# Hong Kong Mathematics Olympiad (1989 – 90)
## Heat Event (Individual)

除非特別聲明，答案須用數字表達，並化至最簡。　　　　時限：40 分鐘
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
每題正確答案得一分。Each correct answer will be awarded 1 mark. Time allowed: 40 minutes

1. 求下式的值：$\dfrac{1}{3-\sqrt{8}}-\dfrac{1}{\sqrt{8}-\sqrt{7}}+\dfrac{1}{\sqrt{7}-\sqrt{6}}-\dfrac{1}{\sqrt{6}-\sqrt{5}}+\dfrac{1}{\sqrt{5}-2}$。

   Find the value of $\dfrac{1}{3-\sqrt{8}}-\dfrac{1}{\sqrt{8}-\sqrt{7}}+\dfrac{1}{\sqrt{7}-\sqrt{6}}-\dfrac{1}{\sqrt{6}-\sqrt{5}}+\dfrac{1}{\sqrt{5}-2}$.

2. 若 $b<0$ 及 $2^{2b+4}-20\times 2^{b}+4=0$，求 $b$ 的值。

   If $b < 0$ and $2^{2b+4}-20\times 2^{b}+4=0$, find the value of $b$.

3. 若 $f(a)=a-2$，且 $F(a,b)=a+b^2$，求 $F(3, f(4))$。

   If $f(a) = a - 2$ and $F(a, b) = a + b^2$, find $F(3, f(4))$.

4. 對正整數 $a$ 及 $b$，定義 $a\# b = a^b + b^a$，若 $2\# w = 100$，求 $w$ 的值。

   For positive integers $a$ and $b$, define $a \# b = a^b + b^a$. If $2 \# w = 100$, find the value of $w$.

5. $a$ 及 $b$ 為常數。直線 $2ax + 3by = 4a + 12b$ 恆過一定點 $P$（其座標與 $a$ 和 $b$ 無關）。
   求 $P$ 點的座標。

   $a$ and $b$ are constants. The straight line $2ax + 3by = 4a + 12b$ passes through a fixed point $P$ whose coordinates do not depend on $a$ and $b$. Find the coordinates of $P$.

6. 某三角形各內角正弦的比為 $3:4:5$。若 $A$ 為該三角形的最小內角，且 $\cos A = \dfrac{x}{5}$，

   求 $x$ 的值。

   The sines of the angles of a triangle are in the ratio $3:4:5$. If $A$ is the smallest interior angle of the triangle and $\cos A = \dfrac{x}{5}$, find the value of $x$.

7. 若 $x + y = 9$、$y + z = 11$ 及 $z + x = 10$，求 $xyz$ 的值。

   If $x + y = 9$, $y + z = 11$ and $z + x = 10$, find the value of $xyz$.

8. 若 $\alpha$、$\beta$ 是方程 $2x^2 + 4x - 3 = 0$ 的根，且 $\alpha^2$、$\beta^2$ 是方程 $x^2 + px + q = 0$ 的根，
   求 $p$ 的值。

   If $\alpha, \beta$ are the roots of the equation $2x^2 + 4x - 3 = 0$ and $\alpha^2, \beta^2$ are the roots of the equation $x^2 + px + q = 0$, find the value of $p$.

9. 若 $x^{\log_{10} x} = \dfrac{x^3}{100}$，且 $x > 10$，求 $x$ 的值。

   If $x^{\log_{10} x} = \dfrac{x^3}{100}$ and $x > 10$, find the value of $x$.

10. 已知 $a_0 = 1$，$a_1 = 3$ 及 $a_n^{\ 2} - a_{n-1}a_{n+1} = (-1)^n$，其中 $n$ 為正整數。求 $a_4$ 的值。

    Given that $a_0 = 1$, $a_1 = 3$ and $a_n^{\ 2} - a_{n-1}a_{n+1} = (-1)^n$ for positive integers $n$.
    Find the value of $a_4$.

11. 求 $2137^{754}$ 的個位數。

    Find the units digit of $2137^{754}$.

12. 若 $\left(r+\dfrac{1}{r}\right)^2 = 3$，求 $r^3 + \dfrac{1}{r^3}$ 的值。

    If $\left(r+\dfrac{1}{r}\right)^2 = 3$, find the value of $r^3 + \dfrac{1}{r^3}$.

13. 正整數 $N$ 被 10、9、8、7、6、5、4、3 及 2 除所得的餘數依次是 9、8、7、6、5、4、3、2 及 1，求 $N$ 的最小值。

A positive integer $N$, when divided by 10, 9, 8, 7, 6, 5, 4, 3 and 2, leaves remainders 9, 8, 7, 6, 5, 4, 3, 2 and 1 respectively. Find the least value of $N$.

