Individual Events

I1	a	2	I2	a	16	I3	a	–1	I4	A	16
	b	1		b	160		b	17		b	$-\frac{1}{2}$
	c	–6		c	3		c	8		c	$\frac{3}{2}$
	d	$\frac{50}{11}$		d	$\frac{8}{27}$		d	18		d	6

Group Events

G1	W	$\frac{1+\sqrt{5}}{2}$	G2	R	18434	G3	b	40	G4	x	137
	T	29		x	6		t	$\frac{12}{5}(=2.4)$		R	$\frac{1}{2}$
	S	106		y	12100		x	$10\sqrt{3}$		z	77
	k	4		Q	9		S	25		r	6

Individual Event 1

I1.1 Let a be a real number and $\sqrt{a}=\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}$. Find the value of a.

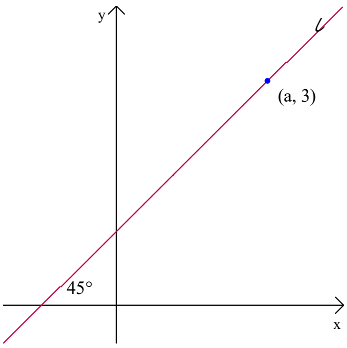
Reference: 2016 FI1.2, 2019 FI2.1

$$\left(\sqrt{a}\right)^2=\left(\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}\right)^2$$

$$a=7+\sqrt{13}-2\sqrt{7^2-\sqrt{13}^2}+7-\sqrt{13}$$

$$=14-2\sqrt{36}=2$$

I1.2 In Figure 1, the straight line ℓ passes though the point $(a, 3)$, and makes an angle 45° with the x-axis. If the equation of ℓ is $x + my + n = 0$ and $b = |1 + m + n|$, find the value of b.



$$\ell: \frac{y-3}{x-2}=\tan 45^\circ$$

$y - 3 = x - 2$

$x - y + 1 = 0$, $m = -1$, $n = 1$

$b = |1 - 1 + 1| = 1$

I1.3 If $x - b$ is a factor of $x^3 - 6x^2 + 11x + c$, find the value of c.

$f(x) = x^3 - 6x^2 + 11x + c$

$f(1) = 1 - 6 + 11 + c = 0$

$c = -6$

I1.4 If $\cos x + \sin x = -\frac{c}{5}$ and $d = \tan x + \cot x$, find the value of d.

Reference: 1992 HI20, 1993 HG10, 1995 HI5, 2007 HI7, 2014 HG3

$$\cos x + \sin x = \frac{6}{5}$$

$$(\cos x + \sin x)^2 = \frac{36}{25}$$

$$1 + 2\sin x\cos x = \frac{36}{25}$$

$$2\sin x\cos x = \frac{11}{25} \Rightarrow \sin x\cos x = \frac{11}{50}$$

$$d = \tan x + \cot x = \frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}=\frac{1}{\sin x\cos x}=\frac{50}{11}$$

Individual Event 2

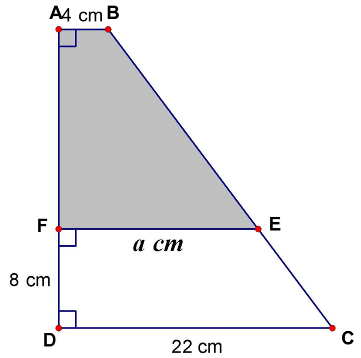
I2.1 Let $n = 1 + 3 + 5 + ... + 31$ and $m = 2 + 4 + 6 ... + 32$. If $a = m - n$, find the value of a.

$a = 2 + 4 + 6 \dots + 32 - (1 + 3 + 5 + \dots + 31)$

$= (2 - 1) + (4 - 3) + \dots + (32 - 31)$

$= 1 + 1 + \dots + 1 = 16$

I2.2 If Figure 1, $ABCD$ is a trapezium, $AB = 4$ cm, $EF = a$ cm, $CD = 22$ cm and $FD = 8$ cm, if the area of $ABEF$ is b cm^2, find the value of b.



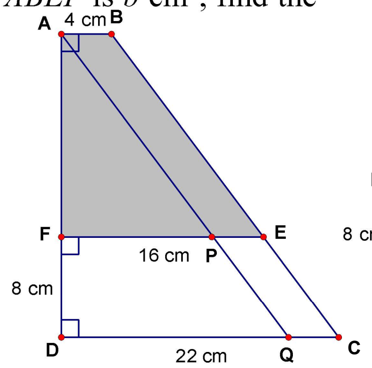
From A, draw a line APQ parallel to BC, cutting FE at P and DC at Q.

