Examples on Mathematical Induction: Sum of product of integers

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1. Prove that $1\cdot3 + 2\cdot3^2 + 3\cdot3^3 +\cdots+ n\cdot3^n = \frac{1}{4}\left[(2n-1)3^{n+1}+3\right]$ for $n = 1, 2, 3, \cdots$

Let $P(n) \equiv$ "$1\cdot3 + 2\cdot3^2 + 3\cdot3^3 +\cdots+ n\cdot3^n = \frac{1}{4}\left[(2n-1)3^{n+1}+3\right]$" for $n = 1, 2, 3, \cdots$"

$n = 1$, L.H.S. $= 1\cdot3 = 3$; R.H.S. $= \frac{1}{4}\left[(1)3^2+3\right] = 3$

$P(1)$ is true.

Suppose $P(k)$ is true. i.e. $1\cdot3 + 2\cdot3^2 + 3\cdot3^3 +\cdots + k\cdot3^k = \frac{1}{4}\left[(2k-1)3^{k+1}+3\right]$ for some $k > 0$

When $n = k + 1$, LHS $= 1\cdot3 + 2\cdot3^2 + 3\cdot3^2 +\cdots + k\cdot3^k + (k + 1)3^{k+1}$

$$= \frac{1}{4}\left[(2k-1)3^{k+1}+3\right] + (k+1)3^{k+1}$$

$$= \frac{1}{4}\left[(2k-1)3^{k+1}+3+4(k+1)3^{k+1}\right]$$

$$= \frac{1}{4}\left[(6k+3)3^{k+1}+3\right]$$

$$= \frac{1}{4}\left[(2k+1)3^{k+2}+3\right] = \text{R.H.S.}$$

$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive n.

2. **AM 2002 Q12**

(a) Prove, by mathematical induction, that

$$2(2) + 3(2^2) + 4(2^3) +\cdots + (n + 1)(2^n) = n(2^{n+1})$$

for all positive integers n.

(b) Show that

$$1(2) + 2(2^2) + 3(2^3) +\cdots+ 98(2^{98}) = 97(2^{99}) + 2$$

(a) When $n = 1$,

L.H.S. $= 2(2) = 4$

R.H.S. $= 1(22) = 4$

L.H.S. = R.H.S.

The statement is true for $n = 1$.

Suppose $2(2) + 3(2^2) +4(2^3) +\cdots + (k + 1)(2^k) = k(2^{k+1})$ for some positive integer k .

When $n = k + 1$,

L.H.S. $= 2(2) + 3(2^2) +4(2^3) +\cdots + (k + 1)(2^k) + (k + 2)(2^{k+1})$

$= k(2^{k+1}) + (k + 2)(2^{k+1})$

$= (2k + 2)(2^{k+1})$

$= (k + 1)(2^{k+2}) =$ R.H.S.

The statement is also true for $n = k + 1$.

By the principle of mathematical induction, it is true for all positive integer n .

(b) $1(2) + 2(2^2)+3(2^3)+ \cdots+98(2^{98})$

$= 2(2) + 3(2^2)+4(2^3)+ \cdots+99(2^{98}) - [2 + 2^2 + 2^3 +\cdots + 2^{98}]$

$= 98(2^{99}) - \dfrac{2\left[2^{98}-1\right]}{2-1}$

$= 98(2^{99}) - 2^{99} + 2$

$= 97(2^{99}) + 2$

3. **2001 Additional Mathematics Q12 Mathematics 1975 Paper 1 Q9**

Prove that $1\times2 + 2\times3 + 3\times4 + \cdots + n(n + 1) = \frac{1}{3}n(n+1)(n+2)$ for all positive integer n.

Hence find the value of

(i) $1\times3 + 2\times4 + 3\times5 + \cdots + 50\times52$.

(ii) $51\cdot52 + 52\cdot53 + \ldots + 100\cdot101$

(iii) $1 + (1 + 2) + (1 + 2 + 3) +\cdots+ (1 + 2 + \cdots + 100)$.

Let $P(n) \equiv$ "$1\times2 + 2\times3 + 3\times4 + \cdots + n(n + 1) = \frac{1}{3}n(n+1)(n+2)$ for $n = 1, 2, 3, \cdots$"

$n = 1$, L.H.S. $= 1\times2 = 2$, R.H.S. $= \frac{1}{3}\cdot1(1+1)(1+2) = 2$

L.H.S. = R.H.S. It is true for $n = 1$

Suppose it is true for $n = k$ for some positive integer k.

i.e. $1\times2 + 2\times3 + 3\times4 + \cdots + k(k + 1) = \frac{1}{3}(k+1)(k+2)(k+3)$

Add $(k + 1)(k + 2)$ to both sides.

L.H.S. $= 1\times2 + 2\times3 + 3\times4 + \cdots + k(k + 1) + (k + 1)(k + 2)$

$= \frac{1}{3}k(k+1)(k+2) + (k + 1)(k + 2)$ by induction assumption

$= \frac{1}{3}(k+1)(k+2)(k+3) =$ R.H.S.

If it is true for $n = k$, then it is also true for $n = k + 1$.

By the principle of mathematical induction, the formula is true for all positive integer n.

(i) $1\times3 + 2\times4 + 3\times5 + \cdots + 50\times52$

$= 1\times(2 + 1) + 2\times(3 + 1) + 3\times(4 + 1) + \cdots + 50\times(51 + 1)$

$= (1\times2 + 2\times3 + 3\times4 + \cdots + 50\times51) + (1 + 2 + 3 + \cdots + 50)$

$= \frac{1}{3}\cdot50\times51\times52 + \frac{1}{2}\cdot50\times51 = \frac{1}{6}\cdot50\times51\times(104+3)$

$= 45475$

(ii) $51\cdot52 + 52\cdot53 + \ldots + 100\cdot101$

$= 1\times2 + 2\times3 + \cdots + 100\cdot101 - (1\times2 + 2\times3 + \cdots + 50\times51)$

$= \frac{1}{3}\cdot100\times101\times102 - \frac{1}{3}\cdot50\times51\times52$

$= 343400 - 44200 = 299200$

(iii) $1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + \ldots + 100)$

$= \frac{1}{2}\cdot1\times2 + \frac{1}{2}\cdot2\times3 + \frac{1}{2}3\times4 + \cdots + \frac{1}{2}\cdot100\times101$

$= \frac{1}{2}\times\frac{1}{3}\cdot100\times101\times102$

$= 171700$

4. Prove that $1\cdot2 + 4\cdot3 + 9\cdot4 + 16\cdot5 + \cdots + n^2(n+1) = \frac{1}{12}n(n+1)(3n^2+7n+2)$ for all positive integer n .

5. Prove that $2\cdot3 + 4\cdot6 + 6\cdot9 + \cdots + 2n(3n) = n(n + 1)(2n + 1)$ for all positive integer n .

6. Prove that $1 + 3 + 6 + 10 + \cdots + \frac{n(n+1)}{2} = \frac{1}{6}n(n + 1)(n + 2)$ for all positive integer n .

7. **2009 Q5 1969 Paper 1 Q1**

Prove that $1\cdot 4 + 2\cdot 5 + \cdots + n(n + 3) = \frac{1}{3}n(n+1)(n+5)$ for all positive integers n.

Let $P(n) \equiv$ "$1\times 4 + 2\times 5 + 3\times 6 + \cdots + n(n + 3) = \frac{1}{3}n(n+1)(n+5)$", where n is a positive integer.

$n = 1$, L.H.S. $= 1\times 4 = 4$, R.H.S. $= \frac{1}{3}\cdot 1\times 2\times 6 = 4$

L.H.S. = R.H.S.

$\therefore$ P(1) is true.

