Supplementary Exercises on Quadratic Inequalities

First created by Mr. Francis Hung in 1995. Last updated 2021-09-01

Example 1 Find the maximum values of $y = -x^2 + 5x - 4$

Solution: $x^2 - 5x + (4 + y) = 0$; $a = 1$, $b = -5$, $c = 4 + y$

For real values of x, $\Delta = b^2 - 4ac \geq 0$

$(-5)^2 - 4(1)(4 + y) \geq 0$

$25 - 16 - 4y \geq 0$

$9 \geq 4y$

$y \leq 2.25$

$\therefore$ The maximum value of y is 2.25

Example 2 Find the maximum and minimum values of $y = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$.

Cross multiplying: $(x^2 + 2x - 7)y = x^2 + 34x - 71$

$x^2(y - 1) + 2(y - 17)x + (71 - 7y) = 0$

$a = y - 1$, $b = 2(y - 17)$, $c = 71 - 7y$

For real values of x, $\Delta = b^2 - 4ac \geq 0$

$4[(y - 17)^2 - (y - 1)(71 - 7y)] \geq 0$

$4[y^2 - 34y + 289 - (71y + 7y - 71 - 7y^2)] \geq 0$

$4(8y^2 - 112y + 360) \geq 0$

$4[8(y^2 - 14y + 45)] \geq 0$

$(y - 5)(y - 9) \geq 0$

$\Rightarrow y \leq 5$ or $y \geq 9$

The minimum value of $y = 9$, whereas the maximum value of $y = 5$ (why?)

Find the corresponding values of x when y attains its maximum/ minimum.

<u>Exercises</u>

1. Find a so that for all real values of x, $(a^2 + a + 1)(x^2 + x + 1) > 9ax$.
2. Prove that for any real value of k,
   the expression $3x^2 + 12x + 7 + k(x^2 - 1)$ cannot be positive for all values of x.
3. Find the range of values of x which satisfy $\dfrac{4x^2 - 20x + 18}{x^2 - 5x + 4} < 3$.
4. Prove that the expression $y = x^2 - 2hkx + h^4 + k^4 - h^2k^2$ can never be negative if x, h, k are all real, and that it cannot be zero unless $h^2 = k^2$; $x = hk$.
5. If $k < -1$, prove that for all real values of x, $\dfrac{x + k}{x^2 + x + 1} < \dfrac{x}{x^2 + 2x + 3}$.
6. (a) If $q < p < r$, show that $y = \dfrac{x - p}{(x - q)(x - r)}$ can take any real values.

   (b) If p does not lies in the above interval, show that $y \leq \alpha$ or $\beta \leq y$, where $\beta - \alpha = \dfrac{4\sqrt{(p - q)(p - r)}}{(q - r)^2}$.

   Illustrate your answer by sketching roughly the graphs of

   $y = \dfrac{x - 4}{(x - 5)(x - 1)}$ and $y = \dfrac{x - 5}{(x - 4)(x - 1)}$.

1. $(a^2 + a + 1)(x^2 + x + 1) > 9ax$

$(a^2 + a + 1)x^2 + (a^2 - 8a + 1)x + (a^2 + a + 1) > 0$ for all x

$\therefore \Delta < 0 \Rightarrow (a^2 - 8a + 1)^2 - 4(a^2 + a + 1)^2 < 0$ and $a^2 + a + 1 > 0$

$(a^2 - 8a + 1 + 2a^2 + 2a + 2)(a^2 - 8a + 1 - 2a^2 - 2a - 2) < 0$ and $a^2 + a + \left(\frac{1}{2}\right)^2 + \frac{3}{4} > 0$

$(3a^2 - 6a + 3)(-a^2 - 10a - 1) < 0$ and $\left(a + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$

$-3(a^2 - 2a + 1)(a^2 + 10a + 1) < 0$ and always true

$(a - 1)^2[(a + 5)^2 - 24] > 0$

$\left(a + 5 + \sqrt{24}\right)\left(a + 5 - \sqrt{24}\right) > 0$ and $a \neq 1$

