**Individual Events**

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|---|---|---|---|---|---|---|---|---|---|---|---|
| **I1** | $A$ | 14 | **I2** | $A$ | 2 | **I3** | $A$ | 18 | **I4** | $A$ | 4 |
| | $B$ | 27 | | $B$ | 10 | | $B$ | 0 | | $B$ | $\frac{3}{4}$ |
| | $C$ | 116 | | $C$ | 40 | | $C$ | 6 | | $C$ | $\sqrt{52}$ |
| | $D$ | 4 | | $D$ | 144 | | $D$ | 2 | | $D$ | 64 |

**Group Events**

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| **G1** | $P$ | 8010 | **G2** | $a$ | $24\sqrt{7}$ | **G3** | $n$ | 14 | **G4** | $a$ | $-14$ |
| | $A$ | 6 | | $b$ | 527 | | $b-a$ | 10 | | $x$ | 6 |
| | numbers | 13696 | | $c$ | 4 | | gain | $\$\frac{7400}{3}=\$2466\frac{2}{3}$ | | equation | $x^2+90x+2=0$ |
| | $m$ | 1 | | $d$ | 2 | | Area | 1000 cm$^2$ | | Area | $4\sqrt{2}-\frac{3\pi}{2}$ see the remark |

**Individual Event 1**

**I1.1** 已知 $m$ 和 $n$ 均為正整數。如果 $m+n+mn=54$ 及 $A=m+n$，求 $A$ 的值。

Given that $m$ and $n$ are positive integers. If $m+n+mn=54$ and $A=m+n$, find the value of $A$.

**Reference: 2006 FG2.4, 2024 FG4.1**

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|---|---|
| $1+m+n+mn=55$<br>$(1+m)(1+n)=55$<br>$1+m=5, 1+n=11$ 或 $1+m=11, 1+n=5$<br>$m=4$，$n=10$ 或 $m=10$，$n=4$<br>$A=m+n=14$ | $1+m+n+mn=55$<br>$(1+m)(1+n)=55$<br>$1+m=5, 1+n=11$ or $1+m=11, 1+n=5$<br>$m=4, n=10$ or $m=10, n=4$<br>$A=m+n=14$ |

**I1.2** 若 $\mathrm{f}(a)=a-2$，$\mathrm{F}(a,b)=b^2+a+A$， $B=\mathrm{F}(4,\mathrm{f}(5))$，求 $B$ 的值。

If $\mathrm{f}(a)=a-2$ , $\mathrm{F}(a,b)=b^2+a+A$ and $B=\mathrm{F}(4,\mathrm{f}(5))$ , find the value of $B$

**Reference: 2023 FI4.2**

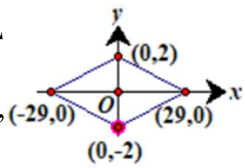
$\mathrm{f}(5)=5-2=3$

$F(4,\mathrm{f}(5))=F(4,3)=3^2+4+14=27$

$B=27$

**I1.3** 在 $x$-$y$ 座標平面上，由 $(B+2, 0)$、$(-B-2, 0)$、$(0, 2)$ 及 $(0, -2)$所形成之菱形的面積為 $C$ 平方單位，求 $C$ 的值。

The area of the rhombus on the $x$-$y$ coordinate plane with vertices $(B+2, 0)$, $(-B-2, 0)$, $(0, 2)$ and $(0, -2)$ is $C$ square units. Find the value of $C$ .




**Reference: 2023 FI3.3**

$$C=\frac{1}{2}\left[2-(-2)\right]\cdot\left[29-(-29)\right]$$
$$=116$$

**I1.4** 如果 $D$ 是正整數且 $\left(\frac{C}{4}+227\right)^{\frac{1}{D}}=D$，求 $D$ 的值。

If $D$ is a positive integer such that $\left(\frac{C}{4}+227\right)^{\frac{1}{D}}=D$, find the value of $D$ .

**Reference: 2023 FI3.4**

$\frac{116}{4}+227=D^D$

$D=4$

**Individual Event 2**

**I2.1** 若 $A$ 是 $2022^{2023^{2024}}$ 的個位數 $A$。

$A$ is the units digit of $2022^{2023^{2024}}$. Find the value of $A$.

| | |
|---|---|
| $2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 8, 2^4 \equiv 6 \pmod{10}$<br>這個數字的規律每隔 4 的倍數重複一次。<br>$2023 \equiv -1 \bmod 4, 2023^2 \equiv 1 \pmod 4$<br>這個數字的規律每隔 2 的倍數重複一次。<br>$2023^{2024} \equiv 1 \pmod 4$<br>$2022^{2023^{2024}} \equiv 2022^1 \equiv 2 \pmod{10}$<br>$A = 2$ | $2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 8, 2^4 \equiv 6 \pmod{10}$<br>This pattern repeats for every multiple of 4.<br>$2023 \equiv -1 \bmod 4, 2023^2 \equiv 1 \pmod 4$<br>This pattern repeats for every multiple of 2.<br>$2023^{2024} \equiv 1 \bmod 4$<br>$2022^{2023^{2024}} \equiv 2022^1 \equiv 2 \pmod{10}$<br>$A = 2$ |

**I2.2** 已知 $(x + 20)^2 + (y - 24)^2$ 的最小值是 $B$，當中 $x$ 和 $y$ 是方程 $19x + 13y = A$ 的整數解。求 $B$ 的值。

$B$ is the minimum value of $(x + 20)^2 + (y - 24)^2$, where $x$ and $y$ are integers that satisfy the equation $19x + 13y = A$. Find the value of $B$.

| | |
|---|---|
| $19x + 13y = 2$<br>$13\times 6 - 19\times 4 = 2$<br>$(-4, 6)$ 為一整數解<br>一般解：$x = -4 + 13k, y = 6 - 19k$<br>$(x + 20)^2 + (y - 24)^2$<br>$= (-4 + 13k + 20)^2 + (6 - 19k - 24)^2$<br>$= (13k + 16)^2 + (-19k - 18)^2$<br>$= 169k^2 + 416k + 256 + 361k^2 + 684k + 324$<br>$= 530k^2 + 1100k + 580$<br>$=530\left(k^2+\frac{110}{53}k+\frac{55^2}{53^2}\right)+580-\frac{55^2\times 10}{53}$<br>$=530\left(k+\frac{55}{53}\right)^2+\frac{490}{53}$<br>$k = 0$，算式$= 580$<br>$k = -1$, 算式$= 10$<br>$k = -2$, 算式$= 500$<br>$B = 10$ | $19x + 13y = 2$<br>$13\times 6 - 19\times 4 = 2$<br>$(-4, 6)$ is an integral solution<br>The general solutions: $x = -4 + 13k, y = 6 - 19k$<br>$(x + 20)^2 + (y - 24)^2$<br>$= (-4 + 13k + 20)^2 + (6 - 19k - 24)^2$<br>$= (13k + 16)^2 + (-19k - 18)^2$<br>$= 169k^2 + 416k + 256 + 361k^2 + 684k + 324$<br>$= 530k^2 + 1100k + 580$<br>$=530\left(k^2+\frac{110}{53}k+\frac{55^2}{53^2}\right)+580-\frac{55^2\times 10}{53}$<br>$=530\left(k+\frac{55}{53}\right)^2+\frac{490}{53}$<br>$k = 0$, expression $= 580$<br>$k = -1$, expression $= 10$<br>$k = -2$, expression $= 500$<br>$B = 10$ |

**I2.3** 在袋中有若干顆紅色和藍色的彈珠，它們的總數量是 $C$。如果加入 $B$ 顆紅色彈珠，紅色和藍色彈珠數量的比例則為 3∶2；如果加入 $B$ 顆藍色彈珠，紅色和藍色彈珠數量的比例則為 2∶3。求 $C$ 的值。

There is $C$ marbles in a bag, which are either red or blue. If we add $B$ red marbles to the bag, the ratio of red marbles to the blue marbles becomes 3 : 2. If we add $B$ blue marbles to the bag, the ratio of red marbles to the blue marbles becomes 2 : 3. Find the value of $C$.

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| 假設原本有 $x$ 顆紅色彈珠。<br>那麼原本有 $(C - x)$ 顆藍色彈珠。<br>$(x + 10) : (C - x) = 3 : 2 \cdots\cdots (1)$<br>$x : (C - x + 10) = 2 : 3 \cdots\cdots(2)$<br>由(1)式：$2x + 20 = 3C - 3x$<br>$5x = 3C - 20 \cdots\cdots(3)$<br>由(2)式：$3x = 2C - 2x + 20$<br>$5x = 2C + 20 \cdots\cdots(4)$<br>$(3) - (4)$: $C = 40$ | Suppose there are $x$ red marbles originally.<br>Then there are $(C - x)$ blue marbles originally.<br>$(x + 10) : (C - x) = 3 : 2 \cdots\cdots (1)$<br>$x : (C - x + 10) = 2 : 3 \cdots\cdots(2)$<br>From (1): $2x + 20 = 3C - 3x$<br>$5x = 3C - 20 \cdots\cdots(3)$<br>From (2): $3x = 2C - 2x + 20$<br>$5x = 2C + 20 \cdots\cdots(4)$<br>$(3) - (4)$: $C = 40$ |

**I2.4** 若 $5\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)=C$ ，求 $D$ 的值。

If $5\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)=C$, find the value of $D$.

$5\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)=40$

$\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)^2=64$

$25+2\sqrt{D}+2\sqrt{625-4D}+25-2\sqrt{D}=64$

$2\sqrt{625-4D}=14$

$625-4D=49$

$D=144$

**Individual Event 3**

**I3.1** 若 $x$ 和 $y$ 為滿足方程 $\frac{1}{x}+\frac{1}{y}=\frac{2}{5}$ 的不同正整數。求 $A=x+y$ 的值。

If $x$ and $y$ are two different positive integers such that $\frac{1}{x}+\frac{1}{y}=\frac{2}{5}$, find the value of $A=x+y$.

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| $\frac{y+x}{xy}=\frac{2}{5}$<br>$2xy=5x+5y$<br>$4xy-10x-10y+25=25$<br>$(2x-5)(2y-5)=25$<br>$(2x-5, 2y-5)=(1, 25), (5, 5)$ 或 $(25, 1)$<br>$\because$ $x$、$y$ 為不同正整數<br>$\therefore (x, y)=(3, 15)$ 或 $(15, 3)$<br>$A=x+y=18$ | $\frac{y+x}{xy}=\frac{2}{5}$<br>$2xy=5x+5y$<br>$4xy-10x-10y+25=25$<br>$(2x-5)(2y-5)=25$<br>$(2x-5, 2y-5)=(1, 25), (5, 5)$ or $(25, 1)$<br>$\because$ $x, y$ are different positive integers<br>$\therefore (x, y)=(3, 15)$ or $(15, 3)$<br>$A=x+y=18$ |

**I3.2** 若 $B$ 是所有正整數 $N$ 使得 7 整除 $2^N+(19-A)$ 的數量，求 $B$ 的值。

If $B$ is the number of positive integers $N$ such that $2^N+(19-A)$ is divisible by 7, find the value of $B$.

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| $2^N+(19-A)=2^N+(19-18)=2^N+1$<br>$2^1+1\equiv 3, 2^2+1\equiv 5, 2^3+1\equiv 2 \pmod 7$<br>這個數字的規律每隔 3 的倍數重複一次。<br>$\therefore$對於每一個正整數 $N$，$2^N+1$ 不能被 7 整除<br>$B=0$ | $2^N+(19-A)=2^N+(19-18)=2^N+1$<br>$2^1+1\equiv 3, 2^2+1\equiv 5, 2^3+1\equiv 2 \pmod 7$<br>This pattern repeats for every multiple of 3.<br>$\therefore$ $2^N+1$ cannot be divisible by 7 for every positive integer $N$<br>$B=0$ |

**I3.3** 已知 $a$ 和 $b$ 為滿足方程組 $a^2-b^2=9$ 及 $ab=3$ 的實數。

若對於正整數 α 和 $C$，$a+b=\sqrt{\sqrt{\alpha}+C-B}$，求 $C$ 的值。

Given that $a$ and $b$ are real numbers such that $a^2-b^2=9$ and $ab=3$.

If $a+b=\sqrt{\sqrt{\alpha}+C-B}$ for positive integers α and $C$, find the value of $C$.

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| 代 $b=\frac{3}{a}$ 入 $a^2-b^2=9$<br>$a^2-\left(\frac{3}{a}\right)^2=9 \Rightarrow a^4-9a^2-9=0$<br>$\because$ $a$ 為實數 $\therefore a^2\ge 0\Rightarrow a^2=\frac{9+\sqrt{117}}{2}$<br>$b^2=\frac{9}{a^2}=\frac{9\times 2}{9+\sqrt{117}}\cdot\frac{\sqrt{117}-9}{\sqrt{117}-9}=\frac{\sqrt{117}-9}{2}$<br>$(a+b)^2=\sqrt{\alpha}+C$ (已知)<br>$a^2+2ab+b^2=\sqrt{\alpha}+C$<br>$\frac{9+\sqrt{117}}{2}+\frac{\sqrt{117}-9}{2}+6=\sqrt{\alpha}+C$<br>$\sqrt{117}+6=\sqrt{\alpha}+C$<br>$\because \sqrt{117}$ 為無理數，6 及 $C$ 為有理數<br>$\therefore C=6$ | Sub. $b=\frac{3}{a}$ into $a^2-b^2=9$<br>$a^2-\left(\frac{3}{a}\right)^2=9 \Rightarrow a^4-9a^2-9=0$<br>$\because$ $a$ is real $\therefore a^2\ge 0\Rightarrow a^2=\frac{9+\sqrt{117}}{2}$<br>$b^2=\frac{9}{a^2}=\frac{9\times 2}{9+\sqrt{117}}\cdot\frac{\sqrt{117}-9}{\sqrt{117}-9}=\frac{\sqrt{117}-9}{2}$<br>$(a+b)^2=\sqrt{\alpha}+C$ (given)<br>$a^2+2ab+b^2=\sqrt{\alpha}+C$<br>$\frac{9+\sqrt{117}}{2}+\frac{\sqrt{117}-9}{2}+6=\sqrt{\alpha}+C$<br>$\sqrt{117}+6=\sqrt{\alpha}+C$<br>$\because \sqrt{117}$ is irrational, 6 and $C$ are rational<br>$\therefore C=6$ |

**I3.4** 若 $x$ 為滿足方程 $(\log_a x)^{\log_a x} = x$ 的實數，其中 $a$ 是常數且 $a > 1$。求 $D = \dfrac{C \log_a x}{3a}$ 的值。

If $x$ is real root of the equation $(\log_a x)^{\log_a x} = x$, where $a$ is a constant and $a > 1$, find the value of $D = \dfrac{C \log_a x}{3a}$.

$$(\log_a x)^{\log_a x} = x$$

$$\log_a \left[(\log_a x)^{\log_a x}\right] = \log_a x$$

$$\log_a x \log_a (\log_a x) = \log_a x$$

$$\log_a (\log_a x) = 1$$

$$\log_a x = a$$

$$x = a^a$$

$$D = \frac{C \log_a x}{3a} = \frac{6 \log_a a^a}{3a}$$

$$= \frac{2a \log_a a}{a} = 2$$

**Individual Event 4**

**I4.1** 如果 $A > 1$ 且 $1+\frac{1}{A}+\frac{1}{A^2}+\frac{1}{A^3}+\cdots=\frac{A}{3}$，求 $A$ 的值。

If $A > 1$ and $1+\frac{1}{A}+\frac{1}{A^2}+\frac{1}{A^3}+\cdots=\frac{A}{3}$, find the value of $A$.

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| 以上是一等比數列的無限項之和，<br>公比$=\frac{1}{A}$， 其中 $0<\frac{1}{A}<1$。<br>$\frac{1}{1-\frac{1}{A}}=\frac{A}{3}$<br>$3A = A(A-1)$<br>$A^2-4A=0$<br>$A = 4$ | The above is an infinite geometric series with common ratio $=\frac{1}{A}$, where $0<\frac{1}{A}<1$.<br>$\frac{1}{1-\frac{1}{A}}=\frac{A}{3}$<br>$3A = A(A-1)$<br>$A^2-4A=0$<br>$A = 4$ |

**I4.2** 如果 $\frac{1}{A}$ 是二次方程 $x^2-Bx+\frac{1}{6}B=0$ 的一個根，求 $B$ 的值。

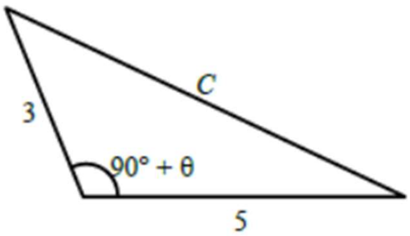
If $\frac{1}{A}$ is a root of the quadratic equation $x^2-Bx+\frac{1}{6}B=0$, find the value of $B$.

$$\left(\frac{1}{4}\right)^2-\frac{B}{4}+\frac{1}{6}B=0$$

$$B=\frac{3}{4}$$

**I4.3** 考慮右圖中的三角形，如果 $\tan\theta=B$，其中 $0°<\theta<90°$，求 $C$ 的值。

Consider the triangle in the figure on the right.
If $\tan\theta = B$, where $0° < \theta < 90°$, find the value of $C$.




$\tan\theta=\frac{3}{4}$, $\sin\theta=\frac{3}{5}$, $\cos\theta=\frac{4}{5}$

$$C^2 = 3^2 + 5^2 - 2\times3\times5\times\cos(90° + \theta)$$

$$C^2 = 34 + 30\sin\theta = 34 + 30\times\frac{3}{5} = 52$$

$$C=\sqrt{52}$$

**I4.4** 設 $d = C^2 - 20$，如果 $D$ 滿足方程 $8^D = D^d$，求 $D$ 的值。

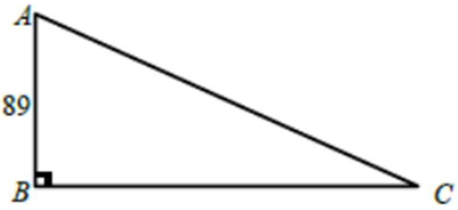
Let $d = C^2 - 20$. If $D$ satisfies the equation $8^D = D^d$, the value of $D$.

| | |
|---|---|
| $d = 52 - 20 = 32$<br>$8^D = D^{32}$<br>$2^{3D} = D^{32}$ …… (1)<br>設 $D = 2^x$<br>比較(1)式的指數：$3\times2^x = 32x$ …… (2)<br>利用嘗試錯誤法，代 $x=3$ 入(2)式：LHS ≠ RHS<br>代 $x = 6$ 入(2)式：LHS = RHS<br>$x = 6$<br>$D = 2^6 = 64$ | $d = 52 - 20 = 32$<br>$8^D = D^{32}$<br>$2^{3D} = D^{32}$ …… (1)<br>Let $D = 2^x$<br>Compare the indices of (1): $3\times2^x = 32x$ …… (2)<br>By trial and error, put $x = 3$ in (2): LHS ≠ RHS<br>Put $x = 6$ in (2): LHS = RHS<br>$x = 6$<br>$D = 2^6 = 64$ |

**Group Event 1**

**G1.1** 若直角三角形 $ABC$ 所有邊長均為正整數，且 $AB = 89$，求三角形 $ABC$ 的周界 $P$。

Find the perimeter $P$ of the right-angled triangle $ABC$ if all the side lengths are positive integers and $AB = 89$.

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|---|---|
| 設 $BC = a$, $AC = b$<br>$a^2 + 89^2 = b^2$<br>$(b + a)(b - a) = 89^2$<br>$\because$ 89 為質數及 $b > a$ 皆為正整數<br>$\therefore (b + a, b - a) = (7921, 1)$<br>$b = 3961$, $a = 3960$<br>$P = 3961 + 3960 + 89 = 8010$ | Let $BC = a$, $AC = b$<br>$a^2 + 89^2 = b^2$<br>$(b + a)(b - a) = 89^2$<br>$\because$ 89 is a prime and $b > a$ are positive integers<br>$\therefore (b + a, b - a) = (7921, 1)$<br>$b = 3961$, $a = 3960$<br>$P = 3961 + 3960 + 89 = 8010$ |

**G1.2** 若 $A$ 是 $8888^{20242024}$ 的個位數。求 $A$ 的值。

If $A$ is the units digit of $8888^{20242024}$. Find the value of $A$.

| | |
|---|---|
| $8^1 \equiv 8$, $8^2 \equiv 4$, $8^3 \equiv 2$, $8^4 \equiv 6 \pmod{10}$<br>這個數字的規律每隔 4 的倍數重複一次。<br>$20242024 \equiv 0 \pmod 4$<br>$8888^{20242024} \equiv 8^4 \equiv 6 \pmod{10}$<br>$A = 6$ | $8^1 \equiv 8$, $8^2 \equiv 4$, $8^3 \equiv 2$, $8^4 \equiv 6 \pmod{10}$<br>This pattern repeats for every multiple of 4.<br>$20242024 \equiv 0 \pmod 4$<br>$8888^{20242024} \equiv 8^4 \equiv 6 \pmod{10}$<br>$A = 6$ |

**G1.3** 有多少個 5 位數包含最少 1 個「1」和最少 1 個「3」？

How many 5-digit numbers contain at least one "1" and at least one "3"?

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| **方法一** 5 位數一共有：$9\times10^4 = 90000$ 個<br>沒有'1'和沒有'3'的 5 位數，<br>一共有 $7\times8^4 = 28672$ 個<br>只有一個'1' 和沒有'3'的 5 位數，<br>一共有 $1\times8^4 + 7\times4\times8^3 = 18432$ 個<br>只有一個'3'和沒有'1'的 5 位數，有 18432 個<br>只有兩個'1' 和沒有'3'的 5 位數，<br>一共有 $= 4\times8^3 + 7\times C_2^4 \times8^2 = 4736$ 個<br>只有兩個'3'和沒有'1'的 5 位數，有 4736 個<br>只有三個'1' 和沒有'3'的 5 位數，<br>一共有 $= 7\times4\times8 + C_2^4 \times8^2 = 608$ 個<br>只有三個'3' 和沒有'1'的 5 位數，有 608 個<br>四個'1'和沒有'3'的 5 位數，有 $7+4\times8 = 39$ 個<br>四個'3'和沒有'1'的 5 位數，有 39 個<br>'11111'有 1 個；'33333'有 1 個<br>總數 $= 90000 - 28672 - (18432+4736+608+39+1)\times2$<br>$= 13696$ 個<br>**方法二** 只有一個'1'和一個'3'的 5 位數，<br>一共有 $= 4\times8^3\times2 + 7\times P_2^4 \times8^2 = 9472$ 個<br>兩個'1' 和一個'3'的 5 位數，<br>共有 $= 7\times8\times C_2^4 \times2 + P_2^4 \times8^2 + C_2^4 \times8^2 = 1824$ 個<br>兩個'3'和一個'1'的 5 位數，有 1824 個<br>兩個'1' 和兩個'3'的 5 位數，<br>共有 $= 7\times C_2^4 + C_2^4 \times2\times8 + C_2^4 \times2\times8 = 234$ 個 | **Method 1** No. of 5-digit numbers $= 9\times10^4 = 90000$<br>If there are no '1' and no '3',<br>numbers $= 7\times8^4 = 28672$<br>If there is only one '1' but no '3',<br>numbers $= 1\times8^4 + 7\times4\times8^3 = 18432$<br>If there is only one '3' but no '1', numbers $= 18432$<br>If there are 2'1's but no '3',<br>numbers $= 4\times8^3 + 7\times C_2^4 \times8^2 = 4736$<br>If there are 2'3's but no '1', numbers $= 4736$<br>If there are 3'1's but no '3',<br>numbers $= 7\times4\times8 + C_2^4 \times8^2 = 608$<br>If there are 3'3's but no '1', numbers $= 608$<br>If there are 4'1's but no '3', numbers $= 7+4\times8 = 39$<br>If there are 4'3's but no '1', numbers $= 39$<br>'11111', number $= 1$; '33333', number $= 1$<br>Total $= 90000 - 28672 - (18432+4736+608+39+1)\times2$<br>$= 13696$<br>**Method 2** If there is only one '1' and one '3',<br>numbers $= 4\times8^3\times2 + 7\times P_2^4 \times8^2 = 9472$<br>If there is 2'1's but only one '3',<br>numbers $= 7\times8\times C_2^4 \times2 + P_2^4 \times8^2 + C_2^4 \times8^2 = 1824$<br>If there is 2'3's but only one '1', numbers $= 1824$<br>If there are 2'3's and 2'1's,<br>numbers $= 7\times C_2^4 + C_2^4 \times2\times8 + C_2^4 \times2\times8 = 234$<br>If there are 3'1's and one '3', |

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| 三個'1'和一個'3'的 5 位數，<br>共有$=C_2^4\times2\times8+4\times8+7\times4=156$ 個<br>三個'3'和一個'1'的 5 位數，有 156 個<br>三個'1'和兩個'3'的 5 位數，共有$=C_2^4+4=10$ 個<br>三個'3'和兩個'1'的 5 位數，有 10 個<br>四個'1'和一個'3'的 5 位數，共有$=1+4=5$ 個<br>四個'3'和一個'1'的 5 位數，有 5 個<br>總數$=9472+1824\times2+156\times2+234+30=13696$ 個 | numbers $=C_2^4\times2\times8+4\times8+7\times4=156$<br>If there are 3'3's and one '1', numbers = 156<br>If there are 3'1's and 2'3's, numbers $=C_2^4+4=10$<br>If there are 3'3's and 2'1's, numbers = 10<br>If there are 4'1's and 1'3', numbers $=1+4=5$<br>If there are 4'3's and 1'1', numbers = 5<br>Total $=9472+1824\times2+156\times2+234+30=13696$ |

**G1.4** 設有 $m$ 對正整數 $a$ 和 $b$，使 $a^4+4b^4$ 為質數，求 $m$ 的值。

Let $m$ be the number of possible pairs of positive integers $a$ and $b$ for which $a^4+4b^4$ is a prime number. Find the value of $m$ .

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| $a^4+4b^4=a^4+4a^2b^2+4b^4-4a^2b^2$<br>$=(a^2+2b^2)^2-(2ab)^2$<br>$=(a^2+2ab+2b^2)(a^2-2ab+2b^2)$<br>$a^2+2ab+2b^2=p$，一個質數及<br>$a^2-2ab+2b^2=1$<br>$(a-b)^2+b^2=1$<br>$a=b$ 及 $b=1$ 或 $a-b=1$ 及 $b=0$<br>$(a,b)=(1,1)$ 或 $(1,0)$<br>代以上答案入 $a^2+2ab+2b^2=p$<br>$1+2+2=p$ (接受)<br>$1+0+0=p$ 不是質數 (捨去)<br>$\therefore (a,b)=(1,1)$<br>$m=1$ | $a^4+4b^4=a^4+4a^2b^2+4b^4-4a^2b^2$<br>$=(a^2+2b^2)^2-(2ab)^2$<br>$=(a^2+2ab+2b^2)(a^2-2ab+2b^2)$<br>$a^2+2ab+2b^2=p$ , a prime number and<br>$a^2-2ab+2b^2=1$<br>$(a-b)^2+b^2=1$<br>Either $a=b$ and $b=1$ or $a-b=1$ and $b=0$<br>$(a,b)=(1,1)$ or $(1,0)$<br>Sub. the solutions into $a^2+2ab+2b^2=p$<br>$1+2+2=p$ (accepted)<br>$1+0+0=p$ , not a prime, (rejected)<br>$\therefore (a,b)=(1,1)$<br>$m=1$ |

**Group Event 2**

**G2.1** 設 $x>0$。已知 $x-\frac{1}{x}=\sqrt{3}$ 且 $a=x^5+x^3+x+\frac{1}{x}+\frac{1}{x^3}+\frac{1}{x^5}$，求 $a$ 的值。

Let $x>0$. Given that $x-\frac{1}{x}=\sqrt{3}$ and $a=x^5+x^3+x+\frac{1}{x}+\frac{1}{x^3}+\frac{1}{x^5}$, find the value of $a$ .

$$x-\frac{1}{x}=\sqrt{3}\Rightarrow x^2-2+\frac{1}{x^2}=3$$

$$x^2+2+\frac{1}{x^2}=7\Rightarrow\left(x+\frac{1}{x}\right)^2=7$$

$$x+\frac{1}{x}=\sqrt{7}$$

$$x^2+\frac{1}{x^2}=5$$

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)=\sqrt{7}(5-1)=4\sqrt{7}$$

$$\left(x^2+\frac{1}{x^2}\right)\left(x^3+\frac{1}{x^3}\right)=(5)\left(4\sqrt{7}\right)\Rightarrow x^5+\frac{1}{x^5}+x+\frac{1}{x}=20\sqrt{7}\Rightarrow x^5+\frac{1}{x^5}=19\sqrt{7}$$

$$a=\left(x^5+\frac{1}{x^5}\right)+\left(x^3+\frac{1}{x^3}\right)+\left(x+\frac{1}{x}\right)=19\sqrt{7}+4\sqrt{7}+\sqrt{7}=24\sqrt{7}$$

**G2.2** 用首2024個正整數：1、2、3、4、5、6、…、2024，造出一個新整數：123456789101112…2024。若$b$是這個整數裡「0」的數量，求 $b$ 的值。

Using the first 2024 positive integers: 1, 2, 3, 4, 5, 6, ⋯ , 2024, a new integer is formed as 123456789101112⋯2024. If $b$ is the number of “0” in this integer, find the value of $b$ .

| | ‘0’ 的數目 | Number of ‘0’s |
|---|---|---|
| 1, 2, ⋯ , 9, 10, 11, ⋯ , 99 | 9 | 9 |
| 100, 211, ⋯ , 999 | (11 + 9)×9 | (11 + 9)×9 |
| 1000, 1001, ⋯ , 1009 | 21 | 21 |
| 1010, 1011, ⋯ , 1099 | 90 + 9 | 90 + 9 |
| 1100, 1101, ⋯ , 1999 | (11 + 9)×9 | (11 + 9)×9 |
| 2000, 2001, ⋯ , 2009 | 21 | 21 |
| 2010, 2011, ⋯ , 2019 | 11 | 11 |
| 2020, 2021, 2022, 2023, 2024 | 6 | 6 |

‘0’ 的總數 Total number of ‘0’= 9 + (11 + 9)×9 + 21 + 90 + 9 + (11 + 9)×9 + 21 + 11 + 6 = 527

**G2.3** $c$ 是 $2024^2-2023^2$ 的正因數的數量。求 $c$ 的值。

$c$ is the number of positive factors of $2024^2-2023^2$ .

| | |
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| $2024^2-2023^2=(2024+2023)(2024-2023)$<br>$=4047=3\times1349$，1349為質數<br>正因數為 1、3、1349、4047。<br>$c=4$ | $2024^2-2023^2=(2024+2023)(2024-2023)$<br>$=4047=3\times1349$, 1349 is a prime<br>The positive factors are 1, 3, 1349, 4047.<br>$c=4$ |

**G2.4** 假設「0」、「1」、「2」、…及「6」分別為星期日、星期一、星期二、…和星期六，今日是星期一，若 $20^{24^{2024}}$ 天後的那一天是星期幾之代號為「$d$」，求 $d$ 的值。

Let "0", "1", "2", $\cdots$ and "6" represent Sunday, Monday, Tuesday, $\cdots$ and Saturday respectively. Today is Monday. If "$d$" represents the day of week that comes after $20^{24^{2024}}$ days. Find the value of $d$.

| | |
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| $20 = 7\times2 + 6 \Rightarrow 20 \equiv -1 \pmod 7$<br>$20^2 \equiv (-1)^2 \equiv 1 \pmod 7$<br>這個數字的規律每隔 2 的倍數重複一次。<br>$24^{2024}$ 是一個偶數。<br>$20^{24^{2024}} \equiv 1 \pmod 7$<br>$20^{24^{2024}}$ 日後為星期二。<br>$d = 2$ | $20 = 7\times2 + 6 \Rightarrow 20 \equiv -1 \pmod 7$<br>$20^2 \equiv (-1)^2 \equiv 1 \pmod 7$<br>This pattern repeats for every multiple of 2.<br>$24^{2024}$ is an even number.<br>$20^{24^{2024}} \equiv 1 \pmod 7$<br>The day after $20^{24^{2024}}$ days is a Tuesday.<br>$d = 2$ |

**Group Event 3**

**G3.1** 試找出最小的正整數 $n$ 使得 $2^{10}+2^{13}+2^n$ 成為一個完全平方數。

Find the smallest positive integer $n$ such that $2^{10}+2^{13}+2^n$ is a perfect square number.

$2^{10}+2^{13}+2^n=m^2$

$2^{10}(1+8)+2^n=m^2$

$2^n=m^2-(2^5\times 3)^2$

$2^a\times 2^b=(m+96)(m-96)$

$m+96=2^a$ , $m-96=2^b$; $a, b\in\mathbb{Z}^+$

$192=2^a-2^b$ , $a>b\in\mathbb{Z}^+$

$2^6\times 3=2^b(2^{a-b}-1)$

$b=6$, $2^{a-b}-1=3\Rightarrow 2^{a-b}=4\Rightarrow a=8$

$2^n=2^8\times 2^6=2^{14}$

$n=14$

**G3.2** 設 $a^2+b^2+6a-14b+58=0$。求 $b-a$ 的值。

Suppose $a^2+b^2+6a-14b+58=0$ . Find the value of $b-a$ .

$a^2+2a(3)+3^2+b^2-2b(7)+7^2=0$

$(a+3)^2+(b-7)^2=0$

$a=-3$, $b=7$

$b-a=7-(-3)=10$

**G3.3** 在正方形土地的某一個角落裡埋著一個裝有\$8,000 的箱子。在一次比賽中，你和另一個叫「倒霉先生」的人一起挖箱子。倒霉先生有一個特點：他總是做出錯誤的選擇。你贏了擲骰子先選。你選了一個角落，倒霉先生選了另一個角落。在你準備開始時，你發現倒霉先生沒有找到箱子。遊戲規則允許你換另一個角落，但要罰\$200。計算換角落的期望收益。

There was a chest containing \$8,000 buried in one of the corners of a square piece of land. In a contest, you and another man called "Mr. Badluck" were digging for the chest. Mr. Badluck had one peculiarity: he always made the wrong choice. You won the toss and chose first. You picked a corner, and Mr. Badluck picked another. Before you started, you observed that Mr. Badluck found no chest. The rules of the game allowed you to make a switch to another corner, but with a penalty of \$200. Calculate the expected gain from making the switch in dollars.

| | |
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| 將正方形的四角命名為 $A$、$B$、$C$ 及 $D$。<br>若不轉換：$\mathrm{P}(A)=\mathrm{P}(B)=\mathrm{P}(C)=\mathrm{P}(D)=\frac{1}{4}$<br>期望收益= $\$\frac{1}{4}\times 8000=\$2000$<br>假設你已選擇了 $A$ 及倒霉先生選擇了 $B$；而 $B$ 沒有箱子。如果你轉換，那麼轉換的條件概率為 $\mathrm{P}(C\mid B$沒有箱子$)=\mathrm{P}(D\mid B$沒有箱子$)=\frac{1}{3}$<br>期望收益=$\$(\frac{1}{3}\times 8000-200)=\$\frac{7400}{3}$. | Label the four corners as $A$, $B$, $C$ and $D$.<br>Without switching, $\mathrm{P}(A)=\mathrm{P}(B)=\mathrm{P}(C)=\mathrm{P}(D)=\frac{1}{4}$<br>Expected gain == $\$\frac{1}{4}\times 8000=\$2000$<br>Suppose you had already chosen $A$ and Mr. Badluck had chosen $B$; while $B$ doesn't contain the chest. If you switch, then the conditional probability $\mathrm{P}(C\mid B$ is not$)=\mathrm{P}(D\mid B$ is not$)=\frac{1}{3}$<br>Expected gain =$\$(\frac{1}{3}\times 8000-200)=\$\frac{7400}{3}$. |

