

Individual Events

SI	k	2	I1	A	15	I2	P	3	I3	A	2	I4	P	4	IS	a	2
	d	1		B	3		Q	4		B	3		Q	300		b	2
	a	-6		C	4		R	10		C	45		R	2		c	-3
	t	$\frac{50}{11}$		D	8		S	112.5		D	7		S	$2\sqrt{3}$		d	35

Group Events

SG	W	$\frac{1+\sqrt{5}}{2}$	G1	m	8	G2	z	540	G3	k	$\sqrt{33}$	G4	m	13	GS	value	-1
	T	29		h	$\sqrt{13}$		R	6		v	6		n	6		$x^4 + \frac{1}{x^4}$	4036079
	S	106		$x+y+z$	11		k	5		value	106		$abc+def$	72		$\cot \alpha$	$\frac{99}{20}$
	k	4		Number	72		xyz	1		r	27405		$p+q$	2		value	$\frac{6023}{6022}$

Sample Individual Event (2007 Final Individual Event 1)

SI.1 Let $\sqrt{k} = \sqrt{7+\sqrt{13}} - \sqrt{7-\sqrt{13}}$, find the value of k .

$$\sqrt{k}^2 = \left(\sqrt{7+\sqrt{13}} - \sqrt{7-\sqrt{13}} \right)^2$$

$$\begin{aligned} k &= 7 + \sqrt{13} - 2\sqrt{7^2 - \sqrt{13}^2} + 7 - \sqrt{13} \\ &= 14 - 2\sqrt{36} = 2 \end{aligned}$$

SI.2 In Figure 1, the straight line ℓ passes through the point $(k, 3)$ and makes an angle 45° with the x -axis. If the equation of ℓ is $x + by + c = 0$ and $d = |1 + b + c|$, find the value of d .

$$\ell: \frac{y-3}{x-2} = \tan 45^\circ$$

$$y - 3 = x - 2$$

$$x - y + 1 = 0, b = -1, c = 1$$

$$d = |1 - 1 + 1| = 1$$

SI.3 If $x - d$ is a factor of $x^3 - 6x^2 + 11x + a$, find the value of a .

$$f(x) = x^3 - 6x^2 + 11x + a$$

$$f(1) = 1 - 6 + 11 + a = 0$$

$$a = -6$$

SI.4 If $\cos x + \sin x = -\frac{a}{5}$ and $t = \tan x + \cot x$, find the value of t .

$$\cos x + \sin x = \frac{6}{5}$$

$$(\cos x + \sin x)^2 = \frac{36}{25}$$

$$1 + 2 \sin x \cos x = \frac{36}{25}$$

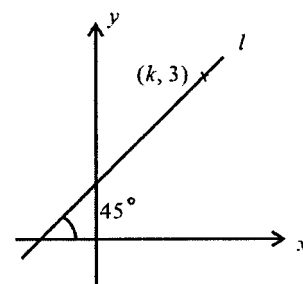
$$2 \sin x \cos x = \frac{11}{25} \Rightarrow \sin x \cos x = \frac{11}{50}$$

$$d = \tan x + \cot x$$

$$= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$

$$= \frac{1}{\sin x \cos x} = \frac{50}{11}$$



Individual Event 1

I1.1 Let $A = 15 \times \tan 44^\circ \times \tan 45^\circ \times \tan 46^\circ$, find the value of A .

Similar question: 2012 FG2.1

$$A = 15 \times \tan 44^\circ \times 1 \times \frac{1}{\tan 44^\circ} = 15$$

I1.2 Let n be a positive integer and $\overbrace{20082008 \cdots 2008}^{n \text{ 2008's}}15$ is divisible by A .

If the least possible value of n is B , find the value of B .

Reference: 2010 HG2

The given number is divisible by 15. Therefore it is divisible by 3 and 5.

The last 2 digits of the given number is 15, which is divisible by 15.

The necessary condition is: $\overbrace{20082008 \cdots 2008}^{n \text{ 2008's}}$ must be divisible by 3.

$2 + 0 + 0 + 8 = 10$ which is not divisible by 3.

The least possible n is 3: $2+0+0+8+2+0+0+8+2+0+0+8 = 30$ which is divisible by 3.

I1.3 Given that there are C integers that satisfy the equation $|x - 2| + |x + 1| = B$, find the value of C

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2012 FG2.3

$$|x - 2| + |x + 1| = 3$$

$$\text{If } x < -1, 2 - x - x - 1 = 3$$

$$\Rightarrow x = -1 \text{ (rejected)}$$

$$\text{If } -1 \leq x \leq 2, 2 - x + x + 1 = 3$$

$$\Rightarrow 3 = 3, \text{ always true } \therefore$$

$$-1 \leq x \leq 2$$

$$\text{If } 2 < x, x - 2 + x + 1 = 3$$

$$\Rightarrow x = 2 \text{ (reject)}$$

$$\therefore -1 \leq x \leq 2 \text{ only}$$

$$\therefore x \text{ is an integer}$$

$$\therefore x = -1, 0, 1, 2; C = 4$$

I1.4 In the coordinate plane, the distance from the point $(-C, 0)$ to the straight line $y = x$ is \sqrt{D} , find the value of D .

The distance from $P(x_0, y_0)$ to the straight line $Ax + By + C = 0$ is $\left| \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \right|$.

