Individual Events

I1	A	14	**I2**	A	2	**I3**	A	18	**I4**	A	4
	B	27		B	10		B	0		B	$\frac{3}{4}$
	C	116		C	40		C	6		C	$\sqrt{52}$
	D	4		D	144		D	2		D	64

Group Events

G1	P	8010	**G2**	a	$24\sqrt{7}$	**G3**	n	14	**G4**	a	-14
A		6	b		527	$b-a$		10		x	6
numbers		13696	c		4	gain		should switch $\$\frac{7400}{3}=\$2466\frac{2}{3}$	equation		$x^2+90x+2=0$
m		1	d		2	Area		1000 cm^2	Area		$4\sqrt{2}-\frac{3\pi}{2}$ see the remark

Individual Event 1

I1.1 已知 m 和 n 均為正整數。如果 $m+n+mn=54$ 及 $A=m+n$，求 A 的值。

Given that m and n are positive integers. If $m+n+mn=54$ and $A=m+n$, find the value of A.

Reference: 2006 FG2.4, 2024 FG4.1

$1+m+n+mn=55$ $(1+m)(1+n)=55$ $1+m=5,\ 1+n=11$ 或 $1+m=11,\ 1+n=5$ $m=4$，$n=10$ 或 $m=10$，$n=4$ $A=m+n=14$	$1+m+n+mn=55$ $(1+m)(1+n)=55$ $1+m=5,\ 1+n=11$ or $1+m=11,\ 1+n=5$ $m=4,\ n=10$ or $m=10,\ n=4$ $A=m+n=14$

I1.2 若 $f(a)=a-2$，$F(a,b)=b^2+a+A$， $B=F(4, f(5))$，求 B 的值。

If $f(a)=a-2$, $F(a,b)=b^2+a+A$ and $B=F(4, f(5))$, find the value of B

Reference: 2023 FI4.2

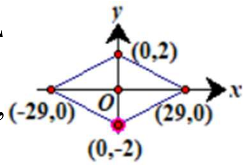
$f(5)=5-2=3$

$F(4, f(5))=F(4,3)=3^2+4+14=27$

$B=27$

I1.3 在 x-y 座標平面上，由 $(B+2, 0)$、$(-B-2, 0)$、$(0, 2)$ 及 $(0, -2)$所形成之菱形的面積為 C 平方單位，求 C 的值。

The area of the rhombus on the x-y coordinate plane with vertices $(B+2, 0)$, $(-B-2, 0)$, $(0, 2)$ and $(0, -2)$ is C square units. Find the value of C.



Reference: 2023 FI3.3

$$C=\frac{1}{2}\left[2-(-2)\right]\cdot\left[29-(-29)\right]$$
$$=116$$

I1.4 如果 D 是正整數且 $\left(\frac{C}{4}+227\right)^{\frac{1}{D}}=D$，求 D 的值。

If D is a positive integer such that $\left(\frac{C}{4}+227\right)^{\frac{1}{D}}=D$, find the value of D.

Reference: 2023 FI3.4

$$\frac{116}{4}+227=D^D$$

$D=4$

Individual Event 2

I2.1 若 A 是 $2022^{2023^{2024}}$ 的個位數 A。

A is the units digit of $2022^{2023^{2024}}$. Find the value of A.

$2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 8, 2^4 \equiv 6 \pmod{10}$ 這個數字的規律每隔 4 的倍數重複一次。 $2023 \equiv -1 \bmod 4, 2023^2 \equiv 1 \pmod 4$ 這個數字的規律每隔 2 的倍數重複一次。 $2023^{2024} \equiv 1 \pmod 4$ $2022^{2023^{2024}} \equiv 2022^1 \equiv 2 \pmod{10}$ $A = 2$	$2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 8, 2^4 \equiv 6 \pmod{10}$ This pattern repeats for every multiple of 4. $2023 \equiv -1 \bmod 4, 2023^2 \equiv 1 \pmod 4$ This pattern repeats for every multiple of 2. $2023^{2024} \equiv 1 \bmod 4$ $2022^{2023^{2024}} \equiv 2022^1 \equiv 2 \pmod{10}$ $A = 2$

I2.2 已知 $(x + 20)^2 + (y - 24)^2$ 的最小值是 B，當中 x 和 y 是方程 $19x + 13y = A$ 的整數解。求 B 的值。

B is the minimum value of $(x + 20)^2 + (y - 24)^2$, where x and y are integers that satisfy the equation $19x + 13y = A$. Find the value of B.

$19x + 13y = 2$ $13\times6 - 19\times4 = 2$ $(-4, 6)$ 為一整數解 一般解：$x = -4 + 13k, y = 6 - 19k$ $(x + 20)^2 + (y - 24)^2$ $= (-4 + 13k + 20)^2 + (6 - 19k - 24)^2$ $= (13k + 16)^2 + (-19k - 18)^2$ $= 169k^2 + 416k + 256 + 361k^2 + 684k + 324$ $= 530k^2 + 1100k + 580$ $=530\left(k^2+\frac{110}{53}k+\frac{55^2}{53^2}\right)+580-\frac{55^2\times10}{53}$ $=530\left(k+\frac{55}{53}\right)^2+\frac{490}{53}$ $k = 0$，算式$= 580$ $k = -1$, 算式$= 10$ $k = -2$, 算式$= 500$ $B = 10$	$19x + 13y = 2$ $13\times6 - 19\times4 = 2$ $(-4, 6)$ is an integral solution The general solutions: $x = -4 + 13k, y = 6 - 19k$ $(x + 20)^2 + (y - 24)^2$ $= (-4 + 13k + 20)^2 + (6 - 19k - 24)^2$ $= (13k + 16)^2 + (-19k - 18)^2$ $= 169k^2 + 416k + 256 + 361k^2 + 684k + 324$ $= 530k^2 + 1100k + 580$ $=530\left(k^2+\frac{110}{53}k+\frac{55^2}{53^2}\right)+580-\frac{55^2\times10}{53}$ $=530\left(k+\frac{55}{53}\right)^2+\frac{490}{53}$ $k = 0$, expression $= 580$ $k = -1$, expression $= 10$ $k = -2$, expression $= 500$ $B = 10$

