

Solve for integral solutions of $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

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Find also the sum of all possible $0 < a \leq 100$.

Clearly a , b and c are non-zero.

Multiply both sides by $a^2 b^2 c^2$.

$$(bc)^2 + (ac)^2 = (ab)^2$$

Let $x = bc$, $y = ac$, $z = ab$, then $x^2 + y^2 = z^2$, where x , y and z are non-zero integers.

$\therefore (x, y, z)$ is a Pythagoreans' triple.

\therefore The solutions are $x = bc = 2kpq$ (1), $y = ac = k(p^2 - q^2)$ (2), $z = ab = k(p^2 + q^2)$... (3)

where k is an integer, $p > q$, p and q are relatively prime positive integers.

$$\frac{(1) \times (2)}{(3)} : c^2 = \frac{2pq(p^2 - q^2)k}{p^2 + q^2} \Rightarrow c = \pm \sqrt{\frac{2pq(p^2 - q^2)k}{p^2 + q^2}}$$

$$\frac{(1) \times (3)}{(2)} : b^2 = \frac{2pq(p^2 + q^2)k}{p^2 - q^2} \Rightarrow b = \pm \sqrt{\frac{2pq(p^2 + q^2)k}{p^2 - q^2}}$$

$$\frac{(2) \times (3)}{(1)} : a^2 = \frac{(p^2 - q^2)(p^2 + q^2)k}{2pq} \Rightarrow a = \pm \sqrt{\frac{(p^2 - q^2)(p^2 + q^2)k}{2pq}}$$

Let $t = \sqrt{\frac{k}{2pq(p^2 - q^2)(p^2 + q^2)}}$, then $a = \pm(p^2 - q^2)(p^2 + q^2)t$, $b = \pm 2pq(p^2 + q^2)t$, $c = \pm 2pq(p^2 - q^2)t$

t	p	q	a	b	c
1	2	1	15	20	12
2	2	1	30	40	24
3	2	1	45	60	36
4	2	1	60	80	48
5	2	1	75	100	60
6	2	1	90	120	72
1	3	1	80	60	48
1	3	2	65	156	60

The highlight triples are repeated. So one of them is rejected.

There is no other triples for which a is less than or equal to 100.

Note that a and b can be interchanged and '60' appeared twice.

Sum of all possible $a = 15 + 30 + 45 + 60 + 75 + 90 + 65 + 20 + 40 + 80 + 100 = 620$