

Integral Cyclic Quadrilateral

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Given a triangle with one angle is θ , $\cos \theta = \frac{3}{5}$. If all sides are integers, find all solutions.

$$c^2 = a^2 + b^2 - 2ab \cos \theta \Rightarrow c^2 = a^2 + b^2 - 2ab \times \frac{3}{5}$$

$$5c^2 = 5a^2 - 10ab + 5b^2 + 4ab$$

$$5[c^2 - (a - b)^2] = 4ab$$

$$\frac{5(c+a-b)}{b} = \frac{4a}{c-a+b} = k$$

$$c = b - a + \frac{bk}{5} \dots\dots(1) \text{ and } 4a = kc - ka + kb \dots\dots(2)$$

$$\text{Sub. (1) into (2): } 4a = \left(b - a + \frac{bk}{5}\right)k - ak + bk$$

$$20a = 10bk - 10ak + bk^2$$

$$(10k + 20)a = (k^2 + 10k)b$$

$$\frac{a}{b} = \frac{k^2 + 10k}{10k + 20}$$

$$\text{Let } a = (k^2 + 10k)p, b = (10k + 20)p$$

$$c = b - a + \frac{bk}{5} = [10k + 20 - k^2 - 10k + (2k + 4)k]p = (k^2 + 4k + 20)p. \text{ Let } p = 1.$$

| k | $a = k^2 + 10k$ | $b = 10k + 20$ | $c = k^2 + 4k + 20$ | Remark |
|-----|-----------------|----------------|---------------------|----------|
| 1 | 11 | 30 | 25 | |
| 2 | 24 | 40 | 32 | 3-4-5 |
| 3 | 39 | 50 | 41 | |
| 4 | 56 | 60 | 52 | 13-14-15 |

Given a triangle with one angle is θ , $\cos \theta = \frac{4}{5}$. If all sides are integers, find all solutions.

$$c^2 = a^2 + b^2 - 2ab \cos \theta \Rightarrow c^2 = a^2 + b^2 - 2ab \times \frac{4}{5}$$

$$5c^2 = 5a^2 - 10ab + 5b^2 + 2ab$$

$$5[c^2 - (a - b)^2] = 2ab$$

$$\frac{5(c+a-b)}{b} = \frac{2a}{c-a+b} = k$$

$$c = b - a + \frac{bk}{5} \dots\dots(1) \text{ and } 2a = kc - ka + kb \dots\dots(2)$$

$$\text{Sub. (1) into (2): } 2a = \left(b - a + \frac{bk}{5}\right)k - ak + bk$$

$$10a = 10bk - 10ak + bk^2$$

$$(10k + 10)a = (k^2 + 10k)b$$

$$\frac{a}{b} = \frac{k^2 + 10k}{10k + 10}$$

$$\text{Let } a = (k^2 + 10k)p, b = (10k + 10)p$$

$$c = b - a + \frac{bk}{5} = [10k + 10 - k^2 - 10k + (2k + 2)k]p = (k^2 + 2k + 10)p. \text{ Let } p = 1.$$

| k | $a = k^2 + 10k$ | $b = 10k + 10$ | $c = k^2 + 2k + 10$ | Remark |
|-----|-----------------|----------------|---------------------|----------|
| 1 | 11 | 20 | 13 | |
| 2 | 24 | 30 | 18 | 3-4-5 |
| 3 | 39 | 40 | 25 | |
| 4 | 56 | 50 | 34 | 17-25-28 |
| 5 | 75 | 60 | 45 | 3-4-5 |
| 6 | 96 | 70 | 58 | 29-35-48 |

Given a triangle with one angle is θ , $\cos \theta = \frac{7}{25}$. If all sides are integers, find all solutions.

$$c^2 = a^2 + b^2 - 2ab \cos \theta \Rightarrow c^2 = a^2 + b^2 - 2ab \times \frac{7}{25}$$

$$25c^2 = 25a^2 - 50ab + 25b^2 + 36ab$$

$$25[c^2 - (a - b)^2] = 36ab$$

$$\frac{25(c+a-b)}{b} = \frac{36a}{c-a+b} = k$$

$$c = b - a + \frac{bk}{25} \dots\dots(1) \text{ and } 36a = kc - ka + kb \dots\dots(2)$$

$$\text{Sub. (1) into (2): } 36a = \left(b - a + \frac{bk}{25}\right)k - ak + bk$$

$$900a = 50bk - 50ak + bk^2$$

$$(50k + 900)a = (k^2 + 50k)b$$

$$\frac{a}{b} = \frac{k^2 + 50k}{50k + 900}$$

$$\text{Let } a = (k^2 + 50k)p, b = (50k + 900)p$$

$$c = b - a + \frac{bk}{25} = [50k + 900 - k^2 - 50k + (2k + 36)k]p = (k^2 + 36k + 900)p. \text{ Let } p = 1.$$

| k | $a = k^2 + 50k$ | $b = 50k + 900$ | $c = k^2 + 36k + 900$ | Remark |
|-----|-----------------|-----------------|-----------------------|-------------|
| 1 | 51 | 950 | 937 | |
| 2 | 104 | 1000 | 976 | 13-122-125 |
| 3 | 159 | 1050 | 1017 | |
| 4 | 216 | 1100 | 1060 | 54-275-265 |
| 5 | 275 | 1150 | 1105 | 55-221-230 |
| 6 | 336 | 1200 | 1152 | 7-24-25 |
| 7 | 399 | 1250 | 1201 | |
| 8 | 464 | 1300 | 1252 | 116-313-325 |
| 9 | 531 | 1350 | 1305 | 59-145-150 |
| 10 | 600 | 1400 | 1360 | 15-34-35 |
| 11 | 671 | 1450 | 1417 | |
| 12 | 744 | 1500 | 1476 | 62-123-125 |
| 13 | 819 | 1550 | 1537 | |
| 14 | 896 | 1600 | 1600 | 14-25-25 |
| 15 | 975 | 1650 | 1665 | 65-110-111 |
| 16 | 1056 | 1700 | 1732 | 264-425-433 |
| 17 | 1139 | 1750 | 1801 | |
| 18 | 1224 | 1800 | 1872 | 17-25-26 |
| 19 | 1311 | 1850 | 1945 | |
| 20 | 1400 | 1900 | 2020 | 70-95-101 |

Given that the three sides of a triangle are positive integers.

Let θ be an acute angle such that $\sin \theta = \frac{p}{r}$, $\cos \theta = \frac{q}{r}$ and $\tan \theta = \frac{p}{q}$ are rational numbers in the simplest forms.

Then (p, q, r) is a primitive Pythagorean triple. $(p, q, r) = (2uv, u^2 - v^2, u^2 + v^2)$ or $(u^2 - v^2, 2uv, u^2 + v^2)$.

Let ABC be a triangle ($BC = a$, $AC = b$, $AB = c$). $\angle ACB = \theta$.

Area of $\triangle ABC = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B$, which must be rational

Denote the area of $\triangle ABC$ by S , then $\sin A = \frac{2S}{bc}$, $\sin B = \frac{2S}{ca}$ and $\sin C = \frac{2S}{ab}$ which are also rational.

By cosine formula, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \frac{c^2 + a^2 - b^2}{2ac}$, $\cos C = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow \cos A, \cos B, \cos C \in \mathbb{Q}$

$\tan A = \frac{\sin A}{\cos A}$, $\tan B = \frac{\sin B}{\cos B}$, $\tan C = \frac{\sin C}{\cos C}$ which are still rational. Find all integral solutions for a, b and c .

$$c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow c^2 = a^2 + b^2 - 2ab \cdot \frac{q}{r}$$

