Individual Events

I1	***a***	2	**I2**	***a***	136	**I3**	***a***	4	**I4**	***a***	8	**I5**	***a***	20
	b	2		***b***	–2620		***b***	24		***b***	9		***b***	2
	c	2		***c***	100		***c***	50		***c***	4		***c***	257
	d	1		***d***	50		***d***	500		***d***	54		***d***	7

Group Events

G6	***p***	–2	**G7**	***a***	36	**G8**	***m***	–2	**G9**	***a***	9	**G10**	***a***	50
	m	8		***b***	18		***d***	3		***b***	3		***b***	10
	r	1		***c***	2		***n***	96		***x***	11		***c***	15
	s	–2		***d***	6		***s***	95856		***y***	10		***d***	60

Individual Event 1

I1.1 Given that $7^{2x} = 36$ and $7^{-x} = (6)^{-\frac{a}{2}}$, find the value of a.

$7^x = 6$

$\Rightarrow 7^{-x} = (6)^{-\frac{a}{2}} = 6^{-1}$

$\Rightarrow a = 2$

I1.2 Find the value of b if $\log_2\{\log_2[\log_2(2b) + a] + a\} = a$.

$\log_2\{\log_2[\log_2(2b) + 2] + 2\} = 2$

$\log_2[\log_2(2b) + 2] + 2 = 2^2 = 4$

$\log_2[\log_2(2b) + 2] = 2$

$\log_2(2b) + 2 = 2^2 = 4$

$\Rightarrow \log_2(2b) = 2$

$2b = 2^2 = 4$

$\Rightarrow b = 2$

I1.3 If c is the total number of positive roots of the equation

$(x - b)(x - 2)(x + 1) = 3(x - b)(x + 1)$, find the value of c.

$(x - 2)(x - 2)(x + 1) - 3(x - 2)(x + 1) = 0$

$(x - 2)(x + 1)[(x - 2) - 3] = 0$

$(x - 2)(x + 1)(x - 5) = 0$

$x = 2, -1$ or 5

$\Rightarrow$ Number of positive roots $= c = 2$

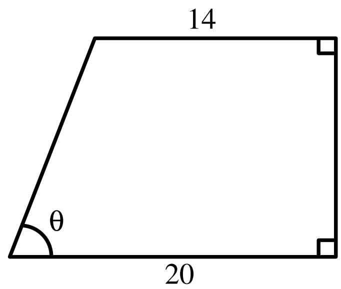
I1.4 If $\sqrt{3-2\sqrt{2}} = \sqrt{c} - \sqrt{d}$, find the value of d.

Reference: 1999 HG3, 2001 FG2.1, 2011 HI7, 2015 FI4.2, 2015 FG3.1

$$\begin{aligned}\sqrt{3-2\sqrt{2}} &= \sqrt{1-2\sqrt{2}+2}\\ &= \sqrt{\left(\sqrt{1}\right)^2 - 2\sqrt{2} + \left(\sqrt{2}\right)^2}\\ &= \sqrt{\left(\sqrt{2}-1\right)^2} = \sqrt{2}-1\end{aligned}$$

$\Rightarrow d = 1$

Individual Event 2



I2.1 If $\sin\theta = \frac{4}{5}$, find a, the area of the quadrilateral.

Let the height be h.

$\tan\theta = \frac{4}{3} = \frac{h}{6}$

$\Rightarrow h = 8$

Area $= \frac{1}{2}(14+20)\cdot 8 = 136$

I2.2 If $b = 126^2 - a^2$, find b.

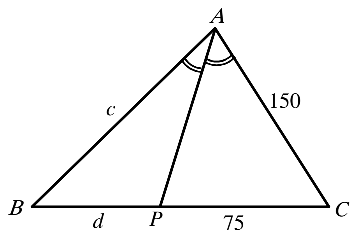
$b = 126^2 - a^2$

$= (126 - 136)(126 + 136) = -2620$

I2.3 Dividing \$$(3000 + b)$ in a ratio 5 : 6 : 8, the smallest part is \$$c$. Find c.

