

22-23 Individual	1	−13	2	7	3	52	4	52	5	9
	6	$16\sqrt{2}$	7	$\frac{4}{5}$	8	4	9	289	10	315
	11	14	12	$\frac{1}{3}$	13	2697	14	100	15	$\frac{2}{7}$
22-23 Group	1	27	2	−1	3	$\frac{67}{25}$	4	86400	5	$\frac{19}{12}$
	6	8	7	401	8	172	9	−5	10	64

Individual Events

- I1** 已知 a 和 b 均為實數。若 $a^2 + b^2 - 8a + 34b + 305 = 0$ ，求 $a + b$ 的值。

Given that a and b are real numbers. If $a^2 + b^2 - 8a + 34b + 305 = 0$, find the value of $a + b$.

Reference: 2021 P1Q15

$$a^2 - 8a + 16 + b^2 + 34b + 289 = 0$$

$$(a - 4)^2 + (b + 17)^2 = 0$$

Sum of two squares = 0 \Leftrightarrow each term = 0

$$a = 4 \text{ and } b = -17$$

$$a + b = 4 - 17 = -13$$

- I2** 若 x 及 y 均為正整數且滿足 $x + 8xy + y = 28$ ，求 $x + 2y$ 的最大可能值。

If x and y are positive integers satisfying $x + 8xy + y = 28$,

find the largest possible value of $x + 2y$.

Reference: 2022 P2Q3

$$x(1 + 8y) + y = 28$$

$$8x(1 + 8y) + 1 + 8y = 225$$

$$(8x + 1)(8y + 1) = 225$$

$8x + 1$	$8y + 1$	x	y	$x + 2y$
1	225	no positive solution		
3	75	no integral solution		
5	45	no integral solution		
9	25	1	3	7
15	15	no integral solution		
25	9	3	1	5

The largest possible value of $x + 2y = 7$

- I3** 設 m 為一個整數常數，其中 $4 < m < 40$ 。若方程 $x^2 - 2(2m - 3)x + 4m^2 - 14m + 8 = 0$ 有兩個整數根，求 x 的最大可能值。

Let m be an integral constant, where $4 < m < 40$. If the equation

$x^2 - 2(2m - 3)x + 4m^2 - 14m + 8 = 0$ has two integral roots, find the largest possible value of x .

Reference: 2011 FI3.1

$$\Delta = 4(2m - 3)^2 - 4(4m^2 - 14m + 8) = P^2, \text{ where } P \text{ is an integer.}$$

$$\left(\frac{P}{2}\right)^2 = 4m^2 - 12m + 9 - 4m^2 + 14m - 8 = 2m + 1, x = -\frac{b}{2} \pm \sqrt{\frac{\Delta}{4}} = (2m - 3) \pm \sqrt{2m + 1}$$

The possible square numbers are 9, 16, 25, 36, 49, 64, 81, ...

$2m + 1$	m	quad. equation	x
9	4	rejected	
25	12	$x^2 - 42x + 416 = 0$	16, 26
49	24	$x^2 - 90x + 1976 = 0$	38, 52
81	40	rejected	

The largest possible value of x is 52.

- I4** 設 a 為一正實數。若 $a^2 + \frac{1}{a^2} = 14$ ，求 $a^3 + \frac{1}{a^3}$ 的值。

Let a be a positive real number. If $a^2 + \frac{1}{a^2} = 14$, find the value of $a^3 + \frac{1}{a^3}$.

Reference: 2017 FI1.4

$$a^2 + \frac{1}{a^2} = 14$$

$$\left(a + \frac{1}{a}\right)^2 = 16$$

$$a + \frac{1}{a} = 4 \text{ or } -4 \text{ (rejected, } \because a > 0)$$

$$\begin{aligned} a^3 + \frac{1}{a^3} &= \left(a + \frac{1}{a}\right) \left(a^2 - 1 + \frac{1}{a^2}\right) \\ &= 4 \times (14 - 1) = 52 \end{aligned}$$

- I5** 若干正整數之和是 60。最大正整數為 15 及其中有一個正整數是 12。除卻這正整數 12，其餘正整數恰好組成一個等差數列。求最小的正整數。

The sum of certain number of positive integers is 60. The largest positive integer is 15 and one of the positive integers is 12. Apart from this positive integer 12, the remaining positive integers form an arithmetic sequence. Find the smallest positive integer.

