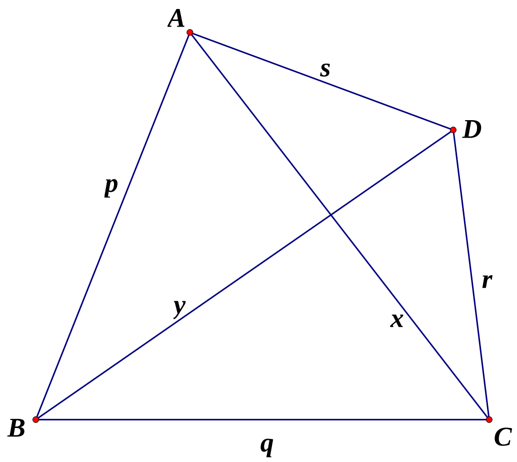
Given 4 sides & a diagonal of a quadrilateral, find the other diagonal

Created by Francis Hung Last updated: 19 February 2024

Let $ABCD$ be a quadrilateral. $AB = p$, $BC = q$, $CD = r$, $DA = s$, $AC = x$, $BD = y$.
Express y in terms of p, q, r, s and x .



$$\cos\angle BAC = \frac{x^2 + p^2 - q^2}{2px}$$

$$\cos\angle DAC = \frac{x^2 + s^2 - r^2}{2sx}$$

$$\cos\angle BAC \cdot \cos\angle DAC$$

$$= \frac{x^2 + p^2 - q^2}{2px} \cdot \frac{x^2 + s^2 - r^2}{2sx}$$

$$= \frac{\left(x^2 + p^2 - q^2\right)\left(x^2 + s^2 - r^2\right)}{4psx^2}$$

$$= \frac{x^4 + \left(p^2 + s^2 - q^2 - r^2\right)x^2 + \left(p^2 - q^2\right)\left(s^2 - r^2\right)}{4psx^2}$$

$$\frac{1}{2}px\sin\angle BAC = \text{area of } \Delta ABC = \Delta_1$$

$$\frac{1}{2}sx\sin\angle DAC = \text{area of } \Delta ACD = \Delta_2$$

$$\sin\angle BAC \cdot \sin\angle DAC = \frac{4\Delta_1\Delta_2}{psx^2}$$

$$\cos\angle BAD = \cos(\angle BAC + \angle DAC) = \cos\angle BAC \cos\angle DAC - \sin\angle BAC \sin\angle DAC$$

$$= \frac{x^4 + \left(p^2 + s^2 - q^2 - r^2\right)x^2 + \left(p^2 - q^2\right)\left(s^2 - r^2\right)}{4psx^2} - \frac{4\Delta_1\Delta_2}{psx^2}$$

$$= \frac{x^4 + \left(p^2 + s^2 - q^2 - r^2\right)x^2 + \left(p^2 - q^2\right)\left(s^2 - r^2\right) - 16\Delta_1\Delta_2}{4psx^2}$$

$$y^2 = p^2 + s^2 - 2ps\cos\angle BAD$$

$$= p^2 + s^2 - \frac{x^4 + \left(p^2 + s^2 - q^2 - r^2\right)x^2 + \left(p^2 - q^2\right)\left(s^2 - r^2\right) - 16\Delta_1\Delta_2}{2x^2}$$

$$= \frac{p^2 + q^2 + r^2 + s^2 - x^2}{2} + \frac{16\Delta_1\Delta_2 - \left(p^2 - q^2\right)\left(s^2 - r^2\right)}{2x^2}$$

$$y = \sqrt{\frac{p^2 + q^2 + r^2 + s^2 - x^2}{2} + \frac{16\Delta_1\Delta_2 - \left(p^2 - q^2\right)\left(s^2 - r^2\right)}{2x^2}}$$

For example: $p = 25$, $q = 39$, $r = 60$, $s = 52$, $x = 56$
By Heron's formula, $\Delta_1 = 420$, $\Delta_2 = 1344$

$$y = \sqrt{\frac{25^2 + 39^2 + 60^2 + 52^2 - 56^2}{2} + \frac{16\times 420\times 1344 - \left(25^2 - 39^2\right)\left(52^2 - 60^2\right)}{2\times 56^2}} = 63$$

Given 3 sides & 2 diagonals of a quadrilateral, find the 4th side

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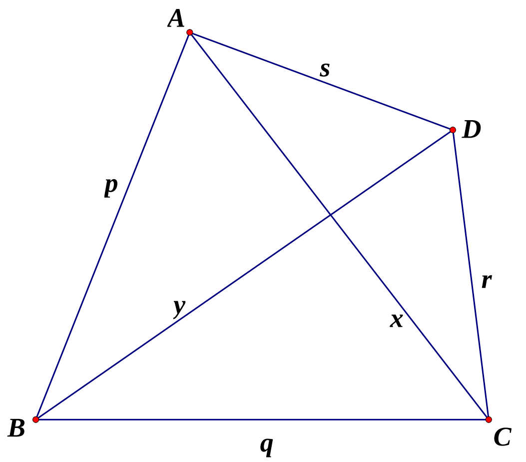
Let $ABCD$ be a quadrilateral. $AB = p$, $BC = q$, $CD = r$, $DA = s$, $AC = x$, $BD = y$.
Express s in terms of p, q, r, x and y.



$$\cos\angle ACB = \frac{q^2 + x^2 - p^2}{2qx}$$

$$\cos\angle BCD = \frac{q^2 + r^2 - y^2}{2qr}$$

$$\cos\angle ACB \cdot \cos\angle BCD$$

$$= \frac{q^2 + x^2 - p^2}{2qx} \cdot \frac{q^2 + r^2 - y^2}{2qr}$$

$$= \frac{\left(q^2 + x^2 - p^2\right)\left(q^2 + r^2 - y^2\right)}{4q^2 rx}$$

$$= \frac{q^4 + \left(x^2 + r^2 - p^2 - y^2\right)q^2 + \left(x^2 - p^2\right)\left(r^2 - y^2\right)}{4q^2 rx}$$

$$\frac{1}{2} qx \sin\angle ACB = \text{area of } \Delta ABC = \Delta_1$$

$$\frac{1}{2} qr \sin\angle BCD = \text{area of } \Delta BCD = \Delta_3$$

$$\sin\angle ACB \cdot \sin\angle BCD = \frac{4\Delta_1\Delta_3}{q^2 rx}$$

$$\cos\angle ACD = \cos(\angle BCD - \angle ACB) = \cos\angle BCD \cos\angle ACD + \sin\angle ACD \sin\angle ACB$$

$$= \frac{q^4 + \left(x^2 + r^2 - p^2 - y^2\right)q^2 + \left(x^2 - p^2\right)\left(r^2 - y^2\right)}{4q^2 rx} + \frac{4\Delta_1\Delta_3}{q^2 rx}$$

$$= \frac{q^4 + \left(x^2 + r^2 - p^2 - y^2\right)q^2 + \left(x^2 - p^2\right)\left(r^2 - y^2\right) + 16\Delta_1\Delta_3}{4q^2 rx}$$

$$s^2 = r^2 + x^2 - 2rx \cos\angle ACD$$

$$= r^2 + x^2 - \frac{q^4 + \left(x^2 + r^2 - p^2 - y^2\right)q^2 + \left(x^2 - p^2\right)\left(r^2 - y^2\right) + 16\Delta_1\Delta_3}{2q^2}$$

$$= \frac{x^2 + y^2 + p^2 + r^2 - q^2}{2} - \frac{\left(x^2 - p^2\right)\left(r^2 - y^2\right) + 16\Delta_1\Delta_3}{2q^2}$$

$$y = \sqrt{\frac{x^2 + y^2 + p^2 + r^2 - q^2}{2} - \frac{\left(x^2 - p^2\right)\left(r^2 - y^2\right) + 16\Delta_1\Delta_3}{2q^2}}$$

For example: $p = 104$, $q = 85$, $r = 195$, $x = 171$, $y = 220$

By Heron's formula, $\Delta_1 = 3420$, $\Delta_3 = 8250$

$$y = \sqrt{\frac{104^2 + 195^2 + 171^2 + 220^2 - 85^2}{2} - \frac{\left(171^2 - 104^2\right)\left(195^2 - 220^2\right) + 16 \times 3420 \times 8250}{2 \times 85^2}} = 204$$