Multiple inequalities

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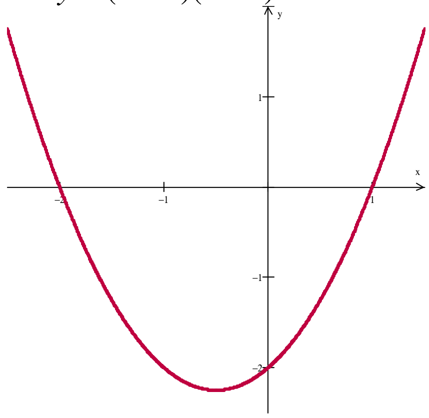
Solve the following inequalities:

1. $(x + 2)(x - 1) \le 0$

Let $y = (x + 2)(x - 1)$



From the graph, $y \le 0$ corresponds to

$x \le -2$ or $1 \le x$

Class work 1

Solve the following quadratic inequalities:

1.1 $(y - 3)(2y + 3) > 0$

1.2 (a) $(1 - z)(z - 4) \ge 0$

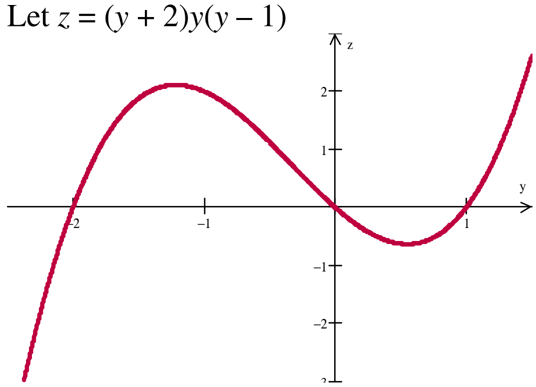
(b) Hence solve $(1 - t^2)(t^2 - 4) \ge 0$

2. $(y + 2)y(1 - y) \le 0$

Multiple by -1 first.

$(y + 2)y(y - 1) \ge 0$

Let $z = (y + 2)y(y - 1)$



From the graph, $z \ge 0$ corresponds to

$-2 \le y \le 0$ or $1 \le y$

Class work 2

Solve the following inequalities:

2.1 $(x - 2)(1 - 2x)(2 - 3x)(3 - 4x) < 0$

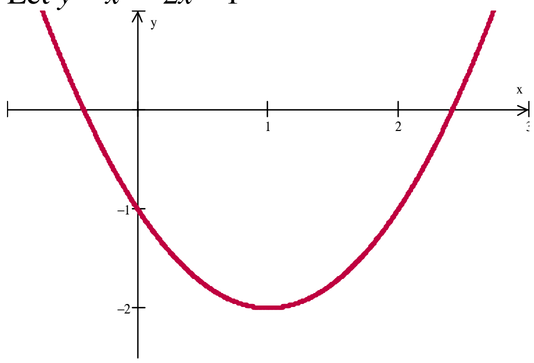
2.2 (a) $(y + 1)(2 - y)(3 - y) > 0$

(b) Hence solve $(\sqrt{t}+1)(2-\sqrt{t})(3-\sqrt{t}) > 0$

3. $x^2 - 2x - 1 < 0$ correct to 2 decimal places.

Let $x^2 - 2x - 1 = 0$, $x = 2.41$ or -0.41 2 d.p.

Let $y = x^2 - 2x - 1$



From the graph, $y < 0$ corresponds to

$-0.41 < x < 2.41$

Class work 3

Solve the following inequalities:

3.1 $a^2 + 5a + 2 > 0$

3.2 $(b - 2)(5 - b^2) > 0$

Correct your answers to 3 significant figures.

4. $x^2 - 2x - 1 > 0$ answer in surd form.

$x^2 - 2x - 1 = 0$, $x = 1-\sqrt{2}$ or $1+\sqrt{2}$

With the same graph as Q3

$x < 1-\sqrt{2}$ or $1+\sqrt{2} < x$

Class work 4

Solve the following inequalities in surd form:

