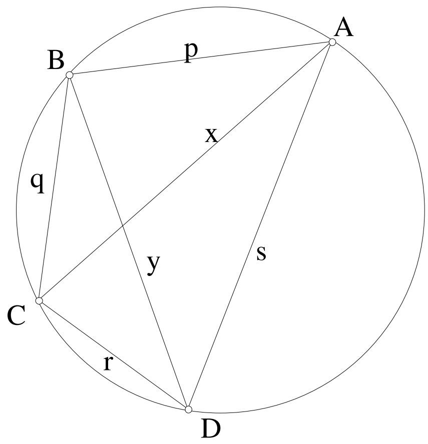
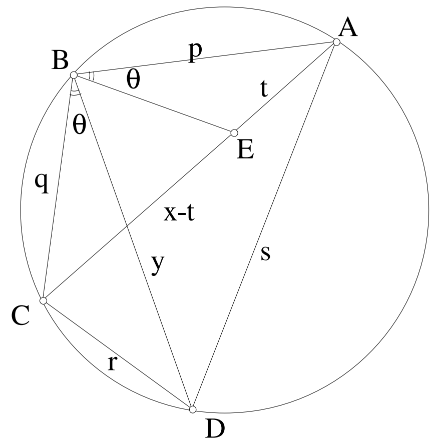
Ptolemy's Theorem First Proof

Created by Francis Hung Last updated: 03 September 2021





Let $ABCD$ be a cyclic quadrilateral. $AB = p$, $BC = q$, $CD = r$, $AD = s$, $AC = x$, $BD = y$, then $pr + qs = xy$.

Proof: Without loss of generality, let $\angle CBD \le \angle ABD$.

On AC locate a point E such that $\angle ABE = \angle CBD$.

Let $\angle ABE = \angle CBD = \theta$, let $AE = t$, then $CE = x - t$.

$\angle BDC = \angle BAE$ (∠s in the same segment)

$\therefore \Delta BCD \sim \Delta BEA$ (equiangular)

$\frac{t}{p} = \frac{r}{y}$ (cor. sides ~ Δs)

$\Rightarrow t\,y = pr \cdots\cdots (1)$

$\angle ABD = \theta + \angle EBD = \angle CBE$

$\angle BCE = \angle BDA$ (∠s in the same segment)

$\therefore \Delta BCE \sim \Delta BDA$ (equiangular)

$\frac{x-t}{q} = \frac{s}{y}$ (cor. sides ~ Δs)

$\Rightarrow xy - ty = qs \cdots\cdots (2)$

(1) + (2) $\Rightarrow xy = pr + qs$.

The proof is complete.

Ptolemy's Theorem Second Proof

HKAL Pure Mathematics 1957 Paper 1 Q6

Created by Mr. Francis Hung

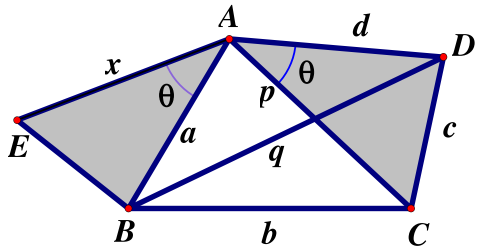
The lengths of sides and diagonals of quadrilateral $ABCD$ are $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = p$, $BD = q$.

If ABE is the triangle similar (and similarly oriented) to triangle ADC with AB and AD as corresponding sides, express EB in terms of a, c and d, and EC in terms of p, q and d. Hence prove

(i) that $pq \le ac + bd$, and

(ii) that if the equality sign holds, $ABCD$ is a cyclic quadrilateral, and conversely.

Deduce a theorem about an equilateral triangle by considering a cyclic quadrilateral $ABCD$ in which ABC is an equilateral triangle.

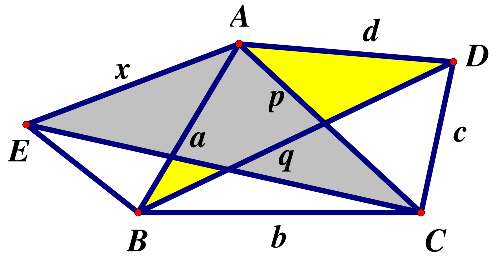


$\because \Delta ADC \sim \Delta ABE \therefore \dfrac{EB}{a} = \dfrac{c}{d} \Rightarrow EB = \dfrac{ac}{d}$

Let $\angle BAE = \theta = \angle CAD$ (corr. $\angle$s. ~ Δs), let $AE = x$

$\dfrac{x}{a} = \dfrac{p}{d}$ and $\angle EAC = \theta + \angle BAC = \angle BAD$

$\therefore \Delta EAC \sim \Delta BAD$ (ratio of 2 sides, included $\angle$)



$\dfrac{EC}{q} = \dfrac{p}{d}$ (cor. sides, ~Δs)

$EC = \dfrac{pq}{d}$

(i) In ΔBCE, $EB + BC \ge EC$ (triangle inequality)

$\dfrac{ac}{d} + b \ge \dfrac{pq}{d}$

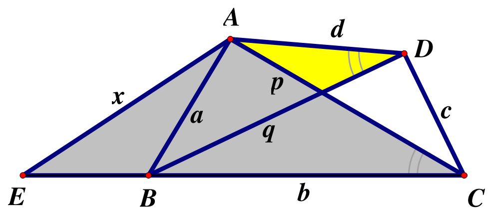
$ac + bd \ge pq$

(ii) If equality holds, then EBC is a straight line.

