Example on Mathematical Induction - inequality

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Prove that $1 > \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} > \frac{1}{2}$ for n > 1

Proof: We shall prove $1 > \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1}$ and

$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} > \frac{1}{2}$ separately.

$n = 2$, LHS = 1 , RHS = $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

Suppose $1 > \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1}$

then $1 + \frac{1}{2k+2} + \frac{1}{2k+3} > \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} + \frac{1}{2k+3}$

$$1 + \frac{1}{2k+2} + \frac{1}{2k+3} - \frac{1}{k+1} > \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} + \frac{1}{2k+3}$$

$$1 - \frac{1}{(2k+2)(2k+3)} > \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} + \frac{1}{2k+3}$$

$$1 > \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} + \frac{1}{2k+3}$$

To prove $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} > \frac{1}{2}$

$n = 2$ LHS = $\frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{1}{2}$ = RHS

Suppose $\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} > \frac{1}{2}$

$$\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{1}{2} - \frac{1}{k+1} + \frac{1}{2k+1} + \frac{1}{2k+2}$$

$$\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{1}{2} + \frac{1}{(2k+1)(2k+2)}$$

$$\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} > \frac{1}{2}$$

By the principle of mathematical Induction, if P(k) is true, then P(k+1) is also true.

Hence we have $1 > \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} > \frac{1}{2}$ for n > 1. Q.E.D.

Method 2

$$1 = \underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ terms}} > \underbrace{\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}}_{n \text{ terms}} > \underbrace{\frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} = \frac{n}{2n} = \frac{1}{2}$$