

| | | | | | | | | | | |
|-------------------------|----------|---|----------|------------------------|----------|--------------------------|----------|------------|-----------|--------------|
| 10-11 Individual | 1 | 11 | 2 | $\frac{26}{5} (= 5.2)$ | 3 | $\frac{1}{2}\sqrt{2010}$ | 4 | 20 | 5 | 237 |
| | 6 | 10 | 7 | $\sqrt{6} - \sqrt{2}$ | 8 | 7 | 9 | 36 | 10 | $24\sqrt{3}$ |
| 10-11 Group | 1 | $a = 1006, b = 1002, c = 998, d = 996, e = 994$ | 2 | 15 | 3 | 5026 | 4 | 4 | 5 | 1005 |
| | 6 | $6\sqrt{2}$ | 7 | 125751 | 8 | 2012 | 9 | 99° | 10 | $2\sqrt{3}$ |

Individual Events

- I1** Find the remainder when 2^{2011} is divided by 13.

Reference: 1972 American High School Mathematics Examination Q31, 2015 FI4.1

$$2^6 = 64 = 13 \times 5 - 1 \equiv -1 \pmod{13}; 2^{12} \equiv 1 \pmod{13}$$

$$2011 = 12 \times 167 + 7$$

$$2^{2011} = 2^{12 \times 167 + 7} = (2^{12})^{167} \times 2^7 \equiv 2^7 \equiv 2^6 \times 2 \equiv -1 \times 2 \equiv -2 \equiv 11 \pmod{13}$$

- I2** Given that $x^2 + y^2 = 1$, find the maximum value of $2x + 5y^2$.

Reference: 2003 FG1.2, 2009 HI5

$$2x + 5y^2 = 2x + 5(1 - x^2) = -5x^2 + 2x + 5$$

$$\text{The maximum value} = \frac{4ac - b^2}{4a} = \frac{4(-5) \cdot 5 - 2^2}{4(-5)} = \frac{26}{5} (= 5.2)$$

- I3** Given that $a + b = \sqrt{\sqrt{2011} + \sqrt{2010}}$ and $a - b = \sqrt{\sqrt{2011} - \sqrt{2010}}$, find the value of ab .
(Give your answer in surd form)

$$(a + b)^2 - (a - b)^2 = \sqrt{2011} + \sqrt{2010} - \sqrt{2011} + \sqrt{2010}$$

$$4ab = 2\sqrt{2010}$$

$$ab = \frac{1}{2}\sqrt{2010}$$

- I4** In $\triangle ABC$, the ratio of the altitudes perpendicular to the three sides AB , BC and CA is $3 : 4 : 5$. If the lengths of the three sides are integers, find the minimum value of AB .

Let $AB = c$, $BC = a$, $CA = b$. By calculating the areas in 3 different ways:

$$3a = 4b = 5c$$

$$a : b : c = 20 : 15 : 12$$

The minimum value of $AB = 20$

- I5** An integer x minus 12 is the square of an integer. x plus 19 is the square of another integer. Find the value of x .

$$x - 12 = n^2 \dots\dots (1); x + 19 = m^2 \dots\dots (2), \text{ where } m, n \text{ are integers.}$$

$$(2) - (1): (m + n)(m - n) = 31$$

$\therefore 31$ is a prime number

$$\therefore m + n = 31 \text{ and } m - n = 1$$

$$m = 16, n = 15$$

$$x = 15^2 + 12 = 237$$

- I6** A , B and C pass a ball among themselves. A is the first one to pass the ball to other one. In how many ways will the ball be passed back to A after 5 passes?

Construct the following table:

| Number of passes | 1 | 2 | 3 | 4 | 5 |
|------------------|---|---------|---------|---------|----------|
| A | 0 | $1+1=2$ | $1+1=2$ | $3+3=6$ | $5+5=10$ |
| B | 1 | $0+1=1$ | $1+2=3$ | $3+2=5$ | $5+6=11$ |
| C | 1 | $0+1=1$ | $1+2=3$ | $3+2=5$ | $5+6=11$ |

There will be 10 ways for the ball to pass back to A .

- I7** Find the value of $\sqrt{7 - \sqrt{12} - \sqrt{13 - 2\sqrt{12}}}$.

Reference: 1993 FI1.4, 1999 HG3, 2001 FG2.1, 2015 FI4.2, 2015 FG3.1

$$\begin{aligned}\sqrt{13 - 2\sqrt{12}} &= \sqrt{12 - 2\sqrt{1 \times 12} + 1} \\ &= \sqrt{(\sqrt{12} - 1)^2} \\ &= \sqrt{12} - 1 \\ \sqrt{7 - \sqrt{12} - \sqrt{13 - 2\sqrt{12}}} &= \sqrt{7 - \sqrt{12} - \sqrt{12} + 1} \\ &= \sqrt{8 - 2\sqrt{12}} \\ &= \sqrt{6 - 2\sqrt{2 \times 6} + 2} \\ &= \sqrt{(\sqrt{6} - \sqrt{2})^2} \\ &= \sqrt{6} - \sqrt{2}\end{aligned}$$

- I8** A school issues 4 types of raffle tickets with face values \$10, \$15, \$25 and \$40. Class A uses several one-hundred dollar notes to buy 30 raffle tickets, including 5 tickets each for two of the types and 10 tickets each for the other two types. How many one-hundred dollars notes Class A use to buy the raffle tickets?

