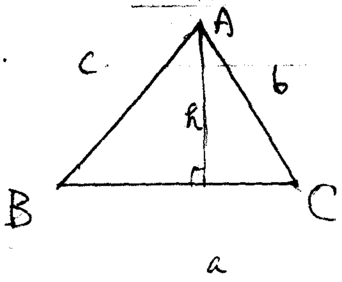
General notes to Trigonometry

Sine Rule and Cosine Rule

Rule

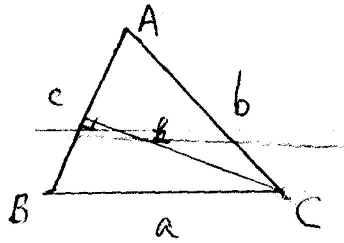
For any $\triangle ABC$ $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$, R = radius of circumcircle.

Proof: Method 1



$$h = c \sin B = b \sin C$$

$$\Rightarrow \quad \frac{b}{\sin B} = \frac{c}{\sin C} \qquad —(1)$$



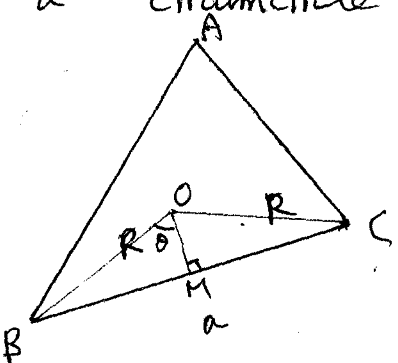
$$k = a \sin B = b \sin A$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} \qquad —(2)$$

Combining (1) and (2) we have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

This method applies to either acute angle or obtuse angle.

1.2 method 2. Draw a circumcircle of $\triangle ABC$



O = centre

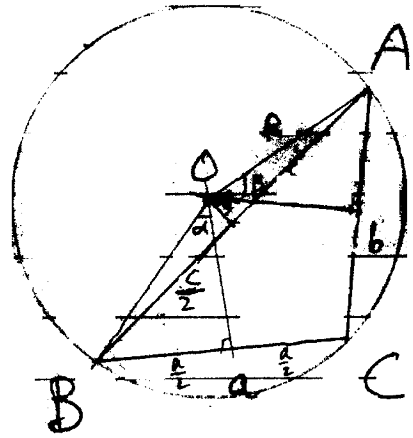
$$BM = MC = \frac{a}{2} = R \sin \theta$$
$$= R \sin \frac{2\theta}{2}$$
$$= R \sin A$$

$$\therefore \quad \frac{a}{\sin A} = 2R$$

Similarly $\dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

For the case:

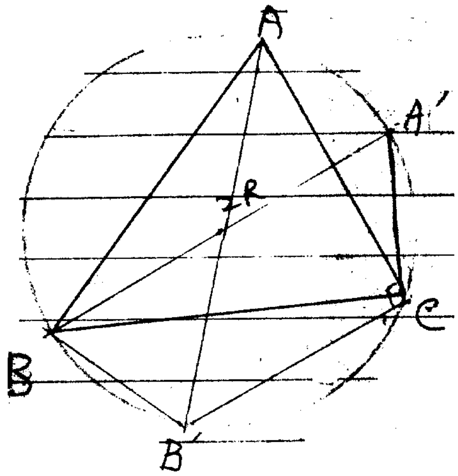
$$\frac{c}{2} = R\sin\theta$$
$$= R\sin\frac{2\theta}{2}$$
$$= R\sin\frac{360^\circ - 2C}{2}$$
$$= R\sin(180^\circ - C)$$
$$= R\sin C$$
$$\Rightarrow 2R = \frac{c}{\sin C}$$



$$\frac{a}{2} = R\sin\alpha$$
$$= R\sin A$$
$$\frac{b}{2} = R\sin\beta$$
$$= R\sin B$$

1.1.1.3 Method 3 Draw a circumcircle of $\triangle ABC$

Case 1



Draw BA' through centre

$$\angle BAC = \angle BA'C$$
$$\frac{a}{\sin A} = \frac{a}{\sin A'}$$

But $\angle A'CB = 90^\circ$ ($\angle$ in semi-circle)

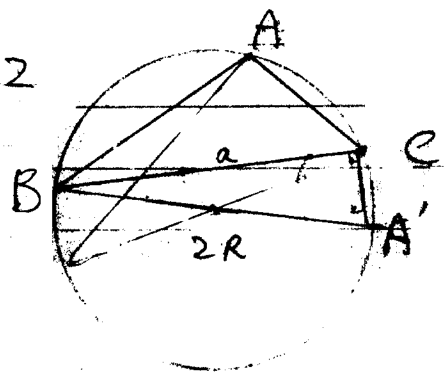
$$\therefore \frac{a}{\sin A'} = 2R$$

Similarily Draw AB' through centre

$$\angle B'CA = 90^\circ$$
$$\therefore \frac{b}{\sin B} = \frac{b}{\sin B'} = 2R$$

similarly $\frac{c}{\sin C} = 2R$

Case 2

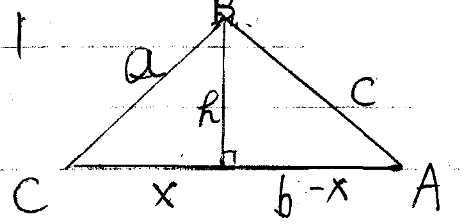


$\angle A + \angle A' = 180^\circ$ (opp $\angle$ of cyclic quad)

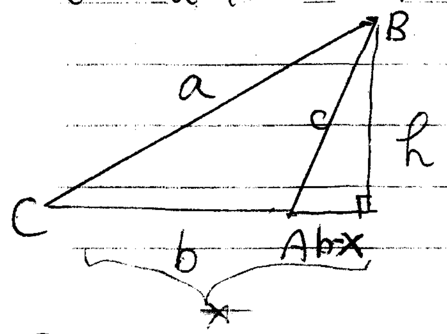
$$\frac{a}{\sin A} = \frac{a}{\sin(180^\circ - A')} = \frac{a}{\sin A'} = 2R$$

1.1.2 Cosine formula. For any $\triangle ABC$ $\quad c^2 = a^2+b^2-2ab\cos C$

1.1.1 Method 1



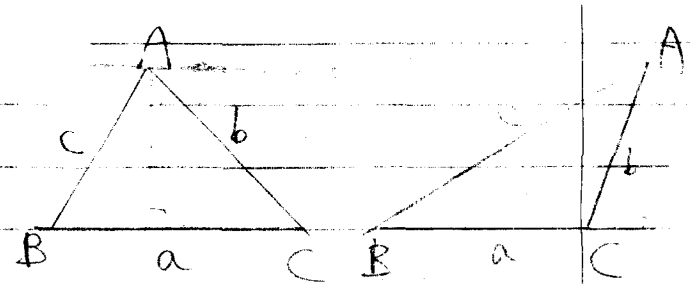
$$h^2 = a^2 - x^2$$
$$= c^2 - (b-x)^2$$
$$\therefore \quad a^2 = c^2 - b^2 + 2bx$$
$$c^2 = a^2+b^2-2ab\cos C$$



$$h^2 = a^2 - x^2$$
$$h^2 = c^2 - (b-x)^2$$
$$a^2 = c^2 - b^2 + 2bx$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

~~1.1.2.2~~

Method 2



$$a = c\cos B + b\cos C \quad -(1)$$
$$b = a\cos C + c\cos A \quad -(2)$$
$$c = a\cos B + b\cos A \quad -(3)$$

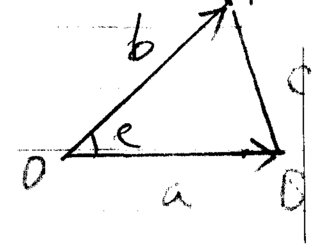
(1): $\quad \cos B = \dfrac{a - b\cos C}{c}$

(2) $\quad \cos A = \dfrac{b - a\cos C}{c}$

$\therefore (3)$: $\quad c = a\,\dfrac{a-b\cos C}{c} + b\,\dfrac{b-a\cos C}{c}$

$$c^2 = a^2+b^2-2ab\cos C$$

1.1.2.3 method 3



$$c^2 = \vec{AB}\cdot\vec{AB} = (\vec{OB}-\vec{OA})(\vec{OB}-\vec{OA})$$
$$= \vec{OB}^2 + \vec{OA}^2 - 2\,\vec{OB}\cdot\vec{OA}$$
$$= a^2+b^2-2ab\cos C$$

1.1.3 Examples

1.1.3.1 If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = k$ constant then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{a_1+a_2+a_3}{b_1+b_2+b_3}$

Futhermore, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{a_1 + r a_2 + r^2 a_3}{b_1 + r b_2 + r^2 b_3}$, for any real number r

Proof: $a_1 = k b_1$, $a_2 = k b_2$, $a_3 = k b_3$

$$\therefore \frac{a_1+a_2+a_3}{b_1+b_2+b_3} = \frac{kb_1+kb_2+kb_3}{b_1+b_2+b_3} = \frac{k(b_1+b_2+b_3)}{b_1+b_2+b_3} = k = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$$\frac{a_1 + r a_2 + r^2 a_3}{b_1 + r b_2 + r^2 b_3} = \frac{k b_1 + k r b_2 + k r^2 b_3}{b_1 + r b_2 + r^2 b_3} = k = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

1.1.3.2 (example 28 G.M P25)

Prove in $\triangle ABC$, that if $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ then $\frac{\sin A}{7} = \frac{\sin B}{6} = \frac{\sin C}{5}$

and $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$

Proof $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = \frac{2(a+b+c)}{11+12+13} = \frac{a+b+c}{18} = k$ (by 1.1.3.1)

$$k = \frac{a+b+c}{18} = \frac{b+c-(c+a)+a+b}{11-12+13}, \quad r=-1, \text{ by } 1.1.3.1$$

$$= \frac{b}{6}$$

$$= \frac{c+a-(b+c)+(a+b)}{12-11+13} = \frac{a}{7}$$

$$= \frac{a+c-(a+b)+(b+c)}{12-13+11} = \frac{c}{5}$$

$$\therefore \frac{a}{7} = \frac{b}{6} = \frac{c}{5} \Rightarrow \frac{\sin A}{7} = \frac{\sin B}{6} = \frac{\sin C}{5}$$

$a = 7k$, $b = 6k$, $c = 5k$

Omitted for cos B, cos C

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{36k^2+25k^2-49k^2}{2(6k)(5k)} = \frac{1}{5}$$

similarly $\cos B = \frac{19}{35}$, $\cos C = \frac{5}{7}$

$$\therefore \cos A : \cos B : \cos C = \frac{1}{5} : \frac{19}{35} : \frac{5}{7} = 7 : 19 : 25$$

i.e.

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$$

1.1.3.3 (Example 29 GM P.26) In ΔABC,

(i) find $a:b:c$ if $A:B:C=1:2:3$

(ii) find $\sin A:\sin B:\sin C$ if $(b+c):(c+a):(a+b)=5:6:7$

(i) $\dfrac{A}{1}=\dfrac{B}{2}=\dfrac{C}{3}=k$

$A=k$, $B=2k$, $C=3k$

$A+B+C=180^\circ$ ($\angle$ sum of Δ)

$k+2k+3k=180^\circ$

$k=30^\circ$

$\therefore A=30^\circ$, $B=60^\circ$, $C=90^\circ$

$a:b:c=\sin A:\sin B:\sin C$

$=\sin 30^\circ:\sin 60^\circ:\sin 90^\circ$

$=\frac{1}{2}:\frac{\sqrt{3}}{2}:1$

$=1:\sqrt{3}:2$

(ii) $\dfrac{b+c}{5}=\dfrac{c+a}{6}=\dfrac{a+b}{7}=k=\dfrac{a+b+c}{9}$ (by 1.1.3.1)

$=\dfrac{b+c-(c+a)+a+b}{5-6+7}=\dfrac{b}{3}$ ($r=-1$, by 1.1.3.1)

$=\dfrac{b+c-(a+b)+c+a}{5-7+6}=\dfrac{c}{2}$

$=\dfrac{c+a-(b+c)+a+b}{6+7-5}=\dfrac{a}{4}$

$\therefore$ $a:b:c=4:3:2=\sin A:\sin B:\sin C$.

or $\sin A:\sin B:\sin C=4:3:2$ //

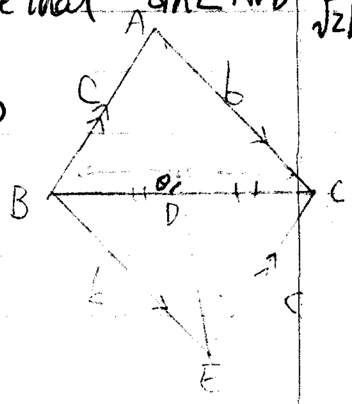
1.1.3.4 (Example 32 GM P29)

In ΔABC, D is the mid-point of BC Prove that $\sin\angle ADB=\dfrac{2b\sin C}{\sqrt{2b^2+2c^2-a^2}}$

Draw a parallelogram ABECA

since diagonals bisect each other, $\therefore$ AE passes D

let $\theta=\angle ADB$



$\dfrac{AD}{\sin C}=\dfrac{b}{\sin\theta}$ (sine rule on ΔADC)

$$\therefore \sin\theta = \frac{b\sin C}{AD}$$

$$= \frac{2b\sin C}{AE}$$

$$= \frac{2b\sin C}{\sqrt{b^2+c^2-2bc\cos\angle ACE}} \quad \text{(Cosine rule on } \triangle ACE)$$

$$= \frac{2b\sin C}{\sqrt{b^2+c^2-2bc\cos(180^\circ-A)}} \quad (\because ABECA \text{ is a // gram})$$

$$= \frac{2b\sin C}{\sqrt{b^2+c^2+(b^2+c^2-a^2)}} \quad \text{(Cosine rule on } \triangle ABC)$$

$$= \frac{2b\sin C}{\sqrt{2b^2+2c^2-a^2}}$$

1.1.3.5 (Example 34 GM P.30)

Let a, b, c be the three sides of $\triangle ABC$ such that $a^2-a-2b-2c=0$ and $a+2b-2c+3=0$ find the greatest angle of the triangle.

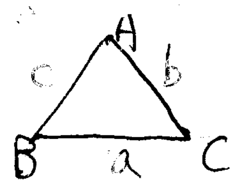
let a be fix, solve $\begin{cases} b+c = \frac{1}{2}(a^2-a) \\ b-c = -\frac{1}{2}(3+a) \end{cases}$

$$\therefore \quad b = \frac{1}{2}\left[\frac{1}{2}(a^2-a)-\frac{1}{2}(3+a)\right]$$

$$= \frac{1}{4}(a^2-2a-3)$$

$$c = \frac{1}{2}\left[\frac{1}{2}(a^2-a)+\frac{1}{2}(3+a)\right]$$

$$= \frac{1}{4}(a^2+3)$$



clearly $c > a, b$

$\therefore$ largest angle is C

$$\cos C = \frac{a^2+\frac{1}{16}(a^2-2a-3)^2-\left[\frac{1}{4}(a^2+3)\right]^2}{2a\times\frac{1}{4}(a^2-2a-3)}$$

$$= \frac{a^2+\frac{1}{16}a^4+\frac{4a^2}{16}+\frac{9}{16}-\frac{4a^3}{16}+\frac{12a}{16}-\frac{6a^2}{16}-\frac{3a^2}{8}-\frac{9}{16}}{\frac{1}{2}a(a-3)(a+1)}$$

$$= \frac{-\frac{1}{4}a^3+\frac{1}{2}a^2+\frac{3}{4}a}{\frac{1}{2}a(a-3)(a+1)} = \frac{-a(a^2-2a-3)}{2a(a^2-2a-3)} = -\frac{1}{2}$$

$$\therefore \quad C = 120^\circ$$

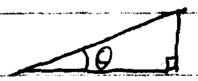
1.2 Section 1.2 Two or Three Dimensional Problems

Details and concepts are assumed to be learned in Form 4

However, much difficult examples involving sine law or cosine law only are given.

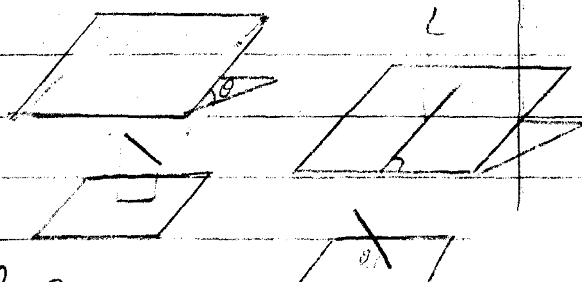
2.1 Terms

slope $m = \tan\theta$



angle between two planes

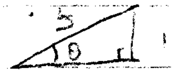
line of (the) greatest slope



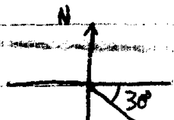
Projection

angle between a line and a plane

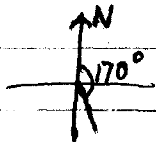
1 in 5 = $\sin\theta = \frac{1}{5}$



E 30°S

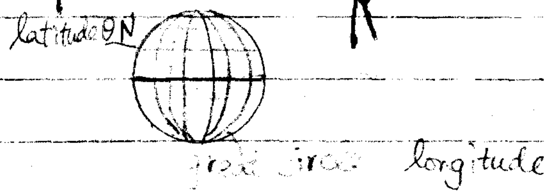


bearing 170°



longitude

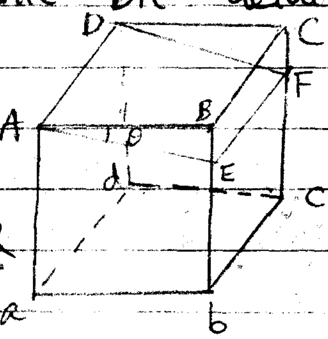
latitude



We explain the projection in a little bit detail:

Given a cube ABCDabcd of sides x

To find the inclined angle between the plane AEFD and abcd



Solution: ADFE is a rectangle of sides x and $x\sec\theta$, i.e, of area $x^2\sec\theta$.

$\therefore$ Area ABCD = (area ADFE) $\cos\theta$

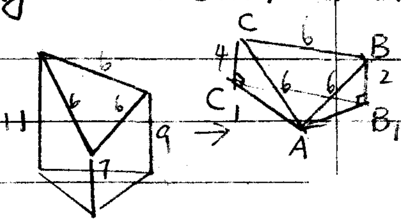
or $\cos\theta = \dfrac{\text{Area ABCD}}{\text{Area ADFE}}$

Humphrey 2 p.1,29

1.22 Examples

1.2.2.1 An equilateral triangle of sides 6 cm has its vertices at heights 7, 9, 11 cm above a horizontal plane. Find the dimensions of the projection of the triangle on the horizontal and the inclination of the equilateral triangle to the horizontal.

Assume the equilateral triangle ABC and its projection AB_1C_1



By Pythagoras' theorem

$$\left.\begin{aligned} AB_1 &= \sqrt{6^2-2^2} = \sqrt{32} \\ AC_1 &= \sqrt{6^2-4^2} = \sqrt{20} \\ B_1C_1 &= \sqrt{6^2-(4-2)^2} = \sqrt{32} \end{aligned}\right\} \text{dimensions of the projection of } \triangle ABC$$

By the formula Area $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$s = (\sqrt{32} + \sqrt{20} + \sqrt{32})/2 = 7.89$$

$$s-a = s-c = \sqrt{5}$$

$$s-b = 3.42$$

$$\therefore \triangle AB_1C_1 = \sqrt{135}$$

clearly, by easy calculation, $\triangle ABC = \frac{1}{2} 6 \times 6 \sin 60^\circ = 9\sqrt{3}$

$\therefore$ if θ = inclination of $\triangle ABC$

$$\cos\theta = \frac{\sqrt{135}}{9\sqrt{3}} = \frac{\cancel{3}\sqrt{5}}{\cancel{9}3}$$

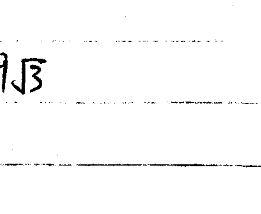
$$\theta = 41.8^\circ$$

F4 maths RE8 10

1.2.2.2 A lamp-shade is made up of 4 equal panes of glass in the shaped of trapezium. The parallel sides of each pane are 28 cm, 8 cm long and each slant size 20 cm. What is the angle between

a) each pane and the vertical; b) two adjacent panes?

Assume V is the virtual vertex, let VA = x



$\triangle VAB \sim \triangle VEF \quad \therefore \quad \dfrac{x}{8} = \dfrac{x+20}{28}$

$$20x = 160$$

$$x = 8$$

Area of projection of $\triangle VEF = \frac{1}{4}$ EFGH (why?)

$$= \frac{1}{4}\ 28 \times 28 = 196$$

$$\text{Area } \triangle VEF = \frac{1}{2} \times 28 \times \sqrt{28^2 - 14^2}$$
$$= 339.5$$

If θ is the angle between VAB and EFGH,

$$\cos\theta = \frac{196}{339.5}$$

$$\theta = 54.7^\circ$$

$\therefore$ angle between each pane and the vertical is $90^\circ - 54.7^\circ = 35.3^\circ$

b) VEF and VFG are two equilateral triangles of sides 28cm

Find a point I on VF such that $EI \perp VF$, $GI \perp VF$.

EI, GI are the heights of $\triangle VEF$, $\triangle VFG$ resp.

$$EI = GI = \frac{2 \times 339.5}{28} = \sqrt{588} \quad \text{(by a)}$$

$$EG = \sqrt{28^2 + 28^2} = 28\sqrt{2}$$

$\therefore$ By cosine law, if α is the angle between VEF and VFG

$$\cos\alpha = \frac{(\sqrt{588})^2 + (\sqrt{588})^2 - (28\sqrt{2})^2}{2(\sqrt{588})^2}$$
$$= -\frac{1}{3}$$
$$\alpha = 109.5^\circ$$

F4maths RE8,16

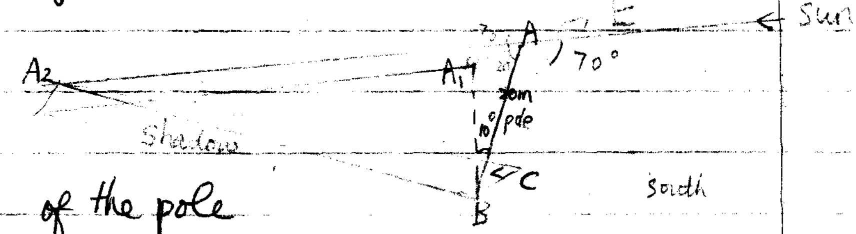
7.2.2.3 A 20-metre pole with one end on level ground is inclined at 10° to the vertical towards the East. At noon one day the angle of elevation of the sun was 70° due South of the pole. What was the length of the shadow of the pole at this time?

let AB be the pole

A_2B is the shadow

AC is the projection of AB

A_1B is the "effective" height of the pole



then $AC = 20\cos 10^\circ$

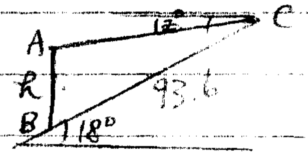
$$A_2C = AC \tan 20^\circ$$
$$= 20\cos 10^\circ \tan 20^\circ$$

$$BC = 20\sin 10^\circ$$

$$A_2B = \sqrt{A_2C^2 + BC^2} = 20\sqrt{\cos^2 10^\circ \tan^2 20^\circ + \sin^2 10^\circ}$$
$$= 7.97\text{m}$$

GM 1D3
1.2.2.4 A tower is on a hillside which slope at 18° to the horizontal. At a point 93.6 m higher up the hill from the foot of the tower, the angle of depression of the top of the tower is 12°. Find the height of the tower.

let AB = Tower



$\angle ACB = 18° - 12° = 6°$

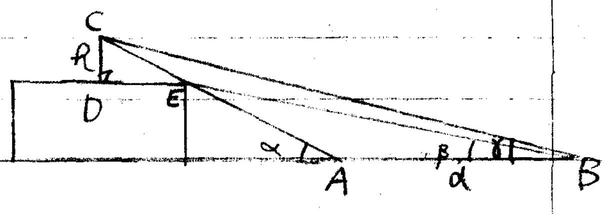
$\angle BAC = 90° + 12° = 102°$

By sine law $\dfrac{h}{\sin 6°} = \dfrac{93.6}{\sin 102°}$

$h = 10.00\text{ m}$

GM 1D7
1.2.2.5 From a point A, the top of a flagstaff of height h can just be seen. The flagstaff stands at the centre of a square tower. B is a point further away from the tower so that AB = d, and the flagstaff, A, B are in the same vertical plane. If the elevation from A and B of the top of the flagstaff are α and γ respectively, and the elevation of the top of the tower from B is β, show that $h = \dfrac{d \sin^2\alpha \sin(\gamma-\beta)}{\sin(\alpha-\gamma)\sin(\alpha-\beta)}$

CD = flagstaff = h



$\angle CBE = \gamma - \beta$

$\angle AEB = \alpha - \beta$ (ext ∠ of △)