I2.3 在袋中有若干顆紅色和藍色的彈珠，它們的總數量是 C。如果加入 B 顆紅色彈珠，紅色和藍色彈珠數量的比例則為 3：2；如果加入 B 顆藍色彈珠，紅色和藍色彈珠數量的比例則為 2：3。求 C 的值。

There is C marbles in a bag, which are either red or blue. If we add B red marbles to the bag, the ratio of red marbles to the blue marbles becomes 3 : 2. If we add B blue marbles to the bag, the ratio of red marbles to the blue marbles becomes 2 : 3. Find the value of C.

假設原本有 x 顆紅色彈珠。 那麼原本有 $(C - x)$ 顆藍色彈珠。 $(x + 10) : (C - x) = 3 : 2 \cdots\cdots (1)$ $x : (C - x + 10) = 2 : 3 \cdots\cdots(2)$ 由(1)式：$2x + 20 = 3C - 3x$ $5x = 3C - 20 \cdots\cdots(3)$ 由(2)式：$3x = 2C - 2x + 20$ $5x = 2C + 20 \cdots\cdots(4)$ $(3) - (4)$: $C = 40$	Suppose there are x red marbles originally. Then there are $(C - x)$ blue marbles originally. $(x + 10) : (C - x) = 3 : 2 \cdots\cdots (1)$ $x : (C - x + 10) = 2 : 3 \cdots\cdots(2)$ From (1): $2x + 20 = 3C - 3x$ $5x = 3C - 20 \cdots\cdots(3)$ From (2): $3x = 2C - 2x + 20$ $5x = 2C + 20 \cdots\cdots(4)$ $(3) - (4)$: $C = 40$

I2.4 若 $5\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)=C$，求 D 的值。

If $5\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)=C$, find the value of D.

$5\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)=40$

$\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)^2=64$

$25+2\sqrt{D}+2\sqrt{625-4D}+25-2\sqrt{D}=64$

$2\sqrt{625-4D}=14$

$625-4D=49$

$D=144$

Individual Event 3

I3.1 若 x 和 y 為滿足方程 $\frac{1}{x}+\frac{1}{y}=\frac{2}{5}$ 的不同正整數，求 $A=x+y$ 的值。

If x and y are two different positive integers such that $\frac{1}{x}+\frac{1}{y}=\frac{2}{5}$, find the value of $A=x+y$.

$\frac{y+x}{xy}=\frac{2}{5}$ $2xy=5x+5y$ $4xy-10x-10y+25=25$ $(2x-5)(2y-5)=25$ $(2x-5, 2y-5)=(1, 25), (5, 5)$ 或 $(25, 1)$ $\because x$、y 為不同正整數 $\therefore (x, y)=(3, 15)$ 或 $(15, 3)$ $A=x+y=18$	$\frac{y+x}{xy}=\frac{2}{5}$ $2xy=5x+5y$ $4xy-10x-10y+25=25$ $(2x-5)(2y-5)=25$ $(2x-5, 2y-5)=(1, 25), (5, 5)$ or $(25, 1)$ $\because x, y$ are different positive integers $\therefore (x, y)=(3, 15)$ or $(15, 3)$ $A=x+y=18$

I3.2 若 B 是所有正整數 N 使得 7 整除 $2^N+(19-A)$ 的數量，求 B 的值。

If B is the number of positive integers N such that $2^N+(19-A)$ is divisible by 7, find the value of B.

$2^N+(19-A)=2^N+(19-18)=2^N+1$ $2^1+1\equiv 3, 2^2+1\equiv 5, 2^3+1\equiv 2 \pmod 7$ 這個數字的規律每隔 3 的倍數重複一次。 $\therefore$ 對於每一個正整數 N，2^N+1 不能被 7 整除 $B=0$	$2^N+(19-A)=2^N+(19-18)=2^N+1$ $2^1+1\equiv 3, 2^2+1\equiv 5, 2^3+1\equiv 2 \pmod 7$ This pattern repeats for every multiple of 3. $\therefore$ 2^N+1 cannot be divisible by 7 for every positive integer N $B=0$

I3.3 已知 a 和 b 為滿足方程組 $a^2-b^2=9$ 及 $ab=3$ 的實數。

若對於正整數 α 和 C，$a+b=\sqrt{\sqrt{\alpha}+C-B}$，求 C 的值。

Given that a and b are real numbers such that $a^2-b^2=9$ and $ab=3$.

If $a+b=\sqrt{\sqrt{\alpha}+C-B}$ for positive integers α and C, find the value of C.

