04-05 Individual	**1**	7	**2**	–20042	**3**	–2	**4**	$\frac{-2+\sqrt{10}}{2}$	**5**	$2\sqrt{6}$
	6	$\frac{2005}{2006}$	**7**	8204	**8**	1	**9**	2	**10**	Tuesday

04-05 Group	**1**	11	**2**	204.1	**3**	4	**4**	8	**5**	970
	6	$\frac{\sqrt{2}}{3}$	**7**	8	**8**	8	**9**	*21 see the remark	**10**	1

Individual Events

I1 Suppose p, q are positive integers and $\frac{96}{35} > \frac{p}{q} > \frac{97}{36}$, find the smallest possible value of q.

Reference: 1996 FG10.3, 2010 HG7

$$3-\frac{9}{35} > \frac{p}{q} > 3-\frac{11}{36}$$

$$\frac{9}{35} < 3-\frac{p}{q} < \frac{11}{36}$$

$$\frac{9}{35} < \frac{3q-p}{q} < \frac{11}{36}$$

$$\frac{35}{9} > \frac{q}{3q-p} > \frac{36}{11}$$

$$4-\frac{1}{9} > \frac{q}{3q-p} > 4-\frac{8}{11}$$

$$\frac{1}{9} < 4-\frac{q}{3q-p} < \frac{8}{11}$$

$$\frac{1}{9} < \frac{11q-4p}{3q-p} = \frac{r}{s} < \frac{8}{11}$$

$\because r$, s are integers and $s > r$.

Minimum $r = 1$, $s = 2$.

$11q - 4p = 1$

$3q - p = 2$

$q = 7$, $p = 19$.

I2 Given that $x = 2005$ and $y = |4x^2 - 5x + 9| - 4|x^2 + 2x + 2| + 3x + 7$, find the value of y.

Reference: 2016 FI4.3

$4x^2 - 5x + 9 = 4(x - 0.625)^2 + 7.4375 > 0$

$\Rightarrow |4x^2 - 5x + 9| = 4x^2 - 5x + 9$

$x^2 + 2x + 2 = (x + 1)^2 + 1 > 0$

$\Rightarrow |x^2 + 2x + 2| = x^2 + 2x + 2$

$y = 4x^2 - 5x + 9 - 4(x^2 + 2x + 2) + 3x + 7$

$\quad = -10x + 8 = -20050 + 8 = -20042$

I3 If x is a real number satisfying the equation $\left(\sqrt{5+2\sqrt{6}}\right)^x+\left(\sqrt{5-2\sqrt{6}}\right)^x = 10$, find the smallest possible value of x.

$\sqrt{5+2\sqrt{6}}=\sqrt{3}+\sqrt{2}$, $\sqrt{5-2\sqrt{6}}=\sqrt{3}-\sqrt{2}=\dfrac{1}{\sqrt{3}+\sqrt{2}}$

Let $y = (\sqrt{3}+\sqrt{2})^x$, then $(\sqrt{3}-\sqrt{2})^x=\dfrac{1}{y}$

The equation is equivalent to $y+\dfrac{1}{y}=10$

$y^2 - 10y + 1 = 0$

$y = 5 \pm 2\sqrt{6}$

When $(\sqrt{3}+\sqrt{2})^x = 5 + 2\sqrt{6} = (\sqrt{3}+\sqrt{2})^2$, $x = 2$.

When $(\sqrt{3}+\sqrt{2})^x = 5 - 2\sqrt{6} = (\sqrt{3}-\sqrt{2})^2 = \dfrac{1}{\left(\sqrt{3}+\sqrt{2}\right)^2}$, $x = -2$.

I4 Let t be a real number satisfying $(1+\sin t)(1+\cos t)=\dfrac{5}{4}$. If $N = \sin t + \cos t$, find the value of N

$N^2 = \sin^2 t + \cos^2 t + 2\sin t\cos t = 1 + 2\sin t\cos t \Rightarrow \sin t\cos t = \dfrac{N^2-1}{2}$ ....(*)

Expand the given equation: $1 + \sin t + \cos t + \sin t\cos t = \dfrac{5}{4}$

$1 + N + \dfrac{N^2-1}{2} = \dfrac{5}{4}$

$\Rightarrow 4 + 4N + 2N^2 - 2 = 5$

$\Rightarrow 2N^2 + 4N - 3 = 0$

$\Rightarrow N = \dfrac{-2+\sqrt{10}}{2}$ or $\dfrac{-2-\sqrt{10}}{2}$ (< -2, rejected because $N = \sin t + \cos t \ge -2$)

I5 In Figure 1, *ABCDEF* is an "L shape" figure formed by six squares. *HAK* is a straight line and the area of the shaded region is equal to $\dfrac{1}{2}$ of the area of *ABCDEF*.

