89-90 Individual	**1**	5	**2**	–2	**3**	7	**4**	6	**5**	(2, 4)
	6	4	**7**	120	**8**	–7	**9**	100	**10**	109
	11	9	**12**	0	**13**	2519	**14**	2	**15**	10 days
	16	5	**17**	$2\sqrt{13}$	**18**	1 : 2	**19**	$\frac{9}{20}$	**20**	58.5

89-90 Group	**1**	275	**2**	73	**3**	2	**4**	0	**5**	1783
	6	26	**7**	$\frac{125}{8}=15\frac{5}{8}$	**8**	4 : 1	**9**	7	**10**	2π

Individual Events

I1 Find the value of $\frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2}$.

$$\frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2}$$
$$=3+\sqrt{8}-\left(\sqrt{8}+\sqrt{7}\right)+\left(\sqrt{7}+\sqrt{6}\right)-\left(\sqrt{6}+\sqrt{5}\right)+\left(\sqrt{5}+2\right)$$
$$= 5$$

I2 If $b < 0$ and $2^{2b+4} - 20\times 2^b + 4 = 0$, find b.

Let $y = 2^b$, then $y^2 = 2^{2b}$, the equation becomes $16y^2 - 20y + 4 = 0$

$4y^2 - 5y + 1 = 0$

$(4y - 1)(y - 1) = 0$

$y = 2^b = \frac{1}{4}$ or $y = 1$

$b = -2$ or 0

$\because b < 0 \therefore b = -2$ only

I3 If $f(a) = a - 2$ and $F(a, b) = a + b^2$, find $F(3, f(4))$.

Reference: 1985 FI3.3, 2013 FI3.2, 2015 FI4.3

$f(4) = 4 - 2 = 2$

$F(3, f(4)) = F(3, 2) = 3 + 2^2 = 7$

I4 For positive integers a and b, define $a\#b = a^b + b^a$. If $2\#w = 100$, find the value of w.

Reference: 1999 FI3.1

$2^w + w^2 = 100$ for positive integer w.

By trail and error, $64 + 36 = 100$

$w = 6$.

I5 a and b are constants. The straight line $2ax + 3by = 4a + 12b$ passes through a fixed point P whose coordinates do not depend on a and b. Find the coordinates of P.

Reference: 1991 HI6, 1996 HI6

$2ax + 3by = 4a + 12b \Rightarrow 2a(x - 2) + 3b(y - 4) = 0$

Put $b = 0 \Rightarrow x = 2$,

Put $a = 0 \Rightarrow y = 4$

$P(2, 4)$

I6 The sines of the angles of a triangle are in the ratio 3 : 4 : 5. If A is the smallest interior angle of the triangle and $\cos A = \frac{x}{5}$, find the value of x.

Reference: 1989 HI10

By Sine rule, $a : b : c = \sin A : \sin B : \sin C = 3 : 4 : 5$

Let $a = 3k$, $b = 4k$, $c = 5k$.

$a^2 + b^2 = (3k)^2 + (4k)^2 = (5k)^2 = c^2$

$\therefore \angle C = 90°$ (converse, Pythagoras' theorem)

$\cos A = \frac{b}{c} = \frac{4}{5}$

$\Rightarrow x = 4$

I7 If $x + y = 9$, $y + z = 11$ and $z + x = 10$, find the value of xyz.

Reference: 1986 FG10.1, 1989 HI15

(1) + (2) – (3): $2y = 10 \Rightarrow y = 5$

(1) + (3) – (2): $2x = 8 \Rightarrow x = 4$

(2) + (3) – (1): $2z = 12 \Rightarrow z = 6$

$\Rightarrow xyz = 120$

I8 If α, β are the roots of the equation $2x^2 + 4x - 3 = 0$ and α^2, β^2 are the roots of the equation $x^2 + px + q = 0$, find the value of p.

