Individual Events

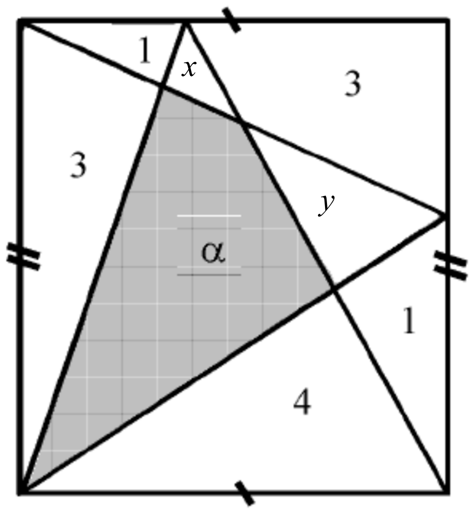
I1	α	5	**I2**	α	$\frac{4}{3}$	**I3**	α	11	**I4**	α	6
	β	55		β	24		β	45		β	5
	γ	6		γ	3		γ	45		γ	7.5
	δ	16		δ	7		δ	81		δ	$-\frac{33}{64}$

Group Events

G1	area	48	**G2**	Product	$\frac{1}{80}$	**G3**	Product	$\frac{11}{20}$	**G4**	PZ	1.6
	minimum	6		$S_{17}+S_{33}+S_{50}$	1		Sum	1		$x^3y+2x^2y^2+xy^3$	5
	remainder	0		Day	5		α	15		d	2
	a_{100}	$\frac{1}{10100}$		α	30		α	5		product	4

Individual Event 1

I1.1 Determine the area of the shaded region, α, in the figure.
(Reference: 2011 FG4.4, 2019 FG3.2)



Label the unmarked regions by x and y respectively.

$3 + \alpha + y = \frac{1}{2}$ area of //-gram $= 4 + \alpha + x$

$\Rightarrow y = x + 1$ ⋯⋯ (1)

$1+x+3+3+\alpha+y+4+1 =$ area of //-gram $= 2(4 + \alpha + x)$

$\Rightarrow 12 + x + y + \alpha = 8 + 2\alpha + 2x$ ⋯⋯ (2)

Sub. (1) into (2): $12 + x + x + 1 + \alpha = 8 + 2\alpha + 2x$

$\Rightarrow \alpha = 5$

I1.2 If the average of 10 distinct positive integers is 2α, what is the largest possible value of the largest integer, β, of the ten integers?

Let the 10 distinct positive integers be $0 < x_1 < x_2 < \cdots < x_{10}$, in ascending order.

$$\frac{x_1 + x_2 + \cdots + x_{10}}{10} = 2\times 5 = 10$$

$x_1 + x_2 + \cdots + x_9 + \beta = 100$

If β is the largest possible, then $x_1, x_2, \cdots, x_9$ must be as small as possible.

The least possible $x_1, x_2, \cdots, x_9$ are 1, 2, 3, ⋯ , 9.

The largest possible $\beta = 100 - (1 + 2 + \ldots + 9) = 100 - 45 = 55$

I1.3 Given that 1, 3, 5, 7, ⋯ , β and 1, 6, 11, 16, ⋯ , β + 1 are two finite sequences of positive integers. Determine γ, the numbers of positive integers common to both sequences.

The two finite sequences are: 1, 3, 5, 7, ⋯ , 55 and 1, 6, 11, 16, ⋯ , 56.

The terms common to both sequences are 1, 11, 21, 31, 41, 51.

$\gamma = 6$

I1.4 If $\log_2 a + \log_2 b \ge \gamma$, determine the smallest positive value δ for $a + b$.

