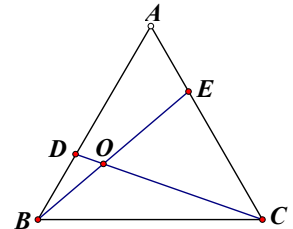


18-19 Individual	1	60	2	$24\sqrt{3}$	3	-4	4	1987	5	516
	6	516	7	$\frac{7\sqrt{5}}{3}$	8	8	9	32	10	4
	11	9	12	$5\sqrt{13}$	13	9	14	3	15	7
18-19 Group	1	1010	2	25	3	30	4	2	5	-1
	6	64	7	120	8	4	9	12	10	$25\sqrt{3} + 37.5$

Individual Events

I1 在圖一中， ABC 是一個等邊三角形。 D 和 E 分別是 AB 和 AC 上的點，使得 $AE = BD$ 。若 CD 和 BE 相交於 O 及 $\angle COE = y^\circ$ ，求 y 的值。

In Figure 1, ABC is an equilateral triangle. D and E are points on AB and AC respectively such that $AE = BD$. If CD and BE intersect at O and $\angle COE = y^\circ$, find the value of y .



Reference: 2000 HG6

$AE = BD$ (已知)	$AE = BD$ (given)
$\angle BAE = \angle CBD = 60^\circ$ (等邊三角形的性質)	$\angle BAE = \angle CBD = 60^\circ$ (prop. of equilateral Δ)
$AB = CB$ (等邊三角形的性質)	$AB = CB$ (prop. of equilateral Δ)
$\therefore \triangle EAB \cong \triangle DBC$ (S.A.S.)	$\therefore \triangle EAB \cong \triangle DBC$ (S.A.S.)
$\angle ABE = \angle BCD = \theta$ (全等三角形對應邊)	$\angle ABE = \angle BCD = \theta$ (cor. sides $\cong \Delta$ s)
$\angle CBE = 60^\circ - \theta$ (等邊三角形的性質)	$\angle CBE = 60^\circ - \theta$ (prop. of equilateral Δ)
$\angle COE = \angle CBE + \angle BCD$ ($\triangle BCO$ 的外角)	$\angle COE = \angle CBE + \angle BCD$ (ext. \angle of $\triangle BCO$)
$= 60^\circ - \theta + \theta = 60^\circ$	$= 60^\circ - \theta + \theta = 60^\circ$
$y = 60$	$y = 60$

I2 設 O 為極座標系統的極點。若 $P(6, 240^\circ)$ 向右平移 16 單位至 Q 而 $\triangle OPQ$ 的面積為 T 平方單位，求 T 的值。

Let O be the pole of the polar coordinate system. If $P(6, 240^\circ)$. If P is translated to the right by 16 units to Q and the area of $\triangle OPQ$ is T square units, find the value of T .

Reference: 2016 HI9

P 的直角座標為 $(6 \cos 240^\circ, 6 \sin 240^\circ) = (-3, -3\sqrt{3})$ 。 Q 的直角座標為 $(13, -3\sqrt{3})$ 。 $T = \frac{1}{2} \left\ \begin{vmatrix} -3 & -3\sqrt{3} \\ 13 & -3\sqrt{3} \end{vmatrix} \right\ = \frac{1}{2} (9\sqrt{3} + 39\sqrt{3}) = 24\sqrt{3}$	The rectangular coordinates of P is $(6 \cos 240^\circ, 6 \sin 240^\circ) = (-3, -3\sqrt{3})$. The rectangular coordinates of Q is $(13, -3\sqrt{3})$. $T = \frac{1}{2} \left\ \begin{vmatrix} -3 & -3\sqrt{3} \\ 13 & -3\sqrt{3} \end{vmatrix} \right\ = \frac{1}{2} (9\sqrt{3} + 39\sqrt{3}) = 24\sqrt{3}$
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I3 已知 x 及 y 均為實數，若 $y^2 - 4xy + 5x^2 - 8x + 16 = 0$ 及 $F = x - y$ ，求 F 的值。

Given that x and y are real numbers.

If $y^2 - 4xy + 5x^2 - 8x + 16 = 0$ and $F = x - y$, find the value of F .

Reference: 2015 HG4

$y^2 - 4xy + 4x^2 + x^2 - 8x + 16 = 0$ $(y - 2x)^2 + (x - 4)^2 = 0$ 兩個平方之和 = 0 \Rightarrow 每一項 = 0 $y - 2x = 0$ 及 $x = 4 \Rightarrow y = 8$ $F = x - y = 4 - 8 = -4$	$y^2 - 4xy + 4x^2 + x^2 - 8x + 16 = 0$ $(y - 2x)^2 + (x - 4)^2 = 0$ sum of two squares = 0 \Rightarrow Each term = 0 $y - 2x = 0$ and $x = 4 \Rightarrow y = 8$ $F = x - y = 4 - 8 = -4$
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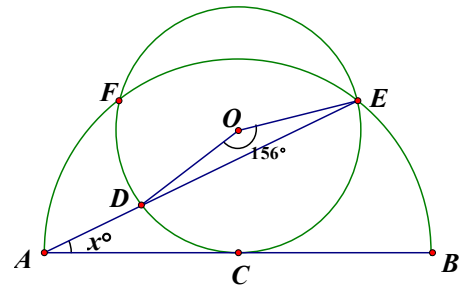
- 14** 設 n 為正整數。若 $a_n = 1 + 2 + \dots + 2^n$ 及 $b = a_{10} - a_5 + a_1$ ，求 b 的值。

Let n be a positive integer. If $a_n = 1 + 2 + \dots + 2^n$ and $b = a_{10} - a_5 + a_1$, find the value of b .

利用等比級數 n 項之和公式： $a_n = 2^{n+1} - 1$ 由 $n = 1, 2, 3, \dots$ $b = a_{10} - a_5 + a_1$ $= (2^{11} - 1) - (2^6 - 1) + (1 + 2)$ $= 2048 - 64 + 3 = 1987$	By the sum to n terms of a geometric series formula, $a_n = 2^{n+1} - 1$ for $n = 1, 2, 3, \dots$ $b = a_{10} - a_5 + a_1$ $= (2^{11} - 1) - (2^6 - 1) + (1 + 2)$ $= 2048 - 64 + 3 = 1987$
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- 15** 在圖二中， AB 為半圓的直徑， C 為半圓的圓心。有一圓形，圓心 O 切 AB 於 C 及交半圓於 E 和 F 。若 AE 交此圓形於 D ， $\angle DOE = 156^\circ$ 及 $\angle BAE = x^\circ$ ，求 x 的值。

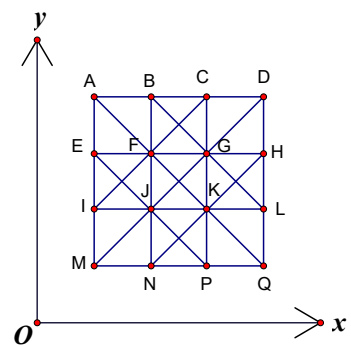
In Figure 2, AB is the diameter of the semi-circle, C is the centre of the semi-circle. A circle with centre at O , touching the semi-circle at C and cutting it at E and F . If AE cuts the circle at D , $\angle DOE = 156^\circ$ and $\angle BAE = x^\circ$, find the value of x .



反角 $\angle DOE = 360^\circ - 156^\circ$ (同頂角) $= 204^\circ$ $\angle DCE = \frac{1}{2}$ 反角 $\angle DOE$ (圓心角兩倍於圓周角) $= 102^\circ$ $\angle ACD = \angle AEC$ (交錯弓形的角) $\angle AEC = x^\circ$ (等腰三角形底角) $\angle BCE = \angle CAE + \angle AEC$ (三角形外角) $= 2x^\circ$ $\angle ACD + \angle DCE + \angle BCE = 180^\circ$ (直線上的鄰角) $x^\circ + 102^\circ + 2x^\circ = 180^\circ$ $x = 26$	Reflex $\angle DOE = 360^\circ - 156^\circ$ (\angle s at a pt.) $= 204^\circ$ $\angle DCE = \frac{1}{2}$ reflex $\angle DOE$ (\angle at centre twice \angle at \odot^{ce}) $= 102^\circ$ $\angle ACD = \angle AEC$ (\angle in alt. segment) $\angle AEC = x^\circ$ (base \angle s isos. Δ) $\angle BCE = \angle CAE + \angle AEC$ (ext. \angle of Δ) $= 2x^\circ$ $\angle ACD + \angle DCE + \angle BCE = 180^\circ$ (adj. \angle s on st. line) $x^\circ + 102^\circ + 2x^\circ = 180^\circ$ $x = 26$
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- 16** 在圖三中，直角座標平面上一個正方形的四個頂點的座標分別為 $(1, 1)$ 、 $(1, 4)$ 、 $(4, 1)$ 及 $(4, 4)$ 。若在該正方形中(包括邊界)選擇任何三個座標均為整數的點，問可組成多少個三角形？

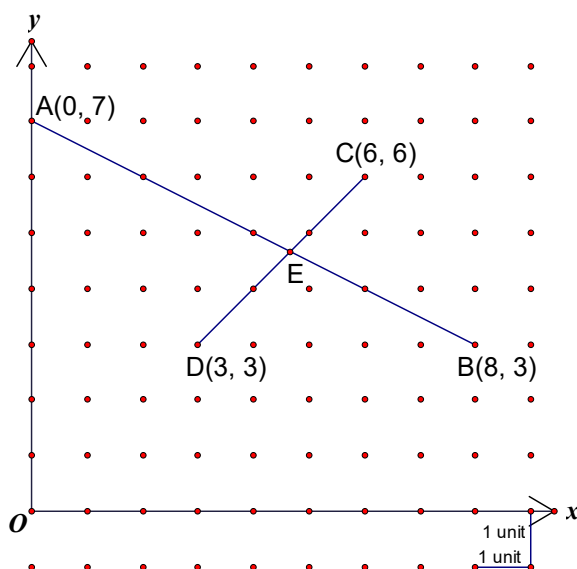
In Figure 3, the vertices of a square in the rectangular coordinate plane are $(1, 1)$, $(1, 4)$, $(4, 1)$ and $(4, 4)$. How many triangles can be formed by selecting any three points in the square (including the boundaries) with integer coordinates?



將這 16 個整數點命名如圖。 其中有 10 條線段穿過 4 點。 另外有 4 條線段穿過 3 點。 三角形的數目 $= C_3^{16} - \text{選中三點在同一直線的數目}$ $= \frac{16 \times 15 \times 14}{1 \times 2 \times 3} - 10 \times C_3^4 - 4 \times C_3^3$ $= 560 - 40 - 4 = 516$	Label the 16 integral points as shown. There are 10 line segments passing through 4 integral points. There are 4 line segments passing through 3 integral points. Number of triangles $= C_3^{16} - \text{number of choices of 3 collinear points}$ $= \frac{16 \times 15 \times 14}{1 \times 2 \times 3} - 10 \times C_3^4 - 4 \times C_3^3$ $= 560 - 40 - 4 = 516$
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I7 在圖四中， AB 與 CD 相交於 E 。設 AE 的長度為 q 單位，求 q 的值。

In Figure 4, AB and CE intersect at E . Let the length of AE be q units. Find the value of q .



定義一個直角座標系統如圖。

A 、 B 、 C 和 D 的座標分別為 $(0, 7)$ 、 $(8, 3)$ 、 $(6, 6)$ 及 $(3, 3)$ 。

$$AB \text{ 的方程為: } y - 7 = \frac{7-3}{0-8} \cdot (x-0)$$

$$y = -\frac{1}{2}x + 7 \cdots (1)$$

$$CD \text{ 的方程為: } y = x \cdots (2)$$

$$\text{代 (2) 入 (1): } x = -\frac{1}{2}x + 7 \Rightarrow x = \frac{14}{3} = y$$

$$q = AE = \sqrt{\left(\frac{14}{3} - 0\right)^2 + \left(\frac{14}{3} - 7\right)^2} = \frac{7\sqrt{5}}{3}$$

Define a rectangular co-ordinates system as shown.

The coordinates of A , B , C and D are $(0, 7)$, $(8, 3)$, $(6, 6)$ and $(3, 3)$ respectively.

$$\text{Equation of } AB: y - 7 = \frac{7-3}{0-8} \cdot (x-0)$$

$$y = -\frac{1}{2}x + 7 \cdots (1)$$

$$\text{Equation of } CD: y = x \cdots (2)$$

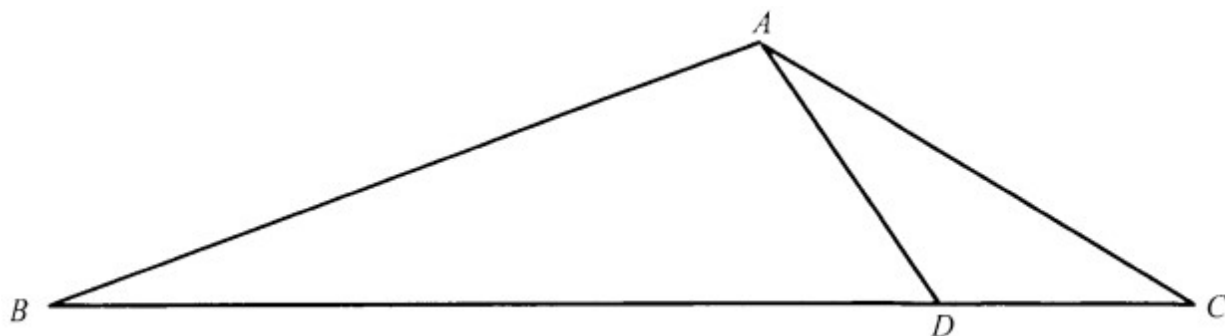
$$\text{Sub. (2) into (1): } x = -\frac{1}{2}x + 7 \Rightarrow x = \frac{14}{3} = y$$

$$q = AE = \sqrt{\left(\frac{14}{3} - 0\right)^2 + \left(\frac{14}{3} - 7\right)^2} = \frac{7\sqrt{5}}{3}$$

- 18 在圖五中， D 是在 BC 上的一點使得 $\angle ABD = \angle CAD$ 及 $\frac{BD}{AC} = \frac{8}{3}$ 。若 $\frac{\Delta ABD \text{ 的面積}}{\Delta ADC \text{ 的面積}} = k$ ，求 k 的值。

In Figure 5, D is a point on BC such that $\angle ABD = \angle CAD$ and $\frac{BD}{AC} = \frac{8}{3}$.

If $\frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ADC} = k$, find the value of k .



$\Delta ACD \sim \Delta BCA$ (A.A.A.)	$\Delta ACD \sim \Delta BCA$ (A.A.A.)
$\frac{AC}{CD} = \frac{BD + DC}{AC}$ (相似三角形對應邊)	$\frac{AC}{CD} = \frac{BD + DC}{AC}$ (corr. sides, $\sim \Delta$ s)
$\frac{AC}{CD} = \frac{BD}{AC} + \frac{DC}{AC}$	$\frac{AC}{CD} = \frac{BD}{AC} + \frac{DC}{AC}$
設 $t = \frac{AC}{CD}$, 則 $\frac{1}{t} = \frac{DC}{AC}$	Let $t = \frac{AC}{CD}$, then $\frac{1}{t} = \frac{DC}{AC}$
$t = \frac{8}{3} + \frac{1}{t}$	$t = \frac{8}{3} + \frac{1}{t}$
$3t^2 - 8t - 3 = 0$	$3t^2 - 8t - 3 = 0$
$(3t + 1)(t - 3) = 0$	$(3t + 1)(t - 3) = 0$
$t = -\frac{1}{3}$ (捨去) 或 $t = 3$	$t = -\frac{1}{3}$ (rejected) or $t = 3$
$CD = \frac{1}{3} AC$	$CD = \frac{1}{3} AC$
$BD = \frac{8}{3} AC$	$BD = \frac{8}{3} AC$
$k = \frac{\Delta ABD \text{ 的面積}}{\Delta ADC \text{ 的面積}} = \frac{BD}{CD} = 8$	$k = \frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ADC} = \frac{BD}{CD} = 8$

- 19 已知 α 及 β 為方程 $x^2 + 32x - 1 = 0$ 的兩個根。

若 $P = (\alpha^2 + 31\alpha - 2)(\beta^2 + 33\beta)$, 求 P 的值。

Given that α and β are the two roots of the equation $x^2 + 32x - 1 = 0$.

If $P = (\alpha^2 + 31\alpha - 2)(\beta^2 + 33\beta)$, find the value of P .

Reference: 2013 HG4

$$\alpha^2 + 32\alpha - 1 = 0 \Rightarrow \alpha^2 + 31\alpha - 2 = -\alpha - 1$$

$$\beta^2 + 32\beta - 1 = 0 \Rightarrow \beta^2 + 33\beta = \beta + 1$$

$$(\alpha^2 + 31\alpha - 2)(\beta^2 + 33\beta) = (-\alpha - 1)(\beta + 1)$$

$$= -(\alpha + 1)(\beta + 1)$$

$$= -(\alpha\beta + \alpha + \beta + 1)$$

$$= -(-1 - 32 + 1)$$

$$= 32$$

$$P = 32$$

I10 設 $c = \sqrt[3]{7+5\sqrt{2}} + \sqrt[3]{7-5\sqrt{2}}$ 。若 $w = c^2$ ，求 w 的值。

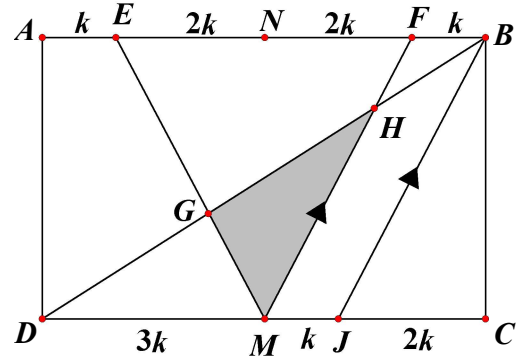
Let $c = \sqrt[3]{7+5\sqrt{2}} + \sqrt[3]{7-5\sqrt{2}}$. If $w = c^2$, find the value of w .