代 $b=\frac{3}{a}$ 入 $a^2-b^2=9$ $a^2-\left(\frac{3}{a}\right)^2=9 \Rightarrow a^4-9a^2-9=0$ $\because a$ 為實數 $\therefore a^2\ge 0 \Rightarrow a^2=\frac{9+\sqrt{117}}{2}$ $b^2=\frac{9}{a^2}=\frac{9\times 2}{9+\sqrt{117}}\cdot\frac{\sqrt{117}-9}{\sqrt{117}-9}=\frac{\sqrt{117}-9}{2}$ $(a+b)^2=\sqrt{\alpha}+C$ (已知) $a^2+2ab+b^2=\sqrt{\alpha}+C$ $\frac{9+\sqrt{117}}{2}+\frac{\sqrt{117}-9}{2}+6=\sqrt{\alpha}+C$ $\sqrt{117}+6=\sqrt{\alpha}+C$ $\because \sqrt{117}$ 為無理數，6 及 C 為有理數 $\therefore C=6$	Sub. $b=\frac{3}{a}$ into $a^2-b^2=9$ $a^2-\left(\frac{3}{a}\right)^2=9 \Rightarrow a^4-9a^2-9=0$ $\because a$ is real $\therefore a^2\ge 0 \Rightarrow a^2=\frac{9+\sqrt{117}}{2}$ $b^2=\frac{9}{a^2}=\frac{9\times 2}{9+\sqrt{117}}\cdot\frac{\sqrt{117}-9}{\sqrt{117}-9}=\frac{\sqrt{117}-9}{2}$ $(a+b)^2=\sqrt{\alpha}+C$ (given) $a^2+2ab+b^2=\sqrt{\alpha}+C$ $\frac{9+\sqrt{117}}{2}+\frac{\sqrt{117}-9}{2}+6=\sqrt{\alpha}+C$ $\sqrt{117}+6=\sqrt{\alpha}+C$ $\because \sqrt{117}$ is irrational, 6 and C are rational $\therefore C=6$

I3.4 若 x 為滿足方程 $\left(\log_a x\right)^{\log_a x} = x$ 的實數，其中 a 是常數且 $a > 1$。求 $D = \dfrac{C \log_a x}{3a}$ 的值。

If x is real root of the equation $\left(\log_a x\right)^{\log_a x} = x$, where a is a constant and $a > 1$, find the value of $D = \dfrac{C \log_a x}{3a}$.

$\left(\log_a x\right)^{\log_a x} = x$

$\log_a \left[\left(\log_a x\right)^{\log_a x}\right] = \log_a x$

$\log_a x \log_a \left(\log_a x\right) = \log_a x$

$\log_a \left(\log_a x\right) = 1$

$\log_a x = a$

$x = a^a$

$D = \dfrac{C \log_a x}{3a} = \dfrac{6 \log_a a^a}{3a}$

$= \dfrac{2a \log_a a}{a} = 2$

Individual Event 4

I4.1 如果 $A>1$ 且 $1+\frac{1}{A}+\frac{1}{A^2}+\frac{1}{A^3}+\cdots=\frac{A}{3}$，求 A 的值。

If $A>1$ and $1+\frac{1}{A}+\frac{1}{A^2}+\frac{1}{A^3}+\cdots=\frac{A}{3}$, find the value of A.

以上是一等比數列的無限項之和， 公比$=\frac{1}{A}$， 其中 $0<\frac{1}{A}<1$。 $\frac{1}{1-\frac{1}{A}}=\frac{A}{3}$ $3A=A(A-1)$ $A^2-4A=0$ $A=4$	The above is an infinite geometric series with common ratio $=\frac{1}{A}$, where $0<\frac{1}{A}<1$. $\frac{1}{1-\frac{1}{A}}=\frac{A}{3}$ $3A=A(A-1)$ $A^2-4A=0$ $A=4$

I4.2 如果 $\frac{1}{A}$ 是二次方程 $x^2-Bx+\frac{1}{6}B=0$ 的一個根，求 B 的值。

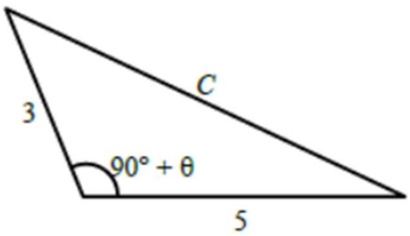
If $\frac{1}{A}$ is a root of the quadratic equation $x^2-Bx+\frac{1}{6}B=0$, find the value of B.

$$\left(\frac{1}{4}\right)^2-\frac{B}{4}+\frac{1}{6}B=0$$

$$B=\frac{3}{4}$$

I4.3 考慮右圖中的三角形，如果 $\tan\theta=B$，其中 $0°<\theta<90°$，求 C 的值。

Consider the triangle in the figure on the right.
If $\tan\theta=B$, where $0°<\theta<90°$, find the value of C.



$\tan\theta=\frac{3}{4}$, $\sin\theta=\frac{3}{5}$, $\cos\theta=\frac{4}{5}$

$$C^2=3^2+5^2-2\times3\times5\times\cos(90°+\theta)$$

$$C^2=34+30\sin\theta=34+30\times\frac{3}{5}=52$$

$$C=\sqrt{52}$$

I4.4 設 $d=C^2-20$，如果 D 滿足方程 $8^D=D^d$，求 D 的值。

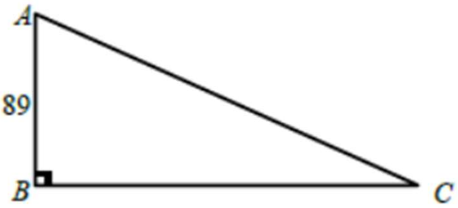
Let $d=C^2-20$. If D satisfies the equation $8^D=D^d$, the value of D.

