

21-22 Paper 1	1	291	2	4	3	281	4	$\frac{1}{10}$	5	16
	6	34	7	7	8	4	9	12.5	10	123
	11	$\frac{1}{6}$	12	6431205	13	23	14	120	15	$\frac{8}{15}$
21-22 Paper 2	1	2022	2	112	3	19	4	$\frac{1}{3}$	5	33
	6	120	7	$\frac{25}{4}$	8	$\frac{1}{1011}$				

Paper 1

1. α 及 β 是方程 $x^2 - 100x + k = 0$ 的實根。若 $\alpha - 7 = 30\beta$ ，求 k 的值。
 α and β are the real roots of the equation $x^2 - 100x + k = 0$. If $\alpha - 7 = 30\beta$, find the value of k .

$$\alpha + \beta = 100 \dots\dots (1), \alpha\beta = k \dots\dots (2)$$

$$\alpha + \beta - 7 = 31\beta$$

$$100 - 7 = 31\beta$$

$$\beta = 3$$

$$\text{Put } \beta = 3 \text{ into } x^2 - 100x + k = 0$$

$$3^2 - 100 \times 3 + k = 0$$

$$k = 291$$

2. 在圖一中， ACD 是一個三角形。 B 是 CD 上的一點使 C
 $AB = AC = 2$ 及 $AD = 4$ 。

若 $BC : BD = 1 : 3$ ，求 CD 的長。

In Figure 1, ACD is a triangle. B is a point on CD such that
 $AB = AC = 2$ and $AD = 4$.

If $BC : BD = 1 : 3$, find the length of CD .

Let $BC = k$, $BD = 3k$, $\angle ADB = \alpha$

Apply cosine formula on $\triangle ABD$ and $\triangle ACD$

$$\cos \alpha = \frac{4^2 + (3k)^2 - 2^2}{2(4)(3k)} = \frac{4^2 + (4k)^2 - 2^2}{2(4)(4k)}$$

$$\frac{12 + 9k^2}{24k} = \frac{12 + 16k^2}{32k}$$

$$4 + 3k^2 = 3 + 4k^2$$

$$k = 1, CD = 4k = 4$$

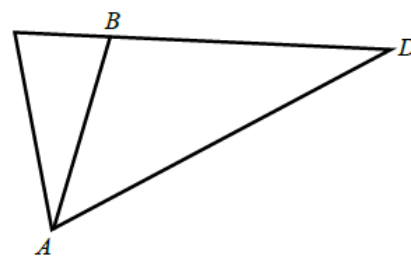


Figure 1 圖一

Method 2 Let $BC = k$, $BD = 3k$.

Produce AB to E so that $BE = 6$, join ED .

$\angle ABC = \angle DBE$ (vert. opp. \angle s)

$$\frac{EB}{AB} = \frac{6}{2} = 3, \quad \frac{BD}{BC} = \frac{3k}{k} = 3$$

$\therefore \triangle BDE \sim \triangle BCA$ (ratio of 2 sides, included \angle s)

$$\therefore \frac{ED}{AC} = 3 \Rightarrow ED = 6$$

$\angle BAD = \angle DAE$ (common \angle s)

$$\frac{AE}{AD} = \frac{2+6}{4} = 2, \quad \frac{AD}{AB} = \frac{4}{2} = 2$$

$\therefore \triangle ADE \sim \triangle ABD$ (ratio of 2 sides, included \angle s)

$$\therefore \frac{DE}{BD} = \frac{6}{3k} = 2 \Rightarrow k = 1$$

$$CD = 4k = 4$$

3. 在圖二中， $ABCD$ 是一個矩形。 E 是 AC 上的一點使 $AE = 25$ 及 $CE = 144$ 。若 $p = AD + DE + CD$ ，求 p 的值。

In Figure 2, $ABCD$ is a rectangle. E is a point on AC such that $AE = 25$ and $CE = 144$. If $p = AD + DE + CD$, find the value of p . **Reference: 1998 FG1.3, 1999 FG5.4**

Let $\angle ADE = \alpha$, $\angle CDE = 90^\circ - \alpha$

$\angle DAE = 90^\circ - \alpha$, $\angle DCE = \alpha$ (\angle sum of Δ)

$$\tan \alpha = \frac{DE}{144} = \frac{25}{DE}$$

$$DE = 60$$

$$AD = \sqrt{25^2 + 60^2} = 65 \text{ (Pythagoras' theorem on } \triangle ADE \text{)}$$

$$CD = \sqrt{60^2 + 144^2} = 156 \text{ (Pythagoras' theorem on } \triangle CDE \text{)}$$

$$p = 65 + 60 + 156 = 281$$

4. 設 x 、 y 及 z 是非零數。若 $2^x = 3^y = 18^z$ ，求 $\frac{xz}{5y(x-z)}$ 的值。

Let x , y and z are non-zero numbers. If $2^x = 3^y = 18^z$, find the value of $\frac{xz}{5y(x-z)}$.

$$2^x = 3^y = 18^z \Rightarrow x \log 2 = y \log 3 = z \log 18 = k$$

$$x = \frac{k}{\log 2}, y = \frac{k}{\log 3}, z = \frac{k}{\log 18}$$

$$\begin{aligned} \frac{xz}{5y(x-z)} &= \frac{\frac{k}{\log 2} \cdot \frac{k}{\log 18}}{\frac{5k}{\log 3} \left(\frac{k}{\log 2} - \frac{k}{\log 18} \right)} \\ &= \frac{\log 3}{5(\log 18 - \log 2)} \\ &= \frac{\log 3}{5 \log 9} = \frac{\log 3}{5 \times 2 \log 3} \\ &= \frac{1}{10} \end{aligned}$$

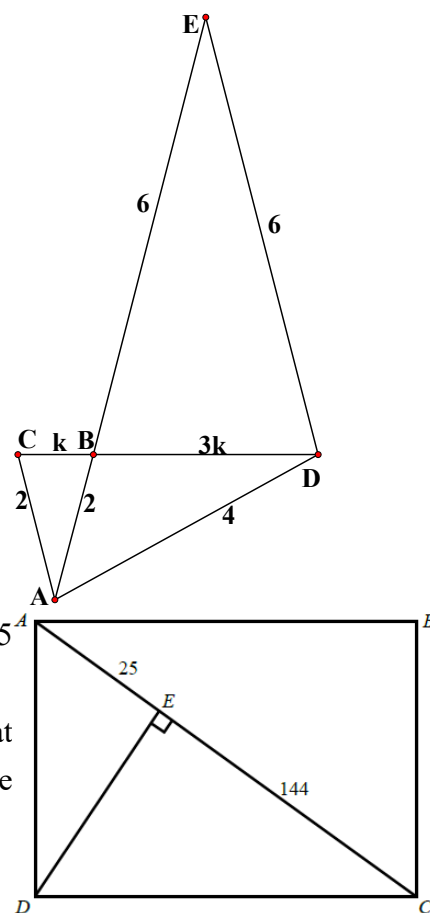


Figure 2 圖二

5. 設 $N = 24x + 216y$ ，其中 x 及 y 均為正整數。若 N 為完全立方數，求 $x + y$ 的最小值。

Let $N = 24x + 216y$, where both x and y are positive integers.

If N is a cube number, find the minimum value of $x + y$.

$N = 24(x + 9y) = 2^3 \times 3(x + 9y) = m^3$, where m is a positive integer.

$3(x + 9y) = k^3$, where k is a positive integer

For the least value of x and y , $x + 9y = 9n^3$, where n is a positive integer

$x = 9, 1 + y = n^3$

$n = 1$, no positive integral solution for y

$n = 2, y = 7$

The minimum value of $x + y = 9 + 7 = 16$

6. 小馬參加數學比賽，解其中一條題目

$$\begin{cases} ax + by = -16 \\ cx + 20y = -224 \end{cases}, \text{ 其中 } a, b \text{ 及 } c \text{ 是實數。}$$

題目的正確答案為 $x = 8$ 及 $y = -10$ 。

怎料小馬抄錯 c 的數值，得出答案 $x = 12$ 及 $y = -13$ 。求原題中 $a^2 + b^2 + c^2$ 的值。

John participated in a mathematics competition, in which one of the questions was to solve

$$\begin{cases} ax + by = -16 \\ cx + 20y = -224 \end{cases}, \text{ where } a, b \text{ and } c \text{ are real numbers.}$$

The correct answer to the question was $x = 8$ and $y = -10$.

However, John copied a wrong value for c and then gave an answer of $x = 12$ and $y = -13$.

Find the value of $a^2 + b^2 + c^2$ in the original question.

Put $x = 8$ and $y = -10$ into $\begin{cases} ax + by = -16 \\ cx + 20y = -224 \end{cases}$.

$$\begin{cases} 8a - 10b = -16 \dots\dots(1) \\ 8c - 200 = -224 \dots\dots(2) \end{cases} \Rightarrow c = -3$$

Let the wrongly copied c be c' .

Put $x = 12$ and $y = -13$ into $\begin{cases} ax + by = -16 \\ c'x + 20y = -224 \end{cases}$.

$$\begin{cases} 12a - 13b = -16 \dots\dots(3) \\ 12c' - 260 = -224 \dots\dots(4) \end{cases} \Rightarrow c' = 3$$

$$(1) \times 13: 104a - 130b = -208 \dots\dots(5)$$

$$(3) \times 10: 120a - 130b = -160 \dots\dots(6)$$

$$(6) - (5): 16a = 48$$

$$a = 3$$

$$\text{Sub. } a = 3 \text{ into (1): } 8 \times 3 - 10b = -16$$

$$b = 4$$

$$a^2 + b^2 + c^2 = 3^2 + 4^2 + 3^2 = 34$$

7. 已知 $459 + x^3 = 3^y$ ，其中 x 及 y 均為正整數。求 y 的最小值。

Given that $459 + x^3 = 3^y$, where both x and y are positive integers. Find the least value of y .

$$3^5 = 243 < 459 < 3^6 = 729$$

$$\text{Try } y = 6, 459 + x^3 = 729$$

$$x^3 = 270, \text{ no solution}$$

$$\text{Try } y = 7, 459 + x^3 = 2187$$

$$x^3 = 1728$$

$$x = 12$$

$$\therefore y = 7$$

8. 在圖三中， D 為四邊形 $ABCE$ 內的一點使得 $AD \parallel BC$ ， $AB \perp AD$ ， $CD \perp DE$ ， $CD = ED$ ， $AD = 4$ cm 及 $BC = 6$ cm。若 $\triangle ADE$ 的面積為 P cm^2 ，求 P 的值。

In Figure 3, D is a point inside the quadrilateral $ABCE$ such that $AD \parallel BC$, $AB \perp AD$, $CD \perp DE$, $CD = ED$, $AD = 4$ cm and $BC = 6$ cm. If P cm^2 is the area of $\triangle ADE$, find the value of P .

