12-13 Individual	**1**	$\sqrt{61}-\sqrt{33}$	**2**	180	**3**	30°	**4**	25	**5**	2012
	6	14	**7**	1	**8**	5	**9**	9	**10**	2
12-13 Group	**1**	13	**2**	192	**3**	6039	**4**	8028	**5**	10
	6	6	**7**	6	**8**	130	**9**	671	**10**	$\frac{2}{2013}$

Individual Events

I1 Simplify $\sqrt{94-2\sqrt{2013}}$.

$$\sqrt{94-2\sqrt{2013}}=\sqrt{61-2\sqrt{61\times 33}+33}$$

$$=\sqrt{\left(\sqrt{61}-\sqrt{33}\right)^2}$$

$$=\sqrt{61}-\sqrt{33}$$

I2 A parallelogram is cut into 178 pieces of equilateral triangles with sides 1 unit. If the perimeter of the parallelogram is P units, find the maximum value of P.

2 equilateral triangles joint to form a small parallelogram.

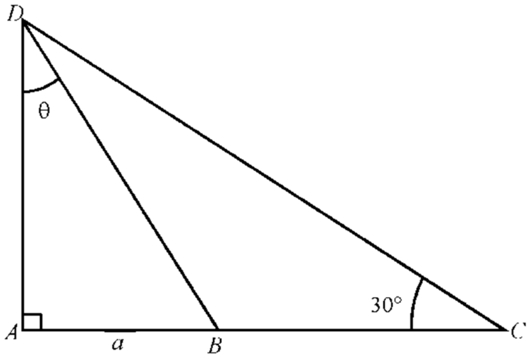
$178 = 2\times 89$

$\therefore$ The given parallelogram is cut into 89 small parallelograms, and 89 is a prime number.

The dimension of the given parallelogram is 1 unit $\times$ 89 units.

$P = 2(1 + 89) = 180$ units

I3 Figure 1 shows a right-angled triangle ACD where B is a point on AC and $BC = 2AB$. Given that $AB = a$ and $\angle ACD = 30°$, find the value of θ.



In ΔABD, $AD=\dfrac{a}{\tan\theta}$

In ΔACD, $AC=\dfrac{AD}{\tan 30^\circ}=\dfrac{\sqrt{3}a}{\tan\theta}$

However, $AC = AB + BC = a + 2a = 3a$

$\therefore \dfrac{\sqrt{3}a}{\tan\theta}=3a$

$\tan\theta=\dfrac{\sqrt{3}}{3}$

$\Rightarrow \theta = 30°$

I4 Given that $x^2 + 399 = 2^y$, where x, y are positive integers. Find the value of x.

Reference: 2018 HG6

$2^9 = 512$, $512 - 399 = 113 \neq x^2$

$2^{10} = 1024$, $1024 - 399 = 625 = 25^2$, $x = 25$

I5 Given that $y = (x + 1)(x + 2)(x + 3)(x + 4) + 2013$, find the minimum value of y.

Reference 1993HG5, 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3

$y = (x + 1)(x + 4)(x + 2)(x + 3) + 2013 = (x^2 + 5x + 4)(x^2 + 5x + 6) + 2013$

$= (x^2 + 5x)^2 + 10(x^2 + 5x) + 24 + 2013 = (x^2 + 5x)^2 + 10(x^2 + 5x) + 25 + 2012$

$= (x^2 + 5x + 5)^2 + 2012 \geq 2012$

The minimum value of y is 2012 .

I6 In a convex polygon with n sides, one interior angle is selected. If the sum of the remaining $n - 1$ interior angle is 2013°, find the value of n.

Reference: 1989 HG2, 1990 FG10.3-4, 1992 HG3, 2002 FI3.4

$1980° = 180°\times(13 - 2) < 2013° < 180°\times(14 - 2) = 2160°$

$n = 14$

I7 Figure 2 shows a circle passes through two points B and C, and a point A is lying outside the circle. Given that BC is a diameter of the circle, AB and AC intersect the circle at D and E respectively and $\angle BAC = 45°$, find $\dfrac{\text{area of } \Delta ADE}{\text{area of } BCED}$.

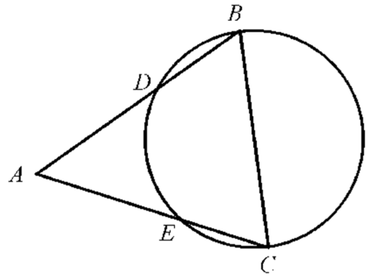
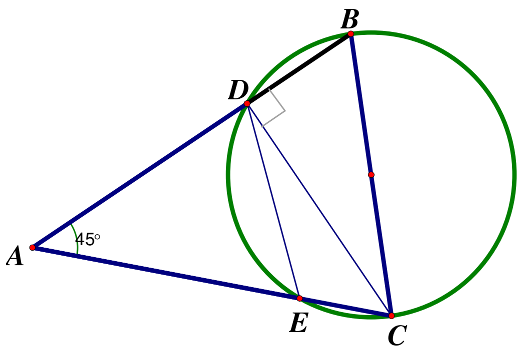


Figure 2



In ΔACD, $\angle BAC = 45°$ (given)

$\angle ADC = 90°$ (adj. ∠ on st. line, ∠ in semi-circle)

$\therefore \dfrac{AD}{AC} = \sin 45° = \dfrac{1}{\sqrt{2}}$ …… (1)

$\angle ADE = \angle ACB$ (ext. ∠ cyclic quad.)