$d=52-20=32$ $8^D=D^{32}$ $2^{3D}=D^{32}$ …… (1) 設 $D=2^x$ 比較(1)式的指數：$3\times2^x=32x$ …… (2) 利用嘗試錯誤法，代 $x=3$ 入(2)式：LHS $\neq$ RHS 代 $x=6$ 入(2)式：LHS = RHS $x=6$ $D=2^6=64$	$d=52-20=32$ $8^D=D^{32}$ $2^{3D}=D^{32}$ …… (1) Let $D=2^x$ Compare the indices of (1): $3\times2^x=32x$ …… (2) By trial and error, put $x=3$ in (2): LHS $\neq$ RHS Put $x=6$ in (2): LHS = RHS $x=6$ $D=2^6=64$

Group Event 1

G1.1 若直角三角形ABC所有邊長均為正整數，且AB = 89，求三角形ABC 的周界P。

Find the perimeter P of the right-angled triangle ABC if all the side lengths are positive integers and $AB = 89$.



設 $BC = a$, $AC = b$ $a^2 + 89^2 = b^2$ $(b + a)(b - a) = 89^2$ $\because$ 89 為質數及 $b > a$ 皆為正整數 $\therefore (b + a, b - a) = (7921, 1)$ $b = 3961$, $a = 3960$ $P = 3961 + 3960 + 89 = 8010$	Let $BC = a$, $AC = b$ $a^2 + 89^2 = b^2$ $(b + a)(b - a) = 89^2$ $\because$ 89 is a prime and $b > a$ are positive integers $\therefore (b + a, b - a) = (7921, 1)$ $b = 3961$, $a = 3960$ $P = 3961 + 3960 + 89 = 8010$

G1.2 若 A 是 $8888^{20242024}$ 的個位數。求 A 的值。

If A is the units digit of $8888^{20242024}$. Find the value of A.

$8^1 \equiv 8$, $8^2 \equiv 4$, $8^3 \equiv 2$, $8^4 \equiv 6 \pmod{10}$ 這個數字的規律每隔 4 的倍數重複一次。 $20242024 \equiv 0 \pmod 4$ $8888^{20242024} \equiv 8^4 \equiv 6 \pmod{10}$ $A = 6$	$8^1 \equiv 8$, $8^2 \equiv 4$, $8^3 \equiv 2$, $8^4 \equiv 6 \pmod{10}$ This pattern repeats for every multiple of 4. $20242024 \equiv 0 \pmod 4$ $8888^{20242024} \equiv 8^4 \equiv 6 \pmod{10}$ $A = 6$

G1.3 有多少個 5 位數包含最少 1 個「1」和最少 1 個「3」？

How many 5-digit numbers contain at least one "1" and at least one "3"?

方法一 5 位數一共有：$9\times10^4 = 90000$ 個 沒有'1'和沒有'3'的 5 位數， 一共有 $7\times8^4 = 28672$ 個 只有一個'1' 和沒有'3'的 5 位數， 一共有 $1\times8^4 + 7\times4\times8^3 = 18432$ 個 只有一個'3'和沒有'1'的 5 位數，有 18432 個 只有兩個'1' 和沒有'3'的 5 位數， 一共有$= 4\times8^3 + 7\times C_2^4\times8^2 = 4736$ 個 只有兩個'3'和沒有'1'的 5 位數，有 4736 個 只有三個'1' 和沒有'3'的 5 位數， 一共有$= 7\times4\times8 + C_2^4\times8^2 = 608$ 個 只有三個'3' 和沒有'1'的 5 位數，有 608 個 四個'1'和沒有'3'的 5 位數，有 7+4×8 =39 個 四個'3'和沒有'1'的 5 位數，有 39 個 '11111'有 1 個；'33333'有 1 個 總數= 90000–28672–(18432+4736+608+39+ 1)×2 = 13696 個 **方法二** 只有一個'1'和一個'3'的 5 位數， 一共有$= 4\times8^3\times2 + 7\times P_2^4\times8^2 = 9472$ 個 兩個'1' 和一個'3'的 5 位數， 共有$= 7\times8\times C_2^4\times2 + P_2^4\times8^2 + C_2^4\times8^2 = 1824$ 個 兩個'3'和一個'1'的 5 位數，有 1824 個 兩個'1' 和兩個'3'的 5 位數， 共有$= 7\times C_2^4 + C_2^4\times2\times8 + C_2^4\times2\times8 = 234$ 個	**Method 1** No. of 5-digit numbers = $9\times10^4 = 90000$ If there are no '1' and no '3', numbers = $7\times8^4 = 28672$ If there is only one '1' but no '3', numbers = $1\times8^4 + 7\times4\times8^3 = 18432$ If there is only one '3' but no '1',numbers = 18432 If there are 2'1's but no '3', numbers = $4\times8^3 + 7\times C_2^4\times8^2 = 4736$ If there are 2'3's but no '1', numbers = 4736 If there are 3'1's but no '3', numbers = $7\times4\times8 + C_2^4\times8^2 = 608$ If there are 3'3's but no '1', numbers = 608 If there are 4'1's but no '3', numbers = 7+4×8 =39 If there are 4'3's but no '1', numbers = 39 '11111', number = 1; '33333', number = 1 Total=90000–28672–(18432+4736+608+39+1)×2 = 13696 **Method 2** If there is only one '1' and one '3', numbers = $4\times8^3\times2 + 7\times P_2^4\times8^2 = 9472$ If there is 2'1's but only one '3', numbers = $7\times8\times C_2^4\times2 + P_2^4\times8^2 + C_2^4\times8^2 = 1824$ If there is 2'3's but only one '1', numbers = 1824 If there are 2'3's and 2'1's, numbers = $7\times C_2^4 + C_2^4\times2\times8 + C_2^4\times2\times8 = 234$ If there are 3'1's and one '3',