$\angle BEC = 180° - (\alpha - \beta)$

$\therefore \angle BCE = \alpha - \beta - \gamma + \beta$
$= \alpha - \gamma$

Consider △ABE $\dfrac{d}{\sin(\alpha-\beta)} = \dfrac{EB}{\sin(180°-\alpha)}$

$EB = \dfrac{d \sin\alpha}{\sin(\alpha-\beta)}$

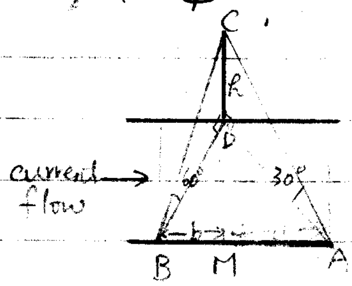
Consider △BCE $\dfrac{EB}{\sin(\alpha-\gamma)} = \dfrac{CE}{\sin(\gamma-\beta)}$, $CE = \dfrac{h}{\sin\alpha}$

$\therefore$
$$\frac{d \sin\alpha}{\sin(\alpha-\beta)\sin(\alpha-\gamma)} = \frac{h}{\sin\alpha \sin(\gamma-\beta)}$$

or $$h = \frac{d\sin^2\alpha \sin(\gamma-\beta)}{\sin(\alpha-\gamma)\sin(\alpha-\beta)}$$

26 The elevation of the top of a vertical mast of height h m, on a straight portion of a bank of a river, is $30°$ from a point A on the opposite bank and a m downstream. From a point B on this bank b m upstream the elevation is $60°$. Prove that $a^2 - b^2 = 8h^2/3$

If $a=50$, $b=10$, calculate the height of the mast, the width of the river, and the elevation of the top of the mast from the point midway between A and B



Let CD be the mast

$$AD = \frac{h}{\tan 30°} = h\sqrt{3}$$

$$BD = \frac{h}{\tan 60°} = \frac{h}{\sqrt{3}}$$

$$DM^2 = AD^2 - a^2 = BD^2 - b^2$$

$$\Rightarrow \quad a^2 - b^2 = (h\sqrt{3})^2 - \left(\frac{h}{\sqrt{3}}\right)^2$$

$$= \frac{8h^2}{3}$$

$a=50$, $b=10$ $\quad h = \left[\frac{3}{8}(50^2 - 10^2)\right]^{\frac{1}{2}}$

$$= 30\text{m}$$

If N is the mid point of AB, $MN = 20$

$$DM = (AD^2 - a^2)^{\frac{1}{2}}$$

$$= (900 \times 3 - 50^2)^{\frac{1}{2}}$$

$$= 10\sqrt{2}$$

$$= 14.14\text{ m}$$

$$DN = (DM^2 + MN^2)^{\frac{1}{2}}$$

$$= (200 + 400)^{\frac{1}{2}}$$

$$= 10\sqrt{6}$$

If θ is the elevation from N,

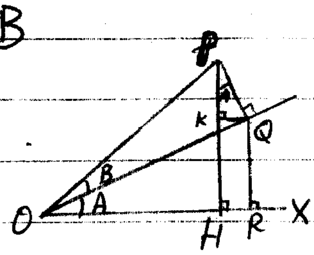
$$\tan\theta = \frac{CD}{DN} = \frac{30}{10\sqrt{6}} = \sqrt{\frac{3}{2}}$$

$$\theta = 50°46'$$

1.3 Addition Formulae

1.3.1 The proofs $\sin(A+B) = \sin A\cos B + \cos A\sin B$

1.3.1.1 method 1 (G.M P1)



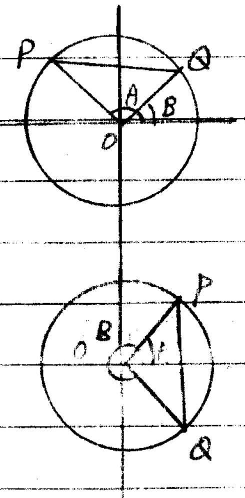
$$\sin(A+B) = \frac{HP}{OP} = \frac{KP+HK}{OP}$$

$$= \frac{RQ+KP}{OP}$$

$$= \frac{OQ\sin A + PQ\cos A}{OP}$$

$$= \sin A\cos B + \cos A\sin B$$

1.3.1.2 method 2 (G.M P.2)



In the following circle of radius 1,

$\angle POQ = A - B$

$P = (\cos A, \sin A)$

$Q = (\cos B, \sin B)$

$$PQ^2 = (\cos B - \cos A)^2 + (\sin B - \sin A)^2$$

$$= 2 - 2(\cos A\cos B + \sin A\sin B)$$

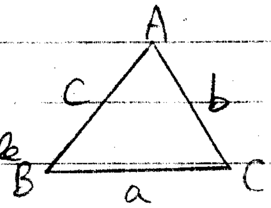
On the other hand, by cosine rule,

$$\cos(A-B) = \frac{OP^2+OQ^2-PQ^2}{2\,OP\,OQ}$$

$$= \frac{1+1-2+2(\cos A\cos B+\sin A\sin B)}{2\times1\times1}$$

$$= \cos A\cos B + \sin A\sin B$$

1.3.1.3 method 3



For any ΔABC, we have sine rule and cosine rule

$$\sin A = \frac{a}{c}\sin C$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\sin B = \frac{b}{c}\sin C$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\therefore \sin A\cos B + \cos A\sin B = \frac{a}{c}\sin C \times \frac{a^2+c^2-b^2}{2ac} + \frac{b^2+c^2-a^2}{2bc}\times\frac{b}{c}\sin C$$

$$= \frac{a^2+c^2-b^2+b^2+c^2-a^2}{2c^2}\sin C$$

$$= \sin C$$

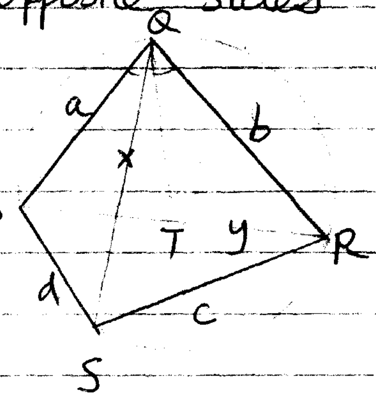
$$= \sin(A+B)$$

1.4 method 4

We first prove the Ptolemy's theorem :

1.4.1 In any cyclic quadrilateral, the product of diagonal is equal to the sum of product of opposite sides

To prove $xy = ac + bd$



Proof WLOG assume $\angle PQS \leq \angle RQS$

Copy $\angle PQS$ so that $\angle RQT = \angle PQS$

$\angle PSQ = \angle TRQ$ ($\angle$ in same seg PQ)

$\therefore \triangle PQS \sim \triangle QTR$ (AAA)

$\therefore \dfrac{x}{b} = \dfrac{d}{TR}$ —— (1)

$\begin{cases} \angle PQT = \angle RQS & (\because \angle RQT = \angle PQS \text{ and } \angle SQT \text{ is common}) \\ \angle QPT = \angle QSR & (\angle \text{ in same seg } PQ) \end{cases}$

$\therefore \triangle PQT \sim \triangle SQR$ (AAA)

$\dfrac{x}{a} = \dfrac{c}{PT}$.

$\therefore \dfrac{x}{a} = \dfrac{c}{y - TR}$ —— (2) $(\because PT + TR = y)$

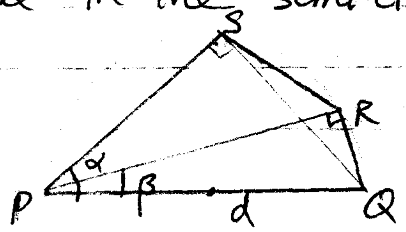
from (1) $TR = \dfrac{bd}{x}$.

Sub into (2) $xy - x\dfrac{bd}{x} = ac$

or $xy = ac + bd$ QED

1.4.2 To prove $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

consider the following cyclic quadrilateral in the semi-circle



By Ptolemy's theorem

$PR \cdot QS = RS \cdot PQ + QR \cdot PS$

$PQ = d$

$PR = d\cos\beta$ $QS = d\sin\alpha$

$RQ = d\sin\beta$ $PS = d\cos\alpha$

$\dfrac{RS}{\sin(\alpha-\beta)} = \dfrac{PR}{\sin\angle PSR}$

$\therefore \dfrac{RS}{\sin(\alpha-\beta)} = \dfrac{PR}{\sin(\angle PQR)}$ $\left(\because \angle PSR + \angle PQR = 180^\circ \text{ opp } \angle \text{ of cyclic quad}\right)$

$= d$

$\therefore \quad RS = d \sin(\alpha-\beta)$

$\therefore \quad d \cos\beta \cdot d \sin\alpha = d \sin(\alpha-\beta) \cdot d + d \sin\beta \cdot d \cos\alpha$

$\therefore \quad \sin(\alpha-\beta) = \sin\alpha \cos\beta - \sin\beta \cos\alpha$ — ✱

1.3.2 Variation

given that formula

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ —(1)

replace B by $-B$

$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$

$\therefore \quad \sin(A-B) = \sin A \cos B - \cos A \sin B$ —(2)

now $\cos(A+B) = \sin[90^\circ - (A+B)]$

$= \sin[(90^\circ - A) - B]$

$= \sin(90^\circ - A) \cos B - \cos(90^\circ - A) \sin B$

$= \cos A \cos B - \sin A \sin B$ —(3)

replace B by $-B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ —(4)

now $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$= \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ —(5)

Similarly $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ —(6)

$\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$ —(7)

$\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$ —(8)

memorize

$S(A \pm B) = S\,C \pm C\,S$

$C(A \pm B) = C\,C \mp S\,S$

$t(A \pm B) = \dfrac{tA \pm tB}{1 \mp tA\,tB}$ This is enough

.3 Examples

3.1 Identity GM 1A 32

If $A+B+C = 180°$, prove $\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C = \cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$

Proof RHS $= \cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$

$= [1 + (\cot\frac{1}{2}A + \cot\frac{1}{2}B)\cot(\frac{1}{2}A+\frac{1}{2}B)]\cot\frac{1}{2}C$

$= [1+(\cot\frac{1}{2}A + \cot\frac{1}{2}B)\cot(90° - \frac{C}{2})]\cot\frac{1}{2}C$

$= \cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C = LHS$.

3.2 GM 1A 29

If $\sin A + \cos B = p$, $\cos A - \sin B = q$, and $A-B = 30°$ prove that $p^2+q^2-3=0$

we have $\sin A = p - \cos B$

$\cos A = q + \sin B$

$\therefore \quad 1 = (p-\cos B)^2 + (q+\sin B)^2$

$\therefore \quad p^2+q^2+2q\sin B - 2p\cos B = 0$

$p^2+q^2 + 2[(\cos A - \sin B)\sin B - (\sin A + \cos B)\cos B] = 0$

$p^2+q^2 - 2 + 2[\sin B\cos A - \cos B\sin A] = 0$

$p^2+q^2 - 2 + 2\sin(B-A) = 0$

$p^2+q^2 - 3 = 0 \qquad (\because A-B = 30°)$

3.3 function of trigonometric functions (GM 1A3)

evaluate $\tan\theta\tan(\theta-60°) + \tan\theta\tan(\theta+60°) + \tan(\theta-60°)\tan(\theta+60°)$

$= \tan\theta(\tan(\theta-60°) + \tan(\theta+60°)) + \tan(\theta-60°)\tan(\theta+60°)$

$= \tan\theta\left[\frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta} + \frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}\right] + \frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}\times\frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}$

$= \tan\theta\ \frac{\tan\theta-\sqrt{3}-\sqrt{3}\tan^2\theta+3\tan\theta+\tan\theta+\sqrt{3}+\sqrt{3}\tan^2\theta+3\tan\theta}{1-3\tan^2\theta} + \frac{\tan^2\theta-3}{1-3\tan^2\theta}$

$= \tan\theta\ \frac{8\tan\theta}{1-3\tan^2\theta} + \frac{\tan^2\theta-3}{1-3\tan^2\theta}$

$= \frac{3(3\tan^2\theta-1)}{1-3\tan^2\theta}$

$= -3$

Any other quicker method?

H.M1A 11

1.3.3.4 Prove that $\tan n\theta - \tan(n-1)\theta = \tan\theta\,[1+\tan n\theta\tan(n-1)\theta]$

Use this to find $\sum_{r=1}^{n} \tan r\theta \tan(r-1)\theta$.