$\alpha + \beta = -2$

$\alpha\beta = -\frac{3}{2}$

$p = -(\alpha^2 + \beta^2) = -(\alpha + \beta)^2 + 2\alpha\beta$

$= -(-2)^2 - 3 = -7$

I9 If $x^{\log_{10} x} = \frac{x^3}{100}$ and $x > 10$, find the value of x. **Reference: 2015 FI4.4, 2023 FG4.1**

Take log on both sides, $\log x \cdot \log x = 3 \log x - \log 100$

$(\log x)^2 - 3 \log x + 2 = 0$

$(\log x - 1)(\log x - 2) = 0$

$\log x = 1$ or $\log x = 2$

$x = 10$ or 100

$\because x > 10 \therefore x = 100$ only

I10 Given that $a_0 = 1$, $a_1 = 3$ and $a_n^2 - a_{n-1}a_{n+1} = (-1)^n$ for positive integers n. Find a_4.

Put $n = 1$, $a_1^2 - a_0 a_2 = (-1)^1 \Rightarrow 3^2 - a_2 = -1 \Rightarrow a_2 = 10$

Put $n = 2$, $a_2^2 - a_1 a_3 = (-1)^2 \Rightarrow 10^2 - 3a_3 = 1 \Rightarrow a_3 = 33$

Put $n = 3$, $a_3^2 - a_2 a_4 = (-1)^3 \Rightarrow 33^2 - 10a_4 = -1 \Rightarrow a_4 = 109$

I11 Find the units digit of 2137^{754}.

Reference 1991 HG1

$7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$

The pattern of units digit repeats for every multiples of 4.

$2137^{754} \equiv (7^4)^{188} \cdot 7^2 \equiv 9 \mod 10$

The units digit is 9.

I12 If $\left(r+\frac{1}{r}\right)^2=3$, find $r^3+\frac{1}{r^3}$.

Reference: 1985 FI1.2, 2017 FI1.4

$r+\frac{1}{r}=\pm\sqrt{3}$

$r^2+\frac{1}{r^2}=\left(r+\frac{1}{r}\right)^2-2=3-2=1$

$r^3+\frac{1}{r^3}=\left(r+\frac{1}{r}\right)\left(r^2-1+\frac{1}{r^2}\right)$

$=\pm\sqrt{3}(1-1)=0$

I13 A positive integer N, when divided by 10, 9, 8, 7, 6, 5, 4, 3 and 2, leaves remainders 9, 8, 7, 6, 5, 4, 3, 2 and 1 respectively. Find the least value of N.

Reference: 1985 FG7.2, 2013FG4.3

$N + 1$ is divisible by 10, 9, 8, 7, 6, 5, 4, 3, 2.

The L.C.M. of 2, 3, 4, 5, 6, 7, 8, 9, 10 is 2520.

$\therefore N = 2520k - 1$, where k is an integer.

The least positive integral of $N = 2520 - 1 = 2519$

I14 If $\frac{1}{A}=\frac{\cos 45^\circ \sin 70^\circ \cos 60^\circ \tan 40^\circ}{\cos 340^\circ \sin 135^\circ \tan 220^\circ}$, find the value of A.

Reference: 1989 HI14

$\frac{1}{A}=\frac{\cos 45^\circ \cos 20^\circ \cos 60^\circ \tan 40^\circ}{\cos 20^\circ \cos 45^\circ \tan 40^\circ}$

$=\cos 60^\circ=\frac{1}{2}$

$A = 2$

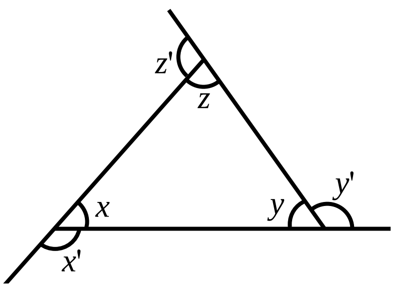
I15 If 10 men can make 20 tables in 5 days, how many days are required to make 60 tables by 15 men?

