Division Problem

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Find all ‘x’ in the division on the right.

$$
\begin{array}{rl}
 & \text{x x 7 x x} \\ \cline{2-2}
\text{x x x x 7 x}\,) & \text{x x 7 x x x x x x x} \\
 & \text{x x x x x x} \\ \cline{2-2}
 & \text{x x x x x 7 x} \\
 & \text{x x x x x x x} \\ \cline{2-2}
 & \quad\text{x 7 x x x x} \\
 & \quad\text{x 7 x x x x} \\ \cline{2-2}
 & \quad\text{x x x x x x x} \\
 & \quad\text{x x x x 7 x x} \\ \cline{2-2}
 & \qquad\text{x x x x x x} \\
 & \qquad\text{x x x x x x} \\ \cline{2-2}
\end{array}
$$

Relabel the ‘x’ as shown.

Let $a = (a_1a_27a_4a_5)_x$

$b = (b_1b_2b_3b_47b_6)_x$

$c = (c_1c_27c_4c_5c_6c_7c_8c_9c_{10})_x$

$d = (d_1d_2d_3d_4d_5d_6)_x$

$e = (e_1e_2e_3e_4e_57e_7)_x$

$f = (f_1f_2f_3f_4f_5f_6f_7)_x$

$g = (g_17g_3g_4g_5g_6)_x$

$h = (h_17h_3h_4h_5h_6)_x$

$i = (i_1i_2i_3i_4i_5i_6i_7)_x$

$j = (j_1j_2j_3j_47j_6j_7)_x$

$k = (k_1k_2k_3k_4k_5k_6)_x$

$m = (m_1m_2m_3m_4m_5m_6)_x$

$$
\begin{array}{rl}
 & \qquad\qquad a_1\ a_2\ 7\ a_4\ a_5 \\ \cline{2-2}
b_1\,b_2\,b_3\,b_4\,7\,b_6\,) & c_1\ c_2\ 7\ c_4\ c_5\ c_6\ c_7\ c_8\ c_9\ c_{10} \\
 & d_1\,d_2\,d_3\,d_4\,d_5\ d_6 \\ \cline{2-2}
 & e_1\ e_2\ e_3\ e_4\ e_5\ 7\ e_7 \\
 & f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7 \\ \cline{2-2}
 & \quad g_1\ 7\ g_3\ g_4\ g_5\ g_6 \\
 & \quad h_1\ 7\ h_3\ h_4\ h_5\ h_6 \\ \cline{2-2}
 & \quad i_1\ i_2\ i_3\ i_4\ i_5\ i_6\ i_7 \\
 & \quad j_1\ j_2\ j_3\ j_4\ 7\ j_6\ j_7 \\ \cline{2-2}
 & \qquad k_1\ k_2\ k_3\ k_4\ k_5\ k_6 \\
 & \qquad m_1\,m_2\,m_3\,m_4\,m_5\,m_6 \\ \cline{2-2}
\end{array}
$$

$b\times 7 = h$, a 6-digits number.

f and j are 7-digits numbers.

so a_2 , a_4 = 8 or 9

$b\times 8$ or $b\times 9$ = 7-digits number

$b\times 7 \le 979999$ and $b\times 9 \ge 1000700$

$111189 \le b \le 139999 \Rightarrow b_1 = 1$

$b_5 = 7 \Rightarrow 111270 \le b \le 139979$ ……..(1)