Suppose x tickets of \$10, y tickets of \$15, z tickets of \$25, t tickets of \$40 are bought.

100 is an even number, the face values \$15 and \$25 are odd numbers.

Then either $(x, y, z, t) = (5, 10, 10, 5)$ or $(x, y, z, t) = (10, 5, 5, 10)$ will make an even sum.

Case 1 $(x, y, z, t) = (5, 10, 10, 5)$

Total cost = $10(5) + 15(10) + 25(10) + 40(5) = 650$, which is not a multiple of 100, rejected.

Case 2 $(x, y, z, t) = (10, 5, 5, 10)$

Total cost = $10(10) + 15(5) + 25(5) + 40(10) = 700$

\therefore Class A uses 7 \$100 notes.

- I9** The length and the width of a rectangle are integers. If its area is larger than its perimeter by 9, find the perimeter.

Let the width be x and the length be y .

$$xy - 2(x + y) = 9$$

$$x(y - 2) - 2y + 4 = 13$$

$$(x - 2)(y - 2) = 13$$

$$x - 2 = 1, y - 2 = 13$$

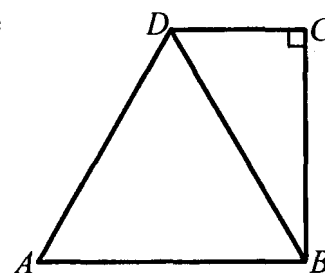
$$x = 3, y = 15$$

$$\text{The perimeter} = 2(3 + 15) = 36$$

- I10** In the figure, $ABCD$ is a trapezium with $\angle C = 90^\circ$. If the area of the equilateral triangle ABD is $16\sqrt{3}$, find the area of trapezium $ABCD$.

$$\text{Area of } \triangle BCD = \frac{1}{2} \text{ area of } \triangle ABD$$

$$\text{Area of trapezium} = 16\sqrt{3} + 8\sqrt{3} = 24\sqrt{3}$$



Group Events

- G1** If $(1000 - a)(1000 - b)(1000 - c)(1000 - d)(1000 - e) = 24^2$, where a, b, c, d and e are even numbers and $a > b > c > d > e$, find the values of a, b, c, d and e .

$$\begin{aligned} 24^2 &= 2^6 \times 3^2 = -6 \times (-2) \times 2 \times 4 \times 6 \\ &= (1000 - 1006)(1000 - 1002)(1000 - 998)(1000 - 996)(1000 - 994) \\ a &= 1006, b = 1002, c = 998, d = 996, e = 994 \end{aligned}$$

- G2** \overline{ab} denotes a two digit number with a as tens digit and b as the unit digit. $R_{\overline{ab}}$ is the remainder when \overline{ab} is divided by $a + b$. Find the maximum value of $R_{\overline{ab}}$.

$$\overline{ab} = 10a + b = (a + b)m + R_{\overline{ab}}, \text{ where } 0 \leq R_{\overline{ab}} < a + b$$

To find the maximum value of $R_{\overline{ab}}$.

We start searching $R_{\overline{ab}}$ from $\overline{ab} = 99$ downwards to 10.

$$\overline{ab} = 99, 99 \div (9 + 9) = 5 \dots\dots 9 = R_{\overline{ab}}$$

$$\overline{ab} = 98, 98 \div (9 + 8) = 5 \dots\dots 13 = R_{\overline{ab}}$$

$$\overline{ab} = 89, 89 \div (9 + 8) = 5 \dots\dots 4 = R_{\overline{ab}}$$

$$\overline{ab} = 97, 97 \div (9 + 7) = 6 \dots\dots 1 = R_{\overline{ab}}$$

$$\overline{ab} = 88, 88 \div (8 + 8) = 5 \dots\dots 8 = R_{\overline{ab}}$$

$$\overline{ab} = 79, 79 \div (7 + 9) = 4 \dots\dots 15 = R_{\overline{ab}}$$

The maximum value of the divisor is 16, which gives the maximum value of remainder = 15

For other divisors, which must be less than 16, the remainders must be less than 15.

\therefore The maximum value of $R_{\overline{ab}}$ is 15.

- G3** Given that a, b and c are integers, and $a + b = 2011$, $c - a = 2010$, $a < b$. Find the greatest possible value of $a + b + c$.

$$a + b + c = 2011 + c$$

$$c = a + 2010$$

$$\text{Maximum } a = 1005, b = 1006, c = 1005 + 2010 = 3015$$

$$\text{Maximum } a + b + c = 2011 + 3015 = 5026$$

- G4** Given that n is a positive integer and $n^4 - 18n^2 + 49$ is a prime number, find the value of n .

$$n^4 - 18n^2 + 49 = (n^2 - 7)^2 - 4n^2 = (n^2 + 2n - 7)(n^2 - 2n - 7)$$

$$\therefore n^2 + 2n - 7 > n^2 - 2n - 7$$

$$n^2 - 2n - 7 = 1 \text{ and } n^2 + 2n - 7 \text{ is a prime}$$

$$n^2 - 2n - 8 = 0$$

$$(n - 4)(n + 2) = 0$$

$$n = 4 \text{ only (verification: } n^2 + 2n - 7 = 17 \text{ which is a prime)}$$