4.1 $x^2 - 4x - 1 \ge 0$

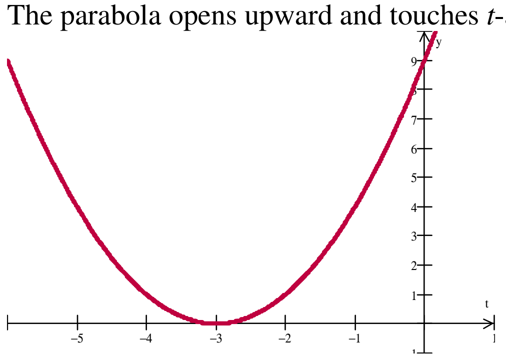
4.2 $1 - 4x - x^2 \ge 0$

4.3 $(1 - 4x - x^2)(x^2 - 4x - 1) \ge 0$

5. $t^2 + 6t + 9 > 0$

Let $y = t^2 + 6t + 9 = (t + 3)^2$

The parabola opens upward and touches t-axis



From the graph, $y > 0$ corresponds to

$t < -3$ or $-3 < t$

Class work 5

Solve the following inequalities:

5.1 $4y^2 - 12y + 9 \le 0$

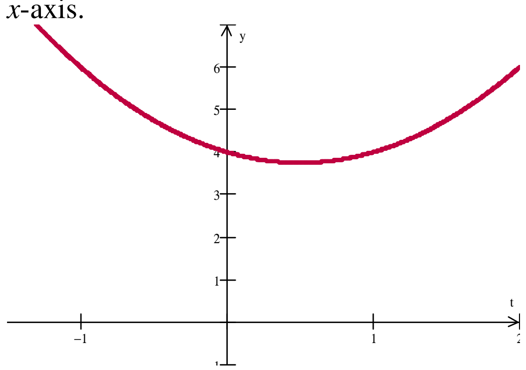
5.2 $(y + 2)(49y^2 - 28y + 4) > 0$

6. $x^2 - x + 4 > 0$

Method 1 Let $y = x^2 - x + 4$

$\Delta = (-1)^2 - 4(1)(4) = -15 < 0$

The parabola opens upward and does not cut the x-axis.



From the graph, $y > 0$ corresponds to

All real numbers of x

Method 2 Completing the squares

$$x^2 - x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 4 > 0$$

$$\left(x - \frac{1}{2}\right)^2 + \frac{15}{4} > 0$$

Always true

x can be any real numbers

Class work 6

Solve the following inequalities:

6.1 $-20 - x^2 + 2x > 0$ by graphical method

6.2 $z^2 - 4z + 9 \ge 0$ by completing the squares

6.3 $(z^2 - 4z + 9)(3z^2 + 6z + 1) > 0$ in surd form.

7. $(x + 2)^{31}x^{28}(x - 3)^{53} < 0$

Note that when $x = -2$, 0 or 3, the inequality does not hold $\therefore$ $x \ne -2$, 0 and 3

$$\frac{(x+2)^{31}x^{28}(x-3)^{53}}{(x+2)^{30}x^{28}(x-3)^{52}} < \frac{0}{(x+2)^{30}x^{28}(x-3)^{52}}$$

$(x + 2)(x - 3) < 0$

(The graph is omitted)

$-2 < x < 3$ and $x \ne 0$

$-2 < x < 0$ or $0 < x < 3$

Class work 7

Solve the following inequalities:

7.1 $(1 - 7t)^{99}(t - 1)^{200}(3 - 2t)^{1002} < 0$

7.2 $(x + 1)^{51}x^{20}(3x^2 + 47x - 2)^{267} < 0$ correct to 3 decimal places

8 $$\frac{(x+2)^{21}}{(x-3)^{97}} \ge 0$$

Note that the inequality does not hold when $x = 3$

$$\frac{(x+2)^{21}}{(x-3)^{97}} \cdot \frac{(x-3)^{98}}{(x+2)^{20}} \ge 0 \cdot \frac{(x-3)^{98}}{(x+2)^{20}}$$

$(x + 2)(x - 3) \ge 0$, $x \ne 3$

(The graph is omitted)

$x \le -2$ or $3 < x$

Class work 8

Solve the following inequalities:

8.1	$\frac{4}{x} \ge 1$	8.5	$\frac{(x^6 - 2x^3 + 1)^3}{x^6 - 1} \le 0$
8.2	$\frac{z-2}{z} \le \frac{z-1}{z}$	8.6	$\frac{(x^3+1)\cdot(x-1)^2}{x^4 - 2x^2 + 1} > 0$
8.3	$\frac{(x+1)^3 \cdot (x-1)^5}{x^{61}} \ge 0$	8.7	$(x + \frac{1}{x})^2 > 2$
8.4	$\frac{x}{x+2} > 1$	8.8	$(x + \frac{1}{x})^2 \ge 6$ in surd form.