$\angle AED = \angle ABC$ (ext. ∠ cyclic quad.)

$\angle DAE = \angle CAB$ (common ∠)

$\therefore \Delta ADE \sim \Delta ACB$ (equiangular)

$\dfrac{AD}{AC} = \dfrac{AE}{AB}$ (corr. of sides, ~Δ's) …… (2)

$$\frac{\text{area of } \Delta ADE}{\text{area of } \Delta ABC} = \frac{\frac{1}{2} AD \cdot AE \sin 45^\circ}{\frac{1}{2} AC \cdot AB \sin 45^\circ} = \left(\frac{AD}{AC}\right)^2 \text{ by (2)}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} \text{ by (1)}$$

$\Rightarrow \dfrac{\text{area of } \Delta ADE}{\text{area of } BCED} = 1$

I8 Solve $\sqrt{31-\sqrt{31+x}}=x$.

$0<\sqrt{31-\sqrt{31+x}}<\sqrt{36}=6\Rightarrow 0<x<6$

Then $31-\sqrt{31+x}=x^2$

$\Rightarrow (31-x^2)^2=31+x$

$x^4-62x^2-x+31^2-31=0$

$(x^4-x)-62x^2+930=0$

$(x^2-x)(x^2+x+1)-62x^2+930=0$

We want to factorise the above equation as $(x^2-x+a)(x^2+x+1+b)=0$

$a(x^2+x)+b(x^2-x)=-62x^2$ …… (1) and $a(1+b)=930$ …… (2)

From (1), $a+b=-62$ …… (3), $a-b=0$ …… (4)

$\therefore a=b=-31$ …… (5), sub. (5) into (2): L.H.S. $=-31(-30)=930=$ R.H.S.

$(x^2-x-31)(x^2+x+1-31)=0$

$x^2-x-31=0$ or $x^2+x-30=0$

$x=\frac{1-5\sqrt{5}}{2},\frac{1+5\sqrt{5}}{2}$, 5, –6.

$\frac{1-5\sqrt{5}}{2}<0$ and $\frac{1+5\sqrt{5}}{2}=\frac{1+\sqrt{125}}{2}>\frac{1+\sqrt{121}}{2}=\frac{1+11}{2}=6$

$\because 0<x<6\ \therefore x=5$ only.

Method 2

Let $\sqrt{31+\sqrt{31-y}}=y$

$\Rightarrow (y^2-31)^2=31-y$, then clearly $y>x$

and $y=\sqrt{31+\sqrt{31-y}}>\sqrt{30.25}=5.5$

$y^4-62y^2+y+930=0$ …… (2)

(2) – (1): $y^4-x^4-62(y^2-x^2)+y+x=0$

$(y+x)(y-x)(y^2+x^2)-62(y+x)(y-x)+(y+x)=0$

$(y+x)[(y-x)(y^2+x^2)-62(y-x)+1]=0$

$\because y+x\neq 0\ \therefore (y-x)(y^2+x^2-62)+1=0$ …… (3)

Assume x and y are positive integers. Then (3) becomes

$(y-x)(y^2+x^2-62)=-1\Rightarrow y-x=1$ …… (4) and $y^2+x^2-62=-1$ …… (5)

From (4), $y=x+1$ …… (6)

Sub. (6) into (5): $(x+1)^2+x^2-62=-1$

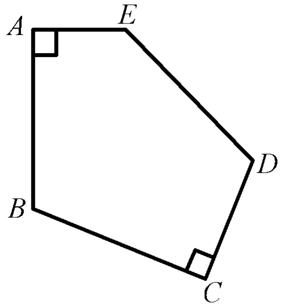
$x^2+2x+1+x^2-61=0$

$2x^2+2x-60=0$

$x^2+x-30=0$

$\Rightarrow x=5$ or -6 (rejected)

I9 Figure 3 shows a pentagon $ABCDE$. $AB = BC = DE = AE + CD = 3$ and $\angle A = \angle C = 90°$, find the area of the pentagon.



Draw the altitude $BN \perp DE$.

Let $AE = y$, $CD = 3 - y$

Cut ΔABE out and then stick the triangle to BC as shown in the figure.

