Supplementary exercise on Mapping

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1. Let $f : A \to B$ be a bijective mapping, x an element of A and y an element of B. Prove that there exists a bijective mapping $g : A \to B$ so that $g(x) = y$.
2. Z is the set of all integers. Give examples of mappings $f : Z \to Z$ which are
   (a) injective but not surjective,
   (b) surjective but not injective.
3. Let $f : A \to B$. Prove that for any subsets X, X_1, X_2 of A and for any subsets Y, Y_1, Y_2 of B,
   (a) $f[X_1 \setminus X_2] \supset f[X_1] \setminus f[X_2]$
   (b) $f[X \cap f^{-1}[Y]] = f[X] \cap Y$
   (c) $f[X_1 \cap X_2] \subset f[X_1] \cap f[X_2]$
   Give an example for which the equality **does not** hold.
   (d) if $Y_1 \subset Y_2$, then $f^{-1}[Y_1] \subset f^{-1}[Y_2]$
   (e) $Y \supset f[f^{-1}[Y]]$
   Give an example for which the equality **does not** hold.
4. Let $f : A \to B$ and $g : B \to C$ be mappings. Prove that
   (a) if f and g are surjective, then $g\circ f : A \to C$ is surjective
   (b) if f and g are injective, then $g\circ f : A \to C$ is injective
   (c) if $g\circ f : A \to C$ is surjective, then g is surjective
   Give counter-examples to show that the converse is not true.
   (d) if $g\circ f : A \to C$ is injective, then f is injective
   Give counter-examples to show that the converse is not true.
   (e) for all subset Z of C, $(g\circ f)^{-1}[Z]=f^{-1}[g^{-1}[Z]]$
5. Prove that, for a mapping $f : A \to B$ is surjective if and only if $Y = f[f^{-1}[Y]]$ for all $Y \subset B$.
6. Prove that, for a mapping $f : A \to B$, the following conditions are equivalent:
   (a) f is injective;
   (b) $X = f^{-1}[f[X]]$ for all $X \subset A$;
   (c) $f[X_1 \cap X_2] = f[X_1] \cap f[X_2]$ for all $X_1, X_2 \subset A$.
7. Let the function $f : S \to T$ be surjective. If $A \subset S$, prove that $T \setminus f[\mathrm{A}] \subset f[S \setminus A]$
8. Let $f : A \to B$ be a function. If $Y \subset B$, prove that $f^{-1}[B \setminus Y] = A \setminus f^{-1}[Y]$
9. Let $f : S \to T$ be a function. If $A, B \subset S$, and $B \subset A$, prove that $f[A \setminus B] = f[A] \setminus f[B]$
10. If $f(x) = x^2 + 2x + 3$, find two functions $g(x)$ for which $f\circ g(x) = x^2 - 6x + 11$.

11. Let A, B, C and D be non-empty sets, $C \subseteq A$, $D \subseteq B$, $f : A \to B$

(*a*) Prove or give a counter-example that $f[C] \subset D$ iff $C \subset f^{-1}[D]$

(*b*) What condition will ensure that $f[C] = D$ iff $C = f^{-1}[D]$? Prove your answer.

12. Let $f : X \to Y$ be a mapping from a set X to a set Y. For any subset S of X, the direct image of S under f is defined by $f[S] = \{y: y = f(x) \text{ for some } x \in S\}$

(*a*) Show that $f[A \cup B] = f[A] \cup f[\mathrm{B}]$ for all subsets A and B of X.

(*b*) Show that if $f[A \cap B] = f[A] \cap f[B]$ for all subsets A and B of X, then f is injective.

(*c*) Show that $f[X \setminus A] = Y \setminus f[A]$ for all subsets A of X if f is bijective.

13. Let A be a set. Suppose f is a mapping from A to itself such that f is injective but not surjective. Let $f[A] = \{f(a): \text{for some } a \in A\}$

(*a*) Show that

(*i*) A is non-empty,

(*ii*) A consists of more than two elements.

(*b*) If $g : A \to A$ is a mapping defined by $g(f(x)) = x$ for all $x \in A$, show that g is not bijective.

(*c*) Show that there exists a unique mapping $h: f[A] \to A$ such that $h(f(x)) = x$ for all $x \in A$.

(*i*) h is well defined.

(*ii*) h is bijective.

1. Let $f : A \to B$ be a bijective mapping, x an element of A and y an element of B. Prove that there exists a bijective mapping $g : A \to B$ so that $g(x) = y$.

If $f(x) = y$ then $g = f$ is the required bijective mapping.

If $f(x) \neq y$,

$\because f$ is bijective,

$\therefore \exists!\ t \in A$ such that $f(t) = y$ and $t \neq x$

Define g: $A \to B$ by $g(a) = \begin{cases} y, & \text{for } a = x \\ f(x), & \text{for } a = t \\ f(a), & \text{for } a \neq x \text{ and } a \neq t \end{cases}$

We are going to show that g is a well-defined function.

$\forall a \in A$, $g(a) = y$ or $f(x)$ or $f(a) \in B$

If $g(a) = y_1$ and $g(a) = y_2$,

Case 1 $a \neq x$ and $a \neq t$

$\Rightarrow f(a) = y_1$ and $f(a) = y_2$

$\because f$ is a well defined function

$\therefore y_1 = y_2$

Case 2 $a = t$

$\Rightarrow g(t) = f(x) = y_1$, $g(t) = f(x) = y_2$

$\because f$ is a well-defined function

$\therefore y_1 = y_2$

Case 3 $a = x$

$\Rightarrow g(x) = y = y_1$, $g(x) = y = y_2$

$\Rightarrow y = y_1 = y_2$

In all 3 cases, g is a well-defined function

We are going to show that g is surjective

$\forall b \in B$, if $b = y$ then $g(x) = y$

if $b = f(x)$ then $g(t) = f(x)$

if $b \neq y$ and $b \neq f(x)$

then $\exists!\ a \in A$ and $a \neq x$ and $a \neq t$ s.t. $f(a) = b$

$\Rightarrow g(a) = b$

$\therefore g$ is surjective

We are going to show that g is injective

If $g(a_1) = g(a_2)$,

Case 1 $g(a_1) = g(a_2) = y$

$\because f(t) = y$ and t is unique

$\therefore t = a_1 = a_2$

Case 2 $g(a_1) = g(a_2) = f(x)$

$\because g(t) = f(x)$

if $a_1 \neq t$ then $g(a_1) = y$ or $f(a_1)$

$\Rightarrow g(a_1) \neq g(t)$

if $a_2 \neq t$ then $g(a_2) = y$ or $f(a_2)$

$\Rightarrow g(a_2) \neq g(t)$

By contrapositivity, $f(x) = g(a_1) = g(a_2) \Rightarrow a_1 = a_2 = t$

Case 3 $g(a_1) = g(a_2) = f(a_1) = f(a_2)$, where $a_1, a_2 \neq x, t$

$\because f$ is injective

$\therefore a_1 = a_2$

Combining the above 3 cases, g is injective.

$\because g$ is both injective and surjective

$\therefore g$ is bijective and is the required function.

2. $f: Z \to Z$

(a) $f(x) = 2x$

then f is a well-defined function. (prove it!)

Suppose $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is an injective function

$5 \in$ Z, but we cannot find an integer such that $2x = 5$

$\therefore f$ is not surjective.

(b) Define $f(x) = \left[\frac{x}{2}\right]$ where $[t]$ is the integer less than or equal to t.

$\left[\frac{3}{2}\right]=1$, $\left[-\frac{5}{2}\right]=-3$, $\left[-\frac{4}{2}\right]=-2$

$\forall y \in Z, f(2y) = \left[\frac{2y}{2}\right] = y$

$\therefore f$ is surjective.

But $f(0) = \left[\frac{0}{2}\right] = 0, f(1) = \left[\frac{1}{2}\right] = 0$

$\therefore f$ is not injective.

3. Let $f: A \to B$. Prove that for any subsets X, X_1, X_2 of A and for any subsets Y, Y_1, Y_2 of B,

(*a*) $f[X_1 \setminus X_2] \supset f[X_1] \setminus f[X_2]$

$\forall f(x) \in f[X_1] \setminus f[X_2]$

$\Rightarrow f(x) \in f[X_1]$ and $f(x) \notin f[X_2]$

$\Rightarrow x \in f^{-1}[f[X_1]]$ and $x \notin X_2$

$\Rightarrow x \in f^{-1}[f[X_1]] \setminus X_2$

$\Rightarrow f(x) \in f[f^{-1}[f[X_1]] \setminus X_2]$ $\quad (\because f[f^{-1}[f[X]]] = f[X])$

$\Rightarrow f(x) \in f[X_1 \setminus X_2]$

$\therefore f[X_1 \setminus X_2] \supset f[X_1] \setminus f[X_2]$

(*b*) $f[X \cap f^{-1}[Y]] = f[X] \cap Y$

$X \cap f^{-1}[Y] \subset X$ and $X \cap f^{-1}[Y] \subset f^{-1}[Y]$

$\Rightarrow f[X \cap f^{-1}[Y]] \subset f[X]$ and $f[X \cap f^{-1}[Y]] \subset f[f^{-1}[Y]] \subset Y$

$\Rightarrow f[X \cap f^{-1}[Y]] \subset f[X] \cap Y$

$\forall f(x) \in f[X] \cap Y$

$\Rightarrow f(x) \in f[X]$ and $f(x) \in Y$

$\Rightarrow x \in f^{-1}[f[X]]$ and $x \in f^{-1}[Y]$

$\Rightarrow x \in f^{-1}[f[X]] \cap f^{-1}[Y]$

$\Rightarrow f(x) \in f[f^{-1}[f[X]] \cap f^{-1}[Y]]$

$\Rightarrow f(x) \in f[X \cap f^{-1}[Y]]$ $\quad (\because f[f^{-1}[f[X]]] = f[X])$

$\therefore f[X \cap f^{-1}[Y]] \supset f[X] \cap Y$

$\because f[X \cap f^{-1}[Y]] \subset f[X] \cap Y$ and $f[X \cap f^{-1}[Y]] \supset f[X] \cap Y$

$\therefore f[X \cap f^{-1}[Y]] = f[X] \cap Y$

(*c*) $f[X_1 \cap X_2] \subset f[X_1] \cap f[X_2]$

$X_1 \cap X_2 \subset X_1$ and $X_1 \cap X_2 \subset X_2$

$\Rightarrow f[X_1 \cap X_2] \subset f[X_1]$ and $f[X_1 \cap X_2] \subset f[X_2]$

$\Rightarrow f[X_1 \cap X_2] \subset f[X_1] \cap f[X_2]$

Give an example for which the equality **does not** hold.

Define $f: \{a, b\} \to \{1\}$ by $f(a) = f(b) = 1$

Let $X_1 = \{a\}$, $X_2 = \{b\}$, $f[X_1 \cap X_2] = f[\phi] = \phi$; $f[X_1] \cap f[X_2] = \{1\}$

$\therefore f[X_1 \cap X_2] \neq f[X_1] \cap f[X_2]$

(*d*) if $Y_1 \subset Y_2$, then $f^{-1}[Y_1] \subset f^{-1}[Y_2]$

$\forall x \in f^{-1}[Y_1] \Rightarrow f(x) \in Y_1$

$\Rightarrow f(x) \in Y_2$ $(\because Y_1 \subset Y_2)$

$\Rightarrow x \in f^{-1}[Y_2]$

$\therefore f^{-1}[Y_1] \subset f^{-1}[Y_2]$

(*e*) $Y \supset f[f^{-1}[Y]]$

$\forall f(x) \in f[f^{-1}[Y]] \Rightarrow x \in f^{-1}[Y]$

$\Rightarrow f(x) \in Y$

$\therefore Y \supset f[f^{-1}[Y]]$

Give an example for which the equality **does not** hold.

Define $f: \{a, b\} \to \{1,2\}$ by $f(a) = f(b) = 1$

Let $Y = \{1,2\}$, $f^{-1}[Y] = \{a, b\}$, $f[f^{-1}[Y]] = \{1\}$

4. Let $f : A \to B$ and $g : B \to C$ be mappings. Prove that

(*a*) if f and g are surjective, then $g \circ f : A \to C$ is surjective

$\forall c \in C$, $\exists\, b \in B$ such that $g(b) = c$ ($\because$ g is surjective)

$\exists\, a \in A$ such that $f(a) = b$ ($\because$ f is surjective)

$\Rightarrow \exists\, a \in A$ such that $g \circ f(a) = g(f(a)) = g(b) = c$

$\therefore$ $g \circ f$ is surjective.

(*b*) if f and g are injective, then $g \circ f : A \to C$ is injective

Suppose $g \circ f(a_1) = g \circ f(a_2)$

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow f(a_1) = f(a_2)$ ($\because$ g is injective)

$\Rightarrow a_1 = a_2$ ($\because$ f is injective)

$\therefore$ $g \circ f$ is injective.

(*c*) if $g \circ f : A \to C$ is surjective, then g is surjective

$\forall c \in C$, $\exists\, a \in A$ such that $g \circ f(a) = c$ ($\because$ $g \circ f$ is surjective)

$\Rightarrow g(f(a)) = c$

Let $b = f(a) \in B$

$\Rightarrow g(b) = c$

$\therefore$ g is surjective.

Give counter-examples to show that the converse is not true.

Define f : R $\to$ R, g: R $\to$ R by

$f(x) = x^2$, $g(x) = x$

$\because$ $g = \boldsymbol{i}$ (identity mapping)

$\therefore g$ is surjective.

$g \circ f(x) = x^2$

$g \circ f$ is not injective because we cannot find $x \in R$ such that $x^2 = -3$.

(*d*) if $g \circ f : A \to C$ is injective, then f is injective

Suppose $f(a_1) = f(a_2)$

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow g \circ f(a_1) = g \circ f(a_2)$

$\Rightarrow a_1 = a_2$ ($\because g \circ f$ is injective)

$\therefore f$ is injective.

Give counter-examples to show that the converse is not true.

Define f : R $\to$ R, g: R $\to$ R by

$f(x) = x$, $g(x) = x^2$

then $f = \boldsymbol{i}$ (identity mapping)

so f is injective

$g \circ f(x) = x^2$

$g \circ f(2) = g \circ f(-2) = 4$

$\therefore$ $g \circ f$ is not injective.

(*e*) for all subset Z of C, $(g \circ f)^{-1}[Z] = f^{-1}[g^{-1}[Z]]$

$\forall a \in f^{-1}[g^{-1}[Z]] \Leftrightarrow f(a) \in g^{-1}[Z]$

$\Leftrightarrow g \circ f(a) \in Z$

$\Leftrightarrow a \in (g \circ f)^{-1}[Z]$

$\therefore$ $(g \circ f)^{-1}[Z] = f^{-1}[g^{-1}[Z]]$

5. Prove that, for a mapping $f: A \to B$ is surjective if and only if $Y = f[f^{-1}[Y]]$ for all $Y \subset B$.

($\Rightarrow$) Given $f: A \to B$ is surjective

By the result of 3(e), $Y \supset f[f^{-1}[Y]]$

$\forall y \in Y \subset B$

$\Rightarrow \exists\, x \in A$ such that $f(x) = y$ ($\because f$ is surjective)

$\Rightarrow x \in f^{-1}[Y]$

$\Rightarrow f(x) \in f[f^{-1}[Y]]$

$\therefore Y \subset f[f^{-1}[Y]]$

$\therefore Y = f[f^{-1}[Y]]$

($\Leftarrow$) Given $Y = f[f^{-1}[Y]]\ \forall Y \subset B$

$\Rightarrow B = f[f^{-1}[B]]$ ($\because B \subset B$)

$\forall y \in B \Rightarrow y \in f[f^{-1}[B]]$

$\Rightarrow \exists\, x \in f^{-1}[B] \subset A$ such that $f(x) = y$

$\Rightarrow f$ is surjective.

6. Prove that, for a mapping $f: A \to B$, the following conditions are equivalent:

(a) f is injective;

(b) $X = f^{-1}[f[X]]$ for all $X \subset A$;

(c) $f[X_1 \cap X_2] = f[X_1] \cap f[X_2]$ for all $X_1, X_2 \subset A$.

We shall prove in the following manner: (a) $\Rightarrow$ (c), (c) $\Rightarrow$ (b), (b) $\Rightarrow$ (a)

(a) $\Rightarrow$ (c)

By 3(c), $f[X_1 \cap X_2] \subset f[X_1] \cap f[X_2]$

$\forall y \in f[X_1] \cap f[X_2]$

$\Rightarrow y = f(x_1)$ and $y = f(x_2)$ for some $x_1 \in X_1$ and $x_2 \in X_2$

$\because f$ is injective $\Rightarrow x_1 = x_2 \in X_1 \cap X_2$

$\Rightarrow f(x) = y \in f[X_1 \cap X_2]$

$\therefore f[X_1 \cap X_2] = f[X_1] \cap f[X_2]$ for all $X_1, X_2 \subset A$.

