# Examples on Mathematical Induction: Recurrence sequence

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1. $\begin{cases} u_1 = u_2 = 1 \\ u_{n+1} = u_n + u_{n-1}\ (n \ge 2) \end{cases}$.

Prove by mathematical induction that $u_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right)$, where $\alpha > \beta$ are roots of $x^2 - x - 1 = 0$

Let $P(n) \equiv$ '$u_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right)$ for all positive integer $n$.'

$\alpha + \beta = 1$, $\alpha\beta = -1$

$u_1 = 1$, $\frac{1}{\sqrt{5}}(\alpha - \beta) = \frac{1}{\sqrt{5}}\left[\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\right]$

$= \frac{1}{\sqrt{5}}\left[\sqrt{1^2 - 4(-1)}\right] = 1$

It is true for $n = 1$

$u_2 = 1$, $\frac{1}{\sqrt{5}}\left(\alpha^2 - \beta^2\right) = \frac{1}{\sqrt{5}}(\alpha+\beta)(\alpha-\beta)$

$= \frac{1}{\sqrt{5}}\left[\sqrt{1^2 - 4(-1)}\right] = 1$

It is true for $n = 2$

Suppose $u_k = \frac{1}{\sqrt{5}}\left(\alpha^k - \beta^k\right)$ and $u_{k+1} = \frac{1}{\sqrt{5}}\left(\alpha^{k+1} - \beta^{k+1}\right)$ for some integer $k > 0$.

$u_{k+2} = u_k + u_{k+1}$

$= \frac{1}{\sqrt{5}}\left(\alpha^k - \beta^k\right) + \frac{1}{\sqrt{5}}\left(\alpha^{k+1} - \beta^{k+1}\right)$

$= \frac{1}{\sqrt{5}}\left(\alpha^k - \beta^k + \alpha^{k+1} - \beta^{k+1}\right)$

$= \frac{1}{\sqrt{5}}\left[\alpha^k(\alpha+1) - \beta^k(\beta+1)\right]$

$= \frac{1}{\sqrt{5}}\left[\alpha^k\left(\alpha^2\right) - \beta^k\left(\beta^2\right)\right]$ ($\because \alpha, \beta$ are roots of $x^2 - x - 1 = 0$, $\alpha^2 - \alpha - 1 = 0$, $\beta^2 - \beta - 1 = 0$)

$= \frac{1}{\sqrt{5}}\left[\alpha^{k+2} - \beta^{k+2}\right]$

$\therefore$ If $P(k)$ and $P(k+1)$ is true then $P(k+2)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive $n$.

2. **1980 Paper 2 Q10**

Let $\alpha$, $\beta$ be the roots of $x^2 - 2x - 1 = 0$, where $\alpha > \beta$ .

For any positive integer $n$, let $U_n = \frac{1}{2\sqrt{2}}\left(\alpha^n - \beta^n\right)$, $V_n = \frac{1}{2\sqrt{2}}\left(\alpha^n + \beta^n\right)$.

(a) Show that
$U_{n+2} = 2U_{n+1} + U_n$ ,
$V_{n+2} = 2V_{n+1} + V_n$ .

(b) (i) Find $U_1$ and $U_2$ .
(ii) Suppose $U_n$ and $U_{n+1}$ are integers, deduce that $U_{n+2}$ is also an integer.
(iii) Is $U_n$ an integer for all positive integers $n$? Give reasons.

(c) Is $V_n$ an integer for all positive integers $n$? Give reasons.

(a)
$$\begin{aligned}2U_{n+1} + U_n &= 2\left[\frac{1}{2\sqrt{2}}\left(\alpha^{n+1} - \beta^{n+1}\right)\right] + \frac{1}{2\sqrt{2}}\left(\alpha^n - \beta^n\right)\\ &= \frac{1}{2\sqrt{2}}\left[2\left(\alpha^{n+1} - \beta^{n+1}\right) + \left(\alpha^n - \beta^n\right)\right]\\ &= \frac{1}{2\sqrt{2}}\left[2\alpha^{n+1} + \alpha^n - \left(2\beta^{n+1} + \beta^n\right)\right]\\ &= \frac{1}{2\sqrt{2}}\left(\alpha^{n+2} - \beta^{n+2}\right) = U_{n+2}\end{aligned}$$