If the length of each small square is 1 cm and the length of *HK* is m cm, find the value of m.

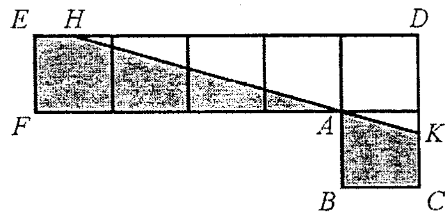


Figure 1

Reference: 2009 HG8, 2012 HI10

It is easy to see that the area of $\Delta DHK = 3$.

Let $DK = x$ cm, $DH = y$ cm; then $xy = 6$ ………… (1)

Join AD, then $\angle ADC = 45° = \angle ADH$, $AD = \sqrt{2}$

Area of ΔADH + area of ΔADK = Area of ΔDHK

$\dfrac{1}{2}y+\dfrac{1}{2}x=3$

$x + y = 6$ ………… (2)

$$\begin{aligned} HK &= \sqrt{x^2+y^2} \\ &= \sqrt{(x+y)^2-2xy} \\ &= \sqrt{6^2-2(6)} \\ &= 2\sqrt{6} \end{aligned}$$

I6 Let n be a natural number, the area of the triangle bounded by the line $nx + (n + 1)y = \sqrt{2}$ and the two ordinate axes is S_n. If $K = S_1 + S_2 + \dots + S_{2005}$, find the value of K.

Reference: 2015 HG10

$nx + (n + 1)y = \sqrt{2}$, x-intercept $= \frac{\sqrt{2}}{n}$, y-intercept $= \frac{\sqrt{2}}{n+1}$; bounded area $= \frac{1}{n(n+1)}$

$$K = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{2005\times 2006} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2005} - \frac{1}{2006} = 1 - \frac{1}{2006} = \frac{2005}{2006}$$

I7 Let $[x]$ be the largest integer not greater than x, for example, $[2.5] = 2$.

If $M = \sum_{n=1}^{1024}[\log_2 n]$, find the value of M. **(Reference: 2005 FG2.4, 2007 FG2.1)**

$\log_2 1 = 0$, $\log_2 2 = 1$, $1 < \log_2 3 < 2$, $\log_2 4 = 2$, $2 < \log_2 5 < 3$, $2 < \log_2 6 < 3$, $2 < \log_2 7 < 3$,

$\log_2 8 = 3$, $3 < \log_2 9 < 4$, ... , $3 < \log_2 15 < 4$,

$\log_2 16 = 4$, $4 < \log_2 17 < 5$, ... , $4 < \log_2 31 < 5$,

....

$\log_2 512 = 9$, $9 < \log_2 513 < 10$, ... , $9 < \log_2 1023 < 10$; $\log_2 1024 = 10$

$$M = 0 + 1 + 1 + 2 + 2 + 2 + 2 + \underbrace{3 + \cdots + 3}_{8\text{ times}} + \underbrace{4 + \cdots + 4}_{16\text{ times}} + \ldots + \underbrace{9 + \cdots + 9}_{512\text{ times}} + 10$$

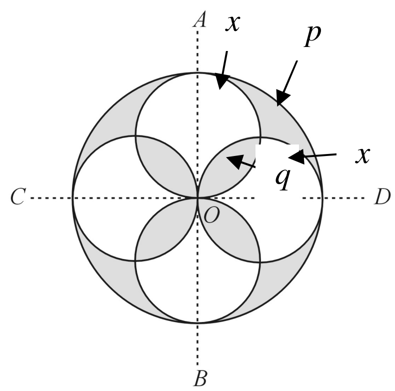
$M = 2 + 4\times 2 + 8\times 3 + 16\times 4 + \ldots + 512\times 9 + 10$ .... (1)

$2M = 4\times 1 + 8\times 2 + 16\times 3 + \ldots + 512\times 8 + 1024\times 9 + 20$ ....(2)

