

88-89 Individual	1	7	2	2	3	24	4	$\frac{1}{2}$	5	12 km/h
	6	45	7	45	8	-4	9	2	10	12
	11	$\frac{5}{12}$	12	147	13	3	14	2	15	24
	16	12	17	6 : 25	18	24	19	230	20	12

88-89 Group	1	$\frac{9}{20}$	2	7	3	6585	4	(12, 20)	5	$\frac{3}{2}$
	6	43	7	480	8	$\frac{12}{5}$	9	5	10	110768

Individual Events

- I1** Given that $x + \frac{1}{x} = 3$, find $x^2 + \frac{1}{x^2}$.

Reference: 1983 FG7.3, 1984 FG10.2, 1985 FI1.2, 1987 FG8.2, 1990 HI12, 1997 HG7

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$$

$$= 3^2 - 2 = 7$$

- I2** If $x \# y = xy - 2x$, find the value of $2 \# 3$.

$$2 \# 3 = 2 \times 3 - 2 \times 2 = 2$$

- I3** Find the number of sides of a regular polygon if an interior angle exceeds an exterior angle by 150° . (**Reference** 1997 HG6)

Let x be the size of each interior angle, y be the size of each exterior angle, n be the number of sides.

$$x = \frac{180^\circ(n-2)}{n}, y = \frac{360^\circ}{n}$$

$$x = y + 150^\circ$$

$$\frac{180^\circ(n-2)}{n} = \frac{360^\circ}{n} + 150^\circ$$

$$180(n-2) = 360 + 150n$$

$$18n - 36 = 36 + 15n$$

$$\Rightarrow n = 24$$

- I4** Find the value of b such that $10^{\log_{10} 9} = 8b + 5$.

$$9 = 8b + 5$$

$$\Rightarrow b = \frac{1}{2}$$

- I5** A man cycles from P to Q with a uniform speed of 15 km/h and then back from Q to P with a uniform speed of 10 km/h. Find the average speed for the whole journey.

Let the distance between P and Q be x km.

$$\text{Total distance travelled} = 2x \text{ km. Total time} = \frac{x}{15} + \frac{x}{10} \text{ hour.}$$

$$\text{Average speed} = \frac{2x}{\frac{x}{15} + \frac{x}{10}} \text{ km/h}$$

$$= \frac{2}{\frac{1}{15} + \frac{1}{10}} = 12 \text{ km/h}$$

- 16** $[x]$ denotes the greatest integer less than or equal to x . For example, $[3] = 3$, $[5.7] = 5$.

If $\left[\sqrt[5]{1}\right] + \left[\sqrt[5]{2}\right] + \cdots + \left[\sqrt[5]{n}\right] = n + 14$, find n .

Reference 1991 HI13

$$\left[\sqrt[5]{1}\right] = 1, \left[\sqrt[5]{2}\right] = 1, \dots, \left[\sqrt[5]{31}\right] = 1; \left[\sqrt[5]{32}\right] = 2, \dots, \left[\sqrt[5]{242}\right] = 2; \left[\sqrt[5]{243}\right] = 3$$

$$\text{If } n \leq 31, \left[\sqrt[5]{1}\right] + \left[\sqrt[5]{2}\right] + \cdots + \left[\sqrt[5]{n}\right] = n$$

$$\text{If } 32 \leq n \leq 242, \left[\sqrt[5]{1}\right] + \left[\sqrt[5]{2}\right] + \cdots + \left[\sqrt[5]{n}\right] = 31 + 2(n - 31) = 2n - 31$$

$$2n - 31 = n + 14$$

$$\Rightarrow n = 45$$

- 17** A boy tries to find the area of a parallelogram by multiplying together the lengths of two adjacent sides. His answer is $\sqrt{2}$ times the correct area. If the acute angle of the parallelogram is x° , find x . (**Reference: 1991 FSG.3-4**)

Let the lengths of two adjacent sides be a and b , where the angle between a and b is x° .

$$ab = \sqrt{2}ab \sin x^\circ$$

$$\sin x^\circ = \frac{1}{\sqrt{2}}$$

$$x = 45$$

- 18** If the points $A(-8, 6)$, $B(-2, 1)$ and $C(4, c)$ are collinear, find c .