Let the smallest positive integer be ℓ and the number of terms be n . then $n < 15$.

$$\frac{n}{2}(15 + \ell) + 12 = 60$$

$$n(15 + \ell) = 96$$

Possible integral values of n are 1, 2, 3, 4, 6, 8 and 12.

The corresponding values of ℓ are 81, 33, 17, 9, -1, -3 and -7.

$$\therefore 1 \leq \ell \leq 15$$

$$\therefore \ell = 9 \text{ only}$$

- I6** 在圖一中，把長方形 $ABCD$ 繞它的中心逆時針轉 45° 得長方形 $EFGH$ 。若 $AB = 4$ ，求陰影部分 $PQRS$ 的面積。

In Figure 1, the rectangle $ABCD$ is rotated about its centre 45° anticlockwise to obtain the rectangle $EFGH$. If $AB = 4$, find the area of the shaded region $PQRS$.

Reference: 2011 FG2.1

Let $PQ = x$. Then $x \cos 45^\circ = 4$

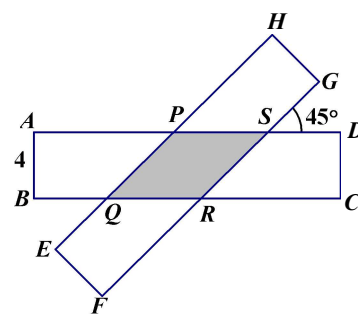
$$x = 4\sqrt{2}$$

$$\therefore EF = AB = 4 \text{ and } \angle FRQ = 45^\circ$$

$$\therefore QR = 4\sqrt{2}$$

$$\therefore PQRS \text{ is a rhombus with length of side } = 4\sqrt{2} \text{ and } \angle PQR = 45^\circ$$

$$\text{Area of } PQRS = (4\sqrt{2})^2 \sin 45^\circ = 16\sqrt{2}$$



圖一 Figure 1

- 17 求 $\left(\frac{1 \times 4 \times 16 \times 64 + 2 \times 8 \times 32 \times 128 + 3 \times 12 \times 48 \times 192 + \cdots + 2023 \times 8092 \times 32368 \times 129472}{1 \times 5 \times 25 \times 125 + 2 \times 10 \times 50 \times 250 + 3 \times 15 \times 75 \times 375 + \cdots + 2023 \times 10115 \times 50575 \times 252875} \right)^{\frac{1}{6}}$ 的值。

Evaluate $\left(\frac{1 \times 4 \times 16 \times 64 + 2 \times 8 \times 32 \times 128 + 3 \times 12 \times 48 \times 192 + \cdots + 2023 \times 8092 \times 32368 \times 129472}{1 \times 5 \times 25 \times 125 + 2 \times 10 \times 50 \times 250 + 3 \times 15 \times 75 \times 375 + \cdots + 2023 \times 10115 \times 50575 \times 252875} \right)^{\frac{1}{6}}$.

Reference: 2000 FI5.1, 2015 FG1.1

$$\begin{aligned} & \left(\frac{1 \times 4 \times 16 \times 64 + 2 \times 8 \times 32 \times 128 + 3 \times 12 \times 48 \times 192 + \cdots + 2023 \times 8092 \times 32368 \times 129472}{1 \times 5 \times 25 \times 125 + 2 \times 10 \times 50 \times 250 + 3 \times 15 \times 75 \times 375 + \cdots + 2023 \times 10115 \times 50575 \times 252875} \right)^{\frac{1}{6}} \\ &= \left[\frac{1 \times 4 \times 16 \times 64 (1^4 + 2^4 + 3^4 + \cdots + 2023^4)}{1 \times 5 \times 25 \times 125 (1^4 + 2^4 + 3^4 + \cdots + 2023^4)} \right]^{\frac{1}{6}} \\ &= \left(\frac{4 \times 4^2 \times 4^3}{5 \times 5^2 \times 5^3} \right)^{\frac{1}{6}} = \left(\frac{4^6}{5^6} \right)^{\frac{1}{6}} = \frac{4}{5} \end{aligned}$$

- 18 若一個等邊三角形的面積與其在周界在數值上相等，求該正三角形的外接圓的半徑。
If the area of an equilateral triangle is numerically equal to its perimeter, find the radius of the circumcircle of this equilateral triangle.

Let the equilateral triangle be ABC with $BC = x$.

