Determinant Examples

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Exercise 8A (Page 253 – 255 Q.1 – Q16)

1. Prove that $\begin{vmatrix} \cos(A+B) & \sin(A+B) & -\cos(A+B) \\ \sin(A-B) & \cos(A-B) & \sin(A-B) \\ \sin 2A & 0 & \sin 2B \end{vmatrix} = \sin 2(A+B)$

2. Solve the equation $\begin{vmatrix} x^2+x+2 & 0 & x^2 \\ x+4 & x^2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$

3. Solve the equation $\begin{vmatrix} 1 & x-5 & 1 \\ -1 & 1 & x-3 \\ x-3 & 1 & -1 \end{vmatrix} = 0$

4. Prove that $\begin{vmatrix} \log a & \log b & \log c \\ \log 2a & \log 2b & \log 2c \\ \log 3a & \log 3b & \log 3c \end{vmatrix} = 0$

5. Prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

6. Evaluate the determinant $\begin{vmatrix} y^2+z^2 & xy & zx \\ xy & z^2+x^2 & yz \\ zx & yz & x^2+y^2 \end{vmatrix}$.

7. Evaluate $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \cdot \begin{vmatrix} b & -a & 0 \\ -a & 0 & b \\ 0 & b & -a \end{vmatrix}$.

8. Factorise $\begin{vmatrix} -2y & x+y & y+z \\ x+y & -2x & z+x \\ y+z & z+x & -2z \end{vmatrix}$.

9. Show that $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

10. Factorise $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$.

11. Factorise $\begin{vmatrix} a^2c^2+b^2d^2 & ac+bd & 1 \\ c^2b^2+a^2d^2 & cb+ad & 1 \\ b^2a^2+c^2d^2 & ba+cd & 1 \end{vmatrix}$.

12. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a-b & b-c & c-a \\ (a-b)^2 & (b-c)^2 & (c-a)^2 \end{vmatrix} = -(2a-b-c)(2b-c-a)(2c-a-b)$.

13. Show that the value of the determinant $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

14. If $A + B + C = 180°$, find the value of the determinant $\begin{vmatrix} \cos(A+B) & \cos(B-C) & \cos(C+A) \\ \cos(A+B) & \cos(B+C) & \cos(C-A) \\ \cos(A-B) & \cos(B+C) & \cos(C+A) \end{vmatrix}$.

15. Show that $\begin{vmatrix} \cos\theta & \cos\theta\cos\phi & \cos(\theta+\phi) \\ \cos(\theta+\phi) & \cos\theta & \cos\phi\cos(\theta+\phi) \\ \cos(\theta+\phi) & \cos\theta\sin^2\phi & -\cos\phi\sin\phi\sin\theta \end{vmatrix} = -\cos^2\phi\sin^2\phi\cos\theta$.

16. Prove that $\begin{vmatrix} \cos(\theta+\alpha) & \sin(\theta+\alpha) & 1 \\ \cos(\theta+\beta) & \sin(\theta+\beta) & 1 \\ \cos(\theta+\gamma) & \sin(\theta+\gamma) & 1 \end{vmatrix}$ is independent of θ.

Third Edition Advanced Level Pure Mathematics Algebra Hung Fung Book Co., Ltd. (p.285 Q6)

17. Factorise $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix}$.

Hence find the general solution of the equation $\begin{vmatrix} 2 & -\sin\theta & \sin^2\theta \\ -\sin\theta+\sin^2\theta & 2+\sin^2\theta & 2-\sin\theta \\ 4 & \sin^2\theta & \sin^4\theta \end{vmatrix} = 0$

New Senior Secondary Mathematics in Action Module 2 Algebra and Calculus 3 (p.16.31 Q24)

18. If A, B and C are interior angles of a triangle, show that

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 4\cos A\cos B\cos C$$

Breakthrough Algebra Pure Mathematics by Y.L. Ng & K.M. Pang **Chapter 3 Example 4.2 p.97**

19. Factorise $\Delta = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$.

1. $$\begin{vmatrix} \cos(A+B) & \sin(A+B) & -\cos(A+B) \\ \sin(A-B) & \cos(A-B) & \sin(A-B) \\ \sin 2A & 0 & \sin 2B \end{vmatrix}$$

$= \sin 2A \begin{vmatrix} \sin(A+B) & -\cos(A+B) \\ \cos(A-B) & \sin(A-B) \end{vmatrix} + \sin 2B \begin{vmatrix} \cos(A+B) & \sin(A+B) \\ \sin(A-B) & \cos(A-B) \end{vmatrix}$ expand along 3rd row

$= \sin 2A\, [\sin(A + B) \sin(A - B) + \cos(A + B) \cos(A - B)]$
$\quad + \sin 2B\, [\cos(A + B) \cos(A - B) - \sin(A + B) \sin(A - B)]$
$= \sin 2A \cos(A + B - A + B) + \sin 2B \cos(A + B + A - B)$
$= \sin 2A \cos 2B + \sin 2B \cos 2A$
$= \sin 2(A + B)$

2. $$\begin{vmatrix} x^2+x+2 & 0 & x^2 \\ x+4 & x^2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\begin{vmatrix} x^2+x+2 & -x^2-x-2 & -x-2 \\ x+4 & x^2-x-4 & -x-2 \\ 1 & 0 & 0 \end{vmatrix} = 0$ ($C_2 - C_1 \to C_2$; $C_3 - C_1 \to C_3$)

$-(x+2)\begin{vmatrix} -x^2-x-2 & 1 \\ x^2-x-4 & 1 \end{vmatrix} = 0$, cofactor expansion along the 3rd row

$-(x + 2)(-x^2 - x - 2 - x^2 + x + 4) = 0$
$(x + 2)(2x^2 - 2) = 0$
$x = -2$ or ± 1

3. $$\begin{vmatrix} 1 & x-5 & 1 \\ -1 & 1 & x-3 \\ x-3 & 1 & -1 \end{vmatrix} = 0$$

$\begin{vmatrix} x-3 & x-3 & x-3 \\ -1 & 1 & x-3 \\ x-3 & 1 & -1 \end{vmatrix} = 0$ ($R_1 + R_2 + R_3 \to R_1$)

$(x-3)\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & x-3 \\ x-3 & 1 & -1 \end{vmatrix} = 0$

$(x-3)\begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & x-2 \\ x-3 & 4-x & 2-x \end{vmatrix} = 0$ ($C_2 - C_1 \to C_2$; $C_3 - C_1 \to C_3$)

$(x-2)(x-3)\begin{vmatrix} 2 & 1 \\ 4-x & -1 \end{vmatrix} = 0$, cofactor expansion along the 1st row

$(x - 2)(x - 3)(x - 6) = 0$
$x = 2, 3$ or 6

4. $$\text{LHS}=\begin{vmatrix}\log a & \log b & \log c\\ \log 2a-\log a & \log 2b-\log b & \log 2c-\log c\\ \log 3a-\log a & \log 3b-\log b & \log 3c-\log c\end{vmatrix}\ (R_2-R_1\to R_2;\ R_3-R_1\to R_3)$$

$$=\begin{vmatrix}\log a & \log b & \log c\\ \log 2 & \log 2 & \log 2\\ \log 3 & \log 3 & \log 3\end{vmatrix}=\log 2\log 3\begin{vmatrix}\log a & \log b & \log c\\ 1 & 1 & 1\\ 1 & 1 & 1\end{vmatrix}=0\ (\text{R}_2 \text{ is identical to R}_3)$$

5. $$\text{LHS}=\begin{vmatrix}a-b & b-c & c-a\\ b-c & c-a & a-b\\ c-a & a-b & b-c\end{vmatrix}=\begin{vmatrix}0 & 0 & 0\\ b-c & c-a & a-b\\ c-a & a-b & b-c\end{vmatrix}=0\ (\text{R}_1+\text{R}_2+\text{R}_3\to\text{R}_1)$$

6. $$\begin{vmatrix}y^2+z^2 & xy & zx\\ xy & z^2+x^2 & yz\\ zx & yz & x^2+y^2\end{vmatrix}=x\begin{vmatrix}\frac{y^2+z^2}{x} & y & z\\ xy & z^2+x^2 & yz\\ zx & yz & x^2+y^2\end{vmatrix}=\begin{vmatrix}y^2+z^2 & y & z\\ x^2y & z^2+x^2 & yz\\ x^2z & yz & x^2+y^2\end{vmatrix}$$

$$=\begin{vmatrix}y^2+z^2 & y^2 & z\\ x^2 & z^2+x^2 & z\\ x^2z & y^2z & x^2+y^2\end{vmatrix}=\begin{vmatrix}y^2+z^2 & y^2 & z^2\\ x^2 & z^2+x^2 & z^2\\ x^2 & y^2 & x^2+y^2\end{vmatrix}$$

$$=\begin{vmatrix}0 & y^2 & z^2\\ -2z^2 & z^2+x^2 & z^2\\ -2y^2 & y^2 & x^2+y^2\end{vmatrix}\ (\text{C}_1-\text{C}_2-\text{C}_3\to\text{C}_1)$$

$$=2\begin{vmatrix}0 & y^2 & z^2\\ -z^2 & z^2+x^2 & z^2\\ -y^2 & y^2 & x^2+y^2\end{vmatrix}=2\begin{vmatrix}0 & y^2 & z^2\\ -z^2 & x^2 & 0\\ -y^2 & 0 & x^2\end{vmatrix}\ (\text{C}_1+\text{C}_2\to\text{C}_2,\ \text{C}_1+\text{C}_3\to\text{C}_3)$$

$= 2(x^2y^2z^2 + x^2y^2z^2) = 4x^2y^2z^2$

7. $$\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}\cdot\begin{vmatrix}b & -a & 0\\ -a & 0 & b\\ 0 & b & -a\end{vmatrix}=\begin{vmatrix}a+b+c & a+b+c & a+b+c\\ b & c & a\\ c & a & b\end{vmatrix}\cdot\begin{vmatrix}b-a & b-a & b-a\\ -a & 0 & b\\ 0 & b & -a\end{vmatrix}\quad \text{R}_1+\text{R}_2+\text{R}_3\to\text{R}_1$$

$$=(b-a)(a+b+c)\begin{vmatrix}1 & 1 & 1\\ b & c & a\\ c & a & b\end{vmatrix}\cdot\begin{vmatrix}1 & 1 & 1\\ -a & 0 & b\\ 0 & b & -a\end{vmatrix}$$

$$=(b-a)(a+b+c)\begin{vmatrix}1 & 0 & 0\\ b & c-b & a-b\\ c & a-c & b-c\end{vmatrix}\cdot\begin{vmatrix}1 & 0 & 0\\ -a & a & a+b\\ 0 & b & -a\end{vmatrix}\ (C_2-C_1\to C_2;\ C_3-C_1\to C_3)$$

$$=(b-a)(a+b+c)\begin{vmatrix}c-b & a-b\\ a-c & b-c\end{vmatrix}\cdot\begin{vmatrix}a & a+b\\ b & -a\end{vmatrix}\ \text{(Cofactor expansion along the 1}^{\text{st}}\text{ row)}$$

