Examples on Mathematical Induction: divisibility 11

Created by Mr. Francis Hung Last updated: September 1, 2021

1. **1977 Paper 2 Q6**

   Let $P(n) \equiv$ "$23^n - 1$ is divisible by 11 for all positive integral values of n ."

   $n = 1$, $23 - 1 = 22 = 11\times 2$, which is divisible by 11

   Suppose it is true for $n = k$

   i.e. $23^k - 1 = 11m$ for some integer m

   $23^k = 11m + 1$

   $23^{k+1} - 1 = 23^k(23) - 1 = (11m + 1)(23) - 1 = 11\times 23m + 22 = 11\times(23m + 2)$

   $23m + 2$ is an integer

   $\therefore\ 23^{k+1} \times 1$ is also divisible by 11.

   If $P(k)$ is true then $P(k+1)$ is also true

   By the principle of mathematics induction, $23^n - 1$ is divisible by 11 for all positive integer n .

2. Prove by mathematical induction $2\cdot 4^{2n+1} + 3^{3n+1}$ is divisible by 11 for all non-negative integers n.

   Let $P(n) \equiv$ "$2\cdot 4^{2n+1} + 3^{3n+1}$ is divisible by 11 for all non-negative integers n."

   $n = 0$, $2\cdot 4 + 3 = 11$ which is divisible by 11.

   Suppose $2\cdot 4^{2k+1} + 3^{3k+1} = 11m$, where m is an integer.

   When $n = k + 1$, $2\cdot 4^{2(k+1)+1} + 3^{3(k+1)+1} = 16\cdot 2\cdot 4^{2k+1} + 27\cdot 3^{3k+1}$

   $= 16\cdot(11m - 3^{3k+1}) + 27\cdot 3^{3k+1}$

   $= 176m - 11\cdot 3^{3k+1}$

   $= 11(16m - 3^{k+1})$

   If $P(k)$ is true then $P(k + 1)$ is also true.

   By the principle of mathematical induction, $P(n)$ is true for all non-negative integers n.

3. Prove that either $12^n - 10^n$ or $12^n + 10^n$ is divisible by 11.

4. Given a 5-digits integer $x = \overline{abcde}$. If $a + c + e - (b + d) = 11k$, where k is an integer, prove that x is divisible by 11.

   Proof: $x = 10000a + 1000b + 100c + 10d + e$

   $= 9999a + a + 1001b - b + 99c + c + 11d - d + e$

   $= 11(909a + 91b + 9c + d) + a - b + c - d + e$

   $= 11(900a + 100b + 9c + d) + 11k$, which is divisible by 11 .

5. HKHLE General Mathematics 1976 Q8(b)

Prove that, for any positive integer n, the integer $10^n + (-1)^{n-1}$ is divisible by 11 .

Hence deduce a necessary and sufficient condition for an integer to be divisible by 11 by considering only the sum and difference of the digits of the integer.

Induction on n .

$n = 1$, $10^1 + (-1)^0 = 11$ which is obviously divisible by 11 .

Suppose $10^k + (-1)^{k-1} = 11m$, where m is an integer, for some positive integer k .

$10^{k+1}+(-1)^k= 10(10^k)+(-1)^k = 10[11m-(-1)^{k-1}]+(-1)^k = 110m +(-1)^k(1 + 10) = 11[10m+(-1)^k]$ which is divisible by 11.

So, by M.I., $10^n + (-1)^{n-1}$ is divisible by 11 for any positive integer n.

The necessary and sufficient condition is: Let S_1 be the sum of all odd digits of an integer N, S_2 be the sum of all even digits of N. S_1-S_2 is divisible by 11 if and only if N is divisible by 11.

Proof: $N = a_n\times10^n + a_{n-1}\times10^{n-1} +\cdots+ a_1\times10 + a_0$, where $0 \le a_r \le 9$ and a_r are integers, $0 \le r \le n$

$S_1 - S_2 = (-1)^n a_n + (-1)^{n-1}a_{n-1} + \cdots - a_1 + a_0$

If N is divisible by 11, $a_n\times10^n + a_{n-1}\times10^{n-1} + \cdots + a_1\times10 + a_0 = 11m$, where m is an integer.

$$N = 11\ m = a_n\times[10^n + (-1)^{n-1}] + a_{n-1}\times[10^{n-1} +(-1)^{n-2}] + \cdots + a_1\times[10 + 1] + a_0\ (1 - 1)$$

$$- [(-1)^{n-1}a_n + (-1)^{n-2}a_{n-1} + \cdots + a_1 - a_0]$$

$$=a_n\times11k_n+a_{n-1}\times11k_{n-1} + \cdots +a_1\times11k_1+a_0\cdot11k_0+S_1-S_2, \text{ where } k_r \text{ are integers, } 0 \le r \le n$$

$$\Rightarrow S_1 - S_2 = 11m - [a_n\times11k_n + a_{n-1}\times11k_{n-1} + \cdots + a_1\times11k_1 + a_0\ 11k_0]$$

$$= 11[m - a_nk_n + a_{n-1}k_{n-1} + \cdots + a_1k_1 + a_0k_0], \text{ which is divisible by 11 .}$$

If $S_1 - S_2$ is divisible by 11, then $(-1)^n a_n+(-1)^{n-1}a_{n-1}+\cdots-a_1+a_0 = 11m$, where m is an integer.

$$N = a_n\times10^n + a_{n-1}\times10^{n-1} + \cdots + a_1\times10 + a_0$$

$$= a_n\times[10^n + (-1)^{n-1}] + a_{n-1}\times[10^{n-1} +(-1)^{n-2}] + \cdots + a_1\times[10 + 1] + a_0\ (1 - 1)$$

$$+ [(-1)^n a_n + (-1)^{n-1}a_{n-1} + \cdots - a_1 + a_0]$$

$$= a_n\times11k_n + a_{n-1}\times11k_{n-1} + \cdots + a_1\times11k_1 + a_0\ 11k_0 + S_1-S_2$$

$$= 11\ [a_nk_n + a_{n-1}k_{n-1} + \cdots + a_1k_1 + a_0k_0] + 11m, \text{ which is divisible by 11 .}$$