1 man can make $\frac{20}{10\times 5}=\frac{2}{5}$ table in 1 day.

15 men can make $\frac{2}{5}\times 15=6$ tables in one day.

They can make 60 tables in 10 days

I16 In figure 1, the exterior angles of the triangle are in the ratio $x' : y' : z' = 4 : 5 : 6$ and the interior angles are in the ratio $x : y : z = a : b : 3$. Find the value of b.



(Figure 1)

Let $x' = 4k$, $y' = 5k$, $z' = 6k$

$4k + 5k + 6k = 360^\circ$ (sum of ext. $\angle$ of polygon)

$15k = 360^\circ$

$\Rightarrow k = 24$

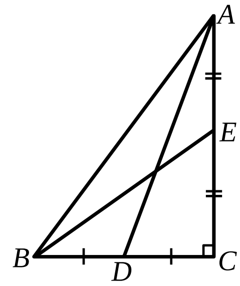
$x' = 96^\circ$, $y' = 120^\circ$, $z' = 144^\circ$

$x = 84^\circ$, $y = 60^\circ$, $z = 36^\circ$ (adj. $\angle$s on st. line)

$x : y : z = 7 : 5 : 3$

$\Rightarrow b = 5$

I17 In ΔABC, $\angle C = 90°$ and D, E are the mid-points of BC and CA respectively. If $AD = 7$ and $BE = 4$, find the length of AB. (See figure 2.) **Reference: 2024 HI5**



(Figure 2)

Let $BD = x = DC$, $AE = y = EC$

$x^2 + (2y)^2 = 7^2$ …… (1)

$(2x)^2 + y^2 = 4^2$ …… (2)

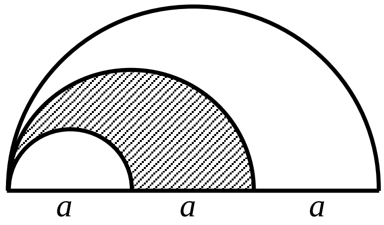
4(1) – (2): $15y^2 = 180 \Rightarrow y^2 = 12$

4(2) – (1): $15x^2 = 15 \Rightarrow x^2 = 1$

$AB^2 = (2x)^2 + (2y)^2 = 4 + 48$

$\Rightarrow AB = \sqrt{52} = 2\sqrt{13}$

I18 Figure 3 shows 3 semi-circles of diameters a, $2a$ and $3a$ respectively. Find the ratio of the area of the shaded part to that of the unshaded part.



(Figure 3)

Area of the shaded part $= \frac{\pi}{2}\cdot a^2 - \frac{\pi}{2}\cdot\left(\frac{a}{2}\right)^2 = \frac{3\pi}{8}a^2$

Area of the unshaded part $= \frac{\pi}{2}\cdot\left(\frac{3a}{2}\right)^2 - \frac{3\pi}{8}\cdot a^2 = \frac{6\pi}{8}\cdot a^2$

The ratio = 3 : 6 = 1 : 2

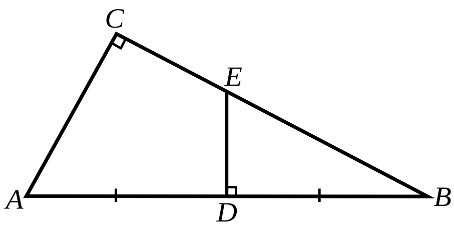
I19 Find the value of $\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots+\frac{1}{19\times 20}$.

$$\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots+\frac{1}{19\times 20}$$

$$=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{19}-\frac{1}{20}\right)$$

$$=\frac{1}{2}-\frac{1}{20}=\frac{9}{20}$$

I20 In figure 4, $\angle C = 90°$, $AD = DB$ and DE is perpendicular to AB. If $AB = 20$ and $AC = 12$, find the area of the quadrilateral $ADEC$.



(Figure 4)

$BD = 10$, $BC = 16$ (Pythagoras' theorem)

$\Delta BDE \sim \Delta BCA$ (equiangular)

$BD : DE : BE = 16 : 12 : 20$ (ratio of sides, ~Δ's)

$DE = 7.5$, $BE = 12.5$

$CE = 16 - 12.5 = 3.5$

$$S_{ADEC} = \frac{1}{2}\cdot 10\cdot 7.5 + \frac{1}{2}\cdot 12\cdot 3.5 = 58.5$$

Method 2

$BD = 10$, $BC = 16$ (Pythagoras' theorem)

$\Delta BDE \sim \Delta BCA$ (equiangular)

$$S_{\Delta BDE} = \left(\frac{BD}{BC}\right)^2 \cdot S_{\Delta ABC} = \left(\frac{10}{16}\right)^2\cdot\frac{1}{2}\cdot 12\cdot 16 = 37.5$$

$$S_{ADEC} = \frac{1}{2}\cdot 12\cdot 16 - 37.5 = 58.5$$

Group Events

G1 If $\frac{1}{a}+\frac{1}{b}=5$ and $\frac{1}{a^2}+\frac{1}{b^2}=13$, find the value of $\frac{1}{a^5}+\frac{1}{b^5}$.

$(1)^2-(2)$: $\frac{2}{ab}=12$

$\Rightarrow ab=\frac{1}{6}$ $\cdots\cdots$ (4)

From (1): $(a+b)\cdot\frac{1}{ab}=5$ $\cdots\cdots$ (5)

Sub. (4) into (5): $6(a+b)=5$

$\Rightarrow a+b=\frac{5}{6}$ $\cdots\cdots$ (6)

From (4) and (6), a and b are roots of $6t^2-5t+1=0$

$(2t-1)(3t-1)=0$

$\Rightarrow t=\frac{1}{2}$ or $\frac{1}{3}$

$\frac{1}{a^5}+\frac{1}{b^5}=2^5+3^5$

$=32+243=275$

G2 There are N pupils in a class.

When they are divided into groups of 4, 1 pupil is left behind.

When they are divided into groups of 5, 3 pupils are left behind.

When they are divided into groups of 7, 3 pupils are left behind.

Find the least value of N.

Reference: 1992 HG4

$N=4p+1$ $\cdots\cdots$ (1), p is an integer

$N=5q+3$ $\cdots\cdots$ (2), q is an integer

$N=7r+3$ $\cdots\cdots$ (3), r is an integer

(3) – (2): $7r=5q$

$r=5k$, $q=7k$, where k is an integer

$N=35k+3=4p+1$

$4p-35k=2$

By trial and error,

$p=18$, $k=2$ is a solution

$N=73$

G3 The coordinates of A, B, C and D are (10, 1), (1, 7), (–2, 1) and (1, 3) respectively. AB and CD meet at P. Find the value of $\frac{AP}{PB}$.

Reference: 1989 HG5

Equation of AB: $\frac{y-1}{x-10}=\frac{1-7}{10-1}$

$\Rightarrow 2x + 3y - 23 = 0$ …… (1)

Equation of CD: $\frac{y-1}{x+2}=\frac{3-1}{1+2}$

$\Rightarrow 2x - 3y + 7 = 0$ …… (2)

(1) + (2): $4x - 16 = 0$

$\Rightarrow x = 4$

(1) – (2): $6y - 30 = 0$

$\Rightarrow y = 5$

Let $\frac{AP}{PB}=r$

$4=\frac{10+r}{1+r}$

$\Rightarrow 4 + 4r = 10 + r$

$\Rightarrow r = 2$

G4 Find the remainder when $2^{1989} + 1$ is divided by 3.

$2^{1989} + 1 = (3 - 1)^{1989} + 1 = 3m - 1 + 1$, binomial theorem, m is an integer.

The remainder is 0.