$= (b-a)(a+b+c)[-(b-c)^2 - (a-b)(a-c)]\cdot(-a^2 - ab - b^2)$

$= (b-a)(a+b+c)(b^2 - 2bc + c^2 + a^2 - ab - ac + bc)\cdot(a^2 + ab + b^2)$

$= (b-a)(a^2 + ab + b^2)(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$

8. Let $a = x + y$, $b = y + z$, $c = z + x$. Then $-2y = c - a - b$, $-2x = b - a - c$, $-2z = a - b - c$.

$$\Delta = \begin{vmatrix} c-a-b & a & b \\ a & b-a-c & c \\ b & c & a-b-c \end{vmatrix}$$

$$= \begin{vmatrix} c & b & a \\ a & b-a-c & c \\ a+b & b-a & a-b \end{vmatrix} \quad (R_1 + R_2 + R_3 \to R_1,\ R_2 + R_3 \to R_3)$$

$$= \begin{vmatrix} c & b+a & a \\ a & b-a & c \\ a+b & 0 & a-b \end{vmatrix} \quad (C_2 + C_3 \to C_2)$$

$$= \begin{vmatrix} a+c & 2b & a+c \\ a & b-a & c \\ a+b & 0 & a-b \end{vmatrix} \quad (R_1 + R_2 \to R_1)$$

$$= \begin{vmatrix} a+c & 2b & 0 \\ a & b-a & c-a \\ a+b & 0 & -2b \end{vmatrix} \quad (C_3 - C_1 \to C_3)$$

$$= \begin{vmatrix} c-b & 2b & 2b \\ a & b-a & c-a \\ a+b & 0 & -2b \end{vmatrix} \quad (R_1 - R_3 \to R_1)$$

$$= \begin{vmatrix} c-b & 2b & 2c \\ a & b-a & c+a \\ a+b & 0 & 2a \end{vmatrix} \quad (2C_1 + C_3 \to C_3)$$

$$= \begin{vmatrix} c-b & 2b & 2c \\ \frac{1}{2}(a-c) & -a & 0 \\ a+b & 0 & 2a \end{vmatrix} \quad (R_2 - \frac{1}{2}R_1 - \frac{1}{2}R_3 \to R_2)$$

$$= 2\begin{vmatrix} c-b & 2b & c \\ \frac{1}{2}(a-c) & -a & 0 \\ a+b & 0 & a \end{vmatrix} = \begin{vmatrix} c-b & 2b & c \\ a-c & -2a & 0 \\ a+b & 0 & a \end{vmatrix} = 2\begin{vmatrix} c-b & b & c \\ a-c & -a & 0 \\ a+b & 0 & a \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & b & c \\ -c & -a & 0 \\ b & 0 & -a \end{vmatrix} \quad (C_1 + C_2 - C_3 \to C_1)$$

$= 4abc$

$= 4(x + y)(y + z)(z + x)$

Method 2 Let f(x, y, z) = $\begin{vmatrix} -2y & x+y & y+z \\ x+y & -2x & z+x \\ y+z & z+x & -2z \end{vmatrix}$, then

f(y, z, x) = $\begin{vmatrix} -2z & y+z & z+x \\ y+z & -2y & x+y \\ z+x & x+y & -2x \end{vmatrix} = \begin{vmatrix} y+z & -2y & x+y \\ z+x & x+y & -2x \\ -2z & y+z & z+x \end{vmatrix} = \begin{vmatrix} -2y & x+y & y+z \\ x+y & -2x & z+x \\ y+z & z+x & -2z \end{vmatrix} = f(x, y, z)$

f(z, x, y) = $\begin{vmatrix} -2x & z+x & x+y \\ z+x & -2z & y+z \\ x+y & y+z & -2y \end{vmatrix} = \begin{vmatrix} x+y & y+z & -2y \\ -2x & z+x & x+y \\ z+x & -2z & y+z \end{vmatrix} = \begin{vmatrix} -2y & x+y & y+z \\ x+y & -2x & z+x \\ y+z & z+x & -2z \end{vmatrix} = f(x, y, z)$

$\therefore$ f(x, y, z) is a cyclic expression of degree 3.

f(x, $-x$, z) = $\begin{vmatrix} 2x & 0 & z-x \\ 0 & -2x & z+x \\ z-x & z+x & -2z \end{vmatrix} = 8x^2z + 2x(z-x)^2 - 2x(z+x)^2 = 0$

$\therefore$ $x + y$ is a factor

By symmetry, $y + z$ and $z + x$ are factors

f(x, y, z) = $k(x + y)(y + z)(z + x)$

f(1, 1, 0) = $2k = \begin{vmatrix} -2 & 2 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2 + 2 + 2 + 2$

$k = 4$

f(x, y, z) = $4(x + y)(y + z)(z + x)$

9. $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = (1 + x)(1 + y)(1 + z) + 1 + 1 - (1 + x + 1 + y + 1 + z)$

$= 1 + x + y + z + yz + xz + xy + xyz - (x + y + z + 1)$

$= yz + xz + xy + xyz$

$= xyz\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$

10. $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$

$= \begin{vmatrix} a^2 & a^2 & bc \\ b^2 & b^2 & ca \\ c^2 & c^2 & ab \end{vmatrix} + \begin{vmatrix} a^2 & -(b-c)^2 & bc \\ b^2 & -(c-a)^2 & ca \\ c^2 & -(a-b)^2 & ab \end{vmatrix}$

$= \begin{vmatrix} a^2 & -(b-c)^2 & bc \\ b^2-a^2 & b^2-a^2+2ac-2bc & c(a-b) \\ c^2-a^2 & c^2-a^2+2ab-2bc & b(a-c) \end{vmatrix}$ ($R_2 - R_1 \to R_2$, $R_3 - R_1 \to R_3$)

$$=(b-a)(c-a)\begin{vmatrix} a^2 & -(b-c)^2 & bc \\ a+b & a+b-2c & -c \\ a+c & a+c-2b & -b \end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix} a^2 & -a^2-b^2-c^2 & bc \\ a+b & 0 & -c \\ a+c & 0 & -b \end{vmatrix} \quad (C_2 - C_1 - 2C_3 \to C_2)$$

$$=(a^2+b^2+c^2)(a-b)(c-a)\begin{vmatrix} a+b & c \\ a+c & b \end{vmatrix} \quad \text{(Cofactor expansion along the 2}^{\text{nd}}\text{ row)}$$

$$= (a^2+b^2+c^2)(a-b)(c-a)\begin{vmatrix} a+b & c \\ c-b & b-c \end{vmatrix} \quad (R_2 - R_1 \to R_2)$$

$$= (a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c)$$

11. $$\begin{vmatrix} a^2c^2+b^2d^2 & ac+bd & 1 \\ c^2b^2+a^2d^2 & cb+ad & 1 \\ b^2a^2+c^2d^2 & ba+cd & 1 \end{vmatrix}$$

$$=\begin{vmatrix} (d^2-a^2)(b^2-c^2) & (d-a)(b-c) & 0 \\ (b^2-d^2)(c^2-a^2) & (b-d)(c-a) & 0 \\ b^2a^2+c^2d^2 & ba+cd & 1 \end{vmatrix} \quad (R_1 - R_3 \to R_1,\ R_2 - R_3 \to R_2)$$

$$=(d-a)(b-c)(b-d)(c-a)\begin{vmatrix} (d+a)(b+c) & 1 \\ (d+b)(c+a) & 1 \end{vmatrix} \quad \text{(Cofactor expansion along the 3}^{\text{rd}}\text{ column)}$$

$$= (d-a)(b-c)(b-d)(c-a)(bd+ab+cd+ac-bc-cd-ab-ad)$$
$$= (d-a)(b-c)(b-d)(c-a)(bd+ac-bc-ad)$$
$$= (d-a)(b-c)(b-d)(c-a)(d-c)(b-a)$$
$$= -(a-b)(b-c)(c-a)(a-d)(b-d)(c-d)$$