Sum of money = \$(3000 – 2620) = \$380

$c = \frac{5}{5+6+8}\cdot 380 = \frac{5}{19}\cdot 380 = 100$



I2.4 In the figure, AP bisects $\angle BAC$. Given that $AB = c$, $BP = d$, $PC = 75$ and $AC = 150$, find d.

Let $\angle BAP = \theta = \angle CAP$, $\angle APC = \alpha$, $\angle BPC = 180° - \alpha$

$\frac{d}{\sin\theta} = \frac{100}{\sin(180^\circ - \alpha)}$ … (1) and $\frac{75}{\sin\theta} = \frac{150}{\sin\alpha}$ … (2)

(1) ÷ (2) $\Rightarrow d = 50$

Individual Event 3

I3.1 If a is the remainder when 2614303940317 is divided by 13, find a.

$2614303939000 = 13\times21100303000$

$2614303940317 = 13\times21100303000 + 1317 = 13\times21100303000 + 1313 + 4$

$a = 4$

I3.2 Let $P(x, b)$ be a point on the straight line $x + y = 30$ such that slope of $OP = a$ (O is the origin). Determine b. **(Reference: 1994 FI1.4)**

$x + b = 30$

$\Rightarrow x = 30 - b$

$m_{OP} = \frac{b}{30-b} = 4$

$\Rightarrow b = 120 - 4b$

$\Rightarrow b = 24$

I3.3 Two cyclists, initially (b + 26) km apart travelling towards each other with speeds 40 km/h and 60 km/h respectively. A fly flies back and forth between their noses at 100 km/h. If the fly flied c km before crushed between the cyclists, find c.

The velocity of one cyclist relative to the other cyclist is (40 + 60) km/h = 100 km/h.

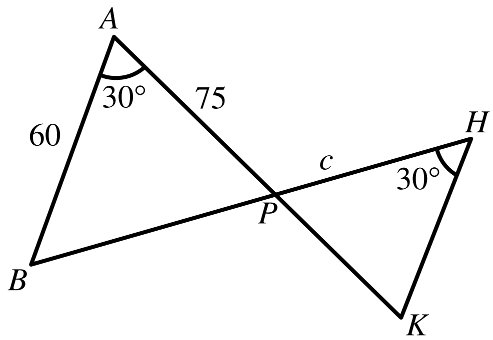
Distance between the two cyclists = (24 + 26) km = 50 km

Time for the two cyclists meet $= \frac{50}{100}$ h $= \frac{1}{2}$ h

The distance the fly flied $= \frac{1}{2}\times100$ km = 50 km

$\Rightarrow c = 50$

I3.4 In the figure, APK and BPH are straight lines. If d = area of triangle HPK, find d.



$\angle BAP = \angle KHP = 30°$ (given)

$\angle APB = \angle KPH$ (vert. opp. $\angle$s)

$\Delta ABP \sim \Delta HKP$ (equiangular)

$\frac{HK}{60} = \frac{50}{75}$

$\Rightarrow HK = 40$

$d = \frac{1}{2}\times50\times40\cdot\sin30^\circ = 500$

Individual Event 4

I4.1 Given that the means of x and y, y and z, z and x are respectively 5, 9, 10. If a is the mean of x, y, z, find the value of a.

$\frac{x+y}{2} = 5$ … (1); $\frac{y+z}{2} = 9$ … (2); $\frac{z+x}{2} = 10$ … (3)

(1) + (2) + (3): $x + y + z = 24$

$\Rightarrow a = 8$

I4.2 The ratio of two numbers is 5 : a. If 12 is added to each of them, the ratio becomes 3 : 4. If b is the difference of the original numbers and $b > 0$, find the value of b.

Let the two numbers be $5k$, $8k$.

$\frac{5k+12}{8k+12} = \frac{3}{4}$

$\Rightarrow 20k + 48 = 24k + 36$

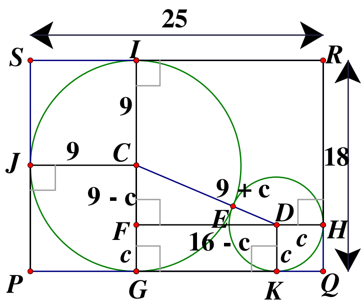
$\Rightarrow 4k = 12$

$\Rightarrow k = 3$

$5k = 15$, $8k = 24$

$b = 24 - 15 = 9$

I4.3 *PQRS* is a rectangle. If c is the radius of the smaller circle, find the value of c.



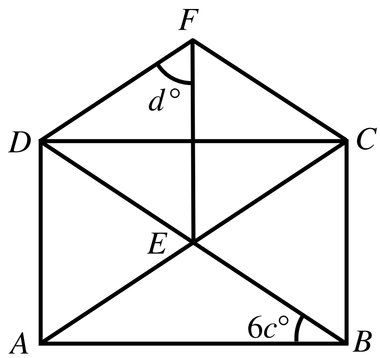
Let the centres of the two circles be C and D, with radius 9 and c respectively.
Suppose the circles touch each other at E.
Further, assume that the circle with centre at C touches SR, PS, PQ at I, J and G respectively. Let the circle with centre at D touches PQ, QR at K and H respectively.
Join CI, CJ, CG, CED, DF, DK, DH.

$CI \perp SR$, $CJ \perp PS$, $CG \perp PQ$, $DK \perp PQ$, $DH \perp PR$ (tangent $\perp$ radius)
DK // HQ (corr. $\angle$s eq.)
$\angle FDK = 90°$ (corr. $\angle$s, DK // HQ)
$DFGK$ is a rectangle (3 angles = 90°)
$\therefore \angle DFG = 90°$ ($\angle$s sum of polygon)
$\angle DFC = 90°$ (adj. $\angle$s on st. line)

C, E, D are collinear ($\because$ the two circles touch each other at E)

$CI = CJ = CG = CE = 9$ (radii of the circle with centre at C)
$DH = DK = DE = c$ (radii of the circle with centre at D)
$CD = c + 9$
$FG = DK = c$ (opp. sides of rectangle $DFGK$)
$CF = 9 - c$
$FD = GK$ (opp. sides of rectangle $DFGK$)
$= PD - PG - KQ$
$= 25 - 9 - c$ (opp. sides of rectangle)
$= 16 - c$
$CF^2 + DF^2 = CD^2$ (Pythagoras' theorem)
$(9 - c)^2 + (16 - c)^2 = (9 + c)^2$
$81 - 18c + c^2 + 256 - 32c + c^2 = 81 + 18c + c^2$
$c^2 - 68c + 256 = 0$
$(c - 4)(c - 64) = 0$
$c = 4$ or 64 (> 18, rejected)

I4.4 *ABCD* is a rectangle and *CEF* is an equilateral triangle, $\angle ABD = 6c°$, find the value of d.



Reference: HKCEE MC 1982 Q51
$\angle ABD = 24°$ (given)
$\angle CAB = 24°$ (diagonals of rectangle)
$\angle AEB = 132°$ ($\angle$s sum of Δ)
$\angle CED = 132°$ (vert. opp. $\angle$s)
$\angle CEF = 60°$ ($\angle$ of an equilateral triangle)
$\angle DEF = 132° - 60° = 72°$
$ED = EC = EF$ (diagonals of rectangle, sides of equilateral Δ)
$\therefore \Delta DEF$ is isosceles (2 sides equal)
$\angle EFD = \angle EDF$ (base $\angle$s isos. Δ)
$d = (180 - 72) \div 2 = 54$ ($\angle$s sum of isos. Δ)

Individual Event 5

I5.1 Two opposite sides of a rectangle are increased by 50% while the other two are decreased by 20%. If the area of the rectangle is increased by a%, find a.

