90-91 Individual	**1**	4	**2**	2	**3**	488	**4**	–17	**5**	4
	6	6	**7**	72.8	**8**	243	**9**	8	**10**	50
	11	–4	**12**	6	**13**	17	**14**	13	**15**	73
	16	338350	**17**	192	**18**	7	**19**	10	**20**	45

90-91 Group	**1**	9	**2**	98	**3**	10	**4**	32767	**5**	174
	6	6	**7**	1601	**8**	41	**9**	25	**10**	110

Individual Events

I1 Find the value of $\log_3 14 - \log_3 12 + \log_3 486 - \log_3 7$.

$\log_3 14 - \log_3 12 + \log_3 486 - \log_3 7$

$= \log_3 \dfrac{14\times 486}{12\times 7}$

$= \log_3 81 = 4$

I2 A scientist found that the population of a bacteria culture doubled every hour. At 4:00 pm, he found that the number of bacteria was 3.2×10^8. If the number of bacteria in that culture at noon on the same day was $N\times 10^7$, find N.

$N\times 10^7\times 2^4 = 3.2\times 10^8$

$16N = 32$

$\Rightarrow N = 2$

I3 If $x+\dfrac{1}{x}=8$, find the value of $x^3+\dfrac{1}{x^3}$. (**Reference: 2018 FI1.4**)

$\left(x+\dfrac{1}{x}\right)^2 = 64$

$\Rightarrow x^2+\dfrac{1}{x^2}=62$

$x^3+\dfrac{1}{x^3}=\left(x+\dfrac{1}{x}\right)\left(x^2-1+\dfrac{1}{x^2}\right)$

$= 8\times(62 - 1) = 488$

I4 If the equations $2x + 3y + a = 0$ and $bx - 2y + 1 = 0$ represent the same line, find the value of $6(a + b)$.

$\dfrac{2}{b}=\dfrac{3}{-2}=\dfrac{a}{1}$

$a=-\dfrac{3}{2}, b=-\dfrac{4}{3}$

$6(a + b) = -9 - 8 = -17$

I5 A boy walks from home to school at a speed of 2 metres per second and runs back at x metres per second. His average speed for the whole journey is $2\dfrac{2}{3}$ metres per second. Find x.

Let the distance between his home and the school be d m.

$\dfrac{2d}{\frac{d}{2}+\frac{d}{x}}=2\dfrac{2}{3}$

$\dfrac{4x}{x+2}=\dfrac{8}{3}$

$3x = 2x + 4$

$x = 4$

I6 The straight line $\frac{ax}{3}-\frac{2by}{5}=2a+b$ passes through a fixed point P. Find the x-coordinate of P.

Reference: 1990 HI5, 1996 HI6

$a\left(\frac{x}{3}-2\right)=b\left(\frac{2y}{5}+1\right)$

Put $b = 0$, $a = 1$, $x = 6$

I7 If the diameter of a sphere is increased by 20%, its volume will be increased by x %. Find x.

Let the radius be r.

When the diameter is increased by 20%, the radius is also increased by 20%

Percentage increase in volume

$$=\frac{\frac{4}{3}\pi(1.2r)^3-\frac{4}{3}\pi r^3}{\frac{4}{3}\pi r^3}\times 100\% \quad = 72.8\%$$

$x = 72.8$

I8 If $\log_7[\log_5(\log_3 x)] = 0$, find x.

$\log_5(\log_3 x) = 1$

$\log_3 x = 5$

$x = 3^5 = 243$

I9 If $\frac{7-8x}{(1-x)(2-x)}=\frac{A}{1-x}+\frac{B}{2-x}$ for all real numbers x where $x \neq 1$ and $x \neq 2$, find $A + B$.

$7 - 8x \equiv A(2 - x) + B(1 - x)$

$2A + B = 7$ …… (1)

$A + B = 8$ …… (2)

(1) – (2): $A = -1$

Put $A = -1$ into (2): $B = 9$

$A + B = 8$

I10 The marked price of an article is p% above its cost price. At a sale, the shopkeeper sells the article at 20% off the marked price. If he makes a profit of 20%, find p.

Let the cost be \$$x$.

