Locus of centre of circles touching two other circles

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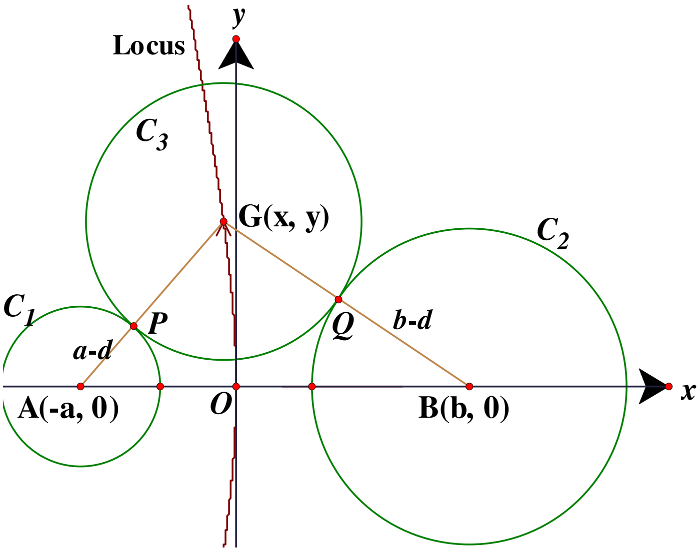
Let C_1: $(x + a)^2 + y^2 = (a - d)^2$, C_2: $(x - b)^2 + y^2 = (b - d)^2$ be two circles, where a, $b > d$ are fixed positive constants. Prove that the locus of <u>centre</u> of circles which touch C_1 and C_2 externally is a branch (which passes through the origin) of the hyperbola: $\dfrac{\left(x-\frac{b-a}{2}\right)^2}{\left(\frac{b-a}{2}\right)^2}-\dfrac{y^2}{ab}=1$.



Suppose 0 is the origin, A is the centre of C_1, B is the centre of C_2, P is the centre of the circle C which touches C_1, C_2 externally.

If r is the radius of the variable circle C,

$PB - PA = (r + b - d) - (r + a - d) = b - a$

Let the coordinates of P be (x, y)

$$\sqrt{(x-b)^2+y^2}-\sqrt{(x+a)^2+y^2} = b-a$$

$$\sqrt{(x-b)^2+y^2}^{\,2} = \left[\sqrt{(x+a)^2+y^2}+b-a\right]^2$$

$$x^2 - 2bx + b^2 + y^2 = x^2 + 2ax + a^2 + y^2 + b^2 - 2ab + a^2 + 2(b-a)\sqrt{(x+a)^2+y^2}$$

$$(a - b)\sqrt{(x+a)^2+y^2} = (a + b)\,x + a^2 - ab$$

$$(a^2 - 2ab + b^2)\,[\,x^2 + 2ax + a^2 + y^2] = (a^2 + 2ab + b^2)\,x^2 + 2(a^2 - ab)\,(a + b)\,x + a^2\,(a^2 - 2ab + b^2)$$

$$4abx^2 + 2ax(a^2 - b^2 - a^2 + 2ab - b^2) - (a - b)^2\,y^2 = 0$$

$$4abx^2 + 4ab(a - b)x - (a - b)^2\,y^2 = 0$$

$$4ab\left[x^2 + (a - b)x + \left(\frac{a-b}{2}\right)^2\right] - (a - b)^2\,y^2 = ab(b - a)^2$$

$$\frac{\left(x-\frac{b-a}{2}\right)^2}{\left(\frac{b-a}{2}\right)^2}-\frac{y^2}{ab}=1$$