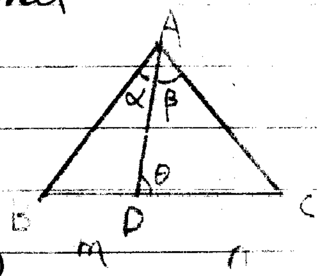
Solution

$$\tan\theta = \tan[n\theta - (n-1)\theta]$$
$$= \frac{\tan n\theta - \tan(n-1)\theta}{1+\tan n\theta\tan(n-1)\theta}$$
$$\therefore \tan n\theta - \tan(n-1)\theta = \tan\theta\,[1+\tan n\theta\tan(n-1)\theta]$$
$$\sum_{r=1}^{n} \tan r\theta\tan(r-1)\theta = \frac{1}{\tan\theta}\left[\sum_{r=1}^{n}\tan\theta\,(1+\tan r\theta\tan(r-1)\theta)\right] - n$$
$$= \frac{1}{\tan\theta}\sum_{r=1}^{n}[\tan r\theta - \tan(r-1)\theta] - n$$
$$= \frac{\tan n\theta}{\tan\theta} - n \quad //$$

GM. P28 ex31

1.3.3.5 The point D divides the side BC of $\triangle ABC$ internally so that $BD:DC = m:n$. If $\angle BAD = \alpha$, $\angle CAD = \beta$ and $\angle CDA = \theta$ prove that

$$m\cot\alpha - n\cot\beta = (m+n)\cot\theta = n\cot B - m\cot C$$



$\triangle ADC$ $\quad \dfrac{n}{\sin\beta} = \dfrac{AC}{\sin\theta}$

$\triangle ABC$ $\quad \dfrac{m+n}{\sin(\alpha+\beta)} = \dfrac{AC}{\sin(\theta-\alpha)}$ $\quad (\because B = \theta - \alpha$ ext $\angle$ of $\triangle)$

dividing $\quad n\sin\theta\sin(\alpha+\beta) = (m+n)\sin(\theta-\alpha)\sin\beta$

$$n\sin\theta\sin(\alpha+\beta) = (m+n)\sin\beta\,(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$
$$[(m+n)\sin\beta\cos\alpha - n\sin(\alpha+\beta)]\sin\theta = (m+n)\sin\alpha\sin\beta\cos\theta$$
$$\therefore (m+n)\cot\theta = \frac{m\sin\beta\cos\alpha - n\cos\beta\sin\alpha}{\sin\alpha\sin\beta}$$
$$= m\cot\alpha - n\cot\beta$$

$\triangle ABD$ $\quad \dfrac{m}{\sin(\theta-B)} = \dfrac{AB}{\sin\theta}$ $\quad (\because \alpha = \theta - B)$

$\triangle ABC$ $\quad \dfrac{m+n}{\sin(B+C)} = \dfrac{AB}{\sin C}$

dividing $\quad m\sin\theta\sin(B+C) = (m+n)\sin C\sin(\theta-B)$

$$m\sin\theta\sin(B+C) = (m+n)\sin C\,(\sin\theta\cos B - \cos\theta\sin B)$$
$$\sin\theta\,[(m+n)\sin C\cos B - m\sin(B+C)] = (m+n)\sin C\sin B\cos\theta$$
$$\therefore (m+n)\cot\theta = n\cot B - m\cot C$$

.4 Multiple-angle and Submultiple angle Formulae

.1 The derivation

$\because \quad \sin(A+B) = \sin A \cos B + \cos A \sin B$

If $A=B$ $\quad \sin 2A = 2\sin A \cos A \qquad -(1)$

similarly $\quad \cos 2A = \cos^2 A - \sin^2 A \qquad -(2)$

$= 2\cos^2 A - 1 \qquad -(3)$

$= 1 - 2\sin^2 A \qquad -(4)$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} \qquad -(5)$$

now $\quad \sin 3A = \sin(2A+A)$

$= (2\sin A \cos A)\cos A + (1-2\sin^2 A)\sin A$

$= 2\sin A(1-\sin^2 A) + \sin A - 2\sin^3 A$

$= 3\sin A - 4\sin^3 A$

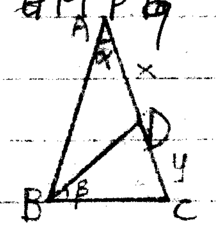
similarly $\quad \cos 3A = 4\cos^3 A - 3\cos A$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

4.2 Examples

2.1 Find $\cos 36^\circ$ without using tables (c.f. GMP 19 ex.10)

Consider the following $\triangle ABC$



BC=X

given $AB=AC$, $AD=BD=BC$

let $\angle A = \alpha$, $\angle CBD = \beta$, $AD = x$, $CD = y$

then $\left.\begin{array}{l} \angle ABD = \alpha \\ \angle ACB = \alpha+\beta \\ \angle BDC = \alpha+\beta \end{array}\right\}$ (base $\angle$ of isos. $\triangle$)

in $\triangle BCD$ $\quad \beta + 2(\alpha+\beta) = 180^\circ$

$\triangle ABC$ $\quad \alpha + 2(\alpha+\beta) = 180^\circ$

solving $\quad \alpha = \beta = 36^\circ$

in $\triangle BCD$ $\quad \dfrac{y}{\sin 36^\circ} = \dfrac{BC}{\sin 72^\circ} = \dfrac{x}{2\sin 36^\circ \cos 36^\circ} \Rightarrow 2\cos 36^\circ = \dfrac{x}{y}$

$\triangle ABC$ $\quad \dfrac{x}{\sin\alpha} = \dfrac{x+y}{\sin 72^\circ} \Rightarrow 2\cos 36^\circ = 1 + \dfrac{y}{x}$

let $t = \dfrac{x}{y}$ $\quad \therefore \quad t^2 - t - 1 = 0 \quad \therefore 2\cos 36^\circ = t = \dfrac{1+\sqrt5}{2}$ or $\dfrac{1-\sqrt5}{2}$ (reject <0)

$\therefore \cos 36^\circ = \dfrac{1+\sqrt5}{4}$

H.M IA 75

1.4.2.2 Prove that $\dfrac{\tan 2\theta + \sec 2\theta - 1}{\tan 2\theta - \sec 2\theta + 1} = \tan(\theta + \frac{\pi}{4})$

Proof: $\dfrac{\tan 2\theta + \sec 2\theta - 1}{\tan 2\theta - \sec 2\theta + 1} = \dfrac{\sin 2\theta - \cos 2\theta + 1}{\sin 2\theta + \cos 2\theta - 1}$

$$= \frac{2\sin\theta\cos\theta - (1 - 2\sin^2\theta) + 1}{2\sin\theta\cos\theta + 1 - 2\sin^2\theta - 1}$$

$$= \frac{\sin\theta\,(\cos\theta + \sin\theta)}{\sin\theta\,(\cos\theta - \sin\theta)}$$

$$= \frac{1 + \tan\theta}{1 - \tan\theta}$$

$$= \tan(\theta + \frac{\pi}{4})$$

H.M IA 17

1.4.2.3 Prove $\dfrac{1}{2\sin\theta}(\operatorname{cosec} 2\theta - \operatorname{cosec} 4\theta) = \dfrac{\cos 3\theta}{\sin 2\theta \sin 4\theta}$

and hence sum to n terms of the series $\dfrac{\cos 3\theta}{\sin 2\theta \sin 4\theta} + \dfrac{\cos 5\theta}{\sin 4\theta \sin 6\theta} + \dfrac{\cos 7\theta}{\sin 6\theta \sin 8\theta} + \dots +$

$$LHS = \frac{1}{2\sin\theta}(\csc 2\theta - \csc 4\theta)$$

$$= \frac{1}{2\sin\theta}\left(\frac{1}{\sin 2\theta} - \frac{1}{\sin 4\theta}\right)$$

$$= \frac{\sin 4\theta - \sin 2\theta}{2\sin\theta\sin 2\theta\sin 4\theta}$$

$$= \frac{2(\sin 2\theta\cos 2\theta - \sin\theta\cos\theta)}{2\sin\theta\sin 2\theta\sin 4\theta}$$

$$= \frac{2\cos\theta(2\cos^2\theta - 1) - \cos\theta}{\sin 2\theta \sin 4\theta}$$

$$= \frac{\cos 3\theta}{\sin 2\theta\sin 4\theta} = RHS.$$

$$\frac{\cos 3\theta}{\sin 2\theta\sin 4\theta} + \frac{\cos 5\theta}{\sin 4\theta\sin 6\theta} + \dots + \frac{\cos(2n+1)\theta}{\sin 2n\theta\sin(2n+2)\theta}$$

$$= \sum_{r=1}^{r} \frac{\cos(2r+1)\theta}{\sin 2r\theta\sin(2r+2)\theta} = \sum_{r=1}^{n} \frac{2\sin\theta\cos(2r+1)\theta}{2\sin\theta\sin 2r\theta\sin(2r+2)\theta}$$

$$= \sum_{r=1}^{n} \frac{\sin(2r+2)\theta + \sin(-2r\theta)}{2\sin\theta\sin 2r\theta\sin(2r+2)\theta} = \sum_{r=1}^{n} \frac{1}{2\sin\theta}\left(\frac{1}{\sin 2r\theta} - \frac{1}{\sin(2r+2)\theta}\right)$$

$$= \frac{1}{2\sin\theta}\left(\frac{1}{\sin 2\theta} - \frac{1}{\sin(2n+2)\theta}\right) \quad //$$

4 Given θ is in third quadrant and $\tan\theta = \frac{4}{3}$, calculate $\tan\frac{1}{2}\theta$ using the formula $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$.

we have $\frac{4}{3} = \frac{2t}{1-t^2}$, $t = \tan\frac{\theta}{2}$

$2t^2 + 3t - 2 = 0$

$(2t-1)(t+2) = 0$

$t = \frac{1}{2}$ or -2

$\frac{1}{2}\theta$ is in 2nd quadrant $\therefore \tan\frac{1}{2}\theta < 0$

P.27 ex4 $\therefore \tan\frac{1}{2}\theta = -2$

25 Calculate $\sin\frac{\pi}{8}$, $\cos\frac{\pi}{8}$, $\tan\frac{\pi}{8}$

let $t = \tan\frac{1}{8}\pi$

then $\frac{2t}{1-t^2} = \tan\frac{1}{4}\pi = 1$

$t^2 + 2t - 1 = 0$

$\therefore t = \sqrt{2} - 1$ // $\quad (-\sqrt{2}-1$ reject$)$

$\sec^2\frac{1}{8}\pi = 1 + t^2$

$= 4 - 2\sqrt{2}$

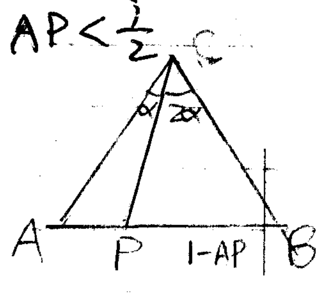
$\cos^2\frac{1}{8}\pi = \frac{1}{4-2\sqrt{2}} = \frac{1}{4}(2+\sqrt{2})$

$\sin^2\frac{1}{8}\pi = 1 - \cos^2\frac{1}{8}\pi = \frac{1}{4}(2-\sqrt{2})$

$\therefore \sin\frac{1}{8}\pi = \frac{1}{2}\sqrt{2-\sqrt{2}} \quad \cos\frac{1}{8}\pi = \frac{1}{2}\sqrt{2+\sqrt{2}}$ //

ex.30

26 In the given figure, $\triangle ABC$ is isos. with $AC = BC = a$, $AB = 1$. P is a point on AB such that $\angle ACP = \alpha$; $\angle BCP = 2\alpha$. Show that $AP = \frac{1}{1+2\cos\alpha}$, Hence deduce $\frac{1}{3} < AP < \frac{1}{2}$



$\triangle APC \quad \frac{AP}{\sin\alpha} = \frac{CP}{\sin A} \quad —(1)$

$\triangle BPC \quad \frac{1-AP}{\sin 2\alpha} = \frac{CP}{\sin B} \quad —(2)$

$A = B \Rightarrow (1) = (2)$

$\therefore \frac{AP}{\sin\alpha} = \frac{1-AP}{2\sin\alpha\cos\alpha}$ or $AP = \frac{1}{1+2\cos\alpha}$

$0^\circ \leq 3\alpha \leq 180^\circ$

$0^\circ \leq \alpha \leq 60^\circ$

$\frac{1}{3} = \frac{1}{1+2(1)} < AP \leq \frac{1}{1+2(\frac{1}{2})} = \frac{1}{2}$