Reference: 1984 FSG.4, 1984 FG7.3, 1986 FG6.2, 1987 FG7.4

$$m_{CB} = m_{BA}$$

$$\frac{c-1}{4+2} = \frac{1-6}{-2+8}$$

$$c = -4$$

- 19** The graphs of $x^2 + y = 8$ and $x + y = 8$ meet at two points. If the distance between these two points is \sqrt{d} , find d .

$$\text{From (1), } y = 8 - x^2 \dots\dots (3)$$

$$\text{From (2), } y = 8 - x \dots\dots (4)$$

$$(3) = (4): 8 - x = 8 - x^2$$

$$x = 0 \text{ or } 1$$

$$\text{When } x = 0, y = 8; \text{ when } x = 1, y = 7$$

$$\text{Distance between the points } (0, 8) \text{ and } (1, 7) = \sqrt{1^2 + (7-8)^2} = \sqrt{2}$$

$$d = 2$$

- 110** The sines of the three angles of a triangle are in the ratio $3 : 4 : 5$. If A is the smallest interior angle of the triangle and $\tan A = \frac{x}{16}$, find x .

Reference: 1990 HI6

$$\text{By Sine rule, } a : b : c = \sin A : \sin B : \sin C = 3 : 4 : 5$$

$$\text{Let } a = 3k, b = 4k, c = 5k.$$

$$a^2 + b^2 = (3k)^2 + (4k)^2 = (5k)^2 = c^2$$

$$\therefore \angle C = 90^\circ \text{ (converse, Pythagoras' theorem)}$$

$$\tan A = \frac{a}{b} = \frac{3}{4} = \frac{12}{16}$$

$$\Rightarrow x = 12$$

- I11** Two dice are thrown. Find the probability that the sum of the two numbers shown is greater than 7.

Reference: 2002 HG7

$$P(\text{sum} > 7) = P(\text{sum} = 8 \text{ or } 9 \text{ or } 10 \text{ or } 11 \text{ or } 12)$$

$$\begin{aligned} &= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} \\ &= \frac{15}{36} = \frac{5}{12} \end{aligned}$$

Method 2 $P(7) = \frac{6}{36} = \frac{1}{6}$

$$P(2, 3, 4, 5, 6) = P(8, 9, 10, 11, 12)$$

$$P(2, 3, 4, 5, 6) + P(7) + P(8, 9, 10, 11, 12) = 1$$

$$2P(8, 9, 10, 11, 12) = 1 - P(7) = \frac{5}{6}$$

$$P(8, 9, 10, 11, 12) = \frac{5}{12}$$

- I12** F is a function defined by $F(x) = \begin{cases} 2x+1, & \text{if } x \leq 3 \\ 3x^2, & \text{if } x > 3 \end{cases}$. Find $F(F(3))$.

$$F(3) = 2 \times 3 + 1 = 7$$

$$F(F(3)) = F(7) = 3 \times 7^2 = 147$$

- I13** If $\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$ and $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 14 \\ y \\ 2 \end{pmatrix} = 26$, find y . (**Reference: 1986 FI3.4**)

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 14 \\ y \\ 2 \end{pmatrix} = 14 + 2y + 6 = 26 \Rightarrow y = 3$$

- I14** If $\frac{1}{B} = \frac{\sin 37^\circ \sin 45^\circ \cos 60^\circ \sin 60^\circ}{\cos 30^\circ \cos 45^\circ \cos 53^\circ}$, find B .

Reference: 1990 HI14

$$\begin{aligned} \frac{1}{B} &= \frac{\sin 37^\circ \sin 45^\circ \cos 60^\circ \sin 60^\circ}{\cos 30^\circ \cos 45^\circ \cos 53^\circ} \\ &= \frac{\sin 37^\circ \sin 45^\circ \cos 60^\circ \sin 60^\circ}{\sin 60^\circ \sin 45^\circ \sin 37^\circ} = \cos 60^\circ = \frac{1}{2} \end{aligned}$$

$$B = 2$$

- I15** If $x + y = -4$, $y + z = 5$ and $z + x = 7$, find the value of xyz .