Let the foot of perpendicular from D to BC be E . Join DE .
 $DE \perp BC$, $AD \perp DE$. Let $AB = DE = h$ cm, $\angle CDE = \theta$.
 $AD = BE = 4$ cm (opp. sides of rectangle)
 $CE = BC - BE = (6 - 4)$ cm = 2 cm

$$\text{In } \triangle CDE, \sin \theta = \frac{2}{CD}$$

$$\angle ADE + 90^\circ + 90^\circ + \theta = 360^\circ \quad (\angle\text{s at a point})$$

$$\angle ADE = 180^\circ - \theta$$

$$P = \frac{1}{2} \cdot 4 \times DE \sin(180^\circ - \theta)$$

$$= 2 \times CD \sin \theta = 2 \times CD \times \frac{2}{CD} = 4$$

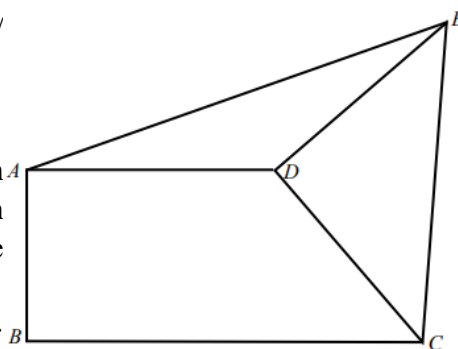


Figure 3 圖三

9. $ABCD$ 是一個圓內接四邊形，其中 $AB = 7$ ， $BC = 15$ ， $CD = 20$ and $DA = 24$ 。求圓 $ABCD$ 的半徑。

$ABCD$ is a cyclic quadrilateral with $AB = 7$, $BC = 15$, $CD = 20$ and $DA = 24$. Find the radius of the circle $ABCD$.

Let $BD = x$, $\angle BAD = \theta$, $\angle BCD = 180^\circ - \theta$ (opp. \angle s cyclic quad.)

Apply cosine rules on $\triangle ABD$ and $\triangle CBD$:

$$x^2 = 7^2 + 24^2 - 2 \times 7 \times 24 \cos \theta = 15^2 + 20^2 - 2 \times 15 \times 20 \cos(180^\circ - \theta)$$

$$625 - 336 \cos \theta = 625 + 600 \cos \theta$$

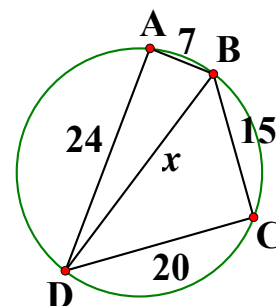
$$\cos \theta = 0, \theta = 90^\circ$$

BD is the diameter (converse, \angle in semi-circle)

$$BD^2 = 7^2 + 24^2 \quad (\text{Pythagoras' theorem})$$

$$BD = 25$$

$$\text{Radius} = 12.5$$



10. 已知 $a^2 + \frac{1}{a^2} = 7$ ，其中 $a > 0$ 。若 $b = a^5 + \frac{1}{a^5}$ ，求 b 的值。

Given that $a^2 + \frac{1}{a^2} = 7$, where $a > 0$. If $b = a^5 + \frac{1}{a^5}$, find the value of b .

$$a^2 + \frac{1}{a^2} + 2 = 7 + 2 = 9$$

$$\left(a + \frac{1}{a}\right)^2 = 3^2 \Rightarrow \because a > 0 \therefore a + \frac{1}{a} = 3$$

$$\left(a + \frac{1}{a}\right)\left(a^2 + \frac{1}{a^2}\right) = 3 \times 7 = 21 \Rightarrow a^3 + \frac{1}{a^3} + a + \frac{1}{a} = 21 \Rightarrow a^3 + \frac{1}{a^3} = 21 - 3 = 18$$

$$\left(a^2 + \frac{1}{a^2}\right)\left(a^3 + \frac{1}{a^3}\right) = 7 \times 18 = 126 \Rightarrow a^5 + \frac{1}{a^5} + a + \frac{1}{a} = 126 \Rightarrow a^5 + \frac{1}{a^5} = 126 - 3 = 123$$

11. x_1 及 x_2 是方程 $(\log 2x)(\log 3x) = a$ 的實根，其中 a 為實數。求 x_1x_2 的值。
 x_1 and x_2 are the real roots of the equation $(\log 2x)(\log 3x) = a^2$, where a is a real number.
 Find the value of x_1x_2 .

$$\begin{aligned}(\log 2 + \log x)(\log 3 + \log x) &= a \\(\log x)^2 + (\log 2 + \log 3) \log x + (\log 2)(\log 3) - a &= 0 \\(\log x)^2 + (\log 6) \log x + (\log 2)(\log 3) - a &= 0 \\\log x_1 + \log x_2 &= -\log 6 \\\log(x_1 x_2) &= \log \frac{1}{6}\end{aligned}$$

$$x_1x_2 = \frac{1}{6}$$

12. 由數字 0, 1, 2, 3, 4, 5, 6 組成一個沒有重複數字的 7 位數。若這個數可以被 55 整除，求這個數的最大值。

A 7-digit number is formed by putting the numerals 0, 1, 2, 3, 4, 5, 6 together without repetition. If this number is divisible by 55, find its largest possible value.