$(1 + p\%)x(1 - 20\%) = (1 + 20\%)x$

$1 + 0.01p = 1.5$

$p = 50$

I11 If $a < 0$ and $2^{2a+4} - 65\times 2^a + 4 = 0$, find a.

$16(2^a)^2 - 65(2^a) + 4 = 0$

$(16\times 2^a - 1)(2^a - 4) = 0$

$2^a = \frac{1}{16}$ or 4

$\because a < 0 \therefore a = -4$

I12 If one root of the equation $(x^2 - 11x - 10) + k(x + 2) = 0$ is zero, find the other root.

Put $x = 0$, $-10 + 2k = 0$

$\Rightarrow k = 5$

$x^2 - 6x = 0$

The other root is 6.

I13 $[x]$ denotes the greatest integer less than or equal to x. For example, $[6] = 6$, $[8.9] = 8$, etc.
If $\left[\sqrt[4]{1}\right]+\left[\sqrt[4]{2}\right]+\cdots+\left[\sqrt[4]{n}\right]=n+2$, find n. **(Reference 1989 HI6)**

$\left[\sqrt[4]{1}\right]=1$, $\left[\sqrt[4]{2}\right]=1$, … , $\left[\sqrt[4]{15}\right]=1$; $\left[\sqrt[4]{16}\right]=2$, … , $\left[\sqrt[4]{80}\right]=2$; $\left[\sqrt[4]{81}\right]=3$

If $n \le 15$, $\left[\sqrt[4]{1}\right]+\left[\sqrt[4]{2}\right]+\cdots+\left[\sqrt[4]{n}\right]=n$

If $16 \le n \le 80$, $\left[\sqrt[4]{1}\right]+\left[\sqrt[4]{2}\right]+\cdots+\left[\sqrt[4]{n}\right]=15+2(n-15)=2n-15$

$2n - 15 = n + 2$

$\Rightarrow n = 17$

I14 a, b are two different real numbers such that $a^2 = 6a + 8$ and $b^2 = 6b + 8$. Find the value of $\left(\frac{4}{a}\right)^2+\left(\frac{4}{b}\right)^2$.

Reference: 1989 HG1

a and b are the roots of $x^2 = 6x + 8$; i.e. $x^2 - 6x - 8 = 0$

$a + b = 6$; $ab = -8$

$$\left(\frac{4}{a}\right)^2+\left(\frac{4}{b}\right)^2=16\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$
$$=\frac{16\left[(a+b)^2-2ab\right]}{(ab)^2}$$
$$=\frac{16\left[6^2-2(-8)\right]}{(-8)^2}=13$$

I15 $3^{12} - 1$ is divisible by an integer which is greater than 70 and smaller than 80. Find the integer.

$3^{12} - 1 = (3^6 + 1)(3^6 - 1) = (3^2 + 1)(3^4 - 3^2 + 1)(3^2 - 1)(3^4 + 3^2 + 1)$

$= 10\times73\times8\times91$

The integer is 73.

I16 It is known that

$$2^3-1^3=3\times1^2+3\times1+1$$
$$3^3-2^3=3\times2^2+3\times2+1$$
$$4^3-3^3=3\times3^2+3\times3+1$$
$$\vdots \qquad \vdots$$
$$101^3-100^3=3\times100^2+3\times100+1$$

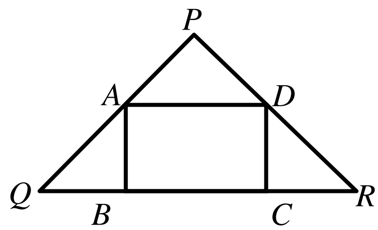
Find the value of $1^2+2^2+3^2+\cdots+100^2$.

Add up these 100 equations: $101^3 - 1 = 3(1^2 + 2^2 + \dots 100^2) + \frac{3}{2}(1+100)\cdot100 + 100$

$1030301 - 1 = 3(1^2 + 2^2 + \dots 100^2) + 15150 + 100$

$1^2 + 2^2 + \dots 100^2 = 338350$

I17 In figure 1, $PQ = PR = 8$ cm and $\angle QPR = 120°$. A, D are the mid-points of PQ, PR respectively. If $ABCD$ is a rectangle of area $\sqrt{x}$ cm^2, find x.