Reference: 1999 FI3.2, 2005 FI2.2, 2016 FG3.3

<p>設 $(a + \sqrt{b})^3 = 7 + 5\sqrt{2}$ $a^3 + 3a^2\sqrt{b} + 3ab + b\sqrt{b} = 7 + 5\sqrt{2}$ $b = 2, a^3 + 3ab = 7, 3a^2 + b = 5 \Rightarrow a = 1$ $(1 + \sqrt{2})^3 = 7 + 5\sqrt{2}$ 及 $(1 - \sqrt{2})^3 = 7 - 5\sqrt{2}$ $c = 1 + \sqrt{2} + (1 - \sqrt{2}) = 2$ $w = c^2 = 4$</p>	<p>Let $(a + \sqrt{b})^3 = 7 + 5\sqrt{2}$ $a^3 + 3a^2\sqrt{b} + 3ab + b\sqrt{b} = 7 + 5\sqrt{2}$ $b = 2, a^3 + 3ab = 7, 3a^2 + b = 5 \Rightarrow a = 1$ $(1 + \sqrt{2})^3 = 7 + 5\sqrt{2}$ and $(1 - \sqrt{2})^3 = 7 - 5\sqrt{2}$ $c = 1 + \sqrt{2} + (1 - \sqrt{2}) = 2$ $w = c^2 = 4$</p>
<p>方法二 $c^3 = 7 + 5\sqrt{2} + 3 \times \sqrt[3]{(7 + 5\sqrt{2})^2 (7 - 5\sqrt{2})}$ $+ 3 \times \sqrt[3]{(7 + 5\sqrt{2})(7 - 5\sqrt{2})^2} + 7 - 5\sqrt{2}$ $= 14 + 3 \times \sqrt[3]{(7 + 5\sqrt{2})(49 - 50)}$ $+ 3 \times \sqrt[3]{(49 - 50)(7 - 5\sqrt{2})}$ $= 14 - 3c$ $c^3 + 3c - 14 = 0$ $(c - 2)(c^2 + 2c + 7) = 0$ $c = 2$ 或 沒有實數解 $w = c^2 = 4$</p>	<p>Method 2 $c^3 = 7 + 5\sqrt{2} + 3 \times \sqrt[3]{(7 + 5\sqrt{2})^2 (7 - 5\sqrt{2})}$ $+ 3 \times \sqrt[3]{(7 + 5\sqrt{2})(7 - 5\sqrt{2})^2} + 7 - 5\sqrt{2}$ $= 14 + 3 \times \sqrt[3]{(7 + 5\sqrt{2})(49 - 50)}$ $+ 3 \times \sqrt[3]{(49 - 50)(7 - 5\sqrt{2})}$ $= 14 - 3c$ $c^3 + 3c - 14 = 0$ $(c - 2)(c^2 + 2c + 7) = 0$ $c = 2$ or no real solution $w = c^2 = 4$</p>

- III** 在圖六中， $ABCD$ 為一個長方形。 M 和 N 分別是 DC 和 AB 的中點且 $AE : EN = BF : FN = 1 : 2$ 。 DB 分別交 EM 和 FM 於 G 及 H 。若長方形 $ABCD$ 及三角形 GHM 的面積分別是 96 和 S ，求 S 的值。

In Figure 6, $ABCD$ is rectangle M and N are the mid-points of DC and AB respectively and $AE : EN = BF : FN = 1 : 2$. DB intersects EM and FM at G and H respectively. If the areas of the rectangle $ABCD$ and the triangle GHM are 96 and S respectively, find the value of S .



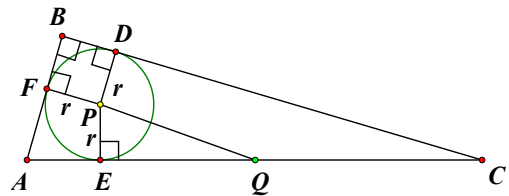
圖六 Figure 6

Reference 1998 HG5, 2016 HI14, 2018 FG3.1

設 $AE = BF = k, EN = NF = 2k, DM = MC = 3k$	Let $AE = BF = k, EN = NF = 2k, DM = MC = 3k$
$\triangle BHF \sim \triangle DHM$ (A.A.A.)	$\triangle BHF \sim \triangle DHM$ (A.A.A.)
$\triangle BGE \sim \triangle DGM$ (A.A.A.)	$\triangle BGE \sim \triangle DGM$ (A.A.A.)
$\frac{DH}{HB} = \frac{DM}{BF} = \frac{3k}{k} = 3$ (相似三角形對應邊)	$\frac{DH}{HB} = \frac{DM}{BF} = \frac{3k}{k} = 3$ (corr. side $\sim \Delta$ s)
$\frac{DG}{GB} = \frac{DM}{BE} = \frac{3k}{5k} = \frac{3}{5}$ (相似三角形對應邊)	$\frac{DG}{GB} = \frac{DM}{BE} = \frac{3k}{5k} = \frac{3}{5}$ (corr. side $\sim \Delta$ s)
$BH = \frac{1}{4} DB, DG = \frac{3}{8} DB$	$BH = \frac{1}{4} DB, DG = \frac{3}{8} DB$
$GH = DB - DG - BH = \left(1 - \frac{3}{8} - \frac{1}{4}\right) DB = \frac{3}{8} DB$	$GH = DB - DG - BH = \left(1 - \frac{3}{8} - \frac{1}{4}\right) DB = \frac{3}{8} DB$
$DG : GH = \frac{3}{8} DB : \frac{3}{8} DB = 1 : 1 \dots\dots (1)$	$DG : GH = \frac{3}{8} DB : \frac{3}{8} DB = 1 : 1 \dots\dots (1)$
過 B 作 $BJ \parallel MF$ ，交 CD 於 J 。	Draw $BJ \parallel MF$, cutting CD at J .
$\frac{DM}{MJ} = \frac{DH}{HB} = 3$ (等比定理)	$\frac{DM}{MJ} = \frac{DH}{HB} = 3$ (theorem of equal ratios)
$MJ = k, JC = 2k$	$MJ = k, JC = 2k$
$\triangle BCD \cong \triangle DAB$ (S.S.S.)	$\triangle BCD \cong \triangle DAB$ (S.S.S.)
$S_{ABCD} = S_{\triangle DAB} = \frac{1}{2} \times 96 = 48$	$S_{ABCD} = S_{\triangle DAB} = \frac{1}{2} \times 96 = 48$
$\frac{S_{\triangle BDJ}}{S_{\triangle BCJ}} = \frac{DJ}{CJ} = \frac{4k}{2k} = 2 \Rightarrow S_{\triangle BDJ} = \frac{2}{3} \times 48 = 32$	$\frac{S_{\triangle BDJ}}{S_{\triangle BCJ}} = \frac{DJ}{CJ} = \frac{4k}{2k} = 2 \Rightarrow S_{\triangle BDJ} = \frac{2}{3} \times 48 = 32$
$\triangle DMH \sim \triangle DJB$ (A.A.A.)	$\triangle DMH \sim \triangle DJB$ (A.A.A.)
$\frac{S_{\triangle DMH}}{S_{\triangle DJB}} = \left(\frac{DM}{DJ}\right)^2 = \left(\frac{3}{4}\right)^2 \Rightarrow S_{\triangle DMH} = \frac{9}{16} \times 32 = 18$	$\frac{S_{\triangle DMH}}{S_{\triangle DJB}} = \left(\frac{DM}{DJ}\right)^2 = \left(\frac{3}{4}\right)^2 \Rightarrow S_{\triangle DMH} = \frac{9}{16} \times 32 = 18$
由 (1), $S_{\triangle GHM} = S_{\triangle GDM} = \frac{1}{2} \times S_{\triangle DMH} = 9$	By (1), $S_{\triangle GHM} = S_{\triangle GDM} = \frac{1}{2} \times S_{\triangle DMH} = 9$

- II2** 在三角形 ABC 中, $AB=14$ 、 $BC=48$ 及 $AC=50$ 。
將 P 及 Q 分別記為 $\triangle ABC$ 的內心及外心。設 PQ
的長度為 d 單位。求 d 的值。

In triangle ABC , $AB = 14$, $BC = 48$ and $AC = 50$.
Denote the in-centre and circumcentre of $\triangle ABC$ by P
and Q respectively. Let the length of PQ be d units.
Find the value of d .



$AB^2 + BC^2 = 14^2 + 48^2 = 196 + 2304 = 2500 = AC^2$ $\angle ABC = 90^\circ$ (畢氏定理的逆定理) AC 是外接圓 ABC 的直徑(半圓上的圓周角的定理) $Q = AC$ 的中點 (外接圓的圓心) $AQ = 25 \dots (1)$ 假設內切圓分別切 BC 、 AC 及 AB 於 D 、 E 及 F 。 $PD \perp BC$, $PE \perp AC$, $PF \perp AB$ (切綫 \perp 半徑) $PDBF$ 是一個長方形 (它有 3 隻直角) 設內切圓的半徑為 r 。 $PD = PE = PF = r$ $PDBF$ 是一個正方形 ($PD = PF$) $BF = BD = r$ $AF = 14 - r$, $CD = 48 - r$ $AE = 14 - r$, $CE = 48 - r$ (由外點引切綫) $AE + EC = AC$ $14 - r + 48 - r = 50$ $r = 6$, $AE = 14 - 6 = 8$ $EQ = AQ - AE = 25 - 8 = 17$ 在 $\triangle PEQ$ 中, $PE^2 + EQ^2 = PQ^2$ (畢氏定理) $6^2 + 17^2 = PQ^2$ $d = \sqrt{325} = 5\sqrt{13}$	$AB^2 + BC^2 = 14^2 + 48^2 = 196 + 2304 = 2500 = AC^2$ $\angle ABC = 90^\circ$ (converse, Pyth. thm.) AC is the diameter of the circumcircle ABC (converse, \angle in semi-circle) $Q = \text{mid-point of } AC$ (centre of circumcircle) $AQ = 25 \dots (1)$ Suppose the in-circle touches BC , AC and AB at D , E and F respectively. $PD \perp BC$, $PE \perp AC$, $PF \perp AB$ (tangent \perp radius) $PDBF$ is a rectangle (it has 3 \perp \angle s) Let r be the radius of the inscribed circle. $PD = PE = PF = r$ $PDBF$ is a square ($PD = PF$) $BF = BD = r$ $AF = 14 - r$, $CD = 48 - r$ $AE = 14 - r$, $CE = 48 - r$ (tangent from ext. pt.) $AE + EC = AC$ $14 - r + 48 - r = 50$ $r = 6$, $AE = 14 - 6 = 8$ $EQ = AQ - AE = 25 - 8 = 17$ In $\triangle PEQ$, $PE^2 + EQ^2 = PQ^2$ (Pythagoras' thm.) $6^2 + 17^2 = PQ^2$ $d = \sqrt{325} = 5\sqrt{13}$
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- II3** 已知正整數 a 、 b 及 c 滿足下列條件：

- $a > b > c$,
- $(a - b)(b - c)(a - c) = 84$,
- $abc < 100$.