- G5** Given that $f(x) = \frac{4^x}{4^x + 2}$, where x is a real number, find the value of

$$f\left(\frac{1}{2011}\right) + f\left(\frac{2}{2011}\right) + f\left(\frac{3}{2011}\right) + \dots + f\left(\frac{2009}{2011}\right) + f\left(\frac{2010}{2011}\right).$$

Reference: 2004 FG4.1, 2012 FI2.2

$$f(x) + f(1 - x) = \frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2} = \frac{4 + 4 + 2 \cdot 4^x + 2 \cdot 4^{1-x}}{4 + 4 + 2 \cdot 4^x + 2 \cdot 4^{1-x}} = 1$$

$$f\left(\frac{1}{2011}\right) + f\left(\frac{2}{2011}\right) + f\left(\frac{3}{2011}\right) + \dots + f\left(\frac{2009}{2011}\right) + f\left(\frac{2010}{2011}\right)$$

$$= f\left(\frac{1}{2011}\right) + f\left(\frac{2010}{2011}\right) + f\left(\frac{2}{2011}\right) + f\left(\frac{2009}{2011}\right) + \dots + f\left(\frac{1005}{2011}\right) + f\left(\frac{1006}{2011}\right) = 1005$$

- G6** In the figure below, M is a point on AC , $AM = MC = BM = 3$. Find the maximum value of $AB + BC$. **Reference: 2007 HG8**
 M is the centre of the circle ABC with AC as diameter.

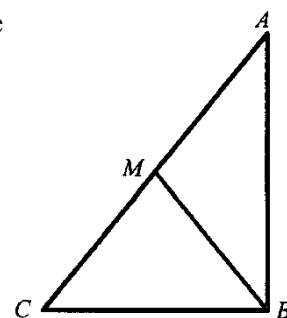
$$\angle ABC = 90^\circ \quad (\angle \text{ in semi-circle})$$

$$\text{Let } \angle ACB = x, AB = 6 \sin \theta, BC = 6 \cos \theta$$

$$AB + BC = 6 \sin \theta + 6 \cos \theta$$

$$= 6\sqrt{2} \sin(\theta + 45^\circ) \leq 6\sqrt{2}$$

$$\therefore \text{The maximum value is } 6\sqrt{2}.$$



- G7** Given that $n! = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1$ and $\frac{2011!}{10^k}$ is an integer, where k is a positive integer. If S is the sum of all possible values of k , find the value of S .

Reference: 1990 HG6, 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2012 FI1.4, 2012 FG1.3

When each factor of 5 is multiplied by 2, a trailing zero will appear in $n!$.

The number of factors of 2 is clearly more than the number of factors of 5 in 2011!

It is sufficient to find the number of factors of 5.

5, 10, 15, ..., 2010; altogether 402 numbers, each have at least one factor of 5.

25, 50, 75, ..., 2000; altogether 80 numbers, each have at least two factors of 5.

125, 250, 375, ..., 2000; altogether 16 numbers, each have at least three factors of 5.

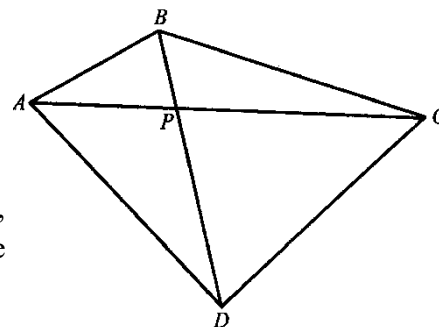
625, 1250, 1875; altogether 3 numbers, each have at least four factors of 5.

$$\therefore \text{Total number of factors of 5 is } 402 + 80 + 16 + 3 = 501$$

There are 501 trailing zeros of 2011! \Rightarrow If $\frac{2011!}{10^k}$ is an integer, then $k = 1, 2, \dots, 501$.

$$S = 1 + 2 + \dots + 501 = \frac{1+501}{2} \times 501 = 125751$$

- G8** Given that a, b, c and d are non-negative integers and $ac + bd + ad + bc = 2011$. Find the value of $a + b + c + d$.
 $(a+b)(c+d) = 2011$, which is a prime number
 $a+b = 2011$ and $c+d = 1$ or vice versa.
 $a+b+c+d = 2012$



- G9** As shown in the figure, $ABCD$ is a convex quadrilateral, $\angle BAC = 27^\circ$, $\angle BCA = 18^\circ$, $\angle BDC = 54^\circ$, $\angle BDA = 36^\circ$. The diagonals AC and BD intersect at P . Find $\angle CPB$.

Reference: 2003 HG8, 2014 FG2.4

Draw a circumscribed circle ABC .

