96-97 Individual	**1**	30	**2**	18	**3**	1854	**4**	–4	**5**	–5050
	6	4:1	**7**	0	**8**	30	**9**	17	**10**	2

96-97 Group	**1**	10000	**2**	$\frac{1}{50}$	**3**	1 : 6	**4**	25	**5**	4
	6	24	**7**	7	**8**	8	**9**	30	**10**	$\frac{1}{12}$

Individual Events

I1 Let n be a positive integer. If $n^2 = 29p + 1$, where p is a prime number, find the value of n.

Reference: 1993 HI7, 1995 HG4

$29p = n^2 - 1$

$29p = (n + 1)(n - 1)$

$\because$ 29 and p are prime number

$\therefore 29 = n + 1, p = n - 1$ or $29 = n - 1$ or $p = n + 1$

Solving the two systems of equations: $p = 27$ or $p = 31$

$\because$ 27 is not a prime number $\therefore p = 27$ is rejected

$p = 31, n = 30$

I2 If the width of a rectangle is increased by $\frac{1}{3}$ m, its area will be increased by $\frac{5}{3}$ m^2. If its length is decreased by $\frac{1}{2}$ m, its area will be decreased by $\frac{9}{5}$ m^2. Let the area of the rectangle be x m^2, find the value of x.

Let the width of the rectangle be a m, its length be b m.

$$\begin{cases} \left(a+\frac{1}{3}\right)b = x+\frac{5}{3} \\ a\left(b-\frac{1}{2}\right) = x-\frac{9}{5} \end{cases} \Rightarrow \begin{cases} ab+\frac{1}{3}b = x+\frac{5}{3} \cdots\cdots(1) \\ ab-\frac{1}{2}a = x-\frac{9}{5} \cdots\cdots(2) \end{cases}$$

$$\because ab = x \ \therefore \begin{cases} x+\frac{1}{3}b = x+\frac{5}{3} \\ x-\frac{1}{2}a = x-\frac{9}{5} \end{cases} \Rightarrow \begin{cases} \frac{1}{3}b = \frac{5}{3} \\ \frac{1}{2}a = \frac{9}{5} \end{cases}$$

$$\Rightarrow a = \frac{18}{5}, b = 5$$

$x = ab = 18$

I3 If s is the sum of all positive factors of 1234, find the value of s.

Reference 1993 HI8, 1994 FI3.2, 1998 HI10, 1998 FI1.4, 2002 FG4.1, 2005 FI4.4

$1234 = 2\times 617$ and 617 is a prime number

All positive factors are 1, 2, 617, 1234.

Sum of all positive factors $= s = 1 + 2 + 617 + 1234 = 1854$

I4 Let $x = \frac{1}{x}$, find the value of $\frac{x^2+2x-3}{x-1} \div \frac{x+5}{x^2+3x-6}$.

$x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = 1$ or -1

However, $\frac{x^2+2x-3}{x-1}$ is undefined for $x = 1$

$\therefore$ Put $x = -1$ into $\frac{x^2+2x-3}{x-1} \div \frac{x+5}{x^2+3x-6} = \frac{(-1)^2+2(-1)-3}{-1-1} \div \frac{-1+5}{(-1)^2+3(-1)-6} = -4$

I5 Find the value of $1^2 - 2^2 + 3^2 - 4^2 + \dots + 99^2 - 100^2$.

Reference: 2002 FG2.3, 2004 HI1, 2015 FI3.2, 2015 FG4.1

$$\begin{aligned}1^2 - 2^2 + 3^2 - 4^2 + \dots + 99^2 - 100^2 &= (1^2 - 2^2) + (3^2 - 4^2) + \dots + (99^2 - 100^2)\\ &= (1 + 2)(1 - 2) + (3 + 4)(3 - 4) + \dots + (99 + 100)(99 - 100)\\ &= -3 - 7 - \dots - 199 \text{ (sum of 50 terms of A.P., common difference = –4)}\\ &= \frac{-3-199}{2}\times 50\\ &= -5050\end{aligned}$$

I6 If $yz : zx : xy = 1 : 2 : 3$, find $\frac{x}{yz} : \frac{y}{zx}$.

