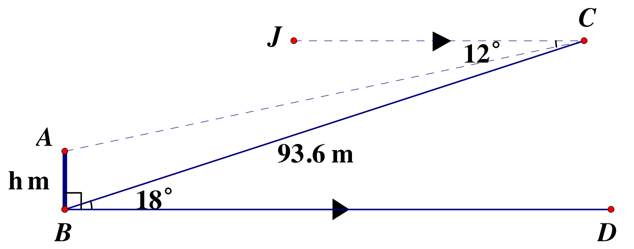
3D notes

Created by Mr. Francis Hung last updated: 2022-08-19

Example 1 A tower is on a hillside which slope at 18° to the horizontal. At a point 93.6 m higher up the hill from the foot of the tower, the angle of depression of the top of the tower is 12°. Find the height of the tower.



Let the tower be $AB = h$ m, with A as the top and B as the foot. BD is the horizontal line.
The slope BC inclines at 18° to BD. $BC = 93.6$ m
JC is another horizontal line. The angle of depression from C to A is $12° = \angle JCA$.
$\angle JCB = 18°$ (alt. $\angle$s $JC \parallel BD$)
$\angle ACB = 18° - 12° = 6°$
$\angle BAC = 90° + 12° = 102°$

By sine rule, $\dfrac{h}{\sin 6^\circ} = \dfrac{93.6}{\sin 102^\circ}$

$h = 10.00$

The height of tower is 10.00 m correct to 2 decimal places.

Example 2 AM 1999 Paper 2 Q11

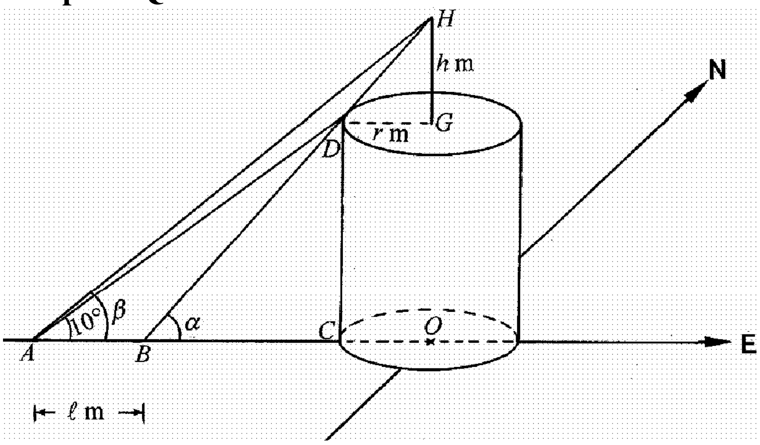


Figure 5 shows a right cylindrical tower with a radius of r m standing on horizontal ground. A vertical pole HG, h m in height, stands at the centre G of the roof of the tower. Let O be the centre of the base of the tower. C is a point on the circumference of the base of the tower due west of O and D is a point on the roof vertically above C. A man stands at a point A due west of O. The angles of elevation of D and H from A are 10° and β respectively. The man walks towards the east to a point B where he can just see the top of the pole H as shown in Figure 5. (Note: If he moves forwards, he can no longer see the pole.) The angle of elevation of H from B is α. Let $AB = \ell$ m.

(a) Show that $AD = \dfrac{\ell \sin\alpha}{\sin(\alpha - 10^\circ)}$ m.

Hence (i) express CD in terms of ℓ and α.

(ii) show that $h = \dfrac{\ell \sin^2\alpha \sin(\beta - 10^\circ)}{\sin(\alpha - 10^\circ)\sin(\alpha - \beta)}$.

(Hint: You may consider ΔADH.)

(b) **In this part, numerical answers should be given correct to two significant figures.**

Suppose $\alpha = 15°$, $\beta = 10.2°$ and $\ell = 97$.

(i) Find
  (1) the height of the pole HG.
  (2) the height and radius of the tower.

(ii) P is a point south-west of O. Another man standing at P can just see the top of the pole H. Find
  (1) the distance of P from O.
  (2) the bearing of B from P.

(a) In ΔABD, $\angle ABD = 180° - \alpha$ (adj. $\angle$s on st. line)

$\angle ADB = \alpha - 10°$ (ext. $\angle$ of Δ)

$\frac{\ell}{\sin \angle ADE} = \frac{AD}{\sin \angle ABD}$ (Sine rule on ΔABD)

$AD = \frac{\ell \sin\alpha}{\sin(\alpha - 10°)}$ m

(i) In ΔACD, $CD = AD \sin 10° = \frac{\ell \sin\alpha \sin 10°}{\sin(\alpha - 10°)}$ m

(ii) In ΔABH, $\angle AHD = \alpha - \beta$ (ext. $\angle$ of Δ)

$\angle HAD = \beta - 10°$

$\frac{DH}{\sin \angle HAD} = \frac{AD}{\sin \angle AHD}$ (Sine rule on ΔADH)

$$\frac{DH}{\sin(\beta - 10°)} = \frac{\frac{\ell \sin\alpha}{\sin(\alpha - 10°)}}{\sin(\alpha - \beta)}$$

$$DH = \frac{\ell \sin\alpha \sin(\beta - 10°)}{\sin(\alpha - 10°)\sin(\alpha - \beta)}$$

$DG \parallel BC$

$\angle HDG = \alpha$ (corr. $\angle$s $DG \parallel BC$)

In ΔAHG, $h = DH \sin\alpha = \frac{\ell \sin^2\alpha \sin(\beta - 10°)}{\sin(\alpha - 10°)\sin(\alpha - \beta)}$

(b) $\alpha = 15°$, $\beta = 10.2°$ and $\ell = 97$

(i) (1) $h = \frac{97 \sin^2 15° \sin(10.2° - 10°)}{\sin(15° - 10°)\sin(15° - 10.2°)} = 3.110027219$

$HG = 3.1$ m (correct to 2 sig. fig.)

(2) $CD = \frac{\ell \sin\alpha \sin 10°}{\sin(\alpha - 10°)}$ m (by (a) (i))

$= \frac{97 \sin 15° \sin 10°}{\sin(15° - 10°)}$ m $= 50.01982714$ m

The height of the tower = 50 m (correct to 2 significant figures)

In ΔDHG, $r = \frac{h}{\tan\alpha} = \frac{3.110027219}{\tan 15°} = 11.60677959$

The radius of the tower = 12 m (correct to 2 significant figures)

(ii) (1) The distance of P from $O = BO = \frac{CD + h}{\tan\alpha} = \frac{50.01982714 + 3.110027219}{\tan 15°}$ m

$= 198.2833159$ m

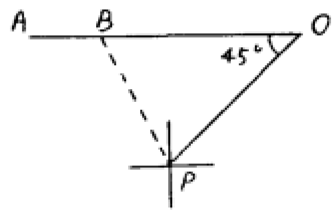
$= 200$ m (correct to 2 sig. fig.)

(2) $\angle BOP = 45°$ and $BO = PO$

ΔBOP is an isosceles triangle

$\angle OBP = \frac{180° - 45°}{2} = 67.5°$ (base $\angle$s isos. Δ)

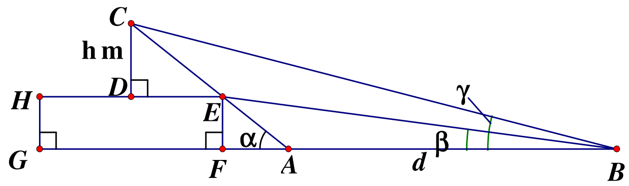
The bearing B from P is N22.5°W.



Class work 1

From a point A, the top of a flagstaff of height h m can just be seen. The flagstaff stands at the centre of a square tower B is a point further away from the tower so that $AB = d$ m and the flagstaff, A, B are in the same vertical plane. If the elevation from A and B of the top of the flagstaff are α and γ respectively, and the elevation of the top of the tower from B is β;

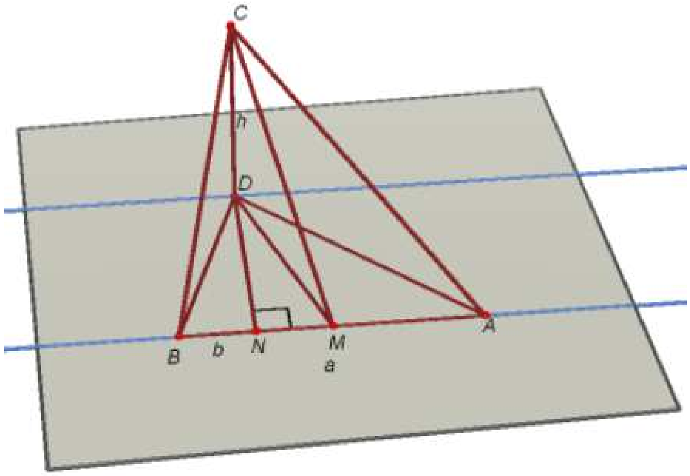
show that $h = \dfrac{d\sin^2\alpha\sin(\gamma-\beta)}{\sin(\alpha-\gamma)\sin(\alpha-\beta)}$.



D is the mid-point of EH

Example 3 The elevation of the top of a vertical mast of height h m on a straight portion of a bank of a river, is 30° from a point A on the opposite bank and a m downstream. From a point B on this bank b m upstream, the elevation is 60°. Prove that $a^2 - b^2 = \frac{8h^2}{3}$.

If $a = 50$, $b = 10$, calculate the height of the mast, the width of the river, and the elevation of the top of the mast from the point mid-way between A and B.



The blue lines in the figure are the river with parallel bank. CD = vertical mast, C = top, D = bottom, $\angle CBD = 60°$, $\angle CAD = 30°$, N is the point nearest to D across the river. $BN = b$, $AN = a$, M is the mid-point of AB

$$BD = \frac{h}{\tan 60^\circ} = \frac{h}{\sqrt{3}}$$

$$AD = \frac{h}{\tan 30^\circ} = \sqrt{3}h$$

Apply Pythagoras' theorem on ΔBDN, ΔADN

$$DN^2 = \left(\frac{h}{\sqrt{3}}\right)^2 - b^2 = \left(\sqrt{3}h\right)^2 - a^2$$

$$a^2 - b^2 = \frac{8h^2}{3}$$

$a = 50$, $b = 10$, $50^2 - 10^2 = \frac{8h^2}{3}$

$h = 30$

The height of the mast is 30 m.

The width of the river $= \sqrt{\left(\sqrt{3}h\right)^2 - a^2}$ m $= \sqrt{\left(\sqrt{3}\times 30\right)^2 - 50^2}$ m $= 10\sqrt{2}$ m = 14.14 m

$$MN = BM - BN = \frac{a+b}{2} - b = \frac{a-b}{2} = \frac{50-10}{2} = 20 \text{ m}$$

$$DM = \sqrt{DN^2 + NM^2} = \sqrt{\left(10\sqrt{2}\right)^2 + 20^2} \text{ m} = 10\sqrt{6} \text{ m}$$

$$\tan \angle CMD = \frac{CD}{DM} = \frac{30}{10\sqrt{6}} = \frac{\sqrt{6}}{2}$$

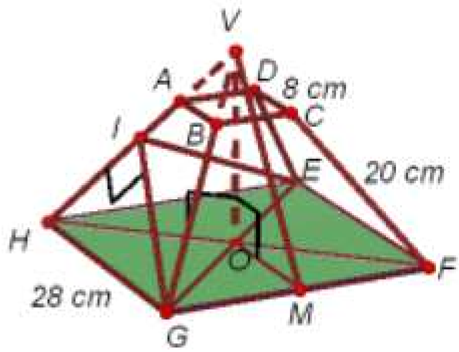
$\angle CMD = 50.8°$

The angle of elevation from M to C = 50.8°

Example 4 A lamp-shade is made of 4 equal panes of glass in the shaped of trapezium. The parallel sides of each pane are 28 cm and 8 cm long and each slant side 20 cm. What is the angle between

(a) each pane and the vertical;

(b) two adjacent panes?



(a) Let the lamp-shade be $ABCDEFGH$ with a virtual vertex V as shown. $ABCD$ is a square with side = 8 cm, $EFGH$ is a square with side = 28 cm and centre at O.

$AH = BG = CF = DE = 20$ cm. The projection of V on the plane $EFGH$ is O.

$EG = FH = \sqrt{28^2 + 28^2}$ cm $= 28\sqrt{2}$ cm

$EO = FO = GO = HO = 14\sqrt{2}$ cm

$\Delta VAB \sim \Delta VHG$ (equiangular)

$\frac{VH}{VA} = \frac{HG}{AB}$ (corr. sides, $\sim\Delta$s)

$\frac{VA+20}{VA} = \frac{28}{8}$

$VA = 8$ cm, similarly, $VB = VC = VD = 8$ cm

$\therefore$ ΔVAB, ΔVBC, ΔVCD, ΔVDA are equilateral triangles with sides = 8 cm

ΔVEF, ΔVFG, ΔVGH, ΔVHE are equilateral triangles with sides = 28 cm

In ΔVOH, $VO \perp HO$, $VH = 28$ cm, $HO = 14\sqrt{2}$ cm, $VO^2 = VH^2 - HO^2$ (Pythagoras' theorem)

$VO = \sqrt{28^2 - \left(14\sqrt{2}\right)^2}$ cm $= 14\sqrt{2}$ cm

Let M be the mid-point of FG. $FM = MG = 14$ cm

$OM = 14$ cm

$\tan\angle VMO = \frac{VO}{OM} = \frac{14\sqrt{2}}{14} = \sqrt{2}$

The angle between each pane and the vertical = $\angle VMO = 54.7°$ (correct to 3 sig. figures)

(b) Let the mid-point of VH be I. Join EG, GI and EI.

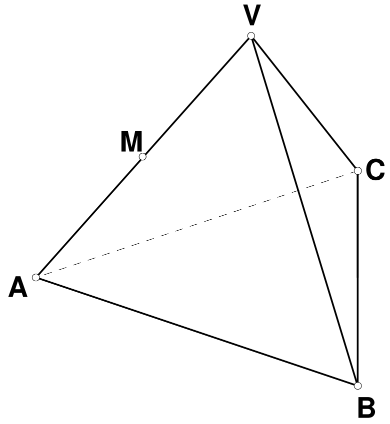
The angle between two adjacent panes is $\angle EIG$.

$EI = GI = 28 \sin 60°$ cm $= 14\sqrt{3}$ cm, $EG = 28\sqrt{2}$ cm

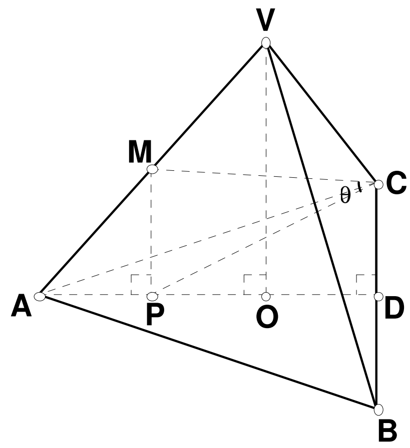
$\cos\angle EIG = \frac{\left(14\sqrt{3}\right)^2 + \left(14\sqrt{3}\right)^2 - \left(28\sqrt{2}\right)^2}{2\left(14\sqrt{3}\right)\left(14\sqrt{3}\right)} = -\frac{1}{3}$

$\angle EIG = 109.47°$

Example 5 In the figure, *VABC* is a regular tetrahedron with side = 2. *M* is the mid-point of *VA*. Find the angle between *CM* and the plane *ABC*.



Solution: Let D be the mid-point of BC.
Then $VM = MA = CD = DB = 1$. $\Delta ACD \cong \Delta ABD$ (SSS)
$\angle ADC = \angle ADB = 90°$ (corr. $\angle$s $\cong$ Δ's)
$AD = \sqrt{2^2 - 1^2} = \sqrt{3}$
$\Delta VMC \cong \Delta AMC$ (SSS)
$\angle VMC = \angle AMC = 90°$ (corr. $\angle$s $\cong$ Δ's)
$CM = \sqrt{2^2 - 1^2} = \sqrt{3}$
O is the orthogonal projection of V on the plane ABC.
$\angle AOV = 90°$. O is the centroid of ΔABC.
$AO = \frac{2}{3} AD = \frac{2\sqrt{3}}{3}$.



P is the projection of M on the plane ABC. $\angle APM = 90°$.
MP // VO (corr. $\angle$s eq.)
$AP = PO = \frac{\sqrt{3}}{3}$ (intercept theorem)

In ΔAPM, $MP^2 = AM^2 - AP^2 = 1^2 - \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{2}{3}$

$\Rightarrow MP = \sqrt{\frac{2}{3}}$

In ΔCPM, $\angle CPM = 90°$. required angle = $\angle PCM = \theta$

$$\sin\theta = \frac{MP}{CM} = \frac{\sqrt{\frac{2}{3}}}{\sqrt{3}} = \frac{\sqrt{2}}{3}$$