設 M 為 a 的最大值。求 M 的值。

Given that a , b and c are positive integers satisfying the following conditions:

- $a > b > c$,
- $(a - b)(b - c)(a - c) = 84$,
- $abc < 100$.

Let M be the maximum value of a . Find the value of M .

84 的正因子包括 1、2、3、4、6、7、12、14、21、28、42 及 84。 $(a - b) + (b - c) = a - c$ $(a - b, b - c, a - c)$ 的可能值 = (3, 4, 7) 或 (4, 3, 7) $(a, b, c) = (a, a - 3, a - 7)$ 或 $(a, a - 4, a - 7)$ 為了使得 a 為最大, b 和 c 必須盡量小 $\therefore (a, b, c) = (a, a - 4, a - 7)$ $9 \times 5 \times 2 = 90$, $10 \times 6 \times 3 = 180$ $M = 9$	Positive factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42 and 84. $(a - b) + (b - c) = a - c$ Possible $(a - b, b - c, a - c) = (3, 4, 7)$ or $(4, 3, 7)$ $(a, b, c) = (a, a - 3, a - 7)$ or $(a, a - 4, a - 7)$ For largest a , b and c must be as small as possible $\therefore (a, b, c) = (a, a - 4, a - 7)$ $9 \times 5 \times 2 = 90$, $10 \times 6 \times 3 = 180$ $M = 9$
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- I14** 已知 $3 \sin x + 2 \sin y = 4$ 。設 N 為 $3 \cos x + 2 \cos y$ 的最大值。求 N 的值。
 Given that $3 \sin x + 2 \sin y = 4$. Let N be the maximum value of $3 \cos x + 2 \cos y$.
 Find the value of N .

The following method is provided by Ms. Wong Ka Man from St. Mark's College.

$$\begin{aligned} & (3 \cos x + 2 \cos y)^2 \\ &= 9 \cos^2 x + 12 \cos x \cos y + 4 \cos^2 y \\ &= 9(1 - \sin^2 x) + 12(\cos x \cos y + \sin x \sin y) + 4(1 - \sin^2 y) - 12 \sin x \sin y \\ &= 13 + 12 \cos(x - y) - (3 \sin x + 2 \sin y)^2 \\ &= 13 + 12 \cos(x - y) - 4^2 = 12 \cos(x - y) - 3 \\ &\leq 12 - 3 = 9 \\ &\therefore 3 \cos x + 2 \cos y \leq 3 \\ &N = 3 \end{aligned}$$

- I15** 已知 x, y 及 z 為正實數且滿足 $\begin{cases} x^2 + xy + y^2 = 7 \\ y^2 + yz + z^2 = 21 \\ x^2 + xz + z^2 = 28 \end{cases}$ 。若 $a = x + y + z$ ，求 a 的值。

Given that x, y and z are positive real numbers satisfying $\begin{cases} x^2 + xy + y^2 = 7 & \dots(1) \\ y^2 + yz + z^2 = 21 & \dots(2) \\ x^2 + xz + z^2 = 28 & \dots(3) \end{cases}$

If $a = x + y + z$, find the value of a .

$\begin{cases} (x-y)(x^2 + xy + y^2) = 7(x-y) \\ (y-z)(y^2 + yz + z^2) = 21(y-z) \\ (z-x)(x^2 + xz + z^2) = 28(z-x) \end{cases}$ $\begin{cases} x^3 - y^3 = 7x - 7y \\ y^3 - z^3 = 21y - 21z \\ z^3 - x^3 = 28z - 28x \end{cases}$ <p>將以上三題方程相加: $0 = -21x + 14y + 7z$ $z = 3x - 2y \dots (4)$</p> <p>$(1) + (2) - (3): 2y^2 + (x+z)y - xz = 0 \dots (5)$ 代(4)入(5): $2y^2 + (x + 3x - 2y)y - x(3x - 2y) = 0$ $2y^2 + (4x - 2y)y - (3x^2 - 2xy) = 0$ $3x^2 - 6xy = 0$ $x = 0$ (x 為正實數, 捨去) 或 $x = 2y \dots (6)$ 代(6)入(1): $4y^2 + 2y^2 + y^2 = 7$ $y = 1$ 或 -1 (y 為正實數, 捨去) $x = 2$ $z = 3x - 2y = 4$ $a = x + y + z = 7$</p>	$\begin{cases} (x-y)(x^2 + xy + y^2) = 7(x-y) \\ (y-z)(y^2 + yz + z^2) = 21(y-z) \\ (z-x)(x^2 + xz + z^2) = 28(z-x) \end{cases}$ $\begin{cases} x^3 - y^3 = 7x - 7y \\ y^3 - z^3 = 21y - 21z \\ z^3 - x^3 = 28z - 28x \end{cases}$ <p>Add up these equations: $0 = -21x + 14y + 7z$ $z = 3x - 2y \dots (4)$</p> <p>$(1) + (2) - (3): 2y^2 + (x+z)y - xz = 0 \dots (5)$ Sub. (4) into (5): $2y^2 + (x + 3x - 2y)y - x(3x - 2y) = 0$ $2y^2 + (4x - 2y)y - (3x^2 - 2xy) = 0$ $3x^2 - 6xy = 0$ $x = 0$ (x is real positive, rejected) or $x = 2y \dots (6)$ Sub. (6) into (1): $4y^2 + 2y^2 + y^2 = 7$ $y = 1$ or -1 (y is real positive, rejected) $x = 2$ $z = 3x - 2y = 4$ $a = x + y + z = 7$</p>
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Group Events

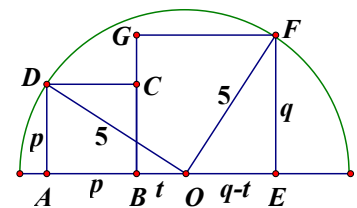
G1 對所有正實數 x ，定義 $f(x) = \log_{2019} x^{2020}$ 。若 $D = f(\sqrt{3}) + f(\sqrt{673})$ ，求 D 的值。

For all positive value real numbers x , define $f(x) = \log_{2019} x^{2020}$. If $D = f(\sqrt{3}) + f(\sqrt{673})$, find the value of D .

$$\begin{aligned} D &= \log_{2019} (\sqrt{3})^{2020} + \log_{2019} (\sqrt{673})^{2020} \\ &= \log_{2019} (\sqrt{3} \times \sqrt{673})^{2020} \\ &= \log_{2019} (2019)^{1010} \\ &= 1010 \end{aligned}$$

G2 圖一所示， $ABCD$ 和 $BEFG$ 是兩個緊貼的正方形，躺臥在一個以 O 為圓心，半徑為 5 cm 的半圓上。其中 A 、 B 和 E 在半圓的直徑， D 和 F 在半圓的弧上。設 $ABCD$ 與 $BEFG$ 的面積之和為 $S\text{ cm}^2$ ，求 S 的值。

Figure 1 shows two adjacent squares $ABCD$ and $BEFG$ lying on a semi-circle with centre O and radius 5 cm . A , B and E lie on the diameter of the semi-circle, D and F lie on the semi-circular arc. Let the sum of areas of $ABCD$ and $BEFG$ be $S\text{ cm}^2$, find the value of S .