Produce BD to cut the circle at E .

$$\angle BEA = 18^\circ = \angle BCA \quad (\angle \text{ s in the same segment})$$

$$\angle BEC = 27^\circ = \angle BAC \quad (\angle \text{ s in the same segment})$$

$$\angle DAE = 36^\circ - 18^\circ = 18^\circ \quad (\text{ext. } \angle \text{ of } \triangle ADE)$$

$$\angle DCE = 54^\circ - 27^\circ = 27^\circ \quad (\text{ext. } \angle \text{ of } \triangle ADE)$$

$$\therefore DA = DE \text{ and } DC = DE \quad (\text{sides opp. eq. } \angle \text{ s})$$

$$\therefore D \text{ is the centre of the circle}$$

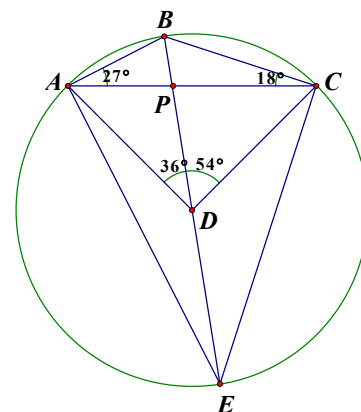
$$\therefore DA = DB = DC \quad (\text{radii})$$

$$\angle DCB = \angle DBC = (180^\circ - 54^\circ) \div 2 = 63^\circ$$

$$(\angle \text{ s sum of isos. } \triangle BCD)$$

$$\angle DCP = 63^\circ - 18^\circ = 45^\circ$$

$$\angle CPB = 45^\circ + 54^\circ = 99^\circ \quad (\text{ext. } \angle \text{ of } \triangle CDP)$$



G10 As shown in the figure, $AC = 3$, $BC = 4$ and $\angle C = 90^\circ$. M is a point on BC such that the radii of the incircles in $\triangle ABM$ and $\triangle ACM$ are equal. Find the length of AM .

Let D, F be the centres of the circles, with radii r , the points of contact be E, G, H, I, J, K as shown. Let $AM = x$, $AB = 5$, $CH = CE = r$.

$AE = 3 - r = AG$ (tangent from ext. point)

$$S_{\triangle ABF} + S_{\triangle AMF} + S_{\triangle BMF} + S_{\triangle AMD} + S_{\triangle ACD} + S_{\triangle CMD} = S_{\triangle ABC}$$

$$\frac{3+4+5}{2} \cdot r + \frac{2x \cdot r}{2} = \frac{3 \times 4}{2}$$

$$r(6+x) = 6$$

$$r = \frac{6}{6+x} \quad \dots\dots (1)$$

In $\triangle ACM$, $MG = x + r - 3 = MH$

$$MC = x + 2r - 3 = x + 2 \times \frac{6}{6+x} - 3 \quad \text{by (1)}$$

$$= \frac{(x-3)(6+x)+12}{6+x} = \frac{x^2+3x-6}{6+x}$$

$$AM^2 - AC^2 = MC^2$$

$$x^2 - 9 = \left(\frac{x^2 + 3x - 6}{6+x} \right)^2$$

$$(x^2 - 9)(x+6)^2 = (x^2 + 3x - 6)^2$$

$$(x^2 - 9)(x^2 + 12x + 36) = x^4 + 6x^3 - 3x^2 - 36x + 36$$

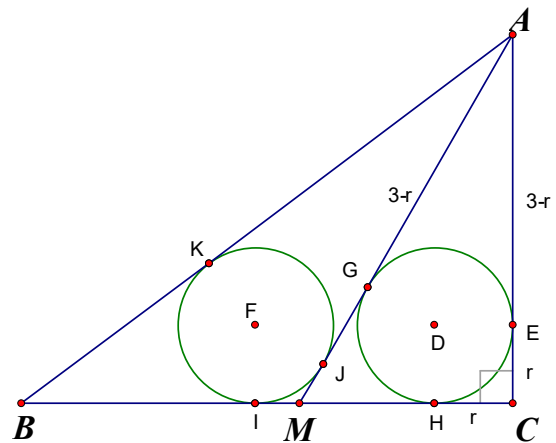
$$6x^3 + 30x^2 - 72x - 360 = 0$$

$$x^3 + 5x^2 - 12x - 60 = 0$$

$$x(x^2 - 12) + 5(x^2 - 12) = 0$$

$$(x+5)(x^2 - 12) = 0$$

$$AM = x = 2\sqrt{3}$$



(tangent from ext. point)

(Pythagoras' Theorem)

Method 2 Lemma In the figure, given a triangle ABC .

$AD \perp BC$, $AD = h$, O is the centre of the inscribed circle with

radius r . $s = \frac{1}{2}(a+b+c)$

$$\text{Then } r = \frac{\text{Area of } \triangle ABC}{s} = \frac{ah}{a+b+c}$$

Proof: Area $\triangle OBC$ + area $\triangle OCA$ + area $\triangle OAB$ = Area of $\triangle ABC$

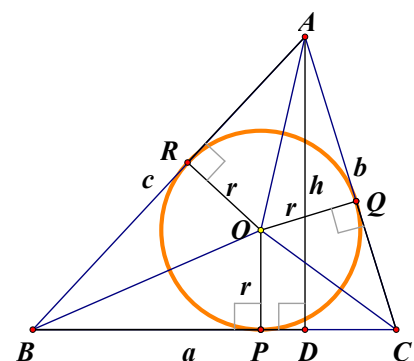
$$\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}ah$$

$$\therefore r = \frac{\text{Area of } \triangle ABC}{s} = \frac{ah}{a+b+c}$$

Let $CM = t$, then $BM = 4 - t$. Now apply the above lemma to $\triangle ACM$ and $\triangle ABM$ respectively.