1.5 The Circular Functions of θ as Rational Functions of $\tan\frac{1}{2}\theta$

1.5.1 The Formulae. (G.M P14, 15)

let $t = \tan\frac{1}{2}\theta$

then $\tan\theta = \tan 2(\tfrac{1}{2})\theta$

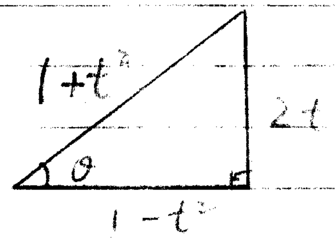
$$= \frac{2t}{1-t^2}$$

$$\sin\theta = 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}$$

$$= \frac{2t}{1+t^2}$$

memorize the Δ



$$\cos\theta = \frac{\sin\theta}{\tan\theta}$$

$$= \frac{1-t^2}{1+t^2}$$

1.5.2 Examples

1.5.2.1 (G.M P.16 ex.4)

Show that if the equation $\sin\theta + k\cos\theta = \sqrt{2}+1$ has a solution then $k^2 \geqslant 2\sqrt{2}+2$

We use another method

$$\sin\theta + k\cos\theta = \sqrt{2}+1$$

$$\sin\theta \times \frac{1}{\sqrt{1+k^2}} + \frac{k}{\sqrt{1+k^2}}\cos\theta = \frac{\sqrt{2}+1}{\sqrt{1+k^2}}$$

$$\sin(\theta+\alpha) = \frac{\sqrt{2}+1}{\sqrt{1+k^2}}, \qquad \cos\alpha = \frac{1}{\sqrt{1+k^2}}$$

$\because$ The equation has a solution

$$\therefore \left|\frac{\sqrt{2}+1}{\sqrt{1+k^2}}\right| \leqslant 1$$

$$\Rightarrow (\sqrt{2}+1)^2 \leqslant 1+k^2$$

$$\therefore k^2 \geqslant 2+1+2\sqrt{2}-1 = 2\sqrt{2}+2$$

1.5 The Circular Functions of θ as Rational Functions of $\tan\frac{1}{2}\theta$

1.5.2 Examples

1.5.2.2 (c.f. G.M. P.49 ex54)

(H.M.2A.13) Prove that for all real values of x.

$$-7 \leq \sin^2 x - 24\sin x\cos x + 11\cos^2 x \leq 19$$

Proof: $\sin^2 x - 24\sin x\cos x + 11\cos^2 x$

$$= \frac{1-\cos 2x}{2} - 12\sin 2x + 11\left[\frac{1+\cos 2x}{2}\right]$$

$$= 6 + 5\cos 2x - 12\sin 2x$$

$$= 6 + 13\sin(\alpha - 2x) \quad , \sin\alpha = \frac{5}{13}$$

$\therefore$ maximum $= 6 + 13 = 19$

minimum $= 6 - 13 = -7$ //

GM P50 ex50b

1.5.2.3 Equations of the Form $a\cos\theta + b\sin\theta = c$

$5\cos\theta - 12\sin\theta + 4 = 0 \qquad 0° \leq \theta \leq 360°$

If $t = \tan\frac{\theta}{2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$, $\sin\theta = \dfrac{2t}{1+t^2}$

$$\therefore 5\left(\frac{1-t^2}{1+t^2}\right) - 12\frac{(2t)}{1+t^2} + 4 = 0$$

$$t^2 + 24t - 9 = 0$$

$$\tan\frac{\theta}{2} = t = \frac{-24 \pm \sqrt{576+36}}{2} = 0.37 \text{ or } -24.37$$

$$\frac{\theta}{2} = 20°18', 200°18' \text{ or } 92°21', 272°21'$$

$0° \leq \theta \leq 360° \Rightarrow \theta = 40°36'$ or $184°42'$ //

1.5.2.6 Express $(3+\cos\theta)\csc\theta$ in terms of $\tan\frac{1}{2}\theta$. Hence show that this expression cannot have value between $-2\sqrt{2}$ and $2\sqrt{2}$

$$E = (3+\cos\theta)\csc\theta$$

$$= \left(3 + \frac{1-t^2}{1+t^2}\right)\frac{1+t^2}{2t}$$

$$= \frac{(2+t^2)}{t} \quad //$$

$$tE = 2 + t^2$$

$$t^2 - tE + 2 = 0$$

t is a real number $\Rightarrow \Delta \geq 0$

$$E^2 - 8 \geq 0$$

$$\therefore \quad E \geq 2\sqrt{2} \text{ or } E \leq -2\sqrt{2}$$

1.5.2.4 Prove that, for real x, the maximum value $c\sin(x+A) + d\sin(x+B)$ is $\sqrt{c^2+d^2+2cd\cos(A-B)}$

Proof: $c\sin(x+A) + d\sin(x+B)$

$= c\sin x\cos A + c\cos x\sin A + d\sin x\cos B + d\cos x\sin B$

$= (c\cos A + d\cos B)\sin x + (c\sin A + d\sin B)\cos x$

$= u\left[\frac{(c\cos A + d\cos B)}{u}\sin x + \frac{c\sin A + d\sin B}{u}\cos x\right]$

where $u = \sqrt{(c\cos A + d\cos B)^2 + (c\sin A + d\sin B)^2}$

$= u[\sin x\cos\alpha + \cos x\sin\alpha]$

where $\cos\alpha = \frac{c\cos A + d\cos B}{u}$

$= u\sin(x+\alpha)$

$\therefore$ maximum $= u = \sqrt{c^2+d^2+2dc\cos(A-B)}$

Tranter Ch5. P133.26

1.5.2.5 Solve $x^3 - 3x - 1 = 0$

let $x = 2\cos\theta$ so that

$8\cos^3\theta - 6\cos\theta - 1 = 0$

using the formula $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

we have $2\cos 3\theta = 1$

and $\cos 3\theta = \frac{1}{2}$

$3\theta = 2n\pi \pm \frac{\pi}{3}$

$\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9}$

$= \frac{\pi}{9}, \frac{7\pi}{9}, \frac{13\pi}{9}$

So $x = 2\cos\theta = 1.879$

or -1.532 $\left(= 2\cos\frac{11\pi}{9}\right)$

or -0.347 $\left(= 2\cos\frac{5\pi}{9}\right)$

1.6 Half-angle Formulae

1.6.1 The Heron's formula

F4 maths P.249 From the cosine formula

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\sin^2 A = 1 - \cos^2 A$$

$$= (1-\cos A)(1+\cos A)$$

$$= \left(1 - \frac{b^2+c^2-a^2}{2bc}\right)\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \frac{2bc-b^2-c^2+a^2}{2bc} \quad \frac{2bc+b^2+c^2-a^2}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc} \cdot \frac{(b+c)^2-a^2}{2bc}$$

$$= \frac{(a+b-c)(a-b+c)}{2bc} \cdot \frac{(b+c+a)(b+c-a)}{2bc}$$

let $2S = a+b+c$

$$2S-2a = b+c-a$$

$$2S-2b = a+c-b$$

$$2S-2c = a+b-c$$

$$\therefore \sin^2 A = \frac{2(S-c)\cdot 2(S-b)}{2bc} \cdot \frac{2S\cdot 2(S-a)}{2bc}$$

$$= \frac{4S(S-a)(S-b)(S-c)}{b^2c^2}$$

for ΔABC, $0^\circ \leq A \leq 180^\circ$ $\therefore$, $\sin A \geq 0$

$$\sin A = +\frac{2}{bc}\sqrt{S(S-a)(S-b)(S-c)} \quad \text{—(1)}$$

$$\text{Area } \Delta = \frac{1}{2}bc\sin A$$

$$= \sqrt{S(S-a)(S-b)(S-c)} \quad \text{—(2)}$$

c.f. G.M P.3) Cosine formula gives $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$$\Rightarrow \quad 2\cos^2\frac{A}{2} - 1 = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}} \qquad \left(\because 0^\circ \leqslant \frac{A}{2} \leqslant 90^\circ \quad \cos\frac{A}{2} \geqslant 0\right)$$

$$= \sqrt{\frac{1+\frac{b^2+c^2-a^2}{2bc}}{2}}$$

$$= \sqrt{\frac{(b^2+c^2+2bc)-a^2}{4bc}}$$

$$= \sqrt{\frac{(b+c)^2 - a^2}{4bc}}$$

$$= \sqrt{\frac{(b+c+a)(b+c-a)}{4bc}}$$

$$= \sqrt{\frac{2s \cdot 2(s-a)}{4bc}}$$

$$= \sqrt{\frac{s(s-a)}{bc}} \quad \text{——(3)}$$

$\because \quad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

by (1) and (3) $\quad \sin\frac{A}{2} = \frac{1}{2} \times \frac{\sin A}{\cos\frac{A}{2}}$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{bc}\sqrt{s(s-a)(s-b)(s-c)} \times \sqrt{\frac{bc}{s(s-a)}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \text{——(4)}$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad \text{——(5)}$$

The formulae for $\sin\frac{B}{2}$, $\sin\frac{C}{2}$ etc are symmetric.

1.6.2 Examples

1.6.2.1 GM P.32 ex.36

For $\triangle ABC$, show that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$

We use another method:

$$\sin\tfrac{1}{2}A\ \sin\tfrac{B}{2}\sin\tfrac{C}{2} = \tfrac{1}{2}\sin\tfrac{A}{2}\left[\cos\tfrac{B-C}{2} - \cos\tfrac{B+C}{2}\right]$$

$$\le \tfrac{1}{2}\sin\tfrac{A}{2}\left[1 - \sin\tfrac{A}{2}\right], \quad \tfrac{B+C}{2} = 90^\circ - \tfrac{A}{2}$$

$$\le \tfrac{1}{2}\left(\frac{\sin\frac{A}{2} + 1 - \sin\frac{A}{2}}{2}\right)^2, \quad (AM \ge GM)$$

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$$= \tfrac{1}{8}$$

1.6.2.2 Given θ acute angle, if, in $\triangle ABC$, $(b+c)\cos\theta = 2\sqrt{bc}\,\cos\frac{1}{2}A$ prove $a = (b+c)\sin\theta$

Hence find a, given $b=18.7$, $c=16.4$, $A=57^\circ$

Proof: use the formula $\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$

$$(b+c)\cos\theta = 2\sqrt{bc}\sqrt{\frac{s(s-a)}{bc}}$$

you cannot use other method

$$= 2\sqrt{s(s-a)}$$

$$= \sqrt{(a+b+c)(b+c-a)}$$

$$\cos\theta = \sqrt{\left(1+\tfrac{a}{b+c}\right)\left(1-\tfrac{a}{b+c}\right)}$$

$$= \sqrt{1 - \left(\tfrac{a}{b+c}\right)^2}$$

$$\cos^2\theta = 1 - \left(\tfrac{a}{b+c}\right)^2$$

$$\therefore \sin\theta = \frac{a}{b+c} \quad (\because \sin\theta \ge 0)$$

$b=18.7 \quad c=16.4 \quad A=57^\circ$

$$\cos\theta = \frac{2\sqrt{bc}\,\cos\frac{1}{2}A}{b+c}$$

$$= \frac{2\sqrt{18.7\times16.4}\,\cos 28.5^\circ}{18.7+16.4}$$

$$= 0.8769$$

$$\theta \doteq 28.7^\circ$$

$$a = (b+c)\sin\theta = (18.7+16.4)\sin 28.7^\circ = 16.87$$

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16.2.3 In $\triangle ABC$, prove that $a^2-(b-c)^2\cos^2\frac{1}{2}A=(b+c)^2\sin^2\frac{1}{2}A$

$$LHS = a^2-(b-c)^2\frac{s(s-a)}{bc}$$

$$=\frac{a^2bc-(b-c)^2\times\frac{1}{4}(a+b+c)(b+c-a)}{bc}$$

$$=\frac{a^2bc-(b-c)^2\times\frac{1}{4}((b+c)^2-a^2)}{bc}$$

$$=\frac{a^2bc+\frac{1}{4}(b^2+c^2-2bc)a^2-\frac{1}{4}(b-c)^2(b+c)^2}{bc}$$

$$=\frac{\frac{1}{4}(b^2+c^2+2bc)a^2-\frac{1}{4}(b-c)^2(b+c)^2}{bc}$$

$$=\frac{\frac{1}{4}(b+c)^2(a^2-(b-c)^2)}{bc}$$

$$=\frac{(b+c)^2(s-c)(s-b)}{bc}$$

$$=(b+c)^2\sin^2\frac{1}{2}A = RHS$$

1.7 Sum and Product Formulae

1.7.1 The Formulae

1.7.1.1 Type 1 Transformation of Products into Sum or Differences

see GM $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

P. 19 other formulae can be easily derive

memorize

$$S + S \rightarrow 2SC$$
$$S - S \rightarrow 2CS$$
$$C + C \rightarrow 2CC$$
$$C - C \rightarrow -2SS$$

Is there a formula for $S + C$? (No)

1.7.1.2 Type 2 Transformation of Sums or Differences into Product

$$\sin C + \sin D = 2\sin\tfrac{1}{2}(C+D)\cos\tfrac{1}{2}(C-D)$$

proof: let $\left.\begin{array}{l} C = x+y \\ D = x-y \end{array}\right\} \Rightarrow \begin{array}{l} x = \frac{C+D}{2} \\ y = \frac{C-D}{2} \end{array}$

$$\begin{aligned} \sin C + \sin D &= \sin(x+y) + \sin(x-y) \\ &= 2\sin x\cos y \\ &= 2\sin\tfrac{1}{2}(C+D)\cos\tfrac{1}{2}(C-D) \end{aligned}$$

The other formulae are proved similarly.

