number of 1's = 6 and the number of 0's = 5,

then the number is said to be a "good number".

(For example, 2016 is a "good number" as $2016 = 11111100000_2$.)

Find the sum of all "good numbers".

Let the 11-digit binary number be $X = \overline{abcdefghijk}$, where $a = 1$ and all other digits are either 0 or 1. If X is a "good number", then, discard the leftmost digit, there are 5 1's and 5 0's.

The number of "good numbers" is $C_5^{10} = \frac{10\times 9\times 8\times 7\times 6}{1\times 2\times 3\times 4\times 5} = 252$

Starting from rightmost digit to 2^9-digit, each digit has 126 1's and 126 0's

Sum of all "good numbers" is $252\times 2^{10} + 126\times 2^9 + 126\times 2^8 + \cdots + 126$

$= 126\times 2^{10} + 126\times \frac{2^{11}-1}{2-1}$

$= 126\times(1024 + 2047)$

$= 126\times 3071 = 386946$

G9 設整數 a、b 及 c 為三角形的邊長。已知 $f(x) = x(x - a)(x - b)(x - c)$，且 x 為一個大於 a、b 及 c 的整數。若 $x = (x - a) + (x - b) + (x - c)$及 $f(x) = 900$，求該三角形三條垂高的總和。

Let the three sides of a triangle are of lengths a, b and c where all of them are integers. Given that $f(x) = x(x - a)(x - b)(x - c)$ where x is an integer of size greater than a, b and c.

If $x = (x - a) + (x - b) + (x - c)$ and $f(x) = 900$, find the sum of the lengths of the three altitudes of this triangle.

$x = (x - a) + (x - b) + (x - c) \Rightarrow 2x = a + b + c$

$2x - 2a = b + c - a$, $2x - 2b = a + c - b$, $2x - 2c = a + b - c$

$16\ f(x) = 2x(2x - 2a)(2x - 2b)(2x - 2c) = (a + b + c)(b + c - a)(a + c - b)(a + b - c) = 16\times 900$

Let $a + b + c = p \cdots (1)$, $b + c - a = q \cdots (2)$, $a + c - b = r \cdots (3)$, $a + b - c = s \cdots (4)$

16×900 is even $\Rightarrow$ at least one of p, q, r, s is even.

If p is even, then (1) – (2): $2a = p - q$, L.H.S. is even $\Rightarrow$ R.H.S. is even $\Rightarrow$ q is even

(1) – (3): $2b = p - r \Rightarrow r$ is even, (1) – (4): $2c = p - s \Rightarrow s$ is even

Similarly, if q is even, then p, r and s must be even.

Conclusion: $p = 2j$, $q = 2k$, $r = 2m$, $s = 2n$ and $jkmn = 900 = 2^2\times 3^2\times 5^2 \cdots (5)$

$a + b + c = 2j$ is the largest, without loss of generality, assume $j > k \geq m \geq n$

$a = j - k$, $b = j - m$, $c = j - n \Rightarrow c \geq b \geq a$

$2j = a + b + c = j - k + j - m + j - n = 3j - (k + m + n) \Rightarrow j = k + m + n \cdots (6)$

Sub. (6) into (5): $(k + m + n)kmn = 900$

$j^4 > jkmn = 900 \Rightarrow j > \sqrt{30} > 5 \cdots (7)$

If $j > 30$, $\because$ (5) $jkmn = 900 > 30kmn$, then $kmn < 30$

$j = k + m + n > 30 \Rightarrow 3k \geq k + m + n > 30 \Rightarrow k > 10$

$(k, m, n) = (12, 1, 1)$, but $(k + m + n)kmn \neq 900$, rejected

$(k, m, n) = (15, 1, 1)$, but $(k + m + n)kmn \neq 900$, rejected

$(k, m, n) = (18, 1, 1)$, but $(k + m + n)kmn \neq 900$, rejected

$(k, m, n) = (20, 1, 1)$, but $(k + m + n)kmn \neq 900$, rejected

$(k, m, n) = (25, 1, 1)$, but $(k + m + n)kmn \neq 900$, rejected

Conclusion: by (7), $6 \leq j = k + m + n \leq 30$

When $j = 30$, $kmn = 30 \Rightarrow (k, m, n) = (30, 1, 1), (15, 2, 1), (10, 3, 1), (6, 5, 1)$ or $(5, 3, 2)$