$$r = \frac{3t}{t+3+\sqrt{9+t^2}} \quad \dots\dots (1)$$

$$r = \frac{3(4-t)}{4-t+5+\sqrt{9+t^2}} \quad \dots\dots (2)$$



$$\begin{aligned}
(1) = (2) &\Rightarrow \frac{3t}{t+3+\sqrt{9+t^2}} = \frac{3(4-t)}{9-t+\sqrt{9+t^2}} \\
\frac{t}{4-t} &= \frac{t+3+\sqrt{9+t^2}}{9-t+\sqrt{9+t^2}} \\
\frac{2t-4}{4} &= \frac{2t-6}{12+2\sqrt{9+t^2}} \quad (\because \frac{A}{B} = \frac{C}{D} \Rightarrow \frac{A-B}{A+B} = \frac{C-D}{C+D}) \\
\frac{t-2}{2} &= \frac{t-3}{6+\sqrt{9+t^2}} \\
(t-2)(6+\sqrt{9+t^2}) &= 2t-6 \\
6t-12+(t-2)\sqrt{9+t^2} &= 2t-6 \\
(t-2)\sqrt{9+t^2} &= 6-4t \\
(t^2-4t+4)(9+t^2) &= 36-48t+16t^2 \\
t^4-4t^3+4t^2+9t^2-36t+36 &= 36-48t+16t^2 \\
t^4-4t^3-3t^2+12t &= 0 \\
t[t^2(t-4)-3(t-4)] &= 0 \\
t(t-4)(t^2-3) &= 0 \\
\therefore t=0 \text{ (rejected), } 4 \text{ (rejected because } CM < BC=4) \text{ or } \sqrt{3}. \\
AM &= \sqrt{AC^2 + CM^2} = \sqrt{3^2 + \sqrt{3}^2} = 2\sqrt{3}
\end{aligned}$$

Geometrical Construction

- Given a straight line L , and two points P and Q lying on the same side of L . Mark a point T on L so that the sum of the lengths of PT and QT is minimal. (Hint: Consider the reflection image of P about the line L .)

Reference: 2008 Heat Sample construction Q2



Figure 1

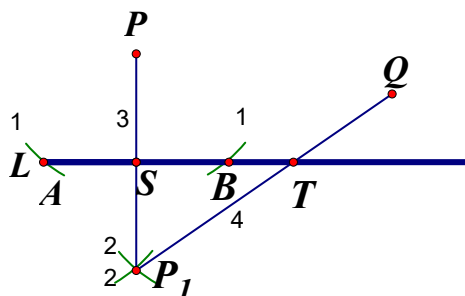


Figure 2

Let A be one of the end point of L nearer to P .

- Use P as centre, PA as radius to draw an arc, which intersects L at A and B .
- Use A as centre, AP as radius to draw an arc. Use B as centre, BP as radius to draw an arc. The two arcs intersect at P_1 .
- Join PP_1 which intersects L at S .
- Join P_1Q which intersects L at T .

$$AP = AP_1 \quad (\text{same radii})$$

$$AB = AB \quad (\text{common side})$$

$$BP = BP_1 \quad (\text{same radii})$$

$$\triangle APB \cong \triangle AP_1B \quad (\text{S.S.S.})$$

$$\angle PBA = \angle P_1BA \quad (\text{corr. sides } \cong \Delta\text{'s})$$

$$BS = BS \quad (\text{common side})$$

$$\triangle PBS \cong \triangle P_1BS \quad (\text{S.A.S.})$$

$$\angle BSP = \angle BSP_1 \quad (\text{corr. sides } \cong \Delta\text{'s})$$

$$= 90^\circ \quad (\text{adj. } \angle\text{s on st. line})$$

$$SP = SP_1 \quad (\text{corr. sides } \cong \Delta\text{'s})$$

$$ST = ST \quad (\text{common side})$$

$$\triangle PST \cong \triangle P_1ST \quad (\text{S.A.S.})$$

$$PT = P_1T \quad (\text{corr. sides } \cong \Delta\text{'s})$$

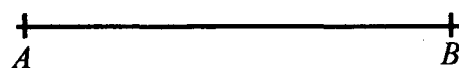
$$PT + QT = P_1T + QT$$

It is known that $P_1T + QT$ is a minimum when P_1, T, Q are collinear.

$\therefore T$ is the required point.

2. The figure shows a line segment AB . D is a fixed point such that A, B, D are not collinear. Construct a triangle ABC so that C, B and D are collinear and $AC - BC = BD$.

如圖所示為一線段 AB 。 D 為一固定點，且 A, B, D 不共線。試作 $\triangle ABC$ ，使得 C, B 及 D 共線，及 $AC - BC = BD$ 。



Remark: The wording in both versions are ambiguous, so I have changed it.

The original question is:

$\times D$

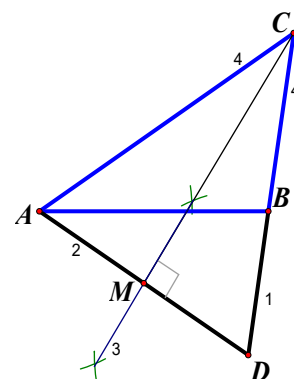
Figure 2 shows a line segment AB which is a $\triangle ABC$ and D is any point not lying on AB . If the difference between the other two sides of $\triangle ABC$ (i.e. $AC - BC$) is equal to BD and C, B and D are collinear, construct $\triangle ABC$.