$\log_2 a + \log_2 b \ge 6$

$ab \ge 2^6 = 64$

$a + b = \left(\sqrt{a} - \sqrt{b}\right)^2 + 2\sqrt{ab} \ge 0 + 2\times\sqrt{64} = 16$

The smallest positive value of δ = 16

Individual Event 2

I2.1 Determine the positive real root, α, of $\sqrt{\left(x+\sqrt{x}\right)}-\sqrt{\left(x-\sqrt{x}\right)}=\sqrt{x}$.

$$\left[\sqrt{\left(x+\sqrt{x}\right)}-\sqrt{\left(x-\sqrt{x}\right)}\right]^2=x$$

$$x+\sqrt{x}-2\sqrt{x^2-x}+x-\sqrt{x}=x$$

$$x=2\sqrt{x^2-x}$$

$x^2=4(x^2-x)$

$3x^2=4x$

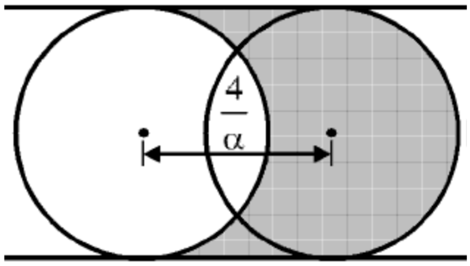
$x=0$ (rejected) or $\frac{4}{3}$

Check: When $x=\frac{4}{3}$,

$$\text{L.H.S.}=\sqrt{\left(\frac{4}{3}+\sqrt{\frac{4}{3}}\right)}-\sqrt{\left(\frac{4}{3}-\sqrt{\frac{4}{3}}\right)}=\sqrt{\frac{4+2\sqrt{3}}{3}}-\sqrt{\frac{4-2\sqrt{3}}{3}}=\frac{\sqrt{3}+1}{\sqrt{3}}-\frac{\sqrt{3}-1}{\sqrt{3}}=\sqrt{\frac{4}{3}}=\text{R.H.S.}$$

$\therefore \alpha=\frac{4}{3}$

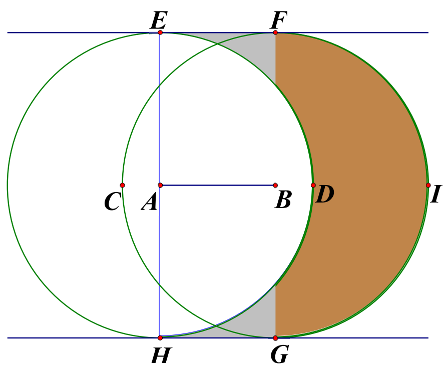
I2.2 In the figure, two circles of radii 4 with their centres placed apart by $\frac{4}{\alpha}$. Determine the area β, of the shaded region.



Let the centres of circles be A and B as shown. $AB = 3$

Suppose the two circles touches the two given line segments at E, F, G, H as shown. Then $EFGH$ is a rectangle with $FE = AB = GH = 3$, $EH = FG = 8$

β = Area of semi-circle FIG + area of rectangle $EFGH$
– area of semi-circle EDH
= Area of rectangle $EFGH$
= 3×8 = 24



I2.3 Determine the smallest positive integer γ such that the equation $\sqrt{x}-\sqrt{\beta\gamma}=4\sqrt{2}$ has an integer solution in x. **Reference: 2019 FG3.1**

$$\sqrt{x}-\sqrt{24\gamma}=4\sqrt{2}$$

$$\sqrt{x}=2\sqrt{6\gamma}+4\sqrt{2}$$

The smallest positive integer γ = 3.

$(\sqrt{x}=2\sqrt{6\times3}+4\sqrt{2}=6\sqrt{2}+4\sqrt{2}=10\sqrt{2}\ \Rightarrow x=200)$

I2.4 Determine the units digit, δ, of $\left(\left(\gamma^\gamma\right)^\gamma\right)^\gamma$.

$$\left(\left(3^3\right)^3\right)^3=\left(3^9\right)^3=3^{27}$$

The units digit of 3, 3^2, 3^3, 3^4 are 3, 9, 7, 1 respectively.

This pattern repeats for every multiples of 4.

