Introduction to Probability

2.1 Random Experiment (隨機實驗): an experiment which has more than one result, the results cannot be predicted.

e.g. Random experiment = drawing a deck of 52 playing cards.

The possible results are called <u>outcomes</u>.

e.g.1 Random experiment = throw a die (骰子)

Outcomes = 1, 2, 3, 4, 5 or 6

The **set** of all possible outcomes is called the **sample space** (S.S. or S in short)

e.g.2 Random experiment = flipping a coin (擲毫)

Outcomes = Head or Tail (公, 字)

Sample Space S = {H, T}

A **subset** of the sample space is called an <u>**event**</u> (事件).

e.g.3 Random Experiment = throwing 2 coins

Sample space = {HH, HT, TH, TT}

E = event of throwing 2 different faces = {HT, TH}

F = event that the first throw is a Head = {HT, HH}

Certainty event (必然事件): an event which is **certain to happen**.

Impossible event (不可能事件): an event which will **definitely not happen**.

Complement event of E (互補事件): the event that E will not happen = E' = S \ E (read as S except E)

Simple event: an event has only **one element**.

e.g.4 Random experiment = select a number from 0, 1, 2, $\cdots$, 9

S = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

E = {any number between 0 to 9} = S = certainty event

F = {11} = ϕ = impossible event

G = event of drawing a prime number = {2, 3, 5, 7}

G’ = S \ G = Complementary event of G = {1, 4, 6, 8, 9, 0}

H = event of drawing the biggest element = {9} = simple event

2.2 Definition of Probability

Let R.E. be a random experiment. S be the sample space. E be an event.

Suppose S is an non-empty set. If all elements of S are equal likely to happen,

then the probability of an event E to be happened is $P(E)=\frac{n(E)}{n(S)}=\frac{\text{number of favourable outcomes in E}}{\text{total number of outcomes in S}}$.

e.g.5 R.E. = throw a die

S = {1, 2, 3, 4, 5, 6}, n(S) = 6

E = event of an odd number ={1, 3, 5}, n(E) = 3

$$P(E)=\frac{n(E)}{n(S)}=\frac{3}{6}=\frac{1}{2}$$

e.g.6 R.E. = throw 2 dice

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}, n(S) = 36

E = event that the sum is greater than 9

= { ____ }, n(E) = __________

$\therefore$ P(sum greater than 9) = $\frac{n(E)}{n(S)}$ =

Exercise 1 When two dice are thrown the probability that the total score is a multiple of 2 is $\frac{1}{2}$. For how many other values of n is it true that, when two dice are thrown, the probability that the total score is a multiple of n is equal to $\frac{1}{n}$?

Ans. 5 (n = 1, 3, 4, 6, 9)

2.3 Experimental Probability (實驗概率)

Suppose you throw an unbiased coin, the probability of getting a head is $\frac{1}{2}$. This is known as the **theoretical probability**. (理論概率)

Suppose you throw a coin twice. Do you expect a Head (H) and a tail (T) must come out ?
The answer is 'No' as the outcomes of each toss is independent.
Suppose you toss a coin 5 times and suppose the outcomes are H, T, T, H, T.
There are two 'H' and three 'T', the ratio $\frac{2}{5}$ is called the experimental probability of getting a head.

In general, the **experimental probability** of an event = $\frac{\text{number of times the event happen}}{\text{total number of trails in the experinment}}$.

Experimental Probability is also known as **empirical probability**.
If the tossed coin is **unbiased**, we may expect that as the number of trials increase,

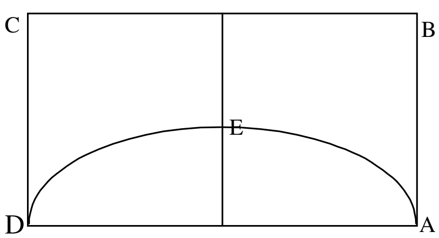
Experimental Probability $\rightarrow$ Theoretical Probability

Class Activity 1 From "Data Handling Dimension KS 3, Mathematics Section, EMB", Exemplar 12
Use the Excel file "dh12_e.xls" to *simulate* the tossing of a coin several hundreds times. Then find the experiment probability of getting a head.