Let the length and width be x and y respectively.

$1.5x \times 0.8y = 1.2xy$

$\Rightarrow a = 20$

I5.2 Let $f(x) = x^3 - 20x^2 + x - a$ and $g(x) = x^4 + 3x^2 + 2$. If $h(x)$ is the highest common factor of $f(x)$ and $g(x)$, find $b = h(1)$.

Reference: 1992 HI5, 2001 FI1.2, 2011 FI3.2

$f(x) = x^3 - 20x^2 + x - 20 = (x^2 + 1)(x - 20)$

$g(x) = x^4 + 3x^2 + 2 = (x^2 + 1)(x^2 + 2)$

$h(x) = \text{H.C.F.} = x^2 + 1$

$b = h(1) = 2$

I5.3 It is known that $b^{16} - 1$ has four distinct prime factors, determine the largest one, denoted by c

$2^{16} - 1 = (2 - 1)(2 + 1)(2^2 + 1)(2^4 + 1)(2^8 + 1) = 3\times5\times17\times257$

$c = 257$

I5.4 When c is represented in binary scale, there are d '0's. Find d.

$257_{(x)} = 256 + 1$

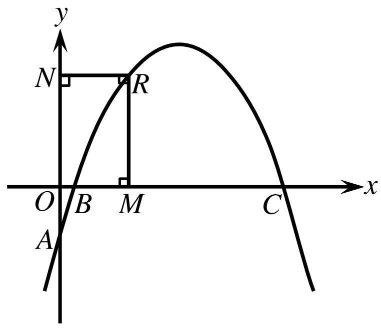
$= 2^8 + 1$

$=100000001_{(ii)}$

$d = 7$

Group Event 6

The following shows the graph of $y = px^2 + 5x + p$. $A = (0, -2)$, $B = \left(\frac{1}{2}, 0\right)$, $C = (2, 0)$, $O = (0, 0)$.



G6.1 Find the value of p.

$y = p\left(x - \frac{1}{2}\right)(x - 2)$

It passes through $A(0, -2)$: $-2 = p\left(-\frac{1}{2}\right)(-2)$.

$p = -2$

G6.2 If $\frac{9}{m}$ is the maximum value of y, find the value of m.

$y = -2x^2 + 5x - 2$

$\frac{9}{m} = \frac{4(-2)(-2) - 5^2}{4(-2)}$

$\Rightarrow m = 8$

G6.3 Let R be a point on the curve such that $OMRN$ is a square. If r is the x-coordinate of R, find the value of r.

$R(r, r)$ lies on $y = -2x^2 + 5x - 2$

$r = -2r^2 + 5r - 2$

$2r^2 - 4r + 2 = 0$

$\Rightarrow r = 1$

G6.4 A straight line with slope $= -2$ passes through the origin cutting the curve at two points E and F. If $\frac{7}{s}$ is the y-coordinate of the midpoint of EF, find the value of s.

Sub. $y = -2x$ into $y = -2x^2 + 5x - 2$

$-2x = -2x^2 + 5x - 2$

$2x^2 - 7x + 2 = 0$

Let $E = (x_1, y_1)$, $F = (x_2, y_2)$.

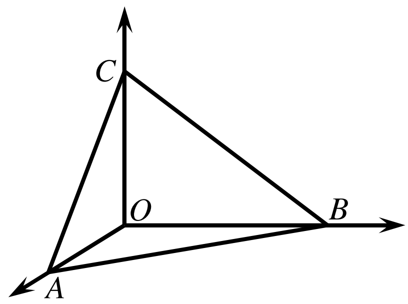
$x_1 + x_2 = \frac{7}{2}$

$\frac{7}{s} = \frac{y_1 + y_2}{2} = \frac{-2x_1 - 2x_2}{2} = -(x_1 + x_2) = \frac{7}{-2}$

$s = -2$

Group Event 7

OABC is a tetrahedron with *OA*, *OB* and *OC* being mutually perpendicular. Given that $OA = OB = OC = 6x$.