$FP = 12$ cm, $DQ = 18$ cm

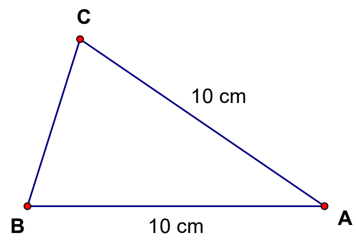
$\Delta AFP \sim \Delta ADQ$

Let $AF = x$ cm $\Rightarrow \dfrac{x}{12} = \dfrac{x+8}{18}$, $x = 16$

$b = \dfrac{(4+16)16}{2} = 160$



I2.3 In Figure 2, ΔABC is a triangle, $AB = AC = 10$ cm and $\angle ABC = b° - 100°$. If ΔABC has c axis of symmetry, find the value of c.



$\angle ABC = 160° - 100° = 60° = \angle ACB = \angle BAC$

ΔABC is an equilateral triangle.

It has 3 axis of symmetry.

I2.4 Let d be the least real root of the $cx^{\frac{2}{3}} - 8x^{\frac{1}{3}} + 4 = 0$, find the value of d.

$3x^{\frac{2}{3}} - 8x^{\frac{1}{3}} + 4 = 0 \Rightarrow \left(3x^{\frac{1}{3}} - 2\right)\left(x^{\frac{1}{3}} - 2\right) = 0$

$x^{\frac{1}{3}} = \dfrac{2}{3}$ or 2

$x = \dfrac{8}{27}$ or 8, the least real root is $\dfrac{8}{27}$.

Individual Event 3

I3.1 Suppose that $a = \cos^4\theta - \sin^4\theta - 2\cos^2\theta$, find the value of a.

$a = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) - 2\cos^2\theta$

$= \cos^2\theta - \sin^2\theta - 2\cos^2\theta = -(\sin^2\theta + \cos^2\theta) = -1$

I3.2 If $x^y = 3$ and $b = x^{3y} + 10a$, find the value of b.

$b = (x^y)^3 - 10 = 3^3 - 10 = 27 - 10 = 17$

I3.3 If there is (are) c positive integer(s) n such that $\dfrac{n+b}{n-7}$ is also a positive integer, find the value of c.

$\dfrac{n+17}{n-7} = 1 + \dfrac{24}{n-7}$

$n - 7 = 1, 2, 3, 4, 6, 8, 12, 24$

$c = 8$

I3.4 Suppose that $d = \log_4 2 + \log_4 4 + \log_4 8 + \cdots + \log_4 2^c$, find the value of d.

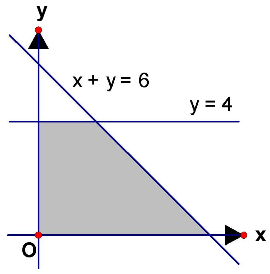
$d = \log_4 2 + \log_4 4 + \log_4 8 + \cdots + \log_4 2^8$

$= \log_4(2\times4\times8\times\cdots\times2^8) = \log_4(2^{1+2+3+\dots+8})$

$= \log_4(2^{36}) = \dfrac{\log 2^{36}}{\log 4} = \dfrac{36\log 2}{2\log 2} = 18$

Individual Event 4

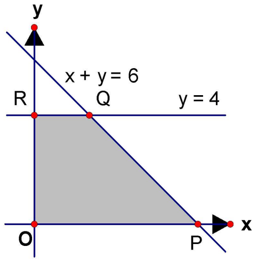
I4.1 In Figure 1, let the area of the closed region bounded by the straight line $x + y = 6$ and $y = 4$, $x = 0$ and $y = 0$ be A square units, find the value of A.



As shown in the figure, the intersection points $P(6, 0)$, $Q(2, 4)$, $R(0, 6)$

$OP = 6$, $OR = 4$, $QR = 2$

Area $= A = \frac{1}{2}(6+2)\cdot 4 = 16$



I4.2 Let $[x]$ be the largest integer not greater than x. For example, $[2.5] = 2$.

If b satisfies the system of equations $\begin{cases} Ax^2 - 4 = 0 \\ 3 + 2(x + [x]) = 0 \end{cases}$, find the value of b.