圖 2 所示為 $\triangle ABC$ 的其中一條邊 AB 及 D 為一非線段 AB 上的任意點。若 $\triangle ABC$ 的其餘兩條邊(即 $AC - BC$)的長度差距等如 BD ，且 C, B 及 D 共線，試構作 $\triangle ABC$ 。

- (1) Join DB and produce it further.
- (2) Join AD .
- (3) Construct the perpendicular bisector of AD , cutting DB produced at C . Let M be the mid-point of AD .
- (4) Join AC, BC .

$\triangle ABC$ is the required triangle.

Proof: $\triangle ACM \cong \triangle DCM$ (S.A.S.)
 $AC = DC$ (corr. sides, $\cong \Delta$ s)
 $BC = DC - BD = AC - BD$
 $\therefore BD = AC - BC$.



Discussion: There are some positions on the plane for which $\triangle ABC$ is not constructible.

Use B as centre BA as radius to draw a circle.

Let AE be the diameter of this circle. $AB = BE$.

Draw 2 circles with AB, BE as diameters.

Let the region (including the boundary) bounded by the circles with AB, BE as diameters be α and β respectively.

Let the region inside the great circle with AE as diameter but not in α and β be γ .

Let the region on or outside the great circle be ω .

When D lies on ω , then $BD \geq AB$

So $AC - BC \geq AB$

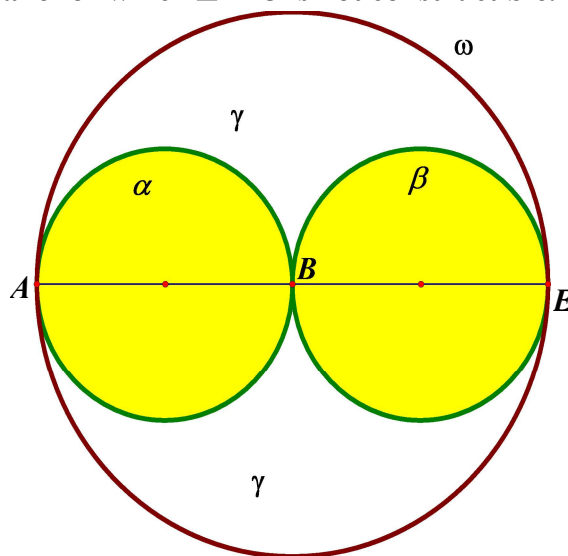
$AC \geq AB + BC$

This inequality violates the triangle law:

The sum of 2 sides of a triangle is larger than the third side.

i.e. $AC \geq AB + BC > AC$, which is false.

\therefore When D lies on or outside the circle, $\triangle ABC$ is not constructible.



When D lies on α ,

$$\triangle AC_1M \cong \triangle DC_1M \quad (\text{S.A.S.})$$

$$\angle C_1AM = \angle C_1DM \quad (\text{corr. } \angle\text{s, } \cong \Delta\text{s})$$

$$\begin{aligned} \angle ADC_1 &= \angle BAD + \angle ABD \quad (\text{ext. } \angle \text{ of } \triangle ABD) \\ &> \angle ABD \end{aligned}$$

$$\therefore \angle C_1AM = \angle ADC_1 > \angle ABD$$

$$\angle C_1AB > \angle ABD$$

$$AC_1 < BC_1 \quad (\text{greater sides opp. greater } \angle\text{s})$$

The contradicts to the fact that $BD = AC_1 - BC_1$

$$\text{because } AC_1 = BD + BC_1 > BC_1$$

In this case the steps are as follows:

- (1) Join DB and produce it further.
- (2) Use B as centre, BD as radius to draw a circle, cutting DB produced at F .
- (3) Join AF .
- (4) Construct the perpendicular bisector of AF , cutting DB at C_2 . Let M_2 be the mid-point of AF .
- (5) Join AC_2, BC_2 .

$\triangle ABC_2$ is the required triangle.

$$\angle ABC_2 = \angle BAF + \angle AFB \quad (\text{ext. } \angle \text{ of } \triangle ABF)$$

$$> \angle AFB$$

$$= \angle FAC_2$$

$$(\text{corr. } \angle\text{s, } \cong \Delta\text{s})$$

$$> \angle BAC_2$$

$$\therefore AC_2 > BC_2 \quad (\text{greater sides opp. greater } \angle\text{s})$$

The triangle ABC is constructible.

When D lies on β , relabel D as F and F as D , the construction steps are shown above.

There is only one possible triangle.