27 = 6×4 + 3

δ = 7

Individual Event 3

I3.1 If the product of numbers in the sequence $10^{\frac{1}{11}}$, $10^{\frac{2}{11}}$, $10^{\frac{3}{11}}$, $\cdots$, $10^{\frac{\alpha}{11}}$ is 1 000 000, determine the value of the positive integer α.

$10^{\frac{1}{11}}\times 10^{\frac{2}{11}}\times 10^{\frac{3}{11}}\times\cdots\times 10^{\frac{\alpha}{11}} = 10^6$

$\frac{1}{11}+\frac{2}{11}+\frac{3}{11}+\cdots+\frac{\alpha}{11}=6$

$\frac{1}{2}(1+\alpha)\alpha=66$

$\alpha^2+\alpha-132=0$

$(\alpha-11)(\alpha+12)=0$

$\alpha=11$

I3.2 Determine the value of β if $\frac{\beta}{1\times2\times3}+\frac{\beta}{2\times3\times4}+\cdots+\frac{\beta}{8\times9\times10}=\alpha$.

Reference: 2003 HG1

$\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}=\frac{(r+2)-r}{r(r+1)(r+2)}=2\cdot\frac{1}{r(r+1)(r+2)}$

Put $r = 1$, $\frac{1}{1\times2}-\frac{1}{2\times3}=2\cdot\frac{1}{1\times2\times3}$

Put $r = 2$, $\frac{1}{2\times3}-\frac{1}{3\times4}=2\cdot\frac{1}{2\times3\times4}$

……………………………………

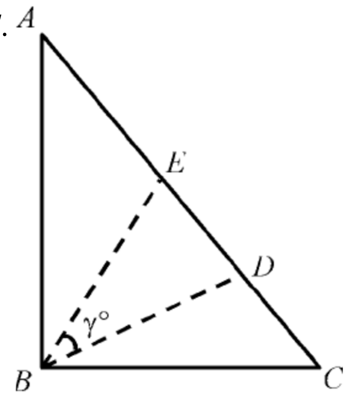
Put $r = 8$, $\frac{1}{8\times9}-\frac{1}{9\times10}=2\cdot\frac{1}{8\times9\times10}$

Add these equations together and multiply both sides by β and divide by 2:

$\frac{\beta}{2}\left[\frac{1}{2}-\frac{1}{9\times10}\right]=\frac{\beta}{1\times2\times3}+\frac{\beta}{2\times3\times4}+\cdots+\frac{\beta}{8\times9\times10}=\alpha=11$

$\beta=45$

I3.3 In the figure, triangle ABC has $\angle ABC = 2\beta°$, $AB = AD$ and $CB = CE$. If $\gamma° = \angle DBE$, determine the value of γ.



Let $\angle ABE = x$

$\angle ABC = 90°$

$\angle CBE = 90° - x$

$\angle ADB = x + \gamma°$ (base $\angle$s isos. Δ)

$\angle CEB = \angle CBE = 90° - x$ (base $\angle$s isos. Δ)

In ΔBDE, $\gamma° + x + \gamma° + 90° - x = 180°$ ($\angle$s sum of Δ)

$\gamma = 45$

I3.4 For the sequence 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, $\cdots$, determine the sum δ of the first γ terms.

$\delta = 1 + 2 + 1 + 2 + 2 + 1 + 2 + 2 + 2 + 1 + 2 + 2 + 2 + 2 + 1 + 2 +\cdots+ 2$ + (5 terms) + $1 + 2 + \ldots + 2$ (6 terms) + $1 + 2 +\cdots+ 2$ (7 terms) + $1 + 2 + \cdots + 2$ (8 terms) + 1

$= 9 + 2\times36 = 81$

Individual Event 4

I4.1 If $\frac{6\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}=3\sqrt{\alpha}-6$, determine the value of α.