$$\begin{aligned}2V_{n+1} + V_n &= 2\left[\frac{1}{2\sqrt{2}}\left(\alpha^{n+1} + \beta^{n+1}\right)\right] + \frac{1}{2\sqrt{2}}\left(\alpha^n + \beta^n\right)\\ &= \frac{1}{2\sqrt{2}}\left[2\left(\alpha^{n+1} + \beta^{n+1}\right) + \left(\alpha^n + \beta^n\right)\right]\\ &= \frac{1}{2\sqrt{2}}\left[2\alpha^{n+1} + \alpha^n + \left(2\beta^{n+1} + \beta^n\right)\right]\\ &= \frac{1}{2\sqrt{2}}\left(\alpha^{n+2} + \beta^{n+2}\right) = V_{n+2}\end{aligned}$$

(b) (i) $\alpha + \beta = 2$, $\alpha\beta = -1$
$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 2^2 - 4(-1) = 8$
$\alpha - \beta = \sqrt{8} = 2\sqrt{2}$ ($\because \alpha > \beta$)

$$U_1 = \frac{1}{2\sqrt{2}}(\alpha - \beta) = \frac{1}{2\sqrt{2}}\cdot\left(2\sqrt{2}\right) = 1$$

$$\begin{aligned}U_2 &= \frac{1}{2\sqrt{2}}\left(\alpha^2 - \beta^2\right)\\ &= \frac{1}{2\sqrt{2}}\cdot(\alpha + \beta)(\alpha - \beta) = \frac{1}{2\sqrt{2}}\cdot(2)\left(2\sqrt{2}\right) = 2\end{aligned}$$

(ii) Suppose $U_n$ and $U_{n+1}$ are integers
$U_{n+2} = 2U_{n+1} + U_n$ by (a)
which is a sum of two integers
$\therefore$ $U_{n+2}$ is also an integer

(iii) By (b)(i), $U_1 = 1$, $U_2 = 2$, both of them are integers
Also by (b)(ii), if $U_n$ and $U_{n+1}$ are integers, then $U_{n+2}$ is an integer
By the principle of induction, $U_n$ an integer for all positive integers $n$

(c) $V_1 = \frac{1}{2\sqrt{2}}(\alpha + \beta) = \frac{1}{2\sqrt{2}}(2) = \frac{\sqrt{2}}{2}$ , which is not an integer

$V_n$ is not an integer for some positive integer $n$ (e.g. $n = 1$ .)

3. If $u_1 = 3$, $u_2 = 5$ and $u_{k+2} = 3u_{k+1} - 2u_k$ for $k \ge 1$. Prove that $u_n = 2^n + 1$.
$a_1 = 3$, R.H.S. $= 2^1 + 1 = 3$, It is true for $n = 1$.
$a_2 = 5$, R.H.S. $= 2^2 + 1 = 5$, It is true for $n = 2$.
Suppose $u_k = 2^k + 1$, $u_{k+1} = 2^{k+1} + 1$ for some positive integer $k$.
$$\begin{aligned}u_{k+2} &= 3u_{k+1} - 2u_k = 3(2^{k+1} + 1) - 2(2^k + 1)\\ &= 3\cdot 2^{k+1} + 3 - 2^{k+1} - 2\\ &= 2\cdot 2^{k+1} + 1\\ &= 2^{k+2} + 1 = \text{R.H.S.}\end{aligned}$$
$\therefore$ If it is true for $n = k$ and $n = k + 1$, then it is also true for $n = k + 2$.
By the principle of mathematical induction, it is true for all positive $n$.

4. If $u_1 = 1$, $u_2 = 2$ and $u_{n+2} = u_{n+1} + u_n$ for $n \ge 1$.
Prove that $u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$ for all positive integer $n$.

5. If $u_1 = 1$, $u_2 = 2$ and $u_{n+2} = u_{n+1} + u_n$ for $n \ge 1$.
Prove that $u_1 + u_2 + \cdots + u_n = u_{n+2} - 2$ for all positive integer $n$.
Let $P(n) \equiv$ "$u_1 + u_2 + \cdots + u_n = u_{n+2} - 2$ for all positive integer $n$."
$n = 1$, L.H.S. $= u_1 = 1$, R.H.S. $= u_3 - 2 = u_1 + u_2 - 2 = 1 + 2 - 2 = 1$, $P(1)$ is true.
Suppose $u_1 + u_2 + \cdots + u_k = u_{k+2} - 2$ for some positive integer $k$.
$$\begin{aligned}u_1 + u_2 + \cdots + u_k + u_{k+1} &= u_{k+1} + u_{k+2} - 2\\ &= u_{k+3} - 2\end{aligned}$$
If $P(k)$ is true, then $P(k + 1)$ is also true.
By the principle of mathematical induction, $P(n)$ is true for all positive integer $n$.

6. If $u_1 = 1$, $u_2 = 2$ and $u_{n+2} = u_{n+1} + u_n$ for $n \ge 1$.
Prove that $u_1 + u_3 + u_5 + \cdots + u_{2n-1} = u_{2n} - 1$ for all positive integer $n$.
Let $P(n) \equiv$ "$u_1 + u_3 + u_5 + \cdots + u_{2n-1} = u_{2n} - 1$ for all positive integer $n$."
$n = 1$, L.H.S. $= u_1 = 1$, R.H.S. $= u_2 - 1 = 2 - 1 = 1$, $P(1)$ is true.
Suppose $u_1 + u_3 + u_5 + \cdots + u_{2k-1} = u_{2k} - 1$ for some positive integer $k$.
$$u_1 + u_3 + u_5 + \cdots + u_{2k-1} + u_{2k+1} = u_{2k} + u_{2k+1} - 1 = u_{2k+2} - 1$$
If $P(k)$ is true, then $P(k + 1)$ is also true.
By the principle of mathematical induction, $P(n)$ is true for all positive integer $n$.

7. If $u_1 = 1$, $u_2 = 2$ and $u_{n+2} = u_{n+1} + u_n$ for $n \ge 1$.
Prove that $u_2 + u_4 + u_6 + \cdots + u_{2n} = u_{2n+1} - 1$ for all positive integer $n$.
Let $P(n) \equiv$ "$u_2 + u_4 + u_6 + \cdots + u_{2n} = u_{2n+1} - 1$ for all positive integer $n$."
$n = 1$, L.H.S. $= u_2 = 2$, R.H.S. $= u_3 - 1 = 3 - 1 = 2$, $P(1)$ is true.
Suppose $u_2 + \cdots + u_{2k} = u_{2k+1} - 1$ for some positive integer $k$.
$$u_2 + \cdots + u_{2k} + u_{2k+2} = u_{2k+1} + u_{2k+2} - 1 = u_{2k+3} - 1$$
If $P(k)$ is true, then $P(k + 1)$ is also true.
By the principle of mathematical induction, $P(n)$ is true for all positive integer $n$.

8. If $u_1 = 1$, $u_2 = -1$ and $u_{n+2} = 2u_{n+1} + 4u_n$ for $n \ge 1$ .
Prove that $u_n = \frac{1}{5\cdot 2^n}\left[\left(5-3\sqrt{5}\right)\alpha^{n-1} + \left(5+3\sqrt{5}\right)\beta^{n-1}\right]$, where $\alpha > \beta$ are roots of $x^2 - 2x - 4 = 0$ .

9. If $u_1 = 1$, $u_2 = 2$ and $u_{n+2} = u_n + u_{n+1}$ for $n \ge 1$. Prove that $u_{n+1}^2 - u_n u_{n+2} = (-1)^{n-1}$.
$u_2^2 - u_1 u_3 = 2^2 - 1\times 3 = 1 = (-1)^0$, it is true for $n = 1$
Suppose $u_{k+1}^2 - u_k u_{k+2} = (-1)^{k-1}$ for some positive integer $k$.
$u_{k+2}^2 - u_{k+1} u_{k+3} = (u_{k+1} + u_k)^2 - u_{k+1} (u_{k+1} + u_{k+2})$
$= (u_{k+1} + u_k)^2 - u_{k+1} (2u_{k+1} + u_k)$
$= u_{k+1}^2 + 2 u_k u_{k+1} + u_k^2 - 2 u_{k+1}^2 - u_k u_{k+1}$
$= u_k^2 + u_k u_{k+1} - u_{k+1}^2$
$= u_k^2 + u_k u_{k+1} - u_k u_{k+2} - (-1)^{k-1}$ (by induction assumption)
$= u_k^2 + u_k u_{k+1} - u_k (u_{k+1} + u_k) + (-1)^k$
$= (-1)^k$
If it is true for $n = k$, then it is also true for $n = k + 1$ .
By the principle of mathematical induction, it is true for all positive integer $n$ .

10. If $n$ is a positive integer, prove that there exists one and only one set of integers $\{a_n, b_n\}$ such that $\left(\sqrt{3}+1\right)^n = a_n\sqrt{3}+b_n$. Hence prove that

(a) $a_{n+2} = 2(a_{n+1} + a_n)$, $b_{n+2} = 2(b_{n+1} + b_n)$

(b) $\left(\sqrt{3}-1\right)^n = (-1)^{n-1}\left(a_n\sqrt{3}-b_n\right)$

We first prove the uniqueness of $a_n$ and $b_n$.

Assume that $\left(\sqrt{3}+1\right)^n = a_n\sqrt{3}+b_n = u_n\sqrt{3}+v_n$, where $a_n$, $b_n$, $u_n$, $v_n$ are integers.

$\therefore\ (a_n-u_n)\sqrt{3} = v_n - b_n$

If $a_n \neq u_n$, then $\sqrt{3} = \dfrac{v_n-b_n}{a_n-u_n}$, which means that $\sqrt{3}$ is a rational number, contradiction.

$\therefore\ a_n = u_n$, $b_n = v_n$

Secondly, we prove $\left(\sqrt{3}+1\right)^n = a_n\sqrt{3}+b_n$ by mathematical induction.

$n = 1$, $\sqrt{3}+1 = a_n\sqrt{3}+b_n$, $a_1 = 1$, $b_1 = 1$, it is true for $n = 1$.

Assume that $\left(\sqrt{3}+1\right)^k = a_k\sqrt{3}+b_k$ where $a_k$, $b_k$ are integers for some positive integer $k$.

$$\begin{aligned}\left(\sqrt{3}+1\right)^{k+1} &= \left(a_k\sqrt{3}+b_k\right)\left(\sqrt{3}+1\right)\\ &= 3a_k+b_k+\left(a_k+b_k\right)\sqrt{3}\end{aligned}$$

$\therefore\ a_{k+1} = a_k + b_k$, $b_{k+1} = 3\,a_k + b_k$, which are integers.

If it is true for $n = k$, then it is also true for $n = k + 1$.

By the principle of mathematical induction, it is true for all positive integer $n$.

(a) $\because\ a_{k+1} = a_k + b_k$, $b_{k+1} = 3\,a_k + b_k$

$$\begin{aligned}a_{n+2} &= a_{n+1} + b_{n+1} = a_{n+1} + 3\,a_n + b_n\\ &= a_{n+1} + 2\,a_n + (a_n + b_n)\\ &= a_{n+1} + 2\,a_n + a_{n+1}\\ &= 2(a_{n+1} + a_n)\end{aligned}$$

$$\begin{aligned}b_{n+2} &= 3\,a_{n+1} + b_{n+1} = 3(a_n + b_n) + b_{n+1}\\ &= 3a_n + b_n + 2\,b_n + b_{n+1}\\ &= b_{n+1} + 2\,b_n + b_{n+1}\\ &= 2(b_{n+1} + b_n)\end{aligned}$$

(b) We prove the argument $\left(\sqrt{3}-1\right)^n = (-1)^{n-1}\left(a_n\sqrt{3}-b_n\right)$ by mathematical induction.