You are advised <u>not</u> to memorize this part, only memorize type I.

Each time you use the formula(e), derive it by yourself.

1.7.2 Examples

HM P16 ex 9

1.7.2.1 If $A+B+C = 180°$ prove $\cos^2 A + (\cos B + \cos C)^2 - 1 = 4\sin^2\tfrac{1}{2}A\cos B\cos C$

Hence if also $\cos B + \cos C - \cos A = 1$ prove that

$\sec A - \sec B - \sec C = 1$

Proof

$$\begin{aligned} LHS &= \cos^2 A + \left(2\cos\tfrac{B+C}{2}\cos\tfrac{B-C}{2}\right)^2 - 1 \\ &= 4\sin^2\tfrac{A}{2}\cos^2\tfrac{B-C}{2} - \sin^2 A \qquad \left(\because \cos\tfrac{B+C}{2} = \sin\tfrac{A}{2}\right) \\ &= 4\sin^2\tfrac{A}{2}\left[\cos^2\tfrac{B-C}{2} - \cos^2\tfrac{A}{2}\right] \end{aligned}$$

$$LHS = 4\sin^2\frac{A}{2}\left[\frac{1+\cos(B-C)}{2} - \frac{1-\cos(B+C)}{2}\right] \quad (\because \cos\frac{A}{2} = \sin\frac{B+C}{2})$$
$$= 4\sin^2\frac{A}{2}\cos B\cos C = RHS \;//$$

If also $\cos B + \cos C - \cos A = 1$

$$\Rightarrow \cos B + \cos C = 1 + \cos A.$$

$\therefore$ Expression becomes:

$$\cos^2 A + (1+\cos A)^2 - 1 = 4\sin^2\tfrac{1}{2}A\cos B\cos C$$
$$2\cos^2 A + 2\cos A = 4\sin^2\tfrac{1}{2}A\cos B\cos C$$
$$\cos^2 A + \cos A = (1-\cos A)\cos B\cos C$$
$$\cos A(\cos B + \cos C) = (1-\cos A)\cos B\cos C$$
$$\sec B + \sec C = \sec A - 1$$
$$\therefore \sec A - \sec B - \sec C = 1 \;//$$

H.M P.20 ex 13

1.7.2.2 In $\triangle ABC$ if $\sin A$, $\sin B$, $\sin C$ form an A.P. prove $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ form an H.P.

Prof: $2\sin B = \sin A + \sin C$ (given AP)

$$2\sin B = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$$
$$\sin B = \cos\frac{B}{2}\cos\frac{A-C}{2} \quad (\because \frac{A+C}{2} = 90^\circ - \frac{B}{2})$$
$$2\sin\frac{B}{2} = \cos\frac{A-C}{2} \quad (\text{clearly } \cos\frac{B}{2} \neq 0)$$

Now
$$\frac{1}{\tan\frac{A}{2}} + \frac{1}{\tan\frac{C}{2}} = \frac{\tan\frac{A}{2} + \tan\frac{C}{2}}{\tan\frac{A}{2}\tan\frac{C}{2}}$$
$$= \frac{\sin\frac{A}{2}\cos\frac{C}{2} + \cos\frac{A}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}}$$
$$= \frac{\sin\frac{A+C}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}}$$
$$= \frac{\cos\frac{B}{2}}{\frac{1}{2}(\cos\frac{A-C}{2} - \cos\frac{A+C}{2})}$$
$$= \frac{2\cos\frac{B}{2}}{2\sin\frac{B}{2} - \cos\frac{A+C}{2}} = \frac{2}{\tan\frac{B}{2}} \quad \#$$

$\therefore \tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ form an H.P.

H.M. P.22 ex ~~19~~, 22

1.7.2.3 Prove the identities:

(a) $\sin^2\theta + \sin^2\phi - \sin^2(\theta-\phi) = 2\sin\theta\sin\phi\cos(\theta-\phi)$

(b) $\dfrac{\tan 3\theta - 2\tan 2\theta + \tan\theta}{4(\tan 3\theta - \tan\theta)} = \sin^2\theta$.

c) $\sin^2(2\theta+\phi) + \sin^2(2\phi+\theta) - \sin^2(\theta-\phi) = 2\cos(\theta-\phi)\sin(2\theta+\phi)\sin(2\phi+\theta)$

a) $$\begin{aligned} RHS &= 2\sin\theta\sin\phi\cos(\theta-\phi) \\ &= \sin\theta\,(\sin\theta + \sin(2\phi-\theta)) \\ &= \sin^2\theta + \sin\theta\sin(2\phi-\theta) \\ &= \sin^2\theta + \tfrac{1}{2}[\cos 2(\theta-\phi) - \cos 2\phi] \\ &= \sin^2\theta + \tfrac{1}{2}[1 - 2\sin^2(\theta-\phi) - 1 + 2\sin^2\phi] \\ &= LHS \end{aligned}$$

always find the <u>abnormal</u> method!

b) $$LHS = \frac{1}{4} - \frac{\tan 2\theta - \tan\theta}{4(\tan 3\theta - \tan 2\theta)}$$ (think why this step is necessary)

$$= \frac{1}{4} - \frac{\frac{1}{\cos 2\theta}\frac{1}{\cos\theta}\sin\theta}{4\frac{1}{\cos 3\theta}\frac{1}{\cos 2\theta}\sin\theta}$$ (why?)

$$\begin{aligned} &= \frac{1}{4} - \frac{1}{4}\frac{\cos 3\theta}{\cos\theta}. \\ &= \frac{1}{4} - \frac{1}{4}\frac{(4\cos^3\theta - 3\cos\theta)}{\cos\theta}. \\ &= 1 - \cos^2\theta \\ &= \sin^2\theta \end{aligned}$$ (miracle!)

c) Only one line is allowed — try! replace θ by $2\theta+\phi$, ϕ by $2\phi+\theta$ in a) and get the result!

E.M 1B 15

1.7.2.4 If $A+B+C=180°$, prove that

deferred until $\sin^3 A\sin(B-C)+\sin^3 B\sin(C-A)+\sin^3 C\sin(A-B)=0$

H.W are collected but $\sin^2 A\sin(B-C)+\sin^2 B\sin(C-A)+\sin^2 C\sin(A-B)=0$

only if $\triangle ABC$ is isos.

Proof: note that $\sin^3 A\sin(B-C)$ (always find the symmetry first)

$=\sin^2 A\ \frac{1}{2}[\cos(A+C-B)-\cos(A+B-C)]$

$=\frac{1}{2}\sin^2 A[\cos 2C-\cos 2B]$ ($\because A+B+C=180°$)

$=\sin^2 A(\sin^2 B-\sin^2 C)$

$\therefore LHS=\sin^2 A(\sin^2 B-\sin^2 C)+\sin^2 B(\sin^2 C-\sin^2 A)+\sin^2 C(\sin^2 A-\sin^2 B)$

$=0$ (That's why symmetry works!)

now $\sin^2 A\sin(B-C)+\sin^2 B\sin(C-A)+\sin^2 C\sin(A-B)=0$ (given)

$\Rightarrow \sin A(\sin^2 B-\sin^2 C)+\sin B(\sin^2 C-\sin^2 A)+\sin C(\sin^2 A-\sin^2 B)=0$

(why? Use the result in first part)

$\sin A(\sin^2 B-\sin^2 C)+\sin B\sin C(\sin C-\sin B)+\sin^2 A(\sin C-\sin B)=0$

$(\sin B-\sin C)[(\sin A)(\sin B+\sin C)-\sin B\sin C-\sin^2 A]=0$

$(\sin B-\sin C)(\sin A-\sin B)(\sin C-\sin A)=0$

$\Rightarrow A=B$ or $B=C$ or $C=A$

$\therefore \triangle ABC$ is isos.

Remark: p only if q means if p then q

but not confuse: if q then p ✗

Q: What is mean by: p provided q ?

P. 13 ex5
Greenwood I

1.7.2.5 If n is an odd integer, and A, B, C are the angles of a triangle, prove that
Cf. P.21 ex24

$$\sin nA + \sin nB + \sin nC = 4 \sin\frac{n\pi}{2} \cos\frac{nA}{2} \cos\frac{nB}{2} \cos\frac{nC}{2}$$

Proof: $LHS = (\sin nA + \sin nB) + \sin nC$

$$= 2 \sin\frac{n(A+B)}{2} \cos\frac{n(A-B)}{2} + \sin nC$$

$$= 2 \sin\frac{n(\pi-C)}{2} \cos\frac{n(A-B)}{2} + 2\sin\frac{nC}{2}\cos\frac{nC}{2}$$

$$= 2 \sin\left[k\pi + \left(\frac{\pi}{2} - \frac{nC}{2}\right)\right] \cos\frac{n(A-B)}{2} + 2\sin\frac{nC}{2}\cos\frac{nC}{2}, \quad n = 2k+1$$

$$= (-1)^k\, 2\cos\frac{nC}{2} \cos\frac{n(A-B)}{2} + 2\sin\frac{nC}{2}\cos\frac{nC}{2}.$$

$$= 2\sin\left(k\pi + \frac{\pi}{2}\right) \cos\frac{nC}{2} \cos\frac{n(A-B)}{2} + 2\sin\frac{nC}{2}\cos\frac{nC}{2}$$

$$= 2\cos\frac{nC}{2}\left[\sin\frac{n\pi}{2} \cos\frac{n(A-B)}{2} + \sin\frac{nC}{2}\right]$$

$$= 2\cos\frac{nC}{2}\left[\sin\frac{n\pi}{2} \cos\frac{n(A-B)}{2} + \sin\frac{n(\pi-(A+B))}{2}\right]$$

$$= 2\cos\frac{nC}{2}\left\{\sin\frac{n\pi}{2} \cos\frac{n(A-B)}{2} + \sin\left[k\pi + \frac{\pi}{2} - \frac{n(A+B)}{2}\right]\right\}$$

$$= 2\cos\frac{nC}{2}\left[\sin\frac{n\pi}{2} \cos\frac{n(A-B)}{2} + (-1)^k \cos\frac{n(A+B)}{2}\right]$$

$$= 2\cos\frac{nC}{2}\left[\sin\frac{n\pi}{2}\right]\left[\cos\frac{n(A-B)}{2} + \cos\frac{n(A+B)}{2}\right]$$

$$= 4\sin\frac{n\pi}{2} \cos\frac{nA}{2} \cos\frac{nB}{2} \cos\frac{nC}{2} \qquad = RHS$$

Remark: Find a <u>necessary</u> condition that: In $\triangle ABC$, n is even and $\sin nA + \sin nB + \sin nC = 0$ (why?)
Is this condition <u>sufficient</u>?

"Necessary" $\sin nA + \sin nB + \sin nC = 0$

$$\Rightarrow \sin 2kA + \sin 2kB + \sin 2kC = 0, \quad n = 2k$$

$$\Rightarrow 2\sin k(A+B)\cos k(A-B) + 2\sin k(\pi-(A+B))\cos k(\pi-(A+B)) = 0$$

$$\Rightarrow \sin k(A+B)\cos k(A-B) + (-1)^{k+1}\sin k(A+B)\,(-1)^k\cos k(A+B) = 0$$

$$\Rightarrow \sin k(A+B)\left[\cos k(A-B) - \cos k(A+B)\right] = 0$$

$$\Rightarrow \sin k(A+B)\,\sin kA\,\sin kB = 0$$

$$\Rightarrow k(A+B) = m\pi \text{ or } kA = m\pi \text{ or } kB = m\pi$$

$\therefore$ necessary condition is: Any one angle is $\frac{m\pi}{k}$ or any sum of two angle is $\frac{m\pi}{k}$.

The condition is "<u>sufficient</u>"
If $A = \frac{m\pi}{k}$ or $(A+B) = \frac{m\pi}{k}$ trace back the '$\Rightarrow$' sign.

H.M IA ex3

1.72.6 If $\sin\theta+\sin\phi=a$, $\tan\theta+\tan\phi=b$, $\sec\theta+\sec\phi=c$

prove that $8bc=a[4b^2+(b^2-c^2)^2]$

Proof: $b^2-c^2=(\tan\theta+\tan\phi)^2-(\sec\theta+\sec\phi)^2$

$=-2+2\tan\theta\tan\phi-2\sec\theta\sec\phi$.

