Coordinates of Incentre of a triangle

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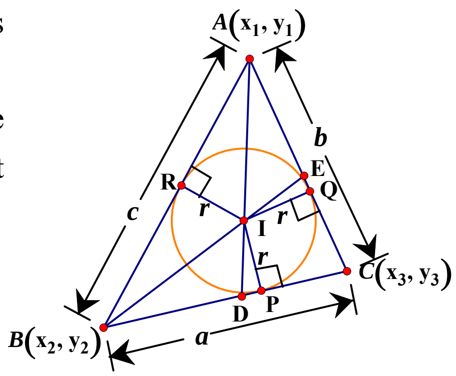

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the coordinates of the vertices of ΔABC. $BC = a$, $CA = b$, $AB = c$. Let I be the incentre.

Let the radius of the inscribed circle be r. The inscribed circle touches ΔABC at P, Q and R. Join AI and produce it to cut BC at D. Denote the areas by S.

BC, CA and AB are tangents to the inscribed circle.

$IP \perp BC$, $IQ \perp AC$, $IR \perp AB$ (tangent $\perp$ radii)

$IP = IQ = IR = r$

$S_{\Delta IBC} + S_{\Delta ICA} + S_{\Delta IAB} = S_{\Delta ABC}$

$\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \sqrt{s(s-a)(s-b)(s-c)}$ Heron's formula, where $s = \frac{1}{2}(a+b+c)$

$sr = \sqrt{s(s-a)(s-b)(s-c)}$

$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$, where $s = \frac{1}{2}(a+b+c)$

Let $\angle BAD = \angle CAD = \theta$ (AD is the $\angle$ bisector of $\angle BAC$)

Let $\angle ADC = \alpha$, $\angle ADB = 180° - \alpha$ (adj. $\angle$s on st. line)

Apply sine rule on ΔABD and ΔACD respectively.

$\frac{BD}{\sin\theta} = \frac{c}{\sin(180^\circ - \alpha)}$ $\cdots\cdots$ (1), $\frac{DC}{\sin\theta} = \frac{b}{\sin\alpha}$ $\cdots\cdots$ (2)

Using the fact that $\sin(180° - \alpha) = \sin\alpha$ and (1)÷(2), we have

$\frac{BD}{DC} = \frac{c}{b} \Rightarrow BD : DC = c : b$

By section formula, $D = \left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c}\right)$

A, I, D are collinear. Let $AI : ID = m : 1 - m$. Then by section formula again,

$$I = \left((1-m)x_1 + \frac{m(bx_2 + cx_3)}{b+c}, (1-m)y_1 + \frac{m(by_2 + cy_3)}{b+c}\right)$$

$$= \left((1-m)x_1 + \frac{mbx_2}{b+c} + \frac{mcx_3}{b+c}, (1-m)y_1 + \frac{mby_2}{b+c} + \frac{mcy_3}{b+c}\right)$$

Replace a by b, b by c and c by a, x_1 by x_2, x_2 by x_3, x_3 by x_1, y_1 by y_2, y_2 by y_3, the coordinates should be the same.

$I = \left((1-n)x_2 + \frac{ncx_3}{c+a} + \frac{nax_1}{c+a}, (1-n)y_2 + \frac{ncy_3}{c+a} + \frac{nay_1}{c+a}\right)$ for some $0 < n < 1$

Compare coefficients of x_1: $1 - m = \frac{na}{c+a}$ $\cdots\cdots$ (3)

Compare coefficients of x_2: $1 - n = \frac{mb}{b+c}$ $\cdots\cdots$ (4)

Compare coefficients of x_3: $\frac{mc}{b+c} = \frac{nc}{c+a} \Rightarrow \frac{m}{b+c} = \frac{n}{c+a}$ $\cdots\cdots$ (5)

Sub. (5) into (3): $1 - m = \frac{ma}{b+c} \Rightarrow b + c - (b+c)m = ma \Rightarrow m = \frac{b+c}{a+b+c}$

$$I = \left(\left(1 - \frac{b+c}{a+b+c}\right)x_1 + \frac{b+c}{a+b+c}\cdot\frac{bx_2}{b+c} + \frac{b+c}{a+b+c}\cdot\frac{cx_3}{b+c}, \left(1 - \frac{b+c}{a+b+c}\right)y_1 + \frac{b+c}{a+b+c}\cdot\frac{by_2}{b+c} + \frac{b+c}{a+b+c}\cdot\frac{cy_3}{b+c}\right)$$

$$= \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$