The distance from $(-4, 0)$ to $x - y = 0$ is

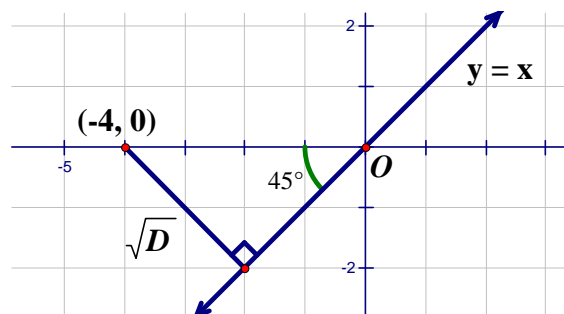
$$\sqrt{D} = \left| \frac{-4 - 0 + 0}{\sqrt{1^2 + (-1)^2}} \right| = \frac{4}{\sqrt{2}} = 2\sqrt{2} = \sqrt{8}$$

$$D = 8$$

Method 2

$$\sqrt{D} = 4 \sin 45^\circ = \frac{4}{\sqrt{2}}$$

$$D = \frac{16}{2} = 8$$



Individual Event 2

I2.1 Given that $P = \left[\sqrt[3]{6} \times \left(\sqrt[3]{\frac{1}{162}} \right) \right]^{-1}$, find the value of P .

$$\begin{aligned} P &= \left[\sqrt[3]{\frac{6}{162}} \right]^{-1} \\ &= \sqrt[3]{\frac{162}{6}} \\ &= \sqrt[3]{27} = 3 \end{aligned}$$

I2.2 Let a , b and c be real numbers with ratios $b : (a + c) = 1 : 2$ and $a : (b + c) = 1 : P$.

If $Q = \frac{a+b+c}{a}$, find the value of Q .

$$2b = a + c \dots\dots (1), 3a = b + c \dots\dots (2)$$

$$(1) - (2): 2b - 3a = a - b \Rightarrow 3b = 4a \Rightarrow a : b = 3 : 4$$

Let $a = 3k$, $b = 4k$, sub. into (1): $2(4k) = 3k + c \Rightarrow c = 5k$

$$Q = \frac{a+b+c}{a} = \frac{3k+4k+5k}{3k} = 4$$

I2.3 Let $R = \left(\sqrt{\sqrt{3} + \sqrt{2}} \right)^{\varrho} + \left(\sqrt{\sqrt{3} - \sqrt{2}} \right)^{\varrho}$. Find the value of R .

$$\begin{aligned} R &= \left(\sqrt{\sqrt{3} + \sqrt{2}} \right)^4 + \left(\sqrt{\sqrt{3} - \sqrt{2}} \right)^4 \\ &= (\sqrt{3} + \sqrt{2})^2 + (\sqrt{3} - \sqrt{2})^2 \\ &= 3 + 2\sqrt{6} + 2 + 3 - 2\sqrt{6} + 2 = 10 \end{aligned}$$

I2.4 Let $S = (x - R)^2 + (x + 5)^2$, where x is a real number. Find the minimum value of S .

$$\begin{aligned} S &= (x - 10)^2 + (x + 5)^2 \\ &= x^2 - 20x + 100 + x^2 + 10x + 25 \\ &= 2x^2 - 10x + 125 \\ &= 2(x^2 - 5x) + 125 \\ &= 2(x - 2.5)^2 + 125 - 2 \times 2.5^2 \\ &= 2(x - 2.5)^2 + 112.5 \geq 112.5 \end{aligned}$$

The minimum value of S is 112.5.

Method 2

$$S = (10 - x)^2 + (x + 5)^2$$

Let $a = 10 - x$, $b = x + 5$

$a + b = 15$, which is a constant

$\therefore a^2 + b^2$ reaches its minimum when $a = b = 7.5$

$$\begin{aligned} \therefore \text{Minimum } S &= 7.5^2 + 7.5^2 \\ &= 112.5 \end{aligned}$$

Individual Event 3

- I3.1** Given that $\frac{1-\sqrt{3}}{2}$ satisfies the equation $x^2 + px + q = 0$, where p and q are rational numbers.

If $A = |p| + 2|q|$, find the value of A .

For an equation with rational coefficients, conjugate roots occur in pairs.

That is, the other root is $\frac{1+\sqrt{3}}{2}$.

$$\frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2} = -p$$

$$\Rightarrow p = -1$$

$$\frac{1-\sqrt{3}}{2} \times \frac{1+\sqrt{3}}{2} = q$$

$$\Rightarrow q = -\frac{1}{2}$$

$$A = |-1| + 2\left|-\frac{1}{2}\right| = 2$$

- I3.2** Two bags U_1 and U_2 contain identical red and white balls. U_1 contains A red balls and 2 white balls. U_2 contains 2 red balls and B white balls. Take two balls out of each bag. If the probability of all four balls are red is $\frac{1}{60}$, find the value of B .

U_1 contains 2 red and 2 white, total 4 balls. U_2 contains 2 red and B white, total $2 + B$ balls.

$$P(\text{all 4 are red}) = \frac{2}{4} \times \frac{1}{3} \times \frac{2}{2+B} \times \frac{1}{1+B} = \frac{1}{60}$$

$$20 = (2+B)(1+B)$$

$$B^2 + 3B - 18 = 0$$

$$(B-3)(B+6) = 0$$

$$B = 3$$

- I3.3** Figure 1 is formed by three identical circles touching one another, the radius of each circle is B cm. If the perimeter of the shaded region is C cm, find the value of C . (Take $\pi = 3$)

Let the centres of the circles be P ,

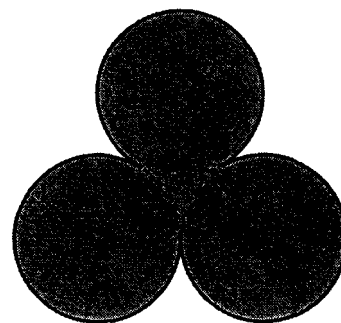
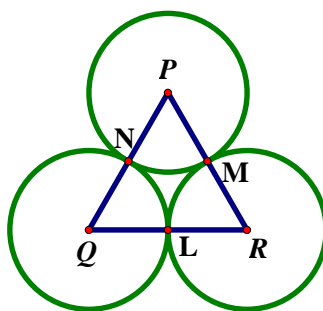
Q and R respectively.

Then $PQ = 2B = 6 = QR = PR$

$\triangle PQR$ is an equilateral \triangle .

$$\angle P = \angle Q = \angle R = 60^\circ$$

$$\begin{aligned} \text{Perimeter} &= 3 \times 2\pi \times 3 - 3 \times \widehat{MN} \\ &= 18\pi - 3 \times 2\pi \times 3 \times \frac{60}{360} \\ &= 15\pi = 45 \end{aligned}$$



- I3.4** Let D be the integer closest to \sqrt{C} , find the value of D .

$$6 = \sqrt{36} < \sqrt{45} < \sqrt{49} = 7$$

$$6 < D < 7$$

$$6.5^2 = 42.25 < 45$$

$$6.5 < D < 7$$

$$D = 7$$

Method 2

$$6 = \sqrt{36} < \sqrt{45} < \sqrt{49} = 7 \Rightarrow 6 < D < 7$$

$$\therefore |45 - 36| = 9, |45 - 49| = 4$$

$\therefore 45$ is closer to 49

$\therefore \sqrt{45}$ is closer to 7

$$D = 7$$

Individual Event 4

- I4.1** Given that x and y are real numbers such that $|x| + x + y = 10$ and $|y| + x - y = 10$. If $P = x + y$, find the value of P .

$$\text{If } x \geq 0, y \geq 0; \begin{cases} 2x + y = 10 \\ x = 10 \end{cases}, \text{ solving give } x = 10, y = -10 \text{ (reject)}$$

$$\text{If } x \geq 0, y < 0; \begin{cases} 2x + y = 10 \\ x - 2y = 10 \end{cases}, \text{ solving give } x = 6, y = -2$$

$$\text{If } x < 0, y \geq 0; \begin{cases} y = 10 \\ x = 10 \end{cases}, \text{ reject}$$

$$\text{If } x < 0, y < 0; \begin{cases} y = 10 \\ x - 2y = 10 \end{cases}, \text{ solving give } x = 30, y = 10 \text{ (reject)}$$

$$\therefore x = 6, y = -2$$

$$P = x + y = 6 - 2 = 4$$

- I4.2** In Figure 1, the shaded area is formed by two concentric circles and has area $96\pi \text{ cm}^2$. If the two radii differ by $2P \text{ cm}$ and the large circle has area $Q \text{ cm}^2$, find the value of Q . (Take $\pi = 3$)

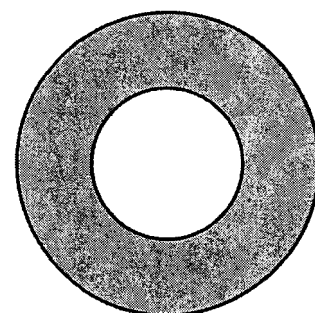
Let the radii of the large and small circles be R and r respectively.

$$\pi(R^2 - r^2) = 96\pi \text{ and } R - r = 8 \dots (1)$$

$$(R + r)(R - r) = 96 \Rightarrow (R + r) \times 8 = 96 \Rightarrow R + r = 12 \dots (2)$$

$$(2) + (1): 2R = 20 \Rightarrow R = 10$$

$$\Rightarrow Q = \pi(10)^2 = 300$$



- I4.3** Let R be the largest integer such that $R^Q < 5^{200}$, find the value of R .

Reference: 1996 HI4, 1999 FG5.3, 2008 FG2.4

$$R^{300} < 5^{200} \Rightarrow R^3 < 25, \text{ the largest integer is } 2.$$

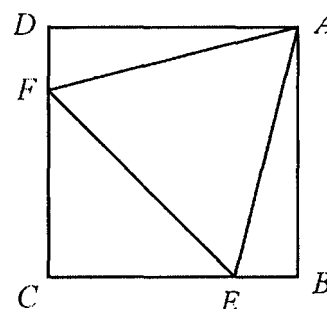
- I4.4** In Figure 2, there are a square $ABCD$ with side length $(R - 1) \text{ cm}$ and an equilateral triangle AEF . (E and F are points on BC and CD respectively). If the area of $\triangle AEF$ is $(S - 3) \text{ cm}^2$, find the value of S .

Reference: 1995 HG7

Let $AF = x \text{ cm} = FE = AE$

$$\angle FAE = 60^\circ, \angle DAF = \angle BAE = 15^\circ$$

$$AD = 1 = AF \cos 15^\circ = x \cos 15^\circ \Rightarrow x = \sec 15^\circ$$



$$S - 3 = \frac{1}{2} x^2 \sin 60^\circ = \frac{1}{2} \cdot \frac{1}{\cos^2 15^\circ} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \cdot \frac{1}{1 + \cos 30^\circ} = \frac{\sqrt{3}}{2} \cdot \frac{1}{1 + \frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2} \cdot \frac{2}{2 + \sqrt{3}} = 2\sqrt{3} - 3$$

$$S = 2\sqrt{3}$$

Method 2

Let $CE = CF = x \text{ cm}$

Then $BE = DF = (1 - x) \text{ cm}$

By Pythagoras' theorem,

$$AF = FE \Rightarrow 1 + (1 - x)^2 = x^2 + x^2$$

$$2 - 2x + x^2 = 2x^2$$

$$x^2 + 2x - 2 = 0 \Rightarrow x^2 = 2 - 2x$$

$$x = -1 + \sqrt{3}$$

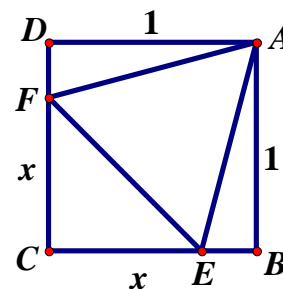
Area of $\triangle AFE$ = Area of square - area of $\triangle CEF$ - 2 area of $\triangle ADF$

$$= 1 - \frac{x^2}{2} - 2 \times \frac{1 \times (1 - x)}{2} = x - \frac{x^2}{2} = x - \frac{2 - 2x}{2} = 2x - 1$$

$$= 2(-1 + \sqrt{3}) - 1 = 2\sqrt{3} - 3$$

$$S - 3 = 2\sqrt{3} - 3$$

$$S = 2\sqrt{3}$$



Individual Spare

IS.1 If all the positive factors of 28 are d_1, d_2, \dots, d_n and $a = \frac{1}{d_1} + \frac{1}{d_2} + \dots + \frac{1}{d_n}$, find the value of a .

Positive factors of 28 are 1, 2, 4, 7, 14, 28. $a = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = 2$

IS.2 Given that x is a negative real number that satisfy $\frac{1}{x + \frac{1}{x+2}} = a$. If $b = x + \frac{7}{2}$, find the value of b

$$\frac{1}{x + \frac{1}{x+2}} = 2 \Rightarrow \frac{x+2}{x(x+2)+1} = 2$$

$$\Rightarrow x + 2 = 2x^2 + 4x + 2$$

$$\Rightarrow 2x^2 + 3x = 0$$

$$\Rightarrow x = -1.5 \text{ or } 0 \text{ (reject)}$$

$$b = -1.5 + 3.5 = 2$$

IS.3 Let α and β be the two roots of the equation $x^2 + cx + b = 0$, where $c < 0$ and $\alpha - \beta = 1$. Find the value of c .

Reference: 2016 FI1.3

$$\alpha\beta = b = 2; (\alpha - \beta)^2 = 1$$

$$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$$

$$\Rightarrow (-c)^2 - 4 \times 2 = 1$$

$$c = -3$$

IS.4 Let d be the remainder of $(196c)^{2008}$ divided by 97. Find the value of d .

$$\begin{aligned} [196 \times (-3)]^{2008} &= 588^{2008} = (97 \times 6 + 6)^{2008} \\ &= (97 \times 6)^{2008} + {}_{2008}C_1 \cdot (97 \times 6)^{2007} \times 6 + \dots + 6^{2008} \\ &= 97m + 6^{2008}, \text{ where } m \text{ is an integer.} \end{aligned}$$

Note that $2^5 \times 3 = 96 = 97 - 1 \equiv -1 \pmod{97}$;

$$2 \times 3^5 = 486 = 97 \times 5 + 1 \equiv 1 \pmod{97};$$

$$\therefore 6^6 = (2^5 \times 3) \times (2 \times 3^5) \equiv -1 \pmod{97}$$

$$6^{2008} = (6^6)^{334} \times 6^4$$

$$\equiv (-1)^{334} \times 1296$$

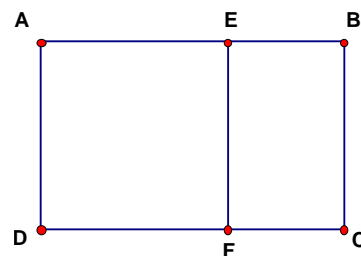
$$\equiv 97 \times 13 + 35$$

$$\equiv 35 \pmod{97}$$

$$\therefore d = 35$$

Sample Group Event (2007 Final Group Event 1)

SG.1 In Figure 1, $AEFD$ is a unit square. The ratio of the length of the rectangle $ABCD$ to its width is equal to the ratio of the length of the rectangle $BCFE$ to its width. If the length of AB is W units, find the value of W .



$$\frac{W}{1} = \frac{1}{W-1}$$

$$W^2 - W - 1 = 0$$

$$\Rightarrow W = \frac{1+\sqrt{5}}{2}$$

SG.2 On the coordinate plane, there are T points (x, y) , where x, y are integers, satisfying $x^2 + y^2 < 10$, find the value of T . (Reference: 2002 FI4.3)

T = number of integral points inside the circle $x^2 + y^2 = 10$.