but $j = k + m + n \neq 30$, rejected

When $j = 25$, $kmn = 36 \Rightarrow (k, m, n) = (18, 2, 1), (12, 3, 1), (9, 4, 1), (9, 2,2), (6,6,1)$,

or $(6, 3, 2)$ but $j = k + m + n \neq 25$, rejected

When $j = 20$, $kmn = 45 \Rightarrow (k, m, n) = (15, 3, 1), (9, 5, 1)$ or $(5, 3, 3)$

but $j = k + m + n \neq 20$, rejected

When $j = 18$, $kmn = 50 \Rightarrow (k, m, n) = (10, 5, 1)$ or $(5, 5, 2)$

but $j = k + m + n \neq 18$, rejected

When $\underline{\mathbf{j = 15}}$, $kmn = 60 \Rightarrow (k, m, n) = (10, 6, 1), (10, 3, 2), (6, 5, 2), (5, 4, 3)$

only $\underline{\mathbf{(10, 3, 2)}}$ satisfies $j = k + m + n = 15$

When $j = 12$, $kmn = 75 \Rightarrow (k, m, n) = (5, 5, 3)$, but $j = k + m + n \neq 12$, rejected

When $j = 10$, $kmn = 90 \Rightarrow (k, m, n) = (9, 5, 2)$ or $(6, 5, 3)$, but $j = k + m + n \neq 10$, rejected

When $j = 9$, $kmn = 100 \Rightarrow (k, m, n) = (5, 5, 4)$, but $j = k + m + n \neq 9$, rejected

When $j = 6$, $kmn = 150 \Rightarrow$ no integral solution, rejected

$a = j - k$, $b = j - m$, $c = j - n \Rightarrow a = 5$, $b = 12$, $c = 13$, a right-angled triangle

The three altitudes of the triangle are: $12, 5, \frac{60}{13}$.

Sum of all altitudes $= 12 + 5 + \frac{60}{13} = \frac{281}{13}$.

G10 求 $\dfrac{1^4+2015^4+2016^4}{1^2+2015^2+2016^2}$ 的值。

Find the value of $\dfrac{1^4+2015^4+2016^4}{1^2+2015^2+2016^2}$.

Reference: 2008 FGS.4, IMO HK Preliminary Selection Contest 2009 Q1

Let $x = 2015.5$, then $2015 = x - 0.5$, $2016 = x + 0.5$

$$\frac{1^4+2015^4+2016^4}{1^2+2015^2+2016^2}=\frac{1+(x-0.5)^4+(x+0.5)^4}{1+(x-0.5)^2+(x+0.5)^2}$$

$$=\frac{1+2\left[x^4+6(0.5)^2x^2+0.5^4\right]}{1+2\left(x^2+0.5^2\right)}$$

$$=\frac{1+2x^4+3x^2+\frac{1}{8}}{1+2x^2+\frac{1}{2}}$$

$$=\frac{2x^4+3x^2+\frac{9}{8}}{2x^2+\frac{3}{2}}$$

$$=\frac{16x^4+24x^2+9}{4\left(4x^2+3\right)}$$

$$=\frac{\left(4x^2+3\right)^2}{4\left(4x^2+3\right)}$$

$$=\frac{4x^2+3}{4}$$

$$=\frac{(2\times 2015.5)^2+3}{4}$$

$$=\frac{4031^2+3}{4}$$

$$=\frac{(4000+31)^2+3}{4}$$

$$=\frac{16000000+248000+961+3}{4}$$

$$=\frac{16248964}{4}=4062241$$

Method 2 (provided by Mr. Mak Hugo Wai Leung)

In general, we have

$$\frac{1+x^4+(x+1)^4}{1+x^2+(x+1)^2}=\frac{2\left(x^4+2x^3+3x^2+2x+1\right)}{2\left(x^2+x+1\right)}=\frac{\left(x^2+x+1\right)^2}{x^2+x+1}=x^2+x+1$$