Let the 7-digit number be $\overline{abcdefg}$

$$55 = 5 \times 11, \text{ the units digit} = 0 \text{ or } 5. \quad g = 0 \text{ or } 5$$

This number is divisible by 11 $\Rightarrow a + c + e + g - (b + d + f) = 11m$, where m is an integer

To maximize the number, let $a = 6$.

$$\text{If } g = 0, b = 5, \text{ then } 6 + c + e + 0 - (5 + d + f) = 11m$$

$$1 + c + e - (d + f) = 11m, \text{ where } c, d, e, f = 1, 2, 3, 4$$

$$\therefore c + d + e + f = 10$$

$$\therefore 1 + c + e - [10 - (c + d)] = 11m$$

$$2(c + d) = 11m + 9$$

L.H.S. is even, R.H.S. is odd, which is a contradiction

$$\text{If } g = 5, b = 4, \text{ then } 6 + c + e + 5 - (4 + d + f) = 11m$$

$$c + e - (d + f) = 11(m - 1) + 4, \text{ where } c, e, d, f = 0, 1, 2, 3$$

By trial and error, $c = 3, e = 2, d = 1, f = 0$

$$3 + 2 - (1 + 0) = 4, \text{ which satisfies the equation}$$

The largest possible 7-digit number is 6431205.

13. 已知 $a^{2x} - b^{2y} = 1672$ ，其中 a, b, x 及 y 為正整數。求 $ax + by$ 的最小值。

Given that $a^{2x} - b^{2y} = 1672$, where a, b, x and y are positive integers.

Find the minimum value of $ax + by$.

$$(a^x - b^y)(a^x + b^y) = 2^3 \times 11 \times 19 = m \times n, \text{ where } m < n \text{ are integers}$$

To minimize $ax - by$, m, n must be as close as possible

$$\text{Let } m = 2 \times 19 = 38, n = 2^2 \times 11 = 44$$

$$a^x - b^y = 38 \dots\dots (1)$$

$$a^x + b^y = 44 \dots\dots (2)$$

$$[(1) + (2)] \div 2 \quad a^x = 41, \text{ no integral solution}$$

$$\text{Let } m = 2 \times 11 = 22, n = 2^2 \times 19 = 76$$

$$a^x - b^y = 22 \dots\dots (3)$$

$$a^x + b^y = 76 \dots\dots (4)$$

$$[(3) + (4)] \div 2 \quad a^x = 49$$

$$a = 7, x = 2$$

$$[(4) - (3)] \div 2 \quad b^y = 27$$

$$b = 3, y = 3$$

The minimum value of $ax + by = 7 \times 2 + 3 \times 3 = 23$.

14. 設 a 、 b 及 c 為非零數字。有多少個三位數 \overline{abc} 使得 $\overline{ab} < \overline{bc} < \overline{ca}$?

Let a , b and c are non-zero digits. How many three digit numbers \overline{abc} are there such that $\overline{ab} < \overline{bc} < \overline{ca}$?

$$\overline{abc} = 112, 113, \dots, 119 \quad (8)$$

$$334, 335, \dots, 339 \quad (6)$$

$$123, 124, \dots, 129 \quad (7)$$

$$\dots\dots\dots$$

$$134, 135, \dots, 129 \quad (6)$$

$$389 \quad (1)$$

$$145, 146, \dots, 149 \quad (5)$$

$$445, \dots, 449 \quad (5)$$

$$\dots\dots\dots$$

$$\dots\dots\dots$$

$$189 \quad (1)$$

$$489 \quad (1)$$

$$223, 224, \dots, 229 \quad (7)$$

$$\dots\dots\dots$$

$$234, 235, \dots, 239 \quad (6)$$

$$889 \quad (1)$$

$$\dots\dots\dots$$

$$289 \quad (1)$$

$$\begin{aligned} \text{Total} &= (8 + 7 + 6 + \dots + 1) + (7 + 6 + \dots + 1) + (6 + 5 + \dots + 1) + \dots + 1 \\ &= 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 \\ &= 120 \end{aligned}$$

15. PQR 是一個等腰三角形，其中 $PQ = PR = 17$ and $QR = 16$ 。將 I 及 H 分別記為 PQR 的內心及垂心。求 IH 長度的值。

PQR is an isosceles triangle with $PQ = PR = 17$ and $QR = 16$. Denote the in-centre and the orthocentre of PQR by I and H respectively. Find the length of HI .

Let A be the mid-point of QR . $QA = AR = 8$. $PA \perp QR$

$$PA = \sqrt{17^2 - 8^2} = 15 \quad (\text{Pythagoras' theorem})$$

I and H lies on PA (altitude of the isosceles triangle)

Draw the inscribed circle with radius r , touching ΔPQR at A , B and C respectively as shown in the figure.

$IA \perp QR$, $IB \perp PR$, $IC \perp PQ$ (tangent \perp radius)

$$\text{Area of } \Delta IQR + \text{area of } \Delta IPR + \text{area of } \Delta IPQ = \text{Area of } \Delta PQR$$

$$\frac{1}{2} \cdot 16r + \frac{1}{2} \cdot 17r + \frac{1}{2} \cdot 17r = \frac{1}{2} \cdot 16 \times 15$$

$$r = IA = \frac{24}{5}$$

Join QH and produce it to cut PR at D . $QD \perp PR$.

$ADPQ$ is a cyclic quadrilateral (converse, \angle s in the same segment)

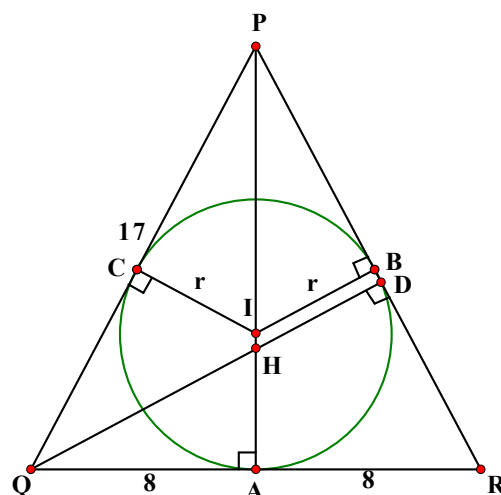
Let $\angle AQH = \theta$, then $\angle APR = \theta$ (\angle s in the same segment)

In ΔAHQ , $HA = 8 \tan \theta$

$$\text{In } \Delta APR, \tan \theta = \frac{8}{15}$$

$$\therefore HA = 8 \times \frac{8}{15} = \frac{64}{15}$$

$$IH = IA - HA = \frac{24}{5} - \frac{64}{15} = \frac{8}{15}$$



Paper 2

1. 設 $\frac{A}{2022} = \frac{1}{1+1 \times 2 \times 3 \times \cdots \times 2022} + \frac{1}{1 + \frac{1}{1 \times 2 \times 3 \times \cdots \times 2022}}$ 。求 A 的值。

Let $\frac{A}{2022} = \frac{1}{1+1 \times 2 \times 3 \times \cdots \times 2022} + \frac{1}{1 + \frac{1}{1 \times 2 \times 3 \times \cdots \times 2022}}$. Find the value of A .

$$\frac{A}{2022} = \frac{1}{1+1 \times 2 \times 3 \times \cdots \times 2022} + \frac{1 \times 2 \times 3 \times \cdots \times 2022}{1 + 1 \times 2 \times 3 \times \cdots \times 2022} = 1$$

$$A = 2022$$

2. \overline{AB} 和 \overline{CB} 均為兩位正整數，其中 A 、 B 和 C 是不同的數字。設 $d = \overline{AB} + \overline{CB}$ 。若 $\overline{AB} \times \overline{CB} = \overline{BCBB}$ 是四位數，求 d 的值。

Both \overline{AB} and \overline{CB} are two-digit positive integers, where A , B and C are different digits.

Let $d = \overline{AB} + \overline{CB}$. If $\overline{AB} \times \overline{CB} = \overline{BCBB}$ is a four-digit number, find the value of d .

$B \neq 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$ $\therefore B = 1, 5$ or 6

$$(10A + B)(10C + B) = 1000B + 100C + 10B + B$$

$$100AC + 10(A + C)B + B^2 = 1000B + 100C + 10B + B$$

$$100(A - 1)C = [1011 - 10(A + C) - B]B$$

L.H.S. is multiple of 100 $\Rightarrow 1011 - 10(A + C) - B$ is a multiple of 100

$$\Rightarrow B = 1$$

$$100(A - 1)C = 1011 - 10(A + C) - 1 = 10(101 - A - C)$$

$$10(A - 1)C = 101 - A - C$$

$$A - 9C + 10AC = 101$$

$$A(1 + 10C) - 9C = 101$$

$$10A(1 + 10C) - 90C = 1010$$

$$10A(1 + 10C) - 9(1 + 10C) = 1001 = 7 \times 11 \times 13$$

$$(10A - 9)(10C + 1) = 7 \times 11 \times 13$$

$$(10A - 9, 10C + 1) = (11, 91) \Rightarrow A = 2, C = 9$$

$$(10A - 9, 10C + 1) = (91, 11) \Rightarrow A = 10, C = 1 \text{ (rejected)}$$

$$d = 21 + 91 = 112$$

3. 假設方程 $x^2y - 2x^2 - 3y - 13 = 0$ 只有一對正整數解 (x_0, y_0) 。若 $a = y_0 - x_0$ ，求 a 的值。
Suppose the equation $x^2y - 2x^2 - 3y - 13 = 0$ has only one pair of positive integral solution (x_0, y_0) . If $a = y_0 - x_0$, find the value of a .

$$x^2y - 2x^2 - 3y - 13 = 0$$

$$x^2(y - 2) - 3(y - 2) = 19$$

$$(x^2 - 3)(y - 2) = 19$$

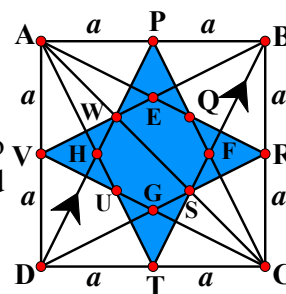
$$\begin{cases} x^2 - 3 = 1 \\ y - 2 = 19 \end{cases} \quad \text{or} \quad \begin{cases} x^2 - 3 = 19 \\ y - 2 = 1 \end{cases}$$

$x = 2, y = 21$ or no positive integral solution

$$a = 21 - 2 = 19$$

4. 圖一所示為一正方形。每一條邊的中點都連接對邊的兩端點，由此形成一個四角星(著色部分)。求 $\frac{\text{四角星的面積}}{\text{正方形的面積}}$ 的值。

Figure 1 shows a square. The mid-point of each side is joined to the two end points of the opposite side and a four-pointed star is thus formed (the shaded part). Find the value of $\frac{\text{Area of the four point star}}{\text{Area of the square}}$.