Reference: 1989 FG10.1

$$\frac{6\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}\cdot\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}=3\sqrt{\alpha}-6$$

$$6\sqrt{3}\cdot\frac{3\sqrt{2}-2\sqrt{3}}{18-12}=3\sqrt{\alpha}-6$$

$$3\sqrt{6}-6=3\sqrt{\alpha}-6$$

$\alpha=6$

I4.2 Consider fractions of the form $\frac{n}{n+1}$, where n is a positive integer. If 1 is subtracted from both the numerator and the denominator, and the resultant fraction remains positive and is strictly less than $\frac{\alpha}{7}$, determine, β, the number of these fractions.

$$0<\frac{n-1}{n}<\frac{6}{7}$$

$7n-7<6n$ and $n>1$

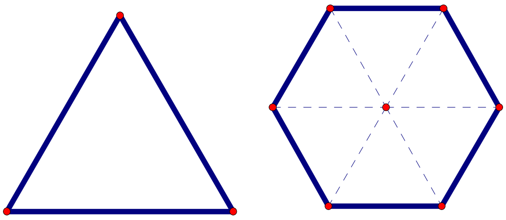
$1<n<7$

Possible n = 2, 3, 4, 5, 6

$\beta=5$

I4.3 The perimeters of an equilateral triangle and a regular hexagon are equal. If the area of the triangle is β square units, determine the area, γ, of the hexagon in square units.

Reference: 1996 FI1.1, 2016 FI2.1



Let the length of the equilateral triangle be x, and that of the regular hexagon be y.

Since they have equal perimeter, $3x=6y$

$\therefore x=2y$

The hexagon can be divided into 6 identical equilateral triangles.

Ratio of areas $=\frac{1}{2}x^2\sin 60^\circ : 6\times\frac{1}{2}y^2\sin 60^\circ = 2:3$

$\beta=5$

$\gamma=5\times\frac{3}{2}=7.5$

I4.4 Determine the value of $\delta=\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\frac{17}{16}+\frac{33}{32}+\frac{65}{64}-\gamma$.

$$\delta=\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\frac{17}{16}+\frac{33}{32}+\frac{65}{64}-7\frac{1}{2}$$

$$=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}-1\frac{1}{2}$$

$$=-\frac{33}{64}$$

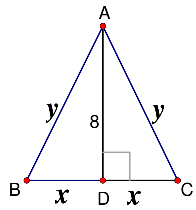
Group Event 1

G1.1 If an isosceles triangle has height 8 from the base, not the legs, and perimeters 32, determine the area of the triangle.

Let the isosceles triangle be ABC with $AB = AC = y$

D is the midpoint of the base BC, $AD \perp BC$, $AD = 8$

Let $BD = DC = x$



Perimeter $= 2x + 2y = 32 \Rightarrow y = 16 - x$ …… (1)

$x^2 + 8^2 = y^2$ …… (2)

Sub. (1) into (2): $x^2 + 64 = 256 - 32x + x^2$

$x = 6$, $y = 10$

Area of the triangle = 48

G1.2 If $f(x) = \dfrac{\left(x+\frac{1}{x}\right)^6 - \left(x^6+\frac{1}{x^6}\right) - 2}{\left(x+\frac{1}{x}\right)^3 + \left(x^3+\frac{1}{x^3}\right)}$ where x is a positive real number, determine the minimum value of f(x).

Reference: 1979 American High School Mathematics Examination Q29

$$f(x) = \frac{\left[\left(x+\frac{1}{x}\right)^3\right]^2 - \left(x^3+\frac{1}{x^3}\right)^2}{\left(x+\frac{1}{x}\right)^3 + \left(x^3+\frac{1}{x^3}\right)}$$

$$= \left(x+\frac{1}{x}\right)^3 - \left(x^3+\frac{1}{x^3}\right)$$

$$= 3\left(x+\frac{1}{x}\right)$$

$$f(x) = 3\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 + 6 \geq 6$$

G1.3 Determine the remainder of the 81-digit integer $\overline{111\cdots1}$ divided by 81.

$$\underbrace{\overline{11\cdots1}}_{81\text{ digits}} = 10^{80} + 10^{79} + \cdots + 10 + 1$$

$$= (10^{80} - 1) + (10^{79} - 1) + \cdots + (10 - 1) + 81$$

$$= \underbrace{\overline{99\cdots9}}_{80\text{ digits}} + \underbrace{\overline{99\cdots9}}_{79\text{ digits}} + \cdots + 9 + 81$$

$$= 9\times\left(\underbrace{\overline{11\cdots1}}_{80\text{ digits}} + \underbrace{\overline{11\cdots1}}_{79\text{ digits}} + \cdots + 1\right) + 81$$

Let $x = \underbrace{\overline{11\cdots1}}_{80\text{ digits}} + \underbrace{\overline{11\cdots1}}_{79\text{ digits}} + \cdots + 1 \equiv 80 + 79 + \cdots + 1 \pmod 9$

$x \equiv \frac{81}{2}\cdot 80 \equiv 9\times40 \equiv 0 \bmod 9$

$x = 9m$ for some integer m

$\underbrace{\overline{11\cdots1}}_{81\text{ digits}} = 9\times9m + 81 \equiv 0 \pmod{81}$

The remainder = 0

G1.4 Given a sequence of real numbers $a_1, a_2, a_3, \cdots$ that satisfy

1) $a_1 = \frac{1}{2}$, and

2) $a_1 + a_2 + \cdots + a_k = k^2 a_k$, for $k \geq 2$.

Determine the value of a_{100}.

Reference: 2013 HG10, 2022 P2Q8

$\frac{1}{2} + a_2 = 2^2 a_2 \Rightarrow a_2 = \frac{1}{2\times 3} = \frac{1}{6}$

$\frac{1}{2} + \frac{1}{6} + a_3 = 3^2 a_3 \Rightarrow a_3 = \frac{1}{3\times 4} = \frac{1}{12}$

Claim: $a_n = \frac{1}{n\times(n+1)}$ for $n \geq 1$

Pf: By M.I., $n = 1, 2, 3$, proved already.

Suppose $a_k = \frac{1}{k\times(k+1)}$ for some $k \geq 1$

$\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{k\times(k+1)} + a_{k+1} = (k+1)^2 a_{k+1}$

$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(k^2 + 2k\right) a_{k+1}$

$a_{k+1} = \frac{1}{k(k+2)} \cdot \left(1 - \frac{1}{k+1}\right) = \frac{1}{(k+1)(k+2)}$

$\therefore$ The statement is true for all $n \geq 1$

$a_{100} = \frac{1}{100\times 101} = \frac{1}{10100}$

Group Event 2

G2.1 By removing certain terms from the sum, $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}$, we can get 1. What is the product of the removed term(s)?

$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}+\frac{1}{4}=1$

The removed terms are $\frac{1}{8};\frac{1}{10}$.

Product $=\frac{1}{80}$

G2.2 If $S_n = 1 - 2 + 3 - 4 + \ldots + (-1)^{n-1} n$, where n is a positive integer, determine the value of $S_{17} + S_{33} + S_{50}$.

If $n = 2m$, where m is a positive integer,

$S_{2m} = (1 - 2) + (3 - 4) + \cdots + (2m - 1 - 2m) = -m$

$S_{2m+1} = -m + 2m + 1 = m + 1$

$S_{17} + S_{33} + S_{50} = 9 + 17 - 25 = 1$

G2.3 Six persons A, B, C, D, E and F are to rotate for night shifts in alphabetical order with A serving on the first Sunday, B on the first Monday and so on. In the fiftieth week, which day does A serve on? (Represent Sunday by 0, Monday by 1, $\cdots$, Saturday by 6 in your answer.)

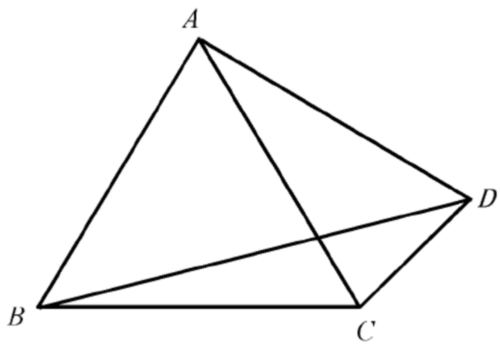
$50\times7 = 350 = 6\times58 + 2$

B serves on Saturday in the fiftieth week.

