Examples on Mathematical Induction: Generalisation

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1. Given that $\begin{cases} 1 = 0 + 1 \\ 2 + 3 + 4 = 1 + 8 \\ 5 + 6 + 7 + 8 + 9 = 8 + 27 \end{cases}$

(a) Deduce the n^{th} statement.

(b) Prove the n^{th} statement by mathematical induction.

(a) $(n - 1)^2 + 1 + (n - 1)^2 + 2 + \cdots + n^2 = (n - 1)^3 + n^3$

(b) Let $P(n) \equiv$ "$(n - 1)^2 + 1 + (n - 1)^2 + 2 + \cdots + n^2 = (n - 1)^3 + n^3$ for all positive integer n."

$n = 1$, L.H.S. $= 1 = 0^3 + 1^3 =$ R.H.S., $P(1)$ is true.

Assume $[(k - 1)^2 + 1] + [(k - 1)^2 + 2] + \cdots + k^2 = (k - 1)^3 + k^3$

LHS is the sum of $2k - 1$ terms

When $n = k + 1$,

L.H.S. $= (k^2 + 1) + (k^2 + 2) + \cdots + (k + 1)^2$ (the sum of $2k + 1$ terms)

$= (k^2 + 1) + (k^2 + 2) + \cdots + (k^2 + 2k - 1) + (k^2 + 2k) + (k + 1)^2$

$= [(k - 1 + 1)^2 + 1] + [(k - 1 + 1)^2 + 2] + \cdots + [(k - 1 + 1)^2 + 2k - 1] + (k^2 + 2k) + (k + 1)^2$

$= \{[(k - 1)^2 + 1] + [(k - 1)^2 + 2] + \cdots + k^2\} + 2[(k - 1) + (k - 1) + \cdots + (k - 1)]_{(2k-1)\text{ terms}}$

$+ (1 + 1 + \cdots + 1)_{(2k-1)\text{ terms}} + (k^2 + 2k) + (k + 1)^2$

$= (k - 1)^3 + k^3 + 2(k - 1)(2k - 1) + 2k - 1 + (k^2 + 2k) + (k^2 + 2k + 1)$

$= k^3 - 3k^2 + 3k - 1 + k^3 + 4k^2 - 6k + 2 + 2k - 1 + k^2 + 2k + k^2 + 2k + 1$

$= 2k^3 + 3k^2 + 3k + 1$

R.H.S. $= k^3 + (k + 1)^3$

$= k^3 + k^3 + 3k^2 + 3k + 1$

$= 2k^3 + 3k^2 + 3k + 1 =$ L.H.S.

It is also true for $n = k + 1$ if it is true for $n = k$.

By the principle of mathematical induction, it is true for all positive integer n.

2. Given that $\begin{cases} 1 - \frac{1}{2} = \frac{1}{2} \\ \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = \frac{1}{3} \\ \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{1}{4} \end{cases}$.

(a) Deduce the n^{th} statement.

(b) Prove the n^{th} statement by mathematical induction.

3. Given that $\begin{cases} 1=1 \\ 1-4=-(1+2) \\ 1-4+9=1+2+3 \\ 1-4+9-16=-(1+2+3+4) \end{cases}$

(a) Deduce the n^{th} statement.

(b) Prove the n^{th} statement by mathematical induction.

(a) Let $P(n) \equiv$ "$1^2 - 2^2 + \cdots + (-1)^{n-1}\cdot n^2 = (-1)^{n-1}(1 + 2 + \cdots + n)$ for all positive integer n ."

(b) $n = 1$, L.H.S. $= 1^2 = 1$, R.H.S. $= 1$, $P(1)$ is true.

Suppose $1^2 - 2^2 + \cdots + (-1)^{k-1}\cdot k^2 = (-1)^{k-1}(1 + 2 + \cdots + k)$ for some positive integer k .

$1^2 - 2^2 + \cdots + (-1)^{k-1}\cdot k^2 + (-1)^k\cdot(k + 1)^2$

$= (-1)^{k-1}(1 + 2 + \cdots + k) + (-1)^k\cdot(k + 1)^2$

$=(-1)^k\left[-\frac{k(k+1)}{2}+(k+1)^2\right]$

$=(-1)^k(k+1)\frac{2k+2-k}{2}=\frac{(-1)^k}{2}(k+1)(k+2)$

R.H.S. $= (-1)^k(1 + 2 + \cdots + k + k + 1)$

$=(-1)^k\left[\frac{(k+1)(k+2)}{2}\right]$

$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive n.

4. Let $S_n = 1 + 2 + 3 + \cdots + (n - 1) + n + (n - 1) + \cdots + 3 + 2 + 1$, where n is a positive integer.

(a) Find the values of the following terms.

(i) S_1 (*ii*) S_2 (*iii*) S_3

(b) Deduce a formula for evaluating S_n, and prove that the formula is true for all positive integers n by mathematical induction.

(a) $S_1 = 1$ (*ii*) $S_2 = 1 + 2 + 1 = 4$ (*iii*) $1 + 2 + 3 + 2 + 1 = 9$

(b) $S_n = n^2$

Let $P(n) \equiv$ "$1 + 2 + 3 + \cdots + (n - 1) + n + (n - 1) + \cdots + 3 + 2 + 1 = n^2$."

$n = 1$, LHS $= 1 =$ RHS, $P(1)$ is true.

Assume $1 + 2 + 3 + \cdots + (k - 1) + k + (k - 1) + \cdots + 3 + 2 + 1 = k^2$ for some integer k.

When $n = k + 1$,

LHS $= 1 + 2 + 3 + \cdots + k + (k + 1) + k + \cdots + 3 + 2 + 1$

$= [1 + 2 + 3 + \cdots + (k - 1) + k + (k - 1) + \cdots + 3 + 2 + 1] + k + (k + 1)$

$= k^2 + 2k + 1$ (induction assumption)

$= (k + 1)^2 =$ RHS

If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $S_n = n^2$ is true for all positive integer n.

5. Given that $\begin{cases} 1+\frac{1}{2}=2-\frac{1}{2} \\ 1+\frac{1}{2}+\frac{1}{4}=2-\frac{1}{4} \\ 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=2-\frac{1}{8} \end{cases}$.

(a) Deduce the n^{th} statement.

(b) Prove the n^{th} statement by mathematical induction.

6. It is given that $S_n=\frac{8\times 1}{1^2\times 3^2}+\frac{8\times 2}{3^2\times 5^2}+\cdots+\frac{8\times n}{(2n-1)^2\times(2n+1)^2}$, where n is a positive integer, and $S_1=\frac{8}{9}$, $S_2=\frac{24}{25}$, $S_3=\frac{48}{49}$, $S_4=\frac{80}{81}$.

(a) Deduce the formula for evaluating S_n,

(b) Prove, by mathematical induction, that the formula obtained in (a) is true for all positive integers n.

(a) $S_1=\frac{8}{9}=\frac{3^2-1}{3^2}$, $S_2=\frac{24}{25}=\frac{5^2-1}{5^2}$, $S_3=\frac{48}{49}=\frac{7^2-1}{7}$, $S_4=\frac{80}{81}=\frac{9^2-1}{9^2}$

$S_n=\frac{(2n+1)^2-1}{(2n+1)^2}=1-\frac{1}{(2n+1)^2}$

(b) Let $P(n)\equiv$ "$\frac{8\times 1}{1^2\times 3^2}+\frac{8\times 2}{3^2\times 5^2}+\cdots+\frac{8\times n}{(2n-1)^2\times(2n+1)^2}=1-\frac{1}{(2n+1)^2}$"

$n=1$, L.H.S. $=\frac{8\times 1}{1^2\times 3^2}=\frac{8}{9}=1-\frac{1}{3^2}$

Suppose $\frac{8\times 1}{1^2\times 3^2}+\frac{8\times 2}{3^2\times 5^2}+\cdots+\frac{8\times k}{(2k-1)^2\times(2k+1)^2}=1-\frac{1}{(2k+1)^2}$

Add $\frac{8\times(k+1)}{(2k+1)^2\times(2k+3)^2}$ to both sides.

$$\frac{8\times 1}{1^2\times 3^2}+\frac{8\times 2}{3^2\times 5^2}+\cdots+\frac{8\times k}{(2k-1)^2\times(2k+1)^2}+\frac{8\times(k+1)}{(2k+1)^2\times(2k+3)^2}$$

$$=1-\frac{1}{(2k+1)^2}+\frac{8\times(k+1)}{(2k+1)^2\times(2k+3)^2}=1-\frac{(2k+3)^2}{(2k+1)^2(2k+3)^2}+\frac{8(k+1)}{(2k+1)^2(2k+3)^2}$$

$$=1-\frac{(2k+3)^2-8(k+1)}{(2k+1)^2(2k+3)^2}=1-\frac{4k^2+12k+9-8k-8}{(2k+1)^2(2k+3)^2}$$

$$=1-\frac{4k^2+4k+1}{(2k+1)^2(2k+3)^2}$$

$$=1-\frac{(2k+1)^2}{(2k+1)^2(2k+3)^2}$$

$$=1-\frac{1}{(2k+3)^2}$$

$\therefore$ If $P(k)$ is true, then $P(k+1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n.