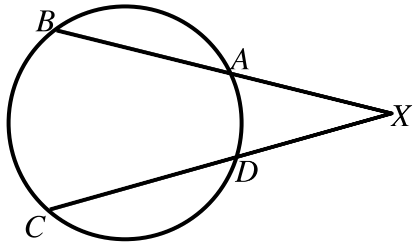
(Figure 1)

Fold ΔPAD along AD, ΔQAB along AB, ΔRCD along DC.

It is easy to show the area of $ABCD = \frac{1}{2}$ area of ΔPQR

$$=\frac{1}{2}\cdot\frac{1}{2}\cdot8\cdot8\cdot\sin 120° = 8\sqrt{3} = \sqrt{192} \Rightarrow x = 192$$

I18 In figure 2, $XA = 10$ cm, $AB = 2$ cm, $XD = 8$ cm and $DC = x$ cm. Find the value of x.



(Figure 2)

$\angle XAD = \angle XCB$ (ext. $\angle$, cyclic quad.)

$\angle AXD = \angle CXB$ (common)

$\angle XDA = \angle XBC$ (ext. $\angle$, cyclic quad.)

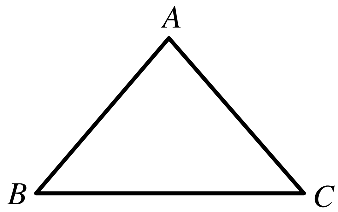
$\Delta XAD \sim \Delta XCB$ (equiangular)

$XA : XD = XC : XB$ (ratio of sides, ~Δ)

$10\times(10 + 2) = 8\times(8 + x)$

$\Rightarrow x = 7$

I19 In figure 3, $AB = AC = 6$ cm and $BC = 9.6$ cm. If the diameter of the circumcircle of ΔABC is x cm, find x.



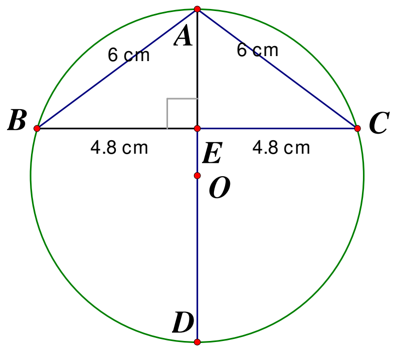
(Figure 3)

$\cos B = \dfrac{9.6 \div 2}{6} = 0.8$

$\Rightarrow \sin B = 0.6$

By sine rule, $\dfrac{b}{\sin B} = 2R$

$\Rightarrow 2R = \dfrac{6}{0.6} = 10$



Method 2 Let the perpendicular bisector of BC intersects the circumscribed circle at A and D. E is the mid point of BC, O is the centre of the circle.

$AE = 3.6$ cm (Pythagoras' Theorem)

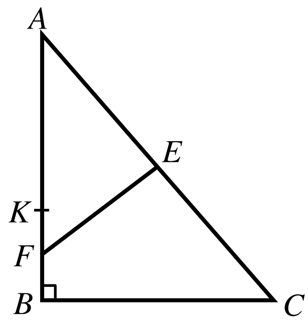
$AE\times ED = BE\times EC$ (intersecting chords theorem)

$ED = 6.4$ cm

$\Rightarrow AD = 10$ cm

$\Rightarrow x = 10$

I20 In figure 4, $\angle ABC = 90°$, $AK = BC$ and E, F are the mid-points of AC, KB respectively. If $\angle AFE = x°$, find x.

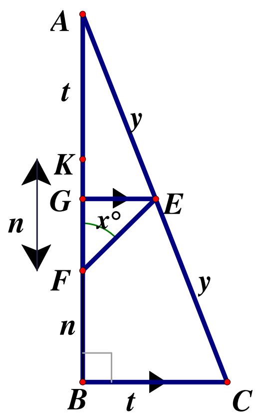


(Figure 4)

Let $AE = y = EC$, $AK = t = BC$, $KF = n = FB$.

Draw EG // CB, cutting AB at G.