Reference: 1986 FG10.1, 1990 HI7

$$(1) + (2) - (3): 2y = -6 \Rightarrow y = -3$$

$$(1) + (3) - (2): 2x = -2 \Rightarrow x = -1$$

$$(2) + (3) - (1): 2z = 16 \Rightarrow z = 8$$

$$xyz = 24$$

- I16** α, β are the roots of the equation $x^2 - 10x + c = 0$. If $\alpha\beta = -11$ and $\alpha > \beta$, find the value of $\alpha - \beta$.

$$\alpha + \beta = 10$$

$$\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$= \sqrt{10^2 - 4(-11)} = 12$$

- I17** In figure 1, $FE \parallel BC$ and $ED \parallel AB$. If $AF : FB = 3 : 2$, find the ratio area of $\triangle DEF$: area of $\triangle ABC$. **Reference: 1990 HG8**
 $BDEF$ is a parallelogram formed by 2 pairs of parallel lines
 $\triangle DEF \cong \triangle FBD$ (A.S.A.)

Let $S_{\triangle DEF} = x = S_{\triangle FBD}$ (where S stands for the area)

$\triangle AEF \sim \triangle ACB$ ($\because FE \parallel BC$, equiangular)

$$\frac{S_{\triangle AEF}}{S_{\triangle ACB}} = \left(\frac{3}{2+3}\right)^2 = \frac{9}{25} \dots\dots (1)$$

$\therefore AE : EC = AF : FB = 3 : 2$ (theorem of equal ratio)

$\therefore DE \parallel AB$

$\therefore AE : EC = BD : DC = 3 : 2$ (theorem of equal ratio)

$\triangle CDE \sim \triangle CBA$ ($\because DE \parallel BA$, equiangular)

$$\frac{S_{\triangle CDE}}{S_{\triangle CBA}} = \left(\frac{2}{2+3}\right)^2 = \frac{4}{25} \dots\dots (2)$$

Compare (1) and (2) $S_{\triangle AEF} = 9k$, $S_{\triangle CDE} = 4k$, $S_{\triangle ABC} = 25k$

$$9k + 4k + x + x = 25k$$

$$x = 6k$$

\Rightarrow area of $\triangle DEF$: area of $\triangle ABC = 6 : 25$

- I18** In figure 2, a regular hexagon $ABCDEF$ is inscribed in a circle centred at O . If the distance of O from AB is $2\sqrt{3}$ and p is the perimeter of the hexagon, find p .

Let H be the foot of perpendiculars drawn from O onto AB .

$$\angle AOB = 360^\circ \div 6 = 60^\circ \text{ (}\angle\text{s at a point)}$$

$$\angle AOH = 30^\circ$$

$$AH = OH \tan 30^\circ = 2\sqrt{3} \times \frac{1}{\sqrt{3}} = 2$$

$$\Rightarrow AB = 4$$

$$\text{Perimeter} = 6 \times 4 = 24$$

- I19** In figure 3, $ABCD$ and $ACDE$ are cyclic quadrilaterals. Find the value of $x + y$.

Reference: 1992 FI2.3

$$\angle ADC = 180^\circ - x^\circ \text{ (opp. } \angle\text{s cyclic quad.)}$$

$$\angle ACD = 180^\circ - y^\circ \text{ (opp. } \angle\text{s cyclic quad.)}$$

$$180^\circ - y^\circ + 180^\circ - x^\circ + 50^\circ = 180^\circ \text{ (}\angle\text{s sum of } \triangle\text{)}$$

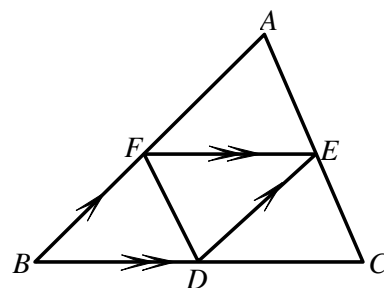
$$x + y = 230$$

- I20** Find the value of a in figure 4.