G7.1 If the volume of *OABC* is ax^3, find a.

$$ax^3 = \frac{1}{3}\cdot\frac{1}{2}(6x)^2\cdot(6x) = 36x^3$$

$\Rightarrow a = 36$

G7.2 If the area of ΔABC is $b\sqrt{3}x^2$, find b.

$$AB = BC = AC = \sqrt{(6x)^2+(6x)^2} = 6x\sqrt{2}$$

ΔABC is equilateral

$\angle BAC = 60°$

Area of $\Delta ABC = b\sqrt{3}x^2 = \frac{1}{2}\left(6x\sqrt{2}\right)^2\sin 60^\circ = 18\sqrt{3}x^2$

$b = 18$

G7.3 If the distance from O to ΔABC is $c\sqrt{3}x$, find c.

By finding the volume of *OABC* in two different ways.

$$\frac{1}{3}\cdot 18\sqrt{3}x^2\times\left(c\sqrt{3}x\right) = 36x^3$$

$c = 2$

G7.4 If θ is the angle of depression from C to the midpoint of AB and $\sin\theta = \frac{\sqrt{d}}{3}$, find d.

$$\frac{1}{3}\cdot 18\sqrt{3}x^2\times\left(c\sqrt{3}x\right) = 36x^3$$

Let the midpoint of *AB* be *M*.

$OC = 6x$, $\frac{OM\times AB}{2} = \frac{OA\times OB}{2}$

$\Rightarrow 6x\sqrt{2}\cdot OM = (6x)^2$

$\Rightarrow OM = 3\sqrt{2}x$

$$CM = \sqrt{OM^2 + OC^2}$$

$$= \sqrt{\left(3\sqrt{2}x\right)^2 + (6x)^2}$$

$$= 3\sqrt{6}x$$

$$\sin\theta = \frac{\sqrt{d}}{3} = \frac{OC}{CM}$$

$$= \frac{6x}{3\sqrt{6}x} = \frac{\sqrt{6}}{3}$$

$d = 6$

Group Event 8

Given that the equation $x^2 + (m + 1)x - 2 = 0$ has 2 integral roots $(\alpha + 1)$ and $(\beta + 1)$ with $\alpha < \beta$ and $m \neq 0$. Let $d = \beta - \alpha$.

G8.1 Find the value of m.

$(\alpha + 1)(\beta + 1) = -2$

$\Rightarrow \alpha + 1 = -1, \beta + 1 = 2$ or $\alpha + 1 = -2, \beta + 1 = 1$

$\Rightarrow (\alpha, \beta) = (-2, 1), (-3, 0)$

When $(\alpha, \beta) = (-3, 0)$, sum of roots $= (\alpha + 1) + (\beta + 1) = -(m + 1) \Rightarrow m = 0$ (rejected)

When $(\alpha, \beta) = (-2, 1)$, sum of roots $= (\alpha + 1) + (\beta + 1) = -(m + 1) \Rightarrow m = -2$

G8.2 Find the value of d.

$d = \beta - \alpha = 1 - (-2) = 3$

Let n be the total number of integers between 1 and 2000 such that each of them gives a remainder of 1 when it is divided by 3 or 7.

Reference: 1994 FG8.1-2, 1998 HI6, 2015 FI3.1

G8.3 Find the value of n.

These numbers give a remainder of 1 when it is divided by 21.

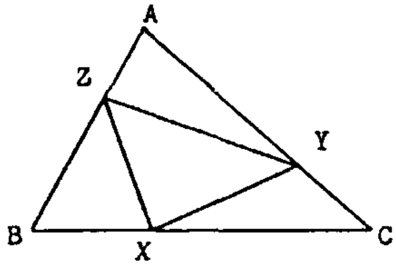
They are $1, 21 + 1, 21\times2 + 1, \dots, 21\times95 + 1$ (= 1996)

$n = 96$

G8.4 If s is the sum of all these n integers, find the value of s.