We first count the number of integral points in the first quadrant:

$$x = 1; y = 1, 2$$

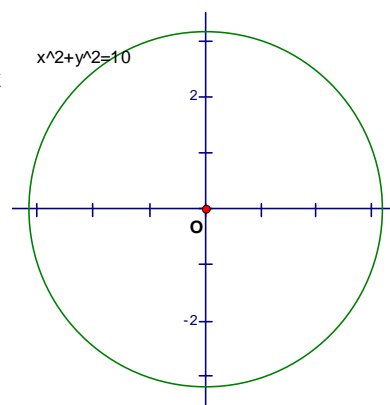
$$x = 2; y = 1, 2$$

Next, the number of integral points on the x -axis and y -axis

$$= 3 + 3 + 3 + 3 + 1 = 13$$

$$T = 4 \times 4 + 3 + 3 + 3 + 3 + 1$$

$$= 29$$



SG.3 Let P and $P + 2$ be both prime numbers satisfying $P(P + 2) \leq 2007$. If S represents the sum of such possible values of P , find the value of S .

$$P^2 + 2P - 2007 \leq 0$$

$$(P + 1)^2 - 2008 \leq 0$$

$$(P + 1 + \sqrt{2008})(P + 1 - \sqrt{2008}) \leq 0$$

$$(P + 1 + 2\sqrt{502})(P + 1 - 2\sqrt{502}) \leq 0$$

$$-1 - 2\sqrt{502} \leq P \leq -1 + 2\sqrt{502}$$

$$P \text{ is a prime} \Rightarrow 0 < P \leq -1 + 2\sqrt{502}$$

$$22 = \sqrt{484} < \sqrt{502} < \sqrt{529} = 23$$

$$43 < -1 + 2\sqrt{502} < 45$$

$$\therefore (P, P + 2) = (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43)$$

$$S = 3 + 5 + 11 + 17 + 29 + 41 = 106$$

SG.4 It is known that $\log_{10}(2007^{2006} \times 2006^{2007}) = a \times 10^k$, where $1 \leq a < 10$ and k is an integer. Find the value of k .

$$a \times 10^k = 2006 \log 2007 + 2007 \log 2006 = 2006 \times (\log 2007 + \log 2006) + \log 2006$$

$$2006 \times (\log 2006 + \log 2006) + \log 2006 < a \times 10^k < 2006 \times (\log 2007 + \log 2007) + \log 2007$$

$$4013 \log 2006 < a \times 10^k < 4013 \log 2007$$

$$4013 \log(2.006 \times 10^3) < a \times 10^k < 4013 \log(2.007 \times 10^3)$$

$$4013 (\log 2.006 + 3) < a \times 10^k < 4013 (\log 2.007 + 3)$$

$$4013 \log 2 + 4013 \times 3 < a \times 10^k < 4013 \log 3 + 3$$

$$1.32429 \times 10^4 = 4013 \times 0.3 + 4013 \times 3 < a \times 10^k < 4013 \times 0.5 + 4013 \times 3 = 1.40455 \times 10^4$$

$$k = 4$$

Group Event 1

G1.1 Given that there are three points on the coordinate plane: $O(0, 0)$, $A(12, 2)$ and $B(0, 8)$. A reflection of $\triangle OAB$ along the straight line $y = 6$ creates $\triangle PQR$. If the overlapped area of $\triangle OAB$ and $\triangle PQR$ is m square units, find the value of m .

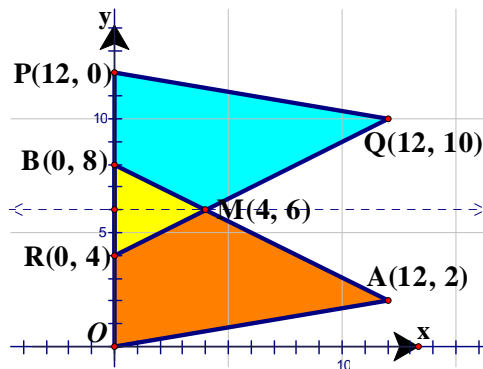
$P(12, 0)$, $Q(12, 10)$, $R(0, 4)$.

Suppose AB intersects QR at $M(x, 6)$.

Slope of MR = slope of QR

$$\frac{6-4}{x} = \frac{10-4}{12} \Rightarrow x = 4; M(4, 6)$$

$$\text{Area of the overlap } \triangle BMR = \frac{1}{2} \cdot (8-4) \times 4 = 8; m = 8$$



G1.2 In Figure 1, $ABCD$ is a parallelogram with $BA = 3$ cm, $BC = 4$ cm and $BD = \sqrt{37}$ cm. If $AC = h$ cm, find the value of h .

$CD = BA = 3$ cm (opp. sides, // -gram)

$$\text{In } \triangle BCD, \cos C = \frac{4^2 + 3^2 - \sqrt{37}^2}{2 \times 3 \times 4} = -\frac{1}{2}$$

$$\cos B = \cos(180^\circ - C) = -\cos C = \frac{1}{2} \text{ (int. } \angle s \text{ } AB \parallel DC)$$

$$\text{In } \triangle ABC, AC = \sqrt{3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cos B} = \sqrt{13}; h = \sqrt{13}$$

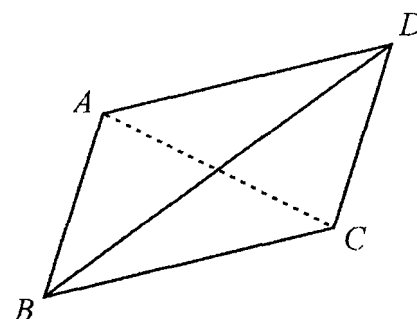
Method 2

By [Apollonius theorem](#),

$$2BA^2 + 2BC^2 = AC^2 + BD^2$$

$$2 \times 3^2 + 2 \times 4^2 = h^2 + 37$$

$$h = \sqrt{13}$$



G1.3 Given that x , y and z are positive integers and the fraction $\frac{151}{44}$ can be written in the form of

$$3 + \frac{1}{x + \frac{1}{y + \frac{1}{z}}}. \text{ Find the value of } x + y + z.$$

$$\frac{151}{44} = 3 + \frac{19}{44} = 3 + \frac{1}{\frac{44}{19}} = 3 + \frac{1}{2 + \frac{6}{19}} = 3 + \frac{1}{2 + \frac{1}{\frac{19}{6}}} = 3 + \frac{1}{2 + \frac{1}{3 + \frac{1}{6}}}$$

$$x = 2, y = 3, z = 6; x + y + z = 11$$

G1.4 When 491 is divided by a two-digit integer, the remainder is 59. Find this two-digit integer.

Let the number be $10x + y$, where $0 < x \leq 9$, $0 \leq y \leq 9$.

$$491 = (10x + y) \cdot Q + 59; 59 < 10x + y$$

$$491 - 59 = 432 = (10x + y) \cdot Q; 432 = 72 \times 6; \text{ the number is } 72.$$

Group Event 2

G2.1 In Figure 1, BD , FC , GC and FE are straight lines.

If $z = a + b + c + d + e + f + g$, find the value of z .

$$a^\circ + b^\circ + g^\circ + \angle BHG = 360^\circ \quad (\angle \text{sum of polygon } ABHG)$$

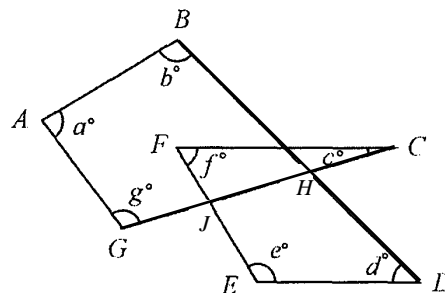
$$c^\circ + f^\circ = \angle CJE \quad (\text{ext. } \angle \text{ of } \triangle CFJ)$$

$$c^\circ + f^\circ + e^\circ + d^\circ + \angle JHD = 360^\circ \quad (\angle \text{sum of polygon } JHDE)$$

$$a^\circ + b^\circ + g^\circ + \angle BHG + c^\circ + f^\circ + e^\circ + d^\circ + \angle JHD = 720^\circ$$

$$a^\circ + b^\circ + c^\circ + d^\circ + e^\circ + f^\circ + g^\circ + 180^\circ = 720^\circ$$

$$z = 540$$



G2.2 If R is the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7, find the value of R .

$$x^6 + y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5) + 2y^6$$

$$6^6 + 1^6 = 7Q_1 + 2; 5^6 + 2^6 = 7Q_2 + 2 \times 2^6; 4^6 + 3^6 = 7Q_3 + 2 \times 3^6$$

$$2 + 2 \times 2^6 + 2 \times 3^6 = 2(1 + 64 + 729) = 1588 = 7 \times 226 + 6; R = 6$$

$$\text{Method 2 } 1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6 \equiv 1^6 + 2^6 + 3^6 + (-3)^6 + (-2)^6 + (-1)^6 \pmod{7}$$

$$\equiv 2(1^6 + 2^6 + 3^6) \equiv 2(1 + 64 + 729) \pmod{7}$$

$$\equiv 2(1 + 1 + 1) \pmod{7} \equiv 6 \pmod{7}$$

G2.3 If $14!$ is divisible by 6^k , where k is an integer, find the largest possible value of k .

We count the number of factors of 3 in $14!$. They are 3, 6, 9, 12. So there are 5 factors of 3.

$$k = 5$$

G2.4 Let x , y and z be real numbers that satisfy $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = \frac{7}{3}$.

Find the value of xyz . (Reference 2010 FG2.2, 2017 FG1.1)

Method 1

$$\text{From (1), } x = 4 - \frac{1}{y} = \frac{4y-1}{y}$$

$$\Rightarrow \frac{1}{x} = \frac{y}{4y-1} \quad \dots\dots (4)$$

$$\text{Sub. (4) into (3): } z + \frac{y}{4y-1} = \frac{7}{3}$$

$$z = \frac{7}{3} - \frac{y}{4y-1} \quad \dots\dots (5)$$

$$\text{From (2): } \frac{1}{z} = 1 - y$$

$$z = \frac{1}{1-y} \quad \dots\dots (6)$$

$$(5) = (6): \frac{1}{1-y} = \frac{7}{3} - \frac{y}{4y-1}$$

$$\frac{1}{1-y} = \frac{28y-7-3y}{3(4y-1)}$$

$$3(4y-1) = (1-y)(25y-7)$$

$$12y-3 = -25y^2-7+32y$$

$$25y^2-20y+4=0$$

$$(5y-2)^2=0 \Rightarrow y=\frac{2}{5}$$

$$\text{Sub. } y=\frac{2}{5} \text{ into (6): } z=\frac{1}{1-\frac{2}{5}}=\frac{5}{3}$$

$$\text{Sub. } y=\frac{2}{5} \text{ into (1): } x+\frac{5}{2}=4 \Rightarrow x=\frac{3}{2}$$

$$xyz=\frac{2}{5} \times \frac{5}{3} \times \frac{3}{2}=1$$

Method 2

$$\begin{cases} x + \frac{1}{y} = 4 \dots\dots (1) \\ y + \frac{1}{z} = 1 \dots\dots (2) \\ z + \frac{1}{x} = \frac{7}{3} \dots\dots (3) \end{cases}$$

$$(1) \times (2): xy + 1 + \frac{x}{z} + \frac{1}{yz} = 4$$

$$x\left(y + \frac{1}{z}\right) + \frac{1}{yz} = 3$$

$$\text{Sub. (2) into the eqt.: } x + \frac{x}{xyz} = 3$$

$$\text{Let } a = xyz, \text{ then } x + \frac{x}{a} = 3 \dots\dots (4)$$

$$(2) \times (3): y\left(\frac{7}{3} + \frac{1}{a}\right) = \frac{4}{3} \dots\dots (5)$$

$$(1) \times (3): z\left(4 + \frac{1}{a}\right) = \frac{25}{3} \dots\dots (6)$$

$$(4) \times (5) \times (6): a\left(1 + \frac{1}{a}\right)\left(\frac{7}{3} + \frac{1}{a}\right)\left(4 + \frac{1}{a}\right) = \frac{100}{3}$$

$$\frac{(a+1)(7a+3)(4a+1)}{3a^2} = \frac{100}{3}$$

$$\text{which reduces to } 28a^3 - 53a^2 + 22a + 3 = 0$$

$$\Rightarrow (a-1)^2(28a+3) = 0$$

$$\therefore a = 1$$

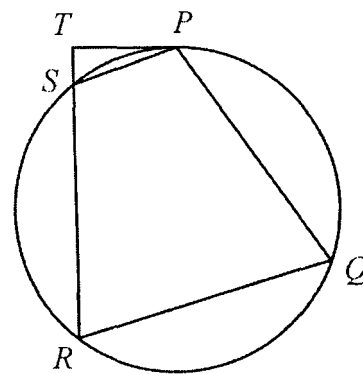
Method 3 $(1) \times (2) \times (3) - (1) - (2) - (3):$

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} - \left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{28}{3} - \frac{22}{3} \Rightarrow xyz + \frac{1}{xyz} = 2$$

$$xyz = 1$$

Group Event 3

G3.1 In Figure 1, $PQRS$ is a cyclic quadrilateral, where S is on the straight line RT and TP is tangent to the circle. If $RS = 8$ cm, $RT = 11$ cm and $TP = k$ cm, find the value of k .



Join PR . $\angle SPT = \angle PRS$ (\angle in alt. seg.)

$\angle STP = \angle PTR$ (common \angle)

$\triangle STP \sim \triangle PTR$ (equiangular)

$$\frac{TP}{TR} = \frac{TS}{TP} \quad (\text{ratio of sides, } \sim \triangle) \quad \frac{k}{11} = \frac{11-8}{k}; k = \sqrt{33}$$

G3.2 The layout in Figure 2 can be used to fold a polyhedron. If this polyhedron has v vertices, find the value of v .

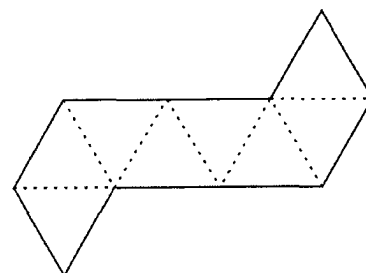
There are 8 faces. $f = 8$.

There are 8 equilateral \triangle s, no of sides $= 8 \times 3 = 24$

Each side is shared by 2 faces. Number of edge $e = 12$

By Euler formula, $v - e + f = 2$

$$v - 12 + 8 = 2 \Rightarrow v = 6$$



G3.3 For arbitrary real number x , define $[x]$ to be the largest integer less than or equal to x . For instance, $[2] = 2$ and $[3.4] = 3$. Find the value of $[1.008^8 \times 100]$.

$$1.008^8 \times 100 = (1 + 0.008)^8 \times 100 = 100(1 + 8 \times 0.008 + 28 \times 0.008^2 + \dots) \approx 106.4$$

The integral value $= 106$

G3.4 When choosing, without replacement, 4 out of 30 labelled balls that are marked from 1 to 30, there are r combinations. Find the value of r .

$$r = C_4^{30} = \frac{30 \times 29 \times 28 \times 27}{1 \times 2 \times 3 \times 4} = 27405$$

Remark: If the question is changed to: Choose 4 out of 30 labelled balls that are marked from 1 to 30 with repetition is allowed, there are r combinations. Find the value of r .

We shall divide into 5 different cases:

Case 1: All 4 balls are the same number, 30 combinations

Case 2: XXXY, where X, Y are different numbers, $30 \times 29 = 870$ combinations

Case 3 XXYY, where X, Y are different numbers, $C_2^{30} = 435$ combinations

Case 4 XXYZ, where X, Y, Z are different numbers, $30 \times C_2^{29} = 12180$ combinations

Case 5 XYZW, where X, Y, Z, W are different numbers, $C_4^{30} = 27405$ combinations

Total number of combinations $= 30 + 870 + 435 + 12180 + 27405 = 40920$

Group Event 4

G4.1 Regular tessellation is formed by identical regular m -polygons for some fixed m .