($a < -5 - 2\sqrt{6}$ or $-5 + 2\sqrt{6} < a$) and $a \neq 1$

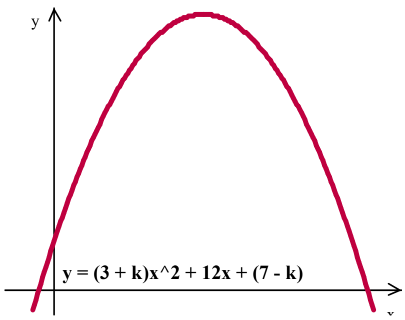
$a < -5 - 2\sqrt{6}$ or $-5 + 2\sqrt{6} < a < 1$ or $1 < a$

2. $y = 3x^2 + 12x + 7 + k(x^2 - 1)$

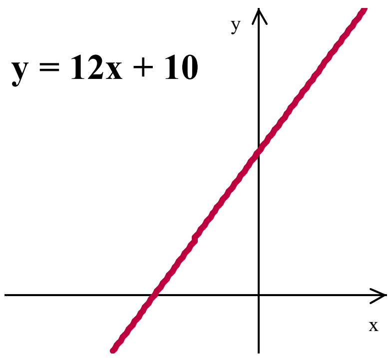
$y = (3 + k)x^2 + 12x + (7 - k)$

If $3 + k < 0$, then the graph opens downward.

Therefore y cannot be positive for all x .



If $k = -3$, $y = 12x + 10$. Therefore y cannot be positive for all x .



If $3 + k > 0$, $\Delta = 12^2 - 4(3 + k)(7 - k)$

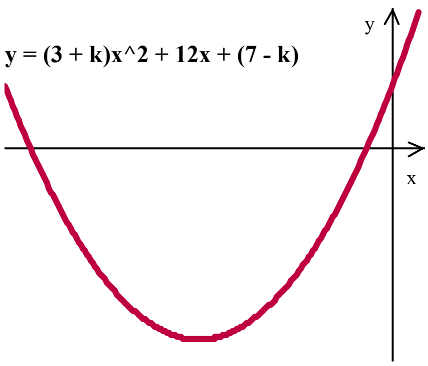
$= 4(36 + k^2 - 4k - 21)$

$= 4(k^2 - 4k + 15)$

$= 4[(k - 2)^2 + 11] > 0$ for all x .

$\therefore$ The graph must cut x-axis somewhere.

Therefore y cannot be positive for all x .



3. $\dfrac{4x^2-20x+18}{x^2-5x+4}<3$

$\dfrac{4x^2-20x+18}{x^2-5x+4}-3<0$

$\dfrac{4x^2-20x+18-3x^2+15x-12}{x^2-5x+4}<0$

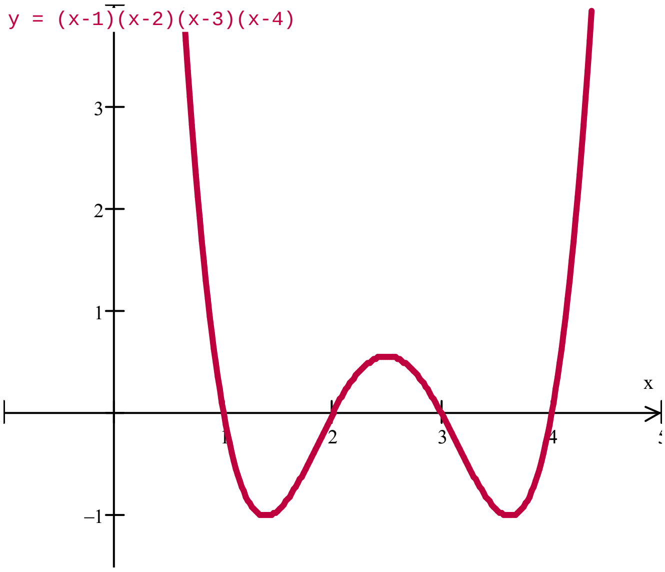
$\dfrac{x^2-5x+6}{x^2-5x+4}<0$

$\dfrac{(x-2)(x-3)}{(x-1)(x-4)}<0$

$(x-1)^2(x-4)^2\cdot\dfrac{(x-2)(x-3)}{(x-1)(x-4)}<0$

$(x-1)(x-2)(x-3)(x-4)<0$

$1<x<2$ or $3<x<4$



4. $y = x^2 - 2hkx + h^4 + k^4 - h^2k^2$

The graph open upwards and

$\Delta = 4h^2k^2 - 4(h^4 + k^4 - h^2k^2)$

$\Delta = -4(h^4 + k^4 - 2h^2k^2)$

$\Delta = -4(h^2 - k^2)^2$

$\Delta \le 0$

$\therefore$ The graph cuts x-axis at most once.

