Algebraic Inequality

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1. Let α, β, γ be the positive roots of $ax^3 + bx^2 + cx + d = 0$, $a \neq 0$.
   (a) Prove that $\alpha + \beta + \gamma = -\frac{b}{a}$; $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$; $\alpha\beta\gamma = -\frac{d}{a}$
   (b) If $a > 0$, use the results of (a) to show that (i) $b^3 \leq 27a^2 d$; (ii) $b^2 \geq 3ac$
   (Hint: (i) Consider $\frac{\alpha+\beta+\gamma}{3} \geq \sqrt[3]{\alpha\beta\gamma}$; (ii) Consider $(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 \geq 0$)
2. Let a, b, c, d be any real numbers. Prove that $(a^2+b^2)\cdot(c^2+d^2) \geq (ac+bd)^2$
   Hence prove that (a) $\sqrt{a^2+b^2} + \sqrt{c^2+d^2} \geq \sqrt{(a+c)^2 + (b+d)^2}$
   (b) if $\sqrt{a^2+b^2} \leq 1$; $\sqrt{c^2+d^2} \leq 1$, then $|ac+bd| \leq 1$
3. (a) Let $a > b$, $c > d$. By using the fact that $(a - b)(c - d) > 0$, prove the Tchebychef Inequality: $\frac{(ac+bd)}{2} > \frac{(a+b)}{2} \cdot \frac{(c+d)}{2}$
   (b) Let $p > q > r$, $s > t > u$. By using the fact that $(p-q)(s-t) + (p-r)(s-u) + (q-r)(t-u) > 0$, prove that $\frac{(ps+qt+ru)}{3} > \frac{(p+q+r)}{3} \cdot \frac{(s+t+u)}{3}$
4. (a) For any real numbers x, y, and z, prove that $x^2 + y^2 + z^2 \geq xy + yz + zx$
   (b) Let a, b and c be the angles of a triangle. Prove that
   $$\tan\frac{a}{2}\tan\frac{b}{2} + \tan\frac{b}{2}\tan\frac{c}{2} + \tan\frac{c}{2}\tan\frac{a}{2} = 1$$
   (c) By using the results of (a) and (b), prove that $\tan^2\frac{a}{2} + \tan^2\frac{b}{2} + \tan^2\frac{c}{2} \geq 1$.
5. If $\frac{a}{b} \leq \frac{c}{d} \leq \frac{e}{f}$, and b, d, f are positive numbers; prove that $\frac{a}{b} \leq \frac{a+c+e}{b+d+f} \leq \frac{e}{f}$.
6. (a) Show that for any real numbers x, y and z:
   $$x^2 + y^2 + z^2 \geq xy + yz + zx$$
   and determine the condition for which the equality holds.
   (b) Let x, y, z be three non-zero real numbers such that $x + y + z = xyz$
   Using (a) or otherwise, show that
   $$x^2 + y^2 + z^2 \geq (x+y+z)\left(\frac{x^2-1}{2x} + \frac{y^2-1}{2y} + \frac{z^2-1}{2z}\right)$$
   and that the equality holds iff $x = y = z$.
7. (a) If $x + y + z = a$, prove that $x^2 + y^2 + z^2 \geq \frac{a^2}{3}$
   (b) If $a \geq 0$; $x + y + z = a$ and $x^2 + y^2 + z^2 = \frac{a^2}{2}$; prove that $0 \leq x, y, z \leq \frac{2a}{3}$
8. If a, b, $c > 0$, prove that
   (a) $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$
   (b) $\frac{b+c-a}{a} + \frac{c+a-b}{b} + \frac{a+b-c}{c} \geq 3$
9. If a, b, $c > 0$ and $a + b + c = 1$, prove that
   (a) $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{9}{2}$
   (b) $a^2 + b^2 + c^2 \geq \frac{1}{3}$

1. (a) $ax^3 + bx^2 + cx + d \equiv a(x - \alpha)(x - \beta)(x - \gamma)$

$ax^3 + bx^2 + cx + d \equiv a[x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma]$

Compare coefficients

$b = -a(\alpha + \beta + \gamma)$

$c = a(\alpha\beta + \beta\gamma + \alpha\gamma)$

$d = -a\alpha\beta\gamma$

$\Rightarrow \alpha + \beta + \gamma = -\frac{b}{a}$; $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$; $\alpha\beta\gamma = -\frac{d}{a}$