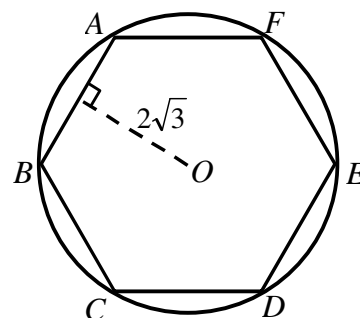
$\triangle AOB \sim \triangle DOC$ (equiangular)

$$\frac{a}{4} = \frac{6}{2}$$

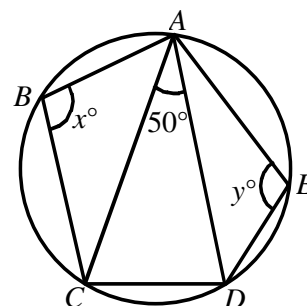
$$a = 12$$



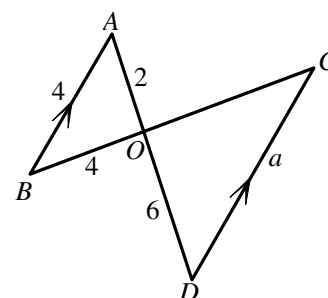
(Figure 1)



(Figure 2)



(Figure 3)



(Figure 4)

Group Events

- G1** Given a and b are distinct real numbers satisfying $a^2 = 5a + 10$ and $b^2 = 5b + 10$. Find the value of $\frac{1}{a^2} + \frac{1}{b^2}$. (**Reference: 1991 HI14**)

a and b are the roots of $x^2 = 5x + 10$; i.e. $x^2 - 5x - 10 = 0$

$$a + b = 5; ab = -10$$

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{a^2 + b^2}{a^2 b^2} = \frac{(a+b)^2 - 2ab}{(ab)^2} = \frac{5^2 - 2(-10)}{(-10)^2} = \frac{9}{20}$$

- G2** An interior angle of an n -sided convex polygon is x° while the sum of other interior angles is 800° . Find the value of n . (**1990 FG10.3-4, 1992 HG3, 2002 FI3.4, 2013 HI6**)

$$800 = 180 \times 4 + 80$$

$$800 + x = 180(n - 2) \quad \angle\text{s sum of polygon}$$

$$\therefore 0 < x < 180$$

$$\therefore 800 + x = 180 \times 5 = 180(n - 2)$$

$$n = 7$$

- G3** It is known that $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n .

Find the value of $21^2 + 22^2 + \dots + 30^2$.

Reference: 1993 HI6

$$\begin{aligned} 21^2 + 22^2 + \dots + 30^2 &= 1^2 + 2^2 + \dots + 30^2 - (1^2 + 2^2 + \dots + 20^2) \\ &= \frac{1}{6} \cdot 30 \cdot 31 \cdot 61 - \frac{1}{6} \cdot 20 \cdot 21 \cdot 41 = 9455 - 2870 = 6585 \end{aligned}$$

- G4** One of the positive integral solutions of the equation $19x + 88y = 1988$ is given by $(100, 1)$. Find another positive integral solution. (**Reference: 1991 HG8**)

$$\text{The line has a slope of } -\frac{19}{88} = \frac{y_2 - y_1}{x_2 - x_1}$$

Given that $(100, 1)$ is a solution.

$$-\frac{19}{88} = \frac{y_2 - 1}{x_2 - 100}$$

Let $y_2 - 1 = -19t$; $x_2 - 100 = 88t$, where t is an integer.

$$y_2 = 1 - 19t, x_2 = 100 + 88t$$

For positive integral solution of (x_2, y_2) , $1 - 19t > 0$ and $100 + 88t > 0$

$$-\frac{25}{22} < t < \frac{1}{19}$$

$\therefore t$ is an integer $\therefore t = 0$ or -1

When $t = -1$, $x_2 = 12$, $y_2 = 20 \Rightarrow$ another positive integral solution is $(12, 20)$.

- G5** The line joining $A(2, 3)$ and $B(17, 23)$ meets the line $2x - y = 7$ at P . Find the value of $\frac{AP}{PB}$.