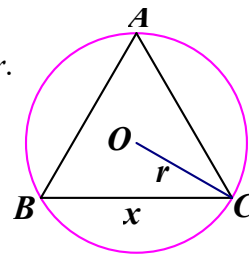
Let O be the centre of the circumscribed circle with $OC = r$.

Perimeter $= 3x$, area $= \frac{1}{2} \cdot x^2 \sin 60^\circ = \frac{\sqrt{3}}{4} x^2$

$3x = \frac{\sqrt{3}}{4} x^2 \Rightarrow x = 4\sqrt{3}$

Apply cosine rule on $\triangle BOC$: $2r^2 - 2r^2 \cos 120^\circ = x^2$

$r^2 \left[2 - 2 \cdot \left(-\frac{1}{2} \right) \right] = (4\sqrt{3})^2 \Rightarrow r = 4$



- 19 在圖二中， P 為正方形 $ABCD$ 內的一點使得 $\triangle ABP \cong \triangle ADP$ ， $AP = 5\sqrt{2}$ 及 $BP = 13$ 。求正方形 $ABCD$ 的面積。

In Figure 2, P is a point inside the square $ABCD$ such that

$\triangle ABP \cong \triangle ADP$. $AP = 5\sqrt{2}$ and $BP = 13$.

Find the area of square $ABCD$.

$\angle BAD = 90^\circ$ (property of square)

$\angle BAP = \angle DAP$ (corr. \angle s, \cong triangles)

$= 45^\circ$

Let E be the foot of perpendicular from P to AB .

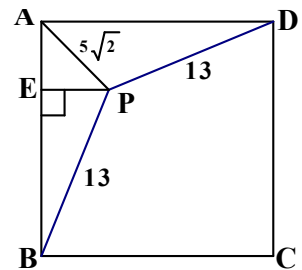
$AE = AP \cos 45^\circ = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5$, $EP = AP \sin 45^\circ = 5$

$EP^2 + BE^2 = BP^2$ (Pythagoras' theorem)

$BE = \sqrt{13^2 - 5^2} = 12$

$AB = AE + EB = 5 + 12 = 17$

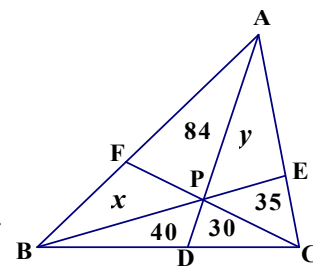
Area of $ABCD = 17^2 = 289$



圖二 Figure 2

- I10** 在圖三中， D 、 E 及 F 分別為 BC 、 AC 及 AB 上的點。 AD 、 BE 及 CF 相交於 P 使得 $\triangle APF$ 的面積 = 84、 $\triangle BPD$ 的面積 = 40、 $\triangle CPD$ 的面積 = 30 及 $\triangle CPE$ 的面積 = 35。求 $\triangle ABC$ 的面積。

In Figure 3, D , E and F are points lying on BC , AB and AC respectively. AD , BE and CF intersect at P such that such that area of $\triangle APF = 84$, area of $\triangle BPD = 40$, area of $\triangle CPD = 30$ and area of $\triangle CPE = 35$. Find the area of $\triangle ABC$.



圖三 Figure 3

Let S represents the area(s), $S_{\triangle BPF} = x$, $S_{\triangle APE} = y$

$\triangle BPD$ and $\triangle CPD$ have the same height but different bases

$$\frac{S_{\triangle BPD}}{S_{\triangle CPD}} = \frac{BD}{DC} = \frac{4}{3}$$

$\triangle ABD$ and $\triangle ACD$ have the same height but different bases

$$\frac{S_{\triangle ABD}}{S_{\triangle ACD}} = \frac{BD}{DC} = \frac{84 + x + 40}{y + 35 + 30} = \frac{4}{3}$$

$$3(124 + x) = 4(y + 65)$$

$$372 + 3x = 4y + 260$$

$$112 + 3x = 4y \dots\dots (1)$$

$\triangle BPC$ and $\triangle CPE$ have the same height but different bases

$$\frac{S_{\triangle BPC}}{S_{\triangle CPE}} = \frac{BP}{PE} = \frac{40 + 30}{35} = \frac{2}{1}$$

$\triangle ABP$ and $\triangle APE$ have the same height but different bases

$$\frac{S_{\triangle ABP}}{S_{\triangle APE}} = \frac{BP}{PE} = \frac{84 + x}{y} = \frac{2}{1}$$

$$84 + x = 2y \dots\dots (*)$$

$$168 + 2x = 4y \dots\dots (2)$$

$$(1) = (2) \quad 112 + 3x = 168 + 2x$$

$$x = 56$$

$$\text{Sub. } x = 56 \text{ into } (*): 84 + 56 = 2y$$

$$y = 70$$

$$\text{Area of } \triangle ABC = 40 + 30 + 35 + 70 + 84 + 56 = 315$$

- I11** 已知 n 是一個少於 2023 正整數。若 n 只有三個不同的因數，求 n 的可能性的總數。

Given that n is a positive integer less than 2023.