**G3.4** 一個凸六邊形有以下性質：

(i) 由任意頂點與相鄰兩個頂點組成的三角形的面積都是1,000 $cm^2$；及

(ii) $CH = DI$。

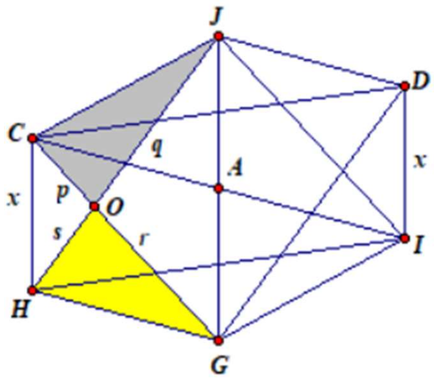
求六邊形的面積。

A convex hexagon has the following property:

(i) all the triangles formed from any vertex with the two adjacent vertices have an area of 1,000 $cm^2$ ; and

(ii) $CH = DI$.

Find the area of the hexagon.




| | |
|---|---|
| 假設$CG$與$JH$相交於$O$，$CI$與$DH$相交於$A$。 | Suppose $CG$ meets $JH$ at $O$, $CI$ meets $DH$ at $A$. |
| 假設$OC = p$、$OJ = q$、$OG = r$、$OH = s$。 | Let $OC = p$, $OJ = q$, $OG = r$, $OH = s$ . |
| $S_{\Delta CHI} = S_{\Delta CHG} \Rightarrow S_{\Delta OCH} + S_{\Delta OCJ} = S_{\Delta OCH} + S_{\Delta OHG}$ | $S_{\Delta CHI} = S_{\Delta CHG} \Rightarrow S_{\Delta OCH} + S_{\Delta OCJ} = S_{\Delta OCH} + S_{\Delta OHG}$ |
| $S_{\Delta OCJ} = S_{\Delta OHG} \Rightarrow \frac{1}{2}\cdot pq\sin\angle COJ = \frac{1}{2}\cdot rs\sin\angle GOH$ | $S_{\Delta OCJ} = S_{\Delta OHG} \Rightarrow \frac{1}{2}\cdot pq\sin\angle COJ = \frac{1}{2}\cdot rs\sin\angle GOH$ |
| $\because \angle COJ = \angle GOH$ (對頂角) | $\because \angle COJ = \angle GOH$ (vert. opp. $\angle$s) |
| $\therefore pq = rs \Rightarrow \frac{OJ}{OH} = \frac{OC}{OG}$ | $\therefore pq = rs \Rightarrow \frac{OJ}{OH} = \frac{OC}{OG}$ |
| $\angle JOG = \angle COH$ (對頂角) | $\angle JOG = \angle COH$ (vert. opp. $\angle$s) |
| $\Delta COH \sim \Delta GOJ$ (兩邊成比例，一夾角相等) | $\Delta COH \sim \Delta GOJ$ (ratio of 2 sides, inc. $\angle$) |
| $\angle CHO = \angle GJO$ (相似三角形對應角) | $\angle CHO = \angle GJO$ (corr. $\angle$s, $\sim\Delta$s) |
| $\therefore CH \,//\, JG$ (交錯角相等) | $\therefore CH \,//\, JG$ (alt. $\angle$s eq.) |
| $\therefore CH \,//\, DI \,//\, JG$, $CJ \,//\, HD \,//\, GI$, $JD \,//\, CI \,//\, HG$ | $\therefore CH \,//\, DI \,//\, JG$, $CJ \,//\, HD \,//\, GI$, $JD \,//\, CI \,//\, HG$ |
| $CDHI$是一個平行四邊形(對邊相等且平行) | $CDHI$ is a //-gram (opp. sides eq. and //) |
| $ACJD$是一個平行四邊形(由兩組平行邊組成) | $ACJD$ is a //-gram (formed by 2 pairs of // lines) |
| $AHGI$是一個平行四邊形(由兩組平行邊組成) | $AHGI$ is a //-gram (formed by 2 pairs of // lines) |
| $S_{\Delta AHI} = S_{\Delta GHI} = 1000$ $cm^2$ | $S_{\Delta AHI} = S_{\Delta GHI} = 1000$ $cm^2$ |
| $S_{\Delta ACD} = S_{\Delta JCD} = 1000$ $cm^2$ | $S_{\Delta ACD} = S_{\Delta JCD} = 1000$ $cm^2$ |
| $ACHG$是一個平行四邊形(由兩組平行邊組成) | $ACHG$ is a //-gram (formed by 2 pairs of // lines) |
| $ADGI$是一個平行四邊形(由兩組平行邊組成) | $ADGI$ is a //-gram (formed by 2 pairs of // lines) |
| $S_{\Delta ACH} = S_{\Delta CGH} = 1000$ $cm^2$ | $S_{\Delta ACH} = S_{\Delta CGH} = 1000$ $cm^2$ |
| $S_{\Delta ADI} = S_{\Delta GDI} = 1000$ $cm^2$ | $S_{\Delta ADI} = S_{\Delta GDI} = 1000$ $cm^2$ |
| 六邊形的面積= $1000\times6$ $cm^2$ = 6000 $cm^2$ | Area of hexagon = $1000\times6$ $cm^2$ = 6000 $cm^2$ |

**Group Event 4**

**G4.1** 設 $a$、$b$ 為非零整數，且滿足方程 $a - ab + b = 18$。求 $a + b$ 的值。

Let $a$ , $b$ be non-zero integers satisfying the equation $a - ab + b = 18$. Find the value of $a + b$ .