$\because \Delta EAC \sim \Delta BAD \therefore \angle ACE = \angle ADB$

$ABCD$ is a cyclic quadrilateral

(converse, $\angle$s in the same segment)



Converse, if $ABCD$ is a cyclic quadrilateral

$\angle ACB = \angle ADB$ ($\angle$s in the same segment)

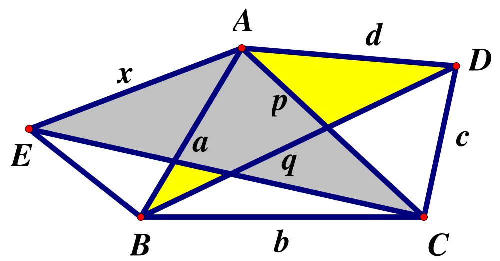
$\because \Delta EAC \sim \Delta BAD \therefore \angle ACE = \angle ADB$

$\therefore \angle ACB = \angle ADB = \angle ACE$

EBC is a straight line

$EB + BC = EC$

$ac + bd = pq$

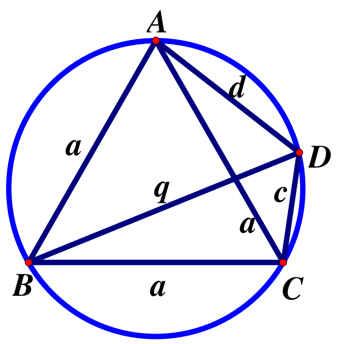


If ΔABC is an equilateral triangle, let $AB = BC = CA = a$

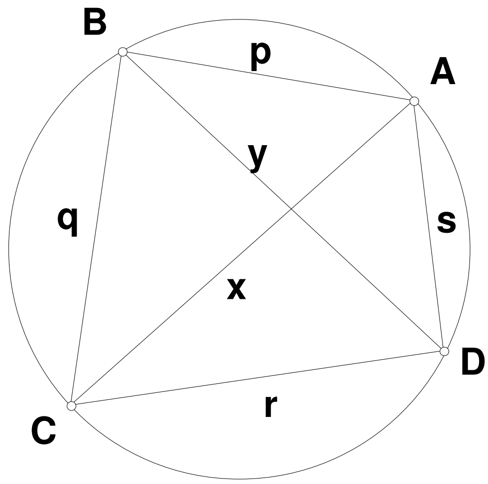
By the above result, $ac + ad = aq$

$c + d = q$

$\therefore BD = AD + CD$



Ptolemy's theorem Extension



In a circle, there is a cyclic quadrilateral *ABCD*.

Let $AB = p$, $BC = q$, $CD = r$, $AD = s$,

Let the diagonals $AC = x$, $BD = y$.

Then $pr + qs = xy$ and $x(pq + rs) = y(ps + qr)$

Proof: In ΔABC, $x^2 = p^2 + q^2 - 2\,pq\cos B$

In ΔADC, $x^2 = r^2 + s^2 - 2\,rs\cos D$

$\therefore x^2 = p^2 + q^2 - 2\,pq\cos B = r^2 + s^2 - 2\,rs\cos D$

$\because \angle B + \angle D = 180°$ $\quad \therefore \cos D = -\cos B$.

$p^2 + q^2 - 2\,pq\cos B = r^2 + s^2 + 2\,rs\cos B$

$$\cos B = \frac{p^2 + q^2 - r^2 - s^2}{2(pq + rs)}$$

$$x^2 = p^2 + q^2 - 2\,pq\,\frac{p^2 + q^2 - r^2 - s^2}{2(pq + rs)}$$

$$= \frac{p^3q + p^2rs + pq^3 + q^2rs - p^3q - pq^3 + pqr^2 + pqs^2}{pq + rs}$$

$$= \frac{pr(ps + qr) + qs(qr + ps)}{pq + rs}$$

$$x^2 = \frac{(ps + qr)(pr + qs)}{pq + rs} \quad \cdots\cdots (3)$$

In ΔABD, $y^2 = p^2 + s^2 - 2\,ps\cos A$

In ΔBCD, $y^2 = q^2 + r^2 - 2\,qr\cos C$

$\therefore y^2 = q^2 + r^2 - 2\,qr\cos C = p^2 + s^2 - 2\,ps\cos A$

$\because \angle A + \angle C = 180°$ $\quad \therefore \cos A = -\cos C$

$q^2 + r^2 - 2\,rq\cos C = p^2 + s^2 + 2\,ps\cos C$

$$\cos C = \frac{q^2 + r^2 - p^2 - s^2}{2(ps + qr)}$$

$$y^2 = q^2 + r^2 - 2\,qr\,\frac{q^2 + r^2 - p^2 - s^2}{2(ps + qr)}$$

$$= \frac{qr^3 + pr^2s + q^3r + pq^2s - q^3r - qr^3 + p^2qr + qrs^2}{ps + qr}$$

$$= \frac{pq(pr + qs) + rs(pr + qs)}{ps + qr}$$

$$y^2 = \frac{(pq + rs)(pr + qs)}{ps + qr} \quad \cdots\cdots (4)$$

$(3) \times (4)$

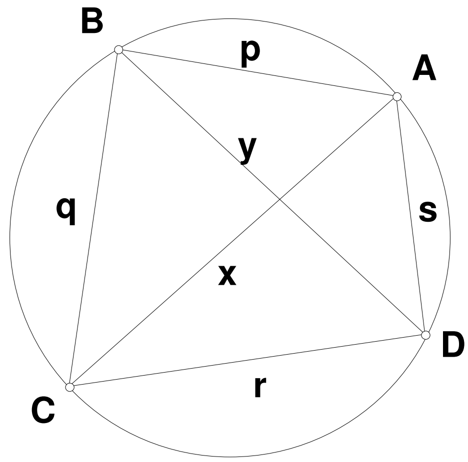
$$x^2\,y^2 = \frac{(ps + qr)(pr + qs)}{pq + rs} \cdot \frac{(pq + rs)(pr + qs)}{ps + qr}$$

$(xy)^2 = (pr + qs)^2$

$\therefore xy = pr + qs$

The theorem is proved.

Ptolemy's Theorem Extension



$x(pq+rs)=\sqrt{\frac{(ps+qr)(pr+qs)}{pq+rs}}\cdot(pq+rs)$ by (3) on page 3

$=\sqrt{(ps+qr)(pr+qs)(pq+rs)}$

$y(ps+qr)=\sqrt{\frac{(pq+rs)(pr+qs)}{ps+qr}}\cdot(ps+qr)$ by (4) on page 3

$=\sqrt{(ps+qr)(pr+qs)(pq+rs)}$

$\therefore x(pq+rs)=y(ps+qr)$

Converse of Ptolemy's theorem Extension

If $pr+qs=xy$ $\cdots\cdots$ (5) and $x(pq+rs)=y(ps+qr)$ $\cdots\cdots$ (6)

for positive quantities p, q, r, s, x and y, then they are the sides of a cyclic quadrilateral.

Proof: (6) implies that $\frac{x}{ps+qr}=\frac{y}{pq+rs}=k$ $\cdots\cdots$ (7), where k is a constant

$x=(ps+qr)k$, $y=(pq+rs)k$ $\cdots\cdots$ (8)

Sub. (8) into (5): $xy=(ps+qr)(pq+rs)k^2=pr+qs$

$k=\sqrt{\frac{(pr+qs)}{(ps+qr)(pq+rs)}}$ $\cdots\cdots$ (9)

Sub. (9) into (8): $x=(ps+qr)\sqrt{\frac{(pr+qs)}{(ps+qr)(pq+rs)}}=\sqrt{\frac{(ps+qr)(pr+qs)}{(pq+rs)}}$

Sub. (9) into (8): $y=(pq+rs)\sqrt{\frac{(pr+qs)}{(ps+qr)(pq+rs)}}=\sqrt{\frac{(ps+qr)(pq+rs)}{(pr+qs)}}$

$$\cos B=\frac{p^2+q^2-x^2}{2pq}=\frac{p^2+q^2-\frac{(ps+qr)(pr+qs)}{(pq+rs)}}{2pq}=\frac{(p^2+q^2)(pq+rs)-(ps+qr)(pr+qs)}{2pq(pq+rs)}$$

$$=\frac{p^3q+pq^3+p^2rs+q^2rs-(p^2rs+pqr^2+pqs^2+q^2rs)}{2pq(pq+rs)}$$

$$=\frac{p^3q+pq^3-pqr^2-pqs^2}{2pq(pq+rs)}=\frac{p^2+q^2-r^2-s^2}{2(pq+rs)}$$

$$\cos D=\frac{r^2+s^2-x^2}{2rs}=\frac{r^2+s^2-\frac{(ps+qr)(pr+qs)}{(pq+rs)}}{2rs}=\frac{(r^2+s^2)(pq+rs)-(ps+qr)(pr+qs)}{2rs(pq+rs)}$$

$$=\frac{pqr^2+pqs^2+r^3s+rs^3-(p^2rs+pqr^2+pqs^2+q^2rs)}{2rs(pq+rs)}$$

$$=\frac{r^3s+rs^3-p^2rs-q^2rs}{2rs(pq+rs)}=\frac{r^2+s^2-p^2-q^2}{2(pq+rs)}=-\cos B$$

$\therefore B+D=180°$

ABCD is a cyclic quadrilateral