Examples on Mathematical Induction: divisibility surd

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1. Given the statement: $(1+\sqrt{5})^n - (1-\sqrt{5})^n$ is divisible by $2^n\sqrt{5}$.
   Prove that the statement is true for $n = 1$ and 2.
   Hence prove that the statement is true for all positive integer n.
2. Given the statement: $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is divisible by 2^n.
   Prove that the statement is true for $n = 1$ and 2.
   Hence prove that the statement is true for all positive integer n.

2. $n = 1$, $(3+\sqrt{5})+(3-\sqrt{5}) = 6 = 2\times3$, which is divisible by 2.
   $n= 2$, $(3+\sqrt{5})^2+(3-\sqrt{5})^2 = 9+6\sqrt{5}+5+9-6\sqrt{5}+5 = 28 = 4\times7$, which is divisible by 2^2.
   Suppose $(3+\sqrt{5})^{k-1}+(3-\sqrt{5})^{k-1} = 2^{k-1}Q$, where Q is an integer, for some integer $k-1$, and suppose $(3+\sqrt{5})^k+(3-\sqrt{5})^k = 2^kR$, where R is an integer, for some integer k.

$$(3+\sqrt{5})^{k+1}+(3-\sqrt{5})^{k+1}$$
$$=(3+\sqrt{5})^k\cdot(3+\sqrt{5})+(3-\sqrt{5})^k\cdot(3-\sqrt{5})$$
$$=\left[(3+\sqrt{5})^k+(3-\sqrt{5})^k\right]\cdot(3+\sqrt{5})+\left[(3+\sqrt{5})^k+(3-\sqrt{5})^k\right]\cdot(3-\sqrt{5})-\left[(3-\sqrt{5})^k\cdot(3+\sqrt{5})+(3+\sqrt{5})^k\cdot(3-\sqrt{5})\right]$$
$$=2^kR\cdot(3+\sqrt{5})+2^kR\cdot(3-\sqrt{5})-(3+\sqrt{5})(3-\sqrt{5})\left[(3-\sqrt{5})^{k-1}+(3+\sqrt{5})^{k-1}\right]$$
$$=2^kR\cdot(3+\sqrt{5}+3-\sqrt{5})-(9-5)(2^{k-1}Q)$$
$$=2^kR\cdot6-4(2^{k-1}Q)=2^{k+1}(3R-Q),\ 3R-Q \text{ is an integer.}$$

which is divisible by 2^{k+1}.
By the principle of mathematical induction, the statement is true for all positive integer n.

3. Given the statement: $(\sqrt{3}+1)^{2n+1}-(\sqrt{3}-1)^{2n+1}$ is divisible by 2^{n+1}.
   Prove that the statement is true for $n = 1$ and 2.
   Hence prove that the statement is true for all positive integer n.

4. Let $a = \frac{1}{2}\left(3+\sqrt{13}\right)$, $b = \frac{1}{2}\left(3-\sqrt{13}\right)$, $f(x) = \frac{a^x - b^x}{\sqrt{13}}$.

Prove that f(3n) is divisible by 10 using $(a^{3n} - b^{3n})(a^3 + b^3)$.

4. $a^3 + b^3 = \frac{\left(3+\sqrt{13}\right)^3 + \left(3-\sqrt{13}\right)^3}{8} = \frac{\left(3+\sqrt{13}+3-\sqrt{13}\right)\left[\left(3+\sqrt{13}\right)^2 - \left(3+\sqrt{13}\right)\left(3-\sqrt{13}\right) + \left(3-\sqrt{13}\right)^2\right]}{8}$

$= \frac{3\left[\left(9+6\sqrt{13}+13\right) - (9-13) + \left(9-6\sqrt{13}+13\right)\right]}{4}$

$= \frac{3(48)}{4} = 36$

$f(3) = \frac{a^3 - b^3}{\sqrt{13}} = \frac{\left(3+\sqrt{13}\right)^3 - \left(3-\sqrt{13}\right)^3}{8\sqrt{13}} = \frac{\left(3+\sqrt{13}-3+\sqrt{13}\right)\left[\left(3+\sqrt{13}\right)^2 + \left(3+\sqrt{13}\right)\left(3-\sqrt{13}\right) + \left(3-\sqrt{13}\right)^2\right]}{8\sqrt{13}}$

$= \frac{\left(9+6\sqrt{13}+13\right) + (9-13) + \left(9-6\sqrt{13}+13\right)}{4} = 10$, which is divisible by 10

$f(6) = \frac{a^6 - b^6}{\sqrt{13}} = \frac{\left(a^3+b^3\right)\left(a^3-b^3\right)}{\sqrt{13}} = 36\times 10$, which is divisible by 10

Suppose $f(3(k-1)) = \frac{a^{3(k-1)} - b^{3(k-1)}}{\sqrt{13}} = 10p$ for some positive integer k–1, where p is an integer.

Suppose $f(3k) = \frac{a^{3k} - b^{3k}}{\sqrt{13}} = 10q$ for some positive integer k, where q is an integer.

$f(3(k+1)) = \frac{a^{3k+3} - b^{3k+3}}{\sqrt{13}} = \frac{\left(a^3+b^3\right)\left(a^{3k}-b^{3k}\right) - \left(a^{3k}b^3 - a^3b^{3k}\right)}{\sqrt{13}}$

$= \frac{\left(a^3+b^3\right)\left(a^{3k}-b^{3k}\right) - a^3b^3\left(a^{3(k-1)} - b^{3(k-1)}\right)}{\sqrt{13}}$

$= 36\times 10q - \left(\frac{9-13}{4}\right)^3 \cdot 10p = 10(36q + p)$, which is divisible by 10

By the principle of mathematical induction, the statement is true for all positive integer n.