12. $$\begin{vmatrix} 1 & 1 & 1 \\ a-b & b-c & c-a \\ (a-b)^2 & (b-c)^2 & (c-a)^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & 0 \\ a-b & 2b-c-a & b+c-2a \\ (a-b)^2 & (2b-c-a)(a-c) & (b+c-2a)(c-b) \end{vmatrix} \quad (C_2 - C_1 \to C_2,\ C_3 - C_1 \to C_3)$$

$$=(2b-c-a)(b+c-2a)\begin{vmatrix} 1 & 1 \\ a-c & c-b \end{vmatrix}$$

$$= (2b-c-a)(b+c-2a)(2c-a-b) = -(2a-b-c)(2b-c-a)(2c-a-b)$$

13. $$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = \begin{vmatrix} a & b+c & c+a \\ b & c+a & a+b \\ c & a+b & b+c \end{vmatrix} + \begin{vmatrix} b & b+c & c+a \\ c & c+a & a+b \\ a & a+b & b+c \end{vmatrix}$$

$$=\begin{vmatrix} a & b & c+a \\ b & c & a+b \\ c & a & b+c \end{vmatrix} + \begin{vmatrix} a & c & c+a \\ b & a & a+b \\ c & b & b+c \end{vmatrix} + \begin{vmatrix} b & b & c+a \\ c & c & a+b \\ a & a & b+c \end{vmatrix} + \begin{vmatrix} b & c & c+a \\ c & a & a+b \\ a & b & b+c \end{vmatrix}$$

$$=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}+\begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix}+\begin{vmatrix} a & c & c \\ b & a & a \\ c & b & b \end{vmatrix}+\begin{vmatrix} a & c & a \\ b & a & b \\ c & b & c \end{vmatrix}+0+\begin{vmatrix} b & c & c \\ c & a & a \\ a & b & b \end{vmatrix}+\begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$$=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}+\begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}=2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

14. $$\begin{vmatrix} \cos(A+B) & \cos(B-C) & \cos(C+A) \\ \cos(A+B) & \cos(B+C) & \cos(C-A) \\ \cos(A-B) & \cos(B+C) & \cos(C+A) \end{vmatrix}$$

$$=\begin{vmatrix} -\cos C & \cos(B-C) & -\cos B \\ -\cos C & -\cos A & \cos(C-A) \\ \cos(A-B) & -\cos A & -\cos B \end{vmatrix}=-\begin{vmatrix} \cos C & -\cos(B-C) & \cos B \\ \cos C & \cos A & -\cos(C-A) \\ -\cos(A-B) & \cos A & \cos B \end{vmatrix}$$

$$=-\begin{vmatrix} \cos C & -\cos(B-C) & \cos B \\ 0 & 2\cos\frac{A+B-C}{2}\cos\frac{A-B+C}{2} & -2\cos\frac{B+C-A}{2}\cos\frac{A+B-C}{2} \\ -2\cos\frac{A+C-B}{2}\cos\frac{A-B-C}{2} & 2\cos\frac{A+B-C}{2}\cos\frac{A+C-B}{2} & 0 \end{vmatrix}$$

($R_2 - R_1 \to R_2$, $R_3 - R_1 \to R_3$)

$$=-\begin{vmatrix} \cos C & -\cos(B-C) & \cos B \\ 0 & 2\sin C\sin B & -2\sin A\sin C \\ -2\sin B\sin A & 2\sin C\sin B & 0 \end{vmatrix}$$

$$=-4\sin B\sin C\begin{vmatrix} \cos C & -\cos(B-C) & \cos B \\ 0 & \sin B & -\sin A \\ -\sin A & \sin C & 0 \end{vmatrix}$$

$$=-4\sin^2 A\sin B\sin C\begin{vmatrix} \cos C & -\cos(B-C) & \cos B \\ 0 & \frac{\sin B}{\sin A} & -1 \\ -1 & \frac{\sin C}{\sin A} & 0 \end{vmatrix}$$

$$=-4\sin A\sin B\sin C\begin{vmatrix} \cos C & -\sin A\cos(B-C) & \cos B \\ 0 & \sin B & -1 \\ -1 & \sin C & 0 \end{vmatrix}$$

$$=-4\sin A\sin B\sin C\begin{vmatrix} \cos C & -\sin B\cos B-\sin C\cos C & \cos B \\ 0 & \sin B & -1 \\ -1 & \sin C & 0 \end{vmatrix}$$

$$=-4\sin A\sin B\sin C\begin{vmatrix} 0 & 0 & 0 \\ 0 & \sin B & -1 \\ -1 & \sin C & 0 \end{vmatrix}=0$$ ($R_1 + \cos C\, R_3 + \cos B\, R_2 \to R_1$)

15. $\begin{vmatrix} \cos\theta & \cos\theta\cos\phi & \cos(\theta+\phi) \\ \cos(\theta+\phi) & \cos\theta & \cos\phi\cos(\theta+\phi) \\ \cos(\theta+\phi) & \cos\theta\sin^2\phi & -\cos\phi\sin\phi\sin\theta \end{vmatrix}$

$$=\cos\theta\begin{vmatrix} \cos\theta & \cos\phi & \cos(\theta+\phi) \\ \cos(\theta+\phi) & 1 & \cos\phi\cos(\theta+\phi) \\ \cos(\theta+\phi) & \sin^2\phi & -\cos\phi\sin\phi\sin\theta \end{vmatrix}=\cos\theta\begin{vmatrix} \cos\theta\cos\phi & \cos^2\phi & \cos(\theta+\phi) \\ \cos(\theta+\phi) & 1 & \cos(\theta+\phi) \\ \cos(\theta+\phi) & \sin^2\phi & -\sin\theta\sin\phi \end{vmatrix}$$

$$=\cos\theta\begin{vmatrix} \cos\theta\cos\phi & \cos^2\phi & \cos(\theta+\phi) \\ -\cos\theta\cos\phi & 0 & \sin\theta\sin\phi \\ \cos(\theta+\phi) & \sin^2\phi & -\sin\theta\sin\phi \end{vmatrix} \quad (R_2 - R_1 - R_3 \to R_2)$$

$$=\cos\theta\begin{vmatrix} 0 & \cos^2\phi & \cos\theta\cos\phi \\ -\cos\theta\cos\phi & 0 & \sin\theta\sin\phi \\ -\sin\theta\sin\phi & \sin^2\phi & 0 \end{vmatrix} \quad (R_1 + R_2 \to R_1,\ R_2 + R_3 \to R_3)$$

$$=-\cos\theta\sin\phi\cos\phi\begin{vmatrix} 0 & \cos\phi & \cos\theta \\ \cos\theta\cos\phi & 0 & \sin\theta\sin\phi \\ \sin\theta & \sin\phi & 0 \end{vmatrix}$$

$= -\cos\theta \sin\phi \cos\phi (\sin^2\theta \sin\phi \cos\phi + \cos^2\theta \sin\phi \cos\phi)$

$= -\cos\theta \sin^2\phi \cos^2\phi (\sin^2\theta + \cos^2\theta)$

$= -\cos\theta \sin^2\phi \cos^2\phi$

16. $\begin{vmatrix} \cos(\theta+\alpha) & \sin(\theta+\alpha) & 1 \\ \cos(\theta+\beta) & \sin(\theta+\beta) & 1 \\ \cos(\theta+\gamma) & \sin(\theta+\gamma) & 1 \end{vmatrix}$ (Cofactor expansion along the 3rd column)

$= \sin(\gamma - \beta) + \sin(\alpha - \gamma) + \sin(\beta - \alpha)$ which is independent of θ.