Class Activity 2 Buffon's Needle Model (1707 - 1788)
A smooth table is ruled with parallel lines at distance a apart. A needle of length $\ell \le a$ is dropped on the table. It can be proved by calculus[1] that the probability that it will cross one of the lines is $\frac{2\ell}{a\pi}$

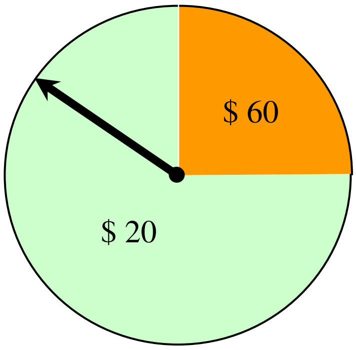
Go to the url: http://www.mste.uiuc.edu/reese/buffon/bufjava.html to do the experiment.
A "hit" occurs when a stick lands on or across one of the lines. Count the number of hits m. Let the total number of throws be N. If $\ell = a$, and as N becomes sufficiently large, $\frac{m}{N} = \frac{2}{\pi}$. Then we can estimate π.

[1] Take one of the parallel lines for the x–axis and any perpendicular to it for y–axis. The probability that the centre of the needle has an ordinate lying between the limits y and y + dy is $\frac{dy}{a}$; and the probability that the inclination of the needle to Oy should be between θ and θ + dθ is $\frac{d\theta}{\pi}$. Hence the probability that the needle will cross Ox is $\iint \frac{dy d\theta}{a\pi}$, where the double integral is taken oven the range of values of y and θ for which the needle will cross Ox. The possible values of y are evidently given by $|y| \le \frac{1}{2}\ell\cos\theta$, and θ lies in the range $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$. Thus, from the figure, where DEA is the curve $y = \frac{1}{2}\ell\cos\theta$ and AB is of length $\frac{1}{2}\ell$, the required probability is $\frac{\text{area } AED}{\text{area of } ABCD} = \frac{\ell}{\frac{a}{2}\pi} = \frac{2\ell}{a\pi}$.



Class Activity 3 From "Data Handling Dimension KS 3, Mathematics Section, EDB", Exemplar 13

Jack usually receives \$30 as the pocket money from his father Donald everyday. One day Jack asks his father if he could have more pocket money. Donald makes the following suggestion. He uses a spinner to give Jack a choice. See the figure below.



Jack spins the pointer. If the pointer stops at the \$60 portion, Jack could get \$60. Otherwise, Jack could get \$20.

Question 1: How much would you expect Jack to get in a month? (Assume 30 days for convenience.)

Use the Excel file "dh13_e.xls" to spin 30 times. Record the results.

Question 2: How much does Jack actually get in this month?

The expected value can be introduced as an application of a weighted mean as follows.

Suppose in N trails, there are n_1 times to get \$60 and n_2 for \$20 (so $N = n_1 + n_2$).

The mean of pocket money can be found as follows.

$$\frac{60n_1 + 20n_2}{N} = 60\times\frac{n_1}{N} + 20\times\frac{n_2}{N}$$

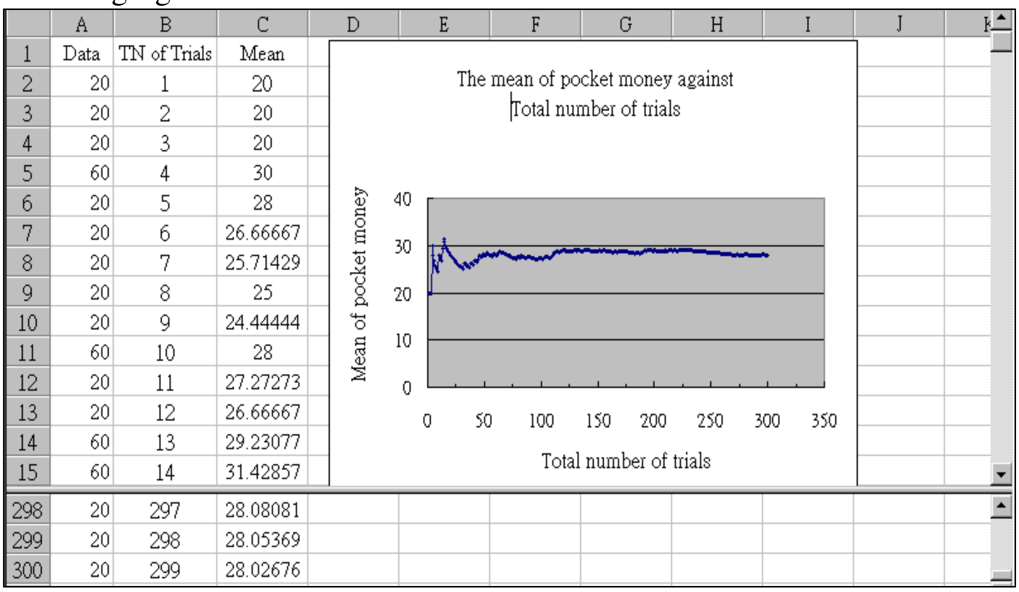
$$\approx 60 \times \text{P (getting \$60)} + 20 \times \text{P (getting \$20)} \quad \text{(provided that } N \text{ is sufficiently large.)}$$