$\begin{cases} 16x^2 - 4 = 0 \\ 3 + 2(x + [x]) = 0 \end{cases}$ from the first equation $x = \frac{1}{2}$ or $-\frac{1}{2}$.

Substitute $x = \frac{1}{2}$ into the second equation: LHS $= 3 + 2(\frac{1}{2} + 0) = 4 \neq$ RHS

Substitute $x = -\frac{1}{2}$ into the second equation: LHS $= 3 + 2(-\frac{1}{2} - 1) = 0 =$ RHS

$\therefore b = -\frac{1}{2}$

I4.3 Let c be the constant term in the expansion of $\left(2x + \frac{b}{\sqrt{x}}\right)^3$. Find the value of c.

$$\left(2x + \frac{b}{\sqrt{x}}\right)^3 = 8x^3 + 12bx\sqrt{x} + 6b^2 + \frac{b^3}{x\sqrt{x}}$$

c = the constant term

$= 6b^2$

$= 6\left(-\frac{1}{2}\right)^2 = \frac{3}{2}$

I4.4 If the number of integral solutions of the inequality $\left|\frac{x}{2} - \sqrt{2}\right| < c$ is d, find the value of d.

$\left|\frac{x}{2} - \sqrt{2}\right| < \frac{3}{2}$

$-\frac{3}{2} < \frac{x}{2} - \sqrt{2} < \frac{3}{2}$

$2\sqrt{2} - 3 < x < 2\sqrt{2} + 3$

$2(1.4) - 3 < x < 2(1.4) + 3$

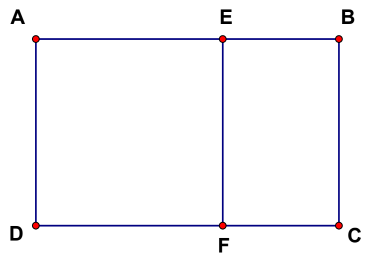
$-0.2 < x < 5.8$

$x = 0, 1, 2, 3, 4, 5$

$d = 6$

Group Event 1

G1.1 In Figure 1, *AEFD* is a unit square. The ratio of the length of the rectangle *ABCD* to its width is equal to the ratio of the length of the rectangle *BCFE* to its width. If the length of *AB* is *W* units, find the value of *W*.

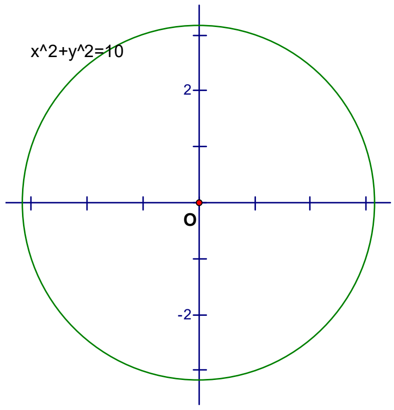


$$\frac{W}{1} = \frac{1}{W-1}$$

$$W^2 - W - 1 = 0 \Rightarrow W = \frac{1+\sqrt{5}}{2}$$

G1.2 On the coordinate plane, there are T points (x, y), where x, y are integers, satisfying $x^2 + y^2 < 10$, find the value of T. **(Reference: 2002 FI4.3)**

T = number of integral points inside the circle $x^2 + y^2 = 10$.



We first count the number of integral points in the first quadrant:

$x = 1$; $y = 1, 2$

$x = 2$; $y = 1, 2$

Next, the number of integral points on the x-axis and y-axis

$= 3 + 3 + 3 + 3 + 1 = 13$

$T = 4\times4 + 3 + 3 + 3 + 3 + 1$

$= 29$

G1.3 Let P and $P + 2$ be both prime numbers satisfying $P(P + 2) \le 2007$.

If S represents the sum of such possible values of P, find the value of S.

$P^2 + 2P - 2007 \le 0$

$(P + 1)^2 - 2008 \le 0$

$(P + 1 + \sqrt{2008})(P + 1 - \sqrt{2008}) \le 0$

$(P + 1 + 2\sqrt{502})(P + 1 - 2\sqrt{502}) \le 0$

$-1 - 2\sqrt{502} \le P \le -1 + 2\sqrt{502}$

P is a prime $\Rightarrow 0 < P \le -1 + 2\sqrt{502}$

$22 = \sqrt{484} < \sqrt{502} < \sqrt{529} = 23$

$43 < -1 + 2\sqrt{502} < 45$

$\therefore (P, P + 2) = (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43)$

$S = 3 + 5 + 11 + 17 + 29 + 41 = 106$

G1.4 It is known that $\log_{10}\left(2007^{2006} \times 2006^{2007}\right) = a\times10^k$, where $1 \le a < 10$ and k is an integer.