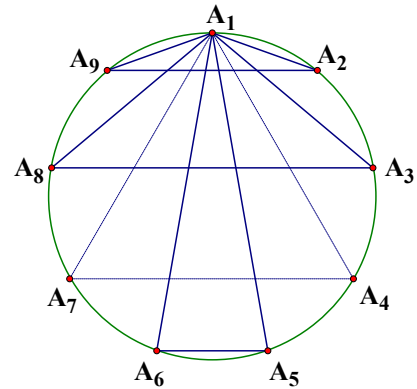


圖一 Figure 1

$OD = OE = 5\text{ cm}$ 。設 $AD = p$ 、 $EF = q$ 。 不妨假設 $q > p$ 。 設 $OB = t$ ，則 $OE = q - t$ 。 $AD \perp AB$ ， $FE \perp BE$ $AD^2 + AO^2 = OE^2 + EF^2 = OF^2$ (畢氏定理) $p^2 + (p + t)^2 = (q - t)^2 + q^2 = 5^2$ $p^2 + p^2 + 2pt + t^2 = q^2 - 2qt + t^2 + q^2$ $2p^2 + 2pt = 2q^2 - 2qt$ $p^2 + pt = q^2 - qt$ $(q + p)t = q^2 - p^2$ $t = q - p$ $OA = p + t = p + q - p = q$ $AD^2 + AO^2 = OD^2$ (畢氏定理) $p^2 + q^2 = 5^2$ $S = p^2 + q^2 = 25$	$OD = OE = 5\text{ cm}$. Let $AD = p$, $EF = q$. Without loss of generality, assume $q > p$. Let $OB = t$, then $OE = q - t$. $AD \perp AB$, $FE \perp BE$ $AD^2 + AO^2 = OE^2 + EF^2 = OF^2$ (Pythagoras' theorem) $p^2 + (p + t)^2 = (q - t)^2 + q^2 = 5^2$ $p^2 + p^2 + 2pt + t^2 = q^2 - 2qt + t^2 + q^2$ $2p^2 + 2pt = 2q^2 - 2qt$ $p^2 + pt = q^2 - qt$ $(q + p)t = q^2 - p^2$ $t = q - p$ $OA = p + t = p + q - p = q$ $AD^2 + AO^2 = OD^2$ (Pythagoras' theorem) $p^2 + q^2 = 5^2$ $S = p^2 + q^2 = 25$
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G3 若從一個正九邊形的 9 個頂點中選 3 點，共可組成多少個等腰三角形？

If three vertices are chosen from the nine vertices of a regular nonagon, how many possible isosceles triangles are there ?

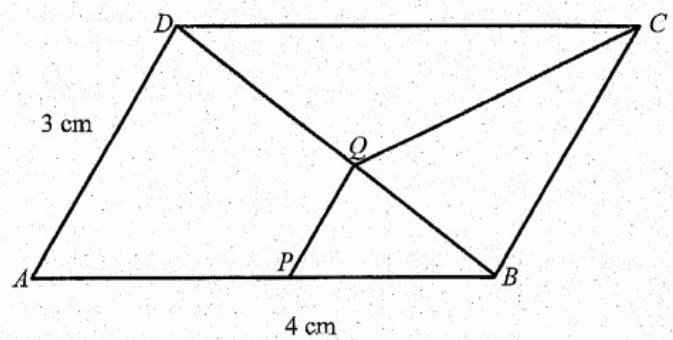


將 9 個頂點依次序命名為 A_1, A_2, \dots, A_9 。
其中有 4 個等腰三角形通過 $A_iA_1A_j$ 及 $A_1A_i = A_1A_j$ 。當中 $A_4A_1A_7$ 是一個等邊三角形。
若果不計算等邊三角形，所有等腰三角形的總數為 $3 \times 9 = 27$ 。
若果包括了所有等邊三角形，所有等腰三角形的總數為 $27 + 3 = 30$ 。

Label the 9 vertices as A_1, A_2, \dots, A_9 in order.
There are 4 isosceles triangles in the form $A_iA_1A_j$ such that $A_1A_i = A_1A_j$. Amongst these 4 isosceles triangles, $A_4A_1A_7$ is an equilateral triangle.
If we do not count these equilateral triangles, the total number of isosceles triangles are $3 \times 9 = 27$
If we include these equilateral triangles, the total number of isosceles triangles = $27 + 3 = 30$

G4 在圖二中， $ABCD$ 為一個平行四邊形，其中 $AB = 4 \text{ cm}$ 、 $AD = 3 \text{ cm}$ 及 $\sin A = \frac{2}{3}$ 。 P 和 Q 分別是 AB 和 BD 上的點，使得 $PQ \parallel AD$ ，且四邊形 $PBCQ$ 的面積為 3 cm^2 。設 AP 的長度為 $q \text{ cm}$ ，求 q 的值。

圖二 Figure 2



In Figure 2, $ABCD$ is a parallelogram, where $AB = 4 \text{ cm}$, $AD = 3 \text{ cm}$ and $\sin A = \frac{2}{3}$. P and Q are points on AB and BD respectively such that $PQ \parallel AD$, and the area of the quadrilateral $PBCQ$ is 3 cm^2 . Let the length of AP be $q \text{ cm}$, find the value of q .

設 S 表示面積。
 $S_{\triangle ABD} = S_{\triangle CDB} = \frac{1}{2} AB \cdot AD \sin A = \frac{1}{2} \cdot 3 \cdot 4 \cdot \frac{2}{3} = 4$
 $\triangle BPQ \sim \triangle BAD$ (A.A.A.)
設 $BQ : QD = k : (1 - k)$
 $\frac{S_{\triangle BPQ}}{S_{\triangle BAD}} = k^2 \Rightarrow S_{\triangle BPQ} = 4k^2$
 $\triangle BCQ$ 及 $\triangle BCD$ 有相同高度
 $\frac{S_{\triangle BCQ}}{S_{\triangle BCD}} = k \Rightarrow S_{\triangle BCQ} = 4k$
 $S_{PBCQ} = 3 \Rightarrow 4k^2 + 4k = 3$
 $(2k + 3)(2k - 1) = 0$
 $k = -1.5$ (捨去) 或 0.5
 $BP : PA = BQ : QD = 0.5 : (1 - 0.5) = 1 : 1$
 $\Rightarrow q = 2$

Let S denote the area.
 $S_{\triangle ABD} = S_{\triangle CDB} = \frac{1}{2} AB \cdot AD \sin A = \frac{1}{2} \cdot 3 \cdot 4 \cdot \frac{2}{3} = 4$
 $\triangle BPQ \sim \triangle BAD$ (A.A.A.)
Let $BQ : QD = k : (1 - k)$
 $\frac{S_{\triangle BPQ}}{S_{\triangle BAD}} = k^2 \Rightarrow S_{\triangle BPQ} = 4k^2$
 $\triangle BCQ$ and $\triangle BCD$ have the same height
 $\frac{S_{\triangle BCQ}}{S_{\triangle BCD}} = k \Rightarrow S_{\triangle BCQ} = 4k$
 $S_{PBCQ} = 3 \Rightarrow 4k^2 + 4k = 3$
 $(2k + 3)(2k - 1) = 0$
 $k = -1.5$ (rejected) or 0.5
 $BP : PA = BQ : QD = 0.5 : (1 - 0.5) = 1 : 1$
 $\Rightarrow q = 2$

G5 已知 $f(x) - 2f\left(\frac{1}{x}\right) = x$, 其中 $x \neq 0$ 。設 y 為滿足方程 $f(x) = 1$ 的 x 的最大值。求 y 的值。

Given that $f(x) - 2f\left(\frac{1}{x}\right) = x$, where $x \neq 0$. Let y be the maximum value of x that satisfies the equation $f(x) = 1$. Find the value of y . **Reference: 2018 HG4**

$f(x) - 2f\left(\frac{1}{x}\right) = x \quad \cdots (1)$ $f\left(\frac{1}{x}\right) - 2f(x) = \frac{1}{x} \quad \cdots (2)$ $(1) + 2(2): -3f(x) = x + \frac{2}{x}$ $\Rightarrow f(x) = -\frac{1}{3}\left(x + \frac{2}{x}\right)$ $f(x) = 1 \Rightarrow -\frac{1}{3}\left(x + \frac{2}{x}\right) = 1$ $x^2 + 2 = -3x$ $x^2 + 3x + 2 = 0$ $x = -1$ 或 -2 $y = -1$	$f(x) - 2f\left(\frac{1}{x}\right) = x \quad \cdots (1)$ $f\left(\frac{1}{x}\right) - 2f(x) = \frac{1}{x} \quad \cdots (2)$ $(1) + 2(2): -3f(x) = x + \frac{2}{x}$ $\Rightarrow f(x) = -\frac{1}{3}\left(x + \frac{2}{x}\right)$ $f(x) = 1 \Rightarrow -\frac{1}{3}\left(x + \frac{2}{x}\right) = 1$ $x^2 + 2 = -3x$ $x^2 + 3x + 2 = 0$ $x = -1$ or -2 $y = -1$
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G6 設 a_k 為多項式 $(2x-2)^3(2x+2)^3(2x+1)^3$ 中 x^k 的係數。

若 $Q = a_2 + a_4 + a_6 + a_8$, 求 Q 的值。

Let a_k be the coefficient of x^k in the polynomial $(2x-2)^3(2x+2)^3(2x+1)^3$.

If $Q = a_2 + a_4 + a_6 + a_8$, find the value of Q .

$$(2x-2)^3(2x+2)^3(2x+1)^3 = 64(x^2-1)^3(2x+1)^3 = 64(x^6-3x^4+3x^2-1)(8x^3+12x^2+6x+1)$$

$$a_2 = 64(3-12) = 64 \times (-9)$$

$$a_4 = 64(-3+3 \times 12) = 64 \times 33$$

$$a_6 = 64(1-3 \times 12) = 64 \times (-35)$$

$$a_8 = 64 \times 12$$

$$Q = a_2 + a_4 + a_6 + a_8 = 64 \times (-9) + 64 \times 33 + 64 \times (-35) + 64 \times 12 = 64 \times (-9 + 33 - 35 + 12) = 64$$

G7 設 $f(x) = -6x^2 + 4x \cos \theta + \sin \theta$, 其中 $0^\circ \leq \theta \leq 360^\circ$ 。已知對所有實數 x , $f(x) \leq 0$ 。若 θ 的最大值與最小值之差為 d° , 求 d 的值。

Let $f(x) = -6x^2 + 4x \cos \theta + \sin \theta$, where $0^\circ \leq \theta \leq 360^\circ$. If is given that $f(x) \leq 0$ for all real numbers x . If d° is the difference between the greatest and the least values of θ , find the value of d .