$\Delta ABE \cong \Delta CBF$ (by construction)

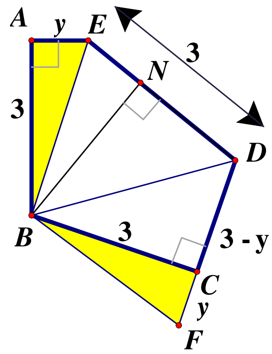
$CF = AE = y$ (corr. sides, $\cong \Delta$'s)

$\therefore DE = DF = 3$

$BD = BD$ (common side)

$BE = BF$ (corr. sides, $\cong \Delta$'s)

$\therefore \Delta BDE \cong \Delta BDF$ (S.S.S.)



$\therefore$ The area of the pentagon = area of ΔBDE + area of ΔBDF

$= 2\times$ area of ΔBDF

$= 2\times\frac{1}{2}\times 3\times 3$

$= 9$ sq. units

I10 If a and b are real numbers, and $a^2 + b^2 = a + b$. Find the maximum value of $a + b$.

$a^2 + b^2 = a + b \Rightarrow \left(a-\frac{1}{2}\right)^2+\left(b-\frac{1}{2}\right)^2=\frac{1}{2}$ …… (1)

$\left[\left(a-\frac{1}{2}\right)-\left(b-\frac{1}{2}\right)\right]^2 \ge 0$

$\Rightarrow \left(a-\frac{1}{2}\right)^2+\left(b-\frac{1}{2}\right)^2-2\left(a-\frac{1}{2}\right)\left(b-\frac{1}{2}\right)\ge 0$

By (1), $\frac{1}{2}-2\left(a-\frac{1}{2}\right)\left(b-\frac{1}{2}\right)\ge 0$

$\Rightarrow \frac{1}{2}\ge 2\left(a-\frac{1}{2}\right)\left(b-\frac{1}{2}\right)$ …… (2)

$(a + b - 1)^2 = \left[\left(a-\frac{1}{2}\right)+\left(b-\frac{1}{2}\right)\right]^2$

$=\left(a-\frac{1}{2}\right)^2+\left(b-\frac{1}{2}\right)^2+2\left(a-\frac{1}{2}\right)\left(b-\frac{1}{2}\right)$

$\le\frac{1}{2}+\frac{1}{2}= 1$ (by (1) and (2))

$\therefore (a + b - 1)^2 \le 1$

$\Rightarrow -1 \le a + b - 1 \le 1$

$\Rightarrow 0 \le a + b \le 2$

The maximum value of $a + b = 2$.

Group Events

G1 Given that the length of the sides of a right-angled triangle are integers, and two of them are the roots of the equation $x^2 - (m + 2)x + 4m = 0$. Find the maximum length of the third side of the triangle. **Reference: 2000 FI5.2, 2001 FI2.1, 2010 FI2.2, 2011 FI3.1**

Let the 3 sides of the right-angled triangle be a, b and c.

If a, b are the roots of the quadratic equation, then $a + b = m + 2$ and $ab = 4m$

$4a + 4b = 4m + 8 = ab + 8$

$4a - ab + 4b = 8$

$a(4 - b) - 16 + 4b = -8$

$a(4 - b) - 4(4 - b) = -8$

$(a - 4)(4 - b) = -8$

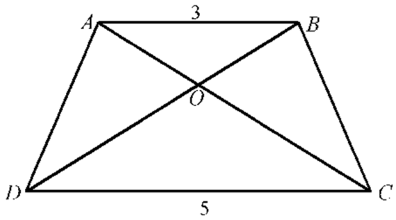
$(a - 4)(b - 4) = 8$

$a - 4$	$b - 4$	a	b	c
1	8	5	12	13
2	4	6	8	10

$\therefore$ The maximum value of the third side is 13.

G2 Figure 1 shows a trapezium $ABCD$, where $AB = 3$, $CD = 5$ and the diagonals AC and BD meet at O. If the area of ΔAOB is 27, find the area of the trapezium $ABCD$.

Reference 1993 HI2, 1997 HG3, 2000 FI2.2, 2002 FI1.3, 2004 HG7, 2010 HG4



$AB \parallel DC$

$\therefore \Delta AOB \sim \Delta COD$ (equiangular)

$$\frac{\text{Area of } \Delta COD}{\text{Area of } \Delta AOB} = \left(\frac{5}{3}\right)^2 \Rightarrow \frac{\text{Area of } \Delta COD}{27} = \frac{25}{9}$$

$\Rightarrow$ Area of $\Delta COD = 75$

$$\frac{\text{Area of } \Delta AOD}{\text{Area of } \Delta AOB} = \frac{DO}{OB} \Rightarrow \frac{\text{Area of } \Delta AOD}{27} = \frac{5}{3}$$

$\Rightarrow$ Area of $\Delta AOD = 45$

$$\frac{\text{Area of } \Delta BOC}{\text{Area of } \Delta AOB} = \frac{CO}{OA} \Rightarrow \frac{\text{Area of } \Delta BOC}{27} = \frac{5}{3}$$

$\Rightarrow$ Area of $\Delta BOC = 45$

The area of the trapezium $ABCD = 27 + 75 + 45 + 45 = 192$

G3 Let x and y be real numbers such that $x^2 + xy + y^2 = 2013$.

