98-99 Individual	**1**	1	**2**	8	**3**	56	**4**	405	**5**	100000
	6	2401	**7**	9	**8**	36	**9**	11	**10**	9

98-99 Group	**1**	3	**2**	–24	**3**	$\frac{1}{2}$	**4**	$\frac{1}{2}$	**5**	6
	6	12	**7**	4	**8**	7	**9**	12	**10**	135

Individual Events

I1 The circumference of a circle is 14π cm. Let X cm be the length of an arc of the circle, which subtends an angle of $\frac{1}{7}$ radian at the centre. Find the value of X.

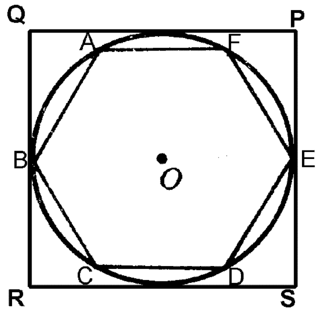
Let r be the radius of the circle.

$2\pi r = 14\pi$

$\Rightarrow r = 7$

$X = r\theta = 7\times\frac{1}{7} = 1$

I2 In Figure 1, $ABCDEF$ is a regular hexagon with area equal to $3\sqrt{3}$ cm^2. Let X cm^2 be the area of the square $PQRS$, find the value of X.



Area of the hexagon = 6×areas of ΔAOB

$3\sqrt{3} = 6\cdot\frac{1}{2}\cdot OB^2\sin 60^\circ = \frac{3\sqrt{3}}{2}\cdot OB^2$

$OB^2 = 2$

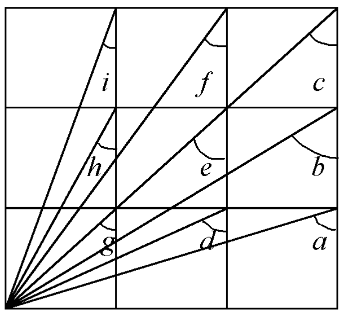
Area of the square = $(2OB)^2 = 4\times 2 = 8$

I3 8 points are given and no three of them are collinear. Find the number of triangles formed by using any 3 of the given points as vertices.

The number of triangles formed

$= {}_8C_3 = \frac{8\times7\times6}{1\times2\times3} = 56$

I4 In Figure 2, there is a 3×3 square.



Let $\angle a + \angle b + \ldots + \angle i = X^\circ$, find the value of X.

Reference: <u>廣州、武漢、福州、重慶、洛陽 初中數學聯賽</u>

$\angle c = \angle e = \angle g = 45^\circ$

$\angle a + \angle i = 90^\circ$, $\angle b + \angle f = 90^\circ$, $\angle d + \angle h = 90^\circ$

$\angle a + \angle b + \ldots + \angle i = 45^\circ\times3 + 90^\circ\times3 = 405^\circ$

$X = 405$

I5 How many integers n are there between 0 and 10^6, such that the unit digit of n^3 is 1?

$1^3 = 1$, the unit digit of n must be 1

There are $10^6 \div 10 = 100000$ possible integers.

I6 Given that a, b, c are positive integers and $a < b < c = 100$, find the number of triangles formed with sides equal a cm, b cm and c cm.

By triangle inequality: $a + b > c = 100$

Possible pairs of (a, b): (2, 99), (3, 98), (3, 99), (4, 97), (4, 98), (4, 99), … ,

(50, 51), (50, 52), … , (50, 99), ...

(98, 99)

Total number of triangles = 1 + 2 + … + 48 + 49 + 48 + … + 2 + 1

$$= \frac{1+49}{2} \times 49 \times 2 - 49 = 2401$$

I7 A group of youngsters went for a picnic. They agreed to share all expenses. The total amount used was \$288. One youngster had no money to pay his share, and each of the others had to pay \$4 more to cover the expenses. How many youngsters were there in the group?

Let the number of youngsters be n.

$$\frac{288}{n-1} - \frac{288}{n} = 4$$

$72 = n^2 - n$

$n = 9$

I8 A two-digit number is equal to 4 times the sum of the digits, and the number formed by reversing the digits exceeds 5 times the sum of the digits by 18. What is the number?

Let the unit digits of the original number be x and the tens digit by y.

$10y + x = 4(x + y)$ ....(1)

$10x + y - 5(x + y) = 18$ ......(2)

From (1), $6y = 3x \Rightarrow x = 2y$ ....(3)

Sub. (3) into (2): $20y + y - 5(2y + y) = 18$

$\Rightarrow y = 3, x = 6$

The number is 36.

I9 Given that the denominator of the 1001^{th} term of the following sequence is 46, find the numerator of this term. $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots$

Suppose the numerator of the 1001^{th} term is n.

