Examples on Mathematical Induction: divisibility 3

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1. Prove by mathematical induction $7^n + 5$ is divisible by 3 for all non-negative integer n .

1. Let P(n) be the statement "$7^n + 5$ is divisible by 3 for all positive integers n ."

$n = 1$, $7^1 + 5 = 12$ which is divisible by 3. P(1) is true.

Suppose P(k) is true for some positive integer k.

i.e. $7^k + 5 = 3m$ for some integer m.

$7^k = 3m - 5$

when $n = k + 1$, $7^{k+1} + 5 = 7(7^k) + 5$

$= 7(3m - 5) + 5$

$= 21m - 30$

$= 3(7m - 10)$

since $7m - 10$ is an integer, $7^{k+1} + 5$ is divisible by 3

P(k + 1) is true when P(k) is true.

By the principle of Mathematical Induction, P(n) is true for all positive integer n.

2. Prove by mathematical induction $2^{2n} - 1$ is divisible by 3 for all non-negative integer n.

3. **1971 AM Paper 1 Q11(a)**

Prove by mathematical induction $n(n^2 + 2)$ is divisible by 3 for all non-negative integer n .
$n = 1$, $1(1^2 + 2) = 3$, which is divisible by 3 .
Suppose $k(k^2 + 2) = 3m$ for some positive integer k, where m is an integer.
$(k + 1)[(k + 1)^2 + 2]$
$= (k + 1)(k^2 + 2k + 3)$
$= k(k^2 + 2) + (k^2 + 2k + 3) + k(2k + 1)$
$= 3m + 3k^2 + 3k + 3$
$= 3(m + k^2 + k + 1)$
$\because$ $m + k^2 + k + 1$ is an integer.
$\therefore$ $(k + 1)[(k + 1)^2 + 2]$ is divisible by 3.
If it is true for $n = k$, then it is also true for $n = k + 1$.
By the principle of mathematical induction, it is true for all positive integer n .

4. **1984 Paper 2 Q2**

Prove by mathematical induction that, for all positive integers n, $4n^3 - n$ is divisible by 3 .
Let $P(n) \equiv$ "$4n^3 - n$ is divisible by 3 for all positive integer n ."
$n = 1$, $4(1)^3 - 1 = 3$, which is divisible by 3
Suppose $P(k)$ is true
i.e. $4k^3 - k$ is divisible by 3 for some positive integer k
$4k^3 - k = 3m$, where m is an integer
When $n = k + 1$,
$4(k + 1)^3 - (k + 1) = 4(k^3 + 3k^2 + 3k + 1) - k - 1$
$= 4k^3 - k + 3k^2 + 3k$
$= 3m + 3(k^2 + k)$ (induction assumption)
$= 3(m + k^2 + k)$
$\because$ $m + k^2 + k$ is an integer
$\therefore$ $4(k + 1)^3 - (k + 1)$ is also divisible by 3
If $P(k)$ is true then $P(k + 1)$ is also true
By the principle of mathematical induction, $P(n)$ is true for all positive integer n.

5. **1996 Paper 2 Q4**

Prove by mathematical induction, that for all positive integers n , $(2n^3 + n)$ is divisible by 3 .

Let $P(n) \equiv$ "$2n^3 + n$ is divisible by 3 for all positive integer n ."

$n = 1$, $2(1)^3 + 1 = 3$, which is divisible by 3

Suppose $P(k)$ is true

i.e. $2k^3 + k$ is divisible by 3 for some positive integer k

$2k^3 + k = 3m$, where m is an integer

When $n = k + 1$,

$$\begin{aligned} 2(k+1)^3 + (k+1) &= 2(k^3 + 3k^2 + 3k + 1) + k + 1 \\ &= (2k^3 + k) + 3(2)k^2 + 3(2)k + 3 \\ &= 3m + 3(2k^2 + 2k + 1) \text{ (induction assumption)} \\ &= 3(m + 2k^2 + 2k + 1) \end{aligned}$$

$\because$ $m + 2k^2 + 2k + 1$ is an integer

$\therefore$ $2(k + 1)^3 + (k + 1)$ is also divisible by 3

If $P(k)$ is true then $P(k + 1)$ is also true

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

6. **HKCEE Additional Mathematics 2006 Q8**

Prove that $n^3 - n + 3$ is divisible by 3 for all positive integers n .

Let $P(n) \equiv$ "$n^3 - n + 3$ is divisible by 3 for all positive integers n ."

$n = 1$, $1^3 - 1 + 3 = 3$, which is divisible by 3.

Suppose $k^3 - k + 3$ is divisible by 3 for some positive integer k.

i.e. $k^3 - k + 3 = 3m$ for some integer m.

When $n = k + 1$, $(k + 1)^3 - (k + 1) + 3$

$$\begin{aligned} &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= (k^3 - k + 3) + 3(k^2 + k) \\ &= 3m + 3(k^2 + k) \\ &= 3(m + k^2 + k) \end{aligned}$$

which is a multiple of 3.

$\therefore$ If $P(k)$ is true, then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

7. **HKCEE Additional Mathematics 2010 Q8**

Prove, by mathematical induction, that $n(n^2 - 3n + 5)$ is divisible by 3 for all positive integers n.

Let $P(n) \equiv$ "$n(n^2 - 3n + 5)$ is divisible by 3", where n is a positive integer.

$n = 1$, $1\times(1 - 3 + 5) = 3$ which is divisible by 3 .

$\therefore$ P(1) is true.

Suppose $k(k^2 - 3k + 5)$ is divisible by 3 for some positive integer k .

$k(k^2 - 3k + 5) = 3m$, where m is an integer.

When $n = k + 1$,

$(k + 1)[(k + 1)^2 - 3(k + 1) + 5]$

$= (k + 1)(k^2 + 2k + 1 - 3k - 3 + 5)$

$= (k + 1)(k^2 - k + 3)$

$= k^3 - k^2 + 3k + k^2 - k + 3$

$= k^3 + 2k + 3 = (k^3 - 3k^2 + 5k) + (3k^2 - 3k + 3)$

$= 3m + 3(k^2 - k + 1) = 3(m + k^2 - k + 1)$

$\because$ $m + k^2 - k + 1$ is an integer

$\therefore$ $3(m + k^2 - k + 1)$ is divisible by 3

$\therefore$ $(k + 1)[(k + 1)^2 - 3(k + 1) + 5]$ is divisible by 3

$\therefore$ If P(k) is true then P(k +1) is also true.

By M.I., P(n) is true for all positive integer n .

8. **HKCEE Additional Mathematics 2011 Q2**

Prove that $5^n - 2^n$ is divisible by 3 for all positive integers n .

Let $P(n) \equiv$ "$5^n - 2^n$ is divisible by 3", where n is a positive integer.

$n = 1$, $5 - 2 = 3$, which is divisible by 3 .

$P(1)$ is true.

Suppose $P(k)$ is true.

i.e. $5^k - 2^k = 3m$ for some positive integer m.

When $n = k + 1$.

$$
\begin{aligned}
5^{k+1} - 2^{k+1} &= 5\times5^k - 2\times2^k \\
&= 5\times(2^k + 3m) - 2\times2^k \\
&= 5\times2^k - 2\times2^k + 5\times3m \\
&= 3\times2^k + 5\times3m \\
&= 3\times(2^k + 5m)
\end{aligned}
$$

$\because$ $2^k + 5m$ is an integer

$\therefore$ $5^{k+1} - 2^{k+1}$ is divisible by 3

If $P(k)$ is true, then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .