Examples on Mathematical Induction: Sum of powers of integers

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1. **1991 Paper 2 Q7**

(a) Prove, by mathematical induction, that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

(b) Using the formula in (a), find the sum $1\times 2 + 2\times 3 + 3\times 4 + \cdots + n(n+1)$.

(c) Deduce the value of $21^2 + 22^2 + \cdots + 50^2$.

(a) Let $P(n) \equiv$ "$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$."

$n = 1$, L.H.S. $= 1^2 = 1$, R.H.S. $= \frac{1}{6}(1)(1+1)(2+1) = 1$.

L.H.S. = R.H.S.

$P(1)$ is true

Suppose $P(k)$ is true for some positive integer k.

i.e. Assume $1^2 + 2^2 + \cdots\cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$ is true …… (*)

When $n = k + 1$,

L.H.S. $= 1^2 + 2^2 + \cdots\cdots + k^2 + (k+1)^2$

$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$ (induction assumption)

$= \frac{1}{6}k(k+1)(2k+1) + \frac{6}{6}(k+1)(k+1)$

$= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)]$

$= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$

$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)(k+1+1)[2(k+1)+1]$

= R.H.S.

$\therefore$ If it is true for $n = k$, then it is also true for $n = k + 1$.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

(b) $1\times 2 + 2\times 3 + 3\times 4 + \cdots + n(n+1)$

$= 1\times(1+1) + 2\times(2+1) + 3\times(3+1) + \cdots + n(n+1)$

$= (1^2 + 1) + (2^2 + 2) + (3^2 + 3) + \cdots + (n^2 + n)$

$= (1^2 + 2^2 + \cdots + n^2) + (1 + 2 + \cdots + n)$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$

$= \frac{1}{6}n(n+1)(2n+1+3)$

$= \frac{1}{6}n(n+1)(2n+4)$

$= \frac{2}{6}n(n+1)(n+2) = \frac{1}{3}n(n+1)(n+2)$

(c) $21^2 + 22^2 + \cdots + 50^2 = 1^2 + 2^2 + \cdots + 50^2 - (1^2 + 2^2 + \cdots + 20^2)$

$= \frac{1}{6}\cdot 50\cdot 51\cdot 101 - \frac{1}{6}\cdot 20\cdot 21\cdot 41 = 40055$

2. Let $A = 1^2 + 4^2 + \cdots + (3n-2)^2$, $B = 2^2 + 5^2 + \cdots\cdots + (3n-1)^2$

(a) find $A - B$,

Use the formula $1^2 + 2^2 + \cdots\cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ to find

(b) $3^2 + 6^2 + \cdots + (3n)^2$,

(c) $A + B$,

(d) Use (a) and (c) to find A and B .

(a)
$$\begin{aligned} A - B &= (1^2 - 2^2) + (4^2 - 5^2) + \cdots\cdots + [(3n-2)^2 - (3n-1)^2] \\ &= (1-2)(1+2) + (4-5)(4+5) + \cdots\cdots + (3n-2-3n+1)(3n-2+3n-1) \\ &= -3 - 9 - 15 - \cdots\cdots - (6n-3) \\ &= -3(1 + 3 + 5 + \cdots\cdots + 2n - 1) \\ &= -3n^2 \end{aligned}$$

(b)
$$\begin{aligned} 3^2 + 6^2 + \cdots\cdots + (3n)^2 &= 9(1^2 + 2^2 + \cdots\cdots + n^2) \\ &= \frac{9}{6}n(n+1)(2n+1) \\ &= \frac{3}{2}n(n+1)(2n+1) \end{aligned}$$

(c)
$$\begin{aligned} A + B &= 1^2 + 2^2 + 4^2 + 5^2 + \cdots\cdots + (3n-2)^2 + (3n-1)^2 \\ &= 1^2 + 2^2 + \cdots\cdots + (3n)^2 - [3^2 + 6^2 + \cdots\cdots + (3n)^2] \\ &= \frac{1}{6}3n(3n+1)(6n+1) - \frac{3}{2}n(n+1)(2n+1) \\ &= \frac{n}{2}(18n^2 + 9n + 1 - 6n^2 - 9n - 3) \\ &= \frac{n}{2}(12n^2 - 2) \\ &= n(6n^2 - 1) \end{aligned}$$

(d) $\because A + B = n(6n^2 - 1)$, $A - B = -3n^2$

$$\therefore A = \frac{n(6n^2-1) - 3n^2}{2} = \frac{n(6n^2 - 3n - 1)}{2}$$

$$\therefore B = \frac{n(6n^2-1) + 3n^2}{2} = \frac{n(6n^2 + 3n - 1)}{2}$$

3. **1988 Paper 2 Q5**

Prove that $1^2 + 3^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for all positive integer n.

Let $P(n) \equiv$ "$1^2 + 3^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for all positive integer n."

$n = 1$, L.H.S. $= 1^2 = 1$, R.H.S. $= \dfrac{1(2-1)(2+1)}{3} = 1 =$ L.H.S.

$P(1)$ is true

Suppose $P(k)$ is true for some positive intreger k.

i.e. $1^2 + 3^2 + \cdots + (2k-1)^2 = \dfrac{k(2k-1)(2k+1)}{3}$

Add $(2k+1)^2$ to both sides

$$\begin{aligned}1^2 + 3^2 + \cdots + (2k-1)^2 + (2k+1)^2 &= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \\ &= \frac{k(2k-1)(2k+1)}{3} + \frac{3(2k+1)^2}{3} \\ &= \frac{(2k+1)\left[k(2k-1)+3(2k+1)\right]}{3} \\ &= \frac{(2k+1)(2k^2+5k+3)}{3} \\ &= \frac{(k+1)(2k+1)(2k+3)}{3}\end{aligned}$$

R.H.S. $= \dfrac{(k+1)\left[2(k+1)-1\right]\left[2(k+1)+1\right]}{3} =$ L.H.S.

If $P(k)$ is true then $P(k+1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integer n.

4. **2000 Paper 2 Q4**

Prove, by mathematical induction, that

$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \dfrac{n(n+1)}{2}$ for all positive integers n.

Let $P(n) \equiv$ "$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \dfrac{n(n+1)}{2}$ for all positive integers n."

$n = 1$, LH.S. $= 1^2 = 1$, R.H.S. $= (-1)^{1-1} \dfrac{1(1+1)}{2} = 1 =$ L.H.S. $P(1)$ is true

Suppose $P(k)$ is true for some positive integer k.

i.e. $1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{k-1} k^2 = (-1)^{k-1} \dfrac{k(k+1)}{2}$ for all some positive integer k

When $n = k + 1$,

L.H.S. $= 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{k-1} k^2 + (-1)^k (k+1)^2$

$= (-1)^{k-1} \dfrac{k(k+1)}{2} + (-1)^k (k+1)^2$ (induction assumption)

$= (-1)^k \left[(k+1)^2 - \dfrac{k(k+1)}{2}\right] = (-1)^k (k+1)\left[\dfrac{2(k+1)-k}{2}\right] = (-1)^k \dfrac{(k+1)(k+2)}{2} =$ R.H.S.

If $P(k)$ is true then $P(k+1)$ is also true

By the principle of mathematical induction, $P(n)$ is true for all positive integer n.

5. (a) Prove that $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = (1 + 2 + \cdots + n)^2$ for all positive integers n.

(b) Deduce that $(n + 1)^3 + (n + 2)^3 + \cdots\cdots + (2n)^3 = \frac{1}{4}n^2(3n + 1)(5n + 3)$ and

(c) find the value of $2^3 + 4^3 + \cdots\cdots + (2n)^3$.

(d) Using the result of (a), prove that $(1^3 - 1) + (2^3 - 2) + \cdots\cdots + (n^3 - n) = \frac{n(n+1)(Pn^2 + Qn + R)}{4}$.

Find P, Q and R .

(a) When $n = 1$, LHS $= 1^3 = 1$, RHS $= \frac{1^2(1+1)^2}{4} = 1$

$\therefore$ LHS = RHS, it is true for $n = 1$

Suppose it is true for $n = k$, i.e. $1^3 + 2^3 + \cdots\cdots + k^3 = \frac{k^2(k+1)^2}{4}$ ....(*)

When $n = k + 1$, L.H.S. $= 1^3 + 2^3 + \cdots\cdots + k^3 + (k + 1)^3$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 \text{ by } (*)$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4} = \text{R.H.S.}$$

If it is true for $n = k$, then it is also true for $n = k + 1$

By the principle of mathematical induction, it is true for all positive integer n.

(b) $(n + 1)^3 + (n + 2)^3 + \cdots + (2n)^3 = 1^3 + 2^3 + \cdots + (2n)^3 - (1^3 + 2^3 + \cdots + n^3)$

$$= \frac{(2n)^2(2n+1)^2}{4} - \frac{n^2(n+1)^2}{4}$$

$$= \frac{n^2\left[4(4n^2+4n+1) - (n^2+2n+1)\right]}{4}$$

$$= \frac{n^2(15n^2+14n+3)}{4} = \frac{1}{4}n^2(3n+1)(5n+3)$$

(c) $2^3 + 4^3 + \cdots + (2n)^3 = 8(1^3 + 2^3 + \cdots + n^3)$

$$= 8 \cdot \frac{n^2(n+1)^2}{4} = 2n^2(n+1)^2$$

(d) $(1^3 - 1) + (2^3 - 2) + \cdots + (n^3 - n)$

$= 1^3 + 2^3 + 3^3 + \cdots + n^3 - (1 + 2 + 3 + \cdots + n)$

$$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)}{2} = \frac{n(n+1)}{4}\left[n(n+1) - 2\right]$$

$$= \frac{n(n+1)(n^2+n-2)}{4}$$

$P = 1$, $Q = 1$, $R = -2$

6. (a) Prove, by mathematical induction, that for all positive integers n,

$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$.

(b) Hence, or otherwise, evaluate $2^3 + 6^3 + 10^3 + \ldots + 38^3$.

(a) Let $P(n) \equiv$ '$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$'

$n = 1$, L.H.S. $= 1^3 = 1$; R.H.S. $= 1^2(2-1) = 1$

L.H.S. = R.H.S. $P(1)$ is true

Suppose $P(k)$ is true

$1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2 - 1) \cdots\cdots (*)$

When $n = k + 1$

L.H.S. $= 1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 + (2k+1)^3$

$= k^2(2k^2 - 1) + (2k+1)^3$ by (*)

$= 2k^4 - k^2 + 8k^3 + 3(4k^2) + 3(2k) + 1$

$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$

R.H.S. $= (k+1)^2[2(k+1)^2 - 1]$

$= (k^2 + 2k + 1)(2k^2 + 4k + 1)$

$= 2k^4 + 4k^3 + 2k^2 + 4k^3 + 8k^2 + 4k + k^2 + 2k + 1$

$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$

L.H.S. = R.H.S.

$\therefore$ $P(k+1)$ is also true

By M.I., $P(n)$ is true for all positive integers n.

(b) $2^3 + 6^3 + 10^3 + \cdots + 38^3$

$= 8(1^3 + 3^3 + 5^3 + \cdots + 19^3)$, $2n - 1 = 19$, $n = 10$

$= 8(10^2)(2\times10^2 - 1)$

$= 800\times199 = 159200$

7. Prove that $\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$ for $n = 1, 2, 3,\cdots$

Let $P(n) \equiv$ "$\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$ for $n = 1, 2, 3,\cdots$";

When $n = 1$, L.H.S. = $1^4 = 1$, R.H.S. $= \frac{1}{30}(2)(3)(3+3-1) = 1$, $P(1)$ is true.

Suppose $P(k)$ is true, where k is an integer.

i.e. $\sum_{r=1}^{k} r^4 = \frac{1}{30}k(k+1)(2k+1)(3k^2+3k-1)$

When $n = k + 1$, $\sum_{r=1}^{k+1} r^4 = \frac{1}{30}k(k+1)(2k+1)(3k^2+3k-1) + (k+1)^4$

$$= \frac{k+1}{30}\left[k(2k+1)(3k^2+3k-1) + 30(k+1)^3\right]$$

$$= \frac{k+1}{30}\left[(2k^2+k)(3k^2+3k-1) + 30(k^3+3k^2+3k+1)\right]$$

$$= \frac{k+1}{30}\left[6k^4+9k^3-2k^2+3k^2-k+30k^3+90k^2+90k+30\right]$$

$$= \frac{k+1}{30}\left[6k^4+39k^3+91k^2+89k+30\right]$$

R.H.S. $= \frac{1}{30}(k+1)(k+2)(2k+3)\left[3(k+1)^2+3(k+1)-1\right]$

$$= \frac{(k+1)}{30}(2k^2+7k+6)(3k^2+9k+5)$$

$$= \frac{(k+1)}{30}(6k^4+39k^3+91k^2+89k+30)$$

$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive n .

