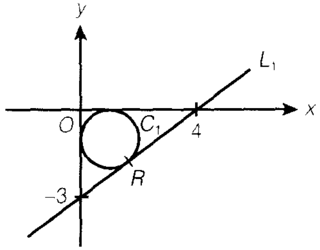
Co-ordinate circle Example

Created by Mr. Francis Hung on 20110904 Last updated: 2021-08-30

Reference Chung Tai Educational Press
New Trend Senior Secondary Mathematics
Essential Drills S.5 Chapter 9 Q23



In the figure, circle C_1 touches L_1 at R.

(a) Find the coordinates of R.

(b) It is given that C_2 is another circle touching the x-axis, the y-axis and L_1. Find the equation of C_2.

(a) Let k be the radius of C_1, if the centre of the circle lies in the fourth quadrant, then the coordinates of the centre of C_1 are $(k, -k)$.

The equation of C_1 is

$(x-k)^2+[y-(-k)]^2=k^2$

$x^2+y^2-2kx+2ky+k^2=0$ …… (1)

The equation of L_1 is: $\frac{x}{4}+\frac{y}{-3}=1$

$\Rightarrow\ y=\frac{3}{4}x-3$ …… (2)

Substitute (2) into (1),

$x^2+(\frac{3}{4}x-3)^2-2kx+2k(\frac{3}{4}x-3)+k^2=0$

$x^2+\frac{9}{16}x^2-\frac{9}{2}x+9-2kx+\frac{3}{2}kx-6k+k^2=0$

$\therefore\ \frac{25}{16}x^2-\frac{k+9}{2}x+k^2-6k+9=0$ ……(3)

$\because$ C_1 touches L_1 at one point.

$\left(-\frac{k+9}{2}\right)^2-4\times\frac{25}{16}\left(k^2-6k+9\right)=0$

$\frac{(k+9)^2}{4}-\frac{25}{4}(k^2-6k+9)=0$

$k^2+18k+81-25k^2+150k-225=0$

$24k^2-168k+144=0$

$k^2-7k+6=0$

$(k-1)(k-6)=0$

$\therefore$ $k=1$ or $k=6$ (rejected)

Substitute $k=1$ into (3),

$\frac{25}{16}x^2-5x+4=0$

$25x^2-80x+64=0$

$(5x-8)^2=0$

$x=\frac{8}{5}$

Substitute $x=\frac{8}{5}$ into (2),

$y=\frac{3}{4}(\frac{8}{5})-3=-\frac{9}{5}$ $\therefore$ The coordinates of R are $(\frac{8}{5}, -\frac{9}{5})$.

(b) When $k=6$, (1) is the equation of C_2.

$\therefore$ The equation of C_2 is

$x^2+y^2-2(6)x+2(6)y+6^2=0$

$x^2+y^2-12x+12y+36=0$

If the centre of the circle lies in the first quadrant, centre = (k, k).

$(x-k)^2+(y-k)^2=k^2$

C_2: $x^2+y^2-2kx-2ky+k^2=0$ …… (4)

Sub. (2) into (4): $x^2+\left(\frac{3}{4}x-3\right)^2-2kx-2k\left(\frac{3}{4}x-3\right)+k^2=0$

$16x^2+(9x^2-72x+144)-32kx-8k(3x-12)+16k^2=0$

$25x^2-(72+56k)x+(16k^2+96k+144)=0$

For tangency, $\Delta=0$

$(72+56k)^2-4(25)(16k^2+96k+144)=0$

$(9+7k)^2-25(k^2+6k+9)=0$

$81+126k+49k^2-25k^2-150k-225=0$

$24k^2-24k-144=0$

$k^2-k-6=0\Rightarrow(k+2)(k-3)=0$

$k=-2$ or $k=3$

When $k=-2$, C_2: $x^2+y^2+4x+4y+4=0$ (rejected, centre not in 1^{st} quad.)

When $k=3$, C_2: $x^2+y^2-6x-6y+9=0$

If the centre of the circle lies in the third quadrant, centre = $(-k, -k)$.

$(x + k)^2 + (y + k)^2 = k^2$

C_2: $x^2 + y^2 + 2kx + 2ky + k^2 = 0$ …… (5)

Sub. (2) into (5): $x^2 + \left(\frac{3}{4}x - 3\right)^2 + 2kx + 2k\left(\frac{3}{4}x - 3\right) + k^2 = 0$

$16x^2 + (9x^2 - 72x + 144) + 32kx + 8k(3x - 12) + 16k^2 = 0$

$25x^2 - (72 - 56k)x + (16k^2 - 96k + 144) = 0$

For tangency, $\Delta = 0$

$(72 - 56k)^2 - 4(25)(16k^2 - 96k + 144) = 0$

$(9 - 7k)^2 - 25(k^2 - 6k + 9) = 0$

$81 - 126k + 49k^2 - 25k^2 + 150k - 225 = 0$

$24k^2 + 24k - 144 = 0$

$k^2 + k - 6 = 0 \Rightarrow (k + 3)(k - 2) = 0$

$k = -3$ or $k = 2$

When $k = -3$, C_2: $x^2 + y^2 - 6x - 6y + 9 = 0$ (rejected, centre not in 3rd quad.)

When $k = 2$, C_2: $x^2 + y^2 + 4x + 4y + 4 = 0$

If the centre of the circle lies in the second quadrant, centre = $(-k, k)$.

$(x + k)^2 + (y - k)^2 = k^2$

C_2: $x^2 + y^2 + 2kx - 2ky + k^2 = 0$ …… (6)

Sub. (2) into (6): $x^2 + \left(\frac{3}{4}x - 3\right)^2 + 2kx - 2k\left(\frac{3}{4}x - 3\right) + k^2 = 0$

$16x^2 + (9x^2 - 72x + 144) + 32kx - 8k(3x - 12) + 16k^2 = 0$

$25x^2 - (72 - 8k)x + (16k^2 + 96k + 144) = 0$

For tangency, $\Delta = 0$

$(72 - 8k)^2 - 4(25)(16k^2 + 96k + 144) = 0$

$(9 - k)^2 - 25(k^2 + 6k + 9) = 0$

$81 - 18k + k^2 - 25k^2 - 150k - 225 = 0$

$24k^2 + 168k + 144 = 0$

$k^2 + 7k + 6 = 0 \Rightarrow (k + 6)(k + 1) = 0$

$k = -6$ or $k = -1$

When $k = -6$, centre = $(6, -6)$, not in 2nd quad, rejected.

When $k = -1$, centre = $(1, -1)$, not in 2nd quad, rejected.

A sketch of four possible circles touching the x-axis, the y-axis and L_1 is drawn below: