Integral Cyclic Quadrilateral

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Given a triangle with one angle is θ, $\cos\theta = \frac{3}{5}$. If all sides are integers, find all solutions.

$c^2 = a^2 + b^2 - 2ab\cos\theta \Rightarrow c^2 = a^2 + b^2 - 2ab \times \frac{3}{5}$

$5c^2 = 5a^2 - 10ab + 5b^2 + 4ab$

$5[c^2 - (a - b)^2] = 4ab$

$$\frac{5(c+a-b)}{b} = \frac{4a}{c-a+b} = k$$

$c = b - a + \frac{bk}{5}$ ……(1) and $4a = kc - ka + kb$ ……(2)

Sub. (1) into (2): $4a = \left(b - a + \frac{bk}{5}\right)k - ak + bk$

$20a = 10bk - 10ak + bk^2$

$(10k + 20)a = (k^2 + 10k)b$

$$\frac{a}{b} = \frac{k^2 + 10k}{10k + 20}$$

Let $a = (k^2 + 10k)p$, $b = (10k + 20)p$

$c = b - a + \frac{bk}{5} = [10k + 20 - k^2 - 10k + (2k + 4)k]p = (k^2 + 4k + 20)p$. Let $p = 1$.

k	$a = k^2 + 10k$	$b = 10k + 20$	$c = k^2 + 4k + 20$	Remark
1	11	30	25	
2	24	40	32	3-4-5
3	39	50	41	
4	56	60	52	13-14-15

Given a triangle with one angle is θ, $\cos\theta = \frac{4}{5}$. If all sides are integers, find all solutions.

$c^2 = a^2 + b^2 - 2ab\cos\theta \Rightarrow c^2 = a^2 + b^2 - 2ab \times \frac{4}{5}$

$5c^2 = 5a^2 - 10ab + 5b^2 + 2ab$

$5[c^2 - (a - b)^2] = 2ab$

$$\frac{5(c+a-b)}{b} = \frac{2a}{c-a+b} = k$$

$c = b - a + \frac{bk}{5}$ ……(1) and $2a = kc - ka + kb$ ……(2)

Sub. (1) into (2): $2a = \left(b - a + \frac{bk}{5}\right)k - ak + bk$

$10a = 10bk - 10ak + bk^2$

$(10k + 10)a = (k^2 + 10k)b$

$$\frac{a}{b} = \frac{k^2 + 10k}{10k + 10}$$

Let $a = (k^2 + 10k)p$, $b = (10k + 10)p$

$c = b - a + \frac{bk}{5} = [10k + 10 - k^2 - 10k + (2k + 2)k]p = (k^2 + 2k + 10)p$. Let $p = 1$.

k	$a = k^2 + 10k$	$b = 10k + 10$	$c = k^2 + 2k + 10$	Remark
1	11	20	13	
2	24	30	18	3-4-5
3	39	40	25	
4	56	50	34	17-25-28
5	75	60	45	3-4-5
6	96	70	58	29-35-48

Given a triangle with one angle is θ, $\cos\theta = \frac{7}{25}$. If all sides are integers, find all solutions.

$c^2 = a^2 + b^2 - 2ab\cos\theta \Rightarrow c^2 = a^2 + b^2 - 2ab \times \frac{7}{25}$

$25c^2 = 25a^2 - 50ab + 25b^2 + 36ab$

$25[c^2 - (a - b)^2] = 36ab$

$$\frac{25(c+a-b)}{b} = \frac{36a}{c-a+b} = k$$

$c = b - a + \frac{bk}{25}$ ……(1) and $36a = kc - ka + kb$ ……(2)

Sub. (1) into (2): $36a = \left(b - a + \frac{bk}{25}\right)k - ak + bk$

$900a = 50bk - 50ak + bk^2$

$(50k + 900)a = (k^2 + 50k)b$

$$\frac{a}{b} = \frac{k^2 + 50k}{50k + 900}$$

Let $a = (k^2 + 50k)p$, $b = (50k + 900)p$

$c = b - a + \frac{bk}{25} = [50k + 900 - k^2 - 50k + (2k + 36)k]p = (k^2 + 36k + 900)p$. Let $p = 1$.

k	$a = k^2 + 50k$	$b = 50k + 900$	$c = k^2 + 36k + 900$	Remark
1	51	950	937	
2	104	1000	976	13-122-125
3	159	1050	1017	
4	216	1100	1060	54-275-265
5	275	1150	1105	55-221-230
6	336	1200	1152	7-24-25
7	399	1250	1201	
8	464	1300	1252	116-313-325
9	531	1350	1305	59-145-150
10	600	1400	1360	15-34-35
11	671	1450	1417	
12	744	1500	1476	62-123-125
13	819	1550	1537	
14	896	1600	1600	14-25-25
15	975	1650	1665	65-110-111
16	1056	1700	1732	264-425-433
17	1139	1750	1801	
18	1224	1800	1872	17-25-26
19	1311	1850	1945	
20	1400	1900	2020	70-95-101

Given that the three sides of a triangle are positive integers.

Let θ be an acute angle such that $\sin\theta = \frac{p}{r}$, $\cos\theta = \frac{q}{r}$ and $\tan\theta = \frac{p}{q}$ are rational numbers in the simplest forms.

Then (p, q, r) is a primitive Pythagorean triple. $(p, q, r) = (2uv, u^2 - v^2, u^2 + v^2)$ or $(u^2 - v^2, 2uv, u^2 + v^2)$.

Let ABC be a triangle ($BC = a$, $AC = b$, $AB = c$). $\angle ACB = \theta$.

Area of $\Delta ABC = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$, which must be rational

Denote the area of ΔABC by S, then $\sin A = \frac{2S}{bc}$, $\sin B = \frac{2S}{ca}$ and $\sin C = \frac{2S}{ab}$ which are also rational.

By cosine formula, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \frac{c^2 + a^2 - b^2}{2ac}$, $\cos C = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow \cos A, \cos B, \cos C \in \mathbb{Q}$

$\tan A = \frac{\sin A}{\cos A}$, $\tan B = \frac{\sin B}{\cos B}$, $\tan C = \frac{\sin C}{\cos C}$ which are still rational. Find all integral solutions for a, b and c.

$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow c^2 = a^2 + b^2 - 2ab\cdot\frac{q}{r}$

Case 1 $q = 2uv$

$(u^2 + v^2)c^2 = (u^2 + v^2)a^2 + (u^2 + v^2)b^2 - 4abuv$

$(u^2 + v^2)[c^2 - (a^2 - 2ab + b^2)] = -4abuv + 2ab(u^2 + v^2)$

$(u^2 + v^2)(c + a - b)(c - a + b) = 2ab(u - v)^2$

$$\frac{(u^2 + v^2)(c + a - b)}{b} = \frac{2(u - v)^2 a}{c - a + b} = k$$

$c + a - b = \frac{bk}{u^2 + v^2}$ ……(1), $c - a + b = \frac{2(u - v)^2 a}{k}$ ……(2)

(1) – (2): $2(a - b) = \frac{bk}{u^2 + v^2} - \frac{2(u - v)^2 a}{k}$

$2(a - b)(u^2 + v^2)k = bk^2 - 2a(u - v)^2(u^2 + v^2)$

$2(u^2 + v^2)ka + 2(u - v)^2(u^2 + v^2)a = 2(u^2 + v^2)kb + k^2 b$

$\frac{a}{b} = \frac{(2u^2 + 2v^2 + k)k}{2(u^2 + v^2)\left[(u - v)^2 + k\right]} \Rightarrow a = (2u^2 + 2v^2 + k)k$, $b = 2(u^2 + v^2)[(u - v)^2 + k]$

Sub. into (2): $c = (2u^2 + 2v^2 + k)k - 2(u^2 + v^2)[(u - v)^2 + k] + 2(u - v)^2(2u^2 + 2v^2 + k)$

$= k^2 + (2u^2 + 2v^2)k - 2(u^2 + v^2)(u - v)^2 - 2(u^2 + v^2)k + 4(u - v)^2(u^2 + v^2) + 2(u - v)^2 k$

$= k^2 + 2(u - v)^2 k + 2(u - v)^2(u^2 + v^2) = k^2 + 2(u - v)^2(u^2 + v^2 + k)$

k	u	v	$a = (2u^2 + 2v^2 + k)k$	$b = 2(u^2 + v^2)[(u - v)^2 + k]$	$c = k^2 + 2(u - v)^2(u^2 + v^2 + k)$	$\cos C = \frac{2uv}{u^2 + v^2}$
1	2	1	11	20	13	$\frac{4}{5}$
1	3	1	21	100	89	$\frac{6}{10} = \frac{3}{5}$
1	3	2	27	52	29	$\frac{12}{13}$
1	4	1	35	340	325	$\frac{8}{17}$
1	4	3	51	100	53	$\frac{24}{25}$
3	2	1	39	40	25	$\frac{4}{5}$

Note that $u > v$ are relatively prime positive integers. $\cos C = \frac{2uv}{u^2 + v^2} > 0$ and so $\angle C$ must be acute.