14. 若 $\dfrac{1}{A} = \dfrac{\cos 45° \sin 70° \cos 60° \tan 40°}{\cos 340° \sin 135° \tan 220°}$ ，求 $A$ 的值。

If $\dfrac{1}{A} = \dfrac{\cos 45° \sin 70° \cos 60° \tan 40°}{\cos 340° \sin 135° \tan 220°}$ , find the value of $A$ .
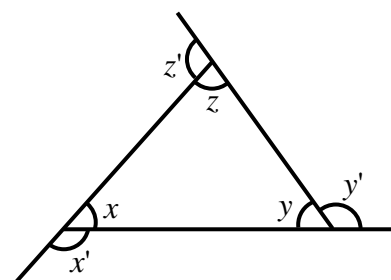
15. 若 10 人需要 5 天製成 20 張檯，請問 15 人需要多少天製成 60 張檯?

If 10 men can make 20 tables in 5 days,

how many days are required to make 60 tables by 15 men ?

16. 圖一的三角形的三個外角的比是 $x' : y' : z' = 4 : 5 : 6$，
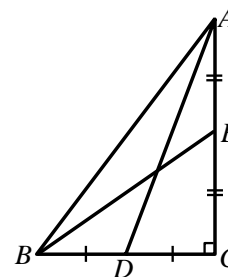而三個內角的比是 $x : y : z = a : b : 3$，求 $b$ 的值。
In figure 1, the exterior angles of the triangle are in the ratio
$x' : y' : z' = 4 : 5 : 6$ and the interior angles are in the ratio
$x : y : z = a : b : 3$. Find the value of $b$ .



(Figure 1)(圖一)

17. 在 $\Delta ABC$ 中，$\angle C = 90°$ 及 $D$、$E$ 分別為 $BC$ 及 $CA$ 的中點。若 $AD = 7$ 及 $BE = 4$，求 $AB$ 的長度。(參考圖二)
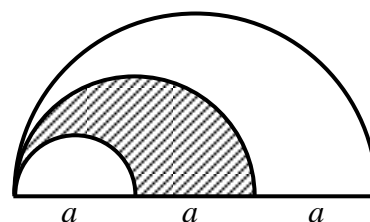In $\Delta ABC$, $\angle C = 90°$ and $D$, $E$ are the mid-points of $BC$ and $CA$ respectively. If $AD = 7$ and $BE = 4$, find the length of $AB$ .
(See figure 2.)



(Figure 2)(圖二)

18. 如圖三，三個半圓的直徑分別為 $a$、$2a$ 及 $3a$。求陰影部分的面積與沒有陰影部分的面積的比值。
Figure 3 shows 3 semi-circles of diameters $a$, $2a$ and $3a$ respectively. Find the ratio of the area of the shaded part to that of the unshaded part.
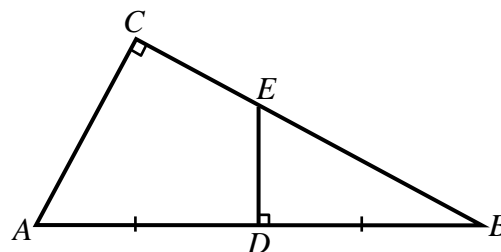


(Figure 3)(圖三)

19. 求 $\dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \dfrac{1}{4 \times 5} + \cdots + \dfrac{1}{19 \times 20}$ 的值

Find the value of $\dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \dfrac{1}{4 \times 5} + \cdots + \dfrac{1}{19 \times 20}$ .

20. 在圖四中，$\angle C = 90°$、$AD = DB$ 及 $DE$ 垂直於 $AB$。若 $AB = 20$ 及 $AC = 12$，
求四邊形 $ADEC$ 的面積。
In figure 4, $\angle C = 90°$, $AD = DB$ and $DE$ is perpendicular to $AB$. If $AB = 20$ and $AC = 12$,
find the area of the quadrilateral $ADEC$ .



(Figure 4)(圖四)

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1. 若 $\dfrac{1}{a}+\dfrac{1}{b}=5$ 及 $\dfrac{1}{a^2}+\dfrac{1}{b^2}=13$，求 $\dfrac{1}{a^5}+\dfrac{1}{b^5}$ 的值。

   If $\dfrac{1}{a}+\dfrac{1}{b}=5$ and $\dfrac{1}{a^2}+\dfrac{1}{b^2}=13$, find the value of $\dfrac{1}{a^5}+\dfrac{1}{b^5}$.

2. 某班有學生 $N$ 人。
   若將學生分為每 4 人一組，有 1 人餘下，
   若將學生分為每 5 人一組，有 3 人餘下，
   若將學生分為每 7 人一組，有 3 人餘下。
   求 $N$ 的最小值。

   There are $N$ pupils in a class.
   When they are divided into groups of 4, 1 pupil is left behind.
   When they are divided into groups of 5, 3 pupils are left behind.
   When they are divided into groups of 7, 3 pupils are left behind.
   Find the least value of $N$.

3. $A$、$B$、$C$ 及 $D$ 的座標依次是$(10, 1)$、$(1, 7)$、$(–2, 1)$及$(1, 3)$。$AB$ 與 $CD$ 相交於 $P$。
   求 $\dfrac{AP}{PB}$ 的值。

   The coordinates of $A$, $B$, $C$ and $D$ are $(10, 1)$, $(1, 7)$, $(–2, 1)$ and $(1, 3)$ respectively. $AB$ and $CD$ meet at $P$. Find the value of $\dfrac{AP}{PB}$.