17. $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix}$

$=\begin{vmatrix} a & b-a & c-a \\ b+c & a-b & a-c \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$ ($C_2 - C_1 \to C_2$, $C_3 - C_1 \to C_3$)

$=(b-a)(c-a)\begin{vmatrix} a & 1 & 1 \\ b+c & -1 & -1 \\ a^2 & a+b & a+c \end{vmatrix}$

$=(b-a)(c-a)\begin{vmatrix} a & 1 & 0 \\ b+c & -1 & 0 \\ a^2 & a+b & c-b \end{vmatrix}$ ($C_3 - C_2 \to C_3$)

$= -(a-b)(b-c)(c-a)(a+b+c)$

Let $a = 2$, $b = -\sin\theta$, $c = \sin^2\theta$, then $\begin{vmatrix} 2 & -\sin\theta & \sin^2\theta \\ -\sin\theta+\sin^2\theta & 2+\sin^2\theta & 2-\sin\theta \\ 4 & \sin^2\theta & \sin^4\theta \end{vmatrix} = 0$

$\Rightarrow -(a-b)(b-c)(c-a)(a+b+c) = 0$

$\Rightarrow a = b$, $b = c$, $c = a$ or $a + b + c = 0$

$\Rightarrow \sin\theta = -2$ (rejected), $\sin\theta(\sin\theta + 1) = 0$, $\sin^2\theta = 2$ (rejected) or $\sin^2\theta - \sin\theta + 2 = 0$

$\Rightarrow \theta = n\pi,\ 2n\pi - \dfrac{\pi}{2}$, where n is an integer or no solution ($\because\ \Delta = -3 < 0$)

18. $A + B + C = 0$

$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$

$=\begin{vmatrix} 1 & \cos C & \cos B \\ 0 & 1-\cos^2 C & \cos A - \cos B\cos C \\ 0 & \cos A - \cos B\cos C & 1-\cos^2 B \end{vmatrix} \begin{matrix} \\ R_2 - \cos C R_1 \\ R_3 - \cos B R_1 \end{matrix}$

$=\begin{vmatrix} \sin^2 C & \cos(180^\circ - B - C) - \cos B\cos C \\ \cos(180^\circ - B - C) - \cos B\cos C & \sin^2 B \end{vmatrix}$ expand along 3rd column

$= \sin^2 B \sin^2 C - [-\cos(B + C) - \cos B \cos C]^2$

$= \sin^2 B \sin^2 C - (\cos B \cos C - \sin B \sin C + \cos B \cos C)^2$

$= \sin^2 B \sin^2 C - (4 \cos^2 B \cos^2 C - 4 \sin B \sin C \cos B \cos C + \sin^2 B \sin^2 C)$

$= 4 (\sin B \sin C - \cos B \cos C) \cos B \cos C$

$= -4 \cos(B + C) \cos B \cos C$

$= -4 \cos(180^\circ - A) \cos B \cos C$

$= 4 \cos A \cos B \cos C$

19. $\Delta = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$

$= \dfrac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}\begin{matrix} aR_1 \\ bR_2 \\ cR_3 \end{matrix}$ (and then divide the whole determinant by *abc*)

$= \dfrac{abc}{abc}\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ (Take out the common factor *abc* from the first column.)

$= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2 - a^2 & b^3 - a^3 \\ 0 & c^2 - a^2 & c^3 - a^3 \end{vmatrix}$ ($R_2 - R_1 \to R_2$, $R_3 - R_1 \to R_3$)

$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2 + ab + a^2 \\ 0 & c+a & c^2 + ca + a^2 \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2 + ab + a^2 \\ 0 & c-b & c^2 - b^2 + ca - ab \end{vmatrix}$ ($R_3 - R_2 \to R_3$)

$= (b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2 + ab + a^2 \\ 0 & c-b & (c-b)(a+b+c) \end{vmatrix}$ (factorization of $c^2 - b^2 + ca - ab$)

$= (b-a)(c-a)(c-b)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2 + ab + a^2 \\ 0 & 1 & a+b+c \end{vmatrix}$

$= (a-b)(b-c)(c-a)\begin{vmatrix} a+b & a^2 + ab + b^2 \\ 1 & a+b+c \end{vmatrix}$ (cofactor expansion about the first column)

$= (a-b)(b-c)(c-a)\left[(a+b)^2 + (a+b)c - a^2 - ab - b^2\right]$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$