Area of $\Delta ABC = \frac{1}{2}(2u^2 + 2v^2 + k)k \cdot 2(u^2 + v^2)\left[(u - v)^2 + k\right] \cdot \frac{u^2 - v^2}{u^2 + v^2} = k(u^2 - v^2)(2u^2 + 2v^2 + k)[(u - v)^2 + k]$

Case 2 $p = 2uv$, $\cos\theta = \frac{q}{r} = \frac{u^2 - v^2}{u^2 + v^2}$

$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow c^2 = a^2 + b^2 - 2ab\cdot\frac{u^2 - v^2}{u^2 + v^2}$

$(u^2 + v^2)c^2 = (u^2 + v^2)(a^2 - 2ab + b^2) + 2ab(u^2 + v^2) - 2ab(u^2 - v^2)$

$(u^2 + v^2)(c + a - b)(c - a + b) = 4v^2ab$

$\frac{(u^2 + v^2)(c + a - b)}{b} = \frac{4av^2}{c - a + b} = k$

$c + a - b = \frac{bk}{u^2 + v^2}$ ……(1) and $c - a + b = \frac{4av^2}{k}$ ……(2)

(1) – (2): $2(a - b) = \frac{bk}{u^2 + v^2} - \frac{4av^2}{k}$

$2k(u^2 + v^2)a - 2k(u^2 + v^2)b = k^2b - 4v^2(u^2 + v^2)a$

$2(u^2 + v^2)(k + 2v^2)a = [k^2 + 2(u^2 + v^2)k]b$

$\frac{a}{b} = \frac{k^2 + 2(u^2 + v^2)k}{2(u^2 + v^2)(k + 2v^2)} \Rightarrow a = k^2 + 2(u^2 + v^2)k$, $b = 2(u^2 + v^2)(k + 2v^2)$

Sub. into (2): $c = k^2 + 2(u^2 + v^2)k - 2(u^2 + v^2)(k + 2v^2) + 4v^2[k + 2(u^2 + v^2)]$

$= k^2 + 4v^2(u^2 + v^2) + 4v^2k = k^2 + 4v^2(u^2 + v^2 + k)$

k	u	v	$a = (2u^2 + 2v^2 + k)k$	$b = 2(u^2 + v^2)(2v^2 + k)$	$c = k^2 + 4v^2(u^2 + v^2 + k)$	$\cos C = \frac{u^2 - v^2}{u^2 + v^2}$
1	1	2	11	90	97	$-\frac{3}{5}$
1	2	1	11	30	25	$\frac{3}{5}$
1	2	3	27	494	505	$-\frac{5}{13}$
1	3	2	27	234	225	$\frac{5}{13}$
1	4	1	35	102	73	$\frac{15}{17}$
1	4	3	51	950	937	$\frac{7}{25}$
3	2	1	39	50	41	$\frac{3}{5}$

Note that u, v are distinct relatively prime positive integers.

If $u > v$, then $\cos C = \frac{u^2 - v^2}{u^2 + v^2} > 0$ and so C is acute. If $u < v$, then $\cos C = \frac{u^2 - v^2}{u^2 + v^2} < 0$ and so C is obtuse.

Area of $\Delta ABC = \frac{1}{2}ab\sin C = \frac{1}{2}(2u^2 + 2v^2 + k)k\cdot 2(u^2 + v^2)(2v^2 + k)\cdot\frac{2uv}{u^2 + v^2}$

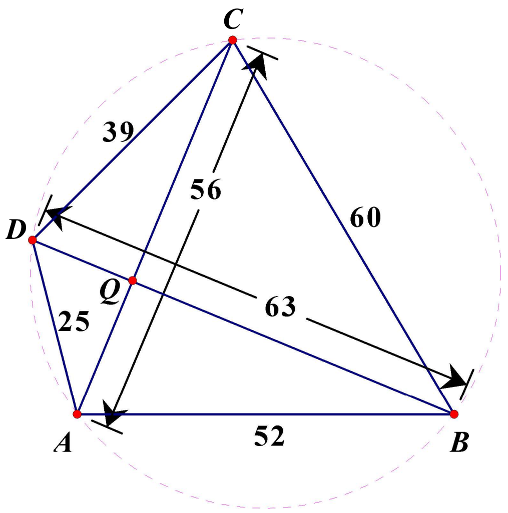
$= 2kuv(2u^2 + 2v^2 + k)(2v^2 + k)$, which is an integer

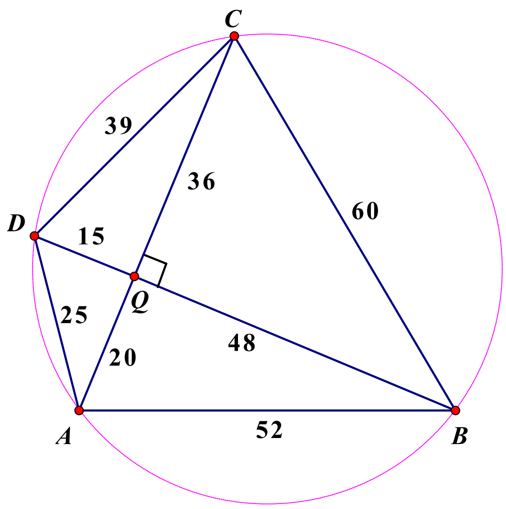
Given that the three sides of a triangle ACB are positive integers. $\angle ACB = \theta$ such that $\sin\theta = \frac{p}{r}$, $\cos\theta = \frac{q}{r}$ and $\tan\theta = \frac{p}{q}$ are all rational numbers. If (p, q, r) is a Pythagorean triple, from the above analysis,

$(a,b,c) = ((2u^2+2v^2+k)k, 2(u^2+v^2)[(u-v)^2+k], k^2+2(u-v)^2(u^2+v^2+k))$ or $((2u^2+2v^2+k)k, 2(u^2+v^2)(2v^2+k), k^2+4v^2(u^2+v^2+k))$

where $k \in \mathbb{Z}^+$, u, v are distinct relatively prime positive integers.

In this section, we are going to find a cyclic quadrilateral for which all sides and all diagonals are integers.
Idea: In February 2008, I asked Dr. Man Keung Siu from the University of Hong Kong about how to find a solution to the above question. He quoted a paper "Normal Trigrade and cyclic quadrilateral with integral sides and diagonals" from April, 1951 American Mathematical Monthly. I didn't understand the content of the paper. However, the author gave an example of integral cyclic quadrilateral $ABCD$, $AB = 52$, $BC = 60$, $CD = 39$, $DA = 25$, $AC = 56$, $BD = 63$.





It takes me years of time to investigate how to find another integral cyclic quadrilateral. Still, I failed to find any other solutions. After retirement, I use cosine formula to find $\cos\angle ACB$.

$\cos\angle ACB = \dfrac{56^2+60^2-52^2}{2\times 56\times 60} = \dfrac{3}{5}$ and so $\sin\angle ACB = \dfrac{4}{5}$, $\tan\angle ACB = \dfrac{4}{3}$ which are all rational.

I started to investigate a triangle with integral sides and rational sines of each angle.