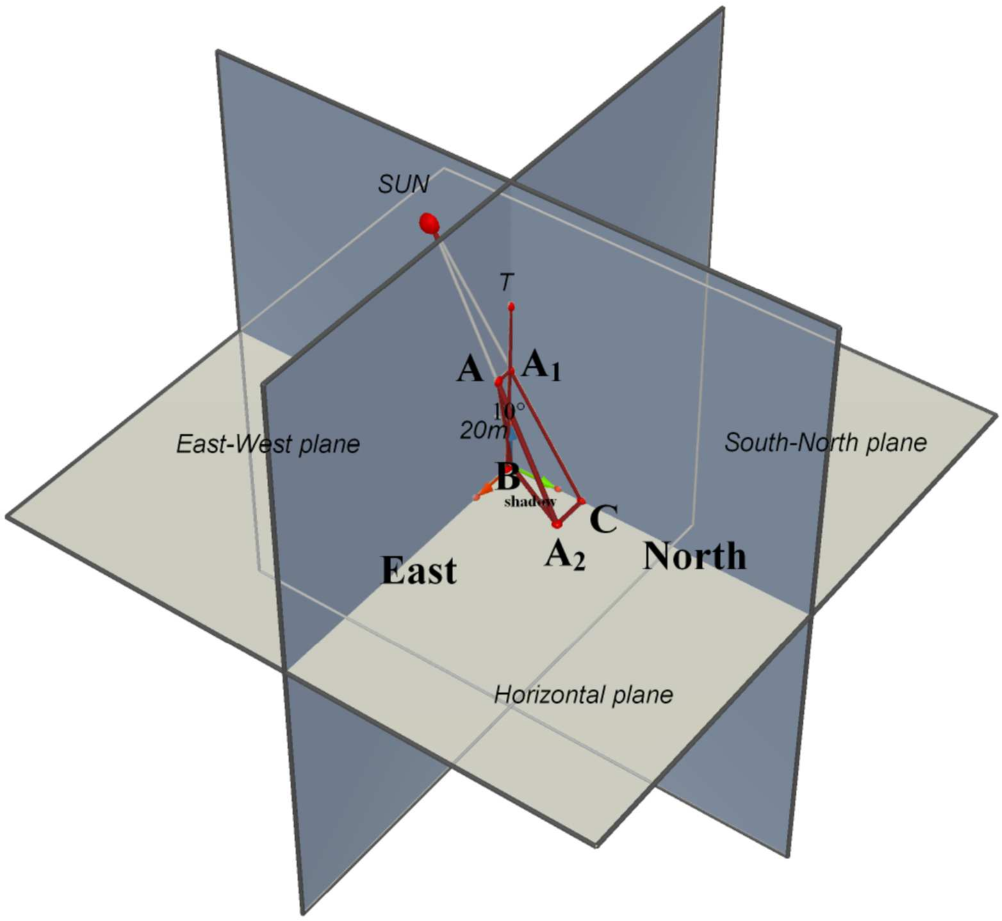
$\theta = 28.1°$

See Example5.cg3.

Example 6

Reference: Certificate Amalgamated Mathematics by W. K. Chow 1981 p.265Q16

A 20-metre pole AB with one end (B) on level ground is inclined at 10° to the vertical towards the East. At noon one day the angle of elevation of the sun was 70° due south of the pole. What was the length of the shadow of the pole at this time?



Construct two imaginary planes which are perpendicular to the horizontal. One plane is across the South-North direction and the other is across the East-West direction.

The foot of the pole, B, is at the intersection of these two imaginary planes.

TB is a vertical line on the intersection of these two planes.

The pole AB lies on the East of the East-West plane such that $\angle ABT = 10°$.

The sun lies on the South of the South-North plane.

A_2B is the shadow of the pole AB on the horizontal ground.

On BT, construct a point A_1 such that AA_1 is parallel to the horizontal, $\angle AA_1B = 90°$.

The sun cast a shadow BC on A_1B with $\angle A_1CB = 70°$, $\angle A_2CB = 90°$.

$A_1B = AB \cos 10° = 20 \cos 10°$ m

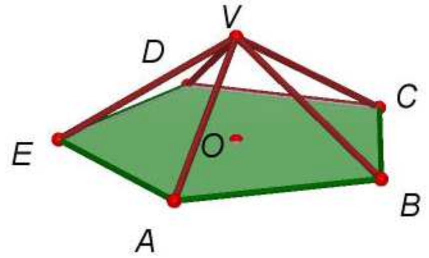
$BC = A_1B \cot 70° = 20 \cos 10° \cot 70°$ m

$A_2C = AA_1 = 20 \sin 10°$ m

$A_2B = \sqrt{A_2C^2 + BC^2} = \sqrt{(20\sin 10°)^2 + (20\cos 10° \cot 70°)^2}$ m = 7.97 m (correct to 3 sig. figures.)

Example 7 In the figure, *VABCDE* is a right pentagonal pyramid with a regular pentagonal base *ABCDE* with side = 2. *V* is the vertex of the pyramid and *O* is the projection of *V* on the base. $VA = VB = VC = VD = VE = 2$. Find the angle between

(a) *VB* and the base,
(b) the plane *VBC* and the base,
(c) the planes *VAB* and *VBC*,
(d) *CB* and the plane *VAB*.



By definition, *O* is the centre of the regular pentagon.

$AB = BC = CD = DE = EA = 2$

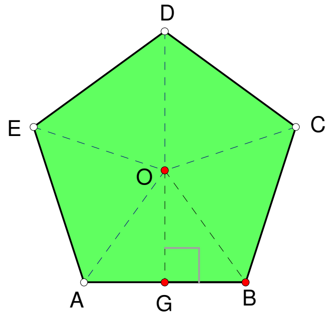
$\angle BAE = \angle ABC = \angle BCD = \angle CDE = \angle AED = 108°$ ($\angle$s sum of polygon)

Join *OA*, *OB*, *OC*, *OD*, *OE*.

$OA = OB = OC = OD = OE$

$\Delta OAB \cong \Delta OBC \cong \Delta OCD \cong \Delta ODE \cong \Delta OEA$ (S.S.S.)

$\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle AOE = 72°$ ($\angle$s at a pt., corr. $\angle$s $\cong$ Δs)



$\angle OAB = \angle OBC$ ($\because$ $OA = OB$, base, $\angle$s isos. Δ)

$$= \frac{180° - \angle AOB}{2} = 54° \text{ (}\angle\text{s sum of }\Delta\text{)}$$

Let *G* be the projection of *O* on *AB*. Then $OG \perp AB$, $\Delta OAG \cong \Delta OBG$ (R.H.S.)

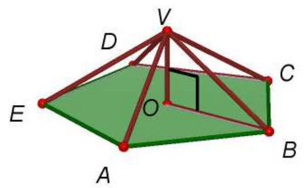
$AG = GB = 1$ (corr. sides $\cong$ Δs), $OG = GB \tan 54° = \tan 54°$, $OA = OB = OC = OD = OE = \sec 54°$

ΔVAB, ΔVBC, ΔVCD, ΔVDE, ΔVAE are congruent equilateral triangles with side = 2.

(a) Join *VO* and *OB*.

$$\cos \angle OBV = \frac{OB}{VB} = \frac{\sec 54°}{2}$$

$\angle OBV = 31.7°$ (correct to 3 sig. fig.)



(b) Let *M* be the mid point of *BC*. $BM = MC = 1$.

$\Delta VBM \cong \Delta VCM$ (S.S.S.)

$\angle BMV = \angle CMV = 90°$ (corr. $\angle$s $\cong$ Δs, adj. $\angle$s on st. line)

By definition, $\angle VOM = 90°$ (*O* is the projection of *V*)

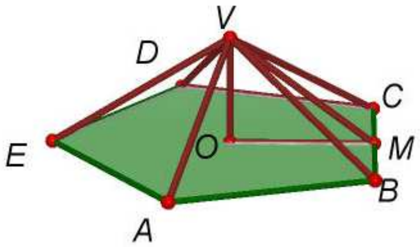
In ΔVBM, $BM^2 + VM^2 = VB^2$ (Pythagoras' theorem)

$\therefore VM = \sqrt{3}$

In ΔOBM, $OM = BM \tan 54° = \tan 54°$

In ΔVOM, $\cos \angle VMO = \dfrac{OM}{VM} = \dfrac{\tan 54°}{\sqrt{3}}$

$\angle VMO = 37.4°$ (correct to 3 sig. fig.)



(c) Let *N* be the mid point of *VB*, $VN = NB = 1$.

Join *AN*, *NC*, *AC*.

$\Delta VAN \cong \Delta BAN$ (S.S.S.)

$\angle VNA = \angle BNA = 90°$ (corr. $\angle$s $\cong$ Δs, adj. $\angle$s on st. line)

$\Delta VCN \cong \Delta BCN$ (S.S.S.)

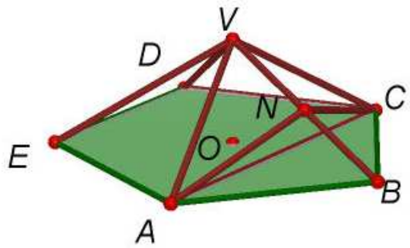
$\angle VNC = \angle BNC = 90°$ (corr. $\angle$s $\cong$ Δs, adj. $\angle$s on st. line)

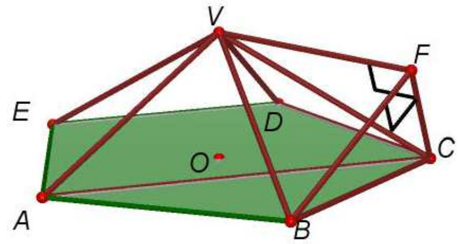
Similar to (*b*), $AN = \sqrt{3} = NC$

In ΔABC, $AC = 2\times 2 \sin 54° = 4 \sin 54°$

In ΔANC, $\sin \dfrac{1}{2}\angle ANC = \dfrac{\frac{1}{2}AC}{AN} = \dfrac{2\sin 54°}{\sqrt{3}}$

$\angle ANC = 138°$ (correct to 3 sig. fig.)