Find the value of k.

$a\times10^k = 2006 \log 2007 + 2007 \log 2006 = 2006\times(\log 2007 + \log 2006) + \log 2006$

$2006\times(\log 2006 + \log 2006) + \log 2006 < a\times10^k < 2006\times(\log 2007 + \log 2007) + \log 2007$

$4013 \log 2006 < a\times10^k < 4013 \log 2007$

$4013 \log(2.006\times10^3) < a\times10^k < 4013 \log(2.007\times10^3)$

$4013 (\log 2.006 + 3) < a\times10^k < 4013(\log 2.007 + 3)$

$4013 \log 2 + 4013\times3 < a\times10^k < 4013 \log 3 + 3$

$1.32429\times10^4 = 4013\times0.3 + 4013\times3 < a\times10^k < 4013\times0.5 + 4013\times3 = 1.40455\times10^4$

$k = 4$

Group Event 2

G2.1 If $R = 1\times2 + 2\times2^2 + 3\times2^3 + \ldots + 10\times2^{10}$, find the value of R.

Reference: 2005 HI7, 2005 FG2.4

$2R = \quad 1\times2^2 + 2\times2^3 + \ldots + 9\times2^{10} + 10\times2^{11}$

$R - 2R = 2 + 2^2 \quad + 2^3 + \ldots\ldots + 2^{10} \quad - 10\times2^{11}$

$-R = \dfrac{a(R^n-1)}{R-1} - 10\times2^{11} = \dfrac{2(2^{10}-1)}{2-1} - 10\times2048$

$R = 20480 - 2(1023) = 18434$

G2.2 If integer x satisfies $x \geq 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}}$, find the minimum value of x.

Let $y = 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}}}$ (to infinity)

$(y-3)^2 = 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}}} = y$

$y^2 - 7y + 9 = 0$

$y = \dfrac{7+\sqrt{13}}{2}$ or $\dfrac{7-\sqrt{13}}{2}$

Clearly $y > 3$ and $\dfrac{7-\sqrt{13}}{2} < 3$

$\therefore y = \dfrac{7+\sqrt{13}}{2}$ only

$5 = \dfrac{7+\sqrt{9}}{2} < \dfrac{7+\sqrt{13}}{2} < \dfrac{7+\sqrt{16}}{2} = 5.5$

$3+\sqrt{3+\sqrt{3}} > 3+\sqrt{3+1.7} > 3+\sqrt{4.41} = 3 + 2.1 = 5.1$

$5.1 < 3+\sqrt{3+\sqrt{3}} < 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}} < 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}}} < 5.5$

$x = 6$

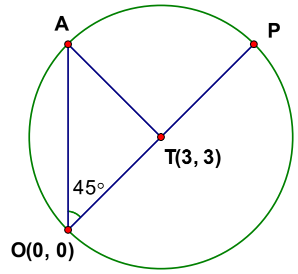
G2.3 Let $y = \dfrac{146410000-12100}{12099}$, find the value of y.

$y = \dfrac{12100^2-12100}{12100-1}$

$= \dfrac{12100(12100-1)}{12100-1}$

$= 12100$

G2.4 On the coordinate plane, a circle with centre $T(3, 3)$ passes through the origin $O(0, 0)$. If A is a point on the circle such that $\angle AOT = 45°$ and the area of ΔAOT is Q square units, find the value of Q.



$OT = \sqrt{3^2+3^2} = 3\sqrt{2}$

$OT = AT$ = radii

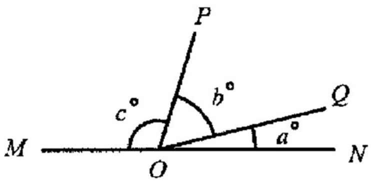
$\angle OAT = 45°$ (side opp. eq. $\angle$s)

$\angle ATO = 90°$ ($\angle$s sum of Δ)

$Q = \dfrac{1}{2}OT\cdot AT = \dfrac{1}{2}\cdot\left(3\sqrt{2}\right)^2 = 9$

Group Event 3

G3.1 In figure 1, MN is a straight line, $\angle QON = a°$, $\angle POQ = b°$ and $\angle POM = c°$. If $b : a = 2 : 1$ and $c : b = 3 : 1$, find the value of b.



$b = 2a$, $c = 3b = 6a$

$a + b + c = 180$ (adj. $\angle$s on st. line)

$a + 2a + 6a = 180 \Rightarrow a = 20$

$b = 2a = 40$

G3.2 It is known that $\sqrt{\frac{50+120+130}{2}\times(150-50)\times(150-120)\times(150-130)}=\frac{50\times 130\times k}{2}$.

If $t=\frac{k}{\sqrt{1-k^2}}$, find the value of t.

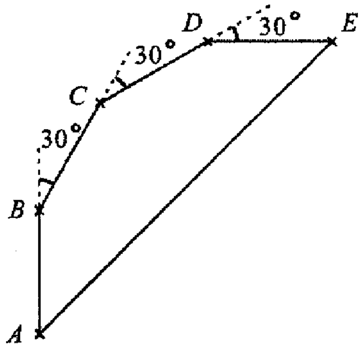
The question is equivalent to: given a triangle with sides 50, 120, 130, find its area.

$\cos C=\frac{50^2+130^2-120^2}{2\cdot 50\cdot 130}=\frac{5}{13}$

Using the formula $\frac{1}{2}ab\sin C=\frac{50\times 130\times k}{2}$, $k = \sin C=\sqrt{1-\cos^2 C}=\frac{12}{13}$

$t=\frac{k}{\sqrt{1-k^2}}=\frac{\sin C}{\cos C}=\tan C=\frac{12}{5}$

G3.3 In Figure 2, an ant runs ahead straightly for 5 sec 15° cm from point A to point B. It then turns 30° to the right and run 5 sec 15° cm to point C. Again it repeatedly turns 30° to the right and run 5 sec 15° cm twice to reach the points D and E respectively. If the distance of AE is x cm, find the value of x.



By symmetry $\angle BAE = \angle DEA = [180°\times(5 - 2) - 150°\times 3] \div 2$

$= 45°$ ($\angle$ sum of polygon)

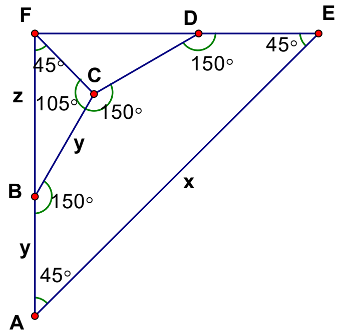
Produce AB and ED to intersect at F.

$\angle AFE = 180° - 45° - 45° = 90°$ ($\angle$s sum of Δ)

By symmetry, $\angle BFC = \angle DFC = 45°$

$\angle BCF = \angle DCF = (360° - 150°) \div 2 = 105°$ ($\angle$s at a pt.)

Let $AB = y = 5 \sec 15°$ cm $= CD = DE$, let $z = BF$.



Apply Sine rule on ΔABC, $\frac{z}{\sin 105°}=\frac{y}{\sin 45°}$

$z=\sqrt{2}\sin 105°\, y$

In ΔAEF, $x = (y + z)\sec 45° = \sqrt{2}\,(y+\sqrt{2}\sin 105°\, y)$

$=y\sqrt{2}\left(1+\sqrt{2}\sin 105°\right)$

$=5\sec 15°\cdot 2\left(\frac{1}{\sqrt{2}}+\sin 105°\right)= 10\sec 15°\left(\sin 105°+\sin 45°\right)$

$= 10 \sec 15°(2 \sin 75° \cos 30°) = 20 \sec 15°\cdot\cos 15°\ \frac{\sqrt{3}}{2}=10\sqrt{3}$

Method 2

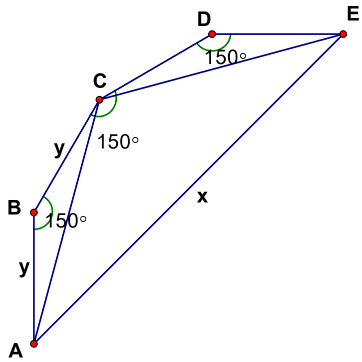
Join AC, CE. With similar working steps, $\angle BAE = \angle DEA = 45°$

$\angle BAC = \angle BCA = 15° = \angle DCE = \angle DEC$ ($\angle$s sum of isos. Δ)

$\angle CAE = 45° - 15° = 30° = \angle CEA$

$AC = CE = 2y \cos 15° = 2\times 5 \sec 15° \times \cos 15° = 10$

$x = 2\,AC \cos 30° = 20\times\frac{\sqrt{3}}{2}=10\sqrt{3}$



G3.4 There are 4 problems in a mathematics competition. The scores of each problem are allocated in the following ways: 2 marks will be given for a correct answer, 1 mark will be deducted from a wrong answer and 0 mark will be given for a blank answer. To ensure that 3 candidates will have the same scores, there should be at least S candidates in the competition. Find the value of S.