Find the maximum value of $x^2 - xy + y^2$.

$2013 = x^2 + xy + y^2 = (x + y)^2 - xy$

$\Rightarrow xy = (x + y)^2 - 2013 \ge 0 - 2013 = -2013$ …… (*)

Let $T = x^2 - xy + y^2$

$2013 = x^2 + xy + y^2 = (x^2 - xy + y^2) + 2xy = T + 2xy$

$2xy = 2013 - T \ge -2013\times 2$ by (*)

$\therefore 2013\times 3 \ge T$

The maximum value of $x^2 - xy + y^2$ is 6039.

G4 If α, β are roots of $x^2 + 2013x + 5 = 0$, find the value of $(\alpha^2 + 2011\alpha + 3)(\beta^2 + 2015\beta + 7)$.

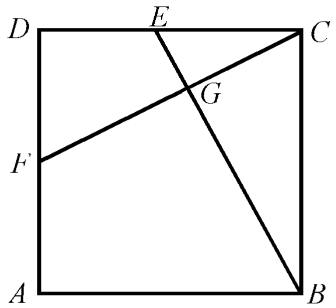
Reference: 1993 HG2, 2010 HI2, 2019 HI9

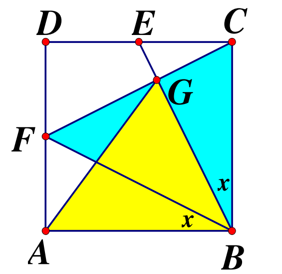
$\alpha^2 + 2013\alpha + 5 = 0 \Rightarrow \alpha^2 + 2011\alpha + 3 = -2\alpha - 2$

$\beta^2 + 2013\beta + 5 = 0 \Rightarrow \beta^2 + 2015\beta + 7 = 2\beta + 2$

$$\begin{aligned}(\alpha^2 + 2011\alpha + 3)(\beta^2 + 2015\beta + 7) &= (-2\alpha - 2)(2\beta + 2)\\ &= -4(\alpha + 1)(\beta + 1)\\ &= -4(\alpha\beta + \alpha + \beta + 1)\\ &= -4(5 - 2013 + 1) = 8028\end{aligned}$$

G5 As shown in Figure 2, $ABCD$ is a square of side 10 units, E and F are the mid-points of CD and AD respectively, BE and FC intersect at G. Find the length of AG.





Join BF and AG.

$CE = DE = DF = FA = 5$

Clearly $\Delta BCE \cong \Delta CDF \cong \Delta BAF$ (S.A.S.)

Let $\angle CBG = x = \angle ABF$ (corr. $\angle$s $\cong \Delta$'s)

$\angle BCE = 90°$

$\angle BCG = 90° - x$

$\angle BGC = 90°$ ($\angle$s sum of Δ)

Consider ΔABG and ΔFBC.

$\frac{AB}{BF} = \cos x$

$\angle ABG = x + \angle FBG = \angle FBC$

$\frac{GB}{BC} = \cos x$

$\therefore$ $\Delta ABG \sim \Delta FBC$ (ratio of 2 sides, included angle)

$\therefore$ $\frac{AB}{FB} = \frac{AG}{FC}$ (corr. sides, ~Δ's)

$\because$ $FB = FC$ (corr. sides, $\Delta CDF \cong \Delta BAF$)

$\therefore$ $AG = AB = 10$

Method 2 Define a rectangle coordinate system with A = origin, AB = positive x-axis, AD = positive y-axis.

$B = (10, 0)$, $C = (10, 10)$, $E = (5, 10)$, $D = (0, 10)$, $F = (0, 5)$

Equation of CF: $y - 5 = \frac{10-5}{10-0}(x-0) \Rightarrow y = \frac{1}{2}x + 5$ …… (1)

Equation of BE: $y - 0 = \frac{10-0}{5-10}(x-10) \Rightarrow y = -2x + 20$ …… (2)

(1) = (2): $\frac{1}{2}x + 5 = -2x + 20 \Rightarrow x = 6$

Sub. $x = 6$ into (1): $y = 8$

$\Rightarrow AG = \sqrt{6^2 + 8^2} = 10$

G6 Let a and b are positive real numbers, and the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ have real roots. Find the minimum value of $a + b$. (**Reference: 1999 FG5.2**)

Discriminants of the two equations ≥ 0

$a^2 - 8b \ge 0$ …… (1)

$(2b)^2 - 4a \ge 0 \Rightarrow b^2 - a \ge 0$ …… (2)

$a^2 \ge 8b \Rightarrow a^4 \ge 64b^2 \ge 64a$

$\Rightarrow a^4 - 64a \ge 0$

$\Rightarrow a(a - 4)(a^2 + 4a + 16) \ge 0$

$\Rightarrow a(a - 4)[(a + 2)^2 + 12] \ge 0$

$\Rightarrow a(a - 4) \ge 0$

$\Rightarrow a \le 0$ or $a \ge 4$

$\because$ $a > 0$ $\therefore$ $a \ge 4$ only

When $a = 4$, sub. into (2): $b^2 - 4 \ge 0$

$\Rightarrow (b + 2)(b - 2) \ge 0$

$\Rightarrow b \le -2$ or $b \ge 2$

$\because$ $b > 0$ $\therefore$ $b \ge 2$ only

The minimum value of $a + b = 4 + 2 = 6$

G7 Given that the length of the three sides of ΔABC form an arithmetic sequence, and are the roots of the equation $x^3 - 12x^2 + 47x - 60 = 0$, find the area of ΔABC.

Let the roots be $a - d$, a and $a + d$.

$a - d + a + a + d = 12 \Rightarrow a = 4$ …… (1)

$(a - d)a + a(a + d) + (a - d)(a + d) = 47$

$\Rightarrow 3a^2 - d^2 = 47 \Rightarrow d = \pm 1$ …… (2)

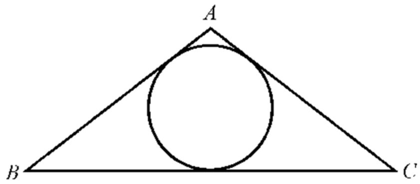
$(a - d)a(a + d) = 60 \Rightarrow a^3 - ad^2 = 60$ …… (3)

Sub. (1) and (2) into (3): L.H.S. = 64 – 4 = 60 = R.H.S.

$\therefore$ The 3 sides of the triangle are 3, 4 and 5.

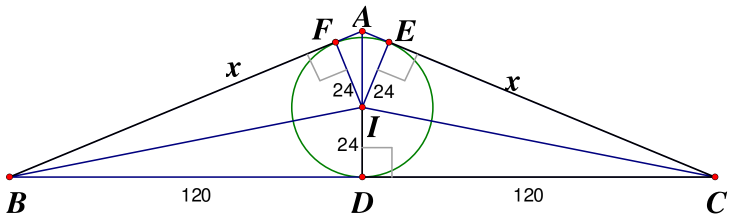
The area of $\Delta ABC = \frac{1}{2}\cdot 3\cdot 4 = 6$ sq. units.

G8 In Figure 3, ΔABC is an isosceles triangle with $AB = AC$, $BC = 240$. The radius of the inscribed circle of ΔABC is 24. Find the length of AB. **Reference 2007 FG4.4, 2022 P1Q15**



Let I be the centre. The inscribed circle touches AB and CA at F and E respectively. Let $AB = AC = x$.

Let D be the mid-point of BC.



$\Delta ABD \cong \Delta ACD$ (S.S.S.)

$\angle ADB = \angle ADC = 90°$

corr. $\angle$s, $\cong$ Δ's, adj. $\angle$s on st. line

BC touches the circle at D

(converse, tangent $\perp$ radius)

$ID = IE = IF$ = radii = 24

$IE\perp AC$, $IF\perp AB$ (tangent$\perp$ radius)

$AD = \sqrt{x^2 - 120^2}$ (Pythagoras' theorem)

$S_{\Delta ABC} = S_{\Delta IBC} + S_{\Delta ICA} + S_{\Delta IAB}$ (where S stands for areas)

$\frac{1}{2}\cdot 240\cdot\sqrt{x^2 - 120^2} = \frac{1}{2}\cdot 240\cdot 24 + \frac{1}{2}\cdot x\cdot 24 + \frac{1}{2}\cdot x\cdot 24$

$\frac{1}{2}\cdot 10\cdot\sqrt{x^2 - 120^2} = \frac{1}{2}\cdot 240 + \frac{1}{2}\cdot x + \frac{1}{2}\cdot x$

$5\sqrt{(x-120)(x+120)} = \sqrt{(120+x)}^2$

$5\sqrt{x-120} = \sqrt{120+x}$

$25(x - 120) = 120 + x$

$24x = 25\times 120 + 120 = 26\times 120$

$AB = x = 26\times 5 = 130$

G9 At most how many numbers can be taken from the set of integers: 1, 2, 3, … , 2012, 2013 such that the sum of any two numbers taken out from the set is not a multiple of the difference between the two numbers?

In order to understand the problem, let us take out a few numbers and investigate the property.

Take 1, 5, 9.

$1 + 5 = 6$, $5 - 1 = 4$, $6 \neq 4k$, for any integer k

$1 + 9 = 10$, $9 - 1 = 8$, $10 \neq 8k$, for any integer k

$5 + 9 = 14$, $9 - 5 = 4$, $14 \neq 4k$, for any integer k

Take 3, 6, 8.

$3 + 6 = 9$, $6 - 3 = 3$, $9 = 3\times 3$

$3 + 8 = 11$, $8 - 3 = 5$, $11 \neq 5k$ for any integer k

$6 + 8 = 14$, $8 - 6 = 2$, $14 = 2\times 7$

Take 12, 28, 40.

