Given 4 sides & a diagonal of a qudrilateral, find the other diagonal

Created by Francis Hung
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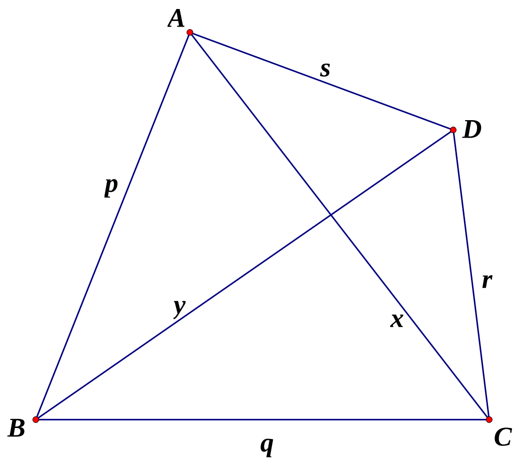
Let $ABCD$ be a quadrilateral. $AB = p$, $BC = q$, $CD = r$, $DA = s$, $AC = x$, $BD = y$.
Express y in terms of p, q, r, s and x .



$$\cos\angle BAC = \frac{x^2+p^2-q^2}{2px}$$

$$\cos\angle DAC = \frac{x^2+s^2-r^2}{2sx}$$

$$\cos\angle BAC \cdot \cos\angle DAC$$
$$= \frac{x^2+p^2-q^2}{2px}\cdot\frac{x^2+s^2-r^2}{2sx}$$
$$= \frac{\left(x^2+p^2-q^2\right)\left(x^2+s^2-r^2\right)}{4psx^2}$$
$$= \frac{x^4+\left(p^2+s^2-q^2-r^2\right)x^2+\left(p^2-q^2\right)\left(s^2-r^2\right)}{4psx^2}$$

$$\frac{1}{2}px\sin\angle BAC = \text{area of } \Delta ABC = \Delta_1$$

$$\frac{1}{2}sx\sin\angle DAC = \text{area of } \Delta ACD = \Delta_2$$

$$\sin\angle BAC \cdot \sin\angle DAC = \frac{4\Delta_1\Delta_2}{psx^2}$$

$$\cos\angle BAD = \cos(\angle BAC + \angle DAC) = \cos\angle BAC\cos\angle DAC - \sin\angle BAC\sin\angle DAC$$
$$= \frac{x^4+\left(p^2+s^2-q^2-r^2\right)x^2+\left(p^2-q^2\right)\left(s^2-r^2\right)}{4psx^2} - \frac{4\Delta_1\Delta_2}{psx^2}$$
$$= \frac{x^4+\left(p^2+s^2-q^2-r^2\right)x^2+\left(p^2-q^2\right)\left(s^2-r^2\right)-16\Delta_1\Delta_2}{4psx^2}$$

$$y^2 = p^2 + s^2 - 2ps\cos\angle BAD$$
$$= p^2+s^2-\frac{x^4+\left(p^2+s^2-q^2-r^2\right)x^2+\left(p^2-q^2\right)\left(s^2-r^2\right)-16\Delta_1\Delta_2}{2x^2}$$
$$= \frac{p^2+q^2+r^2+s^2-x^2}{2}+\frac{16\Delta_1\Delta_2-\left(p^2-q^2\right)\left(s^2-r^2\right)}{2x^2}$$

$$y = \sqrt{\frac{p^2+q^2+r^2+s^2-x^2}{2}+\frac{16\Delta_1\Delta_2-\left(p^2-q^2\right)\left(s^2-r^2\right)}{2x^2}}$$

For example: $p = 25$, $q = 39$, $r = 60$, $s = 52$, $x = 56$

By Heron's formula, $\Delta_1 = 420$, $\Delta_2 = 1344$

$$y = \sqrt{\frac{25^2+39^2+60^2+52^2-56^2}{2}+\frac{16\times420\times1344-\left(25^2-39^2\right)\left(52^2-60^2\right)}{2\times56^2}} = 63$$