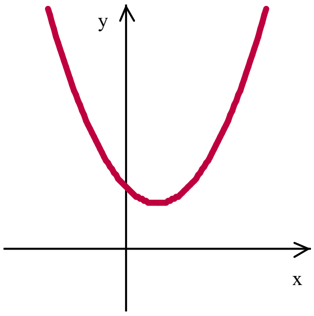
$\therefore y \ge 0$

If $y = 0 \Rightarrow$ double root

$\Rightarrow \Delta = 0 \Rightarrow h^2 = k^2$

In this case, $y=\left(x+\dfrac{b}{2a}\right)^2$, i.e. $y = (x - hk)^2$

When $y = 0$, $x = hk$



5. If $k < -1$, prove that for all real values of x, $\dfrac{x+k}{x^2+x+1}<\dfrac{x}{x^2+2x+3}$.

$x^2 + x + 1 = \left(x+\dfrac{1}{2}\right)^2+\dfrac{3}{4}>0$; $x^2 + 2x + 3 = (x + 1)^2 + 2 > 0$

Let $y = x(x^2 + x + 1) - (x + k)(x^2 + 2x + 3)$

$y = x^3 + x^2 + x - x^3 - 2x^2 - 3x - kx^2 - 2kx - 3k$

$y = -(1 + k)x^2 - 2(1 + k)x - 3k$

$\because (1 + k) < 0 \therefore -(1 + k) > 0 \Rightarrow$ The graph opens upwards

$\Delta = 4[(1 + k)^2 - 3k(1 + k)]$

$\Delta = 4(1 + 2k + k^2 - 3k - 3k^2)$

$\Delta = 4(-2k^2 - k + 1)$

$\Delta=-8\left(k^2+\dfrac{k}{2}-\dfrac{1}{2}\right)$

$$\Delta = -8\left[\left(k+\frac{1}{4}\right)^2 - \frac{9}{16}\right]$$

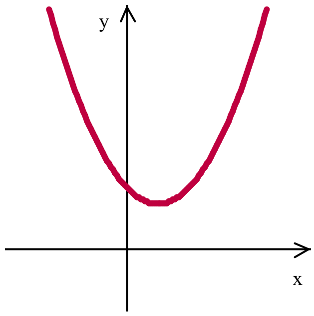
$$k < -1 \Rightarrow k + \frac{1}{4} < -1 + \frac{1}{4} = -\frac{3}{4} < 0$$

$$\left(k+\frac{1}{4}\right)^2 > \left(-\frac{3}{4}\right)^2 = \frac{9}{16}$$

$$\left(k+\frac{1}{4}\right)^2 - \frac{9}{16} > 0$$

$$\Rightarrow \quad \Delta = -8\left[\left(k+\frac{1}{4}\right)^2 - \frac{9}{16}\right] < 0$$

$\Rightarrow$ The curve does not cut x-axis and open upwards



$\Rightarrow y > 0$ for all x

$\therefore\ x(x^2 + x + 1) - (x + k)(x^2 + 2x + 3) > 0$ for all x

$x(x^2 + x + 1) > (x + k)(x^2 + 2x + 3)$

$\therefore$ If $k > -1$, $\dfrac{x+k}{x^2+x+1} < \dfrac{x}{x^2+2x+3}$ for all x.

6. (a) If $q < p < r$, show that $y = \dfrac{x-p}{(x-q)(x-r)}$ can take any real values.