Method 2

$2^1 + 1 = 3 \equiv 0 \bmod 3$, $2^2 + 1 = 5 \equiv 2 \bmod 3$, $2^3 + 1 \equiv 0 \bmod 3$, $2^4 + 1 \equiv 2 \bmod 3$

The pattern of the remainder repeats for every multiples of 2.

$2^{1989} + 1 \equiv 2^1 + 1 \equiv 0 \bmod 3$

$\Rightarrow$ the remainder = 0

G5 Euler was born and died between 1700 A.D. and 1800 A.D. He was $n + 9$ years old in n^3 A.D. and died at the age of 76. Find the year in which Euler died.

Reference: 2024 HI6

Suppose he was born in x years after 1700 A.D.

$1700 + x + n + 9 - 1 = n^3$ ⋯⋯ (1)

$11^3 = 1331$, $12^3 = 1728$, $13^3 > 1800$

$\therefore n = 12$, $x = 1728 - 1700 - 12 - 9 + 1 = 8$

$1700 + x + 76 - 1 = 1783$

$\Rightarrow$ He was died in A.D. 1783.

G6 Let $N!$ denotes the product of the first N natural numbers, i.e. $N! = 1\times2\times3\times\cdots\times N$.

If k is a positive integer such that $30! = 2^k\times$ an odd integer, find k.

Reference: 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FI1.4, 2012 FG1.3

2, 4, 6, 8, $\cdots$, 30 each has at least one factor of 2. Subtotal = 15

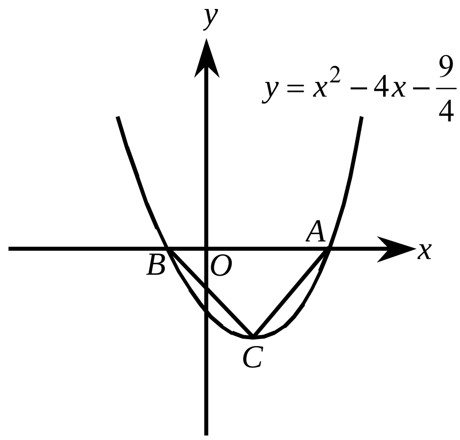
4, 8, $\cdots$, 28 each has at least 2 factors of 2. Subtotal = 7

8, 16, 24 each has at least 3 factors of 2. Subtotal = 3

16 has 4 factors of 2. Subtotal = 1

Total number of factors of 2 = 15 + 7 + 3 + 1 = 26

G7 The graph of the parabola $y = x^2 - 4x - \frac{9}{4}$ cuts the x-axis at A and B (figure 1). If C is the vertex of the parabola, find the area of ΔABC.



(Figure 1)

Let the roots be α, β, where $\alpha > \beta$.

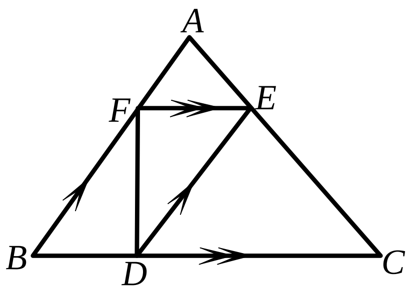
$AB = \alpha - \beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= \sqrt{4^2 + 4\cdot\frac{9}{4}} = 5$

Minimum $= \dfrac{4ac - b^2}{4a} = \dfrac{4\left(-\frac{9}{4}\right) - (-4)^2}{4} = -\dfrac{25}{4}$

Area of $\Delta ABC = \frac{1}{2}\times\frac{25}{4}\times 5 = \frac{125}{8}$

G8 In figure 2, $FE \parallel BC$ and $ED \parallel AB$. If $AF : FB = 1 : 4$, find the ratio of area of ΔEDC : area of ΔDEF.



(Figure 2)

Reference: 1989 HI17

$BDEF$ is a parallelogram formed by 2 pairs of parallel lines

$\Delta DEF \cong \Delta FBD$ (A.S.A.)

Let $S_{\Delta DEF} = x = S_{\Delta FBD}$ (where S stands for the area)

$\Delta AEF \sim \Delta ACB$ ($\because$ $FE \parallel BC$, equiangular)

$$\frac{S_{\Delta AEF}}{S_{\Delta ACB}} = \left(\frac{1}{1+4}\right)^2 = \frac{1}{25} \quad \cdots\cdots (1)$$

$\therefore$ $AE : EC = AF : FB = 1 : 4$ (theorem of equal ratio)

$\because$ $DE \parallel AB$

$\therefore$ $AE : EC = BD : DC = 1 : 4$ (theorem of equal ratio)

$\Delta CDE \sim \Delta CBA$ ($\because$ $DE \parallel BA$, equiangular)

$$\frac{S_{\Delta CDE}}{S_{\Delta CBA}} = \left(\frac{4}{1+4}\right)^2 = \frac{16}{25} \quad \cdots\cdots (2)$$

Compare (1) and (2) $S_{\Delta AEF} = k$, $S_{\Delta CDE} = 16k$, $S_{\Delta ABC} = 25k$

$k + 16k + x + x = 25k$

$x = 4k$

$\Rightarrow$ area of ΔDEF : area of ΔABC = 16 : 4 = 4 : 1

G9 In the attached multiplication (figure 3), the letters O, L, Y, M, P, I, A and D represent different integers ranging from 1 to 9. Find the integer represented by A.

$$\begin{array}{r} O\,L\,Y\,M\,P\,I\,A\,D \\ \times \qquad\qquad\quad D \\ \hline O\,O\,O\,O\,O\,O\,O\,O\,O \end{array}$$

$1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$

Possible $(D,O) = (2,4), (3,9), (4,6), (7,9), (8,4), (9,1)$

When $D = 2$, $O = 4$, $(OLYMPIAD) = 444444444\div2 = 222222222$ rejected

When $D = 3$, $O = 9$, $(OLYMPIAD) = 999999999\div3 = 333333333$ rejected

When $D = 4$, $O = 6$, $(OLYMPIAD) = 666666666\div4 = 166666666.5$ rejected

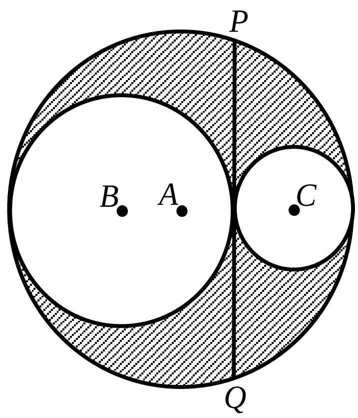
When $D = 7$, $O = 9$, $(OLYMPIAD) = 999999999\div7 = 142857142.7$ rejected

When $D = 8$, $O = 4$, $(OLYMPIAD) = 444444444\div8 = 55555555.5$ rejected

When $D = 9$, $O = 1$, $(OLYMPIAD) = 111111111\div9 = 12345679$

$A = 7$

G10 Three circles, with centres A, B and C respectively, touch one another as shown in figure 4. If A, B and C are collinear and PQ is a common tangent to the two smaller circles, where $PQ = 4$, find the area of the shaded part in terms of π.



(Figure 4)

Reference: 2018FG1.2

Let the radii of the 3 circles with centres A, B and C be a, b, c.

$2b + 2c = 2a \Rightarrow a = b + c$ ⋯⋯ (1)

By intersecting chords theorem, $2c\times2b = 2^2$

$bc = 1$ ⋯⋯ (2)

Shaded area $= \pi a^2 - \pi b^2 - \pi c^2$

$= \pi[a^2 - (b^2 + c^2)]$

$= \pi[a^2 - (b + c)^2 + 2bc]$

$= \pi(a^2 - a^2 + 2)$ by (1) and (2)

$= 2\pi$