4. 求 $2^{1989}+1$ 被 3 除所得的餘數。
   Find the remainder when $2^{1989}+1$ is divided by 3.

5. 歐拉在 1700 A.D.和 1800 A.D.之間出生和去世。在 $n^3$ A.D.時，他剛好 $n+9$ 歲，而他在 76 歲時去世。求歐拉去世的年份。

   Euler was born and died between 1700 A.D. and 1800 A.D. He was $n + 9$ years old in $n^3$ A.D. and died at the age of 76. Find the year in which Euler died.

6. 設 $N!$ 為首 $N$ 個自然數的乘積，即 $N! = 1\times2\times3\times\cdots\times N$。

   若 $k$ 是正整數使得 $30! = 2^k\times$一奇數，求 $k$ 的值。

   Let $N!$ denote the product of the first $N$ natural numbers, i.e. $N! = 1\times2\times3\times\cdots\times N$.

   If $k$ is a positive integer such that $30! = 2^k\times$ an odd integer, find the value of $k$.

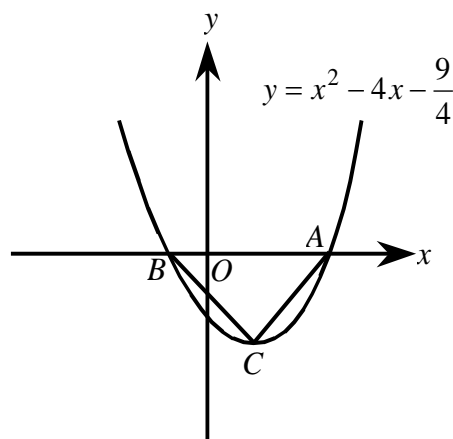7. 拋物綫 $y = x^2 - 4x - \dfrac{9}{4}$ 的圖像交 $x$-軸於 $A$ 及 $B$(圖一)。

若 $C$ 是拋物綫的頂點，求$\Delta ABC$ 的面積。

The graph of the parabola $y = x^2 - 4x - \dfrac{9}{4}$ cuts the $x$-

axis at $A$ and $B$ (figure 1).
If $C$ is the vertex of the parabola,
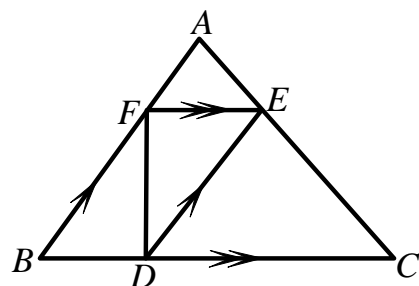find the area of $\Delta ABC$.



(Figure 1)(圖一)

8. 在圖二中，$FE \mathbin{/\mkern-5mu/} BC$ 及 $ED \mathbin{/\mkern-5mu/} AB$。若 $AF : FB = 1 : 4$，
求 $\Delta EDC$ 的面積 $:\Delta DEF$ 的面積。
In figure 2, $FE \mathbin{/\mkern-5mu/} BC$ and $ED \mathbin{/\mkern-5mu/} AB$. If $AF : FB = 1 : 4$,
find the ratio of area of $\Delta EDC$ : area of $\Delta DEF$.



(Figure 2)(圖二)

9. 在所附乘法算式中(圖三)，字母 $O$、$L$、$Y$、$M$、$P$、$I$、$A$ 及 $D$ 代表
由 1 至 9 的不同整數，求 $A$ 所代表的整數。
In the attached multiplication (figure 3), the letters $O$, $L$, $Y$, $M$, $P$, $I$,
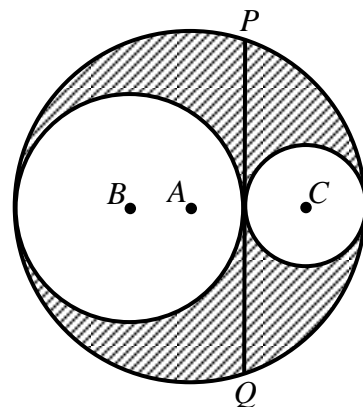$A$ and $D$ represent different integers ranging from 1 to 9. Find the integer
represented by $A$.

$$\begin{array}{r} O\,L\,Y\,M\,P\,I\,A\,D \\ \times \qquad\qquad\quad D \\ \hline O\,O\,O\,O\,O\,O\,O\,O\,O \end{array}$$

(Figure 3)(圖三)

10. 以 $A$、$B$ 及 $C$ 為圓心的三個圓兩兩相切如圖四。若 $A$、$B$ 及
$C$ 共綫，且 $PQ$ 是兩個較小圓的公切綫，其中 $PQ = 4$，試
以 $\pi$ 表陰影面積。
Three circles, with centres $A$, $B$ and $C$ respectively, touch one
another as shown in figure 4. If $A$, $B$ and $C$ are collinear and $PQ$
is a common tangent to the two smaller circles, where $PQ = 4$,
find the area of the shaded part in terms of $\pi$.



(Figure 4)(圖四)

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