$1 + 2 + 3 + \dots + 44 + n = 1001, n \le 45$

$\frac{1}{2}(45)(44) + n = 1001$

$n = 1001 - 990 = 11$

I10 In the following addition, if the letter 'S' represents 4, what digit does the letter 'A' represent?

$$\begin{array}{r} \mathbf{SEE} \\ \mathbf{SEE} \\ \mathbf{SEE} \\ +\ \mathbf{YES} \\ \hline \mathbf{EASY} \end{array} \qquad \begin{array}{r} \mathbf{4EE} \\ \mathbf{4EE} \\ \mathbf{4EE} \\ +\ \mathbf{YE4} \\ \hline \mathbf{EA4Y} \end{array}$$

$3E + 4 = 10a + Y$ ....(1), where a is the carry digit in the tens digit.

$4E + a = 10b + 4$ ......(2), where b is the carry digit in the hundreds digit.

$4\times3 + Y + b = 10E + A$ ....(3)

From (3), $E = 1$ or 2

When $E = 1$, (1) $\Rightarrow Y = 7$, $a = 0$, (2) $\Rightarrow b = 0$, (3) $\Rightarrow A = 9$

When $E = 2$, (2) $\Rightarrow a = 1$, $Y = 0$ reject because $YE4$ is a 3-digit number.

$\therefore A = 9$

Group Events

G1 If a is a prime number and $a^2 - 2a - 15 < 0$, find the greatest value of a.

$(a + 3)(a - 5) < 0$

$\Rightarrow a < 5$

The greatest prime number is 3.

G2 If $a : b : c = 3 : 4 : 5$ and $a + b + c = 48$, find the value of $a - b - c$.

$a = 3k$, $b = 4k$, $c = 5k$; sub. into $a + b + c = 48$

$\Rightarrow 3k + 4k + 5k = 48$

$\Rightarrow k = 4$

$a = 12$, $b = 16$, $c = 20$

$a - b - c = 12 - 16 - 20 = -24$

G3 Find the value of $\log\left(\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}\right)$.

Reference: 1993 FI1.4, 2001 FG2.1, 2011 HI7, 2015 FI4.2, 2015 FG3.1

$$\log\left(\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}\right)=\log\left(\sqrt{\frac{6+2\sqrt{5}}{2}}+\sqrt{\frac{6-2\sqrt{5}}{2}}\right)$$

$$=\log\left(\frac{\sqrt{\left(1+\sqrt{5}\right)^2}+\sqrt{\left(\sqrt{5}-1\right)^2}}{\sqrt{2}}\right)$$

$$=\log\left(\frac{1+\sqrt{5}+\sqrt{5}-1}{\sqrt{2}}\right)=\log\left(\frac{2\sqrt{5}}{\sqrt{2}}\right)=\log\left(\sqrt{2}\sqrt{5}\right)=\log\sqrt{10}=\frac{1}{2}$$

G4 Find the area enclosed by the straight line $x + 4y - 2 = 0$ and the two coordinate axes.

x-intercept = 2, y-intercept $=\frac{1}{2}$; the area $=\frac{1}{2}\times\frac{1}{2}\times 2=\frac{1}{2}$

G5 Natural numbers are written in order starting from 1 until 198th digit as shown $\underbrace{123456789101112\cdots\cdots}_{198\text{ digits}}$. If the number obtained is divided by 9, find the remainder.

123456789 has 9 digits

10111213…9899 has $90\times 2 = 180$ digits

$\therefore$ 1234567891011…9899100101102 has 198 digits.

$1 + 2 + 3 + \ldots + 9 = 45$, $11+12+\ldots+19$ is also divisible by 9,...,

$91+92+\ldots+99$ is divisible by 9.

$10 + 20 + \ldots + 90$ is divisible by 9

$\therefore$ the remainder is the same as 100101102 divided by 9.

$1 + 1 + 1 + 1 + 2 = 6$, the remainder is 6.

G6 The average of 2, a, 5, b, 8 is 6. If n is the average of a, $2a+1$, 11, b, $2b+3$, find the value of n.

$2 + a + 5 + b + 8 = 30$ ....(1), $a + 2a + 1 + 11 + b + 2b + 3 = 5n$ ....(2)

From (1): $a + b = 15$

(2) $5n = 3a + 3b + 15 = 3(a + b) + 15 = 3\times 15 + 15 = 60$

$\Rightarrow n = 12$

G7 If $p = 2x^2 - 4xy + 5y^2 - 12y + 16$, where x and y are real numbers, find the least value of p.