For $n = 1$, $\sqrt{3}-1 = (-1)^{1-1}\left(1\times\sqrt{3}-1\right)$

$a_1 = 1$, $b_1 = 1$, it is true for $n = 1$

Suppose $\left(\sqrt{3}-1\right)^k = (-1)^{k-1}\left(a_k\sqrt{3}-b_k\right)$ for some positive integer $k$.

$$\begin{aligned}\left(\sqrt{3}-1\right)^{k+1} &= (-1)^{k-1}\left(a_k\sqrt{3}-b_k\right)\left(\sqrt{3}-1\right)\\ &= (-1)^k\left(a_k\sqrt{3}-b_k\right)\left(1-\sqrt{3}\right)\\ &= (-1)^k\left(a_k\sqrt{3}-b_k-3a_k+\sqrt{3}b_k\right)\\ &= (-1)^k\left[\left(a_k+b_k\right)\sqrt{3}-\left(b_k+3a_k\right)\right]\\ &= (-1)^k\left(a_{k+1}\sqrt{3}-b_{k+1}\right)\end{aligned}$$

If it is true for $n = k$, then it is also true for $n = k + 1$.

By the principle of mathematical induction, it is true for all positive integer $n$ .

11. The numbers $x_0, x_1, \cdots, x_n, \cdots; y_0, y_1, \cdots, y_n, \cdots; a_0, a_1, \cdots, a_n, \cdots$ satisfy the following conditions:

$x_0 = a_0$, $x_1 = a_1 a_0 + 1$, $\cdots$ , $x_n = a_n x_{n-1} + x_{n-2}$, for $n \ge 2$ ,

$y_0 = 1$, $y_1 = a_1$, $\cdots$ , $y_n = a_n y_{n-1} + y_{n-2}$, for $n \ge 2$ .

Prove by induction that

(a) $x_n y_{n-1} - x_{n-1} y_n = (-1)^{n-1}$, $n \ge 1$

(b) $x_n y_{n-2} - x_{n-2} y_n = (-1)^n a_n$, $n \ge 2$

Deduce that $\dfrac{x_n}{y_n} = a_0 + \dfrac{1}{a_1} - \dfrac{1}{y_1 y_2} + \dfrac{1}{y_2 y_3} - \cdots + (-1)^{n-1}\dfrac{1}{y_{n-1} y_n}$.

(a) $n = 1$, $x_1 y_0 - x_0 y_1 = (a_1 a_0 + 1) - a_0 a_1 = 1 = (-1)^0$, it is true for $n = 1$

Suppose $x_k y_{k-1} - x_{k-1} y_k = (-1)^{k-1}$ for some positive integer $k$.

$$\begin{aligned} x_{k+1} y_k - x_k y_{k+1} &= (a_{k+1} x_k + x_{k-1}) y_k - x_k (a_{k+1} y_k + y_{k-1}) \\ &= -(x_k y_{k-1} - x_{k-1} y_k) \\ &= -(-1)^{k-1} \\ &= (-1)^k \end{aligned}$$

If it is true for $n = k$, then it is also true for $n = k + 1$.

By the principle of mathematical induction, it is true for all positive integer $n$.

(b) $n = 2$, $x_2 y_0 - x_0 y_2 = (a_2 x_1 + x_0) - a_0 (a_2 y_1 + y_0)$

$$\begin{aligned} &= a_2 (a_1 a_0 + 1) + a_0 - a_0 (a_2 a_1 + 1) \\ &= a_2 = (-1)^2 a_2 \end{aligned}$$

It is true for $n = 2$

Suppose $x_k y_{k-2} - x_{k-2} y_k = (-1)^k a_k$ for some positive integer $k \ge 2$.

$$\begin{aligned} x_{k+1} y_{k-1} - x_{k-1} y_{k+1} &= (a_{k+1} x_k + x_{k-1}) y_{k-1} - x_{k-1} (a_{k+1} y_k + y_{k-1}) \\ &= a_{k+1} x_k y_{k-1} - a_{k+1} x_{k-1} y_k \\ &= (x_k y_{k-1} - x_{k-1} y_k) a_{k+1} \\ &= (-1)^{k-1} a_{k+1} \quad \text{by } (a) \end{aligned}$$

If it is true for $n = k$, then it is also true for $n = k + 1$.

By the principle of mathematical induction, it is true for all positive integer $n$.