$AG = GB$ (intercept theorem)



$GB = \dfrac{1}{2}(t + 2n)$

$GF = \dfrac{1}{2}(t + 2n) - n = \dfrac{1}{2}t$

$GE = \dfrac{1}{2}t$ (mid point theorem)

$\angle EGF = 90°$ (int. $\angle$s, EG // CB)

$\therefore$ ΔEGF is a right-angled isosceles triangle

$x = 45$

Group Events

G1 Find the units digit of 1357^{7890}. **(Reference 1990 HI11)**

$7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$

The pattern of units digit repeats for every multiples of 4.

$1357^{7890} \equiv (7^4)^{1972} \cdot 7^2 \equiv 9 \mod 10$

The units digit is 9.

G2 If $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\cdots+\frac{1}{2450}=\frac{x}{100}$, find x.

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\cdots+\frac{1}{2450}$$

$$=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\frac{1}{5\times 6}+\frac{1}{6\times 7}+\cdots+\frac{1}{49\times 50}$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{49}-\frac{1}{50}\right)$$

$$=1-\frac{1}{50}=\frac{49}{50}=\frac{98}{100}$$

$x = 98$

G3 $\frac{a}{3}$, $\frac{b}{4}$ and $\frac{c}{6}$ are three proper fractions in their simplest form, where a, b and c are positive integers. If c is added to the numerator of each fraction, then the sum of the fractions formed will be equal to 6. Find the value of $a + b + c$.

$\because$ $\frac{a}{3}$, $\frac{b}{4}$ and $\frac{c}{6}$ are three proper fractions in their simplest form

$\therefore$ $a = 1$ or 2, $b = 1$ or 3, $c = 1$ or 5 …… (1)

$$\frac{a+c}{3}+\frac{b+c}{4}+\frac{c+c}{6}=6$$

$4(a + c) + 3(b + c) + 2(2c) = 72$

$4a + 3b + 11c = 72$ …… (2)

$a = 2$, $b = 3$, $c = 5$ is a solution

$a + b + c = 10$

G4 Study the Pascal's triangle shown below:

Find the sum of all the numbers from Row 1 to Row 15.

Row 1						1					
Row 2					1		1				
Row 3				1		2		1			
Row 4			1		3		3		1		
Row 5		1		4		6		4		1	
Row 6	1		5		10		10		5		1

$\vdots$

Sum of the first row = 2^0

Sum of the second row = 2^1

Sum of the third row = 2^2

....

Sum of the fifteen row = 2^{14}

Sum of all numbers from row 1 to row 15 = $2^0 + 2^1 + \ldots + 2^{14} = \frac{2^{15}-1}{2-1} = 32767$

G5 In the multiplication □□□ × □□ = □□ × □□ = 5568, each of the above boxes represents an integer from 1 to 9. If the integers for the nine boxes above are all different, find the number represented by □□□ .

$5568 = 2^6 \times 3 \times 29 = 174 \times 32 = 96 \times 58$

□□□ = 174

G6 Find the remainder when $1997^{1990} - 1991$ is divided by 1996.

$1997^{1990} - 1991 = (1996 + 1)^{1990} - 1991$

$= 1996m + 1 - 1991$ (Binomial theorem, m is an integer)

$= 1996(m - 1) + 6$

The remainder is 6.

G7 Find the least positive integral value of n such that $\sqrt{n} - \sqrt{n-1} < \frac{1}{80}$.

$\sqrt{n} - \sqrt{n-1} < \frac{1}{80}$

$\Rightarrow \left(\sqrt{n} - \sqrt{n-1}\right) \cdot \frac{\left(\sqrt{n} + \sqrt{n-1}\right)}{\left(\sqrt{n} + \sqrt{n-1}\right)} < \frac{1}{80}$.

$\frac{1}{\sqrt{n} + \sqrt{n-1}} < \frac{1}{80}$

$\Rightarrow 80 < \sqrt{n} + \sqrt{n-1} < 2\sqrt{n}$

$\Rightarrow 40 < \sqrt{n}$

$1600 < n$

The least positive integral value of $n = 1601$.

G8 One of the solutions of the equation $32x + 59y = 3259$ in positive integers is given by $(x, y) = (100, 1)$. It is known that there is exactly one more pair of positive integers (a, b) ($a \neq 100$ and $b \neq 1$) such that $32a + 59b = 3259$. Find a. **(Reference: 1989 HG4)**

The line has a slope of $-\frac{32}{59} = \frac{y_2 - y_1}{x_2 - x_1}$

Given that (100, 1) is a solution.