$$
\begin{array}{rl}
 & \qquad\qquad a_1\ a_2\ 7\ a_4\ a_5 \\ \cline{2-2}
1\,b_2\,b_3\,b_4\,7\,b_6\,) & c_1\ c_2\ 7\ c_4\ c_5\ c_6\ c_7\ c_8\ c_9\ c_{10} \\
 & d_1\,d_2\,d_3\,d_4\,d_5\ d_6 \\ \cline{2-2}
 & e_1\ e_2\ e_3\ e_4\ e_5\ 7\ e_7 \\
 & f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7 \\ \cline{2-2}
 & \quad g_1\ 7\ g_3\ g_4\ g_5\ g_6 \\
 & \quad h_1\ 7\ h_3\ h_4\ h_5\ h_6 \\ \cline{2-2}
 & \quad 1\ i_2\ i_3\ i_4\ i_5\ i_6\ i_7 \\
 & \quad 1\ j_2\ j_3\ j_4\ 7\ j_6\ j_7 \\ \cline{2-2}
 & \qquad k_1\ k_2\ k_3\ k_4\ k_5\ k_6 \\
 & \qquad m_1\,m_2\,m_3\,m_4\,m_5\,m_6 \\ \cline{2-2}
\end{array}
$$

$778890 \le 7b \le 979853$

h_1 = 7, 8 or 9

when $h_1 = 9$, i_1 has no solution, reject

when $h_1 = 8$, $g_1 = 9$, $i_1 = 1$ …….(2)

when $h_1 = 7$, $g_1 = 9$, i_1 = 1 or 2 ……(3)

when $h_1 = 7$, $g_1 = 8$, $i_1 = 1$ ……..(4)

$b\times a_4 = j$ and a_4 = 8 or 9

$890160\le 8b\le 1119832$, $1001430\le 9b\le 1259811$

$\Rightarrow 1000700 \le j \le 1259799$

$\Rightarrow j_1 = 1, j_2$ = 0, 1 or 2

$\Rightarrow i_1 = 1$, $i_2 = j_2$ or $j_2 + 1$

i_2 = 1, 2 or 3 …….(5)

If there is a borrow digit for $g_3 - h_3$, then $i_2 = 9$
contradict to (5), so there is no borrow digit.
$g_1 - h_1 = 1$ and $g_2 - h_2 = 7 - 7 = 0 = i_2$
$j_2 = 0$
If $a_4 = 9$, $1000700 \le 9b \le 1099799$
$111189 \le b \le 122200$
$b_5 = 7 \Rightarrow 111270 \le b \le 122179$
$778890 \le 7b \le 855253$
$h_2 = 7 \Rightarrow 778890 \le 7b \le 779999$
$111270 \le b \le 111428$
$b_5 = 7 \Rightarrow 111270 \le b \le 111379$

$1001430 \le 9b \le 1002411$
$j_5 = 7 \Rightarrow 1001700 \le 9b \le 1001799$
$111300 \le b \le 111311$
$b_5 = 7 \Rightarrow$ no solution
$\therefore a_4 \ne 9 \Rightarrow a_4 = 8$
By (1): $111270 \le b \le 139979$
$890160 \le 8b \le 1119832$
$j = 8b$, a 7-digits number
$\Rightarrow 1000700 \le 8b \le 1099799$
$125088 \le b \le 137474$
$b_5 = 7 \Rightarrow 125170 \le b \le 137474$

$1001360 \le 8b$
$j_5 = 7 \Rightarrow 1001700 \le 8b$
$125213 \le b$
$b_5 = 7 \Rightarrow 125270 \le b$
$1002160 \le 8b$
$j_5 = 7 \Rightarrow 1002700 \le 8b$
$125338 \le b$
$b_5 = 7 \Rightarrow 125370 \le b$
$1002960 \le 8b$
$j_5 = 7 \Rightarrow 1003700 \le 8b$
$125463 \le b$
$b_5 = 7 \Rightarrow 125470 \le b \le 137474$ .....(6)
$1003760 \le 8b \le 1099792$
$b \times 7 = h$ and $g_1 - h_1 = 1 \Rightarrow h_1 = 7$ or 8
$878290 \le 7b \Rightarrow h_1 = 8$, $g_1 = 9$

$$
\begin{array}{r}
 & & & & & & & a_1 & a_2 & 7 & a_4 & a_5 \\
\hline
1\ b_2\ b_3\ b_4\ 7\ b_6\ ) & c_1 & c_2 & 7 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} \\
 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & & & & \\
\hline
 & e_1 & e_2 & e_3 & e_4 & e_5 & 7 & e_7 & & & \\
 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & & & \\
\hline
 & & & g_1 & 7 & g_3 & g_4 & g_5 & g_6 & & \\
 & & & h_1 & 7 & h_3 & h_4 & h_5 & h_6 & & \\
\hline
 & & & 1 & 0 & i_3 & i_4 & i_5 & i_6 & i_7 & \\
 & & & 1 & 0 & j_3 & j_4 & 7 & j_6 & j_7 & \\
\hline
 & & & & & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\
\hline
 & & & & & m_1 & m_2 & m_3 & m_4 & m_5 & m_6
\end{array}
$$

$$
\begin{array}{r}
 & & & & & & & a_1 & a_2 & 7 & 8 & a_5 \\
\hline
1\ b_2\ b_3\ b_4\ 7\ b_6\ ) & c_1 & c_2 & 7 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} \\
 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & & & & \\
\hline
 & e_1 & e_2 & e_3 & e_4 & e_5 & 7 & e_7 & & & \\
 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & & & \\
\hline
 & & & g_1 & 7 & g_3 & g_4 & g_5 & g_6 & & \\
 & & & h_1 & 7 & h_3 & h_4 & h_5 & h_6 & & \\
\hline
 & & & 1 & 0 & i_3 & i_4 & i_5 & i_6 & i_7 & \\
 & & & 1 & 0 & j_3 & j_4 & 7 & j_6 & j_7 & \\
\hline
 & & & & & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\
\hline
 & & & & & m_1 & m_2 & m_3 & m_4 & m_5 & m_6
\end{array}
$$

$$
\begin{array}{r}
 & & & & & & & a_1 & a_2 & 7 & 8 & a_5 \\
\hline
1\ b_2\ b_3\ b_4\ 7\ b_6\ ) & c_1 & c_2 & 7 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} \\
 & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & & & & \\
\hline
 & e_1 & e_2 & e_3 & e_4 & e_5 & 7 & e_7 & & & \\
 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & & & \\
\hline
 & & 9 & 7 & g_3 & g_4 & g_5 & g_6 & & & \\
 & & 8 & 7 & h_3 & h_4 & h_5 & h_6 & & & \\
\hline
 & & 1 & 0 & i_3 & i_4 & i_5 & i_6 & i_7 & & \\
 & & 1 & 0 & j_3 & j_4 & 7 & j_6 & j_7 & & \\
\hline
 & & & & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & \\
\hline
 & & & & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 &
\end{array}
$$

$h = (87h_3h_4h_5h_6)_x = 7b$

$870000 \le 7b \le 879999$

$124286 \le b \le 125714$

combine with (6): $125470 \le b \le 125714$

$b_5 = 7 \Rightarrow 125470 \le b \le 125679$

$1003760 \le 8b \le 1005432$

$j_5 = 7 \Rightarrow 8b \le 1004799$

$125470 \le b \le 125599$ .....(7)

$878290 \le 7b \le 879193$ .....(8)

$1003760 \le 8b \le 1004792$ .....(9)

$b_2 = 2$, $b_3 = 5$, $b_4 = 4$ or 5

$h_3 = 8$ or 9

$j_3 = 0$, $j_4 = 3$ or 4

when $h_3 = 9$, $g_3 = 9$, $i_3 = 0$, $i_3 - 0 = k_1$, no solution

when $h_3 = 8$, $g_3 = 8$ or 9, $i_3 = 0$ or 1

$i_3 - j_3 = k_1 \Rightarrow k_1 = 1$, $i_3 = 1$, $j_3 = 0 \Rightarrow a_5 = 1$

$g_3 = 9$, $k_1 = m_1 = 1$, $k_2 = m_2 = 2$, $k_3 = m_3 = 5$

By (8) and $h_3 = 8 \Rightarrow 878290 \le 7b \le 878999$

$125470 \le b \le 125571$

$1003760 \le 8b \le 1004568$

$j_5 = 7 \Rightarrow 8b \le 1003799$

$125470 \le b \le 125474$ .....(10)