Suppose the diagonals intersect at Q. Let $\angle BQC = \theta$. It can be proved that $\tan\theta = \dfrac{4\sqrt{(s-a)(s-b)(s-c)(s-d)}}{a^2+c^2-b^2-d^2}$

where $a = 52$, $b = 60$, $c = 39$, $d = 25$, $s = \frac{1}{2}(a+b+c+d)$.

$a^2 + c^2 - b^2 - d^2 = 52^2 + 39^2 - 60^2 - 39^2 = 0 \Rightarrow$ denominator $= 0 \Rightarrow \theta = 90°$

$CQ = BC\cos\angle BCQ = 60\times\frac{3}{5} = 36$, $BQ = BC\sin\angle BCQ = 48$. ΔBCQ is a 3-4-5 Δ.

$\therefore$ $DQ = BD - BQ = 63 - 48 = 15$, $AQ = AC - CQ = 56 - 36 = 20$.

ΔABQ is a 5-12-13 Δ. ΔADQ is a 3-4-5 Δ. ΔCDQ is a 5-12-13 Δ.

This is a **special case** of integral cyclic quadrilateral.

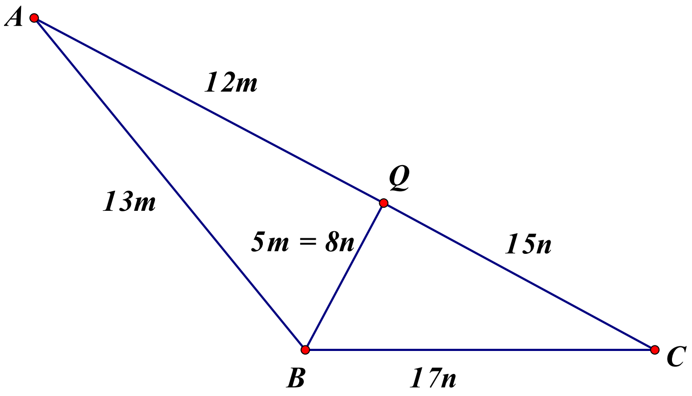
Will there be any other example(s) of integral cyclic quadrilateral *ABCD* with perpendicular diagonals?

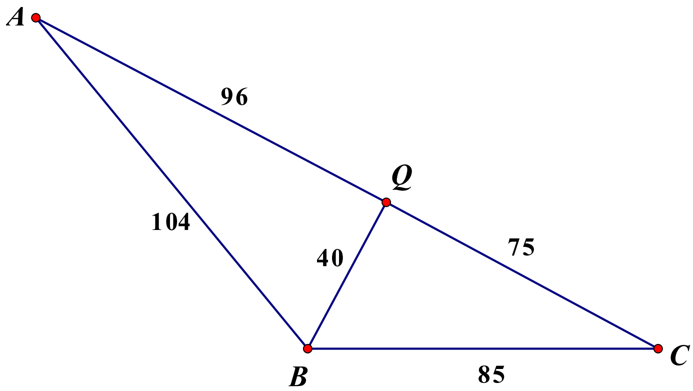
$\angle ACB = \angle ADB$, $\angle CAD = \angle CBD$, $\angle ACD = \angle ABD$, $\angle BDC = \angle BAC$ ($\angle$s in the same segment)

$\therefore$ $\Delta ABQ \sim \Delta DCQ$, $\Delta BCQ \sim \Delta ADQ$ (equiangular)

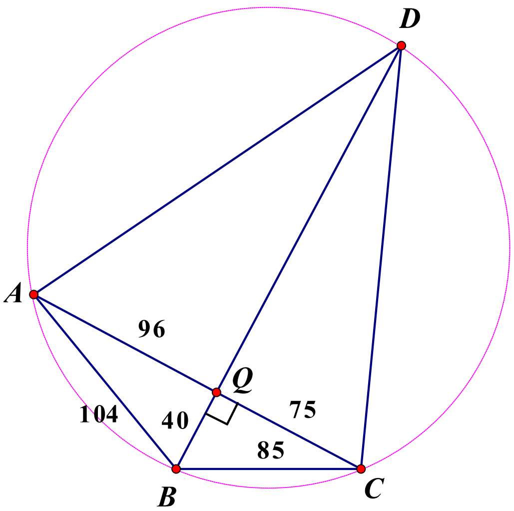
We try to find **two pairs of right-angled triangles with common sides**.

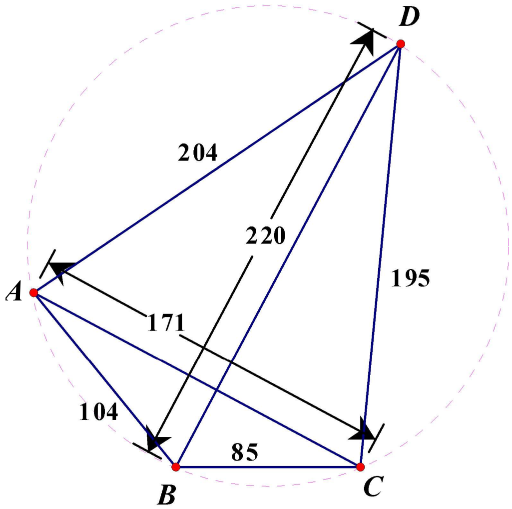
If ΔABQ is a 5-12-13 Δ, ΔBCQ is a 8-15-17 Δ. $AQ = 12m$, $BQ = 5m$, $AB = 13m$, $BQ = 8n$, $CQ = 15n$, $BC = 17n$.





$BC = 5m = 8n$, let $m = 8$, $n = 5$, then $AB = 13\times 8 = 104$, $AQ = 12\times 8 = 96$, $BC = 17\times 5 = 85$, $CQ = 15\times 5 = 75$





Construct the circumscribed circle *ABC*. Extend *BQ* to cut the circumscribed circle at *D*.

$\Delta DCQ \sim \Delta ABQ$ which are 5-12-13 Δs. $DQ = \frac{75}{5}\times 12 = 180$, $CD = \frac{75}{5}\times 13 = 195$.

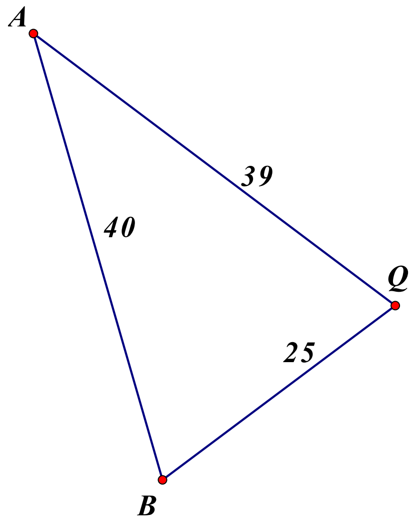
$\Delta ADQ \sim \Delta BCQ$ which are 8-15-17 Δs. $DQ = \frac{96}{8}\times 15 = 180$, $AD = \frac{96}{8}\times 17 = 204$.

$\therefore$ *ABCD* is another integral cyclic quadrilateral with

$AB = 104$, $BC = 85$, $CD = 195$, $DA = 204$, $AC = 96 + 75 = 171$, $BD = 40 + 180 = 220$.

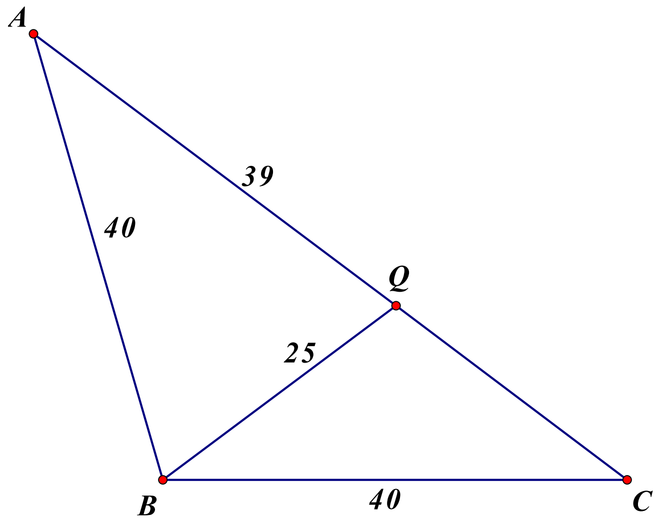
Question: Can we find an integral cyclic quadrilateral $ABCD$ so that the diagonals are not necessarily perpendicular?