When D lies on the boundary of α ,

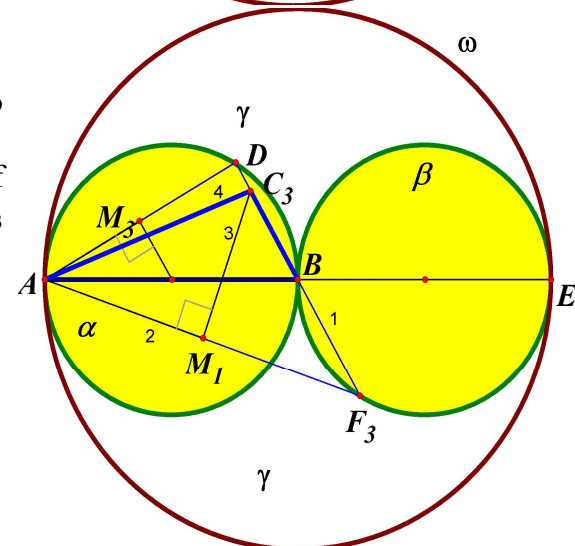
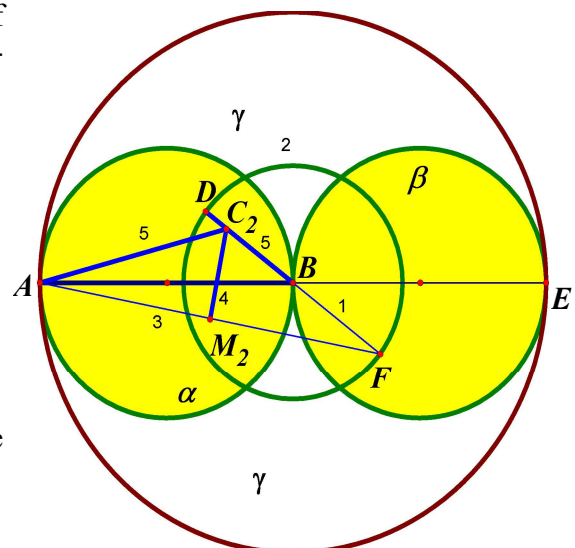
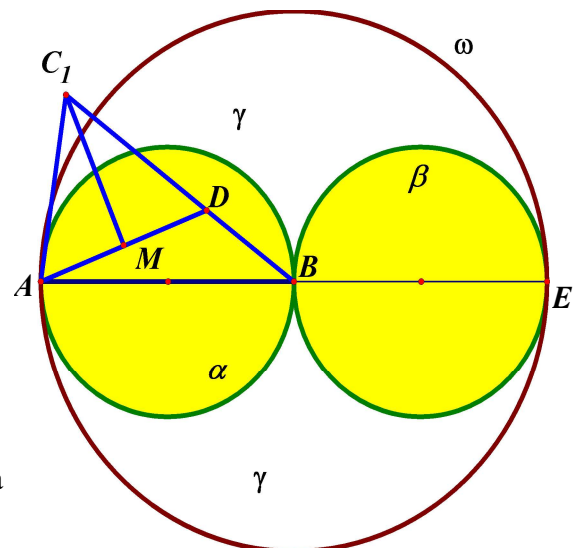
$$\angle ADB = 90^\circ \quad (\angle \text{ in semi-circle})$$

The perpendicular bisector of AD is parallel to DB . \therefore It will not intersect DB produced.

Suppose DB produced intersects the boundary of β at F_3 . Then the perpendicular bisector of AF_3 intersects DB at C_3 .

$\triangle ABC_3$ is the required triangle.

Proof: omitted.

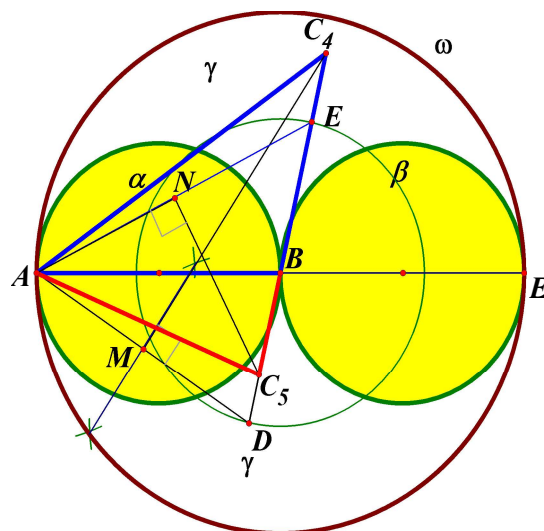


When D lies on γ ,

Use B as centre, BD as radius to draw a circle, cutting DB produced at E on γ .

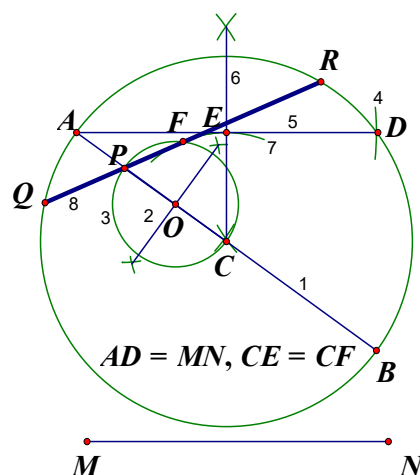
The perpendicular bisector of AD cuts DB produced at C_4 . The perpendicular bisector of AE cut BD at C_5 .

\therefore We can construct 2 possible triangles $\triangle ABC_4$ and $\triangle ABC_5$ satisfying the given conditions.



3. Figure 3 shows a circle of centre C and a line segment MN . P is a point lies inside the circle. Construct a chord QR with points Q and R on the circumference of the circle so that it passes through P and its length is equal to that of MN .