A serves on Friday in the fiftieth week.

Answer 5.

G2.4 In the figure, vertices of equilateral triangle ABC are connected to D in straight line segments with $AB = AD$. If $\angle BDC = \alpha°$, determine the value of α.



Reference: 2003 HG8, 2011 HG9

Use A as centre, AB as radius to draw a circle to pass through B, C, D.

$\angle BAC = 2\angle BDC$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$60° = 2\alpha°$

$\alpha = 30$

Group Event 3

G3.1 Determine the value of the product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{10^2}\right)$.

Reference: 1986 FG10.4, 1999 FIS.4

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{10^2}\right)=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{10}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{10}\right)$$

$$=\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{9}{10}\right)\cdot\left(\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{11}{10}\right)=\frac{1}{10}\times\frac{11}{2}=\frac{11}{20}$$

G3.2 Determine the value of the sum $\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}$,

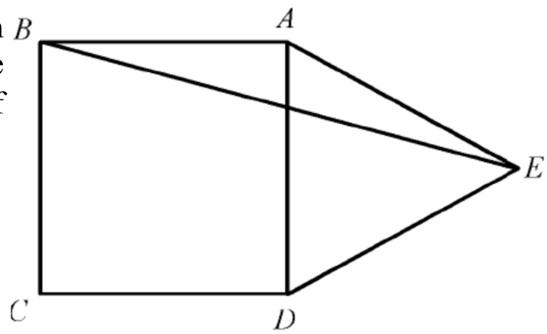
where 100! = 100×99×98×…×3×2×1.

$$\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}$$

$$=\frac{\log 2}{\log 100!}+\frac{\log 3}{\log 100!}+\frac{\log 4}{\log 100!}+\cdots+\frac{\log 100}{\log 100!}$$

$$=\frac{\log 100!}{\log 100!}=1$$

G3.3 In the figure, *ABCD* is a square, *ADE* is an equilateral triangle and *E* is a point outside of the square *ABCD*. If $\angle AEB=\alpha°$, determine the value of α. (**Reference: 1991 FI1.1**)



$\alpha°=\frac{180°-90°-60°}{2}$ (∠s sum of isos. Δ)

$\alpha=15$

G3.4 Fill the white squares in the figure with distinct non-zero digits so that the arithmetical expressions, read both horizontally and vertically, are correct. What is the value of α?

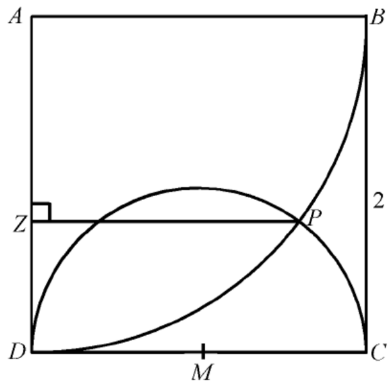
$\alpha=5$

	÷		=	
+		×		
	+		=	*α*
=		=		

6	÷	2	=	3
+		×		
1	+	4	=	5
=		=		
7		8		

Group Event 4

G4.1 In the figure below, $ABCD$ is a square of side length 2. A circular arc with centre at A is drawn from B to D. A semicircle with centre at M, the midpoint of CD, is drawn from C to D and sits inside the square. Determine the shortest distance from P, the intersection of the two arcs, to side AD, that is, the length of PZ.



Join AP, DP, MP.

Let F be the foot of perpendicular from P to CD.