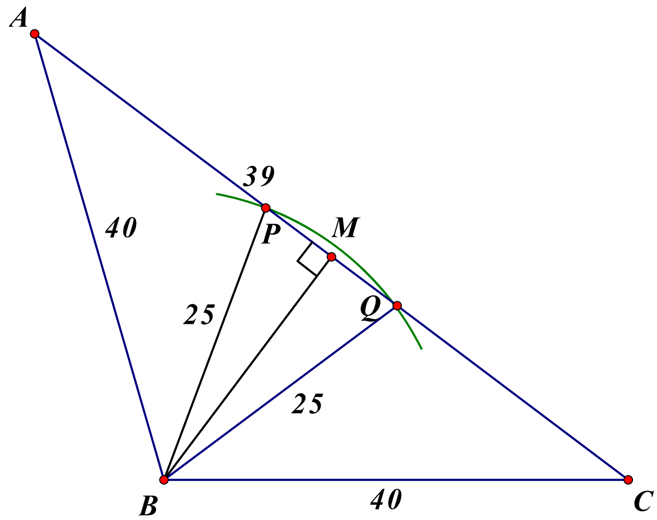
We see from page 3 the last line that 25-39-40 is an integral triangle ΔABQ.



$$\cos \angle BAQ = \frac{40^2 + 39^2 - 25^2}{2 \times 40 \times 39} = \frac{4}{5}.$$

We find another triangle ΔBCQ so that $BC = 40$ and A, Q, C are collinear.





Draw a circular arc with B as centre, radius BQ, cutting AQ at P. $BP = BQ = 25$ (radii)

Let M be the foot of perpendicular from B to PQ. $\Delta BPM \cong \Delta BQM$ (R.H.S.)

$AM = AB \cos \angle BAQ = 40 \times \frac{4}{5} = 32$, $QM = AQ - AM = 39 - 32 = 7 = PM$ (corr. sides, $\cong \Delta$s)

$AP = AQ - QM - PM = 39 - 7 - 7 = 25$

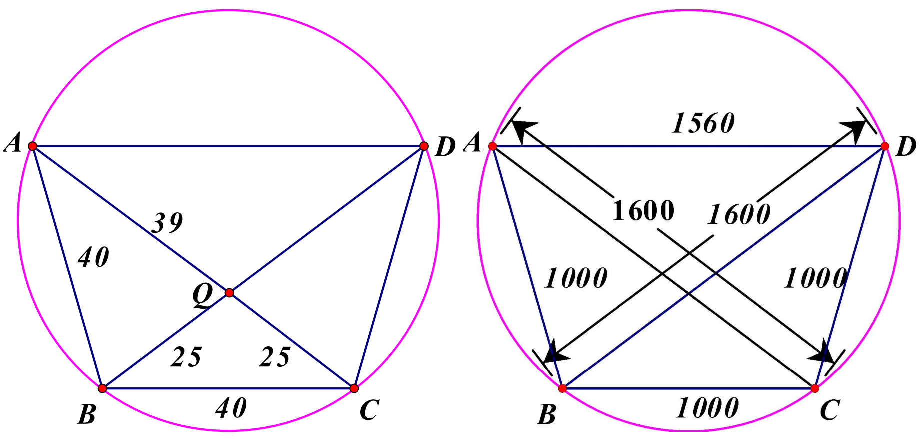
$\angle BAC = \angle BCA$, $\angle BPQ = \angle BQP$ (base $\angle$s, isos. Δ)

$\angle APB = 180° - \angle BPM = 180° - \angle BQM = \angle BQC$ (adj. $\angle$s on st. line)

$\Delta ABP \cong \Delta CBQ$ (A.A.S.)

$CQ = AP = 25$ (corr. sides, $\cong \Delta$s)

Construct a circumscribed circle through A, B and C. Extend BQ to cut the circle again at D. Join AD and CD.



It is easy to show that $\Delta ABQ \cong \Delta DCQ$ (A.A.S.)

$DQ = AQ = 39$, $DC = AB = 40$ (corr. sides, $\cong$ Δs)

$\Delta ADQ \sim \Delta BCQ$ (equiangular)

$$\frac{AD}{BC} = \frac{AQ}{BQ} \quad \text{(corr. sides, } \sim\Delta\text{s)}$$

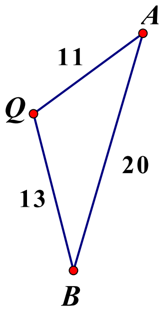
$$AD = 40 \times \frac{39}{25} = \frac{1560}{25}$$

Multiply every side by 25 to give integral sides. $AB = BC = CD = 1000$, $AD = 1560$, $BD = AC = 1600$.

Again, this is a **special case** for **three equal adjacent sides of integral cyclic quadrilateral**.

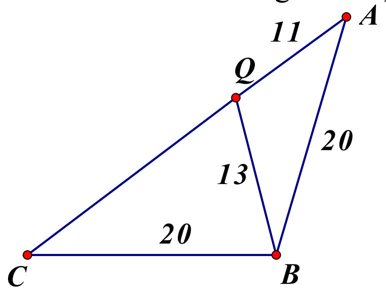
Second example:

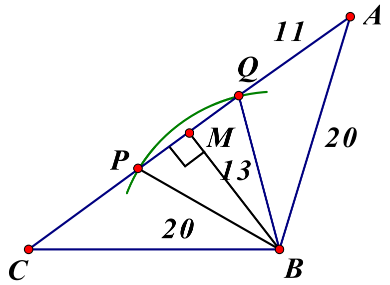
We see from page 3 the first line in the table that 11-13-20 is an integral triangle ΔABQ.



$$\cos \angle BAQ = \frac{11^2 + 20^2 - 13^2}{2 \times 11 \times 20} = \frac{4}{5}.$$

We find another triangle ΔBCQ so that $BC = 20$ and A, Q, C are collinear.





Draw a circular arc with B as centre, radius BQ, cutting AQ at P. $BP = BQ = 25$ (radii)

Let M be the foot of perpendicular from B to PQ. $\Delta BPM \cong \Delta BQM$ (R.H.S.)

$AM = AB \cos \angle BAQ = 20 \times \frac{4}{5} = 16$, $QM = AM - AQ = 16 - 11 = 5 = PM$ (corr. sides, $\cong \Delta$s)

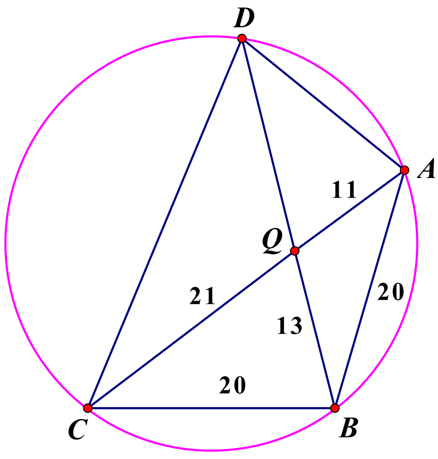
$AP = AM + PM = 16 + 5 = 21$

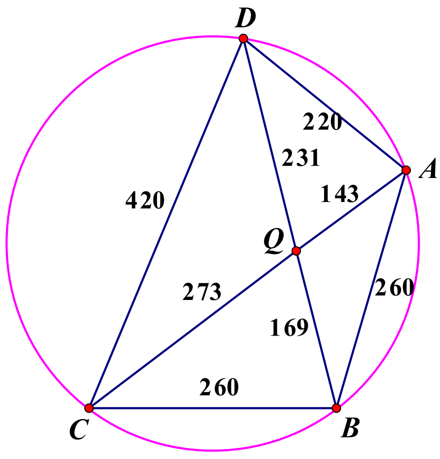
$\angle BAC = \angle BCA$, $\angle BPQ = \angle BQP$ (base $\angle$s, isos. Δ)

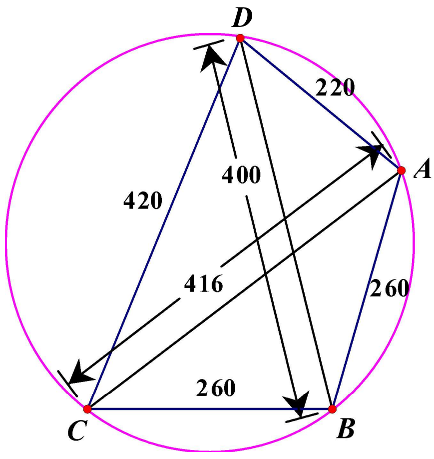
$\Delta ABP \cong \Delta CBQ$ (A.A.S.)