$\Rightarrow b_4 = k_4 = m_4 = 4$, $k_5 = m_5 = 7$

$878290 \le 7b \le 878318$ .....(11)

$\Rightarrow h_4 = 2$ or 3

$1003760 \le 8b \le 1003792$ .....(12)

$\Rightarrow 6 \le j_6 \le 9$

$j_4 = 3$

$j_6 = 7 > 5 = k_3 \Rightarrow$ there is a borrow digit for i_4

$i_4 - 1 - j_4 = 2 \Rightarrow i_4 = 6$

$i_5 = 2$ or 3 .....(13)

when $h_4 = 2$, by (11), $878290 \le 7b \le 878299$

$h_5 = 9$

$g_4 - h_4 = 6 \Rightarrow g_4 = 8$

$\Rightarrow 10 + e_6 - f_6 = g_4$ or $10 + e_6 - 1 - f_6 = g_4$

$\Rightarrow f_6 = 9$ or 8 .....(14)

$125470 \le b \le 125471$, $b_6 = 0$ or 1

$a_2 = 8$ or $9 \Rightarrow f = 8b$ or $9b$

when b =125470, a_2=8, f =1003760 contradict (14)

$$
\begin{array}{rl}
 & a_1\ a_2\ 7\ 8\ a_5 \\
\hline
1\ 2\ 5\ b_4\ 7\ b_6\ ) & c_1\ c_2\ 7\ c_4\ c_5\ c_6\ c_7\ c_8\ c_9\ c_{10} \\
 & d_1\ d_2\ d_3\ d_4\ d_5\ d_6 \\
\hline
 & e_1\ e_2\ e_3\ e_4\ e_5\ 7\ e_7 \\
 & f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7 \\
\hline
 & 9\ 7\ g_3\ g_4\ g_5\ g_6 \\
 & 8\ 7\ h_3\ h_4\ h_5\ h_6 \\
\hline
 & 1\ 0\ i_3\ i_4\ i_5\ i_6\ i_7 \\
 & 1\ 0\ j_3\ j_4\ 7\ j_6\ j_7 \\
\hline
 & k_1\ k_2\ k_3\ k_4\ k_5\ k_6 \\
 & m_1\ m_2\ m_3\ m_4\ m_5\ m_6 \\
\hline
\end{array}
$$

$$
\begin{array}{rl}
 & a_1\ a_2\ 7\ 8\ 1 \\
\hline
1\ 2\ 5\ 4\ 7\ b_6\ ) & c_1\ c_2\ 7\ c_4\ c_5\ c_6\ c_7\ c_8\ c_9\ c_{10} \\
 & d_1\ d_2\ d_3\ d_4\ d_5\ d_6 \\
\hline
 & e_1\ e_2\ e_3\ e_4\ e_5\ 7\ e_7 \\
 & f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7 \\
\hline
 & 9\ 7\ 9\ g_4\ g_5\ g_6 \\
 & 8\ 7\ 8\ h_4\ h_5\ h_6 \\
\hline
 & 1\ 0\ 1\ i_4\ i_5\ i_6\ i_7 \\
 & 1\ 0\ 0\ j_4\ 7\ j_6\ j_7 \\
\hline
 & 1\ 2\ 5\ 4\ 7\ k_6 \\
 & 1\ 2\ 5\ 4\ 7\ m_6 \\
\hline
\end{array}
$$

$$
\begin{array}{rl}
 & a_1\ a_2\ 7\ 8\ 1 \\
\hline
1\ 2\ 5\ 4\ 7\ b_6\ ) & c_1\ c_2\ 7\ c_4\ c_5\ c_6\ c_7\ c_8\ c_9\ c_{10} \\
 & d_1\ d_2\ d_3\ d_4\ d_5\ d_6 \\
\hline
 & e_1\ e_2\ e_3\ e_4\ e_5\ 7\ e_7 \\
 & f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7 \\
\hline
 & 9\ 7\ 9\ g_4\ g_5\ g_6 \\
 & 8\ 7\ 8\ h_4\ h_5\ h_6 \\
\hline
 & 1\ 0\ 1\ 6\ i_5\ i_6\ i_7 \\
 & 1\ 0\ 0\ 3\ 7\ j_6\ j_7 \\
\hline
 & 1\ 2\ 5\ 4\ 7\ k_6 \\
 & 1\ 2\ 5\ 4\ 7\ m_6 \\
\hline
\end{array}
$$