Find the sum of all possible values of m .

$$\text{Each interior angle} = \frac{180^\circ(m-2)}{m} \quad (\angle \text{sum of polygon})$$

Suppose n m -polygons tessellate the space.

$$\frac{180^\circ(m-2)}{m} \cdot n = 360^\circ \quad (\angle \text{s at a point})$$

$$n(m-2) = 2m$$

$$n(m-2) - 2m + 4 = 4$$

$$(n-2)(m-2) = 4$$

$$m-2 = 1, 2 \text{ or } 4$$

$$m = 3, 4 \text{ or } 6$$

$$\text{Sum of all possible } m = 3 + 4 + 6 = 13$$

G4.2 Amongst the seven numbers 3624, 36024, 360924, 3609924, 36099924, 360999924 and 3609999924, there are n of them that are divisible by 38. Find the value of n .

$38 = 2 \times 19$, we need to investigate which number is divisible by 19.

$$19^2 = 361$$

$$3624 = 3610 + 13$$

$$36024 = 36100 - 76 = 100(19^2) - 19 \times 4$$

$$360924 = 361000 - 76$$

$$3609924 = 3610000 - 76$$

$$36099924 = 36100000 - 76$$

$$360999924 = 361000000 - 76$$

$$3609999924 = 3610000000 - 76$$

$$n = 6$$

G4.3 If $208208 = 8^5a + 8^4b + 8^3c + 8^2d + 8e + f$, where a, b, c, d, e , and f are integers and $0 \leq a, b, c, d, e, f \leq 7$, find the value of $a \times b \times c + d \times e \times f$.

Reference: 2011 FI1.2

$$\begin{array}{r} 8 \overline{) 208208} \\ \underline{8 \overline{) 26026}} \dots\dots 0 \\ \underline{8 \overline{) 3253}} \dots\dots 2 \\ \underline{8 \overline{) 406}} \dots\dots 5 \\ \underline{8 \overline{) 50}} \dots\dots 6 \\ 6 \dots\dots 2 \end{array}$$

$$a = 6, b = 2, c = 6, d = 5, e = 2, f = 0$$

$$a \times b \times c + d \times e \times f = 72$$

G4.4 In the coordinate plane, rotate point $A(6, 8)$ about the origin $O(0, 0)$ counter-clockwise for 20070° to point $B(p, q)$. Find the value of $p + q$.

$$20070^\circ = 360^\circ \times 55 + 270^\circ$$

$$\therefore B(8, -6)$$

$$p + q = 2$$

Group Spare**GS.1** Calculate the value of $(\sqrt{2008} + \sqrt{2007})^{2007} \times (\sqrt{2007} - \sqrt{2008})^{2007}$.

$$(\sqrt{2008} + \sqrt{2007})^{2007} \times (\sqrt{2007} - \sqrt{2008})^{2007} = (2007 - 2008)^{2007} = (-1)^{2007} = -1$$

GS.2 If $x - \frac{1}{x} = \sqrt{2007}$, find the value of $x^4 + \frac{1}{x^4}$.

$$\left(x - \frac{1}{x}\right)^2 = 2007$$

$$x^2 - 2 + \frac{1}{x^2} = 2007$$

$$x^2 + \frac{1}{x^2} = 2009$$

$$\left(x^2 + \frac{1}{x^2}\right)^2 = 2009^2 = 4036081$$

$$x^4 + 2 + \frac{1}{x^4} = 4036081$$

$$x^4 + \frac{1}{x^4} = 4036079$$

GS.3 Given that $\cos \alpha = -\frac{99}{101}$ and $180^\circ < \alpha < 270^\circ$. Find the value of $\cot \alpha$.

$$\sec \alpha = -\frac{101}{99}$$

$$\tan^2 \alpha = \sec^2 \alpha - 1$$

$$= \left(-\frac{101}{99}\right)^2 - 1$$

$$= \frac{101^2 - 99^2}{99^2}$$

$$= \frac{(101 - 99) \cdot (101 + 99)}{99^2}$$

$$= \frac{400}{99^2}$$

$$\tan \alpha = \frac{20}{99}$$

$$\cot \alpha = \frac{99}{20} \quad (= 4.95)$$

GS.4 Calculate the value of $\frac{2008^3 + 4015^3}{2007^3 + 4015^3}$.Let $x = 2007.5$, then $2x = 4015$

$$\begin{aligned} \frac{2008^3 + 4015^3}{2007^3 + 4015^3} &= \frac{(x + 0.5)^3 + (2x)^3}{(x - 0.5)^3 + (2x)^3} = \frac{8\left(x + \frac{1}{2}\right)^3 + 8(2x)^3}{8\left(x - \frac{1}{2}\right)^3 + 8(2x)^3} = \frac{(2x + 1)^3 + (4x)^3}{(2x - 1)^3 + (4x)^3} \\ &= \frac{(2x + 1 + 4x)\left[(2x + 1)^2 - 4x(2x + 1) + (4x)^2\right]}{(2x - 1 + 4x)\left[(2x - 1)^2 - 4x(2x - 1) + (4x)^2\right]} \\ &= \frac{(6x + 1)(4x^2 + 4x + 1 - 8x^2 - 4x + 16x^2)}{(6x - 1)(4x^2 - 4x + 1 - 8x^2 + 4x + 16x^2)} \\ &= \frac{(6x + 1)(12x^2 + 1)}{(6x - 1)(12x^2 + 1)} = \frac{6x + 1}{6x - 1} \\ &= \frac{6023}{6022} \end{aligned}$$