<p>設 $a = -6$, $b = 4 \cos \theta$, $c = \sin \theta$</p> <p>$f(x)$ 的最大值 $= \frac{4ac - b^2}{4a} \leq 0$</p> <p>$\frac{4(-6)\sin \theta - (4\cos \theta)^2}{4(-6)} \leq 0$</p> <p>$24 \sin \theta + 16 \cos^2 \theta \leq 0$</p> <p>$3 \sin \theta + 2(1 - \sin^2 \theta) \leq 0$</p> <p>$2 \sin^2 \theta - 3 \sin \theta - 2 \geq 0$</p> <p>$(2 \sin \theta + 1)(\sin \theta - 2) \geq 0$</p> <p>$\sin \theta \leq -0.5$ 或 $\sin \theta \geq 2$ (捨去)</p> <p>$210^\circ \leq \theta \leq 330^\circ \Rightarrow d = 330 - 210 = 120$</p>	<p>Let $a = -6$, $b = 4 \cos \theta$, $c = \sin \theta$</p> <p>Maximum value of $f(x) = \frac{4ac - b^2}{4a} \leq 0$</p> <p>$\frac{4(-6)\sin \theta - (4\cos \theta)^2}{4(-6)} \leq 0$</p> <p>$24 \sin \theta + 16 \cos^2 \theta \leq 0$</p> <p>$3 \sin \theta + 2(1 - \sin^2 \theta) \leq 0$</p> <p>$2 \sin^2 \theta - 3 \sin \theta - 2 \geq 0$</p> <p>$(2 \sin \theta + 1)(\sin \theta - 2) \geq 0$</p> <p>$\sin \theta \leq -0.5$ or $\sin \theta \geq 2$ (rejected)</p> <p>$210^\circ \leq \theta \leq 330^\circ \Rightarrow d = 330 - 210 = 120$</p>
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G8 設 $\{a_n\}$ 為一個正實數序列使當 $n > 1$ 時, $a_n = a_{n-1}a_{n+1} - 1$ 。

已知 2018 在序列中及 $a_2 = 2019$ 。若 a_1 的所有可取的數目為 s , 求 s 的值。

Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n = a_{n-1}a_{n+1} - 1$ for $n > 1$.

It is given that 2018 is in the sequence and $a_2 = 2019$. If the number of all possible values of a_1 is s , find the value of s .

$a_{n+1} = \frac{1+a_n}{a_{n-1}} = \frac{1+\frac{1+a_{n-1}}{a_{n-2}}}{a_{n-1}} = \frac{a_{n-2}+a_{n-1}+1}{a_{n-1}a_{n-2}}$ $= \frac{a_{n-2} + \frac{1+a_{n-2}}{a_{n-3}} + 1}{a_{n-1}} = \frac{a_{n-2}a_{n-3} + 1 + a_{n-2} + a_{n-3}}{(1+a_{n-2}) \cdot a_{n-2}}$ $= \frac{\frac{1+a_{n-2}}{a_{n-3}} \cdot a_{n-2}}{(1+a_{n-2}) \cdot a_{n-2}} = \frac{1+a_{n-3}}{a_{n-2}} = \frac{1+a_{n-3}}{\frac{1+a_{n-3}}{a_{n-4}}}$ $= a_{n-4} \text{ 對於 } n \geq 5$ <p>$\therefore a_1 = a_6 = a_{11} = \dots = a_{5n+1}$</p> <p>$a_2 = a_7 = a_{12} = \dots = a_{5n+2} = 2019$</p> <p>$\therefore a_k = 2018$ 對某些正整數 $k \neq 5n+2$。</p> <p>$\therefore a_n$ 的數值由 a_2 及 a_k 決定。</p> <p>a_1 有 4 種不同的值, $s = 4$</p>	$a_{n+1} = \frac{1+a_n}{a_{n-1}} = \frac{1+\frac{1+a_{n-1}}{a_{n-2}}}{a_{n-1}} = \frac{a_{n-2}+a_{n-1}+1}{a_{n-1}a_{n-2}}$ $= \frac{a_{n-2} + \frac{1+a_{n-2}}{a_{n-3}} + 1}{a_{n-1}} = \frac{a_{n-2}a_{n-3} + 1 + a_{n-2} + a_{n-3}}{(1+a_{n-2}) \cdot a_{n-2}}$ $= \frac{\frac{1+a_{n-2}}{a_{n-3}} \cdot a_{n-2}}{(1+a_{n-2}) \cdot a_{n-2}} = \frac{1+a_{n-3}}{a_{n-2}} = \frac{1+a_{n-3}}{\frac{1+a_{n-3}}{a_{n-4}}}$ $= a_{n-4} \text{ for } n \geq 5$ <p>$\therefore a_1 = a_6 = a_{11} = \dots = a_{5n+1}$</p> <p>$a_2 = a_7 = a_{12} = \dots = a_{5n+2} = 2019$</p> <p>$\therefore a_k = 2018$ for some positive integer $k \neq 5n+2$.</p> <p>$\therefore a_n$ is uniquely determined by a_2 and a_k.</p> <p>a_1 can have 4 different values, $s = 4$</p>
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G9 有多少對正整數 x, y 可滿足 $xy = 6(x + y + \sqrt{x^2 + y^2})$?

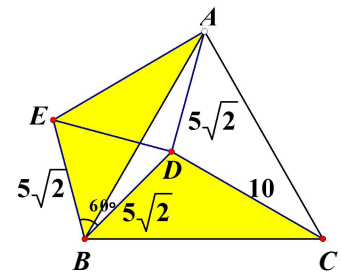
How many pairs of positive integers x, y are there satisfying $xy = 6(x + y + \sqrt{x^2 + y^2})$?

$(xy - 6x - 6y)^2 = 36(x^2 + y^2)$ $x^2y^2 - 12x^2y - 12xy^2 + 72xy = 0$ $xy - 12x - 12y + 72 = 0$ $xy - 12x - 12y + 144 = 72$ $(x - 12)(y - 12) = 72$ $(x - 12, y - 12) = (1, 72), (2, 36), (3, 24), (4, 18),$ $(6, 12), (8, 9), (9, 8), (12, 6), (18, 4), (24, 3),$ $(36, 2), (72, 1)$ 。 一共有 12 對正整數。 註：當 $(x - 12, y - 12) = (-8, -9)$ 或 $(-9, -8)$ $(x, y) = (4, 3)$ 或 $(3, 4)$ 。這兩組答案未能滿足 原方程 \therefore 捨去。	$(xy - 6x - 6y)^2 = 36(x^2 + y^2)$ $x^2y^2 - 12x^2y - 12xy^2 + 72xy = 0$ $xy - 12x - 12y + 72 = 0$ $xy - 12x - 12y + 144 = 72$ $(x - 12)(y - 12) = 72$ $(x - 12, y - 12) = (1, 72), (2, 36), (3, 24), (4, 18),$ $(6, 12), (8, 9), (9, 8), (12, 6), (18, 4), (24, 3),$ $(36, 2), (72, 1)$ 。 There are 12 pairs of positive integers. Remark: When $(x - 12, y - 12) = (-8, -9)$ or $(-9, -8)$ $(x, y) = (4, 3)$ or $(3, 4)$. These two solutions do not satisfy the original equation \therefore rejected.
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G10 D 是等邊三角形 ABC 內的一點使得 $AD = BD = 5\sqrt{2}$ 及 $CD = 10$ 。設 $\triangle ABC$ 的面積為 S ，求 S 的值。
 D is a point inside the equilateral triangle ABC such that $AD = BD = 5\sqrt{2}$ and $CD = 10$.

Let the area of $\triangle ABC$ be S , find the value of S .

Reference: 2014 HI3



如圖所示，將 BD 繞 B 反時針方向轉 60° ，得 BE 。

由作圖所得， $BD = BE = 5\sqrt{2}$ 及 $\angle DBE = 60^\circ$
 $\angle BDE = \angle BED$ (等腰三角形底角)

$$= \frac{180^\circ - 60^\circ}{2} = 60^\circ \quad (\text{三角形內角和})$$

$\triangle BDE$ 是一個等邊三角形。

$DE = BD = 5\sqrt{2}$ (等邊三角形性質)

$AB = AC$ (等邊三角形性質)

$\angle ABC = 60^\circ$ (等邊三角形性質)

$$\begin{aligned} \angle ABE &= \angle DBE - \angle ABD = 60^\circ - \angle ABD \\ &= \angle CBD \end{aligned}$$

$\triangle ABE \cong \triangle CBD$ (S.A.S.)