Using the notations in the figure, the square $ABCD$ has length $= 2a$.

$$\text{Area} = (2a)^2 = 4a^2$$

$$\angle VAW = 45^\circ \quad (\text{Property of a square})$$

$$\angle ADC = 90^\circ \quad (\text{Property of a square})$$

$$AC = 2\sqrt{2}a \quad (\text{Pythagoras' theorem})$$

$$\triangle TDP \cong \triangle CTB \quad (\text{S.A.S.})$$

$$\angle TDP = \angle CTB \quad (\text{corr. } \angle\text{s, } \cong \Delta\text{s})$$

$$PD \parallel BT \quad (\text{corr. } \angle\text{s eq.})$$

$$AW = WS = SC = \frac{2\sqrt{2}a}{3} \quad (\text{Intercept theorem})$$

$$\text{Area of the star} = 4a^2 - 8 \times \text{area of } \triangle AVW$$

$$= 4a^2 - 8 \times \frac{1}{2} a \cdot \frac{2\sqrt{2}}{3} a \sin 45^\circ$$

$$= \frac{4a^2}{3}$$

$$\frac{\text{Area of the four point star}}{\text{Area of the square}} = \frac{\frac{4a^2}{3}}{4a^2} = \frac{1}{3}$$

5. $VABC$ 為一個錐體，其中 $VA = VB = VC$ 及 $AB = BC = CA = a$ m。設它的高為 h m 及它的總表面積及體積相等。若 a 和 h 均為正整數，求 h 的可能值之和。

$VABC$ is a right pyramid with $VA = VB = VC$ and $AB = BC = CA = a$ m. Let its height be h m and its total surface area and volume are the same. If a and h are both positive integers, find the sum of all possible values of h .

$$\text{The base } \triangle ABC \text{ is an equilateral triangle with area} = \frac{1}{2} a^2 \sin 60^\circ \text{ m}^2 = \frac{\sqrt{3}}{4} a^2 \text{ m}^2$$

Let the projection of V on ABC be O , then O is the centroid of $\triangle ABC$.

$$\angle AOB = \angle BOC = \angle COA = 120^\circ \quad (\angle\text{s at a point})$$

$$OA = OB = OC = \frac{\frac{a}{2}}{\cos 30^\circ} = \frac{a}{\sqrt{3}}$$

$VO \perp$ the base $\triangle ABC$.

$$\text{In } \triangle VOA, \angle AOV = 90^\circ, OA^2 + VO^2 = VA^2 \quad (\text{Pythagoras' theorem})$$

$$VA^2 = \left(\frac{a}{\sqrt{3}} \right)^2 + h^2 = \frac{a^2}{3} + h^2$$

Let M be the mid-point of AB . $AM = MB = \frac{a}{2}$, $VM \perp AB$.

In $\triangle VAM$, $VM^2 + AM^2 = VA^2$ (Pythagoras' theorem)

$$VM^2 + \frac{a^2}{4} = \frac{a^2}{3} + h^2$$

$$VM = \sqrt{\frac{a^2}{12} + h^2}$$

$$\text{Area of a lateral face} = \frac{1}{2} AB \times VM = \frac{1}{2} a \times \sqrt{\frac{a^2}{12} + h^2}$$

Total surface area = volume

$$\frac{3}{2} a \times \sqrt{\frac{a^2}{12} + h^2} + \frac{\sqrt{3}}{4} a^2 = \frac{1}{3} \cdot \frac{\sqrt{3}}{4} a^2 h$$

$$18 \cdot \sqrt{\frac{a^2}{12} + h^2} + 3\sqrt{3}a = \sqrt{3}ah$$

$$18 \cdot \sqrt{\frac{a^2}{12} + h^2} = \sqrt{3}(h-3)a$$

$$108 \left(\frac{a^2}{12} + h^2 \right) = (h-3)^2 a^2$$

$$9a^2 + 108h^2 = (h^2 - 6h + 9)a^2$$

$$a^2h^2 - 6a^2h - 108h^2 = 0$$

$$a^2h - 6a^2 - 108h = 0 \dots\dots (*)$$

$$a^2(h-6) = 108h$$

$$a^2 = \frac{108h}{h-6} \dots\dots (1) \Rightarrow h > 6$$

When $h = 7$, no integral solution for a

When $h = 8$, no integral solution for a

$$\text{When } h = 9, a^2 = \frac{108 \times 9}{9-6} \Rightarrow a = 18$$

$$(*) \text{ can be rearranged as } h = \frac{6a^2}{a^2 - 108} \dots\dots (2) \Rightarrow a > 10$$

$$h \geq 9 \Rightarrow 6a^2 \geq 9(a^2 - 108)$$

$$9 \times 36 \geq a^2 \Rightarrow 18 \geq a > 10$$

From (2), a must be a multiple of 6

$$\text{Put } a = 12 \text{ into (2), } h = \frac{6 \times 12^2}{12^2 - 108} = 24$$

Sum of all possible values of $h = 9 + 24 = 33$

6. 圖二中， $ABCD$ 是平行四邊形。 E 為 BC 的中點， AE 和 BD 相交於 H ， AC 和 DE 相交於 F ， AC 和 BD 相交於 G 。若四邊形 $EFGH$ 的面積及 $ABCD$ 的面積分別為 10 cm^2 及 $k \text{ cm}^2$ ，求 k 的值。

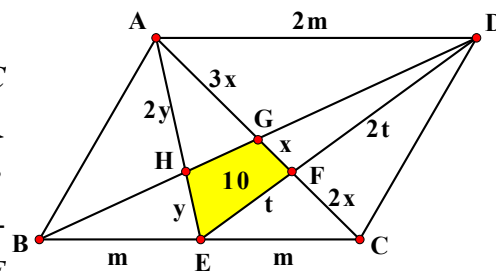


Figure 2 圖二

In Figure 2, $ABCD$ is a parallelogram. E is the mid-point of BC , AE and BD intersect at H , AC and DE intersect at F , AC and BD intersect at G . If the area of the quadrilateral $EFGH$ and $ABCD$ are 10 cm^2 and $k \text{ cm}^2$ respectively, find the value of k .

E is the mid-point of BC . Let $BE = m = EC$, $AD = 2m$

$\triangle ADH \sim \triangle EBH$ (equiangular)

$AH : HE = AD : BE = 2 : 1$ (corr. sides, $\sim \Delta$ s)

Let $AH = 2y$, $HE = y$

$\triangle ADF \sim \triangle CEF$ (equiangular)

$AF : CF = DF : EF = 2 : 1$ (corr. sides, $\sim \Delta$ s)

Let $AF = 4x$, $CF = 2x$, $DF = 2t$, $EF = t$, $AC = AF + FC = 6x$

$AG = GC = 3x$, $GF = GC - FC = x$

Let S represents the area. $\triangle ACD \cong \triangle CAB$ (S.S.S.)

$$S_{\triangle ACD} = S_{\triangle CAB} = \frac{k}{2}$$

$\triangle ADG$, $\triangle GDF$ and $\triangle CDF$ have the same height but different bases AG , GF and FC .

$$S_{\triangle ADG} : S_{\triangle GDF} : S_{\triangle CDF} = AG : GF : FC = 3 : 1 : 2$$

$$S_{\triangle ADG} = \frac{k}{2} \times \frac{3}{6} = \frac{k}{4}, S_{\triangle GDF} = \frac{k}{2} \times \frac{1}{6} = \frac{k}{12}, S_{\triangle CDF} = \frac{k}{2} \times \frac{2}{6} = \frac{k}{6}$$

Height of $\triangle ADE$ = height of \parallel -gram $ABCD$ and they have common base AD .

$$\therefore \text{Area of } \triangle ADE = \frac{1}{2} \times \text{area of } \parallel\text{-gram} = \frac{k}{2}$$

$\triangle ADH$ and $\triangle DEH$ have the same height but different bases AH and HE .

$$S_{\triangle DEH} = \frac{k}{2} \times \frac{1}{3} = \frac{k}{6}$$

$$S_{EFGH} = S_{\triangle DEH} - S_{\triangle GDF}$$

$$10 = \frac{k}{6} - \frac{k}{12}$$

$$k = 120$$

7. 已知 $x + y + z = 1 \dots(1)$, $x^2 + y^2 + z^2 = 2 \dots(2)$ 及 $x^3 + y^3 + z^3 = 3 \dots(3)$ 。求 $x^4 + y^4 + z^4$ 的值。

Given that $x + y + z = 1$, $x^2 + y^2 + z^2 = 2$ and $x^3 + y^3 + z^3 = 3$. Find the value of $x^4 + y^4 + z^4$.

$$(1)^2: (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = 1$$

$$2 + 2(xy + yz + zx) = 1 \Rightarrow xy + yz + zx = -\frac{1}{2} \dots(4)$$

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)[x^2 + y^2 + z^2 - (xy + yz + zx)]$$

$$3 - 3xyz = 1 \times [2 - (-\frac{1}{2})]$$

$$xyz = \frac{1}{6} \dots(5)$$

$$(4)^2: (xy + yz + zx)^2 = \left(-\frac{1}{2}\right)^2$$

$$x^2y^2 + y^2z^2 + z^2x^2 + 2xyz(x + y + z) = \frac{1}{4}$$

$$x^2y^2 + y^2z^2 + z^2x^2 + 2 \times \frac{1}{6} \times 1 = \frac{1}{4}$$

$$x^2y^2 + y^2z^2 + z^2x^2 = -\frac{1}{12} \dots(5)$$

$$(2)^2: (x^2 + y^2 + z^2)^2 = 2^2$$

$$x^4 + y^4 + z^4 + 2(x^2y^2 + y^2z^2 + z^2x^2) = 4$$

$$x^4 + y^4 + z^4 + 2 \times \left(-\frac{1}{12}\right) = 4$$

$$x^4 + y^4 + z^4 = \frac{25}{4}$$

Method 2 (Newton's formulae for the sums of powers of the roots)