$CQ = AP = 21$ (corr. sides, $\cong \Delta$s)

Construct a circumscribed circle through A, B and C. Extend BQ to cut the circle again at D. Join AD and CD.







It is easy to show that $\Delta ABQ \sim \Delta DCQ$ (equiangular)

$$\frac{DQ}{AQ} = \frac{CQ}{BQ} = \frac{CD}{AB} \quad \text{(corr. sides, } \sim\Delta\text{s)}$$

$DQ = 11 \times \frac{21}{13} = \frac{231}{13}$, $CQ = 20 \times \frac{21}{13} = \frac{420}{13}$, $AC = 11 + 21 = 32$, $BD = 13 + \frac{231}{13} = \frac{400}{13}$

$\Delta ADQ \sim \Delta BCQ$ (equiangular)

$$\frac{AD}{BC} = \frac{AQ}{BQ} \quad \text{(corr. sides, } \sim\Delta\text{s)}$$

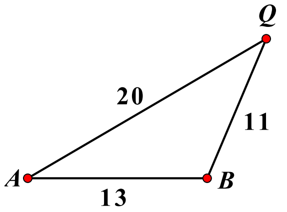
$$AD = 20 \times \frac{11}{13} = \frac{220}{13}$$

Multiply every side by 13 to give integral sides. $AB = BC = 260$, $CD = 420$, $AD = 220$, $AC = 416$, $BD = 400$.

Again, this is a **special case** for **two equal adjacent sides of integral cyclic quadrilateral**.

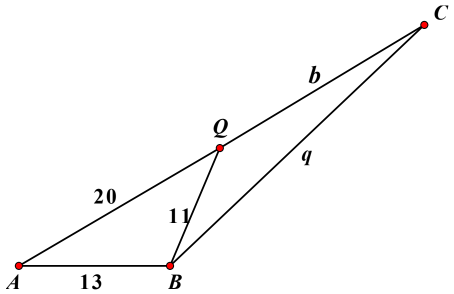
Question: Can find find an integral cyclic quadrilateral *ABCD* so that all adjacent sides are unequal?

We see from page 3 the first line in the table that 11-13-20 is an integral triangle ΔABQ.



$$\cos \angle AQB = \frac{11^2 + 20^2 - 13^2}{2 \times 11 \times 20} = \frac{4}{5}.$$

We find another triangle ΔBCQ so that $BC = q$, $QC = b$ and A, Q, C are collinear.



$$\cos \angle BQC = \cos(180° - \angle AQB) = -\cos \angle AQB = -\frac{4}{5}$$

Apply cosine rule on ΔBCQ: $q^2 = b^2 + 11^2 + 22b \times \frac{4}{5}$

$5q^2 = 5(b^2 + 22b + 11^2) - 110b + 88b$

$22b = 5[(b + 11)^2 - q^2] = 5(b + q + 11)(b - q + 11)$

$$\frac{b+q+11}{b} = \frac{22}{5(b-q+11)} = t$$

$b + q + 11 = bt \cdots\cdots (1)$, $b - q + 11 = \frac{22}{5t} \cdots\cdots (2)$

(1) + (2): $2b + 22 = bt + \frac{22}{5t}$

$10tb + 110t = 5t^2b + 22$

$110t - 22 = (5t^2 - 10t)b$

$$b = \frac{22(5t-1)}{5t(t-2)}$$

(1) – (2): $2q = bt - \frac{22}{5t} = \frac{22t(5t-1)}{5t(t-2)} - \frac{22}{5t}$

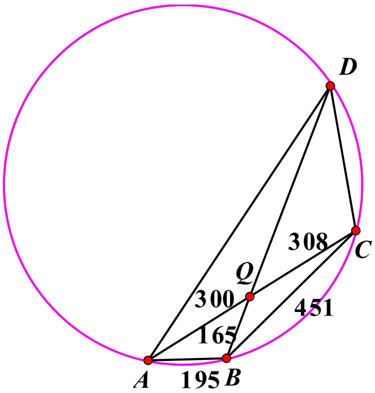
$$q = 11 \cdot \frac{t(5t-1)-(t-2)}{5t(t-2)} = 11 \cdot \frac{5t^2-2t+2}{5t(t-2)}$$

Put $t = 3$, $b = \frac{22 \times 14}{15} = \frac{308}{15}$, $q = 11 \times \frac{41}{15} = \frac{451}{15}$

Multiply every side by 15 to give integral sides.

$AB = 13 \times 15 = 195$, $BQ = 11 \times 15 = 165$, $AQ = 20 \times 15 = 300$.

Construct a circumscribed circle through A, B and C. Extend BQ to cut the circle again at D. Join AD and CD.



It is easy to show that $\Delta ABQ \sim \Delta DCQ$ (equiangular)

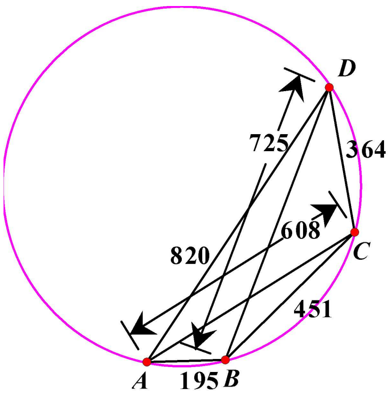
$$\frac{DQ}{AQ}=\frac{CQ}{BQ}=\frac{CD}{AB} \quad \text{(corr. sides, \sim\Delta s)}$$

$DQ = 300\times\frac{308}{165} = 560$, $CD = 195\times\frac{308}{165} = 364$, $AC = 300 + 308 = 608$, $BD = 165 + 560 = 725$

$\Delta ADQ \sim \Delta BCQ$ (equiangular)

$$\frac{AD}{BC}=\frac{AQ}{BQ} \quad \text{(corr. sides, \sim\Delta s)}$$

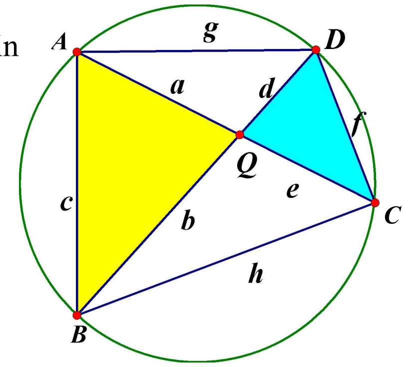
$AD = 451\times\frac{300}{165} = 820$



Question: Can we find a general formula for integral cyclic quadrilateral for which the diagonals are not necessarily perpendicular and the adjacent sides are not necessarily equal?

Let the cyclic quadrilateral be $ABCD$. The diagonals AC and BD intersect at Q.

Let $AQ = a$, $BQ = b$, $AB = c$, $DQ = d$, $CQ = e$, $CD = f$, $AD = g$, $BC = h$ as shown in the figure.



$\Delta ABQ \sim \Delta DCQ$ (equiangular)

$\frac{a}{d} = \frac{b}{e} = \frac{c}{f} = m$, where m is a constant (corr. sides, ~Δs)

$a = dm$, $b = em$, $c = fm$

$\Delta ADQ \sim \Delta BCQ$ (equiangular)

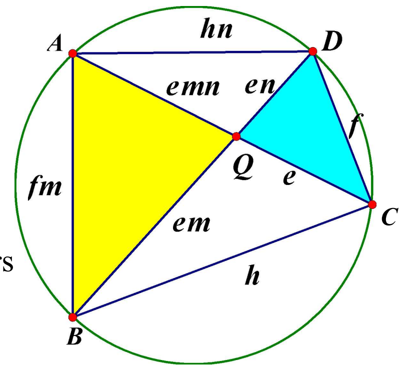
$\frac{a}{b} = \frac{d}{e} = \frac{g}{h} = n$, where n is a constant. (corr. sides, ~Δs)

$a = bn = dmn$, $d = en$, $g = hn$

There are five variables e, f, h, m, n in the figure.

