Examples on Mathematical Induction: divisibility variable

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1. Prove that $x^n - y^n$ is divisible by $x - y$ for all natural numbers n.
2. Prove that $x^n - y^n$ is divisible by $x^2 - y^2$ for all non-negative even integers n.
3. Prove that $x^n - y^n$ is divisible by $x + y$ for all non-negative even integers n.
4. Prove that $a^n - b^n$ is divisible by $a - b$ for all non-negative integers n.
5. (a) Prove that $x^n - nx + n - 1$ is divisible by $(x - 1)^2$ for all non-negative integers n.
   (b) Prove that $x^n - na^{n-1}x + (n - 1)a^n$ is divisible by $(x - a)^2$ for all non-negative integers n.
   (c) Prove that $(x^n - 1)(x^{n+1} - 1)$ is divisible by $(x - 1)(x^2 - 1)$ for all non-negative integers n.
   (d) Prove that $(x^n - 1)(x^{n+1} - 1)(x^{n+2} - 1)$ is divisible by $(x - 1)(x^2 - 1)(x^3 - 1)$ for all $n \ge 0$.
   (e) Deduce the n^{th} statement.
   (f) Prove the n^{th} statement by mathematical induction.
6. Prove, by mathematical induction, that $x^{n+2} + (x + 1)^{2n+1}$ is divisible by $x^2 + x + 1$ for all positive integers n.

Let $P(n) \equiv$ "$x^{n+2} + (x + 1)^{2n+1}$ is divisible by $x^2 + x + 1$ for all positive integers n."

$n = 1$, $x^{n+2} + (x + 1)^{2n+1} = x^3 + (x + 1)^3$

$= (x + x + 1)[x^2 - x(x + 1) + (x + 1)^2]$

$= (2x + 1)(x^2 + x + 1)$, which is divisible by $x^2 + x + 1$.

Suppose $P(k)$ is true for some positive integer k.

i.e. $x^{k+2} + (x + 1)^{2k+1} = (x^2 + x + 1)Q(x)$ for some polynomials $Q(x)$.

When $n = k + 1$

$x^{k+3} + (x + 1)^{2k+3} = x \cdot x^{k+2} + (x + 1)^2 \cdot (x + 1)^{2k+1}$

$= x \cdot x^{k+2} + (x + 1)^2 \cdot [(x^2 + x + 1)Q(x) - x^{k+2}]$, induction assumption

$= (x + 1)^2 \cdot (x^2 + x + 1)Q(x) + x^{k+2} \cdot [x - (x + 1)^2]$

$= (x + 1)^2 \cdot (x^2 + x + 1)Q(x) + x^{k+2} \cdot (-x^2 - x - 1)$

$= (x^2 + x + 1)[(x + 1)^2 \cdot Q(x) - x^{k+2}]$

$\because$ $(x + 1)^2 \cdot Q(x) - x^{k+2}$ is a polynomial in x.

$\therefore$ $x^{k+3} + (x + 1)^{2k+3}$ is divisible by $x^2 + x + 1$.

If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n.