$yz : zx : xy = 1 : 2 : 3 \Rightarrow \frac{xyz}{x} : \frac{xyz}{y} : \frac{xyz}{z} = 1 : 2 : 3$

$\Rightarrow \frac{1}{x} : \frac{1}{y} : \frac{1}{z} = 1 : 2 : 3$

$\Rightarrow x : y : z = 1 : \frac{1}{2} : \frac{1}{3}$

$\Rightarrow x : y : z = 6 : 3 : 2$

Let $x = 6k$, $y = 3k$, $z = 2k$

$\frac{x}{yz} : \frac{y}{zx} = \frac{6k}{3k\cdot 2k} : \frac{3k}{2k\cdot 6k} = 1 : \frac{1}{4} = 4 : 1$

I7 Find the real roots of the equation: $x(x + 1)(x^2 + x + 1) = x$.

$x = 0$ or $(x + 1)(x^2 + x + 1) = 1$

$x = 0$ or $x^3 + 2x^2 + 2x + 1 = 1$

$x = 0$ or $x^3 + 2x^2 + 2x = 0$

$x = 0$ or $x^2 + 2x + 2 = 0$

$x = 0$ or no solution ($\Delta = 2^2 - 4\times 2 = -4 < 0$)

I8 There are 6 students in a class. Everyone sends one Christmas card to each of the rest of the class. Find the total number of cards sent out by the class.

The total number of cards sent out = 5 + 5 + 5 + 5 + 5 + 5 = 30

I9 If $2x^2 - 8x + k \equiv 2(x - 2)^2 + 9$, find the value of k.

Put $x = 0$, $k = 2(2)^2 + 9 = 17$

I10 If the ten-digit number $1357p1357p$ is divisible by 9, find the value of p.

$1 + 3 + 5 + 7 + p + 1 + 3 + 5 + 7 + p = 9m$, where m is an integer.

$32 + 2p = 9m$ …… (*)

$\because 1 + 2\times 4 = 9$

$\therefore 32 + 2\times 128 = 9\times 32$

$p = 128$, $m = 32$ is a solution to (*)

General solution is $p = 128 + 9k$, $m = 32 + 2k$, where k is an integer.

$\because 0 \le p \le 9$

$\therefore 0 \le 128 + 9k \le 9$

$$-\frac{128}{9} \le k \le -\frac{119}{9}$$

$$-14\frac{2}{9} \le k \le -13\frac{2}{9}$$

$k = -14$

$p = 128 - 9\times 14 = 2$

Group Events

G1 If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are arithmetic sequences, where $a_1 = 25$, $b_1 = 75$ and $a_{100} + b_{100} = 100$. Find the sum of the first 100 terms of the sequence $a_1 + b_1, a_2 + b_2, \ldots$.

$a_n = 25 + (n - 1)d_1$, $b_n = 75 + (n - 1)d_2$

$\because a_{100} + b_{100} = 100 \therefore 25 + 99d_1 + 75 + 99d_2 = 100$

$99d_1 + 99d_2 = 0$

$d_1 + d_2 = 0$

$a_n + b_n = 25 + (n - 1)d_1 + 75 + (n - 1)d_2$

$= 100 + (n - 1)(d_1 + d_2)$

$= 100$

Let s_n = sum of the first n terms of $a_1 + b_1, a_2 + b_2, \ldots$

$s_{100} = (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_{100} + b_{100})$

$= 100 + 100 + \ldots + 100 = 10000$

G2 If $f(x) = \dfrac{2x}{x+2}$ and $x_1 = 1$, $x_n = f(x_{n-1})$, find x_{99}. **(Reference: 2018 HI14)**

$x_2 = \dfrac{2}{1+2} = \dfrac{2}{3}$

$x_3 = \dfrac{2 \times \frac{2}{3}}{\frac{2}{3}+2} = \dfrac{1}{2} = \dfrac{2}{4}$

$x_4 = \dfrac{2 \times \frac{1}{2}}{\frac{1}{2}+2} = \dfrac{2}{5}$

Claim: $x_n = \dfrac{2}{n+1}$ for $n = 1, 2, 3, \ldots$

Proof: $n = 1, 2, 3, 4$, proved above.

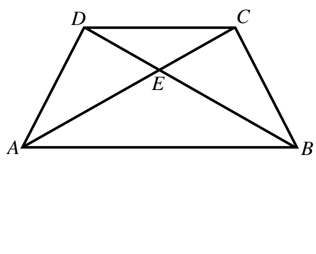
Suppose $x_k = \dfrac{2}{k+1}$ for some positive integer k.

$x_{k+1} = \dfrac{2 \cdot \frac{2}{k+1}}{\frac{2}{k+1}+2} = \dfrac{4}{2k+4} = \dfrac{2}{k+2}$

By M.I., it is true for all positive integer n.

$x_{99} = \dfrac{2}{100} = \dfrac{1}{50}$

G3 $ABCD$ is a trapezium, where $AB // DC$ and area of ΔDCE : area of $\Delta DCB = 1 : 3$, find area of ΔDEC : area of ΔABD.



Reference 1993 HI2, 2000 FI2.2, 2002 FI1.3, 2004 HG7, 2010HG4, 2013 HG2

ΔDCE and ΔDCB have the same base

$\therefore DE : DB = 1 : 3 \Rightarrow DE : EB = 1 : 2$

$\Delta DCE \sim \Delta BAE \Rightarrow$ area of ΔDCE : area of $\Delta BAE = 1 : 4$

ΔADE and ΔABE have the same base

$\therefore$ Area of ΔADE : area of $\Delta ABE = DE : EB = 1 : 2$

Area of ΔABD : area of $\Delta ABE = 3 : 2$

area of ΔDEC : area of $\Delta ABD = \dfrac{1}{4} \div \dfrac{3}{2} = \dfrac{1}{6} = 1 : 6$

G4 Let x be a positive integer. If $\frac{2}{3}\left(\frac{2}{3}\left(\frac{2}{3}(x-1)-1\right)-1\right)$ is divisible by 3, find the least possible value of x.

$\frac{2}{3}\left(\frac{2}{3}\left(\frac{2}{3}(x-1)-1\right)-1\right)=3m$, where m is an integer.

$2\left(\frac{2}{3}\left(\frac{2}{3}x-\frac{2}{3}-1\right)-1\right)=9m$

$\frac{4}{3}\left(\frac{2}{3}x-\frac{5}{3}\right)-2=9m$

$4(2x-5)=9(9m+2)$

$8x-81m=38$ …… (*)

$\because$ $8\times(-10)-81\times(-1)=1$

$\therefore$ $8\times(-380)-81\times(-38)=38$

$x=-380$, $m=-38$ is a solution to (*)

General solution is $x=-380+81k$, $m=-38+8k$, where k is an integer.

For the least positive integral value of x, $-380+81k>0$

$k>\frac{380}{81}=4\frac{56}{81}$

The least integral value of $k=5$

$x=-380+81\times 5=25$

G5 Pipe A alone takes 20 hours to fill a tank and pipe B takes 5 hours to fill the same tank alone. If pipes A and B together take x hours to fill the tank, find the value of x.

$\frac{1}{x}=\frac{1}{20}+\frac{1}{5}$

$\frac{1}{x}=\frac{1}{4}$

$x=4$

G6 Each interior angle of a regular polygon exceeds the exterior angle by 150°. Find the number of sides of the polygon. **Reference: 1989 HI3**

Let x be the size of each interior angle, y be the size of each exterior angle, n be the number of sides.

$x=\frac{180^\circ(n-2)}{n}$, $y=\frac{360^\circ}{n}$

$x=y+150^\circ$

$\frac{180^\circ(n-2)}{n}=\frac{360^\circ}{n}+150^\circ$

$180(n-2)=360+150n$

$18n-36=36+15n$

$n=24$

G7 If $x+\frac{1}{x}=3$, find the value of $x^2+\frac{1}{x^2}$. **(Reference: 1989 HI1)**

$\left(x+\frac{1}{x}\right)^2=9$

$x^2+2+\frac{1}{x^2}=9$

$x^2+\frac{1}{x^2}=7$

G8 Five numbers are in arithmetic progression. If the largest number is 7 times the smallest one and the average of the five numbers is 32, find the smallest number.

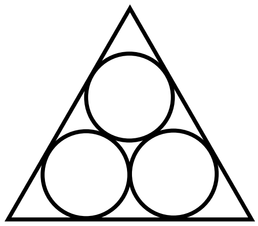
Let the five numbers be $32 - 2d$, $32 - d$, 32, $32 + d$, $32 + 2d$

$32 + 2d = 7(32 - 2d)$

$16d = 192 \Rightarrow d = 12$

The smallest number $= 32 - 2(12) = 8$

G9 In the figure, three identical circles with radius r cm are tightly enclosed in a triangle. If the perimeter of the triangle is $\left(180 + 180\sqrt{3}\right)$cm, find r.



Reference: 1996 FG8.2, 2021 P1Q5

$6r \tan 60° + 6r = 180 + 180\sqrt{3}$

$(\sqrt{3} + 1)r = 30(1 + \sqrt{3})$

$r = 30$

G10 Two fair dice are thrown. Find the probability that the sum is less than 5 and at least one die is a '2'.

Favourable outcome $= \{(2, 1), (2, 2), (1, 2)\}$

Probability $= \frac{3}{36} = \frac{1}{12}$