Let $S_n = x^n + y^n + z^n$, then $S_0 = 3$, $S_1 = 1$, $S_2 = 2$, $S_3 = 3$

Let x , y and z be the roots of $t^3 + a_1t^2 + a_2t + a_3 = 0$

Let $f(t) = (t - x)(t - y)(t - z) \equiv t^3 + a_1t^2 + a_2t + a_3$

$\ln f(t) = \ln(t - x) + \ln(t - y) + \ln(t - z) = \ln(t^3 + a_1t^2 + a_2t + a_3)$

Differentiate both sides w.r.t. t :

$$\frac{f'(t)}{f(t)} = \frac{1}{t - x} + \frac{1}{t - y} + \frac{1}{t - z} = \frac{3t^2 + 2a_1t + a_2}{t^3 + a_1t^2 + a_2t + a_3}$$

$$\text{Replace } t \text{ by } \frac{1}{u}: \frac{u}{1 - ux} + \frac{u}{1 - uy} + \frac{u}{1 - uz} = \frac{(3 + 2a_1u + a_2u^2)u}{1 + a_1u + a_2u^2 + a_3u^3}$$

$$\frac{1}{1 - ux} + \frac{1}{1 - uy} + \frac{1}{1 - uz} = \frac{3 + 2a_1u + a_2u^2}{1 + a_1u + a_2u^2 + a_3u^3}$$

$$\sum_{n=0}^{\infty} (xu)^n + \sum_{n=0}^{\infty} (yu)^n + \sum_{n=0}^{\infty} (zu)^n = \frac{3 + 2a_1u + a_2u^2}{1 + a_1u + a_2u^2 + a_3u^3}$$

$$\sum_{n=0}^{\infty} S_n u^n = \frac{3 + 2a_1u + a_2u^2}{1 + a_1u + a_2u^2 + a_3u^3}$$

$$(1 + a_1u + a_2u^2 + a_3u^3) \left(\sum_{n=0}^{\infty} S_n u^n \right) = 3 + 2a_1u + a_2u^2$$

Compare coefficients of u : $a_1S_0 + S_1 = 2a_1 \Rightarrow a_1 = -1$

Compare coefficients of u^2 : $a_2S_0 + a_1S_1 + S_2 = a_2 \Rightarrow 2a_2 - 1 + 2 = 0 \Rightarrow a_2 = -\frac{1}{2}$

Compare coefficients of u^3 : $a_3S_0 + a_2S_1 + a_1S_2 + S_3 = 0 \Rightarrow 3a_3 - \frac{1}{2} - 1 \times 2 + 3 = 0 \Rightarrow a_3 = -\frac{1}{6}$

Compare coefficients of u^4 : $a_3S_1 + a_2S_2 + a_1S_3 + S_4 = 0$

$$\Rightarrow -\frac{1}{6} - \frac{1}{2} \times 2 - 1 \times 3 + S_4 = 0$$

$$\Rightarrow S_4 = \frac{25}{4}$$

8. 對所有正整數 $n > 1$, 函數 f 定義如下:

$$f(1) = 2021 \text{ 及 } f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n).$$

求 $f(2021)$ 的值。

For all positive integers $n > 1$, a function f is defined as

$$f(1) = 2021 \text{ and } f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n).$$

Find the value of $f(2021)$.

Reference: 2013 HG10, 2014 FG1.4

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n) \Rightarrow f(n) = \frac{f(1) + f(2) + \cdots + f(n-1)}{n^2 - 1}$$

$$f(2) = \frac{f(1)}{3} = \frac{2021}{3}$$

$$f(3) = \frac{f(1) + f(2)}{8} = \frac{2021 + \frac{2021}{3}}{8} = \frac{1 + \frac{1}{3}}{8} \cdot 2021 = \frac{1}{6} \cdot 2021$$

$$f(4) = \frac{f(1) + f(2) + f(3)}{15} = \frac{2021 + \frac{2021}{3} + \frac{2021}{6}}{15} = \frac{\frac{3}{15}}{15} \cdot 2021 = \frac{1}{10} \cdot 2021$$

It is observed that the answer is 2021 divided by the n^{th} triangle number.

$$\text{Claim: } f(n) = \frac{2}{n(n+1)} \cdot 2021 \text{ for } n \geq 1$$

$n = 1, 2, 3, 4$, proved above.

$$\text{Suppose } f(k) = \frac{2}{k(k+1)} \cdot 2021 \text{ for } k = 1, 2, \dots, m \text{ for some positive integer } m.$$

$$f(m+1) = \frac{f(1) + f(2) + \cdots + f(m)}{(m+1)^2 - 1} = \frac{\frac{2}{1 \times 2} + \frac{2}{2 \times 3} + \frac{2}{3 \times 4} + \frac{2}{4 \times 5} + \frac{2}{5 \times 6} + \cdots + \frac{2}{m(m+1)}}{m(m+2)} \cdot 2021$$

$$= 2 \cdot \frac{\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m} - \frac{1}{m+1}\right)}{m(m+2)} \cdot 2021$$

$$= 2 \cdot \frac{1 - \frac{1}{m+1}}{m(m+2)} \cdot 2021 = \frac{2}{(m+1)(m+2)} \cdot 2021$$

\therefore It is also true for m . By the principle of mathematical induction, the formula is true.

$$f(2021) = \frac{2}{2021 \times 2022} \cdot 2021 = \frac{1}{1011}$$