(d) In ΔVOC, $VO^2 = VC^2 - OC^2$ (Pythagoras' theorem)

$VO = \sqrt{4 - \sec^2 54^\circ}$

Area of $\Delta ABC = \frac{1}{2} AB \times BC \sin 108^\circ = 2 \sin 108^\circ$

Volume of the tetrahedron $VABC$

$= \frac{1}{3} \times$ Area of $\Delta ABC \times VO$

$= \frac{1}{3} \times 2 \sin 108^\circ \times \sqrt{4 - \sec^2 54^\circ}$ …… (1)

Let F be the projection of C on the plane VAB.

Area of $\Delta VAB = \frac{1}{2} 2 \times 2 \sin 60^\circ = \sqrt{3}$

Volume of the tetrahedron $VABC$

$= \frac{1}{3} \times$ Area of $\Delta VAB \times CF$

$= \frac{1}{3} \times \sqrt{3} \times CF$ …… (2)

(1) = (2) $\Rightarrow$ $\frac{1}{3} \times 2 \sin 108^\circ \times \sqrt{4 - \sec^2 54^\circ} = \frac{1}{3} \times \sqrt{3} \times CF$

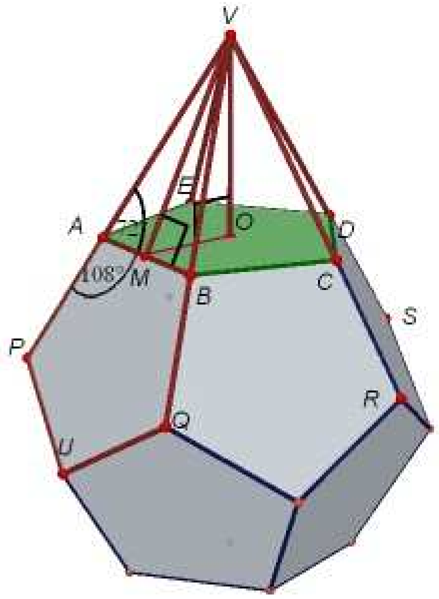
$CF = \frac{2 \sin 108^\circ \sqrt{4 - \sec^2 54^\circ}}{\sqrt{3}}$

In ΔBCF, $\sin \angle CBF = \frac{CF}{BC} = \frac{\sin 108^\circ \sqrt{4 - \sec^2 54^\circ}}{\sqrt{3}}$

$\angle CBF = 35.2^\circ$ (correct to 3 sig. fig.)

See the PowerPoint Example 7

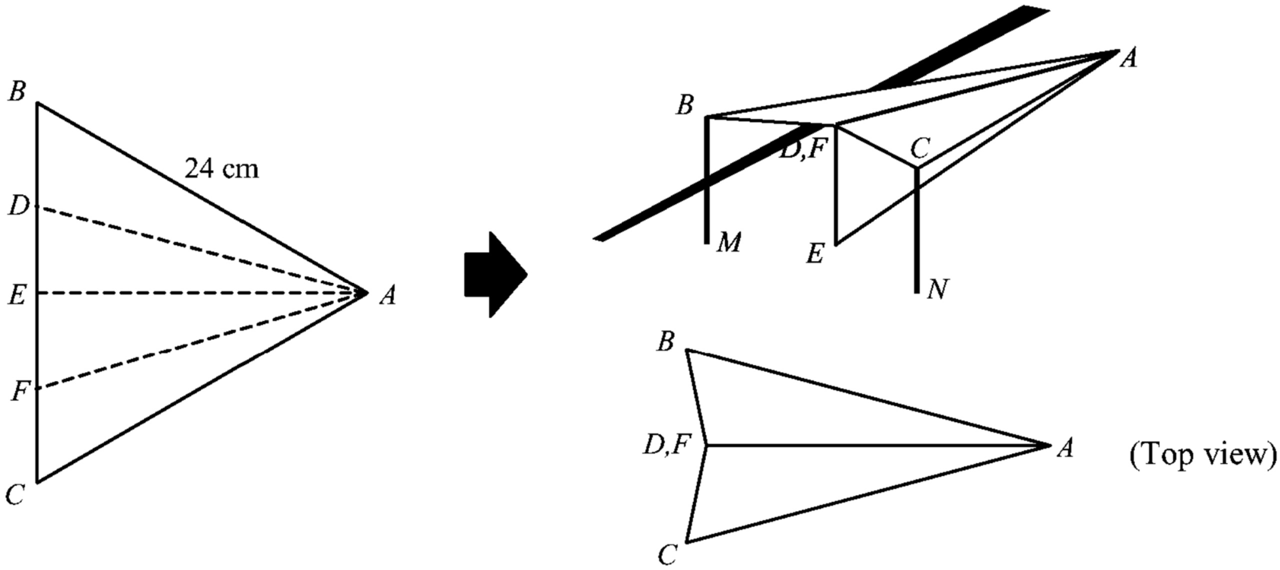
Example 8 The following figure shows a regular dodecahedron with side = 2 (units). $ABCDE$ and $ABQUP$ are two adjacent faces. Find the angle between these two adjacent faces.



$ABCDE$ and $ABQUP$ are regular pentagons with sides = 2.
$\angle PAB = 108°$ ($\angle$s sum of polygon)
Produce PA, QB to meet at V. Join VC, VD, VE.
Then $VABCDE$ is a right pentagonal pyramid with vertex V and a regular pentagonal base $ABCDE$.
$\angle VAB = 72° = \angle VBA$ (adj. $\angle$s on st. line)
$\Delta VAB \cong \Delta VBC \cong \Delta VCD \cong \Delta VDE \cong \Delta VEA$ and $VA = VB = VC = VD = VE$
Let O be the projection of V on the base $ABCDE$. Then O is the centre of $ABCDE$.
Let M be the mid-point of AB. Then $\angle VMB = 90° = \angle VOM = \angle AMO$ and $AM = MB = 1$
In ΔAOM, $\angle BAE = 108°$, $\angle OAM = 54°$, $OM = AM \tan \angle OAM = \tan 54°$
In ΔVAM, $VM = AM \tan \angle VAM = \tan 72°$
In ΔVMO, $\cos \angle VMO = \dfrac{OM}{VM} = \dfrac{\tan 54^\circ}{\tan 72^\circ}$
$\angle VMO = 63.4°$ (correct to 3 significant figures)
The angle between $ABCDE$ and $ABQUP$ is $\angle OMU = 180° - 63.4° = 116.6°$ (adj. $\angle$s on st. line)

Example 9 Mathematics 1999 Q18

In Figure 10, a paper card ABC in the shape of an equilateral triangle of side 24 cm is folded to form a paper aeroplane. D, E and F are points on edge BC so that $BD = DE = EF = FC$. The aeroplane is formed by folding the paper card along the lines AD, AE and AF so that AD and AF coincide. It is supported by two vertical sticks BM and CN of equal length so that A, B, D, F, C lie on the same plane and A, E, M, N lie on the same horizontal ground.



(a) Find the distance between the tips, B and C, of the wings of the aeroplane.
(b) Find the inclination of the wings of the aeroplane to the horizontal ground.
(c) Find the length of the stick CN.

(a) $BD = DE = EF = FC = 6$ cm

$\angle ADB = 60°$ (property of equilateral Δ)

$AD^2 = [24^2 + 6^2 - 2(24)(6) \cos 60°]$ cm^2 = 468 cm^2

$AD = \sqrt{468}$ cm $= AF$

$$\frac{\sin\angle BAD}{6} = \frac{\sin 60^\circ}{\sqrt{468}}$$

$\angle BAD = 13.89788625° = \angle CAF$

BC of the aeroplane = 2(24) sin 13.89788625° cm
= 11.52922707 cm
= 11.5 cm (correct to 3 significant figures)

(b) The inclination of the wings of the aeroplane to the horizontal ground
$= \angle DAE$ of the aeroplane
$= \angle BAE - \angle BAD$
$= 30° - 13.89788625°$
$= 16.10211375° = 16.1°$ (correct to 3 significant figures)

(c) Let H he the mid-point of BC of the aeroplane.
Then $AH \perp BC$.
$AH = AB \cos \angle BAD = 24 \cos 13.89788625°$ cm
= 23.297440824 cm
Let K be the foot of perpendicular from H to the ground.
Then it is easy to show that $\Delta ADE \sim \Delta AHK$.
$HK = AH \cos\angle DAE$
= 23.297440824 sin 16.10211375° cm
= 6.461538462 cm $= CN$
$CN = 6.46$ cm (correct to 3 significant figures)

See Example9.cg3.

3D-problem (vector)

Example 10 Let $O(0, 0, 0)$, $A(1, 0, 0)$, $B(x, y, 0)$ be three points on the x-y plane, where $x, y > 0$.

(a) If OAB is an equilateral triangle, find the coordinates of B.

