Given an acute angled ΔABC. The perimeter of the pedal Δ is less than $\frac{1}{2}\cdot$perimeter of ΔABC.

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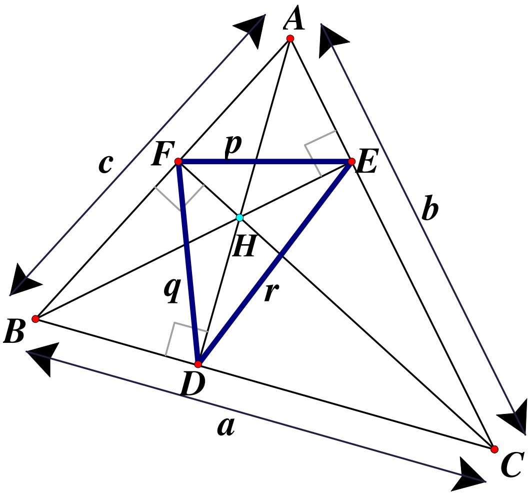


In the figure, ΔABC is an acute-angled triangle.

Let $BC = a$, $AC = b$, $AB = c$.

AD, BE, CF are the altitudes of ΔABC.

They are concurrent at the orthocentre H.

ΔDEF is the pedal triangle of ΔABC.

Let $EF = p$, $DF = q$, $DE = r$.

$\angle BFC = 90° = \angle BEC$ (given)

$\therefore$ $BCEF$ is a cyclic quad. (converse, $\angle$ in semi-circle)

$\angle AEF = \angle B$, $\angle AFE = \angle C$ (ext. $\angle$ cyclic quad.)

$\angle ADC = 90° = \angle AFC$ (given)

$\therefore$ $ACDF$ is a cyclic quad. (converse, $\angle$ in semi-circle)

$\angle BDF = \angle A$, $\angle BFD = \angle C$ (ext. $\angle$ cyclic quad.)

$\angle AEB = 90° = \angle ADB$ (given)

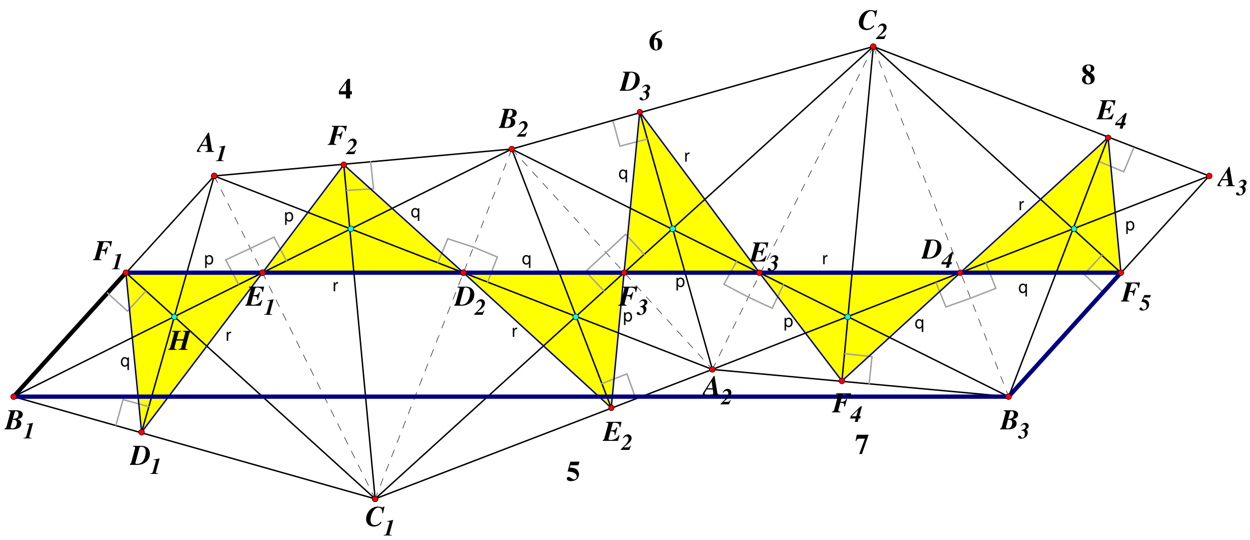
$\therefore$ $AEDB$ is a cyclic quad. (converse, $\angle$ in semi-circle)

$\angle CDE = \angle A$, $\angle CED = \angle B$ (ext. $\angle$ cyclic quad.)