三個'1'和一個'3'的 5 位數， 共有$=C_2^4\times 2\times 8+4\times 8+7\times 4=156$ 個 三個'3'和一個'1'的 5 位數，有 156 個 三個'1'和兩個'3'的 5 位數，共有$=C_2^4+4=10$ 個 三個'3'和兩個'1'的 5 位數，有 10 個 四個'1'和一個'3'的 5 位數，共有$=1+4=5$ 個 四個'3'和一個'1'的 5 位數，有 5 個 總數$=9472+1824\times 2+156\times 2+234+30=13696$ 個	numbers $=C_2^4\times 2\times 8+4\times 8+7\times 4=156$ If there are 3'3's and one '1', numbers = 156 If there are 3'1's and 2'3's, numbers $=C_2^4+4=10$ If there are 3'3's and 2'1's, numbers = 10 If there are 4'1's and 1'3', numbers $=1+4=5$ If there are 4'3's and 1'1', numbers = 5 Total $=9472+1824\times 2+156\times 2+234+30=13696$

G1.4 設有 m 對正整數 a 和 b，使 a^4+4b^4 為質數，求 m 的值。

Let m be the number of possible pairs of positive integers a and b for which a^4+4b^4 is a prime number. Find the value of m.

$a^4+4b^4=a^4+4a^2b^2+4b^4-4a^2b^2$ $=(a^2+2b^2)^2-(2ab)^2$ $=(a^2+2ab+2b^2)(a^2-2ab+2b^2)$ $a^2+2ab+2b^2=p$，一個質數及 $a^2-2ab+2b^2=1$ $(a-b)^2+b^2=1$ $a=b$ 及 $b=1$ 或 $a-b=1$ 及 $b=0$ $(a,b)=(1,1)$ 或 $(1,0)$ 代以上答案入 $a^2+2ab+2b^2=p$ $1+2+2=p$ (接受) $1+0+0=p$ 不是質數 (捨去) $\therefore (a,b)=(1,1)$ $m=1$	$a^4+4b^4=a^4+4a^2b^2+4b^4-4a^2b^2$ $=(a^2+2b^2)^2-(2ab)^2$ $=(a^2+2ab+2b^2)(a^2-2ab+2b^2)$ $a^2+2ab+2b^2=p$, a prime number and $a^2-2ab+2b^2=1$ $(a-b)^2+b^2=1$ Either $a=b$ and $b=1$ or $a-b=1$ and $b=0$ $(a,b)=(1,1)$ or $(1,0)$ Sub. the solutions into $a^2+2ab+2b^2=p$ $1+2+2=p$ (accepted) $1+0+0=p$, not a prime, (rejected) $\therefore (a,b)=(1,1)$ $m=1$

Group Event 2

G2.1 設 $x>0$。已知 $x-\dfrac{1}{x}=\sqrt{3}$ 且 $a=x^5+x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}+\dfrac{1}{x^5}$，求 a 的值。

Let $x>0$. Given that $x-\dfrac{1}{x}=\sqrt{3}$ and $a=x^5+x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}+\dfrac{1}{x^5}$, find the value of a.

$$x-\frac{1}{x}=\sqrt{3}\Rightarrow x^2-2+\frac{1}{x^2}=3$$

$$x^2+2+\frac{1}{x^2}=7\Rightarrow\left(x+\frac{1}{x}\right)^2=7$$

$$x+\frac{1}{x}=\sqrt{7}$$

$$x^2+\frac{1}{x^2}=5$$

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)=\sqrt{7}(5-1)=4\sqrt{7}$$

$$\left(x^2+\frac{1}{x^2}\right)\left(x^3+\frac{1}{x^3}\right)=(5)\left(4\sqrt{7}\right)\Rightarrow x^5+\frac{1}{x^5}+x+\frac{1}{x}=20\sqrt{7}\Rightarrow x^5+\frac{1}{x^5}=19\sqrt{7}$$

$$a=\left(x^5+\frac{1}{x^5}\right)+\left(x^3+\frac{1}{x^3}\right)+\left(x+\frac{1}{x}\right)=19\sqrt{7}+4\sqrt{7}+\sqrt{7}=24\sqrt{7}$$

G2.2 用首2024個正整數：1、2、3、4、5、6、⋯、2024，造出一個新整數：123456789101112⋯2024。若b是這個整數裡「0」的數量，求 b 的值。

Using the first 2024 positive integers: 1, 2, 3, 4, 5, 6, ⋯ , 2024, a new integer is formed as 123456789101112⋯2024. If b is the number of "0" in this integer, find the value of b.