To prove $\dfrac{x_n}{y_n} = a_0 + \dfrac{1}{a_1} - \dfrac{1}{y_1 y_2} + \dfrac{1}{y_2 y_3} - \cdots + (-1)^{n-1}\dfrac{1}{y_{n-1} y_n}$

By (a), $x_n y_{n-1} - x_{n-1} y_n = (-1)^{n-1}$

$$\frac{x_n y_{n-1} - x_{n-1} y_n}{y_n y_{n-1}} = \frac{(-1)^{n-1}}{y_{n-1} y_n}$$

$$\frac{x_n}{y_n} - \frac{x_{n-1}}{y_{n-1}} = \frac{(-1)^{n-1}}{y_{n-1} y_n}$$

$$\frac{x_n}{y_n} = \frac{(-1)^{n-1}}{y_{n-1} y_n} + \frac{x_{n-1}}{y_{n-1}}$$

Inductively, $\dfrac{x_n}{y_n} = \dfrac{(-1)^{n-1}}{y_{n-1} y_n} + \dfrac{(-1)^{n-2}}{y_{n-2} y_{n-1}} + \dfrac{x_{n-2}}{y_{n-2}} = \ldots$

$$= \frac{(-1)^{n-1}}{y_{n-1} y_n} + \frac{(-1)^{n-2}}{y_{n-2} y_{n-1}} + \cdots + \frac{(-1)^2}{y_2 y_3} + \frac{(-1)^1}{y_1 y_2} + \frac{1}{y_0 y_1} + \frac{x_0}{y_0}$$

$\therefore \dfrac{x_n}{y_n} = a_0 + \dfrac{1}{a_1} - \dfrac{1}{y_1 y_2} + \dfrac{1}{y_2 y_3} - \cdots + (-1)^{n-1}\dfrac{1}{y_{n-1} y_n}$

12. The series of numbers 1 + 3 + 7 + 12 + 19 + 27 + 37 + 48 + 61 + ⋯ is defined as follows: $a_{2l} = 3l^2$, $a_{2l-1} = 3l(l-1) + 1$, where $l$ is a positive integer.

Prove that $s_{2l-1} = \frac{1}{2}\ell(4\ell^2 - 3\ell + 1)$, $s_{2l} = \frac{1}{2}\ell(4\ell^2 + 3\ell + 1)$ by mathematical induction.

$l = 1$, $s_1 = a_1 = 3\times 0 + 1 = 1$, RHS $= \frac{1}{2}(4 - 3 + 1) = 1$;

$s_2 = a_1 + a_2 = 1 + 3 = 4$, RHS $= \frac{1}{2}(4 + 3 + 1) = 4$.

∴ It is true for $n = 1$ .

Suppose it is true for $n = k$, i.e. $s_{2k-1} = \frac{1}{2}k(4k^2 - 3k + 1)$, $s_{2k} = \frac{1}{2}k(4k^2 + 3k + 1)$

When $n = k + 1$, $s_{2k+1} = s_{2k} + a_{2k+1} = \frac{1}{2}k(4k^2 + 3k + 1) + 3(k + 1)k + 1$

$$= \frac{1}{2}(4k^3 + 3k^2 + k + 6k^2 + 6k + 2)$$

$$= \frac{1}{2}(4k^3 + 9k^2 + 7k + 2)$$

R.H.S. $= \frac{1}{2}(k+1)\left[4(k+1)^2 - 3(k+1) + 1\right]$

$$= \frac{1}{2}(k+1)\left[4k^2 + 8k + 4 - 3k - 3 + 1\right]$$

$$= \frac{1}{2}(k+1)(4k^2 + 5k + 2)$$

$$= \frac{1}{2}(4k^3 + 9k^2 + 7k + 2)$$

$s_{2k+2} = s_{2k+1} + a_{2k+2} = \frac{1}{2}(k+1)(4k^2 + 5k + 2) + 3(k + 1)^2$

$$= \frac{1}{2}(k+1)(4k^2 + 5k + 2 + 6k + 6)$$

$$= \frac{1}{2}(k+1)(4k^2 + 11k + 8)$$

R.H.S. $= \frac{1}{2}(k+1)\left[4(k+1)^2 + 3(k+1) + 1\right] = \frac{1}{2}(k+1)(4k^2 + 11k + 8)$

If it is true for $n = k$, then it is also true for $n = k + 1$ .

By the principle of mathematical induction, it is true for all positive integer $n$ .