(c) $\Rightarrow$ (b)

We shall use contrapositivity ~(b) $\Rightarrow$ ~(c)

Suppose $X \neq f^{-1}[f[X]]$

$\Rightarrow \exists\, x \in f^{-1}[f[X]]$ and $x \notin X$

$\Rightarrow f(x) \in f[X]$ and $x \notin X$

$\Rightarrow \exists\, x_1 \in X, x \notin X$ such that $f(x) = f(x_1)$

$f[\{x_1\} \cap \{x\}] = f[\phi] = \phi$

$f[\{x_1\}] \cap f[\{x\}] = \{f(x_1)\} \cap \{f(x)\} = \{f(x)\} \neq \phi$

$\therefore$ (c) is not always true for all $X_1, X_2 \subset A$.

(b) $\Rightarrow$ (a)

We use contradiction ~(a) Λ (b) $\Rightarrow$ F

Suppose f is not injective.

$\exists\, x_1 \neq x_2$ such that $f(x_1) = f(x_2)$

$\{x_1\} = f^{-1}[f[\{x_1\}]]$

$= f^{-1}[\{f(x_1)\}]$

$= f^{-1}[\{f(x_2)\}]$

$\Rightarrow x_2 \in f^{-1}[\{f(x_2)\}] = \{x_1\}$ which is false.

7. Let the function $f: S \to T$ be surjective. If $A \subset S$, prove that $T \setminus f[A] \subset f[S \setminus A]$

We use the property $f[X_1 \cup X_2] = f[X_1] \cup f[X_2]$

Given $A \subset S$

$$\begin{aligned} f[A] \cup f[S \setminus A] &= f[A \cup (S \setminus A)] \\ &= f[S] \\ &= T \quad (\because f \text{ is surjective}) \end{aligned}$$

$$\begin{aligned} T \setminus f[A] &= f[A] \cup f[S \setminus A] \setminus f[A] \\ &\subset f[S \setminus A] \end{aligned}$$

8. Let $f: A \to B$ be a function. If $Y \subset B$, prove that $f^{-1}[B \setminus Y] = A \setminus f^{-1}[Y]$

$$\begin{aligned} x \in f^{-1}[B \setminus Y] &\Leftrightarrow f(x) \in B \setminus Y \\ &\Leftrightarrow f(x) \in B \text{ and } f(x) \notin Y \\ &\Leftrightarrow x \in A \text{ and } x \notin f^{-1}[Y] \\ &\Leftrightarrow x \in A \setminus f^{-1}[Y] \end{aligned}$$

$\therefore f^{-1}[B \setminus Y] = A \setminus f^{-1}[Y]$

9. Let $f: S \to T$ be a function. If $A, B \subset S$, and $B \subset A$, prove that $f[A \setminus B] = f[A] \setminus f[B]$

By the result of 3(a), $f[A \setminus B] \supset f[A] \setminus f[B]$

$\forall f(x) \in f[A \setminus B]$

$\Rightarrow \exists\, x \in A \setminus B$ such that $f(x) \in f[A \setminus B]$

$\Rightarrow \exists\, x \in A$ and $x \notin B$

$\Rightarrow \exists f(x)$: $f(x) \in f[A]$ and $f(x) \notin f[B]$

$\Rightarrow f(x) \in f[A] \setminus f[B]$

$\therefore f[A \setminus B] \subset f[A] \setminus f[B]$

$\therefore f[A \setminus B] = f[A] \setminus f[B]$

10. If $f(x) = x^2 + 2x + 3$, find two functions $g(x)$ for which $f\circ g\,(x) = x^2 - 6x + 11$.

$g(x)$ must be a linear function of x.

Let $g(x) = ax + b$, $f(x) = x^2 + 2x + 3$

$$\begin{aligned} f\circ g\,(x) &= f(ax + b) \\ &= (ax + b)^2 + 2(ax + b) + 3 \\ &= ax^2 + 2abx + b^2 + 2ax + 2b + 3 \end{aligned}$$

But $x^2 - 6x + 11 = ax^2 + (2a + 2ab)x + b^2 + 2b + 3$

$-6 = 2a + 2ab$ and $11 = b^2 + 2b + 3$

$a(1 + b) = -3$ and $b^2 + 2b - 8 = 0$

$a(1 + b) = -3$ and ($b = -4$ or $b = 2$)

$\therefore a = 1$, $b = -4$ or $a = -1$, $b = 2$

and $g(x) = x - 4$ or $g(x) = 2 - x$

11. Solution: Note that $f^{-1}[D] = \{x \in A : f(x) \in D\}$

(a) (Proof of only if part) $f[C] \subset D$ (given)

$\forall\, x \in C$

$\Rightarrow f(x) \in f[C]$

$\Rightarrow f(x) \in D$

$\Rightarrow x \in f^{-1}[D]$

$\therefore\; C \subset f^{-1}[D]$

(Proof of if part) $C \subset f^{-1}[D]$ (given)

$\forall\, y \in f[C]$

$\Rightarrow \exists\, x \in C$ such that $f(x) = y \in f[C]$

$\Rightarrow x \in f^{-1}[D]$ and $f(x) = y \in f[C]$

$\Rightarrow y = f(x) \in D$

The proof is completed.

Illustration: $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d, e, f\}$, $C = \{1, 2\}$, $D = \{a, b, c, d\}$

Define $f: A \to B$ by $f(1) = a, f(2) = b, f(3) = c, f(4) = e, f(5) = a$

$f^{-1}[D] = \{1, 2, 3, 5\}, f[C] = \{a, b\} \subset D, C \subset f^{-1}[D]$

(b) The condition is:

$f[C] \subset D$ and there is a bijection $g: C \to D$ defined by $g(x) = f(x)$

Since $C \subseteq A$, $D \subseteq B$ and f is a well-defined function. Also $f[C] \subset D$.

So g is also a well-defined function.

Given the above condition. To prove: $f[C] = D$ iff $C = f^{-1}[D]$

(Proof of only if part) $f[C] = D$ (given)

By the result of (a), $C \subset f^{-1}[D]$

$\forall\, x \in f^{-1}[D]$

$\Rightarrow f(x) = y \in D$

$\because\; g: C \to D$ is a bijection,

$\Rightarrow x \in C$ and x is unique.

$\Rightarrow f^{-1}[D] \subset C$

$\therefore\; C = f^{-1}[D]$

(Proof of if part) $C = f^{-1}[D]$ (given)

By the result of (a), $f[C] \subset D$

$\forall\, y \in D$

$\because g: C \to D$ is a bijection

$\exists$ unique $x \in C$ such that $g(x) = f(x) = y \in D$

$\Rightarrow D \subset f[C]$

$\therefore f[\mathrm{C}] = D$

The proof is completed.

12. (*a*) $\forall\, y \in f[A \cup B] \Leftrightarrow \exists\, x \in A \cup B \wedge f(x) = y$

$\Leftrightarrow (\exists\, x \in A \vee x \in B) \wedge f(x) = y$

$\Leftrightarrow f(x) \in f[A] \vee f(x) \in f[B]$

$\Leftrightarrow f(x) \in f[A] \cup f[B]$

$\therefore f[A \cup B] = f[A] \cup f[\mathrm{B}]\ \forall\, A, B \subset X$

(*b*) Given $f[A \cap B] = f[A] \cap f[B]\ \forall\, A, B \subset X$

Suppose, on the contrary, that f is not injective.

$\exists\, x_1, x_2 \in X$ such that $f(x_1) = f(x_2) \wedge x_1 \neq x_2$

Let $A = \{x_1\}$, $B = \{x_2\}$, then $A, B \subset X$

$\because f[A \cap B] = f[A] \cap f[B]$

$\therefore f[\phi] = \{f(x_1)\} \cap \{f(x_2)\}$

$\Rightarrow \phi = \{f(x_1)\}$, which is false

We have proved that if f is not injective then $f[A \cap B] \neq f[A] \cap f[B]$

$\therefore f[A \cap B] = f[A] \cap f[B]\ \forall\, A, B \subset X \Rightarrow f$ is $1 - 1$.