when b=125470, a_2=9, f=1129230, contradict (14)
when b=125471, a_2=8, f=1003768, contradict (14)
when b=125471, a_2=9, f=1129239, contradict (14)
so $h_4 \neq 2, \Rightarrow h_4 = 3$
$g_4 = 9, f_6 = 8$ or 7 .....(15)
By (11) and $h_4 = 3$, $878300 \le 7b \le 878318$
$125472 \le b \le 125474$
when $a_2 = 9$, $1129248 \le 9b \le 1129266$
$1129248 \le f \le 1129266$
contradict with (15)
so $a_2 = 8$

$1003776 \le 8b \le 1003792, f = 8b$
$f_6 = 8$ or $7 \Rightarrow 1003776 \le 8b \le 1003789$
$125472 \le b \le 125473$
$f_1 = 1, f_2 = 0, f_3 = 0, f_4 = 3, f_5 = 7$
$878304 \le 7b \le 878311$
$1003776 \le 8b \le 1003784$
If $b_6 = 2$, then $k = m = 125472$
$j = 1003776 = f$
$i = j + 12547 = 1016323$
$h = 7b = 878304$
$g = h + 101632 = 979936$
$e = f + 97993 = 1101769$ contradict $e_6 = 7$
Therefore $b_6 = k_6 = m_6 = 3$
$j = 8b = 1003784 = f$
$i = j + 12547 = 1016331$
$h = 7b = 878311$
$g = h + 101633 = 979944$
$e = f + 97994 = 1101778$
$7 - d_3 = 0$ or $6 - d_3 = 0 \Rightarrow d_3 = 6$ or 7
By the method of exhaustion
$125473 \times 3 = 376419$, $125473 \times 5 = 627365$
$a_1 = 3$ or 5
When $a_1 = 3$, $d = 376419$
$c = (d+110177) \times 10000 + 8413 = 4865968413$
contradict $c_3 = 7$
So $a_1 = 5$, $d = 627365$,
$c = (d + 110177) \times 10000 + 8413 = 7375428413$

					a_1	8	7	8	1
1 2 5 4 7 b_6)	c_1	c_2	7	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
	d_1	d_2	d_3	d_4	d_5	d_6				
	e_1	e_2	e_3	e_4	e_5	7	e_7			
	f_1	f_2	f_3	f_4	f_5	f_6	f_7			
			9	7	9	9	g_5	g_6		
			8	7	8	3	h_5	h_6		
		1	0	1	6	i_5	i_6	i_7		
		1	0	0	3	7	j_6	j_7		
				1	2	5	4	7	k_6	
				1	2	5	4	7	m_6	

					a_1	8	7	8	1
1 2 5 4 7 b_6)	c_1	c_2	7	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
	d_1	d_2	d_3	d_4	d_5	d_6				
	e_1	e_2	e_3	e_4	e_5	7	e_7			
	1	0	0	3	7	f_6	f_7			
			9	7	9	9	g_5	g_6		
			8	7	8	3	h_5	h_6		
		1	0	1	6	i_5	i_6	i_7		
		1	0	0	3	7	j_6	j_7		
				1	2	5	4	7	k_6	
				1	2	5	4	7	m_6	

					5	8	7	8	1
1 2 5 4 7 3)	7	3	7	5	4	2	8	4	1	3
	6	2	7	3	6	5				
	1	1	0	1	7	7	8			
	1	0	0	3	7	8	4			
			9	7	9	9	4	4		
			8	7	8	3	1	1		
			1	0	1	6	3	3	1	
			1	0	0	3	7	8	4	
					1	2	5	4	7	3
					1	2	5	4	7	3