$12 + 28 = 40$, $28 - 12 = 16$, $40 \neq 16k$ for any integer k

$28 + 40 = 68$, $40 - 28 = 12$, $68 \neq 12k$ for any integer k

$12 + 40 = 52$, $40 - 12 = 28$, $52 \neq 28k$ for any integer k

$\therefore$ We can take three numbers 1, 5, 9 or 12, 28, 40 (but not 3, 6, 8).

Take the arithmetic sequence 1, 3, 5, … , 2013. (1007 numbers)

The general term $= T(n) = 2n - 1$ for $1 \leq n \leq 1007$

$T(n) + T(m) = 2n + 2m - 2 = 2(n + m - 1)$

$T(n) - T(m) = 2n - 2m = 2(n - m)$

$T(n) + T(m) = [T(n) - T(m)]k$ for some integer k. For example, $3 + 5 = 8 = (5 - 3)\times 4$.

$\therefore$ The sequence 1, 3, 5, … , 2013 does not satisfy the condition.

Take the arithmetic sequence 1, 4, 7, … , 2011. (671 numbers)

The general term $= T(n) = 3n - 2$ for $1 \leq n \leq 671$

$T(n) + T(m) = 3n + 3m - 4 = 3(n + m - 1) - 1$

$T(n) - T(m) = 3n - 3m = 3(n - m) \Rightarrow T(n) + T(m) \neq [T(n) - T(m)]k$ for any non-zero integer k

We can take at most 671 numbers to satisfy the condition.

G10 For all positive integers n, define a function f as

(i) $f(1) = 2012$,

(ii) $f(1) + f(2) + \cdots + f(n-1) + f(n) = n^2 f(n)$，$n > 1$.

Find the value of f(2012) .

Reference: 2014 FG1.4, 2022 P2Q8

$f(1) + f(2) + \cdots + f(n-1) = (n^2 - 1) f(n) \Rightarrow f(n) = \dfrac{f(1)+f(2)+\cdots+f(n-1)}{n^2-1}$

$f(2) = \dfrac{f(1)}{3} = \dfrac{2012}{3}$

$f(3) = \dfrac{f(1)+f(2)}{8} = \dfrac{2012+\frac{2012}{3}}{8} = \dfrac{1+\frac{1}{3}}{8}\cdot 2012 = \dfrac{1}{6}\cdot 2012$

$f(4) = \dfrac{f(1)+f(2)+f(3)}{15} = \dfrac{2012+\frac{2012}{3}+\frac{2012}{6}}{15} = \dfrac{\frac{3}{2}}{15}\cdot 2012 = \dfrac{1}{10}\cdot 2012$

It is observed that the answer is 2012 divided by the n^{th} triangle number.

Claim: $f(n) = \dfrac{2}{n(n+1)}\cdot 2012$ for $n \ge 1$

$n = 1, 2, 3, 4$, proved above.

Suppose $f(k) = \dfrac{2}{k(k+1)}\cdot 2012$ for $k = 1, 2, \cdots, m$ for some positive integer m .

$$f(m+1) = \frac{f(1)+f(2)+\cdots+f(m)}{(m+1)^2-1} = \frac{\frac{2}{1\times 2}+\frac{2}{2\times 3}+\frac{2}{3\times 4}+\frac{2}{4\times 5}+\frac{2}{5\times 6}+\cdots+\frac{2}{m(m+1)}}{m(m+2)}\cdot 2012$$

$$= 2\cdot\frac{\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{m}-\frac{1}{m+1}\right)}{m(m+2)}\cdot 2012$$

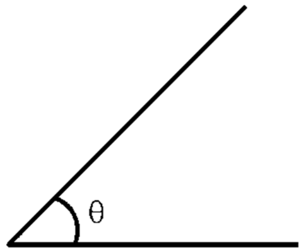
$$= 2\cdot\frac{1-\frac{1}{m+1}}{m(m+2)}\cdot 2012 = \frac{2}{(m+1)(m+2)}\cdot 2012$$

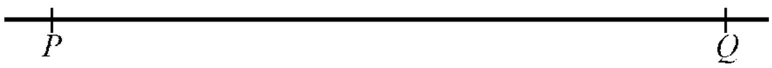
$\therefore$ It is also true for m. By the principle of mathematical induction, the formula is true.