$-\frac{32}{59} = \frac{y_2 - 1}{x_2 - 100}$

Let $y_2 - 1 = -32t$; $x_2 - 100 = 59t$, where t is an integer.

$y_2 = 1 - 32t$, $x_2 = 100 + 59t$

For positive integral solution of (x_2, y_2), $1 - 32t > 0$ and $100 + 59t > 0$

$-\frac{100}{59} < t < \frac{1}{32}$

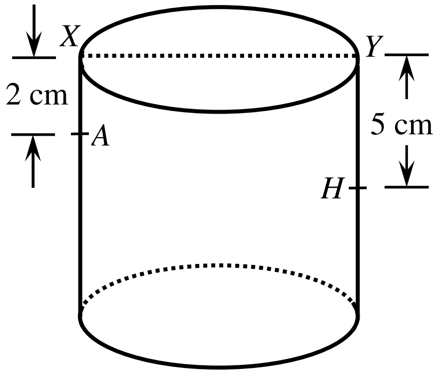
$\because$ t is an integer $\therefore$ $t = 0$ or -1

When $t = -1$, $x_2 = 41$, $y_2 = 33$

$\Rightarrow$ Another positive integral solution is (41, 33)

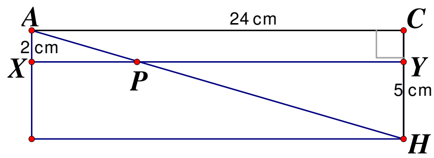
$a = 41$

G9 In figure 1, XY is a diameter of a cylindrical glass, 48 cm in base circumference. On the outside is an ant at A, 2 cm below X and on the inside is a small drop of honey at H, 5 cm below Y. If the length of the shortest path for the ant to reach the drop of honey is x cm, find x. (Neglect the thickness of the glass.)



Reference: 1983 FG8.1, 1993 HI1, 1996 HG9

Cut the cylinder along a plane through XY perpendicular to the base. Unfold the curved surface of the semi-cylinder as a rectangle as shown.



The length of semi-circular arc of the rim XY = 24 cm

When the ant climbs over the rim somewhere at P, and then to H, then AX = 2 cm, YH = 5 cm.

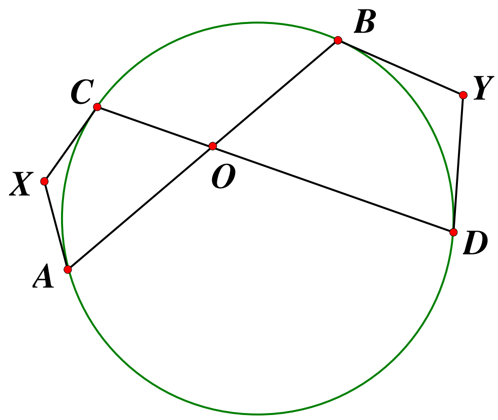
For the shortest distance from A to H, A, P, H must be collinear. Let C be the foot of perpendicular drawn from A onto HY produced. Then $AXYC$ is a rectangle.

AC = 24 cm, CH = (2 + 5)cm = 7 cm

$x^2 = 24^2 + 7^2$ (Pythagoras' theorem)

$x = 25$

G10 In figure 2, two chords AOB, COD cut at O. If the tangents at A and C meet at X, the tangents at B and D meet at Y and $\angle AXC = 130°$, $\angle AOD = 120°$, $\angle BYD = k°$, find k.



$XC = XA$ (tangent from ext. point)

$\therefore$ ΔXAC is an isosceles triangle

$\angle XAC = \angle XCA$ (base $\angle$s isosceles Δ)

$$= \frac{180^\circ - 130^\circ}{2} = 25^\circ \text{ } (\angle \text{ sum of } \Delta XAC)$$

$\angle ADC = \angle XAC = 25°$ ($\angle$ in alt. segment)

$\angle DAO = 180° - 120° - 25° = 35°$ ($\angle$ sum of ΔAOD)

$\angle BDY = \angle DBY = 35°$ ($\angle$ in alt. segment)

$\angle BYD = 180° - 35° - 35° = 110°$ ($\angle$ sum of ΔBDY)

$k = 110$