If n has only 3 distinct factors, find the number of possible values of n .

Reference: 2004 FI1.1

n must be the square of a prime number p , i.e. $n = p^2$, the factors are 1, p and p^2 .

$$44^2 = 1936 < 45^2 = 2025, 44 < \sqrt{2023} < 45$$

Possible n are $2^2, 3^2, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2, 37^2, 41^2$ and 43^2 .

Number of possible n is 14.

- I12** 已知 p 及 q 為正實數。若 $\log_9 p = \log_{15} q = \log_{25} (3p + 2q)$ ，求 $\frac{p}{q}$ 的值。

Given that p and q are positive numbers. If $\log_9 p = \log_{15} q = \log_{25} (3p + 2q)$, find the value of $\frac{p}{q}$.

$$\text{Let } \log_9 p = \log_{15} q = \log_{25} (3p + 2q) = k$$

$$\frac{\log p}{\log 9} = \frac{\log q}{\log 15} = \frac{\log (3p + 2q)}{\log 25} = k$$

$$\begin{aligned}\log 9 &= \frac{\log p}{k}, \log 15 = \frac{\log q}{k}, \log 25 = \frac{\log(3p+2q)}{k} \\ \log 3 &= \frac{\log p}{2k}, \log 3 + \log 5 = \frac{\log q}{k}, \log 5 = \frac{\log(3p+2q)}{2k} \\ \frac{\log p}{2k} + \frac{\log(3p+2q)}{2k} &= \frac{\log q}{k} \\ \log p + \log(3p+2q) &= 2 \log q \\ p(3p+2q) &= q^2 \\ 3p^2 + 2pq - q^2 &= 0 \\ (3p-q)(p+q) &= 0 \\ \frac{p}{q} &= \frac{1}{3} \text{ or } -1 \text{ (rejected)}\end{aligned}$$

- I13** 數列 $\{a_n\}$ 定義為 $a_1 = 1$ 、 $a_2 = \frac{3}{7}$ 及對所有 $n \geq 3$ ， $a_n = \frac{a_{n-2}a_{n-1}}{2a_{n-2} - a_{n-1}}$ 。求 $\frac{1}{a_{2023}}$ 的值。

A sequence of numbers $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = \frac{3}{7}$ and $a_n = \frac{a_{n-2}a_{n-1}}{2a_{n-2} - a_{n-1}}$ for all $n \geq 3$.

Find the value of $\frac{1}{a_{2023}}$.

$$\text{Let } b_1 = \frac{1}{a_1} = 1, b_2 = \frac{1}{a_2} = \frac{7}{3}, b_n = \frac{1}{a_n} = \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}} = 2b_{n-1} - b_{n-2}$$

The characteristics equation for b_n is $\lambda^2 = 2\lambda - 1 \Rightarrow \lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$

\therefore The general solution for $b_n = A + Bn$, where A and B are constants

$$b_1 = A + B = 1 \dots\dots (1)$$

$$b_2 = A + 2B = \frac{7}{3} \dots (2)$$

$$(2) - (1): B = \frac{4}{3}, A = -\frac{1}{3}$$

$$b_n = -\frac{1}{3} + \frac{4n}{3}$$

$$b_{2023} = \frac{1}{a_{2023}} = -\frac{1}{3} + \frac{4 \times 2023}{3} = 2697$$

- I14** ABC 是一個等腰三角形，其中 $AB = AC = 18$ 及 $BC = 12$ 。 P 為 $\triangle ABC$ 內的任意一點使得 $\angle ABP + \angle ACP = 90^\circ$ 及 $AP = 15$ 。
求 $BP^2 + CP^2$ 的值。

ABC is an isosceles triangle with $AB = AC = 18$ and $BC = 12$. P is any interior point of $\triangle ABC$ such that $\angle ABP + \angle ACP = 90^\circ$ and $AP = 15$.

Find the value of $BP^2 + CP^2$.

Let $\angle BAC = \theta$. Rotate $\triangle APC$ clockwise about A through θ to $\triangle AQB$.

$\triangle APQ \sim \triangle ACB$ (S.A.S.) and $\triangle ABQ \cong \triangle ACP$

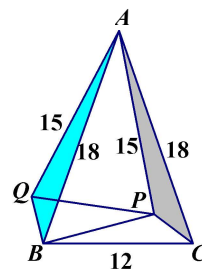
$$QP : 12 = 15 : 18 \text{ (corr. sides, } \sim \Delta \text{s)}$$

$$\Rightarrow QP = 10$$

$$\angle PBQ = \angle ABP + \angle ABQ$$

$$= \angle ABP + \angle ACP = 90^\circ \text{ (corr. } \angle \text{s, } \cong \Delta \text{s)}$$

$$BP^2 + CP^2 = PQ^2 = 100 \quad \text{(Pythagoras' theorem)}$$



115 求方程 $\sqrt[3]{x} + \sqrt[3]{x-4} = \sqrt[3]{x-2}$ 的根之積。

Find the product of roots of the equation $\sqrt[3]{x} + \sqrt[3]{x-4} = \sqrt[3]{x-2}$.

Let $t = x - 2$, then $x = t + 2$, $x - 4 = t - 2$, the equation becomes $\sqrt[3]{t+2} + \sqrt[3]{t-2} = \sqrt[3]{t}$

$$\left(\sqrt[3]{t+2} + \sqrt[3]{t-2}\right)^3 = \left(\sqrt[3]{t}\right)^3$$

$$t+2+3\sqrt[3]{(t+2)^2} \cdot \sqrt[3]{t-2} + 3\sqrt[3]{t+2} \cdot \sqrt[3]{(t-2)^2} + t-2 = t$$

$$t+3\sqrt[3]{(t^2-4)} \cdot \sqrt[3]{t+2} + 3\sqrt[3]{t-2} \cdot \sqrt[3]{(t^2-4)} = 0$$

$$t+3\sqrt[3]{(t^2-4)} \cdot \left(\sqrt[3]{t+2} + \sqrt[3]{t-2}\right) = 0$$

$$t+3\sqrt[3]{(t^2-4)} \cdot \sqrt[3]{t} = 0$$

$$3\sqrt[3]{(t^3-4t)} = -t$$

$$27(t^3-4t) = -t^3$$

$$28t^3 - 108t = 0$$

$$t(7t^2 - 27) = 0$$

$$t = 0, \sqrt{\frac{27}{7}} \text{ or } -\sqrt{\frac{27}{7}}$$

$$x = 2, 2 + \sqrt{\frac{27}{7}} \text{ or } 2 - \sqrt{\frac{27}{7}}$$

$$\begin{aligned} \text{Product of roots} &= 2 \left(2 + \sqrt{\frac{27}{7}} \right) \left(2 - \sqrt{\frac{27}{7}} \right) \\ &= 2 \left(4 - \frac{27}{7} \right) \\ &= \frac{2}{7} \end{aligned}$$

Group Events

G1 求 3^{2023} 的最尾兩位數字。Find the last two digits of 3^{2023} .

Reference 2021 P1Q2

$$3^1 = 03, 3^2 = 09, 3^3 = 27, 3^4 = 81, 3^5 = 43, 3^6 = 29, 3^7 = 87, 3^8 = 61, 3^9 = 83, 3^{10} = 49 \\ 3^{11} = 47, 3^{12} = 41, 3^{13} = 23, 3^{14} = 69, 3^{15} = 07, 3^{16} = 21, 3^{17} = 63, 3^{18} = 89, 3^{19} = 67, 3^{20} = 01 \\ 3^{2023} = (3^{20})^{101} \cdot 3^3 \equiv 27 \pmod{100}$$

Method 2 (provided by Mr. Tam Chi Leung)

$$3^{2023} = 3 \times (3^2)^{1011} = 3 \times (10 - 1)^{1011} \\ = 3 \sum_{k=0}^{1011} C_k^{1011} (-1)^k 10^{1011-k} \\ = 3(10^{1011} - C_1^{1011} \cdot 10^{1010} + C_2^{1011} \cdot 10^{1009} + \dots - C_{1009}^{1011} \cdot 10^2 + C_{1010}^{1011} \cdot 10 - 1) \\ = 3(100m + 10110 - 1), \text{ where } m \text{ is an integer} \\ = 300m + 30327$$

The last two digits = 27.

