Factor

Created by Mr. Francis Hung on 20151229 Last updated: 2019-02-28

Let a, b, m, n be positive integers such that $am = bn$.

Theorem 1 If m and n are relatively prime, then n divides a and m divides b.

Proof: By Euclidean algorithm, there exist integers r and s such that $rm + sn = 1$

$ram + san = a$

$a = rbn + san = (rb + sa)n$

$\therefore$ n divides a

On the other hand, $rm + sn = 1$

$rbm + sbn = b$

$b = rbm + sam = (rb + sa)m$

$\therefore$ m divides b

The result follows.

In addition, we can see that $sa + rb$ is a common factor of a and b $\cdots\cdots$ (1)

Theorem 2 $sa + rb$ is the H.C.F. of a and b .

Proof: Let k be a common factor of a and b.

Then $a = kp$, $b = kq$, where p and q are positive integers.

$sa + rb = kps + kqr = k(ps + qr)$

$\therefore$ k is common factor of $sa + rb$ $\cdots\cdots$ (2)

By (1) and (2), $sa + rb$ is the H.C.F. of a and b.

Example Given $3a = 4b$. 3 and 4 are relatively prime.

Then 4 divides a and 3 divides b.

Let $r = -1$, $s = 1$, $m = 3$, $n = 4$. $rm + sn = -1\times3 + 1\times4 = 1$

$sa + rb = a - b$

e.g. Let $a = 28$, $b = 21$, then $3a = 4b$, $a - b = 28 - 21 = 7$, which is the H.C.F. of a and b.

Example Given $15a = 23b$. 15 and 23 are relatively prime.

Then 23 divides a and 15 divides b.

$23 = 15 + 8$ $\cdots\cdots$ (1)

$15 = 8 + 7$ $\cdots\cdots$ (2)

$8 = 7 + 1$ $\cdots\cdots$ (3)

$1 = 8 - 7$ $\cdots\cdots$ (3)’

$7 = 15 - 8$ $\cdots\cdots$ (2)’

$8 = 23 - 15$ $\cdots\cdots$ (1)’

Sub. (1)’ into (2)’: $7 = 15 - (23 - 15)$

$7 = 2\times15 - 23$ $\cdots\cdots$ (4)

Sub. (1)’ and (4) into (3)’: $1 = 23 - 15 - (2\times15 - 23)$

$23\times2 - 15\times3 = 1$

Let $r = -3$, $s = 2$, $m = 15$, $n = 23$. $rm + sn = -3\times15 + 2\times23 = 1$

$sa + rb = 2a - 3b$

e.g. $a = 46$, $b = 30$, then $15a = 23b$, $2a - 3b = 2\times46 - 3\times30 = 2$, which is the H.C.F. of a and b.

H.C.F. and L.C.M.

Let a and b be two positive integers. If H.C.F. of a and b is c, then $a = cm$, $b = cn$, where m and n are positive integers.

Theorem 3 m and n are relatively prime integers.

Proof: Let k be a common factor of m and n .

$m = kp$, $n = kq$

$a = cm = ckp$, $b = cn = ckq$

$\therefore$ ck is another common factor of a and b .

H.C.F. $= c \ge ck \Rightarrow k = 1$

$\therefore$ m and n are relatively prime integers.

Example $a = 16$, $b = 24$, H.C.F. $= c = 8$, $16 = 8\times2$, $24 = 8\times3$, $m = 2$, $n = 3$

2 and 3 are relatively prime integers.

If L.C.M. of a and b is d , then $d = fa = gb$, where f and g are positive integers.

Theorem 4 f and g are relatively prime integers.

Proof: Let t = H.C.F. of f and g. Then $f = tr$, $g = ts$, where r and s are positive relatively prime

$d = fa = gb = tra = tsb$

$\Rightarrow ra = sb$ is another common multiple of a and b

$d = tra = tsb =$ L.C.M. $\le ra = sb \Rightarrow t = 1$

$\therefore$ f and g are relatively prime integers.

Example $a = 56$, $b = 40$, L.C.M. $= d = 280$, $280 = 56\times5 = 40\times7$, $r = 5$, $s = 7$

5 and 7 are relatively prime integers.

Theorem 5 Let a and b be two positive integers. If H.C.F. = c, L.C.M. = d , then $a\times b = c\times d$.

Proof: Let $a = \left(r_1^{m_1} r_2^{m_2} \cdots r_k^{m_k}\right)\left(s_1^{n_1} s_2^{n_2} \cdots s_i^{n_i}\right)$, $b = \left(r_1^{p_1} r_2^{p_2} \cdots r_k^{p_k}\right)\left(t_1^{q_1} t_2^{q_2} \cdots t_j^{q_j}\right)$ be the prime factorisation

$r_1, r_2, \cdots, r_k, s_1, s_2, \cdots, s_i, t_1, t_2, \cdots, t_j$ are distinct primes.

$m_1, m_2, \cdots, m_k, n_1, n_2, \cdots, n_i, p_1, p_2, \cdots, p_k, q_1, q_2, \cdots, q_j$ are positive indices.

Let $e_1 = \max(m_1, p_1)$, $e_2 = \max(m_2, p_2)$, $\cdots$, $e_k = \max(m_k, p_k)$, and

let $f_1 = \min(m_1, p_1)$, $f_2 = \min(m_2, p_2)$, $\cdots$, $f_k = \min(m_k, p_k)$.

By definition, $c = \left(r_1^{f_1} r_2^{f_2} \cdots r_k^{f_k}\right)$, $d = \left(r_1^{e_1} r_2^{e_2} \cdots r_k^{e_k}\right)\left(s_1^{n_1} s_2^{n_2} \cdots s_i^{n_i}\right)\left(t_1^{q_1} t_2^{q_2} \cdots t_j^{q_j}\right)$

$$\begin{aligned}
cd &= \left(r_1^{f_1} r_2^{f_2} \cdots r_k^{f_k}\right)\left(r_1^{e_1} r_2^{e_2} \cdots r_k^{e_k}\right)\left(s_1^{n_1} s_2^{n_2} \cdots s_i^{n_i}\right)\left(t_1^{q_1} t_2^{q_2} \cdots t_j^{q_j}\right) \\
&= \left(r_1^{e_1+f_1} r_2^{e_2+f_2} \cdots r_k^{e_k+f_k}\right)\left(s_1^{n_1} s_2^{n_2} \cdots s_i^{n_i}\right)\left(t_1^{q_1} t_2^{q_2} \cdots t_j^{q_j}\right) \\
&= \left(r_1^{m_1+p_1} r_2^{m_2+p_2} \cdots r_k^{m_k+p_k}\right)\left(s_1^{n_1} s_2^{n_2} \cdots s_i^{n_i}\right)\left(t_1^{q_1} t_2^{q_2} \cdots t_j^{q_j}\right) \\
&= \left(r_1^{m_1} r_2^{m_2} \cdots r_k^{m_k}\right)\left(s_1^{n_1} s_2^{n_2} \cdots s_i^{n_i}\right)\left(r_1^{p_1} r_2^{p_2} \cdots r_k^{p_k}\right)\left(t_1^{q_1} t_2^{q_2} \cdots t_j^{q_j}\right) \\
&= ab
\end{aligned}$$

Example $a = 56$, $b = 40$, H.C.F. $= c = 8$, L.C.M. $= d = 280$

$a\times b = 56\times40 = 2240 = 8\times280 = c\times d$