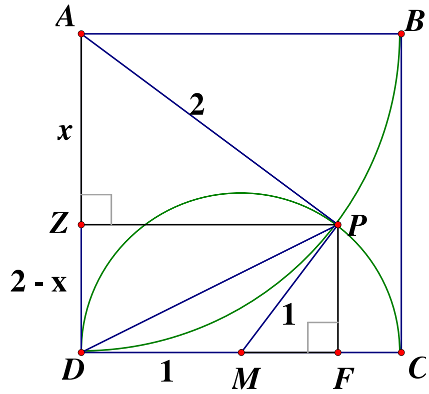
Let $AZ = x$. Then $DZ = 2 - x = PF$, $DM = MP = 1$

In ΔAZP, $AZ^2 + ZP^2 = AP^2$ (Pythagoras' theorem)

$ZP^2 = 4 - x^2$ …… (1)

In ΔPMF, $MF^2 + PF^2 = PM^2$ (Pythagoras' theorem)

$MF^2 = 1 - (2 - x)^2 = 4x - x^2 - 3$ …… (2)



$PZ = DF = 1 + MF$

$4 - x^2 = (1 + \sqrt{4x - x^2 - 3})^2$

$4 - x^2 = 1 + 2\sqrt{4x - x^2 - 3} + 4x - x^2 - 3$

$3 - 2x = \sqrt{4x - x^2 - 3}$

$9 - 12x + 4x^2 = 4x - x^2 - 3$

$5x^2 - 16x + 12 = 0$

$(5x - 6)(x - 2) = 0$

$x = \frac{6}{5}$ or 2 (rejected)

$PZ = \sqrt{4 - x^2} = \sqrt{4 - 1.2^2} = \sqrt{2.56} = 1.6$

Method 2 Let D be the origin, DC be the x-axis, DA be the y-axis.

Equation of circle DPC: $(x - 1)^2 + y^2 = 1 \Rightarrow x^2 - 2x + y^2 = 0$ …… (1)

Equation of circle BPD: $x^2 + (y - 2)^2 = 2^2 \Rightarrow x^2 + y^2 - 4y = 0$ …… (2)

$(1) - (2) \Rightarrow y = \frac{x}{2}$ …… (3)

Sub. (3) into (1): $x^2 - 2x + \frac{x^2}{4} = 0 \Rightarrow x = 0$ (rejected) or 1.6

$\therefore PZ = 1.6$

G4.2 If $x = \frac{\sqrt{5}+1}{2}$ and $y = \frac{\sqrt{5}-1}{2}$, determine the value of $x^3y + 2x^2y^2 + xy^3$.

$xy = \frac{5-1}{4} = 1$

$x^3y + 2x^2y^2 + xy^3 = xy(x^2 + 2xy + y^2)$

$= x^2 + y^2 + 2$

$= \frac{1}{4}(5+1+5+1)+2 = 5$

G4.3 If a, b, c and d are distinct digits and $\begin{array}{r} a\,a\,b\,c\,d \\ -\,d\,a\,a\,b\,c \\ \hline 2\,0\,1\,4\,d \end{array}$, determine the value of d.

Consider the unit digit subtraction, $c = 0$ and there is no borrow digit in the tens digit.

Consider the tens digit, $10 + 0 - b = 4 \Rightarrow b = 6$ and there is a borrow digit in the hundreds.

Consider the hundreds digit, $5 - a = 1 \Rightarrow a = 4$ and there is no borrow digit in the thousands.

Consider the ten thousands digit, $4 - d = 2 \Rightarrow d = 2$

Check: $\begin{array}{r} 44602 \\ -24460 \\ \hline 20142 \end{array}$

G4.4 Determine the product of all real roots of the equation $x^4 + (x - 4)^4 = 32$. **Reference: 2017 FG3.3**

Let $t = x - 2$, then the equation becomes $(t + 2)^4 + (t - 2)^4 = 32$

$2(t^4 + 24t^2 + 16) = 32$

$t^4 + 24t^2 = 0$

$t^2 = 0$ or -24 (rejected)

$t = 0 \Rightarrow x = 2$ (repeated root)

Product of all real roots $= 2\times 2 = 4$