99-00 Individual	**1**	$\frac{170}{891}$	**2**	3	**3**	10	**4**	35	**5**	540
	6	190	**7**	$\frac{1}{3}$	**8**	428571	**9**	24	**10**	0

99-00 Group	**1**	–3	**2**	5	**3**	6	**4**	10	**5**	10
	6	60	**7**	0.93	**8**	421	**9**	12	**10**	0

Individual Events

I1 Let $x = 0.\dot{1}\dot{7} + 0.0\dot{1}\dot{7} + 0.00\dot{1}\dot{7} + \ldots$, find the value of x. (**Reference: 2009 HI1**)

$0.\dot{1}\dot{7} = \frac{17}{99}; 0.0\dot{1}\dot{7} = \frac{17}{990}; 0.00\dot{1}\dot{7} = \frac{17}{9900}$, … It is an infinite geometric series, $a = \frac{17}{99}$, $r = \frac{1}{10}$

$$x = \frac{17}{99} + \frac{17}{990} + \frac{17}{9900} + \cdots$$

$$= \frac{17}{99}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$$

$$= \frac{17}{99} \cdot \frac{10}{9} = \frac{170}{891}$$

I2 Solve the following equation:

$$\frac{1}{x+12} + \frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \frac{1}{(x+3)(x+4)} + \cdots + \frac{1}{(x+10)(x+11)} + \frac{1}{(x+11)(x+12)} = \frac{1}{4}$$

$$\frac{1}{x+12} + \left(\frac{1}{x+1} - \frac{1}{x+2}\right) + \left(\frac{1}{x+2} - \frac{1}{x+3}\right) + \left(\frac{1}{x+3} - \frac{1}{x+4}\right) \cdots + \left(\frac{1}{x+10} - \frac{1}{x+11}\right) + \left(\frac{1}{x+11} - \frac{1}{x+12}\right) = \frac{1}{4}$$

$$\frac{1}{x+1} = \frac{1}{4}$$

$\Rightarrow x = 3$

I3 Using digits 0, 1, 2, and 5, how many 3-digit numbers can be formed, which are divisible by 5? (If no digit may be repeated.)

Possible numbers are: 105, 120, 125, 150, 205, 210, 215, 250, 510, 520.
Altogether 10 numbers.

I4 Figure 1 represents a 4×3 rectangular spiderweb. If a spider walks along the web from A to C and it always walks either due East or due North. Find the total number of possible paths.

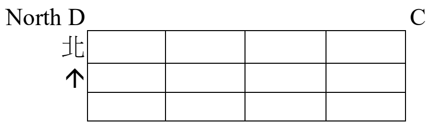
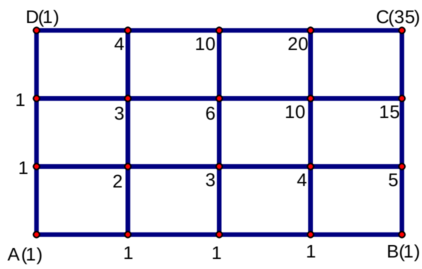


Figure 1 圖一

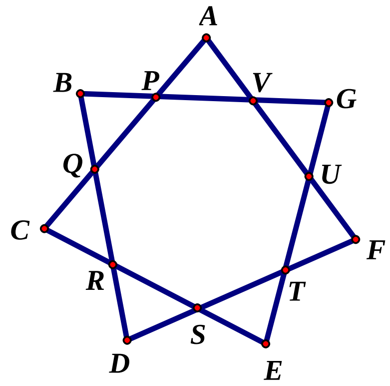
Reference: 1983 FI4.1, 1998 HG6, 2007 HG5

The numbers at each of the vertices of in the following figure show the number of possible ways.



So the total number of ways = 35

I5 In Figure 2, let $\angle A+\angle B+\angle C+\angle D+\angle E+\angle F+\angle G = x°$, find the value of x.



Reference: 1992 HI13, 2012 FG3.2

In the figure, let P, Q, R, S, T, U, V be as shown.