$s = 1 + 22 + 43 + \dots + 1996 = \frac{1}{2}(1+1996)\cdot 96 = 95856$

Group Event 9



BC, CA, AB are divided respectively by the points X, Y, Z in the ratio 1 : 2. Let

area of ΔAZY : area of ΔABC = 2 : a and

area of ΔAZY : area of ΔXYZ = 2 : b.

G9.1 Find the value of a.

area of $\Delta AZY = \frac{2}{3}$ area of ΔACZ (same height)

$= \frac{2}{3} \times \frac{1}{3}$ area of ΔABC (same height)

$\Rightarrow a = 9$

G9.2 Find the value of b.

Reference: 2000 FI5.3

Similarly, area of $\Delta BZX = \frac{2}{9}$ area of ΔABC; area of $\Delta CXY = \frac{2}{9}$ area of ΔABC

area of ΔXYZ = area of ΔABC – area of ΔAZY – area of ΔBZX – area of ΔCXY

$= \frac{1}{3}$ area of ΔABC

2 : b = area of ΔAZY : area of $\Delta XYZ = \frac{2}{9} : \frac{1}{3}$

$\Rightarrow b = 3$

A die is thrown 2 times. Let $\frac{x}{36}$ be the probability that the sum of numbers obtained is 7 or 8 and $\frac{y}{36}$ be the probability that the difference of numbers obtained is 1.

G9.3 Find the value of x.

Favourable outcomes are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6,1), (2,6), (3,5), (4,4), (5,3), (6,2)

P(7 or 8) $= \frac{x}{36}$

$\Rightarrow x = 11$

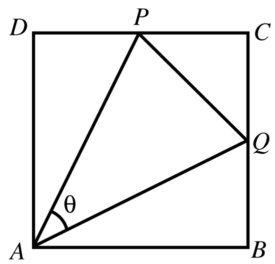
G9.4 Find the value of y.

Favourable outcomes are (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6,5), (5, 4), (4, 3), (3, 2), (2, 1).

P(difference is 1) $= \frac{y}{36}$

$\Rightarrow y = 10$

Group Event 10



ABCD is a square of side length $20\sqrt{5}x$. *P*, *Q* are midpoints of *DC* and *BC* respectively.

G10.1 If $AP = ax$, find a.

$$AP = \sqrt{AD^2 + DP^2}$$
$$= \sqrt{\left(20\sqrt{5}x\right)^2 + \left(10\sqrt{5}x\right)^2} = 50x$$
$\Rightarrow a = 50$

G10.2 If $PQ = b\sqrt{10}x$, find b.

$$PQ = \sqrt{CP^2 + CQ^2} = 10\sqrt{10}x$$
$\Rightarrow b = 10$

G10.3 If the distance from A to PQ is $c\sqrt{10}x$, find c.

$c\sqrt{10}x = AC$ – distance from C to PQ
$$= 20\sqrt{5}x\cdot\sqrt{2} - 10\sqrt{5}x\cdot\left(\frac{1}{\sqrt{2}}\right)$$
$$= 15\sqrt{10}x$$
$\Rightarrow c = 15$

G10.4 If $\sin\theta = \frac{d}{100}$, find d.

Area of $\Delta APQ = \frac{1}{2}\cdot AP\cdot AQ\sin\theta = \frac{1}{2}\cdot PQ\cdot\left(c\sqrt{10}x\right)$

$$\Leftrightarrow \frac{1}{2}\cdot(50x)^2\sin\theta = \frac{1}{2}\cdot 10\sqrt{10}x\cdot 15\sqrt{10}x$$

$$\sin\theta = \frac{d}{100} = \frac{3}{5}$$

$\Rightarrow d = 60$