$AE = CD = 10$ (全等三角形對應邊)

$$DE^2 + DA^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 = 100 = AE^2$$

$\angle ADE = 90^\circ$ (畢氏定理逆定理)

$$\angle ADB = \angle ADE + \angle BDE = 90^\circ + 60^\circ = 150^\circ$$

設 $AB = x$ 。於 $\triangle ABD$ 中應用餘弦公式：

$$x^2 = AD^2 + BD^2 - 2AD \cdot BD \cos \angle ADB$$

$$x^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 - 2(5\sqrt{2})^2 \cos 150^\circ$$

$$x^2 = 100 - 100 \left(-\frac{\sqrt{3}}{2} \right) = 100 + 50\sqrt{3}$$

$$S = \triangle ABC \text{ 的面積} = \frac{1}{2} \cdot AB \cdot BC \sin 60^\circ$$

$$= \frac{1}{2} \cdot (100 + 50\sqrt{3}) \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{25}{2} \cdot (2\sqrt{3} + 3) = 25\sqrt{3} + 37.5$$

As shown in the figure, rotate BD about B anti-clockwise through 60° to BE .

By construction, $BD = BE = 5\sqrt{2}$ and $\angle DBE = 60^\circ$
 $\angle BDE = \angle BED$ (base \angle s isos. \triangle)

$$= \frac{180^\circ - 60^\circ}{2} = 60^\circ \quad (\angle \text{ sum of } \triangle)$$

$\triangle BDE$ is an equilateral triangle.

$DE = BD = 5\sqrt{2}$ (prop. of equil. \triangle)

$AB = AC$ (prop. of equil. \triangle)

$\angle ABC = 60^\circ$ (prop. of equil. \triangle)

$$\begin{aligned} \angle ABE &= \angle DBE - \angle ABD = 60^\circ - \angle ABD \\ &= \angle CBD \end{aligned}$$

$\triangle ABE \cong \triangle CBD$ (S.A.S.)

$AE = CD = 10$ (corr. sides, $\cong \triangle$ s)

$$DE^2 + DA^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 = 100 = AE^2$$

$\angle ADE = 90^\circ$ (converse, Pyth. thm.)

$$\angle ADB = \angle ADE + \angle BDE = 90^\circ + 60^\circ = 150^\circ$$

Let $AB = x$. Apply cosine formula on $\triangle ABD$:

$$x^2 = AD^2 + BD^2 - 2AD \cdot BD \cos \angle ADB$$

$$x^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 - 2(5\sqrt{2})^2 \cos 150^\circ$$

$$x^2 = 100 - 100 \left(-\frac{\sqrt{3}}{2} \right) = 100 + 50\sqrt{3}$$

$$S = \text{area of } \triangle ABC = \frac{1}{2} \cdot AB \cdot BC \sin 60^\circ$$

$$= \frac{1}{2} \cdot (100 + 50\sqrt{3}) \cdot \frac{\sqrt{3}}{2}$$

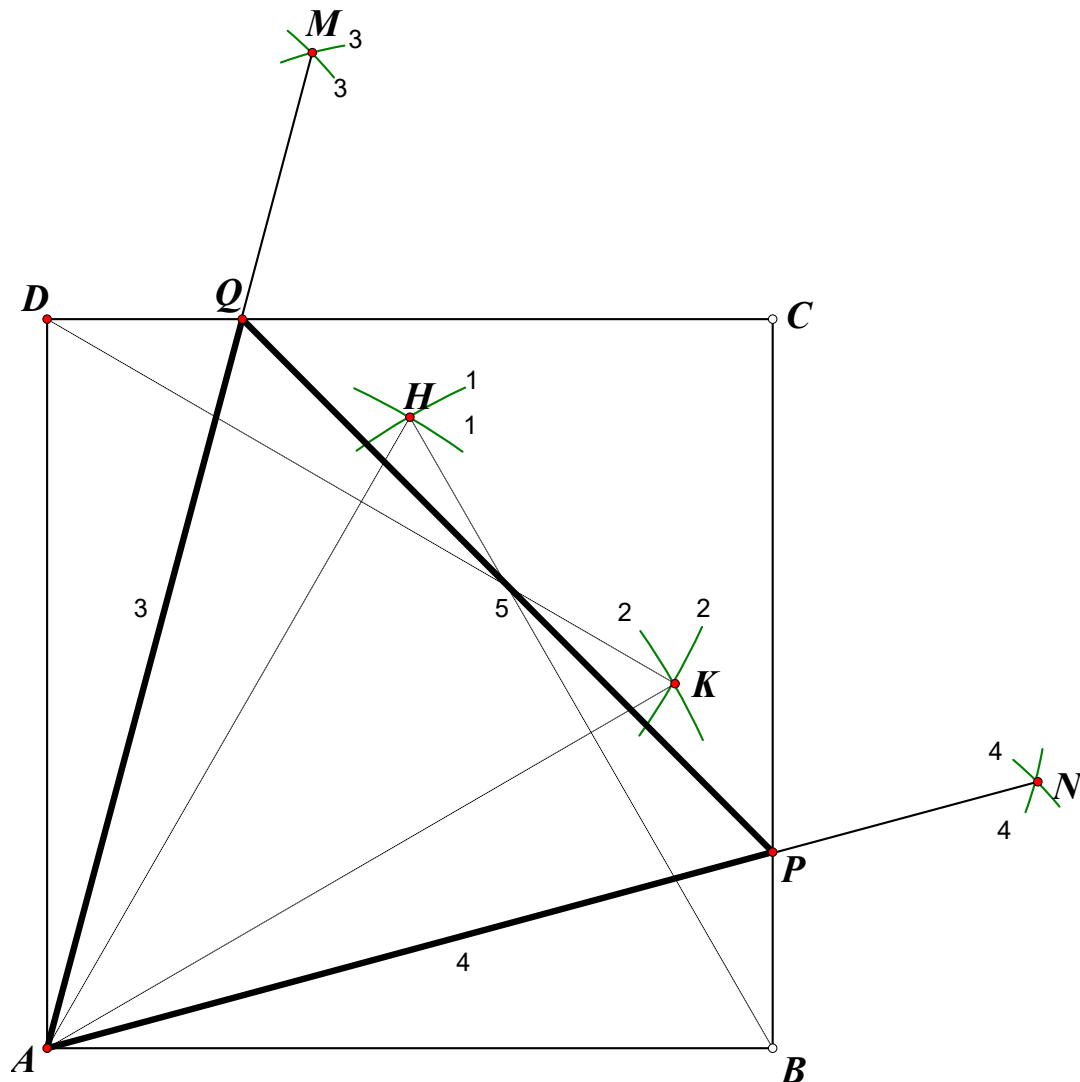
$$= \frac{25}{2} \cdot (2\sqrt{3} + 3) = 25\sqrt{3} + 37.5$$

Geometrical Construction

1. 圖一所示為三個半徑相等且兩兩相切的圓。試作一圓使得它與圖中每一圓相切於一點。
Figure 1 shows three circles with equal radius which are pairwise tangents to each other. Construct a circle which will touch each circle in the figure at a point.

<p>作圖步驟：</p> <ol style="list-style-type: none"> (1) 連接 AB、AC 及 BC。 (2) 作 AC 的垂直平分線。D 為 AC 的中點。此中垂線交以 B 為圓心的圓形於 G。$(OG < AD)$ (3) 作 AB 的垂直平分線。E 為 AB 的中點。兩中垂線相交於 O。 (4) 作圓 $\odot(O, OG)$。 此圓滿足所求。 	<p>Steps:</p> <ol style="list-style-type: none"> (1) Join AB, AC and BC. (2) Draw the perpendicular bisectors of AC. D is the mid-point of AC. It intersects the circle with centre B at G. $(OG < AD)$ (3) Draw the perpendicular bisectors of AB. E is the mid-point of AB. The 2 \perp bisectors intersect at O. (4) Draw a circle $\odot(O, OG)$. This is the required circle.
<p>方法二：</p> <p>於步驟(2)中，中垂線交以 B 為圓心的圓形於 H。 $(OH > AD)$</p> <ol style="list-style-type: none"> (4) 作圓 $\odot(O, OH)$。 此圓亦滿足所求。 	<p>Method 2</p> <p>In step (2), the perpendicular bisector intersects the circle with centre B at H. $(OH > AD)$</p> <ol style="list-style-type: none"> (4) Draw a circle $\odot(O, OH)$. This is another solution.

2. 圖二所示為一個邊長為 1 單位的正方形 $ABCD$ 。試作一個三角形 APQ ，其中 P 、 Q 分別位於 BC 、 CD 上且 $\angle PAB = \angle QAD = 15^\circ$ 。寫出 APQ 是哪一類三角形。
Figure 2 shows a square $ABCD$ with side 1 unit. Construct a triangle APQ , in which P , Q lie on the line segments BC and CD respectively, and $\angle PAB = \angle QAD = 15^\circ$. Write down the type of triangle that APQ is.