$\angle AVP + \angle BPQ + \angle CQR + \angle DRS + \angle EST + \angle FTU + \angle GUV = 360°$

(sum of ext. $\angle$ of polygon)

$\angle A = 180° - (\angle AVP + \angle BPQ)$ ($\angle$s sum of Δ)

$\angle B = 180° - (\angle BPQ + \angle CQR)$ ($\angle$s sum of Δ)

$\angle C = 180° - (\angle CQR + \angle DRS)$ ($\angle$s sum of Δ)

$\angle D = 180° - (\angle DRS + \angle EST)$ ($\angle$s sum of Δ)

$\angle E = 180° - (\angle EST + \angle FTU)$ ($\angle$s sum of Δ)

$\angle F = 180° - (\angle FTU + \angle GUV)$ ($\angle$s sum of Δ)

$\angle G = 180° - (\angle GUV + \angle AVP)$ ($\angle$s sum of Δ)

$\angle A+\angle B+\angle C+\angle D+\angle E+\angle F+\angle G = 180°\times 7 - 2\times 360°$

$x = 540$

I6 Twenty straight lines were drawn on a white paper. Among them, no two or more straight lines are parallel; also no three or more than three straight lines are concurrent. What is the maximum number of intersections that these 20 lines can form?

2 lines give at most 1 intersection.

3 lines give at most 3 intersections.

4 lines give at most 6 intersections. (6 = 1 + 2 + 3)

....

20 lines give at most $1 + 2 + 3 + \cdots + 19$ intersections $= \frac{1+19}{2}\cdot 19 = 190$ intersections

I7 In a family of 2 children, given that one of them is a girl, what is the probability of having another girl? (Assuming equal probabilities of boys and girls.)

Sample space = {(girl, boy), (girl, girl), (boy, girl)} and each outcome is equal probable.

$\therefore$ P(another child is also a girl) $= \frac{1}{3}$

I8 A particular 6-digit number has a unit-digit "1". Suppose this unit-digit "1" is moved to the place of hundred thousands, while the original ten thousand-digit, thousand-digit, hundred-digit, … are moved one digit place to the right. The value of the new 6-digit number is one-third of the value of the original 6-digit number. Find the original 6-digit number.

(**Reference: 1986 FG8**) Let the original number be: $\overline{abcde1}$, and the new number be: $\overline{1abcde}$.

$3\times\overline{1abcde} = \overline{abcde1}$

$3(100000 + 10000a + 1000b + 100c + 10d + e) = 100000a+10000b + 1000c + 100d + 10e + 1$

Compare the unit digit: $e = 7$ with carry digit 2 to the tens digit

Compare the tens digit: $d = 5$ with carry digit 1 to the hundreds digit

Compare the hundreds digit: $c = 8$ with carry digit 2 to the thousands digit

Compare the thousands digit: $b = 2$ with no carry digit to the ten-thousands digit

Compare the ten-thousands digit: $a = 4$ with carry digit 1 to the hundred-thousands digit

The original number is 428571

I9 Find the value of $\frac{12\sin^2 48° + 12\sin^2 42°}{\sin 330° \tan 135° - \sin^2 48° \sin^2 42° \tan 180°}$.

$$\frac{12\sin^2 48° + 12\sin^2 42°}{\sin 330° \tan 135° - \sin^2 48° \sin^2 42° \tan 180°} = \frac{12\sin^2 48° + 12\cos^2 48°}{\left(-\frac{1}{2}\right)(-1) - \sin^2 48° \sin^2 42° \times 0} = \frac{12}{\frac{1}{2}} = 24$$

I10 Find the shortest distance between the line $3x - y - 4 = 0$ and the point (2, 2).

$$d = \left|\frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right| = \left|\frac{3\times 2 - 2 - 4}{\sqrt{3^2 + (-1)^2}}\right| = 0$$

Method 2 Sub. (2, 2) into $3x - y - 4 = 0$, LHS = $3\times 2 - 2 - 4 = 0$ = RHS

$\therefore$ (2, 2) lies on the line, the shortest distance = 0

Group Events

G1 If a is a root of $x^2 + 2x + 3 = 0$, find the value of $\frac{a^5+3a^4+3a^3-a^2}{a^2+3}$.

Reference: 1993 HI9, 2001 FG2.1, 2007 HG3, 2009 HG2

Divide $(a^5 + 3a^4 + 3a^3 - a^2)$ by $(a^2 + 2a + 3)$, quotient $= a^3 + a^2 - 2a$, remainder $= 6a$

$$\frac{a^5+3a^4+3a^3-a^2}{a^2+3}=\frac{(a^2+2a+3)(a^3+a^2-2a)+6a}{(a^2+2a+3)-2a}$$

$$=\frac{6a}{-2a}=-3$$

G2 There are exactly n roots in the equation $(\cos^2\theta - 1)(2\cos^2\theta - 1) = 0$, where $0° < \theta < 360°$. Find the value of n.

$\cos\theta = 1, -1, \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$.

$\theta = 180°, 45°, 315°, 135°, 225°$

$n = 5$

G3 Find the units digit of 2004^{2006}.

$4^1 = 4, 4^2 = 16, 4^3 = 64, 4^4 = 256, \dots$

So the units digit of 2004^{2006} is 6.

G4 Let $x = |y - m| + |y - 10| + |y - m - 10|$, where $0 < m < 10$ and $m \le y \le 10$. Find the minimum value of x.

$x = y - m + 10 - y + 10 - y + m = 20 - y \ge 20 - 10 = 10$

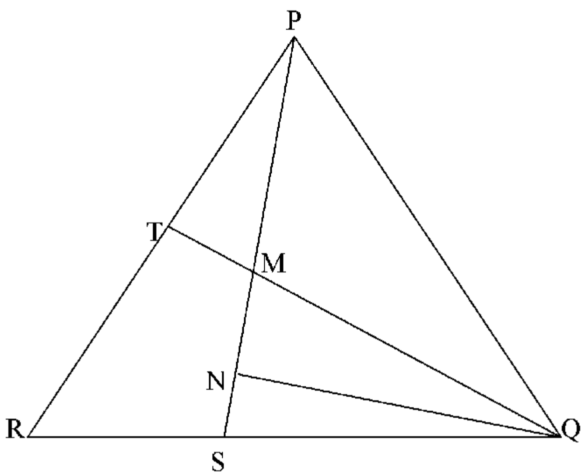
The minimum = 10

G5 There are 5 balls with labels A, B, C, D, E respectively and there are 5 pockets with labels A, B, C, D, E respectively. A ball is put into each pocket. Find the number of ways in which exactly 3 balls have labels that match the labels on the pockets.

First choose any 3 bags out of five bags. Put the balls according to their numbers. The remaining 2 balls must be put in the wrong order.

The number of ways is ${}_5C_3 = 10$.

G6 In Figure 1, ΔPQR is an equilateral triangle, $PT = RS$; PS, QT meet at M; and QN is perpendicular to PS at N. Let $\angle QMN = x°$, find the value of x.



Reference: 2019 HI1

$PT = RS$ (given)

$\angle QPT = 60° = \angle PRS$ ($\angle$ of an equilateral Δ)

$PQ = PR$ (side of an equilateral Δ)

$\Delta PQT \cong \Delta RPS$ (SAS)

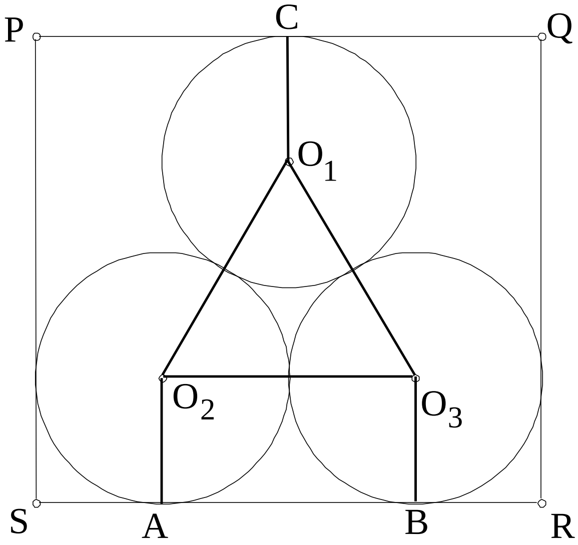
$\therefore \angle PTQ = \angle PSR$ (corr. $\angle$s $\cong \Delta$)

R, S, M, T are concyclic (ext. $\angle$ = int. opp. $\angle$)

$\angle QMN = x° = \angle TRS = 60°$ (ext. $\angle$, cyclic quad.)

$x = 60$

G7 In Figure 2, three equal circles are tangent to each other, and inscribed in rectangle *PQRS*, find the value of $\frac{QR}{SR}$. (Use $\sqrt{3} = 1.7$ and give the answer correct to 2 decimal places)



Let the radii of the circles be r.

Suppose the 3 circles touch the rectangle at A, B and C. Join O_1O_2, O_2O_3, O_1O_3, O_1C, O_2A, O_3B as shown. Then $O_1O_2 = O_2O_3 = O_1O_3 = 2r$

$O_1C = O_2A = O_3B = r$

$O_1O_2O_3$ is an equilateral Δ

$QR = O_1C + O_1O_2 \sin 60° + O_2A$

$= r + 2r\cdot\frac{\sqrt{3}}{2} + r = r(2+\sqrt{3})$

$SR = 4r$, $\frac{QR}{SR} = \frac{r(2+\sqrt{3})}{4r} = \frac{2+1.7}{4} = \frac{37}{40} \approx 0.93$

G8 The sum of two positive integers is 29, find the minimum value of the sum of their squares.

Let the two numbers be a and b.

$a^2 + b^2 = a^2 + (29 - a)^2 = 2a^2 - 58a + 841 = 2(a - 14.5)^2 + 420.5$

$\because$ a and b are integers, the minimum is attained when $a = 15$, $b = 14$

The minimum value of $a^2 + b^2 = 15^2 + 14^2 = 225 + 196 = 421$

G9 Let $x = \sqrt{3+\sqrt{3}}$ and $y = \sqrt{3-\sqrt{3}}$, find the value of $x^2(1 + y^2) + y^2$.

$x^2(1 + y^2) + y^2 = (3+\sqrt{3})(1+3-\sqrt{3}) + 3 - \sqrt{3}$

$= (3+\sqrt{3})(4-\sqrt{3}) + 3 - \sqrt{3}$

$= 12 + 4\sqrt{3} - 3\sqrt{3} - 3 + 3 - \sqrt{3} = 12$

Method 2

$x^2(1 + y^2) + y^2 = (x^2 + 1)(y^2 + 1) - 1$

$= (3+\sqrt{3}+1)(3-\sqrt{3}+1) - 1$

$= 16 - 3 - 1 = 12$

G10 There are nine balls in a pocket, each one having an integer label from 1 to 9. A draws a ball randomly from the pocket and puts it back, then B draws a ball randomly from the same pocket. Let n be the unit digit of the sum of numbers on the two balls drawn by A and B, and P(n) be the probability of the occurrence of n. Find the value of n such that P(n) is the maximum.

P(1) = P((2,9), (3,8), (4,7), (5,6), (6,5), (7,4), (8,3), (9,2))
P(2) = P((1,1), (3,9), (4,8), (5,7), (6,6), (7,5), (8,4), (9,3))
P(3) = P((1,2), (2,1), (4,9), (5,8), (6,7), (7,6), (8,5), (9,4))
P(4) = P((1,3), (2,2), (3,1), (5,9), (6,8), (7,7), (8,6), (9,5))
P(5) = P((1,4), (2,3), (3,2), (4,1), (6,9), (7,8), (8,7), (9,6))
P(6) = P((1,5), (2,4), (3,3), (4,2), (5,1), (7,9), (8,8), (9,7))
P(7) = P((1,6), (2,5), (3,4), (4,3), (5,2), (6,1), (8,9), (9,8))
P(8) = P((1,7), (2,6), (3,5), (4,4), (5,3), (6,2), (7,1), (9,9))
P(9) = P((1,8), (2,7), (3,6), (4,5), (5,4), (6,3), (7,2), (8,1))
P(0) = P((1,9), (2,8), (3,7), (4,6), (5,5), (6,4), (7,3), (8,2), (9,1))

$\therefore$ When $n = 0$, P(n) is a maximum.