In this exemplar, as P (getting \$60) $= \frac{1}{4}$ and P (getting \$20) $= \frac{3}{4}$, the mean of pocket money (in dollars) tends to $60\times\frac{1}{4} + 20\times\frac{3}{4} = 30$ which is called the expected value of the pocket money.

The teacher should remind students that the expected value is a theoretical value rather than an experimental finding.

So, Jack should expect to receive an amount of pocket money of \$30×30 = \$900 in a month.

Students then plot the graph of the mean of the pocket money against the total number of trials. The following figure is an illustration.



	A	B	C
1	Data	TN of Trials	Mean
2	20	1	20
3	20	2	20
4	20	3	20
5	60	4	30
6	20	5	28
7	20	6	26.66667
8	20	7	25.71429
9	20	8	25
10	20	9	24.44444
11	60	10	28
12	20	11	27.27273
13	20	12	26.66667
14	60	13	29.23077
15	60	14	31.42857
298	20	297	28.08081
299	20	298	28.05369
300	20	299	28.02676

The teacher asks students to guess how the mean would change in the graph when the total number of trials increases.

Class Activity 4 From "Data Handling Dimension KS 3, Mathematics Section, EDB", Exemplar 14

1. Throw two dice 25 times.
2. In each throw, subtract the smaller number from the larger number. The answer is called "Dice Difference".
3. If the "Dice Difference" is 0, 1 or 2, then Mr. L wins. If the "Dice Difference" is 3, 4 or 5, Mr. H wins.
4. Put a tick (✓) in the appropriate box for the result of each throw.

No. of Throw	Mr. L wins	Mr. H wins
1		
2		
3		
⋮	⋮	⋮
25		
Total no. of wins		

5. Use the results from the Table of Whole Class from your teacher's transparency to complete the following table.

Group number less than or equal to	Accumulated number of times Mr. L wins	Number of rounds played	Empirical probability that Mr. L wins
1		25	
2		50	
3		75	
4		100	
5		125	
6		150	
7		175	
8		200	
9		225	
10		250	
11		275	
12		300	
13		325	
14		350	
15		375	
16		400	
17		425	
18		450	
19		475	
20		500	

6. Tabulation method

Dice Difference	**Favorable outcomes in ordered pairs**
0	(1,1) (2,2) (3,3) (4,4) (5,5) (6,6)
1	(1,2) (2,3) (3,4) (4,5) (5,6) (2,1) (3,2) (4,3) (5,4) (6,5)
2	(1,3) (2,4) (3,5) (4,6) (3,1) (4,2) (5,3) (6,4)
3	(1,4) (2,5) (3,6) (4,1) (5,2) (6,3)
4	(1,5) (2,6) (5,1) (6,2)
5	(1,6) (6,1)

7. Study the following table.

Dice Difference	**0**	**1**	**2**	**3**	**4**	**5**
Total number of favorable outcomes	6	10	8	6	4	2

This game is not fair since the theoretical probability of Mr. L wins is $\frac{6+10+8}{36}=\frac{2}{3}$ and that of Mr. H is $\frac{6+4+2}{36}=\frac{1}{3}$.

Suggestion for modifying the rules to make the game more fair:

Replace Mr. L and Mr. H by Mr. E and Mr. O. Mr. E wins if the Dice Difference is 0, 2 or 4 and Mr. O wins if the Dice Difference is 1, 3 or 5.

<u>Justification</u>: P (Mr. E wins) = $\frac{6+8+4}{36}=\frac{1}{2}$

P (Mr. O wins) = $\frac{10+6+2}{36}=\frac{1}{2}$

Accept other modifications from students provided that the two players have equal chance of winning.