	‘0’ 的數目	Number of ‘0’s
1, 2, ⋯ , 9, 10, 11, ⋯ , 99	9	9
100, 211, ⋯ , 999	$(11+9)\times 9$	$(11+9)\times 9$
1000, 1001, ⋯ , 1009	21	21
1010, 1011, ⋯ , 1099	90 + 9	90 + 9
1100, 1101, ⋯ , 1999	$(11+9)\times 9$	$(11+9)\times 9$
2000, 2001, ⋯ , 2009	21	21
2010, 2011, ⋯ , 2019	11	11
2020, 2021, 2022, 2023, 2024	6	6

‘0’ 的總數 Total number of ‘0’= $9+(11+9)\times 9+21+90+9+(11+9)\times 9+21+11+6=527$

G2.3 c 是 2024^2-2023^2 的正因數的數量。求 c 的值。

c is the number of positive factors of 2024^2-2023^2.

$2024^2-2023^2=(2024+2023)(2024-2023)$ $=4047=3\times 1349$，1349為質數 正因數為 1、3、1349、4047。 $c=4$	$2024^2-2023^2=(2024+2023)(2024-2023)$ $=4047=3\times 1349$, 1349 is a prime The positive factors are 1, 3, 1349, 4047. $c=4$

G2.4 假設「0」、「1」、「2」、…及「6」分別為星期日、星期一、星期二、…和星期六，今日是星期一，若 $20^{24^{2024}}$ 天後的那一天是星期幾之代號為「d」，求 d 的值。

Let "0", "1", "2", $\cdots$ and "6" represent Sunday, Monday, Tuesday, $\cdots$ and Saturday respectively. Today is Monday. If "d" represents the day of week that comes after $20^{24^{2024}}$ days. Find the value of d.

$20 = 7\times2 + 6 \Rightarrow 20 \equiv -1 \pmod 7$ $20^2 \equiv (-1)^2 \equiv 1 \pmod 7$ 這個數字的規律每隔 2 的倍數重複一次。 24^{2024} 是一個偶數。 $20^{24^{2024}} \equiv 1 \pmod 7$ $20^{24^{2024}}$ 日後為星期二。 $d = 2$	$20 = 7\times2 + 6 \Rightarrow 20 \equiv -1 \pmod 7$ $20^2 \equiv (-1)^2 \equiv 1 \pmod 7$ This pattern repeats for every multiple of 2. 24^{2024} is an even number. $20^{24^{2024}} \equiv 1 \pmod 7$ The day after $20^{24^{2024}}$ days is a Tuesday. $d = 2$

Group Event 3

G3.1 試找出最小的正整數 n 使得 $2^{10}+2^{13}+2^n$ 成為一個完全平方數。

Find the smallest positive integer n such that $2^{10}+2^{13}+2^n$ is a perfect square number.

$2^{10}+2^{13}+2^n=m^2$

$2^{10}(1+8)+2^n=m^2$

$2^n=m^2-(2^5\times 3)^2$

$2^a\times 2^b=(m+96)(m-96)$

$m+96=2^a$, $m-96=2^b$; $a, b\in\mathbb{Z}^+$

$192=2^a-2^b$, $a>b\in\mathbb{Z}^+$

$2^6\times 3=2^b(2^{a-b}-1)$

$b=6$, $2^{a-b}-1=3\Rightarrow 2^{a-b}=4\Rightarrow a=8$

$2^n=2^8\times 2^6=2^{14}$

$n=14$

G3.2 設 $a^2+b^2+6a-14b+58=0$。求 $b-a$ 的值。

Suppose $a^2+b^2+6a-14b+58=0$. Find the value of $b-a$.

$a^2+2a(3)+3^2+b^2-2b(7)+7^2=0$

$(a+3)^2+(b-7)^2=0$

$a=-3$, $b=7$

$b-a=7-(-3)=10$

G3.3 在正方形土地的某一個角落裡埋著一個裝有\$8,000 的箱子。在一次比賽中，你和另一個叫「倒霉先生」的人一起挖箱子。倒霉先生有一個特點：他總是做出錯誤的選擇。你贏了擲骰子先選。你選了一個角落，倒霉先生選了另一個角落。在你準備開始時，你發現倒霉先生沒有找到箱子。遊戲規則允許你換另一個角落，但要罰\$200。你應否更換嗎？計算換角落的期望收益。

There was a chest containing \$8,000 buried in one of the corners of a square piece of land. In a contest, you and another man called "Mr. Badluck" were digging for the chest. Mr. Badluck had one peculiarity: he always made the wrong choice. You won the toss and chose first. You picked a corner, and Mr. Badluck picked another. Before you started, you observed that Mr. Badluck found no chest. The rules of the game allowed you to make a switch to another corner, but with a penalty of \$200. Should you make a switch?

Calculate the expected gain from making the switch in dollars.

將正方形的四角命名為 A、B、C 及 D。 若不轉換：$P(A)=P(B)=P(C)=P(D)=\frac{1}{4}$ 期望收益= $\$\frac{1}{4}\times 8000=\$2000$ 假設你已選擇了 A 及倒霉先生選擇了 B；而 B 沒有箱子。如果你更換，那麼更換的條件概率為 $P(C\mid B$沒有箱子$)=P(D\mid B$沒有箱子$)=\frac{1}{3}$ 期望收益=$\$(\frac{1}{3}\times 8000-200)=\$\frac{7400}{3}>\$2000$。 你應更換。	Label the four corners as A, B, C and D. Without switching, $P(A)=P(B)=P(C)=P(D)=\frac{1}{4}$ Expected gain == $\$\frac{1}{4}\times 8000=\$2000$ Suppose you had already chosen A and Mr. Badluck had chosen B; while B doesn't contain the chest. If you switch, then the conditional probability $P(C\mid B$ is not$)=P(D\mid B$ is not$)=\frac{1}{3}$ Expected gain $=\$(\frac{1}{3}\times 8000-200)=\$\frac{7400}{3}>\$2000$. You should make a switch.