Case 1 $q = 2uv$

$$(u^2 + v^2)c^2 = (u^2 + v^2)a^2 + (u^2 + v^2)b^2 - 4abuv$$

$$(u^2 + v^2)[c^2 - (a^2 - 2ab + b^2)] = -4abuv + 2ab(u^2 + v^2)$$

$$(u^2 + v^2)(c + a - b)(c - a + b) = 2ab(u - v)^2$$

$$\frac{(u^2 + v^2)(c + a - b)}{b} = \frac{2(u - v)^2 a}{c - a + b} = k$$

$$c + a - b = \frac{bk}{u^2 + v^2} \dots\dots(1), c - a + b = \frac{2(u - v)^2 a}{k} \dots\dots(2)$$

$$(1) - (2): 2(a - b) = \frac{bk}{u^2 + v^2} - \frac{2(u - v)^2 a}{k}$$

$$2(a - b)(u^2 + v^2)k = bk^2 - 2a(u - v)^2(u^2 + v^2)$$

$$2(u^2 + v^2)ka + 2(u - v)^2(u^2 + v^2)a = 2(u^2 + v^2)kb + k^2b$$

$$\frac{a}{b} = \frac{(2u^2 + 2v^2 + k)k}{2(u^2 + v^2)[(u - v)^2 + k]} \Rightarrow a = (2u^2 + 2v^2 + k)k, b = 2(u^2 + v^2)[(u - v)^2 + k]$$

$$\begin{aligned} \text{Sub. into (2): } c &= (2u^2 + 2v^2 + k)k - 2(u^2 + v^2)[(u - v)^2 + k] + 2(u - v)^2(2u^2 + 2v^2 + k) \\ &= k^2 + (2u^2 + 2v^2)k - 2(u^2 + v^2)(u - v)^2 - 2(u^2 + v^2)k + 4(u - v)^2(u^2 + v^2) + 2(u - v)^2k \\ &= k^2 + 2(u - v)^2k + 2(u - v)^2(u^2 + v^2) = k^2 + 2(u - v)^2(u^2 + v^2 + k) \end{aligned}$$

| k | u | v | $a = (2u^2 + 2v^2 + k)k$ | $b = 2(u^2 + v^2)[(u - v)^2 + k]$ | $c = k^2 + 2(u - v)^2(u^2 + v^2 + k)$ | $\cos C = \frac{2uv}{u^2 + v^2}$ |
|-----|-----|-----|--------------------------|-----------------------------------|---------------------------------------|----------------------------------|
| 1 | 2 | 1 | 11 | 20 | 13 | $\frac{4}{5}$ |
| 1 | 3 | 1 | 21 | 100 | 89 | $\frac{6}{10} = \frac{3}{5}$ |
| 1 | 3 | 2 | 27 | 52 | 29 | $\frac{12}{13}$ |
| 1 | 4 | 1 | 35 | 340 | 325 | $\frac{8}{17}$ |
| 1 | 4 | 3 | 51 | 100 | 53 | $\frac{24}{25}$ |
| 3 | 2 | 1 | 39 | 40 | 25 | $\frac{4}{5}$ |

Note that $u > v$ are relatively prime positive integers. $\cos C = \frac{2uv}{u^2 + v^2} > 0$ and so $\angle C$ must be acute.

$$\text{Area of } \triangle ABC = \frac{1}{2}(2u^2 + 2v^2 + k)k \cdot 2(u^2 + v^2)[(u - v)^2 + k] \cdot \frac{u^2 - v^2}{u^2 + v^2} = k(u^2 - v^2)(2u^2 + 2v^2 + k)[(u - v)^2 + k]$$

Case 2 $p = 2uv$, $\cos \theta = \frac{q}{r} = \frac{u^2 - v^2}{u^2 + v^2}$

$$c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow c^2 = a^2 + b^2 - 2ab \cdot \frac{u^2 - v^2}{u^2 + v^2}$$

$$(u^2 + v^2)c^2 = (u^2 + v^2)(a^2 - 2ab + b^2) + 2ab(u^2 + v^2) - 2ab(u^2 - v^2)$$

$$(u^2 + v^2)(c + a - b)(c - a + b) = 4v^2ab$$

$$\frac{(u^2 + v^2)(c + a - b)}{b} = \frac{4av^2}{c - a + b} = k$$

$$c + a - b = \frac{bk}{u^2 + v^2} \dots\dots(1) \text{ and } c - a + b = \frac{4av^2}{k} \dots\dots(2)$$

$$(1) - (2): 2(a - b) = \frac{bk}{u^2 + v^2} - \frac{4av^2}{k}$$

$$2k(u^2 + v^2)a - 2k(u^2 + v^2)b = k^2b - 4v^2(u^2 + v^2)a$$

$$2(u^2 + v^2)(k + 2v^2)a = [k^2 + 2(u^2 + v^2)k]b$$

$$\frac{a}{b} = \frac{k^2 + 2(u^2 + v^2)k}{2(u^2 + v^2)(k + 2v^2)} \Rightarrow a = k^2 + 2(u^2 + v^2)k, b = 2(u^2 + v^2)(k + 2v^2)$$

Sub. into (2): $c = k^2 + 2(u^2 + v^2)k - 2(u^2 + v^2)(k + 2v^2) + 4v^2[k + 2(u^2 + v^2)]$
 $= k^2 + 4v^2(u^2 + v^2) + 4v^2k = k^2 + 4v^2(u^2 + v^2 + k)$

| k | u | v | $a = (2u^2 + 2v^2 + k)k$ | $b = 2(u^2 + v^2)(2v^2 + k)$ | $c = k^2 + 4v^2(u^2 + v^2 + k)$ | $\cos C = \frac{u^2 - v^2}{u^2 + v^2}$ |
|-----|-----|-----|--------------------------|------------------------------|---------------------------------|--|
| 1 | 1 | 2 | 11 | 90 | 97 | $-\frac{3}{5}$ |
| 1 | 2 | 1 | 11 | 30 | 25 | $\frac{3}{5}$ |
| 1 | 2 | 3 | 27 | 494 | 505 | $-\frac{5}{13}$ |
| 1 | 3 | 2 | 27 | 234 | 225 | $\frac{5}{13}$ |
| 1 | 4 | 1 | 35 | 102 | 73 | $\frac{15}{17}$ |
| 1 | 4 | 3 | 51 | 950 | 937 | $\frac{7}{25}$ |
| 3 | 2 | 1 | 39 | 50 | 41 | $\frac{3}{5}$ |

Note that u, v are distinct relatively prime positive integers.

If $u > v$, then $\cos C = \frac{u^2 - v^2}{u^2 + v^2} > 0$ and so C is acute. If $u < v$, then $\cos C = \frac{u^2 - v^2}{u^2 + v^2} < 0$ and so C is obtuse.

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2}ab \sin C = \frac{1}{2}(2u^2 + 2v^2 + k)k \cdot 2(u^2 + v^2)(2v^2 + k) \cdot \frac{2uv}{u^2 + v^2} \\ &= 2kuv(2u^2 + 2v^2 + k)(2v^2 + k), \text{ which is an integer} \end{aligned}$$

Given that the three sides of a triangle ACB are positive integers. $\angle ACB = \theta$ such that $\sin \theta = \frac{p}{r}$, $\cos \theta = \frac{q}{r}$ and

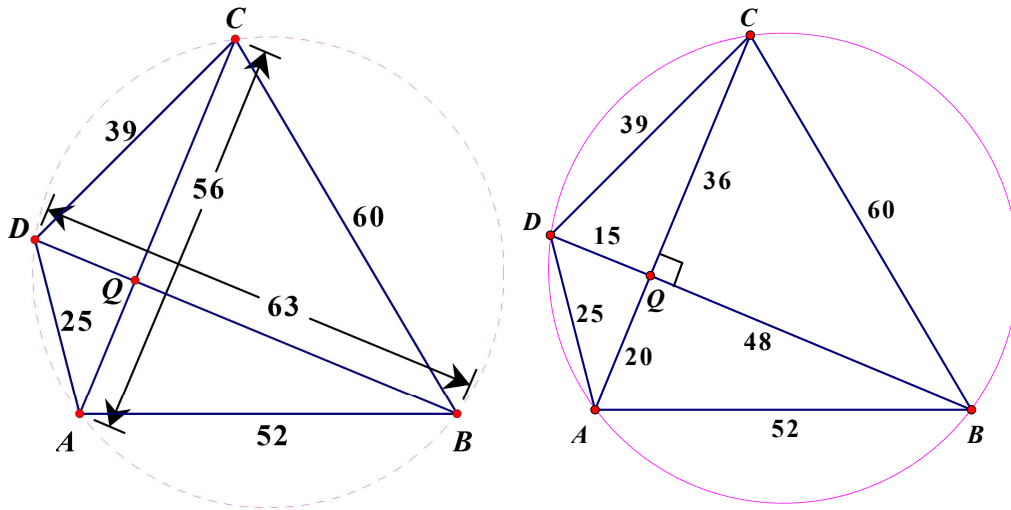
$\tan \theta = \frac{p}{q}$ are all rational numbers. If (p, q, r) is a Pythagorean triple, from the above analysis,

$$(a, b, c) = ((2u^2 + 2v^2 + k)k, 2(u^2 + v^2)[(u-v)^2 + k], k^2 + 2(u-v)^2(u^2 + v^2 + k)) \text{ or } ((2u^2 + 2v^2 + k)k, 2(u^2 + v^2)(2v^2 + k), k^2 + 4v^2(u^2 + v^2 + k))$$

where $k \in \mathbb{Z}^+$, u, v are distinct relatively prime positive integers.

In this section, we are going to find a cyclic quadrilateral for which all sides and all diagonals are integers.

Idea: In February 2008, I asked Dr. Man Keung Siu from the University of Hong Kong about how to find a solution to the above question. He quoted a paper “Normal Trigrade and cyclic quadrilateral with integral sides and diagonals” from April, 1951 American Mathematical Monthly. I didn’t understand the content of the paper. However, the author gave an example of integral cyclic quadrilateral $ABCD$, $AB = 52$, $BC = 60$, $CD = 39$, $DA = 25$, $AC = 56$, $BD = 63$.



It takes me years of time to investigate how to find another integral cyclic quadrilateral. Still, I failed to find any other solutions. After retirement, I use cosine formula to find $\cos \angle ACB$.

$$\cos \angle ACB = \frac{56^2 + 60^2 - 52^2}{2 \times 56 \times 60} = \frac{3}{5} \text{ and so } \sin \angle ACB = \frac{4}{5}, \tan \angle ACB = \frac{4}{3} \text{ which are all rational.}$$

I started to investigate a triangle with integral sides and rational sines of each angle.

Suppose the diagonals intersect at Q . Let $\angle BQC = \theta$. It can be proved that $\tan \theta = \frac{4\sqrt{(s-a)(s-b)(s-c)(s-d)}}{a^2 + c^2 - b^2 - d^2}$

where $a = 52$, $b = 60$, $c = 39$, $d = 25$, $s = \frac{1}{2}(a + b + c + d)$.

$$a^2 + c^2 - b^2 - d^2 = 52^2 + 39^2 - 60^2 - 25^2 = 0 \Rightarrow \text{denominator} = 0 \Rightarrow \theta = 90^\circ$$

$$CQ = BC \cos \angle BCQ = 60 \times \frac{3}{5} = 36, BQ = BC \sin \angle BCQ = 48. \triangle BCQ \text{ is a 3-4-5 } \Delta.$$

$$\therefore DQ = BD - BQ = 63 - 48 = 15, AQ = AC - CQ = 56 - 36 = 20.$$

$\triangle ABQ$ is a 5-12-13 Δ . $\triangle ADQ$ is a 3-4-5 Δ . $\triangle CDQ$ is a 5-12-13 Δ .

This is a **special case** of integral cyclic quadrilateral.

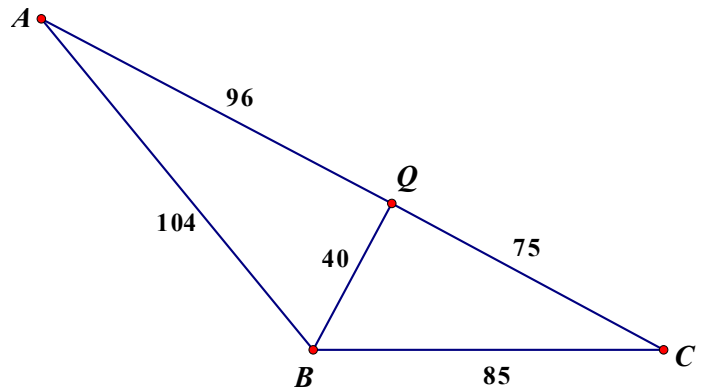
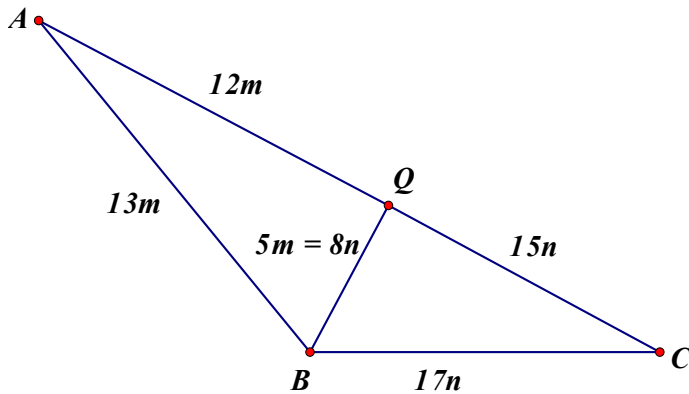
Will there be any other example(s) of integral cyclic quadrilateral $ABCD$ with perpendicular diagonals?

$\angle ACB = \angle ADB$, $\angle CAD = \angle CBD$, $\angle ACD = \angle ABD$, $\angle BDC = \angle BAC$ (\angle s in the same segment)

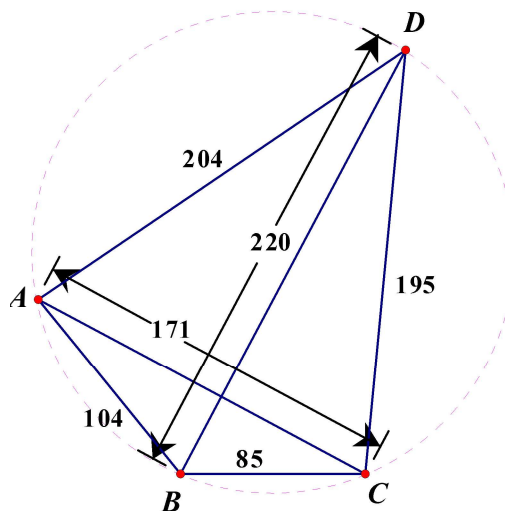
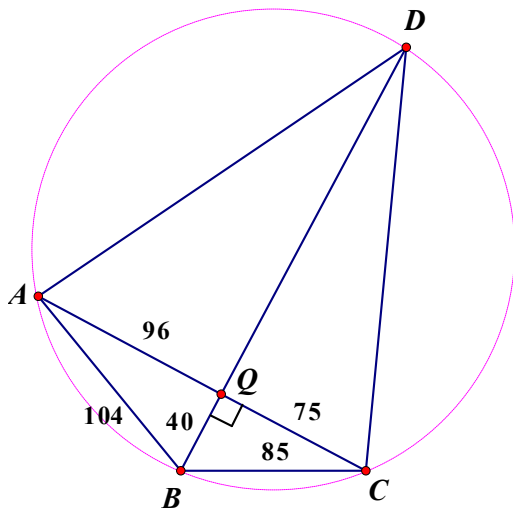
$\therefore \triangle ABQ \sim \triangle DCQ$, $\triangle BCQ \sim \triangle ADQ$ (equiangular)

We try to find **two pairs of right-angled triangles with common sides**.

If $\triangle ABQ$ is a 5-12-13 Δ , $\triangle BCQ$ is a 8-15-17 Δ . $AQ = 12m$, $BQ = 5m$, $AB = 13m$, $BQ = 8n$, $CQ = 15n$, $BC = 17n$.



$BC = 5m = 8n$, let $m = 8$, $n = 5$, then $AB = 13 \times 8 = 104$, $AQ = 12 \times 8 = 96$, $BC = 17 \times 5 = 85$, $CQ = 15 \times 5 = 75$



Construct the circumscribed circle ABC . Extend BQ to cut the circumscribed circle at D .

$\triangle DCQ \sim \triangle ABQ$ which are 5-12-13 Δ s. $DQ = \frac{75}{5} \times 12 = 180$, $CD = \frac{75}{5} \times 13 = 195$.

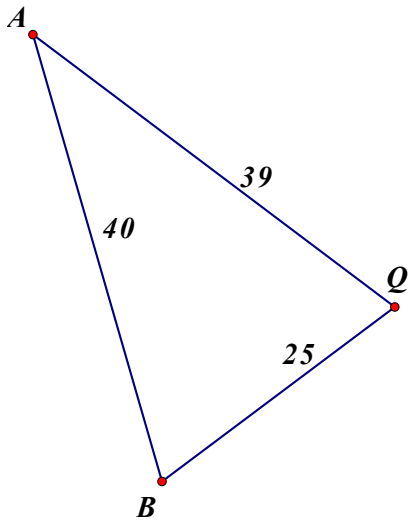
$\triangle ADQ \sim \triangle BCQ$ which are 8-15-17 Δ s. $DQ = \frac{96}{8} \times 15 = 180$, $AD = \frac{96}{8} \times 17 = 204$.