Substituting $x = 2015$ yields

$$\frac{1^4+2015^4+2016^4}{1^2+2015^2+2016^2}=1+2015+2015^2=1+2015+(2000+15)^2$$

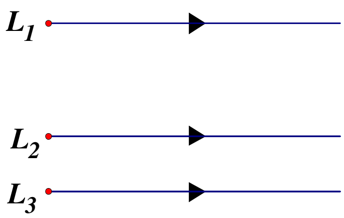
$$= 2016 + 4000000 + 60000 + 225$$

$$= 2016 + 4060225$$

$$= 4062241$$

Geometrical Construction

1. Suppose there are three different parallel lines, L_1, L_2 and L_3. Construct an equilateral triangle with only one vertex lies on each of the three parallel lines.
假設有三條不同的平行綫，L_1、L_2 及 L_3。構作一個等邊三角形，其中每條平行綫只會有一個頂點存在。
Reference:
Dropbox/Data/My%20Web/Home_Page/Geometry/construction/triangle/Equilateral_tri_on_3_parallel_lines.pdf



作圖方法如下(圖一)：

(1) 在 AB 上取任意一點 Y。

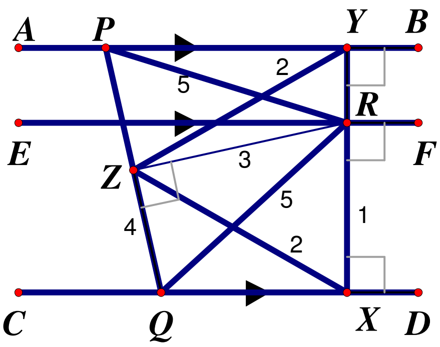
過 Y 作一綫垂直於 AB，交 CD 於 X 及 EF 於 R。

(2) 作一等邊三角形 XYZ。

(3) 連接 ZR。

(4) 過 Z 作一綫垂直於 ZR，交 AB 於 P 及 CD 於 Q。

(5) 連接 PR 及 QR。



圖一

ΔPQR 便是所需的三角形，作圖完畢。

證明如下：

$PQ \perp ZR$ 及 $AB \perp YR$	(由作圖所得)
$\therefore$ P、Y、R、Z 四點共圓	(外角=內對角)
$\angle RPZ = \angle RYZ$	(同弓形上的圓周角)
$= \angle XYZ = 60°$	
$PQ \perp ZR$ 及 $CD \perp RX$	(由作圖所得)
$\therefore$ Q、X、R、Z 四點共圓	(外角=內對角)
$\angle RQZ = \angle RXZ$	(同弓形上的圓周角)
$= \angle YXZ = 60°$	
$\angle PRQ = 180° - 60° - 60° = 60°$	(三角形內角和)

$\therefore$ ΔPQR 為一等邊三角形。
證明完畢。

方法二(由荃灣官立中學徐李炘提供)(圖二)：

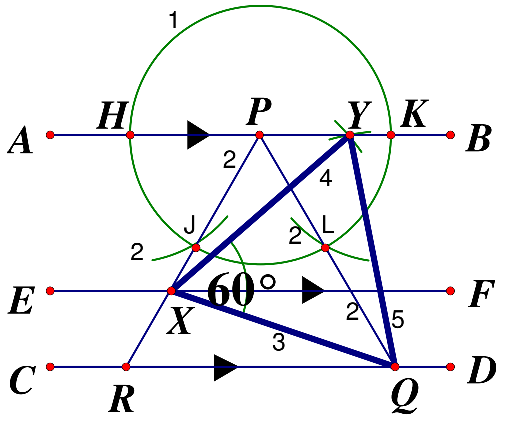
(1) 在 AB 上取任意一點 P。
以 P 為圓心，任意半徑作一圓，交 AB 於 H 及 K。

(2) 以 H 為圓心，半徑為 HP 作一弧，交該圓於 J；以 K 為圓心，半徑為 KP 作一弧，交該圓於 L，使得$\angle JPL = 60°$。連接並延長 PJ，交 EF 於 X，及 CD 於 R。連接並延長 PL，交 CD 於 Q。

(3) 連接 XQ。

(4) 過 X 作一綫段 XY，使得$\angle YXQ = 60°$，且交 AB 於 Y。

(5) 連接 YQ。



圖二

則ΔXYQ 便是一個等邊三角形了。作圖完畢。

證明如下：

ΔHPJ 及ΔKPL 是等邊三角形 (由作圖所得)