We shall tabulate different cases:

case no.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	marks for each question
correct	4	3	3	2	2	2	1	1	1	1	0	0	0	0	0	2
blank	0	1	0	2	0	1	3	2	1	1	4	3	2	1	0	0
wrong	0	0	1	0	2	1	0	1	2	3	0	1	2	3	4	–1
Total	8	6	5	4	2	3	2	1	0	–1	0	–1	–2	–3	–4	

The possible total marks for one candidate to answer 4 questions are:

8, 6, 5, 4, 3, 2, 1, 0, –1, –2, –3, –4; altogether 12 possible combinations.

To ensure **3** candidates will have the same scores, we consider the worst scenario:

Given that there are 24 candidates. 2 candidates score 8 marks, 2 candidates score 6 marks, ..., 2 candidates score –4 marks, then the 25$^{\text{th}}$ candidate will score the same as the other two candidates.

Group Event 4

G4.1 Let x be the number of candies satisfies the inequalities $120 \le x \le 150$. 2 candies will be remained if they are divided into groups of 5 candies each; 5 candies will be remained if they are divided into groups of 6 candies each. Find the value of x.

$x = 5m + 2 = 6n + 5$, where m and n are integers.

$5m - 6n = 3$

$5\times 3 - 6\times 2 = 15 - 12 = 3$

$\therefore m = 3$, $n = 2$ is a pair of solution

The general solution is $m = 3 + 6t$, $n = 2 + 5t$, where t is any integer.

$x = 5m + 2 = 5(3 + 6t) + 2 = 30t + 17$

$120 \le x \le 150 \Rightarrow 120 \le 30t + 17 \le 150$

$103 \le 30t \le 133$

$3.43 < t < 4.43 \Rightarrow t = 4$

$x = 30\times 4 + 17 = 137$

G4.2 On the coordinate plane, the points $A(3, 7)$ and $B(8, 14)$ are reflected about the line $y = kx + c$, where k and c are constants, their images are $C(5, 5)$ and $D(12,10)$ respectively. If $R = \frac{k}{c}$, find the value of R.

By the property of reflection, the line $y = kx + c$ is the perpendicular bisector of A, C and B, D.

That is to say, the mid points of A, C and B, D lies on the line $y = kx + c$

M = mid point of A, C = (4, 6), N = mid point of B, D = (10, 12)

By two points form, $\frac{y-6}{x-4} = \frac{12-6}{10-4}$

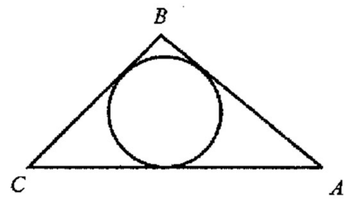
$y = x + 2 \Rightarrow k = 1, c = 2, R = \frac{1}{2}$

G4.3 Given that $z = \sqrt[3]{456533}$ is an integer, find the value of z.

$70 = \sqrt[3]{343000} < \sqrt[3]{456533} < \sqrt[3]{512000} = 80$

By considering the cube of the unit digit, the only possible answer for z is 77.

G4.4 In Figure 1, ΔABC is an isosceles triangle, $AB = BC = 20$ cm and $\tan \angle BAC = \frac{4}{3}$. If the length of radius of the inscribed circle of ΔABC is r cm, find the value of r.



Reference: 2013 HG8, 2022 P1Q15

$\angle BAC = \angle BCA$; $\sin \angle BAC = \frac{4}{5}$, $\cos \angle BAC = \frac{3}{5}$.

$AC = 2\times 20 \cos \angle BAC = 40\times \frac{3}{5} = 24$, the height of ΔABC from $B = 20 \sin \angle BAC = 16$

Area of $\Delta ABC = \frac{1}{2}\cdot 24\cdot 16 = 192 = \frac{r}{2}(20 + 20 + 24)$

$r = 6$