Reference: 2001 HI3, 2012 HG5, 2018 HI1

$p = 2x^2 - 4xy + 2y^2 + 3y^2 - 12y + 16 = 2(x - y)^2 + 3(y^2 - 4y + 4) + 4 = 2(x - y)^2 + 3(y - 2)^2 + 4$

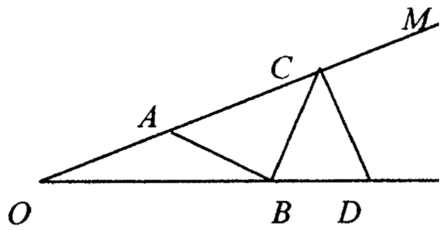
$p \geq 4$, the least value of p is 4.

G8 Find the units digit of 333^{335}.

$3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, the units digit of 3^{4m} is 1, where m is any positive integer.

$333^{335} = 333^{4\times 83+3} = (333^4)^{83}\times 333^3 = (\ldots 1)^{83}\times(\ldots 3^3) = \ldots 7$, the units digit is 7.

G9 In Figure 1, $\angle MON = 20°$, A is a point on OM, $OA = 4\sqrt{3}$, D is a point on ON, $OD = 8\sqrt{3}$, C is any point on AM, B is any point OD. If $\ell = AB + BC + CD$, find the least value of ℓ. (**Reference: 2016 HG5)**



Reflect the figure along the line OM, then reflect the figure between $\angle MON_1$ along the line ON_1.

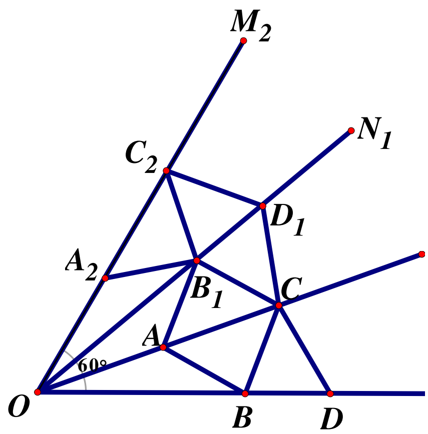
$\angle NOM_2 = 3\times 20° = 60°$

$\ell = AB + BC + CD = AB_1 + B_1C + CD$

$\ell = A_2B_1 + B_1C + CD$

ℓ is the shortest when A_2, B_1, C, D are collinear.

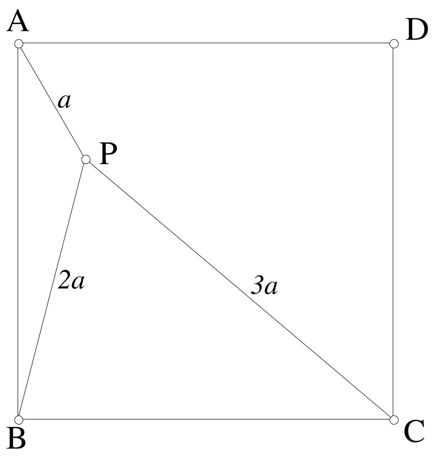
By cosine formula on ΔOA_2D,



Shortest $\ell = A_2D = \sqrt{\left(4\sqrt{3}\right)^2 + \left(8\sqrt{3}\right)^2 - 2\left(4\sqrt{3}\right)\left(8\sqrt{3}\right)\cos 60^\circ}$

$= \sqrt{48 + 192 - 96} = 12$

G10 In figure 2, P is a point inside the square $ABCD$, $PA = a$, $PB = 2a$, $PC = 3a$ ($a > 0$). If $\angle APB = x°$, find the value of x.



Reference: 2014 HG4

Rotate ΔAPB by 90° in anti-clockwise direction about B.

Let P rotate to Q, A rotate to E.

$\Delta APB \cong \Delta EQB$ (by construction)

$EQ = a$, $BQ = 2a = PB$. Join AQ.

$\angle PBQ = 90°$ (Rotation)

$\angle ABQ = 90° - \angle ABP = \angle PBC$

$AB = BC$ (sides of a square)

$\Delta ABQ \cong \Delta CBP$ (SAS)

$AQ = CP = 3a$ (corr. sides $\cong \Delta$'s)

$\because \angle PBQ = 90°$ (Rotation)

$\therefore PQ^2 = PB^2 + QB^2$ (Pyth. Theorem)

$= (2a)^2 + (2a)^2 = 8a^2$

$AP^2 + PQ^2 = a^2 + 8a^2 = 9a^2$

$AQ^2 = (3a)^2$

$\therefore AP^2 + PQ^2 = AQ^2$

$\angle APQ = 90°$

$\because \angle PBQ = 90°$ and $PB = QB$

$\therefore \angle BPQ = 45°$

$\angle APB = 45° + 90° = 135°$