Challenge Problem 3

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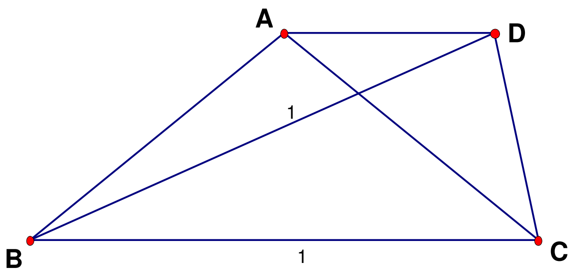
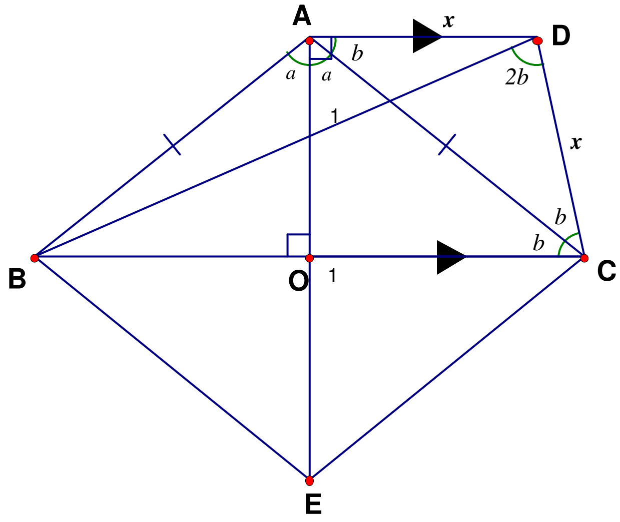
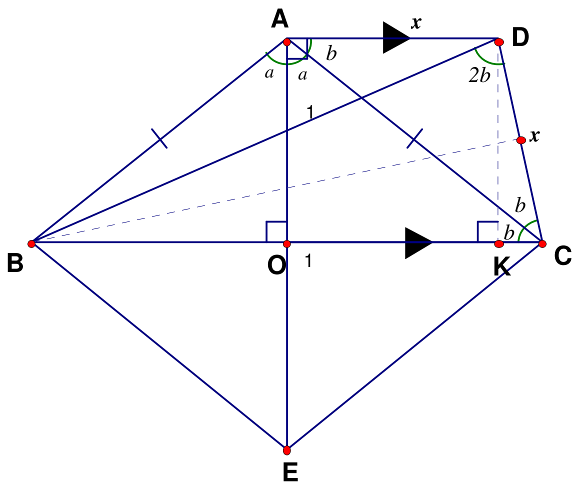
In a trapezium, $AD \parallel BC$, $BC = BD = 1$, $AB = AC$, $CD < 1$, $\angle BAC + \angle BDC = 180°$, find CD.





Reflect ΔABC along BC to ΔEBC. Let $CD = x$.
$ABEC$ is a rhombus since $AB = AC = BE = CE$
Suppose AE intersects BC at O. Then $\angle AOB = 90°$ (property of rhombus)
$\angle DAO = 90°$ (alt. $\angle$s, $AD \parallel BC$)
Let $\angle BAE = a = \angle CAE$ (property of rhombus)
Let $\angle CAD = b = \angle ACB$ (alt. $\angle$s, $AD \parallel BC$)
$a + b = 90°$ (ext. $\angle$ of ΔACO) ……… (1)
$\angle BAC + \angle BDC = 180°$ (given)
$\angle BDC = 180° - 2a$
$\angle BCD = \angle BDC = 180° - 2a$ (base $\angle$s, isos. Δ)
$\angle ACD = \angle BCD - \angle ACB = 180° - 2a - b$
$= 180° - a - b - a$
$= 180° - 90° - a$ (by (1))
$= 90° - a$
$= b$ (by (1))
$= \angle CAD$
$\therefore ACD$ is an isos. Δ (base $\angle$s. eq.)
$AD = CD = x$ (sides opp. eq. $\angle$s)
$\because \Delta BCD$ is an isos. Δ with $BC = BD = 1$
$\therefore x = 2\times 1\times \cos 2b = 2 \cos 2b$ ……… (2)



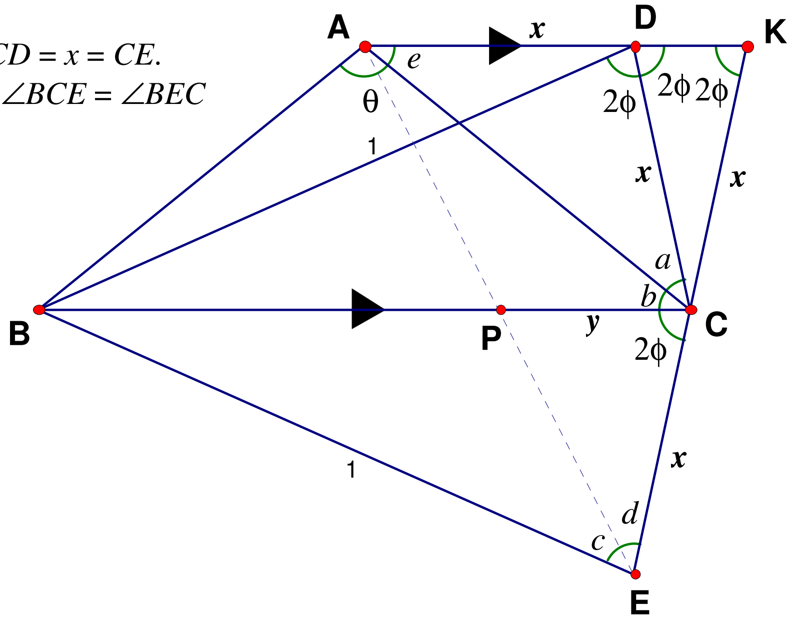
From D, drop a perpendicular line down to K on BC

$$\cos 2b = \frac{KC}{x} = \frac{OC - OK}{x} = \frac{\frac{1}{2} - AD}{x} = \frac{\frac{1}{2} - x}{x} = \frac{1-2x}{2x} \quad \cdots\cdots\cdots (3)$$

Sub. (3) into (2): $x = \dfrac{1-2x}{x}$

$x^2 = 1 - 2x$
$x^2 + 2x - 1 = 0$
$(x + 1)^2 = 2$
$x + 1 = \sqrt{2}$
$x = \sqrt{2} - 1$

Method 2



Reflect ΔBDC along BC to ΔBEC. Let $CD = x = CE$.

Let $\angle BAC = \theta$, $\angle BDC = 2\phi = \angle BCD = \angle BCE = \angle BEC$

$\theta + 2\phi = 180°$ (given)

A, B, E, C are concylic (opp. $\angle$s supp.)

Let $\angle ACD = a$, $\angle ACB = b$, $\angle AEB = c$

$\angle AEC = d$, $\angle CAD = e$

Join AE, suppose it cuts BC at P.

Let $CP = y$

$\because AB = AC$ (given)

$\therefore c = d$ (eq. chords. eq. $\angle$s)

$= \phi$

$b = c$ ($\angle$s in the same seg.)

$= \phi$

$a = 2\phi - \phi = \phi$

$e = b = \phi$ (alt. $\angle$s AD // BC)

$\therefore \Delta ACD$ is isos. (base $\angle$s eq.)

$AD = x$ (sides opp. eq. $\angle$s)

Produce AD and EC to meet at K.

$\angle CDK = a + b = 2\phi$ (alt. $\angle$s AD // BC)

$\angle DKC = 2\phi$ (corr. $\angle$s AK // BC)

$\therefore \Delta CDK$ is isos. (base $\angle$s eq.)

$CK = x$ (sides opp. eq. $\angle$s)

$= CE$

$\because PC$ // AK

$AP = PE$ (intercept theorem)

$AK = 2\,PC = 2y$ (mid point theorem on ΔAEK)

$DK = 2y - x$

It is easy to see that $\Delta AKC \sim \Delta EAK$ (equiangular)

$$\therefore \frac{KC}{AK} = \frac{AK}{EK} \Rightarrow \frac{x}{2y} = \frac{2y}{2x} \Rightarrow x^2 = 2y^2 \cdots\cdots\cdots (1)$$

Further, $\Delta CDK \sim \Delta BCD$ (equiangular)

$$\frac{2y - x}{x} = \frac{x}{1}$$

$$\therefore y = \frac{x^2 + x}{2} \cdots\cdots\cdots (2)$$

Sub. (2) into (1): $x^2 = 2(\frac{x^2 + x}{2})^2$

$$x^2 = \frac{x^2(x+1)^2}{2}$$

$(x + 1)^2 = 2$

$x = \sqrt{2} - 1$