作圖步驟：

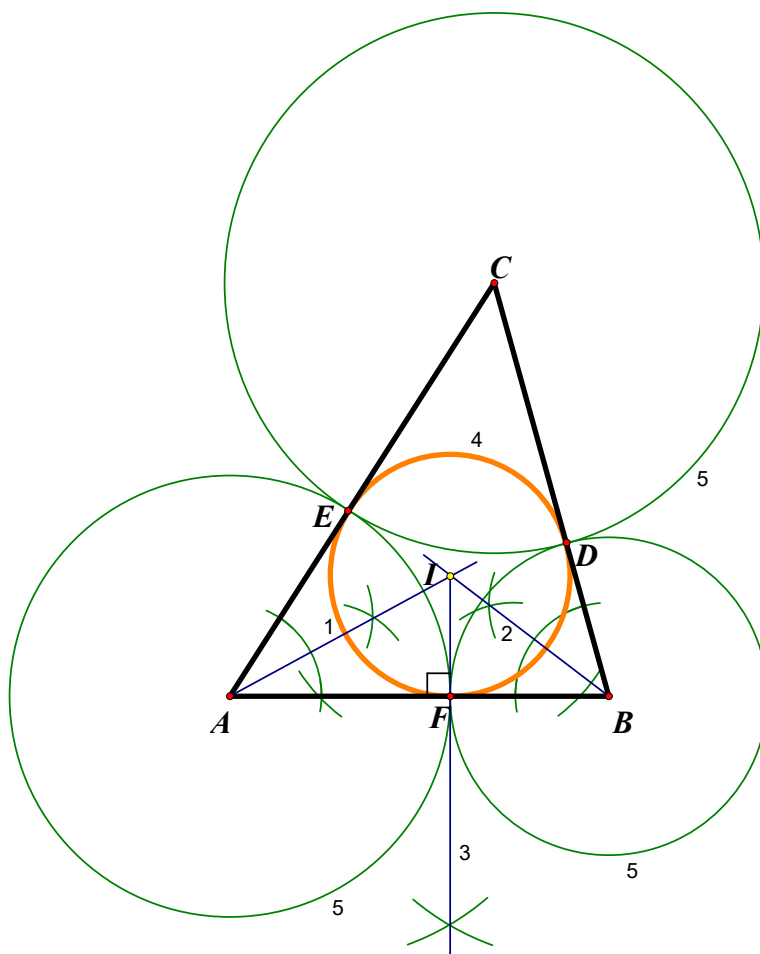
- (1) 作等邊三角形 AHB 。
 $\angle BAH = 60^\circ$, $\angle DAH = 30^\circ$ 。
- (2) 作等邊三角形 AKD 。
 $\angle DAK = 60^\circ$, $\angle BAK = 30^\circ$ 。
- (3) 作 $\angle DAH$ 的角平分綫 AM ，交 CD 於 Q 。
 $\angle DAQ = 15^\circ$ 。
- (4) 作 $\angle BAK$ 的角平分綫 AN ，交 CB 於 P 。
 $\angle BAP = 15^\circ$ 。
- (5) 連接 PQ 。
 $\triangle APQ$ 是一個等邊三角形。

Steps:

- (1) Construct an equilateral triangle AHB .
 $\angle BAH = 60^\circ$, $\angle DAH = 30^\circ$.
- (2) Construct an equilateral triangle AKD .
 $\angle DAK = 60^\circ$, $\angle BAK = 30^\circ$.
- (3) Construct the angle bisector AM of $\angle DAH$, cutting CD at Q . $\angle DAQ = 15^\circ$.
- (4) Construct the angle bisector AN of $\angle BAK$, cutting CB at P . $\angle BAP = 15^\circ$.
- (5) Join PQ .
 $\triangle APQ$ is an equilateral triangle.

3. 圖三所示為一個三角形 ABC 。試以 A 、 B 及 C 為圓心分別構作三個圓，使得它們兩兩相切。

Figure 3 shows a triangle ABC . Use A , B and C as centres to construct three circles respectively that are pairwise tangent to each other. **Reference: 2009 HSC1, 2012HC2, 2014 HC1**

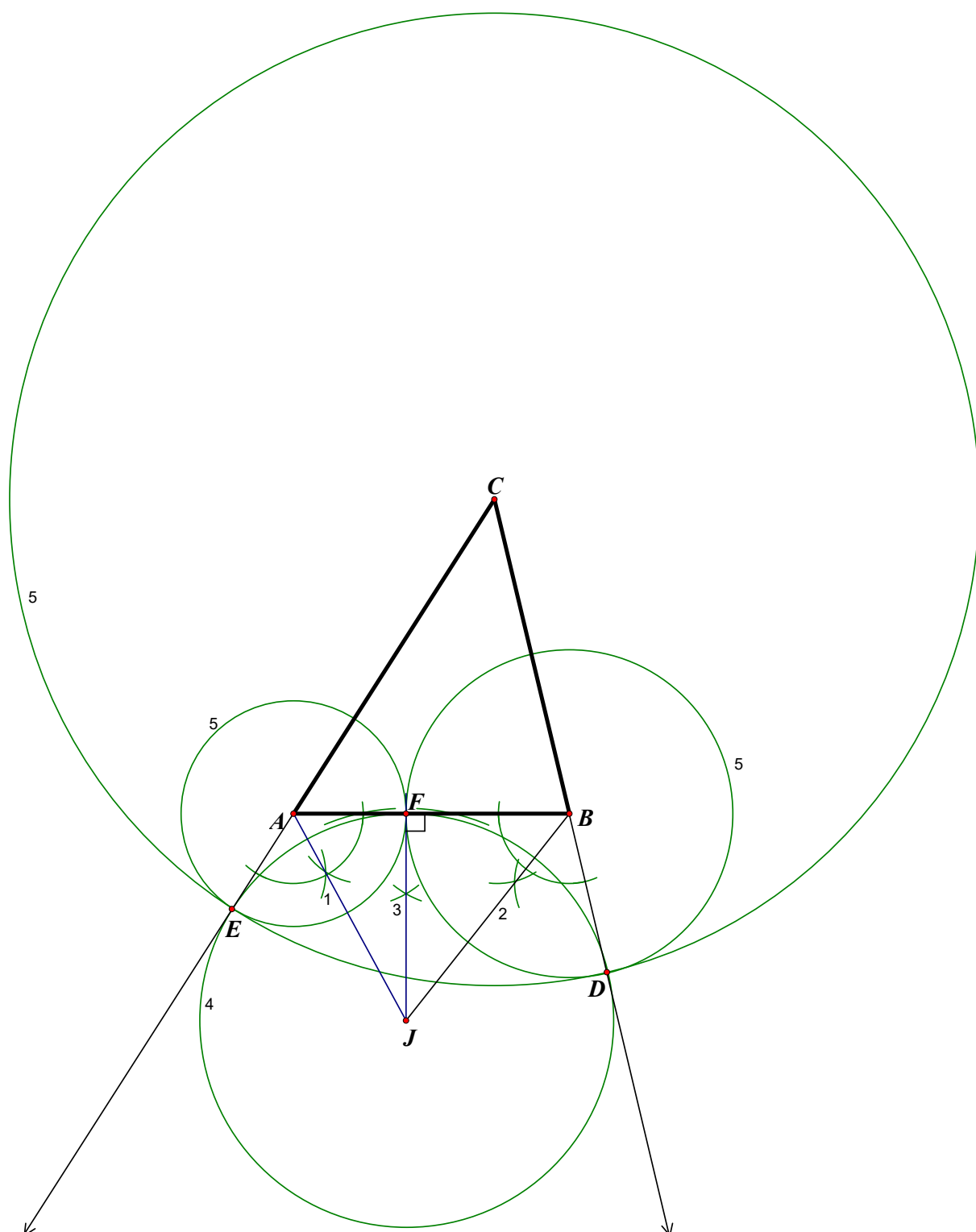


作圖步驟：

- (1) 作 $\angle A$ 的角平分綫。
 - (2) 作 $\angle B$ 的角平分綫。
兩條角平分綫相交於內切圓心 I 。
 - (3) 作綫段 $IF \perp AB$ 。
 - (4) 作內切圓 $\odot(I, IF)$ ，分別切 BC 和 AC 於 D 和 E 。
- 由切綫性質， $AE = AF$ 、 $BD = BF$ 、 $CD = CE$ 。
- (5) 作三圓 $\odot(A, AE)$ 、 $\odot(B, BD)$ 、 $\odot(C, CE)$ 。

Steps:

- (1) Construct the angle bisector of $\angle A$.
- (2) Construct the angle bisector of $\angle B$.
The two \angle bisectors intersect at the incentre I .
- (3) Construct a line $IF \perp AB$.
- (4) Construct the incircle $\odot(I, IF)$, touching BC and AC at D and E respectively.
By tangent property, $AE = AF$, $BD = BF$, $CD = CE$.
- (5) Draw 3 circles $\odot(A, AE)$, $\odot(B, BD)$, $\odot(C, CE)$.



方法二作圖步驟：

- (1) 作 $\angle A$ 的外角平分綫。
- (2) 作 $\angle B$ 的外角平分綫。
兩條角平分綫相交於旁切圓心 J 。
- (3) 作綫段 $JF \perp AB$ 。
- (4) 作旁切圓 $\odot(J, JF)$ ，分別切 CB 和 CA 的
延綫於 D 和 E 。
由切綫性質， $AE = AF$ 、 $BD = BF$ 、 $CD = CE$ 。
- (5) 作三圓 $\odot(A, AE)$ 、 $\odot(B, BD)$ 、 $\odot(C, CE)$ 。

Method 2 Steps:

- (1) Construct the exterior angle bisector of $\angle A$.
- (2) Construct the exterior angle bisector of $\angle B$.
The two \angle bisectors intersect at the excentre J .
- (3) Construct a line $JF \perp AB$.
- (4) Construct the excircle $\odot(J, JF)$, touching
 CB produced and CA produced at D and E
respectively.
By tangent property, $AE = AF$, $BD = BF$, $CD = CE$.
- (5) Draw 3 circles $\odot(A, AE)$, $\odot(B, BD)$, $\odot(C, CE)$.