$\angle ADF = 90° - \angle BDF = 90° - \angle A = 90° - \angle CDE = \angle ADE$ … (1)

$\angle BED = 90° - \angle CED = 90° - \angle B = 90° - \angle AEF = \angle BEF$ … (2)

$\angle CFD = 90° - \angle BFD = 90° - \angle C = 90° - \angle AFE = \angle CFE$ … (3)



Now relabel ΔABC as $\Delta A_1B_1C_1$, ΔDEF as $\Delta D_1E_1F_1$. Shade $\Delta D_1E_1F_1$.

Reflect $\Delta A_1B_1C_1$ along the dotted line A_1C_1 to give $\Delta A_1B_2C_1$, $\Delta D_1E_1F_1 \cong \Delta D_2E_1F_2$. … (4)

Reflect $\Delta A_1B_2C_1$ along the dotted line B_2C_1 to give $\Delta A_2B_2C_1$, $\Delta D_2E_1F_2 \cong \Delta D_2E_2F_3$. … (5)

Reflect $\Delta A_2B_2C_1$ along the dotted line A_2B_2 to give $\Delta A_2B_2C_2$, $\Delta D_2E_2F_3 \cong \Delta D_3E_3F_3$. … (6)

Reflect $\Delta A_2B_2C_2$ along the dotted line A_2C_2 to give $\Delta A_2B_3C_2$, $\Delta D_3E_3F_3 \cong \Delta D_4E_3F_4$. … (7)

Reflect $\Delta A_2B_3C_2$ along the dotted line B_3C_2 to give $\Delta A_3B_3C_2$, $\Delta D_4E_3F_4 \cong \Delta D_4E_4F_5$. … (8)

$\angle B_1E_1B_2 = 90° + 90° = 180°$, $\angle A_1D_2A_2 = 90° + 90° = 180°$, $\angle C_1F_3C_2 = 90° + 90° = 180°$,

$\angle B_2E_3B_3 = 90° + 90° = 180°$, $\angle A_2D_4E_4 = 90° + 90° = 180°$

$\therefore$ $B_1E_1B_2$, $A_1D_2A_2$, $C_1F_3C_2$, $B_2E_3B_3$, $A_2D_4E_4$ are straight lines

By the results of (1), (2), (3), $\angle B_1E_1F_1 = \angle B_2E_1D_2$, $\angle A_1D_2E_1 = \angle A_2D_2F_3$, $\angle C_1F_3D_2 = \angle C_2F_3E_3$, $\angle B_2E_3F_3 = \angle B_3E_3D_4$, $\angle A_2D_4E_3 = \angle A_3D_4F_5$.

$\therefore$ $F_1E_1D_2$, $E_1D_2F_3$, $D_2F_3E_3$, $F_3E_3D_4$, $E_3D_4F_5$ are straight lines (converse, vert. opp. ∠s)

i.e. F_1, E_1, D_2, F_3, E_3, D_4, F_5 are collinear.

$F_1F_5 = p + r + q + p + r + q = 2(p + q + r) = 2$(perimeter of pedal Δ$DEF$)

By the property of reflection, B_1A_1 has turned $2\angle A$ (anti-clockwise) to B_2A_1.

B_2A_1 has turned $2\angle B$ (anti-clockwise) to B_2A_2.

B_2A_2 has turned $2\angle A$ (clockwise) to B_3A_2, B_3A_2 has turned $2\angle B$ (clockwise) to B_3A_3.

Take anti-clockwise rotation as positive and clockwise rotation as negative.

Then the overall angle of rotation of B_1A_1 to B_3A_3 is $2\angle A + 2\angle B - 2\angle A - 2\angle B = 0°$

$\therefore$ B_1A_1 // B_3A_1

Clearly $\Delta B_1C_1F_1 \cong \Delta B_3C_2F_5$ (A.A.S.)

$\therefore$ $B_1F_1 = B_3F_5$ (corr. sides, ≅ Δ's)

$\Rightarrow$ $B_1B_3F_5F_1$ is a //-gram (opp. sides are eq. and //)

$B_1B_3 = F_1F_5 = 2(p + q + r)$ (opp. sides of //-gram)

On the other hand, $B_1B_3 \le B_1C_1 + C_1A_2 + A_2B_3$ (The shortest distance between 2 points is a st. line)

$\therefore$ $2(p + q + r) \le a + b + c$

i.e. **perimeter of pedal ΔDEF $\le \frac{1}{2}$ ·perimeter of ΔABC.** Q.E.D.