Let ΔBCQ be an obtuse-angled triangle with $\angle BQC > 90°$.

Then ΔCDQ is an acute-angled triangle.



By the formula on Page 4, the only possible solution for ΔCDQ is:

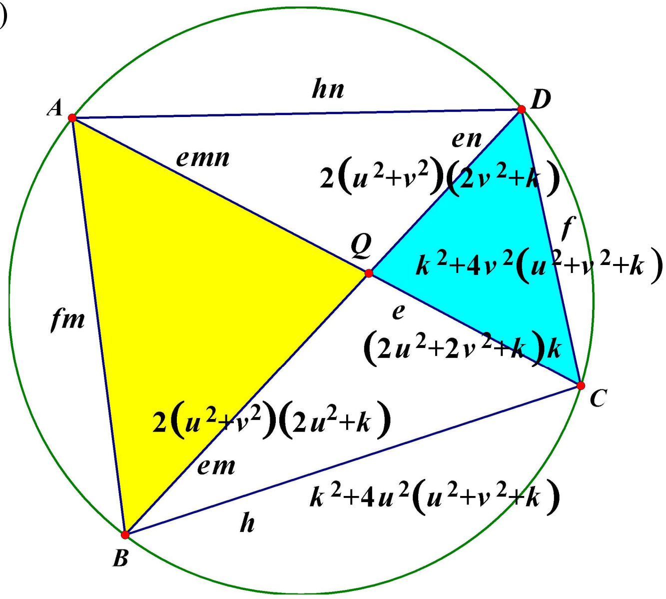
$CD = k^2 + 4v^2(u^2 + v^2 + k)$, $CQ = (2u^2 + 2v^2 + k)k$, $DQ = 2(u^2 + v^2)(2v^2 + k)$

$\cos\angle CQD = \frac{u^2 - v^2}{u^2 + v^2} > 0$, where $u > v$ are distinct relatively prime positive integers

Again, ΔBCQ is another triangle adjacent to ΔCDQ with $\cos\angle BQC = \frac{v^2 - u^2}{u^2 + v^2} < 0$

The roles of ***u* and *v* are interchanged**.

$BC = h = k^2 + 4u^2(u^2 + v^2 + k)$, $BQ = em = 2(u^2 + v^2)(2u^2 + k)$



$$n = \frac{en}{e} = \frac{DQ}{CQ} = \frac{2\left(u^2 + v^2\right)\left(2v^2 + k\right)}{\left(2u^2 + 2v^2 + k\right)k}$$

$AQ = emn = BQ \times n$

$$= 2\left(u^2 + v^2\right)\left(2u^2 + k\right) \cdot \frac{2\left(u^2 + v^2\right)\left(2v^2 + k\right)}{\left(2u^2 + 2v^2 + k\right)k}$$

$$= 4 \cdot \frac{\left(u^2 + v^2\right)^2\left(2u^2 + k\right)\left(2v^2 + k\right)}{\left(2u^2 + 2v^2 + k\right)k}$$

$$AD = hn = \left[k^2 + 4u^2\left(u^2 + v^2 + k\right)\right] \cdot \frac{2\left(u^2 + v^2\right)\left(2v^2 + k\right)}{\left(2u^2 + 2v^2 + k\right)k}$$

$$m = \frac{em}{e} = \frac{BQ}{CQ} = \frac{2\left(u^2 + v^2\right)\left(2u^2 + k\right)}{\left(2u^2 + 2v^2 + k\right)k}$$

$$AB = fm = \left[k^2 + 4v^2\left(u^2 + v^2 + k\right)\right] \cdot \frac{2\left(u^2 + v^2\right)\left(2u^2 + k\right)}{\left(2u^2 + 2v^2 + k\right)k}$$

Multiply every side by $(2u^2 + 2v^2 + k)k$ to give integral sides:

$CQ = (2u^2 + 2v^2 + k)^2k^2$, $CD = (2u^2 + 2v^2 + k)k[k^2 + 4v^2(u^2 + v^2 + k)]$, $DQ = 2k(u^2 + v^2)(2v^2 + k)(2u^2 + 2v^2 + k)$

$BC = (2u^2 + 2v^2 + k)k[k^2 + 4u^2(u^2 + v^2 + k)]$, $BQ = 2(u^2 + v^2)(2u^2 + k)(2u^2 + 2v^2 + k)k$

$AQ = 4(u^2 + v^2)^2(2u^2 + k)(2v^2 + k)$, $AD = 2(u^2 + v^2)(2v^2 + k)[k^2 + 4u^2(u^2 + v^2 + k)]$

$AB = 2(u^2 + v^2)(2u^2 + k)[k^2 + 4v^2(u^2 + v^2 + k)]$

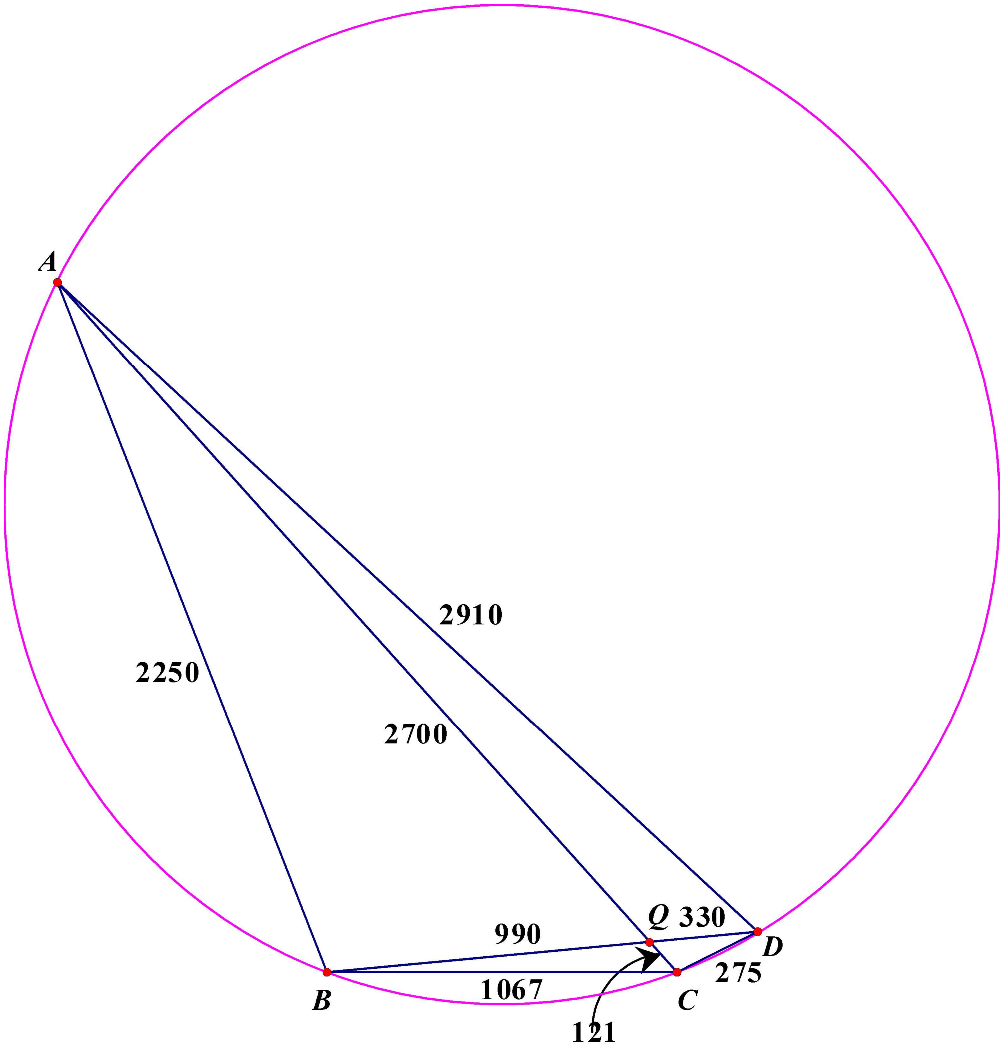
For example, take $k = 1$, $u = 2$, $v = 1$:

$CQ = 121$, $CD = 11\times(1+4\times6) = 275$, $DQ = 2\times5\times3\times11 = 330$

$BC = 11\times(1+16\times6) = 1067$, $BQ = 2\times5\times9\times11 = 990$

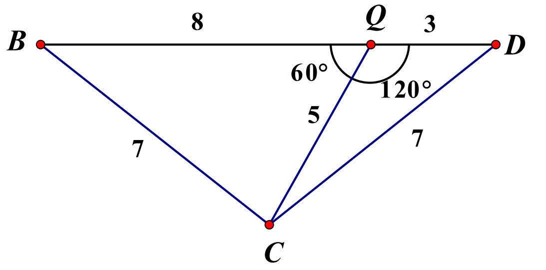
$AQ = 4\times5^2\times9\times3 = 2700$, $AD = 2\times5\times3\times(1+16\times6) = 2910$

$AB = 2\times5\times9\times(1+4\times6) = 2250$, $AC = AQ + CQ = 2700 + 121 = 2821$, $BD = BQ + DQ = 990 + 330 = 1320$



Think further: From the document: https://twhung78.github.io/Number_Theory/120triangle.pdf, we know that 3-5-7 is a 120° triangle, whereas 5-7-8 is a 60° triangle.

Combine the common side '5' to give a bigger triangle as shown:



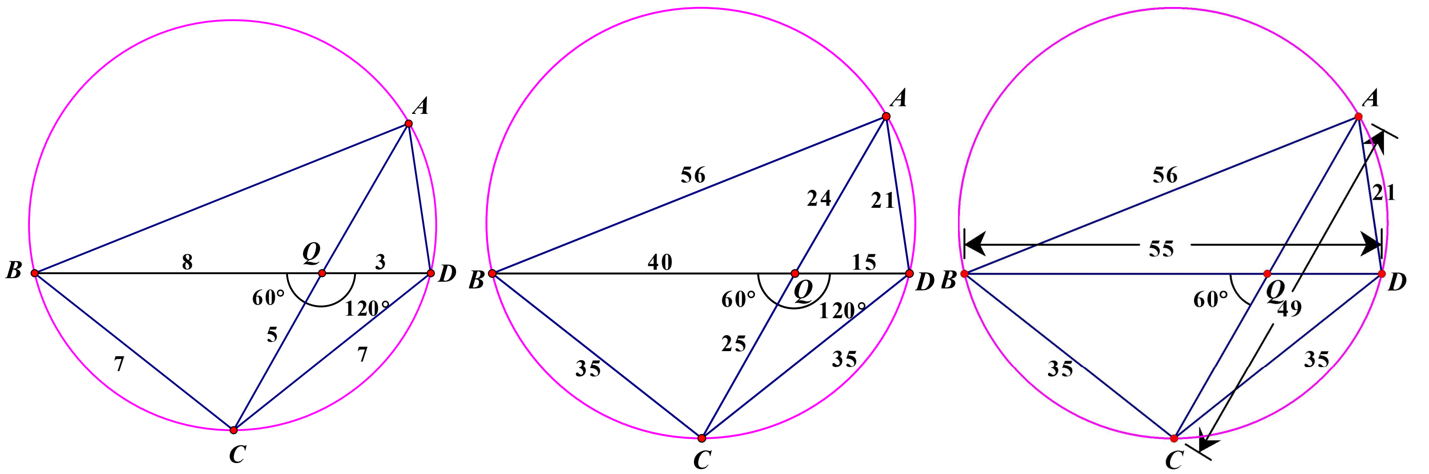
$\cos \angle CQD = \frac{3^2+5^2-7^2}{2\times3\times5} = -\frac{1}{2} \Rightarrow \angle CQD = 120°$

$\cos \angle BQC = \frac{8^2+5^2-7^2}{2\times5\times8} = \frac{1}{2} \Rightarrow \angle BQC = 60°$

$\angle BQC + \angle CQD = 60° + 120° = 180°$

$\therefore$ B, Q, D are collinear.

Construct a circumscribed circle through B, C and D. Extend CQ to cut the circle again at A. Join AB and AD.



It is easy to show that $\Delta ABQ \sim \Delta DCQ$ (equiangular)

$\frac{DQ}{AQ} = \frac{CQ}{BQ} = \frac{CD}{AB}$ (corr. sides, ~Δs)

$AQ = 3 \times \frac{8}{5} = \frac{24}{5}$, $AB = 7 \times \frac{8}{5} = \frac{56}{5}$, $AC = \frac{24}{5} + 5 = \frac{49}{5}$, $BD = 8 + 3 = 11$

$\Delta ADQ \sim \Delta BCQ$ (equiangular)

$\frac{AD}{BC} = \frac{AQ}{BQ}$ (corr. sides, ~Δs)

$AD = 7 \times \frac{3}{5} = \frac{21}{5}$

Multiply every side by 5 to give integral sides. $BC = CD = 35$, $AD = 21$, $AB = 56$, $AC = 49$, $BD = 55$.

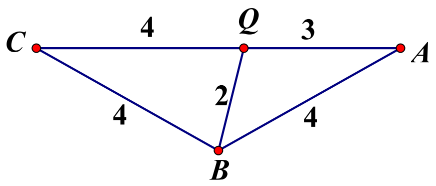
$\therefore$ We can construct another integral cyclic quadrilateral with a simpler formula, but the area of each smaller triangle inside (and hence the cyclic quadrilateral) are not integers.

Again, this is a **special case** for **two equal adjacent sides of integral cyclic quadrilateral and the angle between the two diagonals is 60°**.

Question: Given any triangle ΔABQ with integral sides, can we construct an integral cyclic quadrilateral, while the angle between the diagonals are not necessarily 60° using a similar method?

Let ΔABC be a 2-3-4 triangle with $AB = 4$, $BQ = 2$, $CQ = 3$.

We can construct (method on page 9) another triangle ΔBCQ (with common sides BQ) so that $BC = 4$, $CQ = 4$.



$$\cos \angle AQB = \frac{2^2 + 3^2 - 4^2}{2 \times 2 \times 3} = -\frac{1}{4}$$

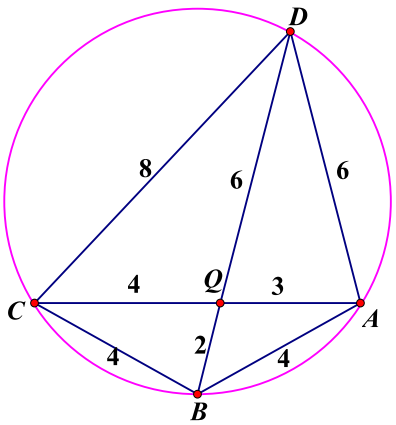
$$\cos \angle BQC = \frac{2^2 + 4^2 - 4^2}{2 \times 2 \times 4} = \frac{1}{4} = -\cos \angle AQB$$

$\angle AQB + \angle BQC = 180°$

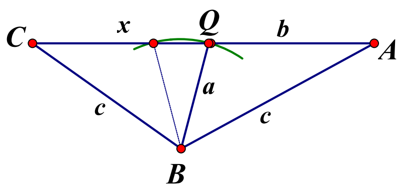
$\therefore$ A, Q, C are collinear.

Construct a circumscribed circle through A, B and C. Extend BQ to cut the circle again at D. Join AD and CD.

Using a similar method, we can prove that $DQ = 6$, $AD = 6$, $CD = 8$.

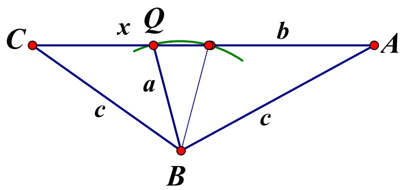


In this case, $AB = BC = 4$ (two equal adjacent sides, whereas the angle between the diagonals $\neq 60°$.)



Given any triangle ΔABQ with integral sides ($AB = c$, $BQ = a$, $AQ = b$). We can use the method on page 9 to find another triangle ΔBCQ with a common side BQ and A, Q, C are collinear. $BC = c$, $CQ = x$.

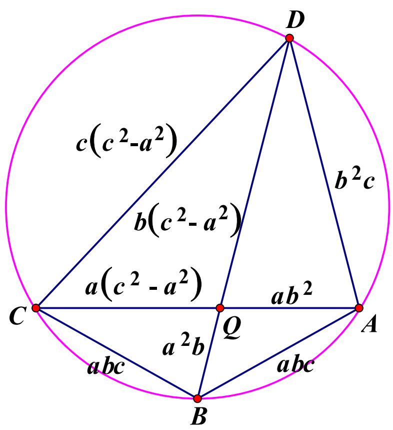
$$\cos \angle AQB = \frac{a^2 + b^2 - c^2}{2ab}, \ \cos \angle BQC = -\frac{a^2 + b^2 - c^2}{2ab}$$



If $\angle AQB$ is obtuse, then $x = b + 2a \cos \angle BQC = b - \dfrac{a^2 + b^2 - c^2}{b} = \dfrac{c^2 - a^2}{b}$.

If $\angle AQB$ is acute, then $x = b - 2a \cos \angle AQB = b - \dfrac{a^2 + b^2 - c^2}{b} = \dfrac{c^2 - a^2}{b}$.

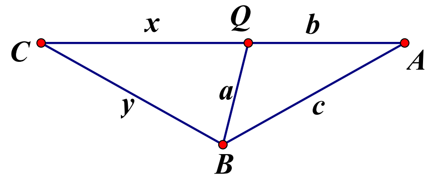
If $\angle AQB = 90°$, then $x = b = \dfrac{c^2 - a^2}{b}$.



Finally, a cyclic quadrilateral with integral sides is formed.

The only necessary condition is $c > a$ and a, b, c obey triangle inequality.

Let $CQ = x$, $BC = y$.



$$y^2 = x^2 + a^2 - 2ax \cos\angle BQC = x^2 + a^2 + 2ax \cos\angle AQB = x^2 + a^2 + 2ax\cdot \frac{a^2+b^2-c^2}{2ab}$$

$$y^2 = x^2 + 2ax + a^2 + \frac{x}{b}\left(a^2+b^2-c^2\right) - 2ax$$

$$(y + x + a)(y - x - a) = \frac{x}{b}\left(a^2-2ab+b^2-c^2\right) = \frac{x}{b}(a-b+c)(a-b-c)$$

$$\frac{y+x+a}{x(a-b+c)} = \frac{a-b-c}{y-x-a} = k$$

$$y + x + a = k(a - b + c)x \cdots\cdots (1),\quad y - x - a = \frac{a-b-c}{k} \cdots\cdots (2)$$

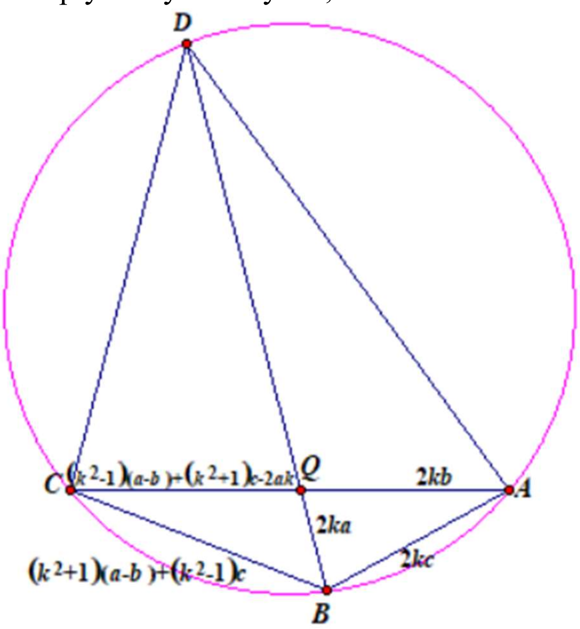
$$(1) + (2):\ 2y = k(a-b+c) + \frac{a-b-c}{k} = \frac{k^2(a-b+c)+a-b-c}{k}$$

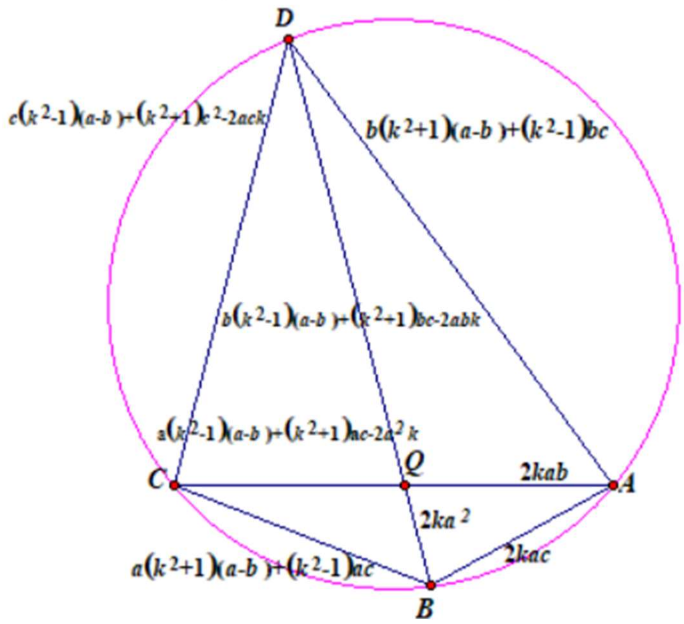
$$y = \frac{\left(k^2+1\right)a - \left(k^2+1\right)b + \left(k^2-1\right)c}{2k} = \frac{\left(k^2+1\right)(a-b)+\left(k^2-1\right)c}{2k}$$

$$(1) - (2):\ 2x + 2a = k(a-b+c) - \frac{a-b-c}{k}$$

$$x = \frac{k^2(a-b+c)-(a-b-c)-2ak}{2k} = \frac{\left(k^2-1\right)(a-b)+\left(k^2+1\right)c-2ak}{2k}$$

Multiply every side by $2k$, construct a circumscribed circle through ABC and extend BQ to cut the circle at D.





$$\frac{DQ}{\left(k^2-1\right)(a-b)+\left(k^2+1\right)c-2ak} = \frac{2kb}{2ka} = \frac{AD}{\left(k^2+1\right)(a-b)+\left(k^2-1\right)c} \quad \text{(corr. sides, } \Delta ADQ \sim \Delta BCQ\text{)}$$

$$DQ = \frac{b}{a}\left[\left(k^2-1\right)(a-b)+\left(k^2+1\right)c-2ak\right],\ AD = \frac{b}{a}\left[\left(k^2+1\right)(a-b)+\left(k^2-1\right)c\right]$$

$$\frac{DQ}{2kb} = \frac{CD}{2kc} \quad \text{(corr. sides, } \Delta CDQ \sim \Delta BAQ\text{)}$$

$$CD = \frac{c}{b}\cdot\frac{b}{a}\left[\left(k^2-1\right)(a-b)+\left(k^2+1\right)c-2ak\right] = \frac{c}{a}\left[\left(k^2-1\right)(a-b)+\left(k^2+1\right)c-2ak\right]$$

Multiply every side again by a to give an integral cyclic quadrilateral (provided that a, b, c and k are +ve integers.)

Reference:

1. Pythagorean Triple: https://twhung78.github.io/Number_Theory/Pythagorean_triple.pdf
2. Angle between two diagonals in a cyclic quadrilateral: https://twhung78.github.io/Geometry/6%20Circles/2%20Cyclic%20quadrilateral/Angle_diagonals_cyclic_quadrilateral.pdf
3. "Normal Trigrade and cyclic quadrilateral with integral sides and diagonals" from April, 1951 American Mathematical Monthly.
4. 120° triangle: https://twhung78.github.io/Number_Theory/120triangle.pdf