(b) $\because$ α, β, γ are the positive roots and $a > 0$,

$\therefore\ \alpha + \beta + \gamma = -\frac{b}{a} > 0 \Rightarrow b < 0$

$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} > 0 \Rightarrow c > 0$

$\alpha\beta\gamma = -\frac{d}{a} > 0 \Rightarrow d < 0$

(i) By the theorem A.M. ≥ G.M.

$$\frac{\alpha + \beta + \gamma}{3} \geq \sqrt[3]{\alpha\beta\gamma}$$

$$\begin{cases} -\frac{b}{3a} \geq \sqrt[3]{\alpha\beta\gamma} > 0 \\ -\frac{b}{3a} \geq \sqrt[3]{\alpha\beta\gamma} > 0 \\ -\frac{b}{3a} \geq \sqrt[3]{\alpha\beta\gamma} > 0 \end{cases}$$

Multiply these three inequalities: $-\frac{b^3}{27a^3} \geq -\frac{d}{a} \Rightarrow \frac{b^3}{27a^3} \leq \frac{d}{a}$

$b^3 \leq 27a^2 d$

(ii) $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \geq 0$

$2[\alpha^2 + \beta^2 + \gamma^2 - (\alpha\beta + \beta\gamma + \alpha\gamma)] \geq 0$

$2[(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \alpha\gamma)] \geq 0$

$\left(-\frac{b}{a}\right)^2 - \frac{3c}{a} \geq 0 \Rightarrow\ b^2 \geq 3ac$

2. $(a^2 + b^2)\cdot(c^2 + d^2) - (ac + bd)^2$

$= a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 - a^2c^2 - 2abcd - a^2d^2$

$= b^2c^2 - 2abcd + a^2d^2$

$= (bc - ad)^2 \geq 0$

$\therefore\ (a^2 + b^2)\cdot(c^2 + d^2) \geq (ac + bd)^2$

(a) $(a^2 + b^2)\cdot(c^2 + d^2) \geq (ac + bd)^2$

$\sqrt{(a^2 + b^2)\cdot(c^2 + d^2)} \geq (ac + bd)$

$2\sqrt{(a^2 + b^2)\cdot(c^2 + d^2)} \geq 2(ac + bd)$

$a^2 + b^2 + 2\sqrt{(a^2 + b^2)\cdot(c^2 + d^2)} + c^2 + d^2 \geq a^2 + b^2 + 2(ac + bd) + c^2 + d^2$

$\left[\sqrt{(a^2 + b^2)} + \sqrt{(c^2 + d^2)}\right]^2 \geq (a + c)^2 + (b + d)^2$

$\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \geq \sqrt{(a + c)^2 + (b + d)^2}$

(b) $\because\ (a^2+b^2)\cdot(c^2+d^2)\ge(ac+bd)^2$ and $\sqrt{a^2+b^2}\le 1;\sqrt{c^2+d^2}\le 1$,

$\therefore\ \sqrt{a^2+b^2}\cdot\sqrt{c^2+d^2}\ge|ac+bd|$

$1\ge\ |ac+bd|$

3. (a) $a>b,\ c>d$

$a-b>0$ and $c-d>0$

$(a-b)(c-d)>0$

$ac+bd-bc-ad>0$

$ac+bd>bc+ad$

$2(ac+bd)>ac+bc+ad+bd$

$2(ac+bd)>(a+b)(c+d)$

$$\frac{(ac+bd)}{2}>\frac{(a+b)}{2}\cdot\frac{(c+d)}{2}$$

(b) $p>q>r,\ s>t>u.$

$p-q>0$ and $s-t>0\Rightarrow(p-q)(s-t)>0$

$p-r>0$ and $s-u>0\Rightarrow(p-r)(s-u)>0$

$q-r>0$ and $t-u>0\Rightarrow(q-r)(t-u)>0$

$(p-q)\,(s-t)+(p-r)\,(s-u)+(q-r)\,(t-u)>0$

$ps+qt+ps+ru+qt+ru>qs+pt+rs+pu+rt+qu$

$2(ps+qt+ru)>pt+pu+qs+qu+rs+rt$

$3(ps+qt+ru)>ps+pt+pu+qs+qt+qu+rs+rt+ru$

$3(ps+qt+ru)>(p+q+r)(s+t+u)$

$$\frac{(ps+qt+ru)}{3}>\frac{(p+q+r)}{3}\cdot\frac{(s+t+u)}{3}$$

4. (a) $x^2+y^2+z^2-(xy+yz+zx)$

$$=\frac{1}{2}\left[2x^2+2y^2+2z^2-2(xy+yz+zx)\right]$$

$$=\frac{1}{2}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]$$

$$=\frac{1}{2}\left[\text{sum of three squares}\right]\ \ge 0$$

$x^2+y^2+z^2\ge xy+yz+zx$

(b) $a+b+c=180°$

$$\frac{a+b+c}{2}=90^\circ$$

$$\frac{c}{2}=90^\circ-\left(\frac{a}{2}+\frac{b}{2}\right)$$

$$\cot\frac{c}{2}=\cot\left[90^\circ-\left(\frac{a}{2}+\frac{b}{2}\right)\right]$$