(2) – (1): $M = 1024\times 9 + 10 - (2 + 4 + 8 + \ldots + 512)$

$M = 1024\times 9 + 10 - 1022 = 8204$

I8 In Figure 2, AB is perpendicular to CD, their intersection point O is the center of the large circle and the centers of the four circles lies on either AB or CD. Given also that the radius of the large circle is 1 cm and the radius of each of the four small circles is $\frac{1}{2}$ cm. If the area of the shaded region is $R\text{ cm}^2$, find the value of R. (take $\pi = 3$)



Reference: 1999 FI2.3

In the first quadrant, let the labelled areas be p, q and x.

$$q = \left[\frac{\pi}{4}\cdot\left(\frac{1}{2}\right)^2 - \frac{1}{2}\cdot\left(\frac{1}{2}\cdot\frac{1}{2}\right)\right]\times 2 = \frac{\pi}{8} - \frac{1}{4} = \frac{\pi - 2}{8} = \frac{1}{8}$$

$x + q$ = area of one semi-circle $= \frac{\pi\left(\frac{1}{2}\right)^2}{2} = \frac{3}{8}$; $2x + p + q$ = area of the quadrant $= \frac{1}{4}\pi(1)^2 = \frac{3}{4}$

$2\times(1) = (2)$, $2x + 2q = 2x + p + q \Rightarrow p = q = \frac{1}{8}$

Shaded area $= 4p + 4q = 4\left(\frac{1}{8} + \frac{1}{8}\right) = 1$

I9 Given that $60^a = 3$, $60^b = 5$. If $R = 12^{\frac{1-a-b}{2(1-b)}}$, find the value of R.

Reference: 2001 HI1, 2003 FG2.2, 2004 FG4.3, 2006 FG4.3

$a = \frac{\log 3}{\log 60}$, $b = \frac{\log 5}{\log 60}$

$$\frac{1-a-b}{2(1-b)} = \frac{1-\frac{\log 3}{\log 60}-\frac{\log 5}{\log 60}}{2\left(1-\frac{\log 5}{\log 60}\right)} = \frac{\log 60-\log 3-\log 5}{2(\log 60-\log 5)} = \frac{\log\frac{60}{15}}{2\left(\log\frac{60}{5}\right)} = \frac{\log 4}{2\log 12} = \frac{2\log 2}{2\log 12} = \frac{\log 2}{\log 12}$$

$$12^{\frac{1-a-b}{2(1-b)}} = 12^{\frac{\log 2}{\log 12}} = \left[(10)^{\log 12}\right]^{\frac{\log 2}{\log 12}} = 10^{\log 2} = 2$$

I10 Given that 29th January 2005 is Saturday, on what day is 29th January 2008?

1 year = 365 days = $7\times52 + 1$ days and there is no leap year in this period.

3 years = $3\times7\times52 + 3$ days, Saturday + 3 = Tuesday; after 3 years, it will be a Tuesday.

Group Events

G1 If $x = \frac{19}{97} + \frac{19}{97}\times2 + \frac{19}{97}\times3 + \ldots + \frac{19}{97}\times10$ and a is the integer that is the closest to x, find the value of a.

$x = \frac{19}{97}(1+2+3+\ldots+10) = \frac{19}{97}\times55 = \frac{1045}{97} = 10.8$, $a = 11$

G2 Given that the area of a square $ABCD$ is equal to 130 cm^2 and a circle O passes through the points A, B, C and D. If the area of the circle O is b cm^2, find the value of b. (Take $\pi = 3.14$)

Let the length of side of the square be x cm ($x = \sqrt{130}$)

$AC = \sqrt{x^2+x^2} = \sqrt{2}x$ = diameter of the circle; the radius of the circle $= \frac{\sqrt{2}x}{2} = \frac{\sqrt{260}}{2}$.

$b = \pi r^2 = 3.14\times\frac{260}{4} = 3.14\times65 = 204.1$

G3 Given that p, q and r are distinct roots of the equation $x^3 - x^2 + x - 2 = 0$. If $Q = p^3 + q^3 + r^3$, find the value of Q.

$p + q + r = 1$, $pq + qr + pr = 1$, $pqr = 2$.