$\angle HPJ = 60° = \angle KPL$ (等邊三角形的性質)

$\angle JPL = 60°$ (直綫上的鄰角)

$\angle PRQ = 60° = \angle PQR$ (AB // CD 的交錯角)

$\therefore \Delta PQR$ 是一個等邊三角形

$\angle QXY = 60° = \angle QPY$ (由作圖所得)

$PXQY$ 為一個圓內接四邊形。 (同弓形上的圓周角的逆定理)

$\angle XYQ = \angle XPQ = 60°$ (同弓形上的圓周角)

$\angle XQY = \angle HPX = 60°$ (圓內接四邊形的外角)

$\therefore \Delta XYQ$ 是一個等邊三角形。

證明完畢。

方法三(由譚志良先生提供)(圖三)：

(1) 在 AB 上取任意一點 P。

以反時針方向，作等邊三角形ΔPEG 及ΔPFH。

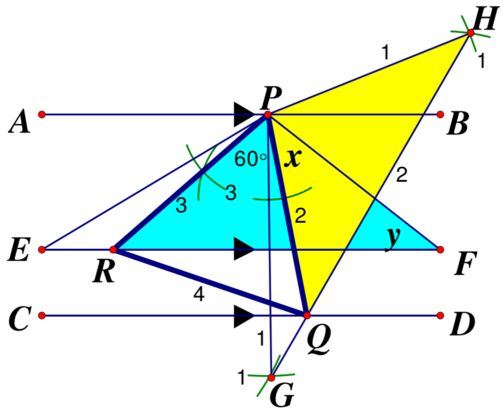
(2) 連接 GH，交 CD 於 Q，連接 PQ。

(3) 以順時針方向，作$\angle QPR = 60°$，交 EF 於 R。

(4) 連接 QR。

則ΔPQR 便是一個等邊三角形了。

作圖完畢。



圖三

證明如下：

設$\angle QPF = x$，$\angle PFE = y$

考慮ΔPEF 及ΔPGH

$PE = PG$，$PF = PH$	(等邊三角形的性質)
$\angle EPG = 60° = \angle FPH$	(等邊三角形的性質)
$\angle EPF = 60° + \angle GPF = \angle GPH$	
$\therefore \Delta PEF \cong \Delta PGH$	(S.A.S.)
$\angle PEF = \angle PHG = y$	(全等三角形的對應角)
$\angle RPF = 60° + x = \angle QPH$	
$PF = PH$	(已證)
$\therefore \Delta RPF \cong \Delta QPH$	(A.S.A.)
$PR = PQ$	(全等三角形的對應邊)
$\therefore \Delta PQR$ 為一等腰三角形	(兩邊相等)
$\angle PQR = \angle PRQ$	(等腰三角形底角相等)
$= (180° - 60°) \div 2$	(三角形內角和)
$= 60°$	