G2 對於 $0 < x < 2$, 求 $\left(\frac{\sqrt{2+x}}{\sqrt{2+x}-\sqrt{2-x}} + \frac{2-x}{\sqrt{4-x^2}+x-2} \right) \left(\sqrt{\frac{4}{x^2}-1} - \frac{2}{x} \right)$ 的值。

For $0 < x < 2$, find the value of $\left(\frac{\sqrt{2+x}}{\sqrt{2+x}-\sqrt{2-x}} + \frac{2-x}{\sqrt{4-x^2}+x-2} \right) \left(\sqrt{\frac{4}{x^2}-1} - \frac{2}{x} \right)$.

Reference: 2017 FG3.2

Expression

$$= \left[\frac{\sqrt{2+x}}{\sqrt{2+x}-\sqrt{2-x}} \cdot \frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}+\sqrt{2-x}} + \frac{2-x}{\sqrt{4-x^2}-(2-x)} \cdot \frac{\sqrt{4-x^2}+(2+x)}{\sqrt{4-x^2}+(2+x)} \right] \left(\frac{\sqrt{4-x^2}}{x} - \frac{2}{x} \right) \\ = \left[\frac{2+x+\sqrt{4-x^2}}{2+x-(2-x)} + \frac{(2-x)(\sqrt{4-x^2}+2-x)}{4-x^2-(2-x)^2} \right] \left(\frac{\sqrt{4-x^2}-2}{x} \right) \\ = \left[\frac{2+x+\sqrt{4-x^2}}{2x} + \frac{\sqrt{4-x^2}+2-x}{2x} \right] \cdot \frac{\sqrt{4-x^2}-2}{x} \\ = \frac{2\sqrt{4-x^2}+4}{2x} \cdot \frac{\sqrt{4-x^2}-2}{x} = \frac{\sqrt{4-x^2}+2}{x} \cdot \frac{\sqrt{4-x^2}-2}{x} = \frac{4-x^2-4}{x^2} = -1$$

G3 已知 $\tan \alpha$ 和 $\tan \beta$ 是二次方程 $x^2 - 4x - 2 = 0$ 的根。

求 $\sin^2(\alpha + \beta) + 2 \sin(\alpha + \beta)\cos(\alpha + \beta) + 3 \cos^2(\alpha + \beta)$ 的值。

Given that $\tan \alpha$ and $\tan \beta$ are the roots of the quadratic equation $x^2 - 4x - 2 = 0$.

Find the value of $\sin^2(\alpha + \beta) + 2 \sin(\alpha + \beta)\cos(\alpha + \beta) + 3 \cos^2(\alpha + \beta)$.

$$\tan \alpha + \tan \beta = 4, \tan \alpha \tan \beta = -2$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{4}{1 - (-2)} = \frac{4}{3} > 0$$

$\alpha + \beta$ lies in the first or third quadrant.

When $\alpha + \beta$ lies in the first quadrant, $\sin(\alpha + \beta) > 0$, $\cos(\alpha + \beta) > 0$

$$\sin(\alpha + \beta) = \frac{4}{5}, \cos(\alpha + \beta) = \frac{3}{5}$$

$$\text{Expression} = \left(\frac{4}{5} \right)^2 + 2 \left(\frac{4}{5} \right) \left(\frac{3}{5} \right) + 3 \left(\frac{3}{5} \right)^2 = \frac{67}{25}$$

When $\alpha + \beta$ lies in the second quadrant, the answer is the same.

G4 排列 5 個不同的單數及 5 個不同的雙數在同一行使得任意兩個相鄰數的積必為雙數。求所有排列的可能性數目。

Five distinct odd numbers and five distinct even numbers are arranged in a row such that the product of any two consecutive numbers is always even.

Find the number of all possible arrangements.

Method 1 Arrange the five odd numbers in a row, number of permutations = $5! = 120$

The following pattern shows one possible arrangement:

	O_2		O_5		O_1		O_4		O_3	
G_1		G_2		G_3		G_4		G_5		G_6

Mark the gaps amongst these odd numbers as G_1, G_2, G_3, G_4, G_5 and G_6 .

If E_1 is put into G_1 , then only the following pattern is possible:

E	O_2	E	O_5	E	O_1	E	O_4	E	O_3
---	-------	---	-------	---	-------	---	-------	---	-------

Number of permutations for putting "E"s = $5! = 120$

If E_5 is put into G_6 , then only the following pattern is possible:

	O_2	E	O_5	E	O_1	E	O_4	E	O_3	E
--	-------	---	-------	---	-------	---	-------	---	-------	---

Number of permutations for putting "E"s = $5! = 120$

If E does not put in G_1 and G_6 , then, the 5 "E"s must be inserted into G_2, G_3, G_4 and G_5 .

One of G_2, G_3, G_4, G_5 must contain 2 "E"s, number of arrangements = $P_2^5 \times 4! = 480$

Total number of permutations = $120 \times (120 + 120 + 480) = 86400$

Method 2 If these 10 numbers are arranged in any order, number of permutation = $10! = 3628800$

Let **O** be an odd number, **E** be an even number. Then $E \times E = O \times E = E \times O = \text{even}$, $O \times O = \text{odd}$

We find the number of ways such **the product of at least one pair of consecutive number is odd.**

Case 1 Divide these five odd numbers into groups of $(O_1O_2), O_3, O_4, O_5$.

Number of ways of arranging these odd numbers = $P_2^5 \times 4! = 480$

Next, arrange the five even numbers in a row, number of permutations = $5! = 120$

Then insert the 4 groups of odd numbers amongst the 6 gaps of even numbers so that every group of odd numbers is separated by even numbers. Number of combinations = $C_4^6 = 15$

Number of different arrangement in case 1 = $480 \times 120 \times 15 = 864000$

Case 2 Divide these five odd numbers into groups of $(O_1O_2), (O_3O_4), O_5$.

Similarly, the number of different arrangements in case 2 = $5 \times P_2^4 \times P_2^2 \div 2 \times 3! \times 5! \times C_3^6 = 864000$

Case 3 Divide these five odd numbers into groups of $(O_1O_2), (O_3O_4O_5)$.

The number of different arrangements in case 3 = $P_3^5 \times P_2^2 \times 2! \times 5! \times C_2^6 = 432000$

Case 4 Divide these five odd numbers into groups of $O_1, O_2, (O_3O_4O_5)$.

The number of different arrangements in case 4 = $P_3^5 \times 3! \times 5! \times C_3^6 = 864000$

Case 5 Divide these five odd numbers into groups of $(O_1O_2O_3O_4), O_5$.

The number of different arrangements in case 5 = $P_4^5 \times 2! \times 5! \times C_2^6 = 432000$

Case 6 Divide these five odd numbers into one group = $(O_1O_2O_3O_4O_5)$.

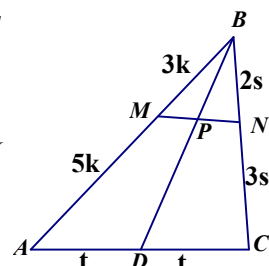
The number of different arrangements in case 6 = $5! \times 5! \times C_1^6 = 86400$

Total number of arrangements = $3628800 - (864000 + 864000 + 432000 + 864000 + 432000 + 86400)$
 $= 3628800 - 3542400 = 86400$

G5 在圖一中， M 和 N 分別是 $\triangle ABC$ 的邊 AB 和 BC 上的點。 MN 與 $\triangle ABC$ 的中線相交於 P 。若 $\frac{AM}{BM} = \frac{5}{3}$ 及 $\frac{CN}{BN} = \frac{3}{2}$ 。求 $\frac{DP}{BP}$ 的值。

In Figure 1, M and N are points on AB and BC of $\triangle ABC$ respectively. MN and the median of $\triangle ABC$ intersect at P .

If $\frac{AM}{BM} = \frac{5}{3}$ and $\frac{CN}{BN} = \frac{3}{2}$, find the value of $\frac{DP}{BP}$.