$p^3 + q^3 + r^3 - 3pqr = (p + q + r)(p^2 + q^2 + r^2 - pq - qr - pr)$

$p^3 + q^3 + r^3 = (p + q + r)[(p + q + r)^2 - 3(pq + qr + pr)] + 3pqr$

$= 1[(1)^2 - 3(1)] + 3\times2 = 4 \Rightarrow Q = 4$

Method 2 $p^3 - p^2 + p - 2 = 0$ … (1), $q^3 - q^2 + q - 2 = 0$ … (2), $r^3 - r^2 + r - 2 = 0$ … (3)

(1) + (2) + (3): $(p^3 + q^3 + r^3) - (p^2 + q^2 + r^2) + (p + q + r) - 6 = 0$

$p^3 + q^3 + r^3 = (p^2 + q^2 + r^2) - (p + q + r) + 6 = (p + q + r)^2 - 2(pq + qr + rp) - 1 + 6 = 4 = Q$

G4 When a 3-digit number minus the sum of the values of the three digits, the difference is a 3-digit number $\overline{46x}$, find the value of x.

Let the 3-digit numbers be $100a + 10b + c$, where a, b and c are integers between 0 to 9.

$100a + 10b + c - (a + b + c) = 400 + 60 + x$

$99a + 9b = 460 + x \Rightarrow 9(11a + b) = 460 + x$, which is divisible by 9.

So $4 + 6 + x$ must be divisible by 9, $x = 8$.

G5 If B is an integer and $B > (\sqrt{2}+\sqrt{3})^6$, find the smallest possible value of B.

Reference: HKAL PM 1991 P1 Q11, 2003 FI3.2

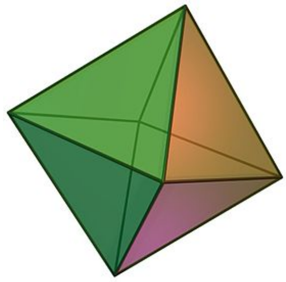
Note that $0 < \sqrt{3}-\sqrt{2} < 1$ and hence $0 < (\sqrt{3}-\sqrt{2})^6 < 1$

$(\sqrt{3}+\sqrt{2})^6 + (\sqrt{3}-\sqrt{2})^6 = 2(\sqrt{3}^6 + 15\times\sqrt{3}^4\times\sqrt{2}^2 + 15\times\sqrt{3}^2\times\sqrt{2}^4 + \sqrt{2}^6)$

$= 2(27 + 270 + 180 + 8) = 970$

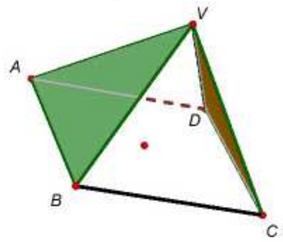
$(\sqrt{3}+\sqrt{2})^6 = 970 - (\sqrt{3}-\sqrt{2})^6 < 970$ = the smallest value of B

G6 Suppose the side of a regular octahedron is equal to 1 cm and the volume is equal to f cm^3, find the value of f.



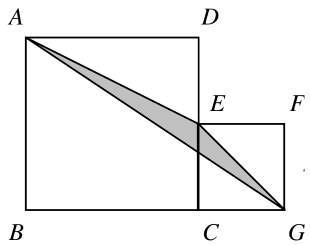
A regular octahedron can be cut into two identical right pyramids with square base (All sides length = 1 cm)

$AC = \sqrt{1^2+1^2} = \sqrt{2}$ cm; Height $= \sqrt{1^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{2}}{2}$ cm



Volume $= \frac{1}{3}\cdot 1^2\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{6}$ cm^3; $f = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$

G7 In Figure 1, $ABCD$ and $CEFG$ are two squares and $FG = 4$ cm. If the area of ΔAEG is equal to g cm^2, find the value of g.



Reference: 2000 FI4.2, 2004 HI9, 2018 HI12

Join AC. $\angle ACB = 45° = \angle EGC$

AC // EG (corr. $\angle$s eq.)

ΔAEG and Δ CEG have the same base and the same height.

Area of ΔAEG = area of $\Delta CEG = \frac{1}{2}\cdot 4\times 4 = 8$ cm^2

$g = 8$

G8 Let x be a real number. If h is the greatest value of x such that $2(\log_{\frac{1}{2}} x)^2 + 9\log_{\frac{1}{2}} x + 9 \le 0$, find the value of h.

$(2\log_{\frac{1}{2}} x + 3)(\log_{\frac{1}{2}} x + 3) \le 0$

$-3 \le \log_{\frac{1}{2}} x \le -1.5$

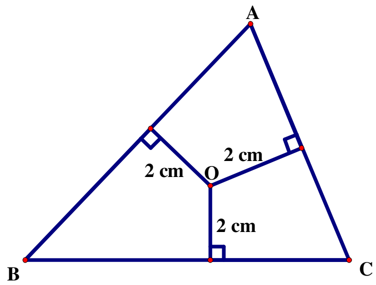
$-3\log\frac{1}{2} \ge \log x \ge -1.5\log\frac{1}{2}$

$\log 8 \ge \log x \ge \log\sqrt{8}$

$\sqrt{8} \le x \le 8$

h = the greatest value of $x = 8$

G9 Given that the perpendicular distances from the point O to three sides of a triangle ABC are all equal to 2 cm and the perimeter of ΔABC is equal to 21 cm. If the area of ΔABC is equal to k cm^2, find the value of k.



Reference 1992 HG8, 2015 HG2, 2021 P1Q6

$k \text{ cm}^2 = (\frac{1}{2}\cdot 2 \text{ cm}\cdot BC + \frac{1}{2}\cdot 2 \text{ cm}\cdot AC + \frac{1}{2}\cdot 2 \text{ cm}\cdot AB)$

$= (BC + AC + AB)$ cm

$= 21 \text{ cm}^2$

$k = 21$

Remark: The old version of the question was:

Given that the perpendicular distances from the point O to three sides of a triangle ABC are all equal to 2 cm and the perimeter of ΔABC is equal to **20 cm**. If the area of ΔABC is equal to k cm^2, find the value of k.

The old version of the question was wrong because it can be proved that the minimum area is $\frac{100\sqrt{3}}{9}$ cm^2 < 20 cm^2

Proof: Let $BC = a$ cm, $AC = b$ cm, $AB = c$ cm,

$s \text{ cm} = \frac{1}{2}(a + b + c) \text{ cm} = 10 \text{ cm}$

$k = \sqrt{s(s-a)(s-b)(s-c)}$ Heron's formula

$= \sqrt{10(10-a)(10-b)(10-c)}$

$\le \sqrt{10}\cdot\left[\frac{(10-a)+(10-b)+(10-c)}{3}\right]^{3/2}$ (A.M. ≥ G.M.)

$= \sqrt{10}\cdot\left[\frac{30-(a+b+c)}{3}\right]^{3/2}$

$= \sqrt{10}\cdot\left(\frac{30-20}{3}\right)^{3/2}$

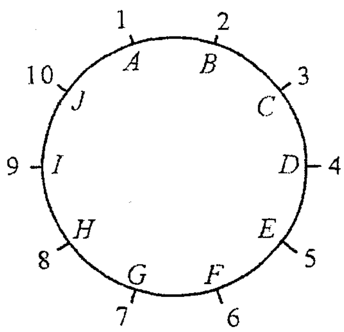
$= \sqrt{10}\cdot\left(\frac{10}{3}\right)^{3/2}$

$= \frac{100}{3\sqrt{3}}$

$k \le \frac{100\sqrt{3}}{9} \approx 19.24 < 20 = k$

which is a contradiction.

G10 In Figure 2, ten people are sitting in a round table with sitting numbers 1, 2, 3, ... , 10 respectively. Each of them chooses an integer A, B, C, ... , J respectively and tells the people on his left and right about his chosen number. Then each of them calculates the average number of the chosen numbers of his two neighborhoods and announces this average numbers are the same as the corresponding sitting numbers, find the value of F.



圖二
Figure 2

$1=\frac{J+B}{2}$ ....(1)

$2=\frac{A+C}{2}$ ......(2)

$3=\frac{B+D}{2}$ ......(3)

$4=\frac{C+E}{2}$ ......(4)

$5=\frac{D+F}{2}$ ......(5)

$6=\frac{E+G}{2}$ ......(6)

$7=\frac{F+H}{2}$ ......(7)

$8=\frac{G+I}{2}$ ....(8)

$9=\frac{H+J}{2}$ ....(9)

$10=\frac{I+A}{2}$ ......(10)

(7)–(9)+(1)–(3)+(5):

$1 = F \Rightarrow F = 1$