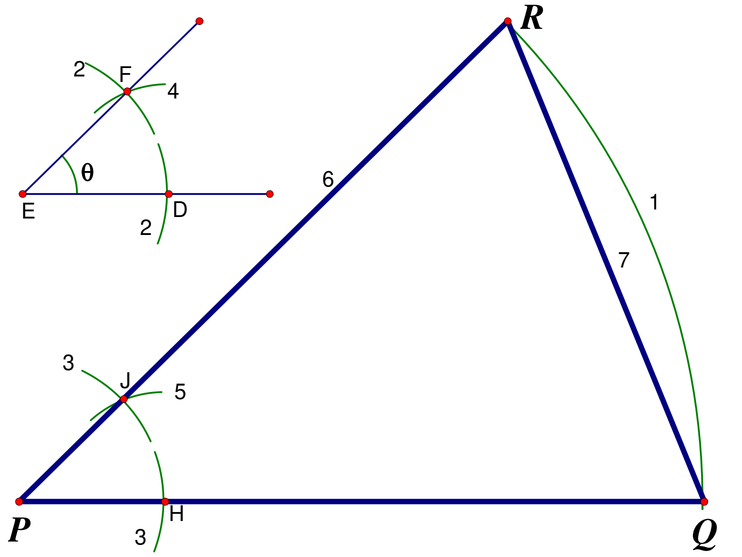
$f(2012) = \dfrac{2}{2012\times 2013}\cdot 2012 = \dfrac{2}{2013}$

Geometrical Construction

1. Line segment PQ and an angle of size θ are given below. Construct the isosceles triangle PQR with $PQ = PR$ and $\angle QPR = \theta$.





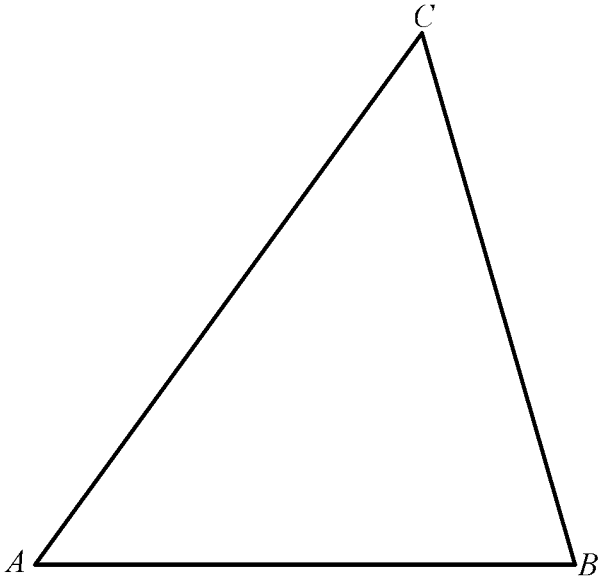


Steps. Let the vertex of the given angle be E.

1. Use P as centre, PQ as radius to draw a circular arc QR.
2. Use E as centre, a certain radius to draw an arc, cutting the given angle at D and F respectively.
3. Use P as centre, the same radius in step 2 to draw an arc, cutting PQ a H.
4. Use D as centre, DF as radius to draw an arc.
5. Use H as centre, DF as radius to draw an arc, cutting the arc in step 3 at J.
6. Join PJ, and extend PJ to cut the arc in step 1 at R.
7. Join QR.

ΔPQR is the required triangle.

2. Construct a rectangle with AB as one of its sides and with area equal to that of ΔABC below.



Theory

Let the height of the rectangle be h.

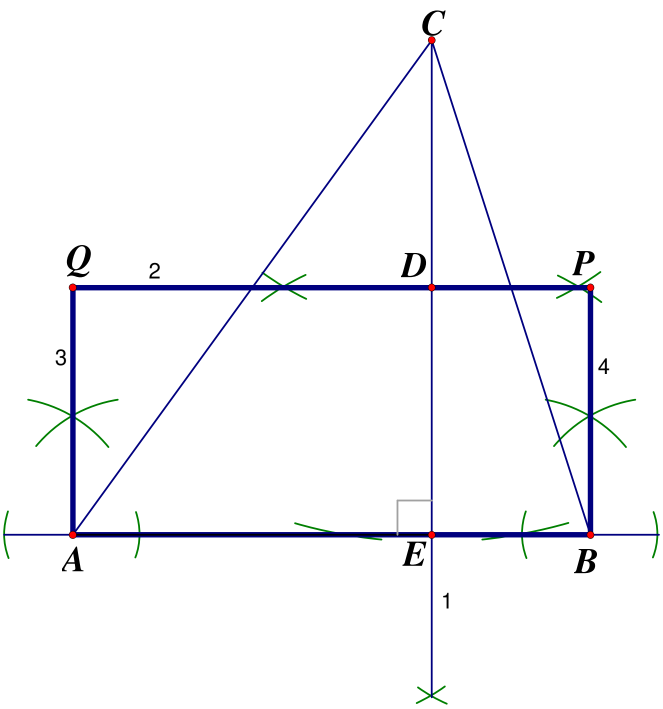
Let the height of the triangle be k.

$\because$ Area of rectangle = area of triangle

$$AB \times h = \frac{1}{2} AB \times k$$

$$h = \frac{1}{2} k$$

$\therefore$ The height of rectangle is half of the height of the triangle.