Reference: 1990 HG3

$$\text{Equation of AB: } \frac{y-3}{x-2} = \frac{23-3}{17-2} \Rightarrow 3y = 4x + 1 \dots\dots (2)$$

$$\text{From (1): } y = 2x - 7 \dots\dots (3)$$

$$\text{Sub. (3) into (2): } 3(2x - 7) = 4x + 1 \Rightarrow x = 11$$

$$\text{Sub. } x = 11 \text{ into (3): } y = 2(11) - 7 = 15$$

The point of intersection is $P(11, 15)$.

$$\frac{AP}{PB} = \frac{11-2}{17-11} = \frac{3}{2}$$

- G6** Find the remainder when 7^{2047} is divided by 100.

Reference: 2002 HG4

The question is equivalent to find the last 2 digits of 7^{2047} .

$$7^1 = 7, 7^2 = 49, 7^3 = 343, 7^4 = 2401$$

The last 2 digits repeats for every multiples of 4.

$$7^{2047} = 7^{4 \times 511 + 3}$$

The last 2 digits is 43.

- G7** If $\log_2[\log_3(\log_7 x)] = \log_3[\log_7(\log_2 y)] = \log_7[\log_2(\log_3 z)] = 0$, find the value of $x + y + z$.

$$\log_2[\log_3(\log_7 x)] = 0$$

$$\Rightarrow \log_3(\log_7 x) = 1$$

$$\Rightarrow \log_7 x = 3$$

$$\Rightarrow x = 7^3 = 343$$

$$\log_3[\log_7(\log_2 y)] = 0$$

$$\Rightarrow \log_7(\log_2 y) = 1$$

$$\Rightarrow \log_2 y = 7$$

$$\Rightarrow y = 2^7 = 128$$

$$\log_7[\log_2(\log_3 z)] = 0$$

$$\Rightarrow \log_2(\log_3 z) = 1$$

$$\Rightarrow \log_3 z = 2$$

$$\Rightarrow z = 3^2 = 9$$

$$x + y + z = 343 + 128 + 9 = 480$$

- G8** In figure 1, $AB \parallel MN \parallel CD$. If $AB = 4$, $CD = 6$ and $MN = x$, find the value of x .

Reference: 1985 FI2.4, 1990 FG6.4

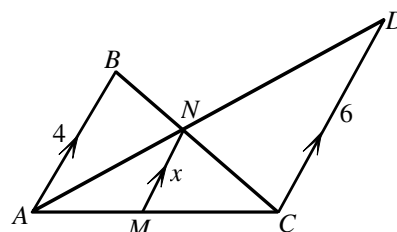
$\triangle AMN \sim \triangle ACD$ (equiangular)

$$\frac{x}{6} = \frac{AM}{AC} \quad \dots (1) \text{ (ratio of sides, } \sim \Delta \text{'s)}$$

$\triangle CMN \sim \triangle CAB$ (equiangular)

$$\frac{x}{4} = \frac{MC}{AC} \quad \dots (2) \text{ (ratio of sides, } \sim \Delta \text{'s)}$$

$$(1) + (2): \frac{x}{6} + \frac{x}{4} = \frac{AM + MC}{AC} = 1 \Rightarrow x = \frac{12}{5}$$



(Figure 1)

- G9** In figure 2, $\angle B = 90^\circ$, $BC = 3$ and the radius of the inscribed circle of $\triangle ABC$ is 1. Find the length of AC .

Reference: 1985 FG9.2

Let the centre be O . Suppose the circle touches BC at P , AC at Q and AB at R respectively. Let $AB = c$ and $AC = b$.

$OP \perp BC$, $OQ \perp AC$, $OR \perp AB$ (tangent \perp radius)

$OPBR$ is a rectangle $\Rightarrow OPBR$ is a square.