$=\dfrac{-2\cos(\theta+\phi)}{\cos\theta\cos\phi}-\dfrac{2}{\cos\theta\cos\phi}$

$4b^2+(b^2-c^2)^2=4(\tan\theta+\tan\phi)^2+4\dfrac{\cos^2(\theta+\phi)+2\cos(\theta+\phi)+1}{\cos^2\theta\cos^2\phi}$

$=4\dfrac{1+2\cos(\theta+\phi)+1}{\cos^2\theta\cos^2\phi}$

$=8\dfrac{1+\cos(\theta+\phi)}{\cos^2\theta\cos^2\phi}$.

$a[4b^2+(b^2-c^2)^2]=(\sin\theta+\sin\phi)\times 8\dfrac{1+\cos(\theta+\phi)}{\cos^2\theta\cos^2\phi}$

$=\dfrac{8}{\cos^2\theta\cos^2\phi}\left(2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right)\left(2\cos^2\frac{\theta+\phi}{2}\right)$

$LHS=8bc=8(\tan\theta+\tan\phi)(\sec\theta+\sec\phi)$

$=8\dfrac{\sin(\theta+\phi)(\cos\theta+\cos\phi)}{\cos^2\theta\cos^2\phi}$

$=\dfrac{8}{\cos^2\theta\cos^2\phi}\left[2\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2}\left(2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right)\right]=RHS$

Greenwood IP.10 ex.1.11

1.7.2.7 Prove that $\sin\theta+\sin(\theta+\alpha)+\dots+\sin(\theta+(n-1)\alpha)=\dfrac{\sin\left(\theta+\frac{(n-1)\alpha}{2}\right)\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$

Proof: $2\sin\frac{\alpha}{2}[\sin\theta+\sin(\theta+\alpha)+\dots+\sin(\theta+(n-1)\alpha)]$

$=\sum_{r=0}^{n-1}\left[2\sin\frac{\alpha}{2}\sin(\theta+r\alpha)\right]$

$=\sum_{r=0}^{n-1}\left[\cos\left(\theta+\frac{2r-1}{2}\alpha\right)-\cos\left(\theta+\left(\frac{2r+1}{2}\right)\alpha\right)\right]$

$=\cos\left(\theta-\frac{\alpha}{2}\right)-\cos\left(\theta+\frac{2n-1}{2}\alpha\right)$

$=2\sin\left(\theta+\frac{n-1}{2}\alpha\right)\sin\frac{n\alpha}{2}$

$\therefore$ dividing by $2\sin\frac{\alpha}{2}$, we get the result.

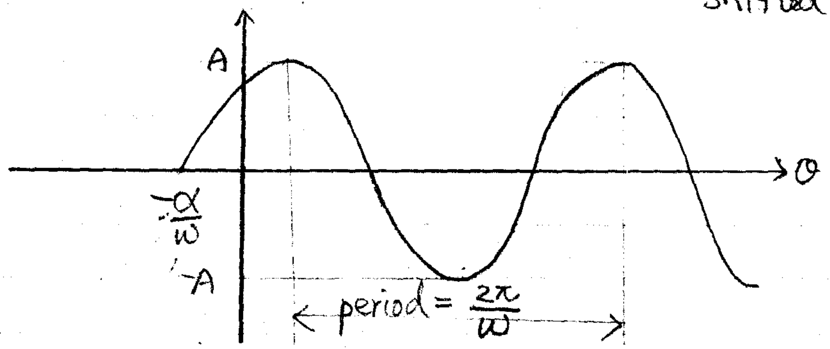
1.8 Solution of Trigonometrical Equation

1.8.1 Graph of trigonometrical Function.

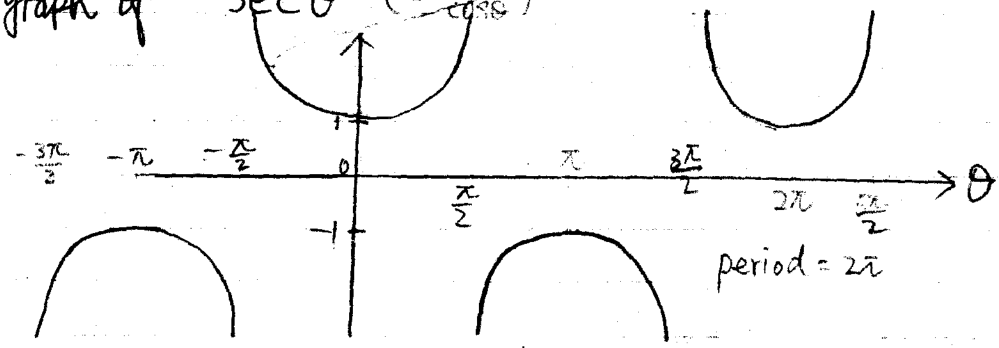
The graph of $\sin\theta$, $\cos\theta$, $\tan\theta$ are omitted

1.8.1.1 graph of $A\sin(\omega\theta+\alpha)$

"shifted sine curve"

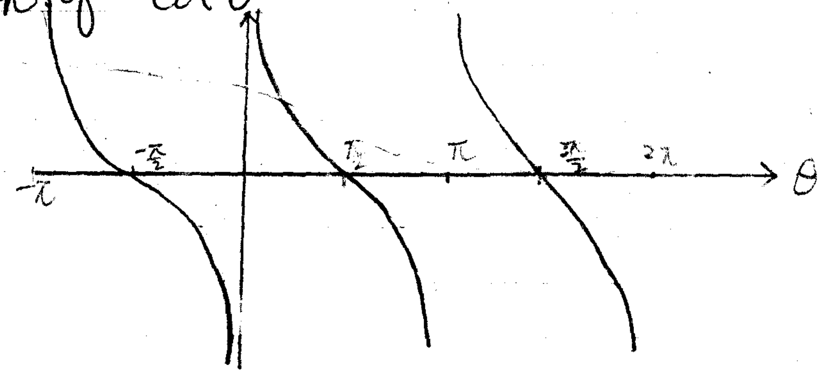


1.8.1.2 graph of $\sec\theta$ $(=\frac{1}{\cos\theta})$

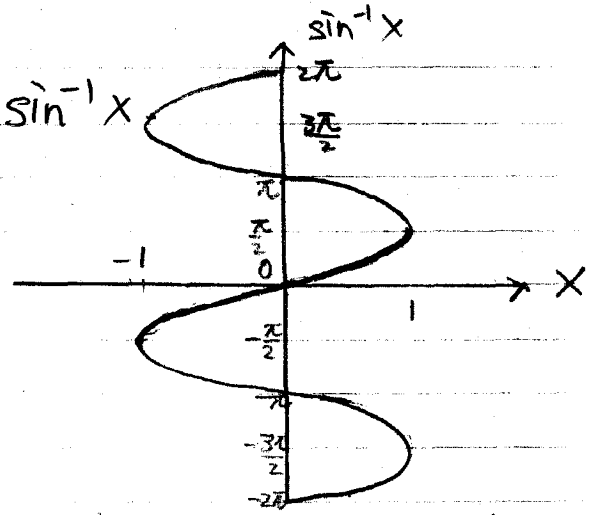


exercise: draw the graph of $\csc\theta$ $(=\frac{1}{\sin\theta})$

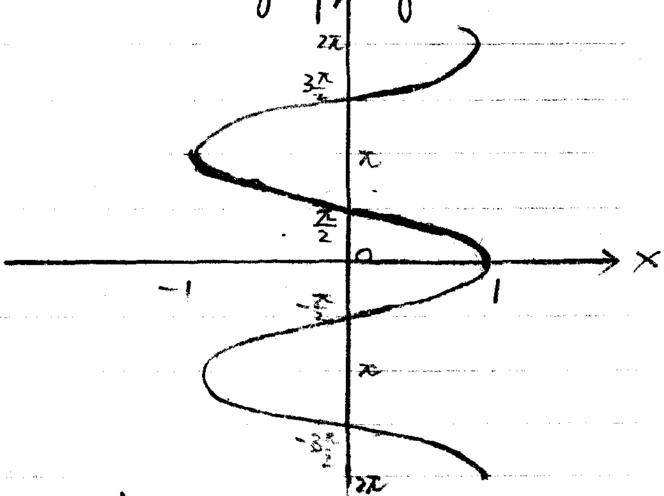
1.8.1.3 graph of $\cot\theta$



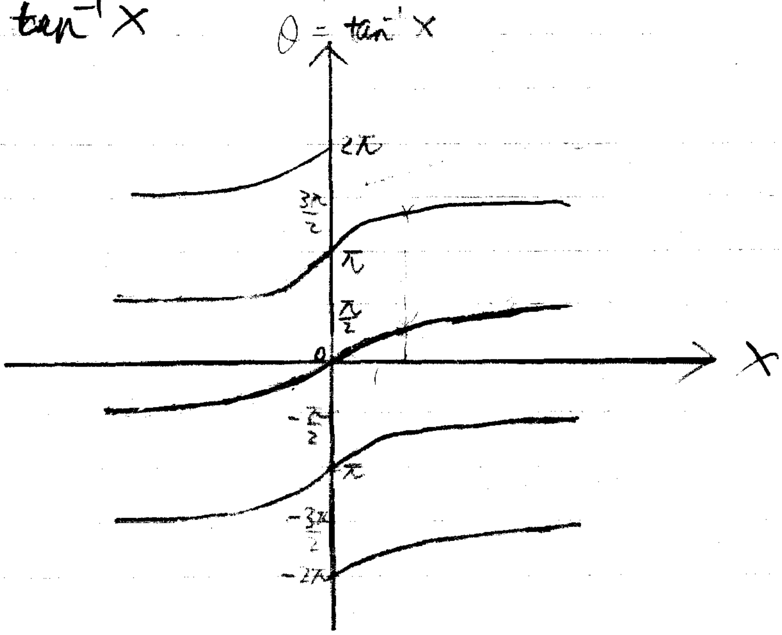
1.8.1.4 graph of $\sin^{-1} x$



1.8.1.5 exercise draw the graph of $\cos^{-1} x$



1.8.1.6 graph of $\tan^{-1} x$



1.8.1.7 Remark: The "blue line" corresponds to the <u>Principal Value</u>

In fact, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ are <u>not functions</u> at all unless principal values are restricted.

$$\therefore \quad \theta = \sin^{-1}x \ , \quad -\tfrac{\pi}{2} \leqslant \theta \leqslant \tfrac{\pi}{2}$$
$$\theta = \cos^{-1}x \ , \quad 0 \leqslant \theta \leqslant \pi \ .$$
$$\theta = \tan^{-1}x \ , \quad -\tfrac{\pi}{2} < \theta \leqslant \tfrac{\pi}{2}$$

1.8.2 General Solution (G.M P.42,43)

$\sin x = \sin\alpha$

$x = n\pi + (-1)^n\alpha$, α is in radian

$= 180^\circ n + (-1)^n\alpha$, α is in degree.

n is any whole number

$\cos x = \cos\alpha$.

$x = 360^\circ n \pm \alpha^\circ$

$= 2n\pi \pm \alpha^c$

$\tan x = \tan\alpha$

$x = 180^\circ n + \alpha^\circ$

$= n\pi + \alpha^c$

1.8.3 Examples

1.8.3.1 Solve $\sin 2^x = -\frac{1}{2}$

Greenwood II P.4 ex.1.4

Solution $2^x = n\pi + (-1)^n(-\frac{\pi}{6})$, $n = 0, \pm1, \pm2, \ldots$

$$x = \frac{\log_{10}[n\pi + (-1)^{n+1}\frac{\pi}{6}]}{\log_{10} 2}$$

$\because (-1)^{n+1}(\frac{\pi}{6}) + n\pi > 0 \quad \therefore \quad n = 1, 2, 3, 4, \ldots$

Greenwood I p.7 ex.1.9

1.8.3.2 Solve simultaneously $\begin{cases} 5^{\tan x}\cdot 7^{\tan y}=175 & —(1) \\ 5^{\tan y}\cdot 7^{\tan x}=245 & —(2) \end{cases}$

Solution: (1) × (2) $(5\cdot 7)^{\tan x+\tan y}=175\cdot 245$

$(35)^{\tan x+\tan y}=35^3$

$\therefore \tan x+\tan y=3$ — (3)

$\frac{(1)}{(2)}$ $5^{\tan x-\tan y}\cdot 7^{\tan y-\tan x}=\frac{175}{245}$

$\left(\frac{5}{7}\right)^{\tan x-\tan y}=\frac{5}{7}$

$\therefore \tan x-\tan y=1$ — (4)

$\frac{(3)+(4)}{2}$ $\tan x=2$

$\frac{(3)-(4)}{2}$ $\tan y=1$

$\therefore \begin{cases} x=n\pi+\tan^{-1}2 \\ y=n\pi+\tan^{-1}1 \end{cases}$ //

Always in radian!