**Reference: 2024 FI1.1**

| | |
|---|---|
| $a - ab + b - 1 = 17$<br>$(1 - a)(b - 1) = 17$<br>$(1 - a, b - 1) = (1, 17)$ 或 $(-1, -17)$<br>$(a, b) = (0, 18)$ (捨去) 或 $(2, -16)$<br>$a + b = -14$ | $a - ab + b - 1 = 17$<br>$(1 - a)(b - 1) = 17$<br>$(1 - a, b - 1) = (1, 17)$ or $(-1, -17)$<br>$(a, b) = (0, 18)$ (rejected) or $(2, -16)$<br>$a + b = -14$ |

**G4.2** 設 $x$ 為一正整數，且滿足 $x(x + 1)(x + 2)(x + 3) = 3024$。求 $x$ 的值。

Let $x$ be a positive integer satisfying $x(x + 1)(x + 2)(x + 3) = 3024$. Find the value of $x$ .

| | |
|---|---|
| $x(x + 1)(x + 2)(x + 3) = 3024$<br>$(x^2 + 3x)(x^2 + 3x + 2) = 3024$<br>$(x^2 + 3x)^2 + 2(x^2 + 3x) + 1 = 3025$<br>$(x^2 + 3x + 1)^2 = 55^2$<br>$x^2 + 3x + 1 = 55$ 或 $x^2 + 3x + 1 = -55$<br>$x^2 + 3x - 54 = 0$ 或 $x^2 + 3x + 56 = 0$<br>$(x - 6)(x + 9) = 0$ 或沒有實數解<br>$\because x > 0 \therefore x = 6$ | $x(x + 1)(x + 2)(x + 3) = 3024$<br>$(x^2 + 3x)(x^2 + 3x + 2) = 3024$<br>$(x^2 + 3x)^2 + 2(x^2 + 3x) + 1 = 3025$<br>$(x^2 + 3x + 1)^2 = 55^2$<br>$x^2 + 3x + 1 = 55$ or $x^2 + 3x + 1 = -55$<br>$x^2 + 3x - 54 = 0$ or $x^2 + 3x + 56 = 0$<br>$(x - 6)(x + 9) = 0$ or no real solution<br>$\because x > 0 \therefore x = 6$ |

**G4.3** 設 $\alpha$、$\beta$ 為二次方程 $x^2 + 6x + 2 = 0$ 的兩個根，

求以 $\dfrac{\alpha^2}{\beta}$ 和 $\dfrac{\beta^2}{\alpha}$ 為根及 $x^2$ 的系數為 1 的二次方程。

Let $\alpha$, $\beta$ be the two roots of the quadratic equation $x^2 + 6x + 2 = 0$.

Find the quadratic equation whose roots are $\dfrac{\alpha^2}{\beta}$ and $\dfrac{\beta^2}{\alpha}$, and coefficient of $x^2$ is 1 .

$\alpha + \beta = -6$, $\alpha\beta = 2$

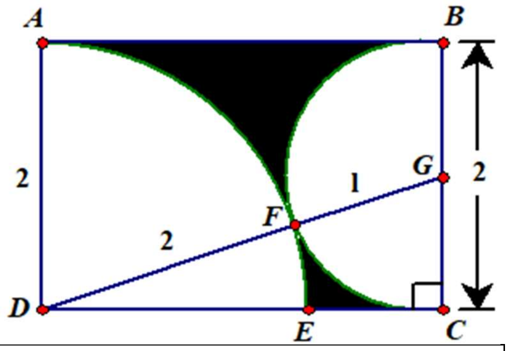
$$\begin{aligned}\alpha^3 + \beta^3 &= (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)\\ &= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]\\ &= -6[(-6)^2 - 3(2)] = -180\end{aligned}$$

$$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{-180}{2} = -90$$

$$\frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha} = \alpha\beta = 2$$

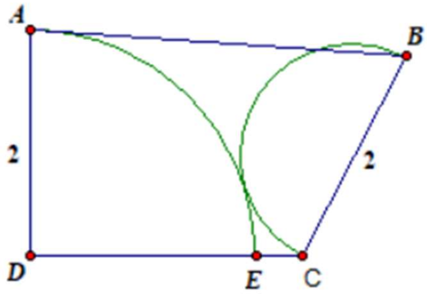
$x^2 + 90x + 2 = 0$

**G4.4** 右圖空白部分由一個四分一圓和一個半圓互相外切組成。*ABCD*是一個長方形。求陰影部分的面積。
The unshaded part in the diagram on the right is made up of a quarter-circle and a semi-circle which touch each other externally. *ABCD* is a rectangle.
Find the area of the shaded part.




| | |
|---|---|
| 假設該四分一圓和半圓互相外切於 $F$ 及 $G$ 為半圓的圓心。那麼 $D$、$F$、$G$ 共綫。<br>$DG = DF + FG = 2 + 1 = 3$<br>$CD^2 + CG^2 = DG^2$ (畢氏定理)<br>$CG = \sqrt{3^2 - 1^2} = 2\sqrt{2}$<br>陰影面積$= 2\sqrt{2}\times 2 - \frac{1}{4}\cdot\pi(2)^2 - \frac{1}{2}\cdot\pi(1)^2$<br>$= 4\sqrt{2} - \frac{3\pi}{2}$ | Suppose the quarter-circle and the semi-circle touch each other at $F$ and $G$ is the centre of the semi-circle. Then $D$, $F$, $G$ are collinear.<br>$DG = DF + FG = 2 + 1 = 3$<br>$CD^2 + CG^2 = DG^2$ (Pythagoras' theorem)<br>$CG = \sqrt{3^2 - 1^2} = 2\sqrt{2}$<br>Shaded area $= 2\sqrt{2}\times 2 - \frac{1}{4}\cdot\pi(2)^2 - \frac{1}{2}\cdot\pi(1)^2$<br>$= 4\sqrt{2} - \frac{3\pi}{2}$ |

**Remark:** the original question was:
右圖空白部分由一個四分一圓和一個半圓組成，求陰影部分的面積。
The unshaded part in the diagram on the right is made up of a quarter-circle and a semi-circle. Find the area of the shaded part.




If *ABCD* is not a rectangle, then it is impossible to find *CD* and hence the area of of the shaded part. Furthermore, the fact that the quarter-circle and the semi-circle touch each other externally must be specified.