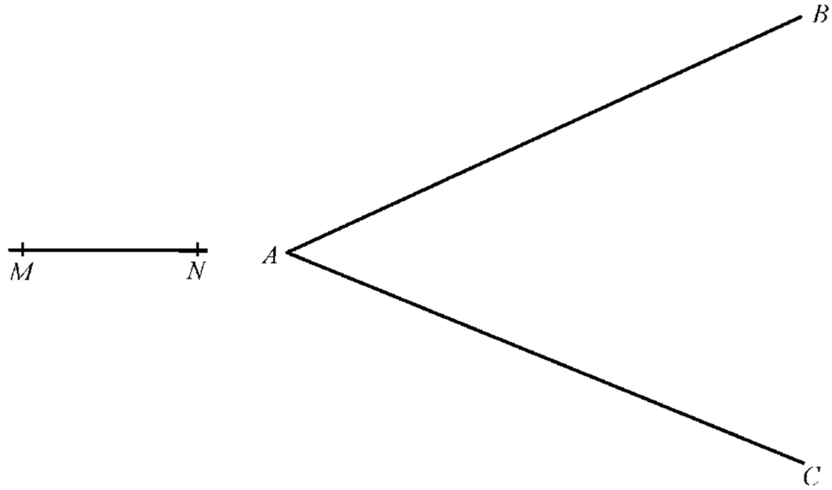


Steps.

1. Draw a line segment $CE \perp AB$. (E lies on AB, CE is the altitude of ΔABC)
2. Draw the perpendicular bisector PQ of CE, D is the mid-point of CE.
3. Draw a line segment $AQ \perp AB$, cutting PQ and Q.
4. Draw a line segment $BP \perp AB$, cutting PQ and P.

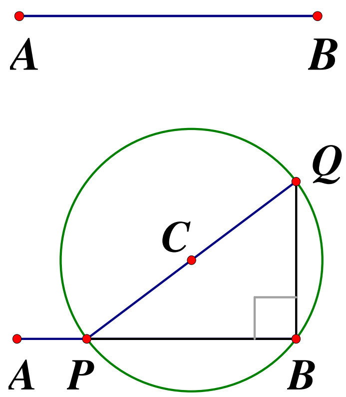
$ABPQ$ is the required rectangle.

3. The figure below shows two straight lines AB and AC intersecting at the point A. Construct a circle with radius equal to the line segment MN so that AB and AC are tangents to the circle.



Lemma:

如圖，已給一綫段 AB，過 $\mathbf{B}$ 作一綫段垂直於 $\mathbf{AB}$。



作圖方法如下：

(1) 取任意點 C(C 在 AB 之間的上方)為圓心，CB 為半徑作一圓，交 AB 於 P。

(2) 連接 PC，其延長綫交圓於 Q；連接 BQ。

BQ 為所求的垂直綫。

作圖完畢。

證明如下：

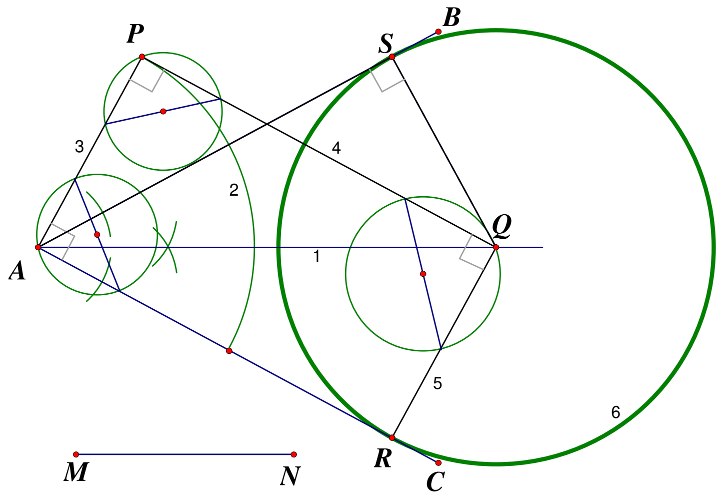
PCQ 為圓之直徑 (由作圖所得)

$\angle PBQ = 90°$ (半圓上的圓周角)

證明完畢。

Steps.

1. Draw the angle bisector AQ
2. Use A as centre, MN as radius to draw an arc.
3. Use the lemma to draw $AP \perp AC$, AP cuts the arc in step 2 at P.
4. Draw $PQ \perp AP$, PQ cuts the angle bisector at Q.
5. Draw $QR \perp PQ$, QR cuts AC at R.
6. Use Q as centre, QR as radius to draw a circle.



This is the required circle.

Proof:

$\angle ARQ = 90°$ ($\angle$s sum of polygon)

$APQR$ is a rectangle.

AC is a tangent touching the circle at R (converse, tangent $\perp$ radius)

Let S be the foot of perpendicular drawn from Q onto AB, $QS \perp AB$.

$\Delta AQR \cong \Delta AQS$ (A.A.S.)

$\therefore$ $SQ = SR$ (corr. sides, $\cong \Delta$'s)

S lies on the circle and $QS \perp AB$

$\therefore$ AB is a tangent touching the circle at S (converse, tangent $\perp$ radius)