$$\frac{1}{\tan\frac{c}{2}}=\tan\left(\frac{a}{2}+\frac{b}{2}\right)$$

$$\frac{1}{\tan\frac{c}{2}}=\frac{\tan\frac{a}{2}+\tan\frac{b}{2}}{1-\tan\frac{a}{2}\tan\frac{b}{2}}$$

$$1-\tan\frac{a}{2}\tan\frac{b}{2}=\tan\frac{a}{2}\tan\frac{c}{2}+\tan\frac{b}{2}\tan\frac{c}{2}$$

$$\tan\frac{a}{2}\tan\frac{b}{2}+\tan\frac{b}{2}\tan\frac{c}{2}+\tan\frac{c}{2}\tan\frac{a}{2}=1$$

(b) Let $x=\tan\frac{a}{2}$, $y=\tan\frac{b}{2}$, $z=\tan\frac{c}{2}$

by (a) $\tan^2\frac{a}{2}+\tan^2\frac{b}{2}+\tan^2\frac{c}{2}\ge\tan\frac{a}{2}\tan\frac{b}{2}+\tan\frac{b}{2}\tan\frac{c}{2}+\tan\frac{a}{2}\tan\frac{c}{2}=1$

$\therefore\tan^2\frac{a}{2}+\tan^2\frac{b}{2}+\tan^2\frac{c}{2}\ge 1$

5. $\frac{a}{b}\le\frac{c}{d}\le\frac{e}{f}\Rightarrow ad\le bc$ and $af\le be$ and $cf\le de$

$(b+d+f)e-(a+c+e)f=(be-af)+(de-cf)\ge 0$

$\therefore (b+d+f)e\ge(a+c+e)f$

$\frac{a+c+e}{b+d+f}\le\frac{e}{f}$ $\because$ b, d, f are positive …… (1)

$(a+c+e)b-(b+d+f)a=(bc-ad)+(be-af)\ge 0$

$\therefore (a+c+e)b\ge(b+d+f)a$

$\frac{a}{b}\le\frac{a+c+e}{b+d+f}$ $\because$ b, d, f are positive …… (2)

Combine (1) and (2), $\frac{a}{b}\le\frac{a+c+e}{b+d+f}\le\frac{e}{f}$.

6. (a) $x^2+y^2+z^2-xy-yz-zx$

$=\frac{1}{2}\left(x^2-2xy+y^2+y^2-2yz+z^2+z^2-2xz+x^2\right)$

$=\frac{1}{2}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]\ge 0$

$\therefore x^2+y^2+z^2\ge xy+yz+zx$, equality holds when $x=y=z$.

(b) Let x, y, z be three non-zero real numbers such that $x+y+z=xyz$

$(x+y+z)\left(\frac{x^2-1}{2x}+\frac{y^2-1}{2y}+\frac{z^2-1}{2z}\right)$

$=\frac{1}{2}(x+y+z)\left[(x+y+z)-\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\right]$

$=\frac{1}{2}\left[(x+y+z)^2-\frac{(x+y+z)(xy+yz+zx)}{xyz}\right]$

$=\frac{1}{2}\left[(x+y+z)^2-(xy+yz+zx)\right]$ ($\because$ Given that $x+y+z=xyz$)

$=\frac{1}{2}\left(x^2+y^2+z^2+xy+yz+zx\right)$

Hence LHS – RHS $=x^2+y^2+z^2-\frac{1}{2}\left(x^2+y^2+z^2+xy+yz+zx\right)$

$=\frac{1}{2}\left(x^2+y^2+z^2-xy-yz-zx\right)\ge 0$, by (a)

and that the equality holds iff $x=y=z\ne 0$.

7. (a) By Tchebychef's Inequality, if $a \ge b \ge c$ and $x \ge y \ge z$, then Product mean ≥ mean product

$$\frac{ax+by+cz}{3} \ge \frac{a+b+c}{3} \cdot \frac{x+y+z}{3}$$

If $x + y + z = a$, prove that $x^2 + y^2 + z^2 \ge \frac{a^2}{3}$

Without loss of generality assume $x \ge y \ge z$, then $\frac{x^2+y^2+z^2}{3} \ge \frac{x+y+z}{3} \cdot \frac{x+y+z}{3}$

$$\frac{x^2+y^2+z^2}{3} \ge \frac{a}{3} \cdot \frac{a}{3}$$

$\therefore x^2 + y^2 + z^2 \ge \frac{a^2}{3}$

(b) If $a \ge 0$; $x + y + z = a$ and $x^2 + y^2 + z^2 = \frac{a^2}{2}$; prove that $0 \le x, y, z \le \frac{2a}{3}$

$y + z = a - x$ …… (1)

$y^2 + z^2 = \frac{a^2}{2} - x^2$ …… (2)

$(1)^2 - (2)$: $2yz = (a - x)^2 - \left(\frac{a^2}{2} - x^2\right)$

$2yz = \frac{a^2}{2} - 2ax + 2x^2$ ⋯ (3)

$2yz = \frac{1}{2}(a-2x)^2 \ge 0$, similarly $2xy = \frac{1}{2}(a-2z)^2 \ge 0$; $2xz = \frac{1}{2}(a-2y)^2 \ge 0$ ⋯ (4)

$\therefore yz \ge 0 \Rightarrow$ either $y \ge 0, z \ge 0$ or $y \le 0, z \le 0$

If $y \le 0, z \le 0$, then by the above result (4), $2xz \ge 0 \Rightarrow x \le 0$

But $x + y + z = a \ge 0$, L.H.S. = sum of 3 negative numbers < 0, while R.H.S. ≥ 0 which leads to a contradiction

$\therefore x \ge 0, y \ge 0$ and $z \ge 0$.

To prove that $x, y, z \le \frac{2a}{3}$

By (3), $\frac{a^2}{2} - 2ax + 2x^2 = 2yz$

$\frac{a^2}{2} - 2ax + 2x^2 \le y^2 + z^2$ $\because y \ge 0, z \ge 0$ and AM ≥ GM

$\frac{a^2}{2} - 2ax + 2x^2 \le \frac{a^2}{2} - x^2$

$3x^2 \le 2ax$

$x \le \frac{2a}{3}$ Cancel x, $x \ge 0$

By symmetry, $0 \le x, y, z \le \frac{2a}{3}$

8. If a, b, $c > 0$, prove that

(a) $a^a\, b^b\, c^c \ge (abc)^{\frac{a+b+c}{3}}$

WLOG assume $a \ge b \ge c > 0$

then $\dfrac{a^{3a}b^{3b}c^{3c}}{a^{a+b+c}b^{c+a+b}c^{a+b+c}} = \dfrac{a^{a-b}b^{b-c}c^{c-a}}{a^{c-a}b^{a-b}c^{b-c}}$

$$= \left(\frac{a}{b}\right)^{a-b}\left(\frac{b}{c}\right)^{b-c}\left(\frac{c}{a}\right)^{c-a} \ge 1$$

$\therefore (a^a\, b^b\, c^c)^3 \ge (abc)^{a+b+c}$

$a^a\, b^b\, c^c \ge (abc)^{\frac{a+b+c}{3}}$

(b) $\dfrac{b+c-a}{a} + \dfrac{c+a-b}{b} + \dfrac{a+b-c}{c}$

$= \left(\dfrac{a}{b} + \dfrac{b}{a}\right) + \left(\dfrac{b}{c} + \dfrac{c}{b}\right) + \left(\dfrac{c}{a} + \dfrac{a}{c}\right) - 3$

$\ge\ 2\sqrt{\dfrac{a}{b}\cdot\dfrac{b}{a}} + 2\sqrt{\dfrac{b}{c}\cdot\dfrac{c}{b}} + 2\sqrt{\dfrac{c}{a}\cdot\dfrac{a}{c}} - 3$ (AM $\ge$ GM)

$= 6 - 3 = 3$, equality holds when $a = b = c$

9. If a, b, $c > 0$ and $a + b + c = 1$, prove that

(a) $\dfrac{1}{a+b} + \dfrac{1}{b+c} + \dfrac{1}{c+a} \ge \dfrac{9}{2}$

Using the result AM $\ge$ GM $\ge$ HM

If x, y, $z > 0$, then $\dfrac{x+y+z}{3} \ge \dfrac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{9}{a+b+b+c+c+a} = \frac{9}{2}$$

(b) $a^2 + b^2 + c^2 \ge \dfrac{1}{3}$; this is a special case for Q7(a).