Suppose $1\times 4 + 2\times 5 + 3\times 6 + \cdots + k(k + 3) = \frac{1}{3}k(k+1)(k+5)$ for some positive integer k.

When $n = k + 1$,

L.H.S. $= 1\times 4 + 2\times 5 + 3\times 6 + \cdots + k(k + 3) + (k + 1)(k + 4)$

$= \frac{1}{3}k(k+1)(k+5) + (k + 1)(k + 4)$

$= \frac{1}{3}(k+1)(k^2+5k+3k+12)$

$= \frac{1}{3}(k+1)(k^2+8k+12)$

$= \frac{1}{3}(k+1)(k+2)(k+6)$

$= \frac{1}{3}(k+1)(k+1+1)(k+1+5)$

$\therefore$ If P(k) is true then P(k +1) is also true.

By M.I., P(n) is true for all positive integer n.

8. Prove that $1\cdot n+2\cdot(n-1)+3\cdot(n-2)+\cdots+(n-1)\cdot 2+n\cdot 1=\frac{1}{6}n(n + 1)(n + 2)$ for all positive integers n.

Let $P(n) \equiv$ "$1\times n+2\times(n-1)+3\times(n-2)+\cdots+(n-1)\times 2+n\times 1=\frac{n}{6}(n+1)(n+2)$"

$n = 1$. L.H.S. $= 1$; R.H.S. $= \frac{1}{6}(2)(3) = 1$

$P(1)$ is true.

Suppose $P(k)$ is true.

$1\times k+2\times(k-1)+3\times(k-2)+\cdots+(k-1)\times 2+k\times 1=\frac{k}{6}(k+1)(k+2)$

$1\times(k+1)+2\times k+3\times(k-1)+\cdots+k\times 2+(k+1)\times 1$

$=1\times k+2\times(k-1)+\cdots+(k-1)\times 2+k\times 1+[1+2+\cdots+k+(k+1)]$

$=\frac{k}{6}(k+1)(k+2)+\frac{1}{2}(k+1)(k+2)$

$=\frac{1}{6}(k+1)(k+2)(k+3)$

$\therefore$ If $P(k)$ is true, then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all natural numbers.

9. Prove that $1\cdot 1 + 2\cdot 3 + 3\cdot 5 +\cdots+ n(2n - 1) = \frac{1}{6}n(n+1)(4n-1)$ for all positive integers n.

Hence simplify $1\times(2n-1) + 2\times(2n-3) + 3\times(2n-5) +\cdots + n[2n - (2n-1)]$.

(a) Let $P(n) \equiv$ "$1\times1 + 2\times3 + 3\times5 +\cdots + n(2n-1) = \frac{n(n+1)(4n-1)}{6}$ for $n = 1, 2, 3, \cdots$"

$n = 1$, L.H.S. = 1, R.H.S. = $\frac{1\cdot 2\cdot 3}{6} = 1$

It is true for $n = 1$

Suppose $1\times1 + 2\times3 + 3\times5 +\cdots+ k(2k - 1) = \frac{k(k+1)(4k-1)}{6}$

When $n = k + 1$,

L.H.S. $= 1\times1 + 2\times3 + 3\times5 +\cdots+ k(2k - 1) + (k + 1)(2k + 1)$

$= \frac{k(k+1)(4k-1)}{6} + (k + 1)(2k + 1)$ (induction assumption)

$= \frac{(k+1)}{6}[k(4k-1)+6(2k+1)]$ $(= \frac{4k^3+15k^2+17k+6}{6})$

$= \frac{(k+1)}{6}[4k^2-k+12k+6]$

$= \frac{(k+1)}{6}(4k^2+11k+6)$

$= \frac{(k+1)(k+2)(4k+3)}{6}$

$= \frac{(k+1)(k+1+1)[4(k+1)-1]}{6}$ = R.H.S.

If $P(k)$ is true, then $P(k + 1)$ is also true.

By the principle of mathematical induction, it is true for all positive integers n.

(b) $1\times(2n-1) + 2\times(2n-3) + 3\times(2n-5) + \cdots + n[2n - (2n-1)]$

$= 1\times2n + 2\times2n + 3\times2n +\cdots+ n\times2n - [1\times1 + 2\times3 + 3\times5 +\cdots+ n(2n - 1)]$

$= 2n(1 + 2 + 3 +\cdots+ n) - \frac{n(n+1)(4n-1)}{6}$

$= 2n\times\frac{n(n+1)}{2} - \frac{n(n+1)(4n-1)}{6}$

$= \frac{n(n+1)}{6}(6n-4n+1)$

$= \frac{n(n+1)(2n+1)}{6}$

10. Prove, by mathematical induction, that

$2\times1 + 3\times3 + 4\times5 + \cdots + (n+1)(2n-1) = \frac{n}{6}\left(4n^2 + 9n - 1\right)$ for all positive integers n.

Let $P(n) \equiv$ "$2\times1 + 3\times3 + 4\times5 + \cdots + (n+1)(2n-1) = \frac{n}{6}\left(4n^2 + 9n - 1\right)$", n is a +ve integer.

$n = 1$, L.H.S. $= 2\times1 = 2$, R.H.S. $= \frac{1}{6}\left(4\cdot1^2 + 9 - 1\right) = 2$

L.H.S. = R.H.S.

$P(1)$ is true.

Suppose that $P(k)$ is true for some positive integer k.

i.e. $2\times1 + 3\times3 + 4\times5 + \cdots + (k+1)(2k-1) = \frac{k}{6}\left(4k^2 + 9k - 1\right)$ …… (*)

When $n = k + 1$,

L.H.S. $= 2\times1 + 3\times3 + 4\times5 + \cdots + (k+1)(2k-1) + (k+2)[2(k+1)-1]$

$= \frac{k}{6}\left(4k^2 + 9k - 1\right) + (k+2)(2k+1)$ (by (*))

$= \frac{1}{6}\left[\left(4k^3 + 9k^2 - k\right) + 6\left(2k^2 + 5k + 2\right)\right]$

$= \frac{1}{6}\left(4k^3 + 9k^2 - k + 12k^2 + 30k + 12\right)$

$= \frac{1}{6}\left(4k^3 + 21k^2 + 29k + 12\right)$

R.H.S. $= \frac{k+1}{6}\left[4(k+1)^2 + 9(k+1) - 1\right]$

$= \frac{k+1}{6}\left(4k^2 + 8k + 4 + 9k + 9 - 1\right)$

$= \frac{k+1}{6}\left(4k^2 + 17k + 12\right)$

$= \frac{1}{6}\left(4k^3 + 17k^2 + 12k + 4k^2 + 17k + 12\right)$

$= \frac{1}{6}\left(4k^3 + 21k^2 + 29k + 12\right)$

If $P(k)$ is true then $P(k+1)$ is also true

By the principle of mathematical induction, $P(n)$ is true for all positive integer n.

11. **1993 Paper 2 Q1 1969 香港中文中學會考高級數學試卷一 Q5**

Prove by mathematical induction, that

$1^2\times 2 + 2^2\times 3 + \cdots + n^2(n+1) = \dfrac{n(n+1)(n+2)(3n+1)}{12}$ for any positive integer n.

Let $P(n) \equiv$ "$1^2\times 2 + 2^2\times 3 + \cdots + n^2(n+1) = \dfrac{n(n+1)(n+2)(3n+1)}{12}$ for any positive integer n."

$n = 1$, L.H.S. $= 1^2\times 2 = 2$, R.H.S. $= \dfrac{1(1+1)(1+2)(3+1)}{12} = 2 =$ L.H.S.

$P(1)$ is true

Suppose $P(k)$ is true,

then $1^2\times 2 + 2^2\times 3 + \cdots + k^2(k+1) = \dfrac{k(k+1)(k+2)(3k+1)}{12}$ for some positive integer k.

When $n = k + 1$

L.H.S. $= 1^2\times 2 + 2^2\times 3 + \cdots + k^2(k+1) + (k+1)^2(k+2)$

$$= \frac{k(k+1)(k+2)(3k+1)}{12} + (k+1)^2(k+2) \text{ (induction assumption)}$$

$$= \frac{k(k+1)(k+2)(3k+1)}{12} + \frac{12(k+1)^2(k+2)}{12}$$

$$= \frac{(k+1)(k+2)[k(3k+1)+12(k+1)]}{12}$$

$$= \frac{(k+1)(k+2)(3k^2+13k+12)}{12}$$

$$= \frac{(k+1)(k+2)(k+3)(3k+4)}{12}$$

$$= \frac{(k+1)(k+1+1)(k+1+2)[3(k+1)+1]}{12} = \text{R.H.S.}$$

If $P(k)$ is true, then $P(k+1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

12. **1998 Paper 2 Q3**

Prove, by mathematical induction, that

$1\times2 + 2\times3 + 2^2\times4 +\cdots + 2^{n-1}(n + 1) = 2^n(n)$ for all positive integers n.

Let $P(n) \equiv$ "$1\times2 + 2\times3 + 2^2\times4 +\cdots + 2^{n-1}(n + 1) = 2^n(n)$", n is a positive integer.

$n = 1$, L.H.S. $= 1\times2 = 2$, R.H.S. $= 2^1(1) = 2$

L.H.S. = R.H.S.

$P(1)$ is true.

Suppose that $P(k)$ is true for some positive integer k.

i.e. $1\times2 + 2\times3 + 2^2\times4 +\cdots + 2^{k-1}(k + 1) = 2^k(k)$ $\cdots\cdots$ (*)

When $n = k + 1$,

L.H.S. $= 1\times2 + 2\times3 + 2^2\times4 +\cdots + 2^{k-1}(k + 1)+ 2^k(k + 2)$

$= 2^k(k) + 2^k(k + 2)$ (by (*))

$= 2^k(k + k + 2) = 2^k(2k + 2)$

$= 2^{k+1}(k + 1) =$ R.H.S.

If $P(k)$ is true then $P(k + 1)$ is also true

By the principle of mathematical induction, $P(n)$ is true for all positive integer n.

13. **1992 Paper 2 Q1 2012 M2 Q3**

Prove by mathematical induction, that

$1\times2 + 2\times5 + 3\times8 +\cdots + n(3n - 1) = n^2(n + 1)$ for all positive integers n.

Let $P(n) \equiv$ "$1\times2 + 2\times5 + 3\times8 +\cdots + n(3n - 1) = n^2(n + 1)$", where n is a positive integer.

$n = 1$, L.H.S. $= 1\times2 = 2$, R.H.S. $= 1^2(1 + 1) = 2$

L.H.S. = R.H.S.

$P(1)$ is true.

Suppose that $P(k)$ is true for some positive integer k.

i.e. $1\times2 + 2\times5 + 3\times8 +\cdots + k(3k - 1) = k^2(k + 1)$ $\cdots\cdots\cdots$ (*)

When $n = k + 1$,

L.H.S. $= 1\times2 + 2\times5 + 3\times8 +\cdots+ k(3k - 1) + (k + 1)[3(k + 1) - 1]$

$= k^2(k + 1) + (k + 1)(3k + 2)$ (by (*))

$= (k + 1)(k^2 + 3k + 2)$

$= (k + 1)^2(k + 2)$

= R.H.S.

If $P(k)$ is true then $P(k + 1)$ is also true

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

14. (a) Prove, by mathematical induction, that for all positive integers n,

$1^2\times4 + 2^2\times5 + 3^2\times6 + \cdots + n^2(n + 3) = \frac{1}{4}n(n + 1)(n^2 + 5n + 2)$.

(b) Hence, according to the formula $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$, deduce the formula for evaluating $1^2\times2 + 2^2\times3 + 3^2\times4 + \cdots + n^2(n + 1)$.

(a) Let $P(n) \equiv$ "$1^2\times4 + 2^2\times5 + 3^2\times6 + \cdots + n^2(n + 3) = \frac{1}{4}n(n + 1)(n^2 + 5n + 2)$."

$n = 1$, L.H.S. $= 1^2\times4 = 4$, R.H.S. $= \frac{1}{4}\cdot1(1 + 1)(1^2 + 5 + 2) = 4$

$\therefore$ L.H.S. = R.H.S. $\Rightarrow P(1)$ is true.

Suppose $P(k)$ is true for some positive integer k.

i.e. $1^2\times4 + 2^2\times5 + 3^2\times6 + \cdots + k^2(k + 3) = \frac{1}{4}k(k + 1)(k^2 + 5k + 2)$

Add $(k + 1)^2[(k + 1) + 3]$ to both sides.

L.H.S. $= 1^2\times4 + 2^2\times5 + 3^2\times6 + \cdots + k^2(k + 3) + (k + 1)^2[(k + 1) + 3]$

$= \frac{1}{4}k(k + 1)(k^2 + 5k + 2) + (k + 1)^2(k + 4)$ (induction assumption)

$= \frac{1}{4}(k + 1)(k^3 + 5k^2 + 2k) + \frac{4}{4}(k + 1)(k^2 + 5k + 4)$

$= \frac{1}{4}(k + 1)(k^3 + 5k^2 + 2k + 4k^2 + 20k + 16)$

$= \frac{1}{4}(k + 1)(k^3 + 9k^2 + 22k + 16)$

R.H.S. $= \frac{1}{4}(k + 1)(k + 1 + 1)[(k + 1)^2 + 5(k + 1) + 2]$

$= \frac{1}{4}(k + 1)(k + 2)(k^2 + 7k + 8)$

$= \frac{1}{4}(k + 1)(k^3 + 7k^2 + 8k + 2k^2 + 14k + 16) = \frac{1}{4}(k + 1)(k^3 + 9k^2 + 22k + 16)$

$\therefore$ L.H.S. = R.H.S.

If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, the formula is true for all positive integer n.

(b) $1^2\times2 + 2^2\times3 + 3^2\times4 + \cdots + n^2(n + 1)$

$= 1^2\times(4 - 2) + 2^2\times(5 - 2) + 3^2\times(6 - 2) + \cdots + n^2(n + 3 - 2)$

$= 1^2\times4 + 2^2\times5 + 3^2\times6 + \cdots + n^2(n + 3) - 2(1^2 + 2^2 + \cdots + n^2)$

$= \frac{1}{4}n(n + 1)(n^2 + 5n + 2) - \frac{2}{6}n(n + 1)(2n + 1)$

$= \frac{1}{12}n(n + 1)(3n^2 + 15n + 6 - 8n - 4)$

$= \frac{1}{12}n(n + 1)(3n^2 + 7n + 2)$

$= \frac{1}{12}n(n + 1)(n + 2)(3n + 1)$

15. Prove that $4\cdot 7\cdot 10+7\cdot 10\cdot 13+\cdots+(3n+1)(3n+4)(3n+7)=\frac{1}{12}[(3n+1)(3n+4)(3n+7)(3n+10)-1\cdot 4\cdot 7\cdot 10]$ for all positive integers n.

Let $P(n)\equiv$"$4\cdot 7\cdot 10+7\cdot 10\cdot 13+\cdots+(3n+1)(3n+4)(3n+7)=\frac{1}{12}[(3n+1)(3n+4)(3n+7)(3n+10)-1\cdot 4\cdot 7\cdot 10]$"

$n=1$, L.H.S. $=4\cdot 7\cdot 10$, R.H.S. $=\frac{1}{12}[4\cdot 7\cdot 10\cdot 13-4\cdot 7\cdot 10]=\frac{4\cdot 7\cdot 10}{12}[13-1]$

L.H.S. = R.H.S., $P(1)$ is true.

Suppose $4\cdot 7\cdot 10+7\cdot 10\cdot 13+\cdots+(3k+1)(3k+4)(3k+7)=\frac{1}{12}[(3k+1)(3k+4)(3k+7)(3k+10)-1\cdot 4\cdot 7\cdot 10]$

When $n=k+1$, LHS $=4\cdot 7\cdot 10+7\cdot 10\cdot 13+\ldots+(3k+1)(3k+4)(3k+7)+(3k+4)(3k+7)(3k+10)$

$$=\frac{1}{12}[(3k+1)(3k+4)(3k+7)(3k+10)-1\cdot 4\cdot 7\cdot 10]+(3k+4)(3k+7)(3k+10)$$

$$=\frac{1}{12}[(3k+1)(3k+4)(3k+7)(3k+10)-1\cdot 4\cdot 7\cdot 10+12(3k+4)(3k+7)(3k+10)]$$

$$=\frac{1}{12}[(3k+1+12)(3k+4)(3k+7)(3k+10)-1\cdot 4\cdot 7\cdot 10]$$

$$=\frac{1}{12}[(3k+4)(3k+7)(3k+10)(3k+13)-1\cdot 4\cdot 7\cdot 10]$$

If $P(k)$ is true, then $P(k+1)$ is also true. By induction, it is true for all positive integers n.

16. **1972 香港中文中學會考高級數學試卷一 Q2(b), 1989 Paper 2 Q2**

(a) Prove that $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\cdots\cdots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$

for all positive integer n.

(b) Hence, or otherwise, find the value of $11\cdot 12\cdot 13+12\cdot 13\cdot 14+\cdots\cdots+25\cdot 26\cdot 27$.

(a) Let $P(n)$ be the statement.

When $n=1$, L.H.S. $=1\cdot 2\cdot 3=6$, R.H.S. $=\frac{1}{4}1(1+1)(1+2)(1+3)=6$. $\therefore$ $P(1)$ is true.

Assume $P(k)$ is true. i.e. $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+k(k+1)(k+2)=\frac{1}{4}k(k+1)(k+2)(k+3)$

When $n=k+1$, L.H.S. $=1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+k(k+1)(k+2)+(k+1)(k+2)(k+3)$

$$=\frac{1}{4}k(k+1)(k+2)(k+3)+(k+1)(k+2)(k+3)$$

$$=(k+1)(k+2)(k+3)(\frac{1}{4}k+1)$$

$$=\frac{1}{4}(k+1)(k+2)(k+3)(k+4)$$

$$=\frac{1}{4}(k+1)(k+1+1)(k+1+2)(k+1+3)=\text{R.H.S.}$$

If $P(k)$ is true then $P(k+1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

(b) $11\cdot 12\cdot 13+12\cdot 13\cdot 14+\cdots\cdots+25\cdot 26\cdot 27$

$=(1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots\cdots+25\cdot 26\cdot 27)-(1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots\cdots+10\cdot 11\cdot 12)$

$=\frac{1}{4}25(25+1)(25+2)(25+3)-\frac{1}{4}10(10+1)(10+2)(10+3)$

$=122850-4290$

$=118560$

17. Prove that $1\cdot2\cdot3\cdot4\cdot5 + 2\cdot3\cdot4\cdot5\cdot6 + 3\cdot4\cdot5\cdot6\cdot7 + \cdots$ to n^{th} term $= \frac{1}{6}n(n+1)(n+2)(n+3)(n+4)(n+5)$ for all positive integers n.