(b) If G is the centroid of ΔOAB., find the coordinates of G.

(c) Let $C(x', y', z')$ be a point such that $z' > 0$. If $OABC$ is a regular tetrahedron, find the coordinates of C.

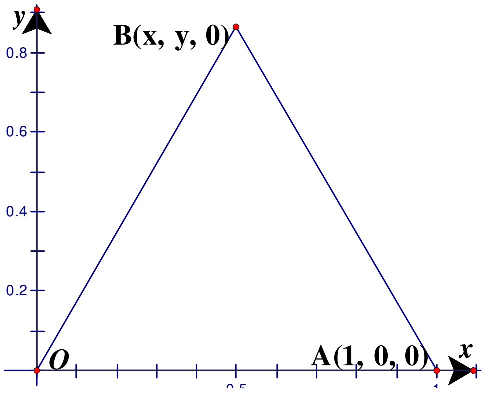
(d) Find the shortest distance between OA and BC.

(a) $OA = 1 = OB$

$\angle AOB = 60°$

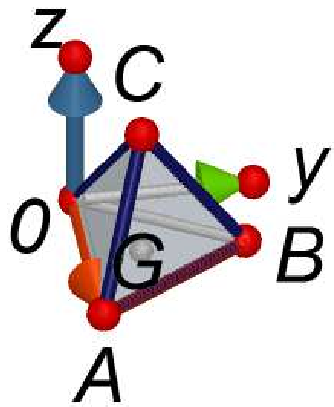
$B = (\cos 60°, \sin 60°, 0)$

$= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$



(b) $G = \left(\frac{0+1+\frac{1}{2}}{3}, \frac{0+0+\frac{\sqrt{3}}{2}}{3}, \frac{0+0+0}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, 0\right)$

(c) $CG \perp$ x-y plane



$C = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, z'\right)$

$OC = 1 \Rightarrow \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{6}\right)^2 + (z')^2} = 1$

$\frac{1}{4} + \frac{1}{12} + z'^2 = 1 \Rightarrow z' = \sqrt{\frac{2}{3}}$

$C = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}\right)$

(d) Let M be the mid-point of OA and N be the mid-point of BC.

$M = \left(\frac{1}{2}, 0, 0\right)$, $N = \left(\frac{\frac{1}{2}+\frac{1}{2}}{2}, \frac{\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{6}}{2}, \frac{0+\frac{\sqrt{6}}{3}}{2}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{6}\right)$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{6}}{6}\mathbf{k}$; $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -\frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{6}}{3}\mathbf{k}$

$\overrightarrow{OA} \cdot \overrightarrow{MN} = \mathbf{i} \cdot \left(\frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{6}}{6}\mathbf{k}\right) = 0$

$\overrightarrow{MN} \cdot \overrightarrow{BC} = \left(\frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{6}}{6}\mathbf{k}\right) \cdot \left(-\frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{6}}{3}\mathbf{k}\right) = -\frac{1}{3} + \frac{1}{3} = 0$

$\therefore MN \perp OA$ and $MN \perp BC$

$MN = \sqrt{\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{6}\right)^2} = \sqrt{\frac{1}{3} + \frac{1}{6}} = \sqrt{\frac{1}{2}}$

The shortest distance between OA and BC is $\frac{\sqrt{2}}{2}$.

Q2 (M2 PP Q12 modified)

$O(0, 0, 0)$, $A(1, 0, 0)$, $B(0, 1, 0)$, $C(1, 1, 1)$. Find the shortest distance between AB and OC.

$\overrightarrow{OA} = \mathbf{i}$, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\mathbf{i} + \mathbf{j}$

$\overrightarrow{OC} \times \overrightarrow{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ = normal vector perpendicular to OC and AB

Let the angle between OA and $\overrightarrow{OC} \times \overrightarrow{AB}$ be θ.

Then the projection of OA on $\overrightarrow{OC} \times \overrightarrow{AB}$ is

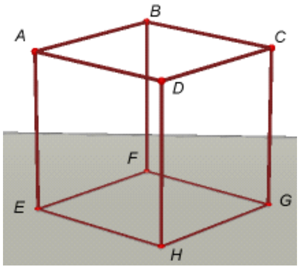
$$OA \cos\theta = \frac{\left|\overrightarrow{OA} \cdot \left(\overrightarrow{OC} \times \overrightarrow{AB}\right)\right|}{\left|\overrightarrow{OC} \times \overrightarrow{AB}\right|}$$

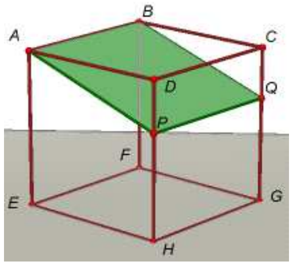
$$= \frac{\left|\mathbf{i} \cdot (-\mathbf{i} - \mathbf{j} + 2\mathbf{k})\right|}{\sqrt{(-1)^2 + 1^2 + 2^2}}$$

$$= \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

$\therefore$ The shortest distance between AB and OC is $\frac{\sqrt{6}}{6}$.

In the figure, *ABCDHGFE* is a rectangular cuboid lying on a horizontal ground.





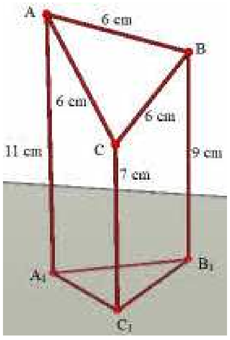
P, *Q* are points lying on the edges *DH* and *CG* respectively such that *DP* = *CQ*.
Let the angle between the inclined plane *APQB* and the base *EFGH* be θ.
Then θ = ∠*PAD*.

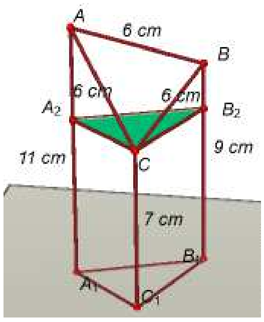
$$\cos\theta = \frac{AD}{AP} = \frac{AD \times AB}{AP \times AB} = \frac{\text{Area of ABCD}}{\text{Area of APQB}} = \frac{\text{Area of shadow}}{\text{Area of object}}$$

Example 11

An equilateral triangle of sides 6 cm has its vertices at heights 7 cm, 9 cm, 11 cm above a horizontal plane. Find the dimensions of the projection of the triangle on the horizontal and the inclination of the equilateral triangle to the horizontal.

Solution.





Assume the equilateral triangle is *ABC* and its projection is $A_1B_1C_1$.
By Pythagoras' theorem, the dimensions of the projection of *ABC* are

$A_1B_1 = \sqrt{6^2 - 2^2} = \sqrt{32}$

$A_1C_1 = \sqrt{6^2 - 4^2} = \sqrt{20}$

$B_1C_1 = \sqrt{6^2 - (4-2)^2} = \sqrt{32}$

By Heron's formula, area of $A_1B_1C_1 = \sqrt{s(s-a)(s-b)(s-c)}$

$s = \dfrac{\sqrt{32} + \sqrt{20} + \sqrt{32}}{2} = 7.89$

$s - a = s - c = \sqrt{5}$

$s - b = 3.42$

Area of $A_1B_1C_1 = \sqrt{135}$

By direct calculation, area of $ABC = \dfrac{1}{2} \cdot 6^2 \cdot \sin 60^\circ = 9\sqrt{3}$

If θ is the inclination of *ABC*,

$\cos\theta = \dfrac{\sqrt{135}}{9\sqrt{3}} = \dfrac{\sqrt{5}}{3}$

$\theta = 41.8°$

Example 12 3D Mathematics 1989

Answers in this question should be given correct to at least 3 significant figures or in surd form.

In figure 3, a triangular board ABC, right-angled at A with $AB = AC = 10$ m, is placed with the vertex A on the horizontal ground. AB and AC make angles 45° and 30° with the horizontal respectively. The sun casts a shadow $AB'C'$ of the board on the ground such that B' and C' are vertically below B and C respectively.

(a) Find the length of AB' and AC'.

(b) Find the lengths of BC, BB' and CC'.

(c) Using the results of (b), or otherwise, find the length of $B'C'$.

(d) Find $\angle B'AC'$. Hence find the area of the shadow.

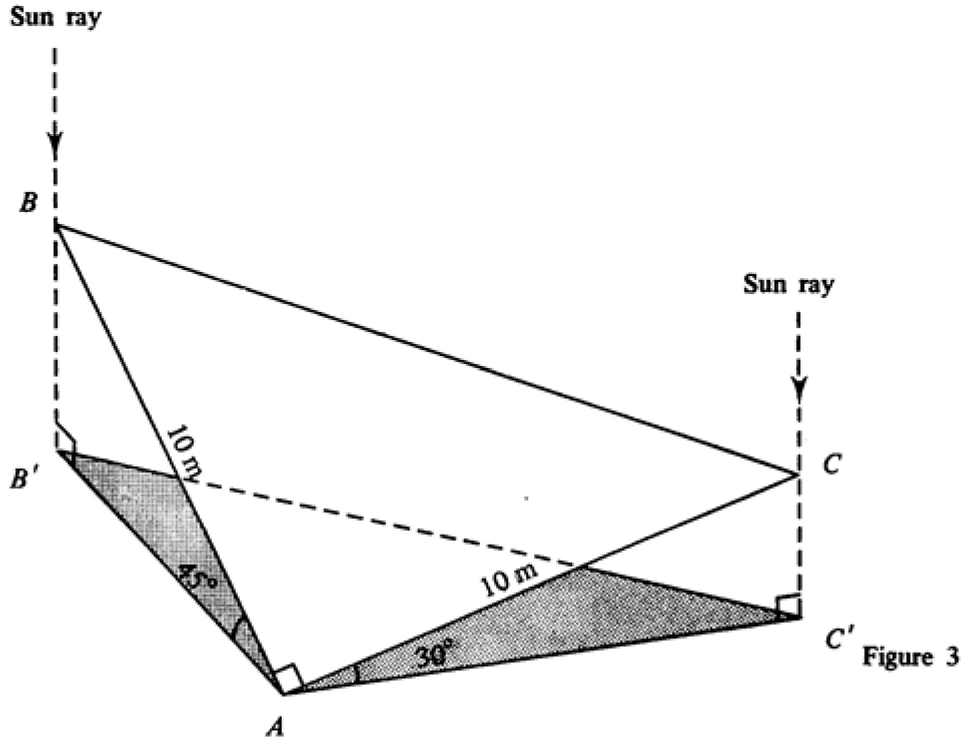


Figure 3

(a) $AB' = AB \cos 45° = 10 \cos 45°$ m $= 5\sqrt{2}$ m

$AC' = AC \cos 30° = 10 \cos 30°$ m $= 5\sqrt{3}$ m

(b) $BC^2 = AB^2 + AC^2 = 10^2 + 10^2$ (Pythagoras' theorem)

$BC = 10\sqrt{2}$ m

$BB' = AB \sin 45° = 10 \sin 45°$ m $= 5\sqrt{2}$ m

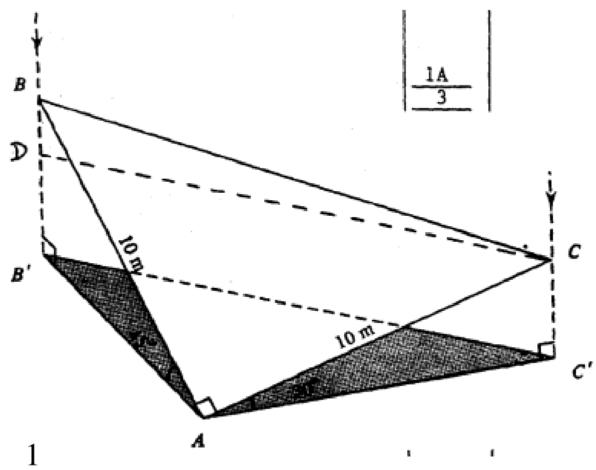
$CC' = AC \sin 30° = 10 \sin 30°$ m $= 5$ m

(c) Let D be the foot of perpendicular from C to BB'.

$BD = \left(5\sqrt{2} - 5\right)$ m

$B'C' = CD = \sqrt{\left(10\sqrt{2}\right)^2 - \left(5\sqrt{2} - 5\right)^2}$ m

$= \sqrt{125 + 50\sqrt{2}}$ m $= 13.9897$ m



(d) By cosine rule, $\cos\angle B'AC' = \dfrac{50 + 75 - \left(125 + 50\sqrt{2}\right)}{2\times 5\sqrt{2}\times 5\sqrt{3}} = -\dfrac{1}{\sqrt{3}}$

$\angle B'AC' = 125.264°$

Area of the shadow $= \dfrac{1}{2}\cdot\left(5\sqrt{2}\right)\cdot\left(5\sqrt{3}\right)\sin 125.264° = 25$ m^2

Example 13 AM 2003 Q18

18. (a) Figure 4 shows a tetrahedron $OPQR$ with RO perpendicular to the plane OPQ. Let θ be the angle between the planes RPQ and OPQ. Show that $\dfrac{\text{Area of } \Delta\text{OPQ}}{\text{Area of } \Delta\text{RPQ}} = \cos\theta$

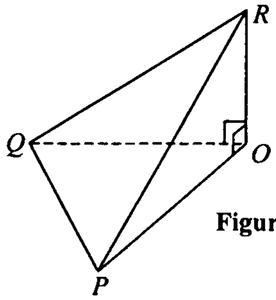


Figure 4

(b) In figure 5, a pole of length 2 m is erected vertically at a point E on the horizontal ground. A triangular board ABC of area 12 m^2 is supported by the pole such that side AB touches the ground and vertex C is fastened to the top of the pole. AB = 6 m, $BC = x$ m and $CA = y$ m, where $6 > x > y$. The sun rays are vertical and cast a shadow of the board on the ground.

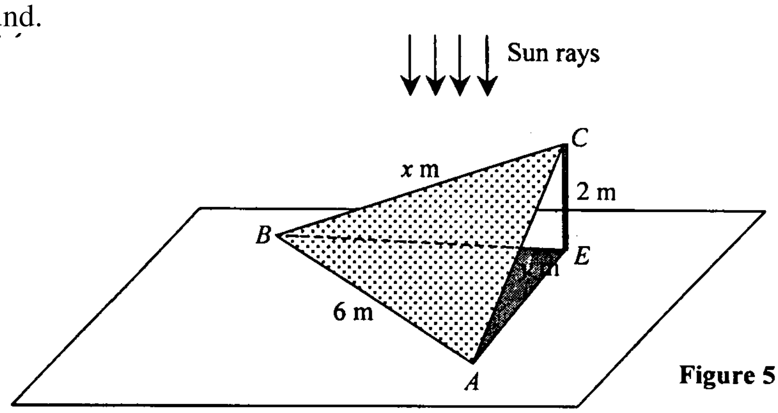


Figure 5

(i) Find the area of the shadow.

(ii) Two other ways of supporting the board with the pole are to fasten vertex A or B to the top of the pole with the opposite touching the ground. Among these three ways, determine which one will give the largest shadow.

(a) Let M be the point on QP such that $MR \perp QP$ and $MO \perp QP$.

$$\frac{\text{Area of } \Delta\text{OPQ}}{\text{Area of } \Delta\text{RPQ}} = \frac{\frac{1}{2}QP \times MR}{\frac{1}{2}QP \times MO}$$

$$= \frac{MR}{MO} = \cos\theta$$

(b) (i) Let M be the point on AB such that $CM \perp AB$ and $EM \perp AB$.

Area of ΔABC = 12 m^2

$\frac{1}{2} \times 6 \times CM = 12$

$CM = 4$

$ME^2 + 2^2 = 4^2$

$ME = 2\sqrt{3}$

Area of $\Delta ABE = \frac{1}{2} \times 6 \times 2\sqrt{3} = 6\sqrt{3}$ m^2

(ii) Let N be the point on BC such that $AN \perp BC$.
Let L be the point on AC such that $BL \perp AC$.
Area of ΔABC = 12 m^2

$\Rightarrow \frac{1}{2}\times x\times AN = \frac{1}{2}\times y\times BL = 12$

$AN = \frac{24}{x}$, $BL = \frac{24}{y}$; $CM = 4$

$6 > x > y \Rightarrow \frac{24}{6} < \frac{24}{x} < \frac{24}{y}$

$\Rightarrow CM < AN < BL$

If the pole is to fasten vertex A, let the inclination be α .
If the pole is to fasten vertex B, let the inclination be β .

$\sin\theta = \frac{2}{CM}$; $\sin\alpha = \frac{2}{AN}$; $\sin\beta = \frac{2}{BL}$

$CM < AN < BL \Rightarrow \sin\beta < \sin\alpha < \sin\theta \Rightarrow \beta < \alpha < \theta$

$\cos\beta > \cos\alpha > \cos\theta$

Area of shadow will be largest when the pole is to fasten vertex B .

Example 14

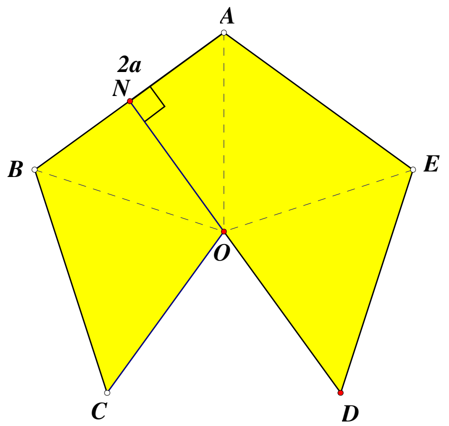


Figure A

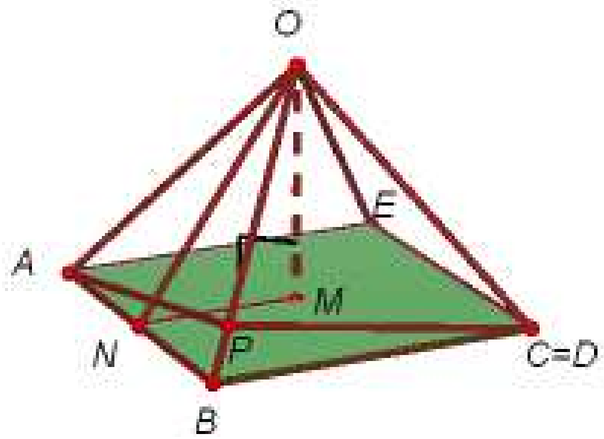


Figure B

In Figure A, $ABCDE$ is a regular pentagon with length of a side = $2a$. O is the centre of the pentagon. ΔOCD is cut off from the pentagon. The remaining hexagon is folded along OB, OA and OE. A right pyramid with square base is formed by sticking OC and OD together in Figure B.