(*c*) Given f is bijective, try to show that $f[X \setminus A] = Y \setminus f[A]\ \forall\, A \subset X$

$\forall\, y \in f[X \setminus A] \Rightarrow \exists\, x \in X \setminus A$ such that $f(x) = y$

$\Rightarrow \exists\, x \in X \wedge x \notin A \wedge f(x) = y \ldots (1)$

Claim: $x \notin A \wedge f(x) \notin f[A]$

Proof: otherwise $f(x) \in f[A]$

$\Rightarrow \exists\, a \in A$ such that $f(a) = f(x) \wedge x \notin A$

$\Rightarrow a \neq x$

$\Rightarrow f$ is not injective

$\Rightarrow f$ is not bijective

cont'd from (1) $\Rightarrow y = f(x) \in f[X] \wedge f(x) \notin f[A]$

$\Rightarrow y \in f[X] \setminus f[A]$

$\Rightarrow y \in Y \setminus f[A]$ ($\because f$ is surjective $\therefore f[X] = Y$)

$\therefore f[X \setminus A] \subset Y \setminus f[A]$

$\forall\, y \in Y \setminus f[A] \Rightarrow y \in f[X] \setminus f[A]$ ($\because f$ is surjective $\therefore f[X] = Y$)

$\Rightarrow \exists\, x \in X$ such that $y = f(x) \in f[X] \wedge f(x) \notin f[A]$

$\Rightarrow \exists\, x \in X \wedge x \notin A$ such that $y = f(x)$

$\Rightarrow \exists\, x \in X \setminus A$ such that $y = f(x)$

$\Rightarrow y = f(x) \in f[X \setminus A]$

$\therefore f[X \setminus A] \supset Y \setminus f[A]$

$\therefore f[X \setminus A] = Y \setminus f[A]$

13. (*a*) (*i*) $\because f$ is not surjective

$\therefore \exists\, b \in A$ such that $f(a) \neq b\ \forall\, a \in A$ … (1)

$\therefore A \neq \phi$

(*ii*) From (*a*), let $a = f(b)$

$\because f$ is not surjective

$\therefore a \neq b$ … (2)

Let $c = f(a)$

If $c = a$, then $f(a) = f(b)$

$\Rightarrow a = b$ ($\because f$ is $1 - 1$)

This result contradicts with (2)

$\therefore c \neq a$

If $c = b$, then $f(a) = b$

This result contradicts with (1)

$\therefore c \neq b$

$\therefore a, b, c$ are distinct elements of A

$\therefore A$ consists of more than two elements.

(*b*) From (*a*), $\exists\, b \in A$ such that $f(a) \neq b\ \forall\, a \in A$

Let $g(b) = c$

However, $g(f(c)) = c \wedge b \neq f(c)$

$\Rightarrow g(b) = g(f(c)) \wedge b \neq f(c)$

$\therefore g$ is not injective

$\therefore g$ is not bijective

(*c*) Suppose there is another bijective function g such that $g: f[A] \to A$ and $g(f(x)) = x$

$\forall\, y \in f[A]$, let $h(y) = x_1$, $g(y) = x_2$

By the definition of h, $y = f(x_1)$

$\therefore x_2 = g(y) = g(f(x_1)) = x_1$

$\therefore g(y) = h(y)\ \forall\, y \in f[A]$

$\Rightarrow g = h$

$\therefore h$ is unique.

(*i*) $h: f[A] \to A$

$\forall\, y \in f[A] \Rightarrow \exists\, x \in A$ such that $f(x) = y$

If $h(y) = x_1 \wedge h(y) = x_2$

$\Rightarrow h(f(x_1)) = x_1 \wedge h(f(x_2)) = x_2$

$\Rightarrow y = f(x_1) \wedge y = f(x_2)$

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$ ($\because f$ is $1 - 1$)

$\therefore h$ is well defined.

(ii) To show that h is injective

Suppose $h(y_1) = h(y_2)$

$\Rightarrow \exists\, x_1 \in A \wedge f(x_1) = y_1;\ \exists\, x_2 \in A \wedge f(x_2) = y_2$

$\Rightarrow h(f(x_1)) = h(y_1) = h(y_2) = h(f(x_2))$

$\Rightarrow x_1 = x_2$

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow y_1 = y_2$

$\Rightarrow f$ is injective

To show that h is surjective

$\forall\, x \in A \quad \exists\, y = f(x) \in f[A]$ such that $h(y) = h(f(x)) = x$

$\therefore$ h is surjective

$\because$ h is both injective and surjective

$\therefore$ h is bijective.