GM IE3a

1.8.3.3 Solve $4\cos\theta-3\sin\theta=1$, $0°\leq\theta\leq 360°$

Solution: $4\cos\theta=1+3\sin\theta$

$16(1-\sin^2\theta)=1+9\sin^2\theta+6\sin\theta$

$25\sin^2\theta+6\sin\theta-15=0$

$\sin\theta=\frac{-3\pm\sqrt{9+375}}{25}=0.66$ or $^-0.90$

$\theta=41.6°$ or $138.4°$ or $244.7°$ or $295.3°$

Check: $\theta=41.6°$ LHS $=1$ O.K.

$\theta=138.4°$ LHS $=-4.98$ reject.

$\theta=244.7°$ LHS $=1$ O.K

$\theta=295.3°$ LHS $=4.42$ reject //

GM P50 ex 15b

1.8.3.4 $x = R\sin(\theta+\beta)$

$y = R\sin(\theta-\beta)$, $R>0$, $0<\beta<\frac{\pi}{2}$.

Find the maximum of $(x+y)^2$ and θ when $(x+y)^2$ is maximum

Solution $x+y = R[\sin(\theta+\beta)+\sin(\theta-\beta)]$

$= 2R\sin\theta\cos\beta$

$\therefore (x+y)^2 = 4R^2\sin^2\theta\cos^2\beta$

$\therefore$ maximum of $(x+y)^2$ is $4R^2\cos^2\beta$ attains at $\sin\theta = \pm 1$

i.e $\theta = n\pi \pm \frac{\pi}{2}$, $n = 0, \pm1, \pm2, \pm3, \ldots$

GMIE ex 15

1.8.3.5 Solve $\cos 3\theta - (\sqrt{3}+1)\cos 2\theta + (\sqrt{3}+3)\cos\theta - \sqrt{3} - 1 = 0$

$(4\cos^3\theta - 3\cos\theta) - (\sqrt{3}+1)(2\cos^2\theta - 1) + (\sqrt{3}+3)\cos\theta - (\sqrt{3}+1) = 0$

$\cos\theta[4\cos^2\theta - 2(\sqrt{3}+1)\cos\theta + \sqrt{3}] = 0$

$\cos\theta(2\cos\theta - 1)(2\cos\theta - \sqrt{3}) = 0$

$\cos\theta = 0$ or $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$.

$\theta = 2n\pi \pm \frac{\pi}{6}$ or $2n\pi \pm \frac{\pi}{3}$, or $2n\pi \pm \frac{\pi}{2}$, $n = 0, \pm1, \pm2$

GM IE28

1.8.3.6 Solve $\tan(\frac{\pi}{4}-x) + \cot(\frac{\pi}{4}-x) = 4$

$\frac{1}{\sin(\frac{\pi}{4}-x)\cos(\frac{\pi}{4}-x)} = 4$ (why?)

$\sin(\frac{\pi}{2}-2x) = \frac{1}{2}$

$\cos 2x = \frac{1}{2}$

$x = n\pi \pm \frac{\pi}{6}$, $n = 0, \pm1, \pm2, \ldots$

The method of solving by graph (eg $\sin x = \frac{1}{2}x$) are omitted.

1.9 Inverse Circular Functions

1.9.1 Definitions $\theta = \sin^{-1} x$, $-1 \le x \le 1$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

only principal value are involved.

$\theta = \cos^{-1} x$ $-1 \le x \le 1$ $0 \le \theta \le \pi$

$\theta = \tan^{-1} x$, any x $-\frac{\pi}{2} < \theta \le \frac{\pi}{2}$

see 1.8.1.4 - 1.8.1.7

1.9.2 Properties :

a) $\sin^{-1}(-x) = -\sin^{-1} x$

b) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

c) $\tan^{-1}(-x) = -\tan^{-1} x$

d) $\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$

e) $\tan^{-1} x = \frac{\pi}{2} - \tan^{-1} \frac{1}{x}$

f) $\cot^{-1} x = \tan^{-1} \frac{1}{x}$

g) For $0 \le x \le 1$ $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$

For $-1 \le x < 0$ $\sin^{-1} x = -\cos^{-1} \sqrt{1-x^2}$.

(This is because the Principal values are different)

h) For $0 \le x \le 1$ $\cos^{-1} x = \sin^{-1} \sqrt{1-x^2}$.

For $-1 \le x < 0$ $\cos^{-1} x = \pi - \sin^{-1} \sqrt{1-x^2}$

1.9.3 Combination of inverse circular functions

Usually, compounded angle formulae are used.

1.9.3.1 If $-\frac{\pi}{2} \le \sin^{-1} x + \sin^{-1} y \le \frac{\pi}{2}$ then $\sin^{-1} x + \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

if $\pi \ge \sin^{-1} x + \sin^{-1} y > \frac{\pi}{2}$ then $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

if $-\pi \le \sin^{-1} x + \sin^{-1} y < -\frac{\pi}{2}$ then $\sin^{-1} x + \sin^{-1} y = -\pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

1.9.3.2 If $0 \le \cos^{-1} x + \cos^{-1} y \le \pi$, $\cos^{-1} x + \cos^{-1} y = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$

exercise : Find out the other relation of $\cos^{-1} x + \cos^{-1} y$ when it lies between $-\pi$ and 0

1.9.3.3 If $-\frac{\pi}{2} \leq \tan^{-1}x + \tan^{-1}y \leq \frac{\pi}{2}$ then $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

exercise find out the other two relation if

1° $\frac{\pi}{2} < \tan^{-1}x + \tan^{-1}y \leq \pi$

2° $-\pi \leq \tan^{-1}x + \tan^{-1}y < -\frac{\pi}{2}$

1.9.4 Examples

1.9.4.1 Prove that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $-1 \leq x \leq 1$

B.Sc P.102 ex12

Proof: $\sin(\sin^{-1}x + \cos^{-1}x) = x^2 + (\cos\sin^{-1}x)(\sin\cos^{-1}x)$

$= x^2 + (\sqrt{1-x^2})^2$ (by 1.9.2 g, h)

$= 1$

$= \sin\frac{\pi}{2}$

$\therefore \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ (clearly $-\frac{\pi}{2} \leq \sin^{-1}x + \cos^{-1}x \leq \frac{3\pi}{2}$)

1.9.4.2 Solve $\begin{cases} \sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3} & -(1) \\ \cos^{-1}x - \cos^{-1}y = \frac{\pi}{3} & -(2) \end{cases}$ $-1 \leq x, y \leq 1$

B.Sc P.33 ex12

(1) + (2) $\frac{\pi}{2} + \sin^{-1}y - \cos^{-1}y = \pi$ (by 1.9.4.1)

$\sin^{-1}y - \cos^{-1}y = \frac{\pi}{2}$ —(3)

$y^2 - (1-y^2) = 1$ (take sine)

$y^2 = 1$

$y = \pm 1$

<u>Check</u> $y = -1$ $\sin^{-1}y - \cos^{-1}y = -\frac{\pi}{2} - \pi \neq \frac{\pi}{2}$ (reject)

$y = 1$ $\sin^{-1}y - \cos^{-1}y = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ (accept)

Sub $y=1$ into (1) $\sin^{-1}x + \frac{\pi}{2} = \frac{2\pi}{3}$

$\sin^{-1}x = \frac{\pi}{6}$

$x = \frac{1}{2}$

$\therefore \begin{cases} x = \frac{1}{2} \\ y = 1 \end{cases}$ //

Note that in the equation involving inverse circular functions, we have to <u>check</u> the result.

HM 2B 21 / GM 1 F 21

1.9.4.3 If p, q, r are all positive, prove that

$$\tan^{-1}\frac{p-q}{1+pq} + \tan^{-1}\frac{q-r}{1+qr} = \tan^{-1}\frac{p-r}{1+rp}$$

Proof: $LHS = \tan^{-1}p - \tan^{-1}q + \tan^{-1}q - \tan^{-1}r$

$= \tan^{-1}p - \tan^{-1}r$

$= \tan^{-1}\frac{p+r}{1-rp} = RHS$ //

HM 2B 27 / GM 1 F 27

1.9.4.4 Solve $\sec^{-1}a + \sec^{-1}\frac{x}{a} = \sec^{-1}b + \sec^{-1}\frac{x}{b}$

Solution $\cos^{-1}\frac{1}{a} - \cos^{-1}\frac{a}{x} = \cos^{-1}\frac{1}{b} - \cos^{-1}\frac{b}{x}$

$\frac{1}{x} - \frac{\sqrt{a^2-1}}{a}\frac{\sqrt{x^2-a^2}}{x} = \frac{1}{x} - \frac{\sqrt{b^2-1}}{b}\frac{\sqrt{x^2-b^2}}{x}$ (take cosine)

clearly $x \neq 0, a \neq 0, b \neq 0, |a| \geq 1, |b| \geq 1, |x| \geq \min(|a|,|b|)$

$b^2(a^2-1)(x^2-a^2) = a^2(b^2-1)(x^2-b^2)$

$[b^2(a^2-1) - a^2(b^2-1)]x^2 = a^2b^2(a^2-1) - a^2b^2(b^2-1)$

$(a^2-b^2)x^2 = a^2b^2(a^2-b^2)$

if $a = b$ the solution is $|x| \geq |a|$

if $a = -b > 0$ (say)

The equation becomes $\cos^{-1}\frac{1}{a} - \cos^{-1}\frac{a}{x} = \cos^{-1}\frac{a}{x} - \cos^{-1}\frac{1}{a}$

$\Rightarrow \frac{1}{a} = \frac{a}{x}$

$x = a^2$

if $a^2 \neq b^2 \Rightarrow x = \pm ab$

<u>Check</u> $x = -ab$

$LHS = \cos^{-1}\frac{1}{a} + \cos^{-1}\frac{a}{x}$

$= \cos^{-1}\frac{1}{a} + \cos^{-1}-\frac{1}{b}$

$= \cos^{-1}\frac{1}{a} - \cos^{-1}\frac{1}{b} + \pi$ (by 1.9.2.b)

$RHS = \cos^{-1}\frac{1}{b} - \cos^{-1}-\frac{b}{x}$

$= \cos^{-1}\frac{1}{b} - \cos^{-1}\frac{1}{a} + \pi$

$\neq LHS$ (reject) $a \neq b$

If $x = ab$

$LHS = \cos^{-1}\frac{1}{a} + \cos^{-1}\frac{a}{x}$

$$LHS = \cos^{-1}\frac{1}{a} + \cos^{-1}\frac{1}{b}$$

$$RHS = \cos^{-1}\frac{1}{b} + \cos^{-1}\frac{1}{a} \qquad a \neq b \qquad (\text{accept}).$$

Conclusion: $\begin{cases} |x| \geq |a| & \text{if } a = b \\ x = a^2 & \text{if } a = -b \\ x = ab & \text{if } a^2 \neq b^2. \end{cases}$

Greenwood P14 ex.2 3

1.9.4.5 Prove that $\tan^{-1}(r+1) - \tan^{-1} r = \cot^{-1}(1+r+r^2)$ and hence sum $\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \cdots + \cot^{-1}(1+n+n^2)$

$$\tan^{-1}(r+1) - \tan^{-1} r = \tan^{-1}\frac{r+1-r}{1+(r+1)r} \qquad \left(\text{clearly } -\frac{\pi}{2} \leq \tan^{-1}(r+1) - \tan^{-1} r \leq \frac{\pi}{2}\right)$$

$$= \tan^{-1}\frac{1}{1+r+r^2}$$

$$= \cot^{-1}(1+r+r^2)$$

$$\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \ldots + \cot^{-1}(1+n+n^2) = \sum_{r=1}^{n} \cot^{-1}(1+r+r^2)$$

$$= \sum_{r=1}^{n}\left[\tan^{-1}(r+1) - \tan^{-1} r\right]$$

$$= \tan^{-1}(n+1) - \frac{\pi}{4}$$

GH IF 14

1.9.4.6 Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Proof: $LHS = \tan^{-1}\dfrac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}} + \tan^{-1}\dfrac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}}$

$$= \tan^{-1}\frac{4}{7} + \tan^{-1}\frac{3}{11}$$

$$= \tan^{-1}\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{12}{77}}$$

$$= \tan^{-1}\frac{65}{65}$$

$$= \frac{\pi}{4} = RHS \;//$$