1.1 $y < -\frac{3}{2}$ or $3 < y$

1.2 (a) $1 \le z \le 4$

(b) $-2 \le t \le -1$ or $1 \le t \le 2$

2.1 $x < \frac{1}{2}$ or $\frac{2}{3} < x < \frac{3}{4}$ or $2 < x$

2.2 (a) $-1 < y < 2$ or $3 < y$

(b) $0 \le t < 4$ or $9 < t$

3.1 $a < -4.56$ or $-0.438 < a$

3.2 $b < -2.24$ or $2 < b < 2.24$

4.1 $x \le 2 - \sqrt{5}$ or $2 + \sqrt{5} \le x$

4.2 $-2 - \sqrt{5} \le x \le -2 + \sqrt{5}$

4.3 $-2 - \sqrt{5} \le x \le 2 - \sqrt{5}$ or $-2 + \sqrt{5} \le x \le 2 + \sqrt{5}$

5.1 $y = \frac{3}{2}$

5.2 $-2 < y < \frac{2}{7}$ or $\frac{2}{7} < y$

6.1 No solution

6.2 z can be all real numbers

6.3 $z < \frac{-3 - \sqrt{6}}{3}$ or $\frac{-3 + \sqrt{6}}{3} < z$

7.1 $\frac{1}{7} < t < 1$ or $1 < t < \frac{3}{2}$ or $\frac{3}{2} < t$

7.2 $x < -15.709$ or $-1 < x < 0$ or $0 < x < 0.042$

8.1 $0 < x \le 4$

8.2 $z > 0$

8.3 $-1 \le x < 0$ or $1 \le x$

8.4 $x < -2$

8.5 $-1 < x < 1$

8.6 $-1 < x < 1$ or $1 < x$

8.7 $x < 0$ or $x > 0$

8.8 $x \le \frac{-\sqrt{2} - \sqrt{6}}{2}$ or $\frac{\sqrt{2} - \sqrt{6}}{2} \le x < 0$ or $0 < x \le \frac{\sqrt{6} - \sqrt{2}}{2}$ or $x \le \frac{\sqrt{6} + \sqrt{2}}{2}$

1.1 $(y-3)(2y+3) > 0$

$y < -\frac{3}{2}$ or $3 < y$

1.2 (a) $(1-z)(z-4) \ge 0$

$(z-1)(z-4) \le 0$

$1 \le z \le 4$

(b) $(1-t^2)(t^2-4) \ge 0$

Let $t^2 = z$, then $(1-z)(z-4) \ge 0$

By (*a*), $1 \le z \le 4$

$1 \le t^2 \le 4$

$-2 \le t \le -1$ or $1 \le t \le 2$

2.1 $(x-2)(1-2x)(2-3x)(3-4x) < 0$

$(2x-1)(3x-2)(4x-3)(x-2) > 0$

$x < \frac{1}{2}$ or $\frac{2}{3} < x < \frac{3}{4}$ or $2 < x$

2.2 (a) $(y+1)(2-y)(3-y) > 0$

$(y+1)(y-2)(y-3) > 0$

$-1 < y < 2$ or $3 < y$

(b) $(\sqrt{t}+1)(2-\sqrt{t})(3-\sqrt{t}) > 0$

Let $y = \sqrt{t}$, then $(y+1)(2-y)(3-y) > 0$

By (*a*), $-1 < y < 2$ or $3 < y$

$-1 < \sqrt{t} < 2$ or $3 < \sqrt{t}$

$0 \le \sqrt{t} < 2$ or $3 < \sqrt{t}$

$0 \le t < 4$ or $9 < t$

3.1 $a^2 + 5a + 2 > 0$

Let $a^2 + 5a + 2 = 0$

$a = -4.56$ or -0.438 (correct to 3 sig. fig.)

The solution is: $a < -4.56$ or $-0.438 < a$

3.2 $(b-2)(5-b^2) > 0$

$(b+\sqrt{5})(b-2)(b-\sqrt{5}) < 0$

$b < -2.24$ or $2 < b < 2.24$

4.1 $x^2 - 4x - 1 \ge 0$

Let $x^2 - 4x - 1 = 0$

$x = 2-\sqrt{5}$ or $2+\sqrt{5}$

The solution is $x \le 2-\sqrt{5}$ or $2+\sqrt{5} \le x$

4.2 $1 - 4x - x^2 \ge 0$

$x^2 + 4x - 1 \le 0$

Let $x^2 + 4x - 1 = 0$

$x = -2-\sqrt{5}$ or $-2+\sqrt{5}$

The solution is $-2-\sqrt{5} \le x \le -2+\sqrt{5}$

4.3 $(1-4x-x^2)(x^2-4x-1) \ge 0$

$(x+2+\sqrt{5})(x+2-\sqrt{5})(x-2+\sqrt{5})(x-2-\sqrt{5}) \le 0$

$-2-\sqrt{5} \le x \le 2-\sqrt{5}$ or $-2+\sqrt{5} \le x \le 2+\sqrt{5}$

5.1 $4y^2 - 12y + 9 \le 0$

$(2y-3)^2 \le 0$

$y = \frac{3}{2}$

5.2 $(y+2)(49y^2 - 28y + 4) > 0$

$(y+2)(7y-2)^2 > 0$

$\frac{(y+2)(7y-2)^2}{(7y-2)^2} > \frac{0}{(7y-2)^2}$, $y \ne \frac{2}{7}$

$y + 2 > 0$ and $y \ne \frac{2}{7}$

$y > -2$ and $y \ne \frac{2}{7}$

$-2 < y < \frac{2}{7}$ or $\frac{2}{7} < y$

6.1 $-20 - x^2 + 2x > 0$ by graphical method

$x^2 - 2x + 20 < 0$

$\Delta = (-2)^2 - 4(1)(20) = -76 < 0$

The graph opens upwards and does not cut x-axis.

$\therefore$ Always false, no solution

6.2 $z^2 - 4z + 9 \ge 0$ by completing the squares

$z^2 - 4z + 4 + 5 \ge 0$

$(z-2)^2 + 5 \ge 0$

Always true, z can be all real numbers

6.3 $(z^2 - 4z + 9)(3z^2 + 6z + 1) > 0$

Let $3z^2 + 6z + 1 = 0$

$z = \frac{-3-\sqrt{6}}{3}$ or $\frac{-3+\sqrt{6}}{3}$

By 6.2, $z^2 - 4z + 9$ is always positive.

$\frac{(z^2-4z+9)(3z^2+6z+1)}{z^2-4z+9} > \frac{0}{z^2-4z+9}$

$3z^2 + 6z + 1 > 0$

The solution: $z < \frac{-3-\sqrt{6}}{3}$ or $\frac{-3+\sqrt{6}}{3} < z$

7.1 $(1-7t)^{99}(t-1)^{200}(3-2t)^{1002} < 0$

$\frac{(1-7t)^{99}(t-1)^{200}(3-2t)^{1002}}{(1-7t)^{98}(t-1)^{200}(3-2t)^{1002}} < \frac{0}{(1-7t)^{98}(t-1)^{200}(3-2t)^{1002}}$

$1 - 7t < 0$, $t \ne 1$ and $t \ne \frac{3}{2}$

$\frac{1}{7} < t < 1$ or $1 < t < \frac{3}{2}$ or $\frac{3}{2} < t$

7.2 $(x+1)^{51}x^{20}(3x^2 + 47x - 2)^{267} < 0$

When $3x^2 + 47x - 2 = 0$, $x = 0.042, -15.709$ (3 d.p.)

$\frac{(x+1)^{51}x^{20}(3x^2+47x-2)^{267}}{(x+1)^{50}x^{20}(3x^2+47x-2)^{266}} < \frac{0}{(x+1)^{50}x^{20}(3x^2+47x-2)^{266}}$

$(x+1)(3x^2+47x-2) < 0$, $x \ne -1, 0, 0.042$ and -15.709.

$(x < -15.709$ or $-1 < x < 0.042)$ and $x \ne 0$

$\therefore$ $x < -15.709$ or $-1 < x < 0$ or $0 < x < 0.042$

8.1 $\frac{4}{x} \ge 1$, $x \ne 0$

$\frac{4}{x} \cdot x^2 \ge x^2$

$4x \ge x^2$

$0 \ge x^2 - 4x = x(x-4)$

$0 < x \le 4$

8.2 $\frac{z-2}{z} \le \frac{z-1}{z}$, $z \ne 0$

$z^2 \cdot \frac{z-2}{z} \le z^2 \cdot \frac{z-1}{z}$, $z \ne 0$

$z(z-2) \le z(z-1)$, $z \ne 0$

$0 \le z$ and $z \ne 0$

$0 < z$

8.3 $\frac{(x+1)^3 \cdot (x-1)^5}{x^{61}} \ge 0$, $x \ne 0$

$\frac{x^{62}}{(x+1)^2(x-1)^4} \cdot \frac{(x+1)^3 \cdot (x-1)^5}{x^{61}} \ge \frac{x^{62} \cdot 0}{(x+1)^2(x-1)^4}$