18. **1997 Paper 2 Q7**
Let $T_n = (n^2 + 1)(n!)$ for any positive integer n.
Prove, by mathematical induction, that
$T_1 + T_2 + \cdots + T_n = n[(n + 1)!]$ for any positive integer n .
[Note: $n! = n(n - 1)(n - 2) \cdots 3\cdot2\cdot1$.]
Let $P(n) \equiv$ "$T_1 + T_2 + \cdots + T_n = n[(n + 1)!]$ for any positive integer n ."
$n = 1$, $T_1 = (1^2 + 1)(1!) = 2$
R.H.S. $= 1[(1 + 1)!] = 2 =$ L.H.S.
$P(1)$ is true
Suppose $P(k)$ is true for some positive integer k .
i.e. $T_1 + T_2 + \cdots + T_k = k[(k + 1)!]$ for some positive integer k .
When $n = k + 1$,
$T_1 + T_2 + \cdots + T_k + T_{k+1}$
$= k[(k + 1)!] + [(k + 1)^2 + 1][(k + 1)!]$ (induction assumption)
$= (k + k^2 + 2k + 2)[(k + 1)!]$
$= (k^2 + 3k + 2)[(k + 1)!]$
$= (k + 1)(k + 2)[(k + 1)!]$
$= (k + 1)[(k + 2)!] =$ R.H.S.
If $P(k)$ is true then $P(k + 1)$ is also true
By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

19. Let $P(n) \equiv$ "$(1)(1!) + (2)(2!) + 3(3!) + \cdots + (n)(n!) = (n + 1)! - 1$."
$n = 1$, L.H.S. $= (1)(1!) = 1$, R.H.S. $= 2! - 1 = 1$
$P(1)$ is true.
Suppose $(1)(1!) + (2)(2!) + 3(3!) + \cdots + (k)(k!) = (k + 1)! - 1$ for some positive integer k.
Add $(k + 1)(k + 1)!$ To both sides
L.H.S. $= (1)(1!) + (2)(2!) + 3(3!) + \cdots + (k)(k!) + (k + 1)(k + 1)!$
$= (k + 1)! - 1 + (k + 1)(k + 1)!$ (induction assumption)
$= (k + 1)!\,(k + 1 + 1) - 1$
$= (k + 1)!\,(k + 2) - 1$
$= (k + 2)! - 1$
$=$ R.H.S.
$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true.
By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

20. Let $U_n(x)=\dfrac{x(x+1)(x+2)\cdots(x+n-1)}{n!}$, prove that $\sum_{n=1}^{p} U_n(x)=U_p(x+1)-1$.

$Q(p) \equiv$ "$\sum_{n=1}^{p} U_n(x)=U_p(x+1)-1$, for all positive integer p ."

$p = 1$, L.H.S. $=\sum_{n=1}^{1} U_n(x)= U_1(x) = x$

R.H.S. $=U_1(x+1)-1=\dfrac{x+1}{1!}-1=x$

$Q(1)$ is true.

Suppose $\sum_{n=1}^{k} U_n(x)=U_k(x+1)-1$ for some positive integer k .

$$\begin{aligned}\sum_{n=1}^{k+1} U_n(x) &= \sum_{n=1}^{k} U_n(x)+U_{k+1}(x)\\ &=U_k(x+1)-1+U_{k+1}(x)\\ &=\frac{(x+1)(x+2)\cdots(x+k)}{k!}-1+\frac{x(x+1)(x+2)\cdots(x+k-1)(x+k)}{(k+1)!}\\ &=\frac{(k+1)(x+1)(x+2)\cdots(x+k)+x(x+1)(x+2)\cdots(x+k-1)(x+k)}{(k+1)!}-1\\ &=\frac{(x+1)(x+2)\cdots(x+k)(x+k+1)}{(k+1)!}-1\\ &=U_{k+1}(x+1)-1\end{aligned}$$

If $Q(k)$ is true, then $Q(k + 1)$ is also true.

By the principle of mathematical induction, it is true for all positive integers n .