$y[x^2 - (q + r)x + qr] = x - p$

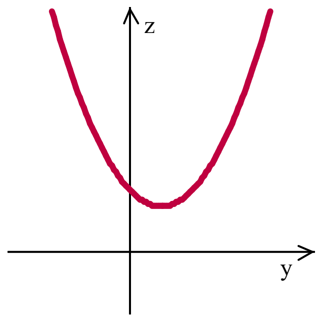
$yx^2 - [(q + r)y + 1]x + qry + p = 0$

For all real x, $\Delta \ge 0$

$\Rightarrow [(q + r)y + 1]^2 - 4y(qry + p) \ge 0$

$(q + r)^2y^2 + 2(q + r)y + 1 - 4qry^2 - 4py \ge 0$

$(q - r)^2y^2 + 2(q + r - 2p)y + 1 \ge 0$ …… (1)



Δ of (1) $= 4[(q + r - 2p)^2 - (q - r)^2]$

$= 4(q + r - 2p + q - r)(q + r - 2p - q + r)$

$= 4(2q - 2p)(2r - 2p)$

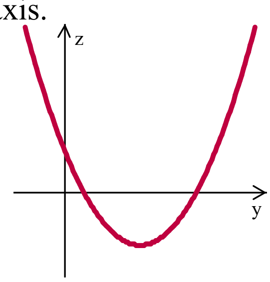
$= 16(q - p)(r - p) < 0 \quad (\because\ q - p < 0$ and $r - p > 0)$

$\therefore$ From (1), the curve $z = (q - r)^2y^2 + 2(q + r - 2p)y + 1$ does not cut y-axis.

To solve $z \ge 0 \Rightarrow y$ can be any real number.

(b) If $p \le q$ or $r \le p$ then Δ of (1) ≥ 0

The graph $z = (q - r)^2y^2 + 2(q + r - 2p)y + 1$ looks like:



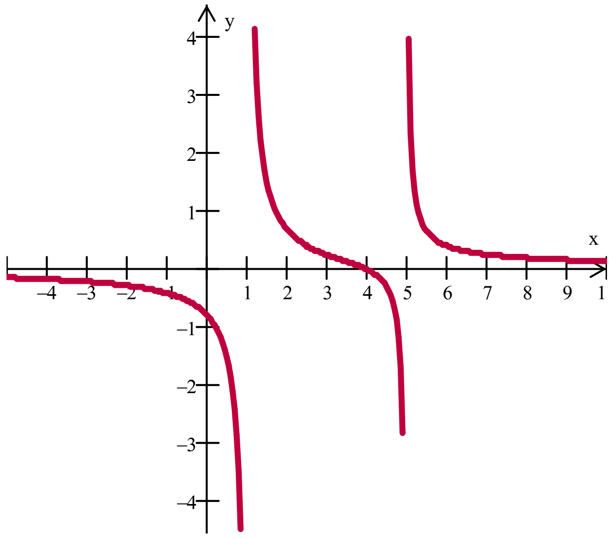
To solve $z \ge 0 \Rightarrow y \le \alpha$ or $\beta \le y$

where $\beta - \alpha = \sqrt{(\beta-\alpha)^2} = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$$\beta - \alpha = \sqrt{\frac{[2(q+r-2p)]^2}{(q-r)^4} - 4\cdot\frac{1}{(q-r)^2}} = \frac{4\sqrt{(p-q)(p-r)}}{(q-r)^2}.$$

$y = \dfrac{x-4}{(x-5)(x-1)}$, $p = 4$, $q = 1$, $r = 5$; $\therefore\ q < p < r$

By 6(a), y can take any values



$y = \dfrac{x-5}{(x-4)(x-1)}$, $p = 5$, $q = 1$, $r = 4$; $\therefore$ $q < r < p$

By 6(b), if α and β are roots of the equations $z = 0$, where

$z = (q - r)^2y^2 + 2(q + r - 2p)y + 1 = 0$

$z = 9y^2 - 10y + 1 = 0$

$(9y - 1)(y - 1) = 0 \Rightarrow y = \dfrac{1}{9}$ or $y = 1$

In this case, $yx^2 - [(q + r)y + 1]x + qry + p = 0$

$yx^2 - (5y + 1)x + 4y + 5 = 0$

When $y = \dfrac{1}{9}$, $\dfrac{1}{9}x^2 - \dfrac{14}{9}x + \dfrac{49}{9} = 0 \Rightarrow x = 7$

When $y = 1$, $x^2 - 6x + 9 = 0 \Rightarrow x = 3$