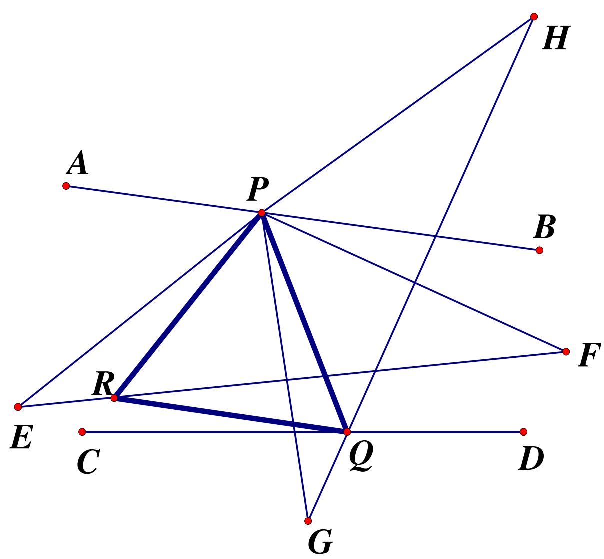
$\therefore \Delta PQR$ 是一個等邊三角形

證明完畢。

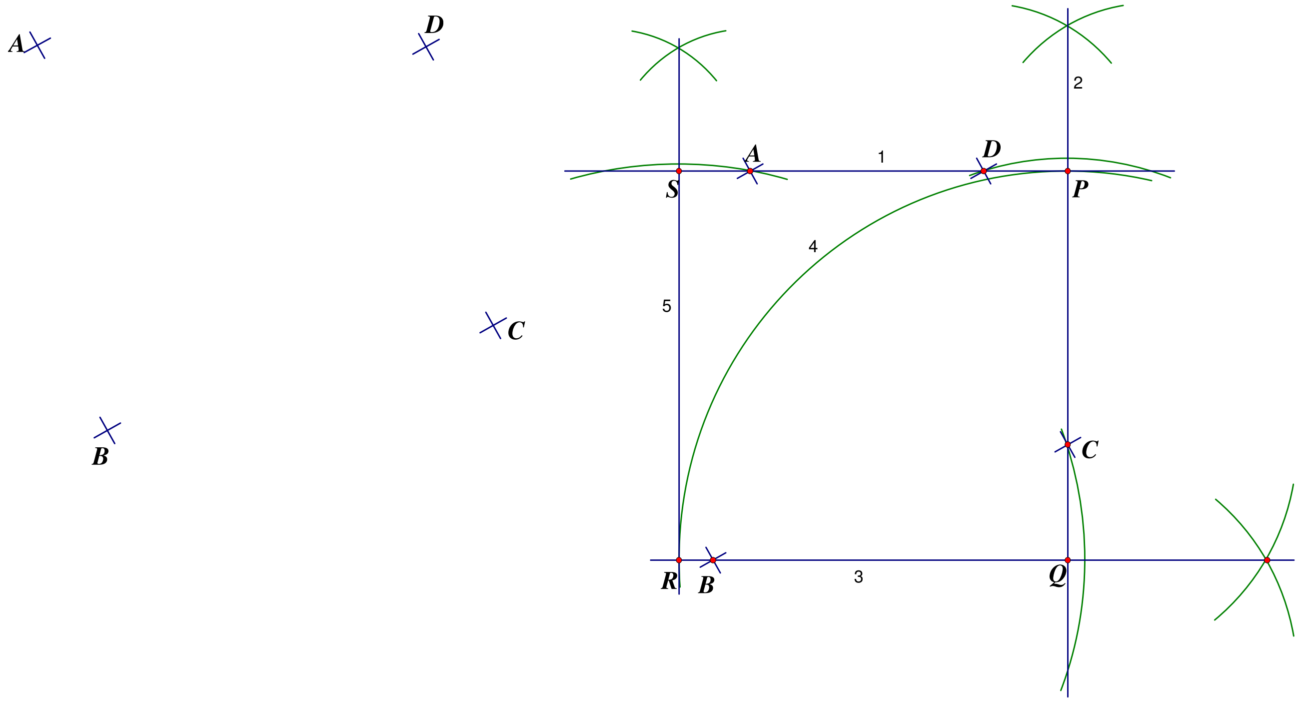
註：

以上證明沒有應用 $AB \mathbin{/\mkern-6mu/} CD \mathbin{/\mkern-6mu/} EF$ 的性質，

所以這個方法可以適用於任意三條綫。



2. Given four points A, B, C and D as shown in the figure below, construct a square which passes through these four points.
下圖所示為四點 A、B、C 及 D，構作一個通過這四點的正方形。



The construction steps are as follows:

(1) Join AD and extend AD to both ends longer.

(2) Construct a line through C and perpendicular to AD which intersects AD produced at P.

(3) Construct a line through B and perpendicular to PC which intersects PC produced at Q.

(4) Use Q as centre and QP as radius to draw an arc, cutting QB produced at R.

(5) Construct a line through R and perpendicular to DA which intersects DA produced at S.

Then $PQRS$ is the required square.

Proof: By construction, $\angle SPQ = \angle PQR = \angle PSR = 90°$

$\angle QRS = 360° - 90° - 90° - 90° = 90°$ ($\angle$s sum of polygon)

$\therefore$ $PQRS$ is a rectangle

By step (4), $PQ = QR$ = radii of the arc.

$\therefore$ $PQRS$ is a square.

Remark: *A*, *B*, *C* and *D* may lie outside the square.