$BP = BR = 1$ (opp. sides of rectangle)

$$CP = 3 - 1 = 2$$

$CQ = CP = 2$ (tangent from ext. point)

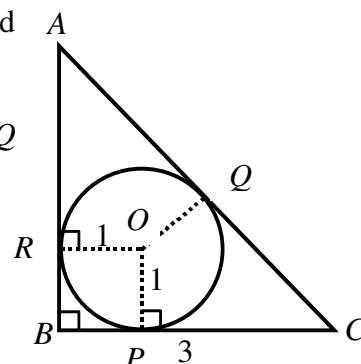
Let $AR = AQ = t$ (tangent from ext. point)

$$3^2 + (1 + t)^2 = (2 + t)^2 \text{ (Pythagoras' theorem)}$$

$$9 + 1 + 2t + t^2 = 4 + 4t + t^2$$

$$6 = 2t$$

$$t = 3 \Rightarrow AC = 2 + 3 = 5$$



G10 In the attached division (see figure 3), the dividend in (a) is divisible by the divisor in line (b). Find the dividend in line (a). (Each asterisk * is an integer from 0 to 9.)

Relabel the '*' as shown.

$$\text{Let } a = (a_1 8 a_3)_x,$$

$$b = (b_1 b_2 b_3)_x$$

$$c = (c_1 c_2 c_3 c_4 c_5 c_6)_x$$

$$d = (d_1 d_2 d_3 d_4)_x$$

$$e = (e_1 e_2 e_3)_x$$

$$f = (f_1 f_2 f_3)_x$$

$$g = (g_1 g_2 g_3 g_4)_x$$

$\therefore 8b = e$ and $a_1 \times b = d$, $a_3 \times b = g$ 4-digit numbers

$$\therefore a_1 = 9 \text{ and } a_3 = 9 \text{ and } d = g$$

$$d = 9b > 1000 \text{ and } f = 8b < 999$$

$$112 \leq b \leq 124 \dots\dots (1)$$

$$b_1 = 1, d_1 = 1$$

$$b_2 = 1 \text{ or } 2, f_1 = 8 \text{ or } 9, d_2 = 0 \text{ or } 1$$

$$c_1 = 1$$

$$e_1 - f_1 = 1$$

$$\therefore f_1 = 8 \text{ and } e_1 = 9$$

$$8b = (8f_2 f_3)_x < 900$$

$$b < 112.5 \dots\dots (2)$$

Combine (1) and (2)

$$b = 112$$

$$c = 989 \times 112 = 110768$$

$$\begin{array}{r} \text{ * 8 * } \\ \hline (b) \dots\dots * * *) \text{ * * * * * } \dots\dots (a) \end{array}$$

$$\begin{array}{r} \text{ * * * } \\ \hline \text{ * * * } \\ \hline \text{ * * * } \\ \hline \text{ * * * } \\ \hline \text{ * * * } \\ \hline \text{ a}_1 \text{ 8 } \text{ a}_3 \\ \hline (b) \dots\dots b_1 \text{ } b_2 \text{ } b_3) \text{ c}_1 \text{ } c_2 \text{ } c_3 \text{ } c_4 \text{ } c_5 \text{ } c_6 \dots\dots (a) \\ \hline \text{ d}_1 \text{ } d_2 \text{ } d_3 \text{ } d_4 \\ \hline \text{ e}_1 \text{ } e_2 \text{ } e_3 \\ \hline \text{ f}_1 \text{ } f_2 \text{ } f_3 \\ \hline \text{ g}_1 \text{ } g_2 \text{ } g_3 \text{ } g_4 \\ \hline \text{ g}_1 \text{ } g_2 \text{ } g_3 \text{ } g_4 \\ \hline \text{ 9 } \text{ 8 } \text{ 9 } \\ \hline (b) \dots\dots 1 \text{ } b_2 \text{ } b_3) \text{ 1 } c_2 \text{ } c_3 \text{ } c_4 \text{ } c_5 \text{ } c_6 \dots\dots (a) \\ \hline \text{ 1 } d_2 \text{ } d_3 \text{ } d_4 \\ \hline \text{ e}_1 \text{ } e_2 \text{ } e_3 \\ \hline \text{ f}_1 \text{ } f_2 \text{ } f_3 \\ \hline \text{ 1 } d_2 \text{ } d_3 \text{ } d_4 \\ \hline \text{ 1 } d_2 \text{ } d_3 \text{ } d_4 \\ \hline \end{array}$$