8. Prove that $(1^3 + 2^3 + \cdots + n^3) + 3(1^5 + 2^5 + \cdots + n^5) = 4(1 + 2 + \cdots + n)^3$.

Note that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, $4(1 + 2 + \cdots + n)^3 = 4[\frac{n(n+1)}{2}]^3 = \frac{n^3(n+1)^3}{2}$

Let $P(n) \equiv$ "$(1^3 + 2^3 + \cdots + n^3) + 3(1^5 + 2^5 + \cdots + n^5) = \frac{n^3(n+1)^3}{2}$ for $n = 1, 2, 3, \cdots$"

$n = 1$, L.H.S. = 1 + 3 = 4, R.H.S. $= \frac{1^3 \cdot 2^3}{2} = 4$

L.H.S. = R.H.S., $P(1)$ is true

Suppose $(1^3 + 2^3 + \cdots + k^3) + 3(1^5 + 2^5 + \cdots + k^5) = \frac{k^3(k+1)^3}{2}$ for some positive integer k.

When $n = k + 1$, L.H.S. $= [1^3 + 2^3 + \cdots + k^3 + (k + 1)^3] + 3[1^5 + 2^5 + \cdots + k^5 + (k + 1)^5]$

$$= \frac{k^3(k+1)^3}{2} + (k+1)^3 + 3(k+1)^5$$

$$= \frac{(k+1)^3}{2}\left[k^3 + 2 + 6(k+1)^2\right]$$

$$= \frac{(k+1)^3}{2}\left[k^3 + 6k^2 + 12k + 8\right]$$

$$= \frac{(k+1)^3(k+2)^3}{2}$$

$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive n .

9. Prove that $1 + (1+9) + (1+9+25) + \cdots + [1^2+3^2+5^2 + \cdots + (2n-1)^2] = \frac{1}{3}\left[n^2(n+1)^2 - \frac{1}{2}n(n+1)\right]$.

Note that RHS $= \frac{n+1}{6}\left[2n^2(n+1) - n\right] = \frac{n+1}{6}\left(2n^3 + 2n^2 - n\right) = \frac{n(n+1)}{6}\left(2n^2 + 2n - 1\right)$

$n = 1$, L.H.S. = 1, R.H.S. $= \frac{1}{3}\left[2^2 - \frac{1}{2}\cdot 2\right] = 1$, it is true for $n = 1$

Assume that $1 + (1+9) + (1+9+25) + \cdots + [1^2+3^2+5^2 + \cdots + (2k-1)^2] = \frac{1}{3}\left[k^2(k+1)^2 - \frac{1}{2}k(k+1)\right]$

When $n = k + 1$,

L.H.S. $= 1 + (1+9) + (1+9+25) + \cdots + [1^2+3^2+5^2 + \cdots + (2k-1)^2] + [1^2+3^2+5^2 + \cdots + (2k+1)^2]$

$= \frac{k+1}{6}\left(2k^3 + 2k^2 - k\right) + [1^2+2^2+3^2 + \cdots + (2k+1)^2] - [2^2 + 4^2 + \cdots + (2k)^2]$

$= \frac{k+1}{6}\left(2k^3 + 2k^2 - k\right) + \frac{(2k+1)}{6}(2k+2)(4k+3) - \frac{4k}{6}(k+1)(2k+1)$

$= \frac{k+1}{6}\left[2k^3 + 2k^2 - k + 2(2k+1)(4k+3) - 4k(2k+1)\right]$

$= \frac{k+1}{6}\left[2k^3 + 2k^2 - k + 2(8k^2 + 10k + 3) - 4(2k^2 + k)\right]$

$= \frac{k+1}{6}\left(2k^3 + 10k^2 + 15k + 6\right)$

R.H.S. $= \frac{(k+1)(k+2)}{6}\left[2(k+1)^2 + 2(k+1) - 1\right]$

$= \frac{(k+1)(k+2)}{6}\left(2k^2 + 6k + 3\right)$

$= \frac{k+1}{6}\left(2k^3 + 10k^2 + 15k + 6\right)$

L.H.S. = R.H.S., it is also true for $n = k + 1$ if it is true for $n = k$.

By the principle of mathematical induction, it is true for all positive integer n.