M is the centre of the squared base $ABCE$ of the pyramid in Figure B. N is the mid-point of AB.

(a) Find OA and ON.

(b) Find OM.

(c) Find the volume of the pyramid.

(d) Find the angle between the faces OAB and OBC in Figure B.

(a) $\angle BAO = \angle ABC = 108°$ ($\angle$ sum of polygon)

$OA = OB = OC = OD = OE$

$\angle OAB = \angle OBA = 54°$

$AN = NB = a$

$\Delta OAB \cong \Delta OBN$ (S.A.S.)

In ΔOAN, $OA = a \sec 54°$

$ON = a \tan 54°$

(b) In Figure B, $BC = 2a$, $NM = a$

In ΔOMN, $OM^2 + NM^2 = ON^2$ (Pythagoras' theorem)

$OM^2 = (a \tan 54°)^2 - a^2 = a^2(\tan^2 54° - 1)$

$OM = a\sqrt{\tan^2 54° - 1}$

Let $\beta = 54°$, then $5\beta = 270°$, $3\beta = 270° - 2\beta$, let $\tan \beta = t$

$\tan 3\beta = \tan(270° - 2\beta) = \cot 2\beta$

$$\frac{3t - t^3}{1 - 3t^2} = \frac{1 - t^2}{2t}$$

$6t^2 - 2t^4 = 1 - t^2 - 3t^2 + 3t^4$

$5t^4 - 10t^2 + 1 = 0$

$$t^2 = \frac{5+\sqrt{20}}{5} = \frac{5+2\sqrt{5}}{5} \quad \text{or} \quad \frac{5-\sqrt{20}}{5} = \frac{5-2\sqrt{5}}{5}$$

$\tan 54° > \tan 45° = 1$

$$t^2 = \frac{5+2\sqrt{5}}{5} \quad \text{only}$$

$$OM = a\sqrt{\frac{5+2\sqrt{5}}{5} - 1} = a\sqrt{\frac{2}{\sqrt{5}}}$$

(c) Volume of the pyramid $= \frac{1}{3}(2a)^2 a\sqrt{\frac{2}{\sqrt{5}}} = \frac{4a^3}{3}\sqrt{\frac{2}{\sqrt{5}}}$

(d) Let P be a point on OB (in Figure B) such that $AP \perp OB$ and $CP \perp OB$.

$AC = \sqrt{2}AB = 2\sqrt{2}a$ (in Figure B)

In ΔABP, $\angle OBA = 54°$

$AP = 2a \sin 54°$

angle between the faces OAB and $OBC = \angle APC$ (in Figure B)

$$\cos \angle APC = \frac{AP^2 + CP^2 - AC^2}{2AP \cdot CP}$$

$$= \frac{(2a\sin 54^\circ)^2 + (2a\sin 54^\circ)^2 - (2\sqrt{2}a)^2}{2(2a\sin 54^\circ)}$$

$$= \frac{2(\sin 54^\circ)^2 - (\sqrt{2})^2}{2(\sin 54^\circ)^2}$$

$$= \frac{(\sin 54^\circ)^2 - 1}{(\sin 54^\circ)^2}$$

$$= \frac{-\cos^2 54^\circ}{\sin^2 54^\circ}$$

$$= -\cot^2 54°$$

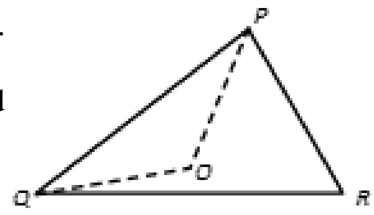
$\angle APC = 121.86°$

Example 15 (CLSMSS 2020-21 S.6 Mock Exam. Paper 1 Q19)

(a) The figure shows a triangle PQR. O is the circumcentre of ΔPQR. Let M be the mid-point of PQ.

(i) By expressing $\angle POQ$, express OP in terms of PQ and $\angle PRQ$.

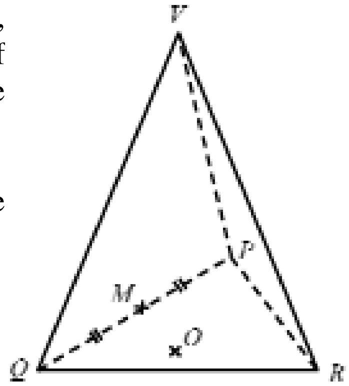
(ii) Let A be the area of ΔPQR. Show that $OP = \dfrac{PQ \cdot QR \cdot RP}{4A}$.



(b) $VPQR$ is a tetrahedron with ΔPQR described in (a) as the base, where $PQ = 7$ cm, $QR = 8$ cm and $RP = 5$ cm. O is the foot of perpendicular from V to the plane PQR and the volume of the tetrahedron $VPQR$ is 80 cm^3.

(i) Find VP.

(ii) By finding VM, or otherwise, evaluate the angle between the side QR and the plane VPQ .



(a) (i) Construct the circumsribed circle PQR. Join OM.

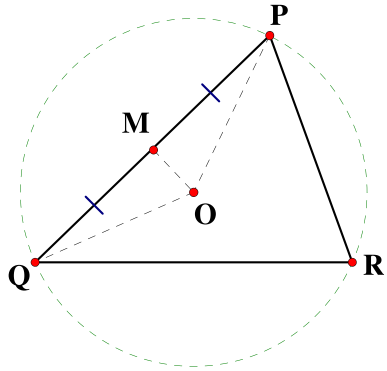
$\Delta POM \cong \Delta QOM$ (S.S.S.)

$\angle POM = \angle QOM$ (corr. $\angle$s, $\cong \Delta$s)

$\angle PRQ = \dfrac{1}{2}\angle POQ$ ($\angle$ at centre twice $\angle$ at $\odot^{\text{ce}}$)

$= \angle POM$

$OM \perp PQ$ (line joining centre to mid-point of chord $\perp$ chord)



In ΔPOM, $\sin \angle POM = \dfrac{PM}{OP}$

$\therefore \sin \angle PRQ = \dfrac{\frac{1}{2}PQ}{OP}$

$OP = \dfrac{PQ}{2\sin \angle PRQ}$

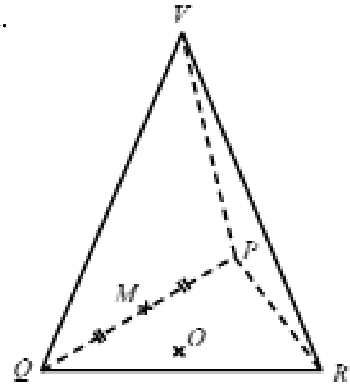
(ii) $\sin \angle PRQ = \dfrac{PQ}{2OP}$ $\cdots$ (*)

$A = \dfrac{1}{2} RP \cdot QR \sin \angle PRQ$

$= \dfrac{1}{2} RP \cdot QR \cdot \dfrac{PQ}{2OP}$ by (*)

$\therefore OP = \dfrac{PQ \cdot QR \cdot RP}{4A}$.

(b) (i) First, we find the area of ΔPQR by Heron's formula.



$s=\frac{1}{2}(5+7+8)=10$ cm

$A=\sqrt{10(10-5)(10-7)(10-8)}=10\sqrt{3}$ cm^2

Volume of the tetrahedron $=\frac{1}{3}A\cdot VO=80\text{ cm}^3$

$\frac{1}{3}\cdot 10\sqrt{3}\cdot VO=80\Rightarrow VO=8\sqrt{3}\text{ cm}$

By (a)(ii), $OP=\frac{PQ\cdot QR\cdot RP}{4A}=\frac{5\cdot 7\cdot 8}{4(10\sqrt{3})}=\frac{7}{\sqrt{3}}$ cm

In ΔVOP, $VO^2+OP^2=VP^2$ (Pythagoras' theorem)

$VP=\sqrt{(8\sqrt{3})^2+\left(\frac{7}{\sqrt{3}}\right)^2}=\sqrt{\frac{625}{3}}=\frac{25\sqrt{3}}{3}$ cm

(ii)

$OP=OQ$	(radii of the circumscribed circle)
$VO=VO$	(common side)
$\angle VOP=\angle VOQ=90^\circ$	(given)
$\Delta VOP\cong\Delta VOQ$	(S.A.S.)
$VP=VQ$	(corr. sides, $\cong\Delta$s)
$VM=VM$	(common sides)
$PM=MQ$	(given that M is the mid-point of PQ)
$\Delta VMP\cong\Delta VMQ$	(S.S.S.)
$\angle VMP=\angle VMQ$	(corr. $\angle$s, $\cong\Delta$s)
$\angle VMP+\angle VMQ=180^\circ$	(adj. $\angle$s on st. line)

$\therefore\ \angle VMP=90^\circ$

In ΔVMP, $VM^2+MP^2=VP^2$ (Pythagoras' theorem)

$VM=\sqrt{\frac{625}{3}-\left(\frac{7}{2}\right)^2}=\sqrt{\frac{2353}{12}}$

$\sin\angle VPM=\frac{VM}{VP}=\sqrt{\frac{2353}{12}}\div\sqrt{\frac{625}{3}}=\frac{\sqrt{2353}}{50}$

Area of $\Delta VPQ=\frac{1}{2}\cdot VP\cdot PQ\cdot\sin\angle VPQ$

$=\frac{1}{2}\cdot\frac{25}{\sqrt{3}}\cdot 7\cdot\frac{\sqrt{2353}}{50}=\frac{7\sqrt{2353}}{4\sqrt{3}}$

Let the perpendicular distance from R to the plane VPQ be h cm. Then

Volume of the tetrahedron $VPQR=\frac{1}{3}\cdot$ Area of $\Delta VPQ\cdot h=80$

$\frac{1}{3}\cdot\frac{7\sqrt{2353}}{4\sqrt{3}}\cdot h=80$

$h=\frac{960\sqrt{3}}{7\sqrt{2353}}$

The angle bewteen QR and the plane VPQ is θ.

$\sin\theta=\frac{h}{QR}=\frac{960\sqrt{3}}{7\sqrt{2353}}\div 8=\frac{120\sqrt{3}}{7\sqrt{2353}}=0.612114785$

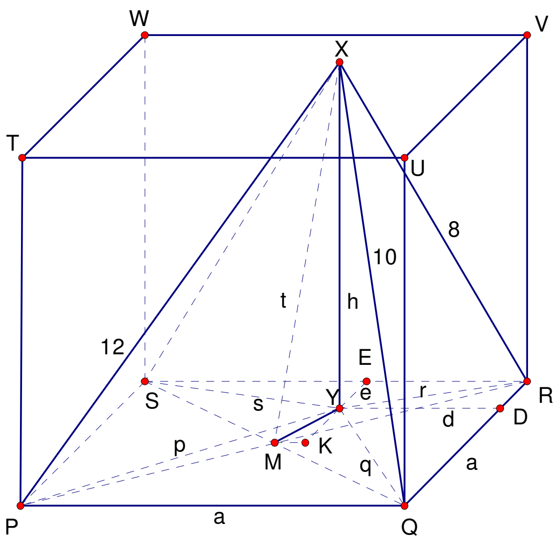
$\theta=37.74257329^\circ\approx 37.7^\circ$ (correct to 3 significant figures)

Example 14 Canadian Mathematics Competition 2015 Cayley Contest (Grade 10) Q25

Rectangular prism $PQRSWTUV$ has a square base $PQRS$. Point X is on the face $TUVW$ so that $PX = 12$, $QX = 10$ and $RX = 8$. The maximum possible area of rectangle $PQUT$ is closest to

(A) 67.84 (B) 67.82 (C) 67.90

(D) 67.86 (E) 67.88



Suppose the diagonals PR and QS intersect at M. Let $PQ = QR = RS = SP = a$.

Then $PR = QS = \sqrt{2}a$

M is the centre of the square.

$PM = MR = QM = MS = \frac{\sqrt{2}a}{2}$ (diagonals of a square)

Let the projection of X on the plane $PQRS$ be Y. (Y lies inside ΔSMR)

XY is the height of the prism. Then XY is perpendicular to the plane $PQRS$.

Let $PY = p$, $QY = q$, $RY = r$, $SY = s$, $XY = h$.

Let $\angle PMX = \theta$, $\angle SMX = \alpha$, $XM = t$.

Apply cosine rule on ΔPXM, $12^2 = \left(\frac{\sqrt{2}a}{2}\right)^2 + t^2 - 2\cdot\frac{\sqrt{2}}{2}at\cos\theta$ …… (1)

$\angle RMX = 180° - \theta$ (adj. $\angle$s on st. line)

Apply cosine rule on ΔRXM, $8^2 = \left(\frac{\sqrt{2}a}{2}\right)^2 + t^2 - 2\cdot\frac{\sqrt{2}}{2}at\cos(180^\circ - \theta)$ …… (2)

Using the fact that $\cos(180° - \theta) = -\cos\theta$

(1) + (2): $208 = a^2 + 2t^2$ …… (3)

$a^2 - 2\sqrt{2}at + 2t^2 + 2\sqrt{2}at = 208$

$(a - \sqrt{2}t)^2 + 2\sqrt{2}at = 208$

$2\sqrt{2}at = 208 - (a - \sqrt{2}t)^2$

$2\sqrt{2}at \le 208$

$a^2t^2 \le \left(\frac{104}{\sqrt{2}}\right)^2 = 5408$ …… (*)

Apply cosine rule on ΔSXM, $SX^2 = \left(\frac{\sqrt{2}a}{2}\right)^2 + t^2 - 2\cdot\frac{\sqrt{2}}{2}at\cos\alpha$ …… (4)

$\angle QMX = 180° - \alpha$ (adj. $\angle$s on st. line)

Apply cosine rule on ΔQXM, $10^2 = \left(\frac{\sqrt{2}a}{2}\right)^2 + t^2 - 2\cdot\frac{\sqrt{2}}{2}at\cos(180^\circ - \alpha)$ …… (5)

Using the fact that $\cos(180° - \alpha) = -\cos\alpha$

(4) + (5): $SX^2 + 100 = a^2 + 2t^2$ …… (6)
(3) = (6): $SX^2 + 100 = 208 = a^2 + 2t^2$
$\therefore SX = \sqrt{108}$ …… (7)

Apply Pythagoras' theorem on ΔPXY, ΔQXY, ΔRXY, ΔSXY

$p^2 = 144 - h^2$ …… (8)

$q^2 = 100 - h^2$ …… (9)

$r^2 = 64 - h^2$ …… (10)

$s^2 = 108 - h^2$ …… (11)

Draw YD // SR, cutting QR at D. Let $YD = d$.
Draw YE // QR, cutting SR at E. Let $YE = e$.
Draw MK // SR, cutting EY produced at K.
Then $\angle MKY = 90°$

$MK + YD = \frac{a}{2}$; $KY + YE = \frac{a}{2}$ ($\because$ M is the centre of the square $PQRS$)

$\therefore MK = \frac{a}{2} - d$ …… (12); $KY = \frac{a}{2} - e$ …… (13)

In ΔYRD, $d^2 + e^2 = r^2$ …… (14) (Pythagoras' theorem)

Sub. (10) into (14), $d^2 + e^2 = 64 - h^2$ …… (15)

In ΔYSE, $(a - d)^2 + e^2 = s^2$ …… (16) (Pythagoras' theorem)

Sub. (11) into (16), $(a - d)^2 + e^2 = 108 - h^2$ …… (17)

In ΔYQD, $(a - e)^2 + d^2 = q^2$ …… (18) (Pythagoras' theorem)

Sub. (9) into (18), $(a - e)^2 + d^2 = 100 - h^2$ …… (19)

(17) – (15): $a^2 - 2ad = 44$

$\Rightarrow a - 2d = \frac{44}{a}$ …… (20)

(19) – (15): $a^2 - 2ae = 36$

$\Rightarrow a - 2e = \frac{36}{a}$ …… (21)

In ΔMKY, $MY^2 = MK^2 + KY^2$ …… (22) (Pythagoras' theorem)

Sub. (12), (13) into (22): $MY^2 = \left(\frac{a}{2} - d\right)^2 + \left(\frac{a}{2} - e\right)^2 = \frac{1}{4}\left[(a - 2d)^2 + (a - 2e)^2\right]$ …… (23)

Sub. (20), (21) into (23): $MY^2 = \frac{1}{4}\left[\left(\frac{44}{a}\right)^2 + \left(\frac{36}{a}\right)^2\right] = \frac{808}{a^2}$ …… (24)

In ΔMXY, $XY^2 = XM^2 - MY^2$ …… (25) (Pythagoras' theorem)

Sub. (24) into (25): $XY^2 = h^2 = t^2 - \frac{808}{a^2}$

$\Rightarrow a^2h^2 = a^2 t^2 - 808$ …… (26)

The area of rectangle $PQUT = PQ \times PT = ah = \sqrt{(at)^2 - 808}$ by (26)

$\leq \sqrt{5408 - 808}$ by (*)

$= \sqrt{4600} = 10\sqrt{46} \approx 67.82$ (correct to 2 decimal places)

Answer (B).