$\therefore ABCD$ is another integral cyclic quadrilateral with

$AB = 104$, $BC = 85$, $CD = 195$, $DA = 204$, $AC = 96 + 75 = 171$, $BD = 40 + 180 = 220$.

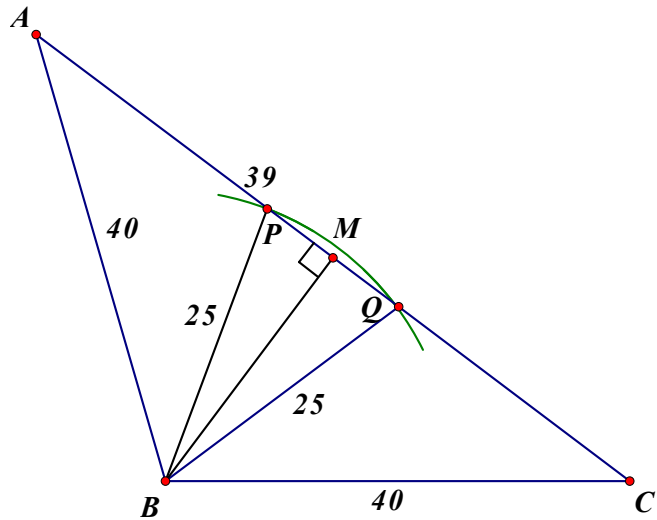
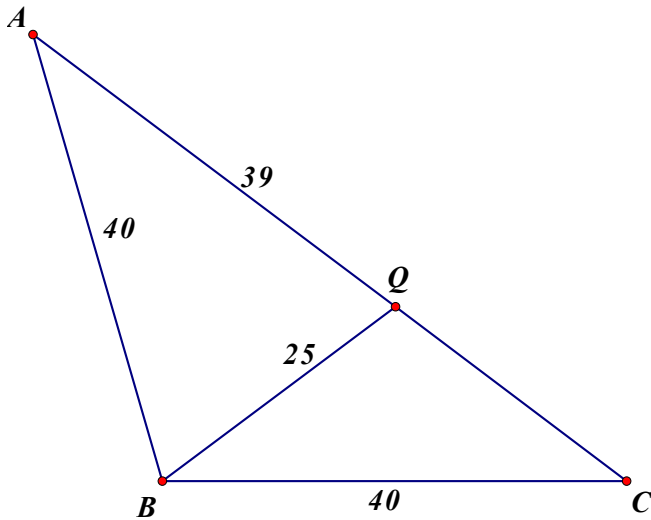
Question: Can we find an integral cyclic quadrilateral $ABCD$ so that the diagonals are not necessarily perpendicular?

We see from page 3 the last line that 25-39-40 is an integral triangle $\triangle ABQ$.



$$\cos \angle BAQ = \frac{40^2 + 39^2 - 25^2}{2 \times 40 \times 39} = \frac{4}{5}.$$

We find another triangle $\triangle BCQ$ so that $BC = 40$ and A, Q, C are collinear.



Draw a circular arc with B as centre, radius BQ , cutting AQ at P . $BP = BQ = 25$ (radii)

Let M be the foot of perpendicular from B to PQ . $\triangle BPM \cong \triangle BQM$ (R.H.S.)

$$AM = AB \cos \angle BAQ = 40 \times \frac{4}{5} = 32, \quad QM = AQ - AM = 39 - 32 = 7 = PM \text{ (corr. sides, } \cong \Delta \text{s)}$$

$$AP = AQ - QM - PM = 39 - 7 - 7 = 25$$

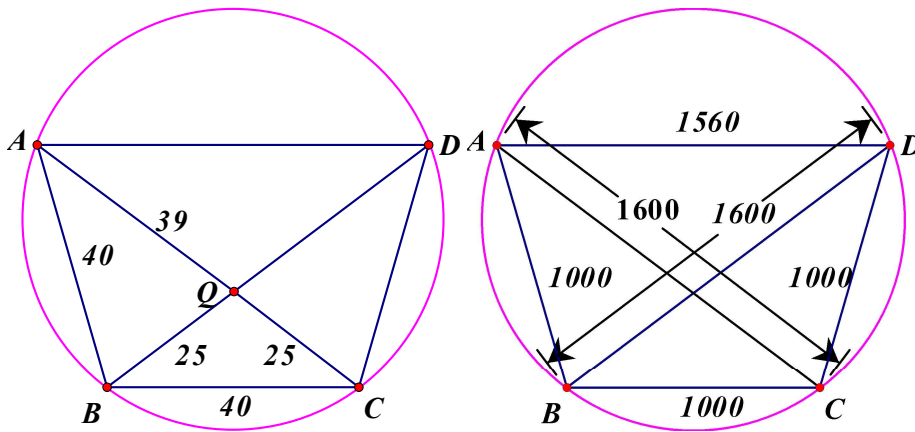
$$\angle BAC = \angle BCA, \quad \angle BPQ = \angle BQP \text{ (base } \angle \text{s, isos. } \Delta \text{)}$$

$$\angle APB = 180^\circ - \angle BPM = 180^\circ - \angle BQM = \angle BQC \text{ (adj. } \angle \text{s on st. line)}$$

$$\triangle ABP \cong \triangle CBQ \text{ (A.A.S.)}$$

$$CQ = AP = 25 \text{ (corr. sides, } \cong \Delta \text{s)}$$

Construct a circumscribed circle through A , B and C . Extend BQ to cut the circle again at D . Join AD and CD .



It is easy to show that $\triangle ABQ \cong \triangle DCQ$ (A.A.S.)

$DQ = AQ = 39$, $DC = AB = 40$ (corr. sides, $\cong \Delta$ s)

$\triangle ADQ \sim \triangle BCQ$ (equiangular)

$$\frac{AD}{BC} = \frac{AQ}{BQ} \quad (\text{corr. sides, } \sim \Delta\text{s})$$

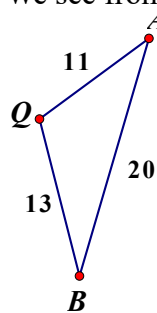
$$AD = 40 \times \frac{39}{25} = \frac{1560}{25}$$

Multiply every side by 25 to give integral sides. $AB = BC = CD = 1000$, $AD = 1560$, $BD = AC = 1600$.

Again, this is a **special case for three equal adjacent sides of integral cyclic quadrilateral**.

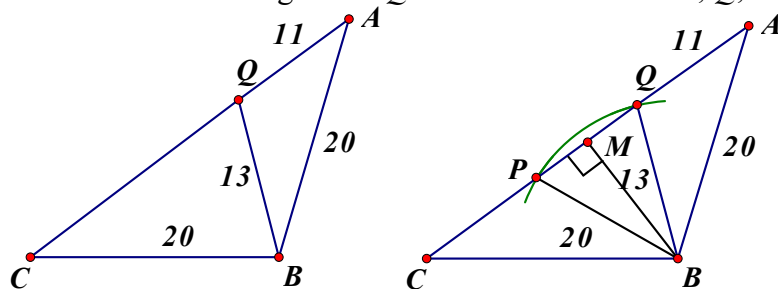
Second example:

We see from page 3 the first line in the table that 11-13-20 is an integral triangle $\triangle ABQ$.



$$\cos \angle BAQ = \frac{11^2 + 20^2 - 13^2}{2 \times 11 \times 20} = \frac{4}{5}.$$

We find another triangle $\triangle BCQ$ so that $BC = 20$ and A, Q, C are collinear.



Draw a circular arc with B as centre, radius BQ , cutting AQ at P . $BP = BQ = 25$ (radii)

Let M be the foot of perpendicular from B to PQ . $\triangle BPM \cong \triangle BQM$ (R.H.S.)

$$AM = AB \cos \angle BAQ = 20 \times \frac{4}{5} = 16, QM = AM - AQ = 16 - 11 = 5 = PM \text{ (corr. sides, } \cong \Delta s)$$

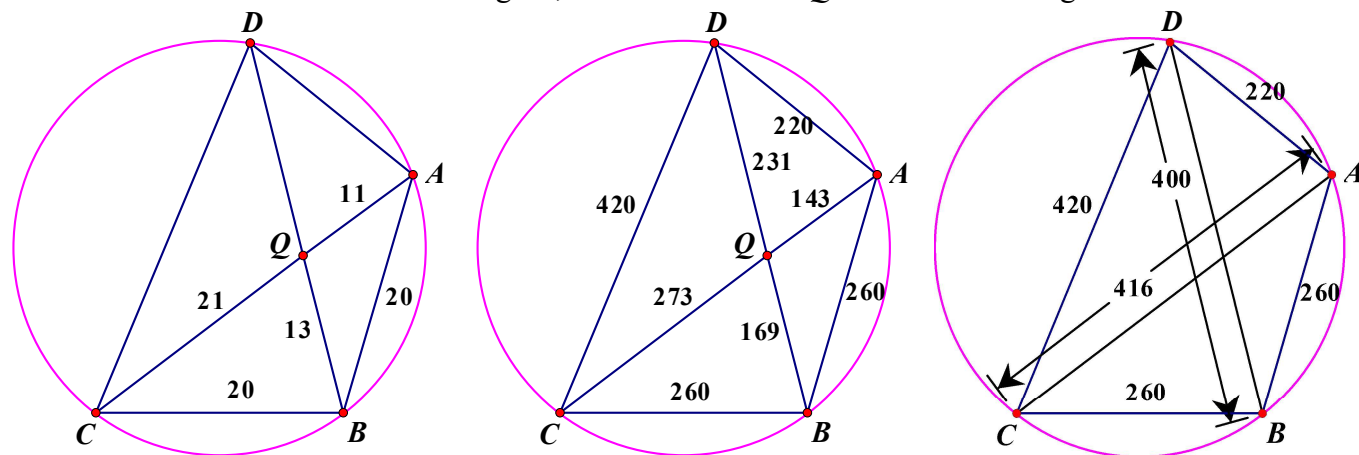
$$AP = AM + PM = 16 + 5 = 21$$

$$\angle BAC = \angle BCA, \angle BPQ = \angle BQP \text{ (base } \angle s, \text{ isos. } \Delta)$$

$$\triangle ABP \cong \triangle CBQ \text{ (A.A.S.)}$$

$$CQ = AP = 21 \text{ (corr. sides, } \cong \Delta s)$$

Construct a circumscribed circle through A, B and C . Extend BQ to cut the circle again at D . Join AD and CD .



It is easy to show that $\triangle ABQ \sim \triangle DCQ$ (equiangular)

$$\frac{DQ}{AQ} = \frac{CQ}{BQ} = \frac{CD}{AB} \text{ (corr. sides, } \sim \Delta s)$$

$$DQ = 11 \times \frac{21}{13} = \frac{231}{13}, CQ = 20 \times \frac{21}{13} = \frac{420}{13}, AC = 11 + 21 = 32, BD = 13 + \frac{231}{13} = \frac{400}{13}$$

$\triangle ADQ \sim \triangle BCQ$ (equiangular)

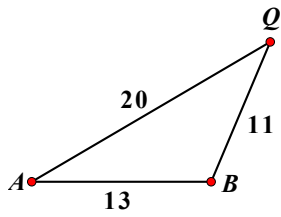
$$\frac{AD}{BC} = \frac{AQ}{BQ} \text{ (corr. sides, } \sim \Delta s)$$

$$AD = 20 \times \frac{11}{13} = \frac{220}{13}$$

Multiply every side by 13 to give integral sides. $AB = BC = 260$, $CD = 420$, $AD = 220$, $AC = 416$, $BD = 400$.
Again, this is a **special case for two equal adjacent sides of integral cyclic quadrilateral**.

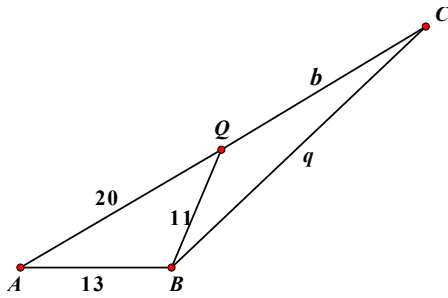
Question: Can find an integral cyclic quadrilateral $ABCD$ so that all adjacent sides are unequal?

We see from page 3 the first line in the table that 11-13-20 is an integral triangle $\triangle ABQ$.



$$\cos \angle AQB = \frac{11^2 + 20^2 - 13^2}{2 \times 11 \times 20} = \frac{4}{5}.$$

We find another triangle $\triangle BCQ$ so that $BC = q$, $QC = b$ and A, Q, C are collinear.



$$\cos \angle BQC = \cos(180^\circ - \angle AQB) = -\cos \angle AQB = -\frac{4}{5}$$

Apply cosine rule on $\triangle BCQ$: $q^2 = b^2 + 11^2 + 22b \times \frac{4}{5}$

$$5q^2 = 5(b^2 + 22b + 11^2) - 110b + 88b$$

$$22b = 5[(b + 11)^2 - q^2] = 5(b + q + 11)(b - q + 11)$$

$$\frac{b + q + 11}{b} = \frac{22}{5(b - q + 11)} = t$$

$$b + q + 11 = bt \dots\dots (1), \quad b - q + 11 = \frac{22}{5t} \dots\dots (2)$$

$$(1) + (2): 2b + 22 = bt + \frac{22}{5t}$$

$$10tb + 110t = 5t^2b + 22$$

$$110t - 22 = (5t^2 - 10t)b$$

$$b = \frac{22(5t - 1)}{5t(t - 2)}$$

$$(1) - (2): 2q = bt - \frac{22}{5t} = \frac{22t(5t - 1)}{5t(t - 2)} - \frac{22}{5t}$$

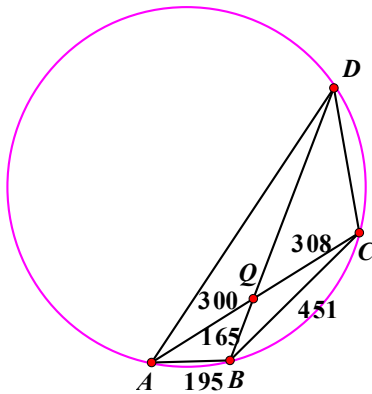
$$q = 11 \cdot \frac{t(5t - 1) - (t - 2)}{5t(t - 2)} = 11 \cdot \frac{5t^2 - 2t + 2}{5t(t - 2)}$$

$$\text{Put } t = 3, \quad b = \frac{22 \times 14}{15} = \frac{308}{15}, \quad q = 11 \times \frac{41}{15} = \frac{451}{15}$$

Multiply every side by 15 to give integral sides.

$$AB = 13 \times 15 = 195, \quad BQ = 11 \times 15 = 165, \quad AQ = 20 \times 15 = 300.$$

Construct a circumscribed circle through A , B and C . Extend BQ to cut the circle again at D . Join AD and CD .



It is easy to show that $\triangle ABQ \sim \triangle DCQ$ (equiangular)

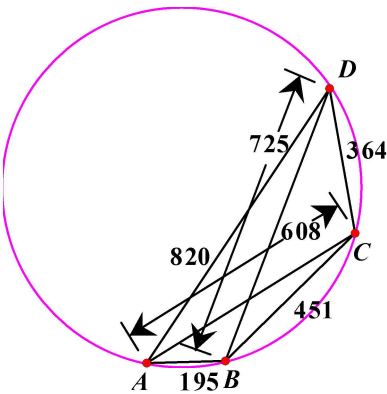
$$\frac{DQ}{AQ} = \frac{CQ}{BQ} = \frac{CD}{AB} \quad (\text{corr. sides, } \sim \Delta s)$$

$$DQ = 300 \times \frac{308}{165} = 560, \quad CD = 195 \times \frac{308}{165} = 364, \quad AC = 300 + 308 = 608, \quad BD = 165 + 560 = 725$$

$\triangle ADQ \sim \triangle BCQ$ (equiangular)

$$\frac{AD}{BC} = \frac{AQ}{BQ} \quad (\text{corr. sides, } \sim \Delta s)$$

$$AD = 451 \times \frac{300}{165} = 820$$



Question: Can we find a general formula for integral cyclic quadrilateral for which the diagonals are not necessarily perpendicular and the adjacent sides are not necessarily equal?

Let the cyclic quadrilateral be $ABCD$. The diagonals AC and BD intersect at Q .

Let $AQ = a$, $BQ = b$, $AB = c$, $DQ = d$, $CQ = e$, $CD = f$, $AD = g$, $BC = h$ as shown in the figure.