G3.4 一個凸六邊形有以下性質：

(i) 由任意頂點與相鄰兩個頂點組成的三角形的面積都是1,000 cm^2；及

(ii) $CH = DI$。

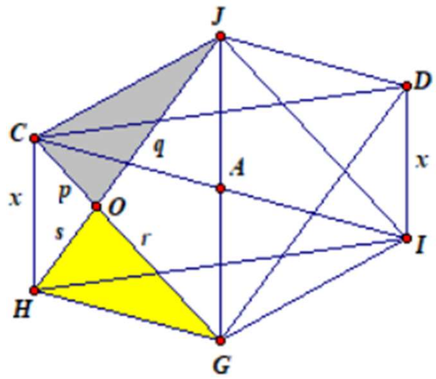
求六邊形的面積。

A convex hexagon has the following property:

(i) all the triangles formed from any vertex with the two adjacent vertices have an area of 1,000 cm^2 ; and

(ii) $CH = DI$.

Find the area of the hexagon.



假設CG與JH相交於O，CI與DH相交於A。 假設$OC = p$、$OJ = q$、$OG = r$、$OH = s$。 $S_{\Delta CHI} = S_{\Delta CHG} \Rightarrow S_{\Delta OCH} + S_{\Delta OCJ} = S_{\Delta OCH} + S_{\Delta OHG}$ $S_{\Delta OCJ} = S_{\Delta OHG} \Rightarrow \frac{1}{2}\cdot pq\sin\angle COJ = \frac{1}{2}\cdot rs\sin\angle GOH$ $\because \angle COJ = \angle GOH$ (對頂角) $\therefore pq = rs \Rightarrow \frac{OJ}{OH} = \frac{OC}{OG}$ $\angle JOG = \angle COH$ (對頂角) $\Delta COH \sim \Delta GOJ$ (兩邊成比例，一夾角相等) $\angle CHO = \angle GJO$ (相似三角形對應角) $\therefore CH \parallel JG$ (交錯角相等) $\therefore CH \parallel DI \parallel JG$, $CJ \parallel HD \parallel GI$, $JD \parallel CI \parallel HG$ $CDHI$是一個平行四邊形(對邊相等且平行) $ACJD$是一個平行四邊形(由兩組平行邊組成) $AHGI$是一個平行四邊形(由兩組平行邊組成) $S_{\Delta AHI} = S_{\Delta GHI} = 1000$ cm^2 $S_{\Delta ACD} = S_{\Delta JCD} = 1000$ cm^2 $ACHG$是一個平行四邊形(由兩組平行邊組成) $ADGI$是一個平行四邊形(由兩組平行邊組成) $S_{\Delta ACH} = S_{\Delta CGH} = 1000$ cm^2 $S_{\Delta ADI} = S_{\Delta GDI} = 1000$ cm^2 六邊形的面積= 1000×6 cm^2 = 6000 cm^2	Suppose CG meets JH at O, CI meets DH at A. Let $OC = p$, $OJ = q$, $OG = r$, $OH = s$. $S_{\Delta CHI} = S_{\Delta CHG} \Rightarrow S_{\Delta OCH} + S_{\Delta OCJ} = S_{\Delta OCH} + S_{\Delta OHG}$ $S_{\Delta OCJ} = S_{\Delta OHG} \Rightarrow \frac{1}{2}\cdot pq\sin\angle COJ = \frac{1}{2}\cdot rs\sin\angle GOH$ $\because \angle COJ = \angle GOH$ (vert. opp. $\angle$s) $\therefore pq = rs \Rightarrow \frac{OJ}{OH} = \frac{OC}{OG}$ $\angle JOG = \angle COH$ (vert. opp. $\angle$s) $\Delta COH \sim \Delta GOJ$ (ratio of 2 sides, inc. $\angle$) $\angle CHO = \angle GJO$ (corr. $\angle$s, $\sim\Delta$s) $\therefore CH \parallel JG$ (alt. $\angle$s eq.) $\therefore CH \parallel DI \parallel JG$, $CJ \parallel HD \parallel GI$, $JD \parallel CI \parallel HG$ $CDHI$ is a //-gram (opp. sides eq. and //) $ACJD$ is a //-gram (formed by 2 pairs of // lines) $AHGI$ is a //-gram (formed by 2 pairs of // lines) $S_{\Delta AHI} = S_{\Delta GHI} = 1000$ cm^2 $S_{\Delta ACD} = S_{\Delta JCD} = 1000$ cm^2 $ACHG$ is a //-gram (formed by 2 pairs of // lines) $ADGI$ is a //-gram (formed by 2 pairs of // lines) $S_{\Delta ACH} = S_{\Delta CGH} = 1000$ cm^2 $S_{\Delta ADI} = S_{\Delta GDI} = 1000$ cm^2 Area of hexagon = 1000×6 cm^2 = 6000 cm^2

Group Event 4

G4.1 設 a、b 為非零整數，且滿足方程 $a-ab+b=18$。求 $a+b$ 的值。

Let a , b be non-zero integers satisfying the equation $a-ab+b=18$. Find the value of $a+b$.

Reference: 2024 FI1.1

$a-ab+b-1=17$ $(1-a)(b-1)=17$ $(1-a, b-1)=(1, 17)$ 或 $(-1,-17)$ $(a, b)=(0, 18)$ (捨去) 或 $(2,-16)$ $a+b=-14$	$a-ab+b-1=17$ $(1-a)(b-1)=17$ $(1-a, b-1)=(1, 17)$ or $(-1,-17)$ $(a, b)=(0, 18)$ (rejected) or $(2,-16)$ $a+b=-14$

G4.2 設 x 為一正整數，且滿足 $x(x+1)(x+2)(x+3)=3024$。求 x 的值。

Let x be a positive integer satisfying $x(x+1)(x+2)(x+3)=3024$. Find the value of x .

$x(x+1)(x+2)(x+3)=3024$ $(x^2+3x)(x^2+3x+2)=3024$ $(x^2+3x)^2+2(x^2+3x)+1=3025$ $(x^2+3x+1)^2=55^2$ $x^2+3x+1=55$ 或 $x^2+3x+1=-55$ $x^2+3x-54=0$ 或 $x^2+3x+56=0$ $(x-6)(x+9)=0$ 或沒有實數解 $\because x>0 \therefore x=6$	$x(x+1)(x+2)(x+3)=3024$ $(x^2+3x)(x^2+3x+2)=3024$ $(x^2+3x)^2+2(x^2+3x)+1=3025$ $(x^2+3x+1)^2=55^2$ $x^2+3x+1=55$ or $x^2+3x+1=-55$ $x^2+3x-54=0$ or $x^2+3x+56=0$ $(x-6)(x+9)=0$ or no real solution $\because x>0 \therefore x=6$

G4.3 設 α、β 為二次方程 $x^2+6x+2=0$ 的兩個根，

求以 $\dfrac{\alpha^2}{\beta}$ 和 $\dfrac{\beta^2}{\alpha}$ 為根及 x^2 的系數為 1 的二次方程。

Let α, β be the two roots of the quadratic equation $x^2+6x+2=0$.

Find the quadratic equation whose roots are $\dfrac{\alpha^2}{\beta}$ and $\dfrac{\beta^2}{\alpha}$, and coefficient of x^2 is 1 .

$\alpha+\beta=-6$, $\alpha\beta=2$

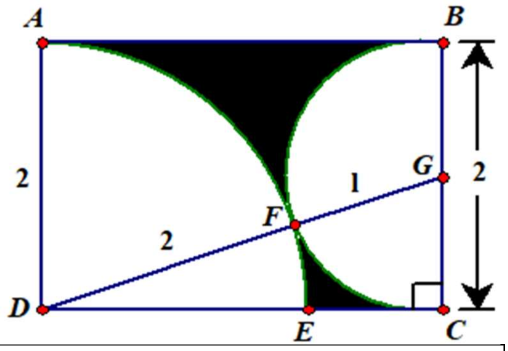
$$\begin{aligned}\alpha^3+\beta^3&=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)\\&=(\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]\\&=-6[(-6)^2-3(2)]=-180\end{aligned}$$

$$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{-180}{2}=-90$$

$$\frac{\alpha^2}{\beta}\times\frac{\beta^2}{\alpha}=\alpha\beta=2$$

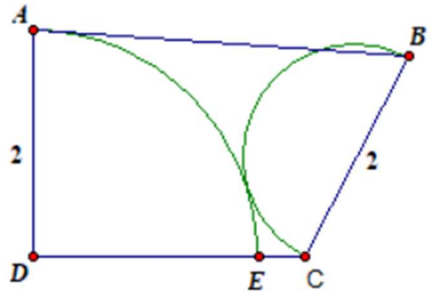
$x^2+90x+2=0$

G4.4 右圖空白部分由一個四分一圓和一個半圓互相外切組成。$ABCD$是一個長方形。求陰影部分的面積。
The unshaded part in the diagram on the right is made up of a quarter-circle and a semi-circle which touch each other externally. $ABCD$ is a rectangle.
Find the area of the shaded part.



假設該四分一圓和半圓互相外切於 F 及 G 為半圓的圓心。那麼 D、F、G 共綫。 $DG = DF + FG = 2 + 1 = 3$ $CD^2 + CG^2 = DG^2$ (畢氏定理) $CG = \sqrt{3^2 - 1^2} = 2\sqrt{2}$ 陰影面積$= 2\sqrt{2} \times 2 - \frac{1}{4}\cdot\pi(2)^2 - \frac{1}{2}\cdot\pi(1)^2$ $= 4\sqrt{2} - \frac{3\pi}{2}$	Suppose the quarter-circle and the semi-circle touch each other at F and G is the centre of the semi-circle. Then D, F, G are collinear. $DG = DF + FG = 2 + 1 = 3$ $CD^2 + CG^2 = DG^2$ (Pythagoras' theorem) $CG = \sqrt{3^2 - 1^2} = 2\sqrt{2}$ Shaded area $= 2\sqrt{2} \times 2 - \frac{1}{4}\cdot\pi(2)^2 - \frac{1}{2}\cdot\pi(1)^2$ $= 4\sqrt{2} - \frac{3\pi}{2}$

Remark: the original question was:
右圖空白部分由一個四分一圓和一個半圓組成，
求陰影部分的面積。
The unshaded part in the diagram on the right is made up of a quarter-circle and a semi-circle. Find the area of the shaded part.



If $ABCD$ is not a rectangle, then it is impossible to find CD and hence the area of of the shaded part. Furthermore, the fact that the quarter-circle and the semi-circle touch each other externally must be specified.