Q3 P is a point inside the equilateral triangle ABC such that $PA = 5$, $PB = 7$, $PC = 8$, find AB.

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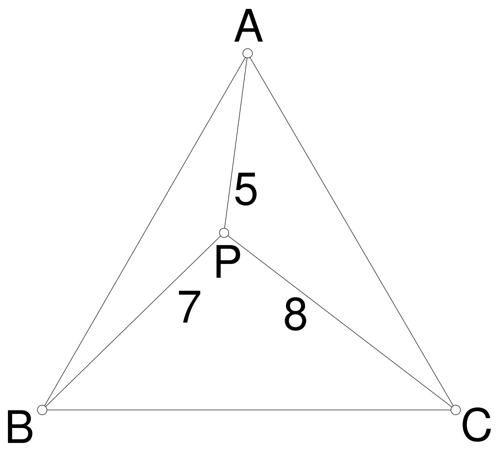

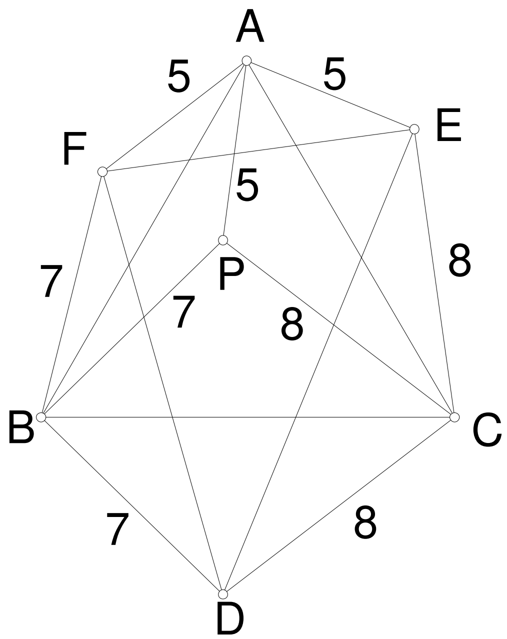

Reflect ΔAPB along the line AB to ΔAFB.
Reflect ΔAPC along the line AC to ΔAEC.
Reflect ΔBPC along the line BC to ΔBDC.
By the property of reflection, $AF = AE = 5$, $BF = BD = 7$, $CD = CE = 8$.
$\angle BAF = \angle BAP$, $\angle ABF = \angle ABP$, $\angle CBD = \angle CBP$, $\angle BCD = \angle BCP$, $\angle ACE = \angle ACP$, $\angle CAE = \angle CAP$
$\angle EAF = 2(\angle BAP + \angle CAP) = 120°$
$\angle DBF = 2(\angle CBP + \angle ABP) = 120°$
$\angle DCE = 2(\angle BCP + \angle ACP) = 120°$
Join DE, EF, DF.
Then $EF = 5\sqrt{3}$, $DF = 7\sqrt{3}$, $DE = 8\sqrt{3}$
By Heron's formula, s = half of perimeter = $(5\sqrt{3} + 7\sqrt{3} + 8\sqrt{3})\div 2 = 10\sqrt{3}$
$s - a = 5\sqrt{3}$, $s - b = 3\sqrt{3}$, $s - c = 2\sqrt{3}$,
Area of $\Delta DEF = \sqrt{10\sqrt{3}\cdot 5\sqrt{3}\cdot 3\sqrt{3}\cdot 2\sqrt{3}} = 30\sqrt{3}$
Area of $AECDBF = S_{\Delta DEF} + S_{\Delta AEF} + S_{\Delta BDF} + S_{\Delta CDE}$

$$= 30\sqrt{3} + \frac{1}{2}\cdot 5\cdot 5\cdot \sin 120° + \frac{1}{2}\cdot 7\cdot 7\cdot \sin 120° + \frac{1}{2}\cdot 8\cdot 8\cdot \sin 120°$$

$$= 30\sqrt{3} + \frac{\sqrt{3}}{4}(25 + \cdot 49 + \cdot 64)$$

$$= \frac{\sqrt{3}}{4}\cdot 258 \text{ ....(1)}$$

On the other hand, Area of $AECDBF = 2(S_{\Delta APB} + S_{\Delta BPC} + S_{\Delta CPA})$

$$= 2\,S_{\Delta ABC}$$

$$= 2\cdot\frac{\sqrt{3}}{4}(AB)^2 \text{ ....(2)}$$

(1) = (2) $\frac{\sqrt{3}}{4}\cdot 258 = 2\cdot\frac{\sqrt{3}}{4}(AB)^2$

$AB = \sqrt{129}$

Method 2

Rotate ΔAPC about A by 60° to ΔADB.
Then $AD = 5$, $BD = 8$ and $\angle DAP = 60°$
ΔADP is an isosceles $\Delta \Rightarrow \angle ADP = \angle APD = 60°$
$\therefore \Delta ADP$ is an equilateral Δ. $DP = 5$
$\cos\angle BDP = \dfrac{5^2 + 8^2 - 7^2}{2\cdot 5\cdot 8} = \dfrac{1}{2} \Rightarrow \angle BDP = 60°$
$\angle ADB = 60° + 60° = 120°$
$AB^2 = 5^2 + 8^2 - 2\cdot 5\cdot 8 \cos 120° = 129$
$AB = \sqrt{129}$