$\triangle ABQ \sim \triangle DCQ$ (equiangular)

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f} = m, \text{ where } m \text{ is a constant (corr. sides, } \sim \Delta s)$$

$$a = dm, b = em, c = fm$$

$\triangle ADQ \sim \triangle BCQ$ (equiangular)

$$\frac{a}{b} = \frac{d}{e} = \frac{g}{h} = n, \text{ where } n \text{ is a constant. (corr. sides, } \sim \Delta s)$$

$$a = bn = dmn, d = en, g = hn$$

There are five variables e, f, h, m, n in the figure.

Let $\triangle BCQ$ be an obtuse-angled triangle with $\angle BQC > 90^\circ$.

Then $\triangle CDQ$ is an acute-angled triangle.

By the formula on Page 4, the only possible solution for $\triangle CDQ$ is:

$$CD = k^2 + 4v^2(u^2 + v^2 + k), CQ = (2u^2 + 2v^2 + k)k, DQ = 2(u^2 + v^2)(2v^2 + k)$$

$$\cos \angle CQD = \frac{u^2 - v^2}{u^2 + v^2} > 0, \text{ where } u > v \text{ are distinct relatively prime positive integers}$$

$$\text{Again, } \triangle BCQ \text{ is another triangle adjacent to } \triangle CDQ \text{ with } \cos \angle BQC = \frac{v^2 - u^2}{u^2 + v^2} < 0$$

The roles of u and v are interchanged.

$$BC = h = k^2 + 4u^2(u^2 + v^2 + k), BQ = em = 2(u^2 + v^2)(2u^2 + k)$$

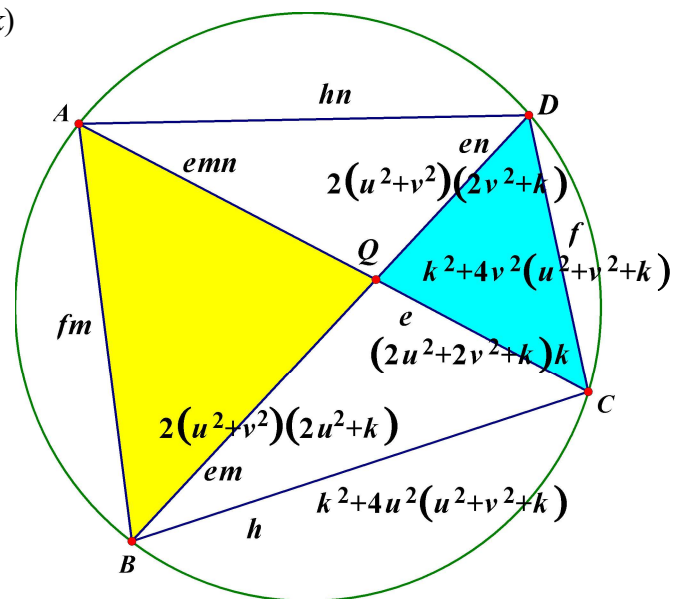
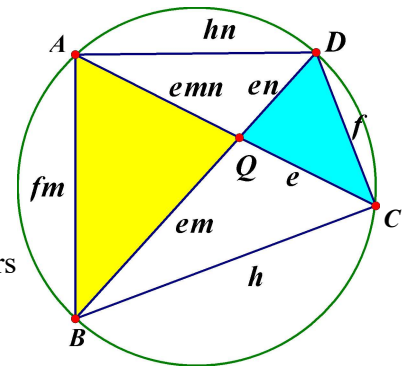
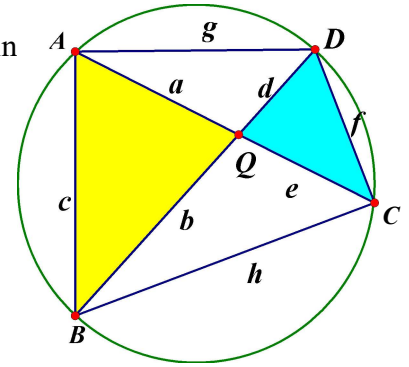
$$n = \frac{en}{e} = \frac{DQ}{CQ} = \frac{2(u^2 + v^2)(2v^2 + k)}{(2u^2 + 2v^2 + k)k}$$

$$AQ = emn = BQ \times n$$

$$\begin{aligned} &= 2(u^2 + v^2)(2u^2 + k) \cdot \frac{2(u^2 + v^2)(2v^2 + k)}{(2u^2 + 2v^2 + k)k} \\ &= 4 \cdot \frac{(u^2 + v^2)^2 (2u^2 + k)(2v^2 + k)}{(2u^2 + 2v^2 + k)k} \end{aligned}$$

$$AD = hn = \left[k^2 + 4u^2(u^2 + v^2 + k) \right] \cdot \frac{2(u^2 + v^2)(2v^2 + k)}{(2u^2 + 2v^2 + k)k}$$

$$m = \frac{em}{e} = \frac{BQ}{CQ} = \frac{2(u^2 + v^2)(2u^2 + k)}{(2u^2 + 2v^2 + k)k}$$



$$AB = fm = \left[k^2 + 4v^2(u^2 + v^2 + k) \right] \cdot \frac{2(u^2 + v^2)(2u^2 + k)}{(2u^2 + 2v^2 + k)k}$$

Multiply every side by $(2u^2 + 2v^2 + k)k$ to give integral sides:

$$CQ = (2u^2 + 2v^2 + k)^2 k^2, CD = (2u^2 + 2v^2 + k)k[k^2 + 4v^2(u^2 + v^2 + k)], DQ = 2k(u^2 + v^2)(2v^2 + k)(2u^2 + 2v^2 + k)$$

$$BC = (2u^2 + 2v^2 + k)k[k^2 + 4u^2(u^2 + v^2 + k)], BQ = 2(u^2 + v^2)(2u^2 + k)(2u^2 + 2v^2 + k)k$$

$$AQ = 4(u^2 + v^2)^2(2u^2 + k)(2v^2 + k), AD = 2(u^2 + v^2)(2v^2 + k)[k^2 + 4u^2(u^2 + v^2 + k)]$$

$$AB = 2(u^2 + v^2)(2u^2 + k)[k^2 + 4v^2(u^2 + v^2 + k)]$$

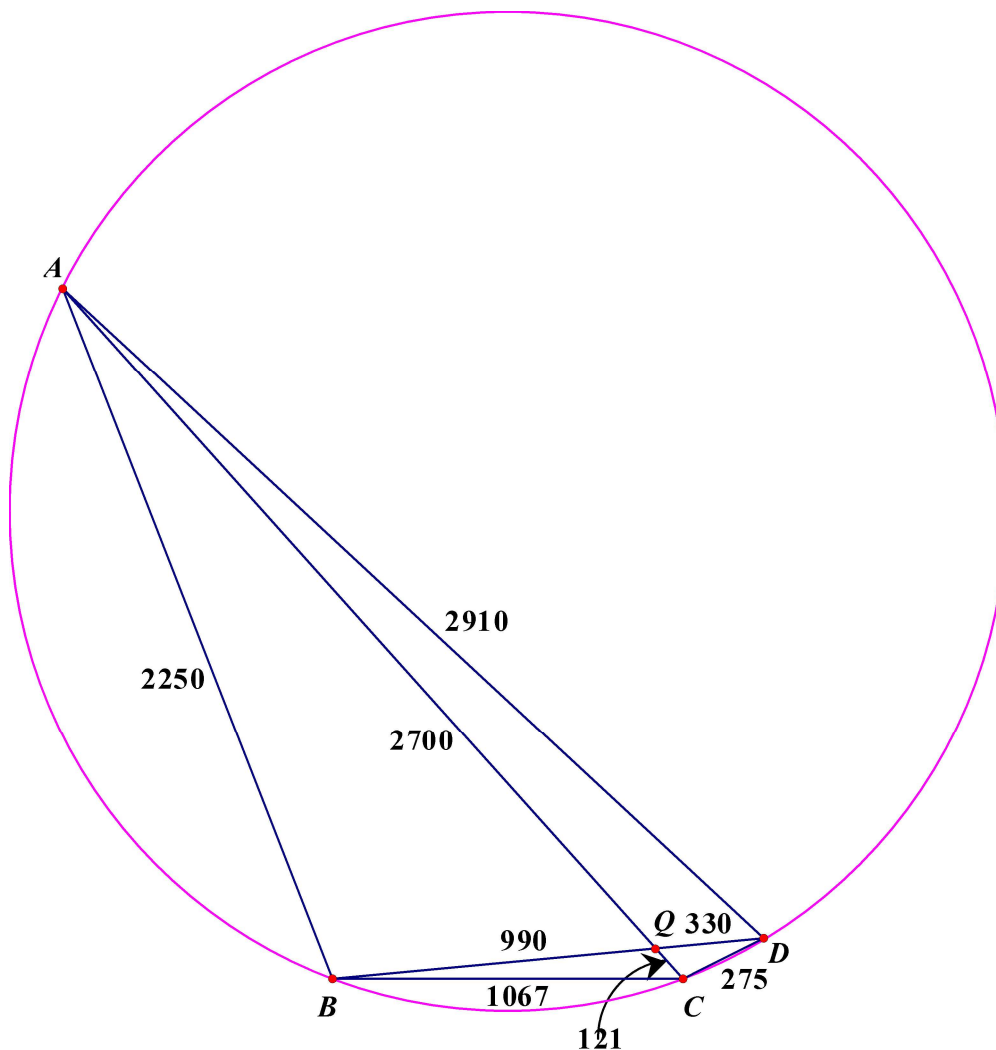
For example, take $k = 1, u = 2, v = 1$:

$$CQ = 121, CD = 11 \times (1 + 4 \times 6) = 275, DQ = 2 \times 5 \times 3 \times 11 = 330$$

$$BC = 11 \times (1 + 16 \times 6) = 1067, BQ = 2 \times 5 \times 9 \times 11 = 990$$

$$AQ = 4 \times 5^2 \times 9 \times 3 = 2700, AD = 2 \times 5 \times 3 \times (1 + 16 \times 6) = 2910$$

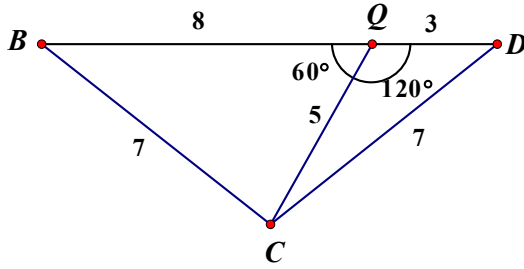
$$AB = 2 \times 5 \times 9 \times (1 + 4 \times 6) = 2250, AC = AQ + CQ = 2700 + 121 = 2821, BD = BQ + DQ = 990 + 330 = 1320$$



Think further: From the document: https://twhung78.github.io/Number_Theory/120triangle.pdf,

we know that 3-5-7 is a 120° triangle, whereas 5-7-8 is a 60° triangle.

Combine the common side '5' to give a bigger triangle as shown:



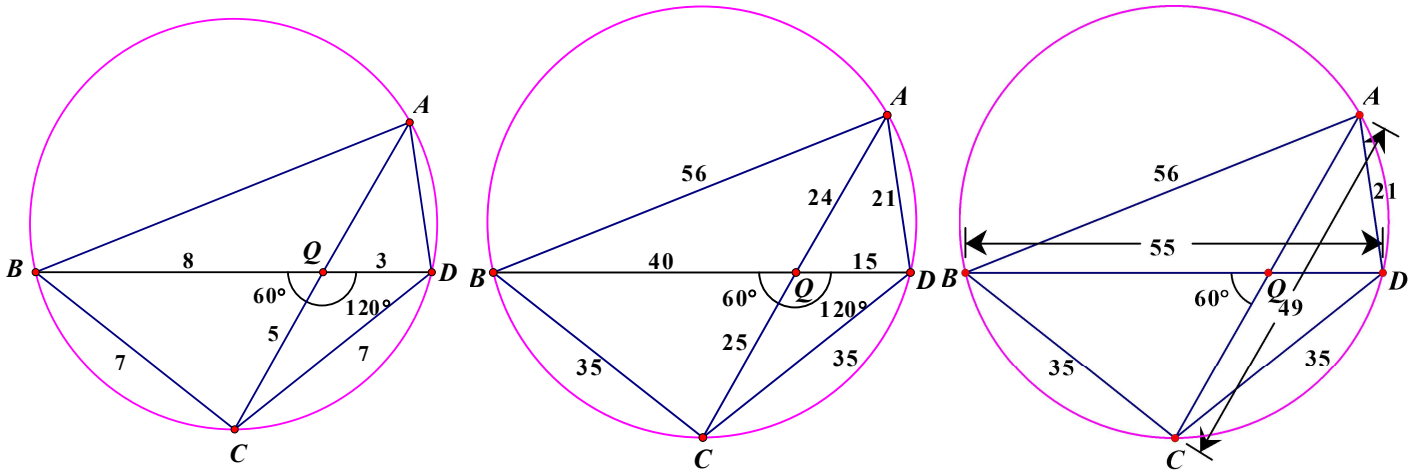
$$\cos \angle CQD = \frac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5} = -\frac{1}{2} \Rightarrow \angle CQD = 120^\circ$$

$$\cos \angle BQC = \frac{8^2 + 5^2 - 7^2}{2 \times 5 \times 8} = \frac{1}{2} \Rightarrow \angle BQC = 60^\circ$$

$$\angle BQC + \angle CQD = 60^\circ + 120^\circ = 180^\circ$$

$\therefore B, Q, D$ are collinear.

Construct a circumscribed circle through B, C and D . Extend CQ to cut the circle again at A . Join AB and AD .



It is easy to show that $\triangle ABQ \sim \triangle DCQ$ (equiangular)

$$\frac{DQ}{AQ} = \frac{CQ}{BQ} = \frac{CD}{AB} \quad (\text{corr. sides, } \sim \Delta s)$$

$$AQ = 3 \times \frac{8}{5} = \frac{24}{5}, AB = 7 \times \frac{8}{5} = \frac{56}{5}, AC = \frac{24}{5} + 5 = \frac{49}{5}, BD = 8 + 3 = 11$$

$\triangle ADQ \sim \triangle BCQ$ (equiangular)

$$\frac{AD}{BC} = \frac{AQ}{BQ} \quad (\text{corr. sides, } \sim \Delta s)$$

$$AD = 7 \times \frac{3}{5} = \frac{21}{5}$$

Multiply every side by 5 to give integral sides. $BC = CD = 35, AD = 21, AB = 56, AC = 49, BD = 55$.

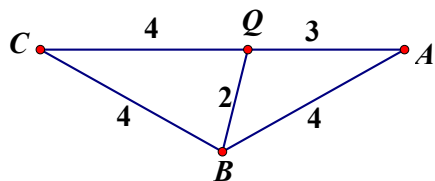
\therefore We can construct another integral cyclic quadrilateral with a simpler formula, but the area of each smaller triangle inside (and hence the cyclic quadrilateral) are not integers.

Again, this is a **special case for two equal adjacent sides of integral cyclic quadrilateral and the angle between the two diagonals is 60° .**

Question: Given any triangle $\triangle ABQ$ with integral sides, can we construct an integral cyclic quadrilateral, while the angle between the diagonals are not necessarily 60° using a similar method?

Let $\triangle ABC$ be a 2-3-4 triangle with $AB = 4$, $BQ = 2$, $CQ = 3$.

We can construct (method on page 9) another triangle $\triangle BCQ$ (with common sides BQ) so that $BC = 4$, $CQ = 4$.



$$\cos \angle AQB = \frac{2^2 + 3^2 - 4^2}{2 \times 2 \times 3} = -\frac{1}{4}$$

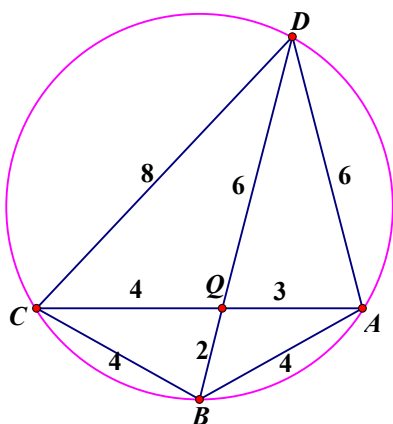
$$\cos \angle BQC = \frac{2^2 + 4^2 - 4^2}{2 \times 2 \times 4} = \frac{1}{4} = -\cos \angle AQB$$

$$\angle AQB + \angle BQC = 180^\circ$$

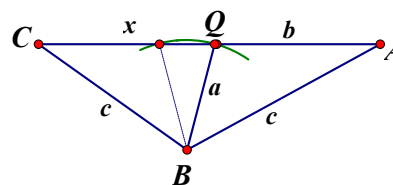
$\therefore A, Q, C$ are collinear.

Construct a circumscribed circle through A, B and C . Extend BQ to cut the circle again at D . Join AD and CD .

Using a similar method, we can prove that $DQ = 6$, $AD = 6$, $CD = 8$.

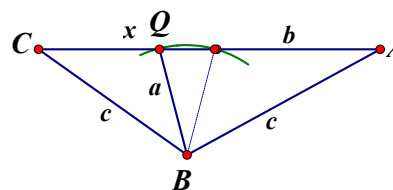


In this case, $AB = BC = 4$ (two equal adjacent sides, whereas the angle between the diagonals $\neq 60^\circ$.)



Given any triangle $\triangle ABQ$ with integral sides ($AB = c$, $BQ = a$, $AQ = b$). We can use the method on page 9 to find another triangle $\triangle BCQ$ with a common side BQ and A, Q, C are collinear. $BC = c$, $CQ = x$.

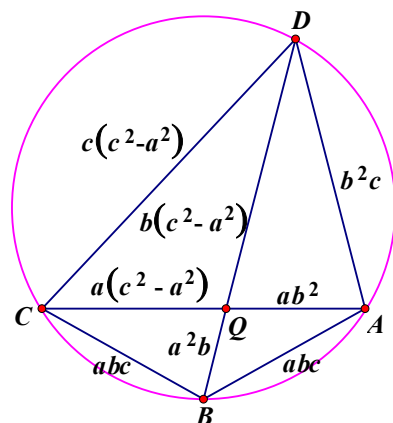
$$\cos \angle AQB = \frac{a^2 + b^2 - c^2}{2ab}, \cos \angle BQC = -\frac{a^2 + b^2 - c^2}{2ab}$$



If $\angle AQB$ is obtuse, then $x = b + 2a \cos \angle BQC = b - \frac{a^2 + b^2 - c^2}{b} = \frac{c^2 - a^2}{b}$.

If $\angle AQB$ is acute, then $x = b - 2a \cos \angle AQB = b - \frac{a^2 + b^2 - c^2}{b} = \frac{c^2 - a^2}{b}$.

If $\angle AQB = 90^\circ$, then $x = b = \frac{c^2 - a^2}{b}$.



Finally, a cyclic quadrilateral with integral sides is formed.

The only necessary condition is $c > a$ and a, b, c obey triangle inequality.

Let $CQ = x$, $BC = y$.

$$y^2 = x^2 + a^2 - 2ax \cos \angle BQC = x^2 + a^2 + 2ax \cos \angle AQB = x^2 + a^2 + 2ax \cdot \frac{a^2 + b^2 - c^2}{2ab}$$

$$y^2 = x^2 + 2ax + a^2 + \frac{x}{b}(a^2 + b^2 - c^2) - 2ax$$

$$(x + y + a)(x - y + a) = \frac{x}{b}[c^2 - (a^2 - 2ab + b^2)] = \frac{x}{b}[c^2 - (a - b)^2]$$

$$\frac{x + y + a}{x} = \frac{c^2 - (a - b)^2}{b(x - y + a)} = k$$

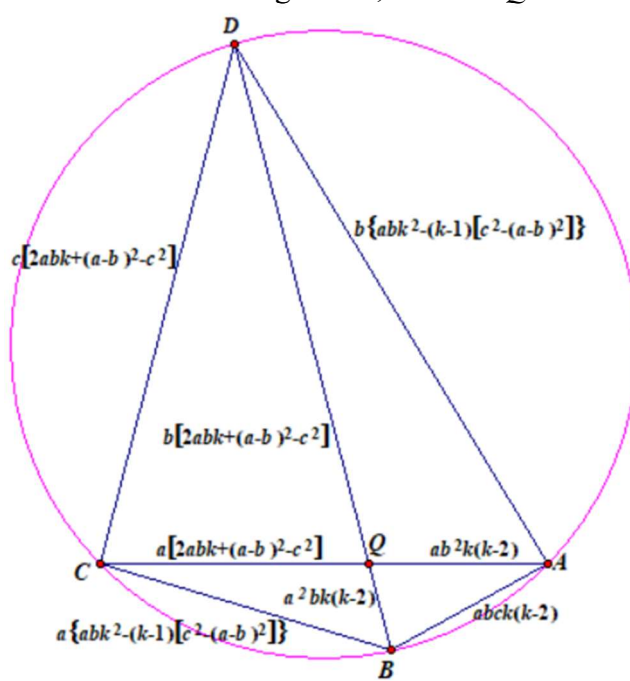
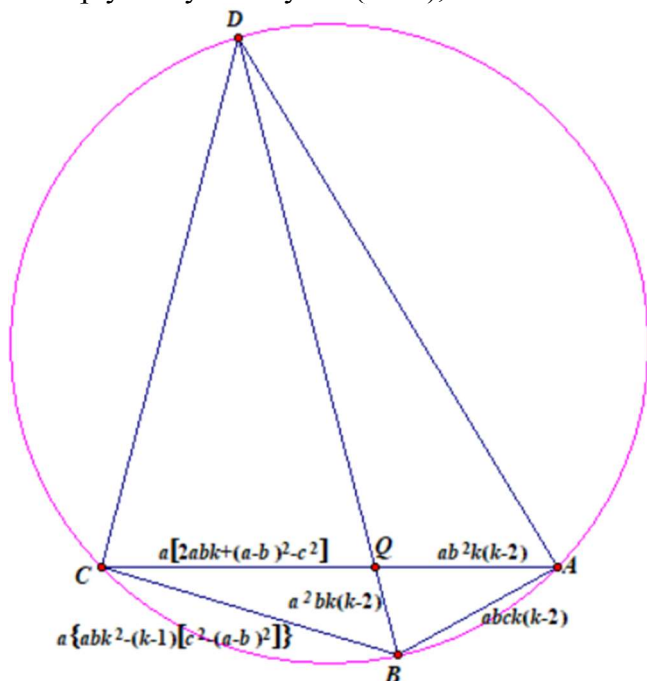
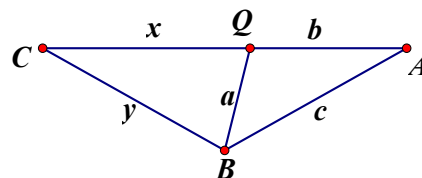
$$x + y + a = kx \dots\dots (1), x - y + a = \frac{c^2 - (a - b)^2}{bk} \dots\dots (2)$$

$$(1) + (2): 2x + 2a = kx + \frac{c^2 - (a - b)^2}{bk} \Rightarrow x = \frac{2abk + (a - b)^2 - c^2}{bk(k - 2)}$$

$$(1) - (2): 2y = kx + \frac{(a - b)^2 - c^2}{bk} = \frac{2abk + (a - b)^2 - c^2}{b(k - 2)} + \frac{(a - b)^2 - c^2}{bk}$$

$$y = \frac{ak}{k - 2} - \frac{k - 1}{bk(k - 2)}[c^2 - (a - b)^2]$$

Multiply every side by $abk(k - 2)$, construct a circumscribed circle through ABC , extend BQ to the circle at D .



$$\frac{DQ}{a[2abk + (a - b)^2 - c^2]} = \frac{ab^2k(k - 2)}{a^2bk(k - 2)} = \frac{AD}{a\{abk^2 - (k - 1)[c^2 - (a - b)^2]\}} \quad (\text{corr. sides, } \triangle ADQ \sim \triangle BCQ)$$

$$DQ = b[2abk + (a - b)^2 - c^2], AD = b\{abk^2 - (k - 1)[c^2 - (a - b)^2]\}$$

$$\frac{DQ}{ab^2k(k - 2)} = \frac{CD}{abck(k - 2)} \quad (\text{corr. sides, } \triangle CDQ \sim \triangle BAQ)$$

$$CD = c[2abk + (a - b)^2 - c^2]$$

Reference:

1. Pythagorean Triple: https://twhung78.github.io/Number_Theory/Pythagorean_triple.pdf
2. Angle between two diagonals in a cyclic quadrilateral:
https://twhung78.github.io/Geometry/6%20Circles/2%20Cyclic%20quadrilateral/Angle_diagonals_cyclic_quad_rilateral.pdf
3. "Normal Trigrade and cyclic quadrilateral with integral sides and diagonals" from April, 1951 American Mathematical Monthly.
4. 120° triangle: https://twhung78.github.io/Number_Theory/120triangle.pdf