- (1) Join CP and produce it to 2 ends A, B of the diameter of the circle as shown.
- (2) Draw the perpendicular bisector of CP , O is the mid-point of CP .
- (3) Use O as centre, OC as radius to draw a circle.
- (4) Use A as centre, MN as radius to draw an arc, cutting the given circle at D .
- (5) Join AD .
- (6) Draw the perpendicular bisector of AD , E is the mid-point of AD .
- (7) Use C as centre, CE as radius to draw an arc, cutting the circle in step (3) at F .
- (8) Join PF and produce it to cut the circle at Q and R . Then QR is the required chord.



Proof: $\angle PFC = 90^\circ$ (\angle in semi-circle)
 $CE = CF$ (by construction)
 $QR = AD = MN$ (chords eq. distance from centre are equal)

Method 2: (Provided by Tsuen Wan Government Secondary School Tam Lok Him)

Let the radius of the given circle be R , the distance between CP be r .

- (1) Use M as centre and R as radius to draw an arc, use N as centre and R as radius to draw an arc. The two arcs intersect at A .
- (2) Use A as centre, R as radius to draw a circle C_1 . The circle C_1 must pass through M, N .
- (3) Use A as centre, r as radius to draw a circle C_2 , which cuts MN at P_1 .
- (4) On the given circle, use P as centre, MP_1 as radius to draw a circle, which cuts the given circle at Q .
- (5) Join QP and produce it to cut the given circle at R .

Then QPR is the required chord.

Proof: $QP = MP_1$ (by construction)

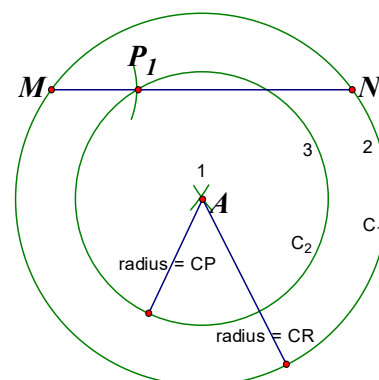
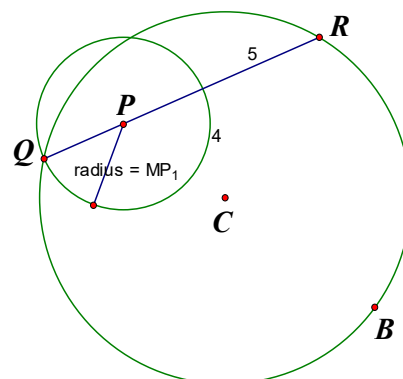
$CP = AP_1$ (by construction)

$CQ = AM$ (by construction)

$\triangle CPQ \cong \triangle AP_1M$ (S.S.S.)

height of $\triangle CPQ$ = height of $\triangle AP_1M$

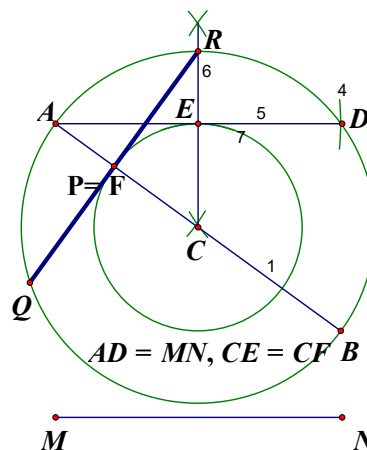
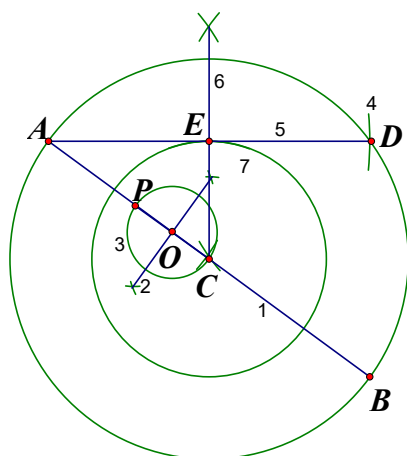
$QR = MN$ (chords eq. distance from centre are equal)



Remark There are some positions of P inside the circle for which the chord is not constructible.

The circle in step 7 may not cut the circle in step 3. At the limiting position, $CQ^2 = PQ^2 + CP^2$

$$4r^2 = QR^2 + 4CP^2 \Rightarrow QR = 2\sqrt{r^2 - CP^2}$$



\therefore For fixed position P , the length of chord must satisfy $2\sqrt{r^2 - CP^2} \leq MN \leq 2r$.

Percentage of correct questions

| | | | | | | | | | |
|----------|--------|----------|--------|----------|--------|----------|--------|-----------|--------|
| 1 | 23.55% | 2 | 26.6% | 3 | 52.16% | 4 | 25.87% | 5 | 60.04% |
| 6 | 31.34% | 7 | 10.83% | 8 | 70.35% | 9 | 41.22% | 10 | 62.88% |
| 1 | 22.18% | 2 | 24.69% | 3 | 40.17% | 4 | 43.1% | 5 | 17.57% |
| 6 | 37.66% | 7 | 11.30% | 8 | 58.16% | 9 | 21.76% | 10 | 3.77% |