Weierstrass' inequalities

Reference: Advanced Level Pure Mathematics Algebra by K. S. Ng, Y. K. Kwok p.131 Created on 20110909

Given $0 < x_1, x_2, \cdots, x_n < 1$, $n \geq 2$ and $S_n = x_1 + x_2 + \cdots + x_n < 1$, then we have the following results:

$$1 + S_n < (1 + x_1)(1 + x_2) \cdots (1 + x_n) < \frac{1}{1 - S_n}$$

and $1 - S_n < (1 - x_1)(1 - x_2) \cdots (1 - x_n) < \dfrac{1}{1 + S_n}$.

This is equivalent to four inequalities:

(A) $1 + S_n < (1 + x_1)(1 + x_2) \cdots (1 + x_n)$

(B) $(1 + x_1)(1 + x_2) \cdots (1 + x_n) < \dfrac{1}{1 - S_n}$

(C) $1 - S_n < (1 - x_1)(1 - x_2) \cdots (1 - x_n)$

(D) $(1 - x_1)(1 - x_2) \cdots (1 - x_n) < \dfrac{1}{1 + S_n}$.

To prove (A):

$(1 + x_1)(1 + x_2) \cdots (1 + x_n) = 1 + (x_1 + x_2 + \cdots + x_n)$ + other positive terms $> 1 + S_n$

To prove (B) by induction.

$n = 2$. $(1 + x_1)(1 + x_2)(1 - x_1 - x_2) = (1 + x_1 + x_2 + x_1x_2)(1 - x_1 - x_2)$

$= 1 - (x_1 + x_2)^2 + x_1x_2(1 - x_1 - x_2)$

$= 1 - (x_1^2 + x_1x_2 + x_2^2) - x_1x_2(x_1 + x_2) < 1$

$\Rightarrow (1 + x_1)(1 + x_2) < \dfrac{1}{1 - S_2}$ $(\because 1 - S_2 > 0)$

Suppose $(1 + x_1)(1 + x_2) \cdots (1 + x_k) < \dfrac{1}{1 - S_k}$

$(1 + x_1)(1 + x_2) \ldots (1 + x_k)(1 + x_{k+1}) < \dfrac{1 + x_{k+1}}{1 - S_k}$ …… (1)

$\Rightarrow -x_{k+1} S_{k+1} < 0$

$\Rightarrow 1 - S_{k+1} + x_{k+1} - x_{k+1} S_{k+1} < 1 - S_k$

$\Rightarrow (1 + x_{k+1})(1 - S_{k+1}) < 1 - S_k$

$\Rightarrow \dfrac{1 + x_{k+1}}{1 - S_k} < \dfrac{1}{1 - S_{k+1}}$ …… (2) $(\because 1 - S_k > 0$ and $1 - S_{k+1} > 0)$

Combine (1) and (2): $(1 + x_1)(1 + x_2) \cdots (1 + x_k)(1 + x_{k+1}) < \dfrac{1}{1 - S_{k+1}}$

By M.I., $(1 + x_1)(1 + x_2) \cdots (1 + x_n) < \dfrac{1}{1 - S_n}$

To prove (C) by induction.

$n = 2$. $(1 - x_1)(1 - x_2) = 1 - x_1 - x_2 + x_1x_2 = 1 - S_2 + x_1x_2 > 1 - S_2$

Suppose $1 - S_k < (1 - x_1)(1 - x_2) \cdots (1 - x_k)$

$(1 - S_k)(1 - x_{k+1}) < (1 - x_1)(1 - x_2) \cdots (1 - x_k)(1 - x_{k+1})$

$1 - S_k - x_{k+1} + x_{k+1}S_k < (1 - x_1)(1 - x_2) \cdots (1 - x_k)(1 - x_{k+1})$

$1 - S_{k+1} = 1 - S_k - x_{k+1} < 1 - S_k - x_{k+1} + x_{k+1}S_k < (1 - x_1)(1 - x_2) \cdots (1 - x_k)(1 - x_{k+1})$

By M.I., $1 - S_n < (1 - x_1)(1 - x_2) \cdots (1 - x_n)$.

Weierstrass' inequalities

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To prove (D):

Now $0 < x_i < 1 \Rightarrow 1 - x_i^2 < 1$, and so $(1 - x_i)(1 + x_i) < 1$.

$0 < 1 - x_i < \frac{1}{1+x_i}$ for $i = 1, 2, \cdots, n$.

$(1 - x_1)(1 - x_2) \cdots (1 - x_n) < \frac{1}{(1+x_1)(1+x_2)\cdots(1+x_n)} < \frac{1}{1+S_n}$