$(x+1)x(x-1) \ge 0$ and $x \ne 0$

$-1 \le x < 0$ or $1 \le x$

8.4 $\frac{x}{x+2} > 1$, $x \ne -2$

$(x+2)^2 \cdot \frac{x}{x+2} > (x+2)^2$

$x(x+2) > (x+2)^2$

$0 > 2(x+2)$

$x < -2$

8.5 $\dfrac{\left(x^6-2x^3+1\right)^3}{x^6-1} \le 0$

$\dfrac{\left(x^3-1\right)^6}{\left(x^3-1\right)\left(x^3+1\right)} \le 0$

$\dfrac{\left(x^3-1\right)^2\left(x^3+1\right)^2}{\left(x^3-1\right)^6} \cdot \dfrac{\left(x^3-1\right)^6}{\left(x^3-1\right)\left(x^3+1\right)} \le \dfrac{\left(x^3-1\right)^2\left(x^3+1\right)^2}{\left(x^3-1\right)^6}$

$(x^3 + 1)(x^3 - 1) \le 0$, $x \ne 1$ and $x \ne -1$

$(x + 1)(x^2 - x + 1)\ (x - 1)(x^2 + x + 1) \le 0$

$x^2 - x + 1 = \left(x-\dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$ for all x.

$x^2 + x + 1 = \left(x+\dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$ for all x.

$\dfrac{(x+1)\left(x^2-x+1\right)(x-1)\left(x^2+x+1\right)}{\left(x^2-x+1\right)\left(x^2+x+1\right)} \le \dfrac{0}{\left(x^2-x+1\right)\left(x^2+x+1\right)}$

$(x + 1)(x - 1) \le 0$, $x \ne 1$ and $x \ne -1$

$-1 < x < 1$

8.6 $\dfrac{\left(x^3+1\right)\cdot(x-1)^2}{x^4-2x^2+1} > 0$

$\dfrac{(x+1)\left(x^2-x+1\right)\cdot(x-1)^2}{\left(x^2-1\right)^2} > 0$, $x \ne \pm 1$

$x^2 - x + 1 = \left(x-\dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$ for all x.

$\dfrac{\left(x^2-1\right)^2}{\left(x^2-x+1\right)(x-1)^2} \cdot \dfrac{(x+1)\left(x^2-x+1\right)\cdot(x-1)^2}{\left(x^2-1\right)^2} > \dfrac{0\cdot\left(x^2-1\right)^2}{\left(x^2-x+1\right)(x-1)^2}$

$x + 1 > 0$, $x \ne \pm 1$

$-1 < x < 1$ or $1 < x$

8.7 $(x + \dfrac{1}{x})^2 > 2$, $x \ne 0$

$x^2 + 2 + \dfrac{1}{x^2} > 2$

$x^2 + \dfrac{1}{x^2} > 0$

L.S. = sum of squares

Always true for $x \ne 0$

$x < 0$ or $x > 0$

8.8 $(x + \dfrac{1}{x})^2 \ge 6$, $x \ne 0$

$(x + \dfrac{1}{x})^2 - (\sqrt{6})^2 \ge 0$

$(x + \dfrac{1}{x} + \sqrt{6})(x + \dfrac{1}{x} - \sqrt{6}) \ge 0$

$x(x + \dfrac{1}{x} + \sqrt{6})\cdot x(x + \dfrac{1}{x} - \sqrt{6}) \ge 0\cdot x^2$

$(x^2 + \sqrt{6}\,x + 1)(x^2 - \sqrt{6}\,x + 1) \ge 0$

$(x + \dfrac{\sqrt{2}+\sqrt{6}}{2})(x + \dfrac{-\sqrt{2}+\sqrt{6}}{2})(x + \dfrac{\sqrt{2}-\sqrt{6}}{2})(x - \dfrac{\sqrt{2}+\sqrt{6}}{2}) \ge 0$, $x \ne 0$

$x \le \dfrac{-\sqrt{2}-\sqrt{6}}{2}$ or $\dfrac{\sqrt{2}-\sqrt{6}}{2} \le x < 0$ or $0 < x \le \dfrac{\sqrt{6}-\sqrt{2}}{2}$ or $x \le \dfrac{\sqrt{6}+\sqrt{2}}{2}$