圖一 Figure 1

Let $AM = 5k$, $BM = 3k$, $CN = 3s$, $BN = 2s$, $AD = CD = t$

Join AP , CP , MD . Let the areas be S . $S_{\triangle BMP} = p$, $S_{\triangle BNP} = q$

$\triangle AMP$ and $\triangle BMP$ have the same height but different bases

$$\frac{S_{\triangle AMP}}{S_{\triangle BMP}} = \frac{AM}{BM} = \frac{5}{3} \Rightarrow S_{\triangle AMP} = \frac{5p}{3}$$

$\triangle CNP$ and $\triangle BNP$ have the same height but different bases

$$\frac{S_{\triangle CNP}}{S_{\triangle BNP}} = \frac{CN}{BN} = \frac{3}{2} \Rightarrow S_{\triangle CNP} = \frac{3q}{2}$$

$\triangle ADP$ and $\triangle CDP$ have the same height and same base

$$\Rightarrow S_{\triangle ADP} = S_{\triangle CDP}$$

$\triangle ABD$ and $\triangle CBD$ have the same height and same base

$$\Rightarrow S_{\triangle ABD} = S_{\triangle CBD}$$

$$S_{\triangle ABD} - S_{\triangle ADP} = S_{\triangle CBD} - S_{\triangle CDP} \Rightarrow S_{\triangle ABP} = S_{\triangle CBP}$$

$$p + \frac{5p}{3} = q + \frac{3q}{2} \Rightarrow \frac{8p}{3} = \frac{5q}{2} \Rightarrow \frac{p}{q} = \frac{15}{16}$$

$\triangle BMP$ and $\triangle BNP$ have the same height but different bases

$$\frac{S_{\triangle BMP}}{S_{\triangle BNP}} = \frac{MP}{NP} \Rightarrow \frac{p}{q} = \frac{MP}{NP} = \frac{15}{16}. \text{ Let } MP = 15x, NP = 16x$$

Draw PE , BF and $NG \parallel MD$, cutting AC at E , F and G respectively.

By the theorem of equal ratios,

$$AD : DF = AM : BM = 5 : 3, FG : GC = BN : CN = 2 : 3, DE : GE = MP : NP = 15 : 16$$

$$DF = \frac{3t}{5} = 0.6t, CF = CD - DF = t - 0.6t = 0.4t$$

$$GC = \frac{3}{5}CF = \frac{3}{5} \times 0.4t = 0.24t, FG = \frac{2}{5}CF = \frac{2}{5} \times 0.4t = 0.16t$$

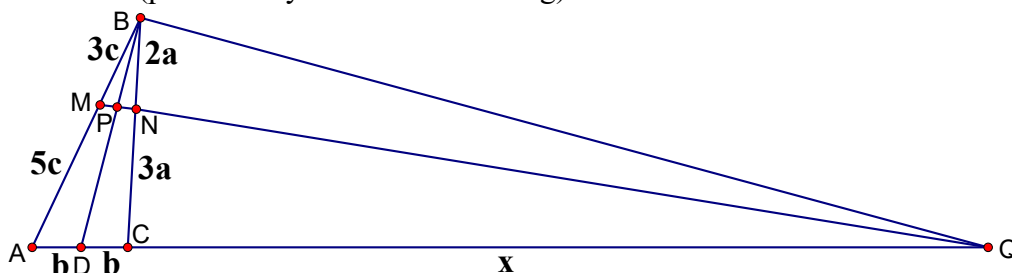
$$DG = DF + FG = 0.6t + 0.16t = 0.76t$$

$$DE = \frac{15}{15+16}DG = \frac{15}{31} \times 0.76t = \frac{11.4}{31}t, GE = \frac{16}{31} \times 0.76t = \frac{12.16}{31}t$$

$$FE = GE - FG = \frac{12.16}{31}t - 0.16t = \frac{7.2}{31}t$$

$$\frac{DP}{BP} = \frac{DE}{FE} = \frac{11.4}{31}t \div \frac{7.2}{31}t = \frac{19}{12}$$

Method 2 (provided by Mr. Tam Chi Leung)



Produce MN and AC to intersect at Q . Let CQ be x .

Apply Menelus theorem, $\triangle ABC$ is cut by MNQ .

$$\frac{AM}{MB} \cdot \frac{BN}{NC} \cdot \frac{CQ}{QA} = 1$$

$$\frac{5}{3} \cdot \frac{2}{3} \cdot \frac{x}{x+2b} = 1$$

$$10x = 9(x + 2b)$$

$$x = 18b$$

Apply Menelus theorem, $\triangle ABD$ is cut by MPQ .

$$\frac{AM}{MB} \cdot \frac{BP}{PD} \cdot \frac{DQ}{QA} = 1$$

$$\frac{5}{3} \cdot \frac{BP}{PD} \cdot \frac{x+b}{x+2b} = 1$$

$$\frac{5}{3} \cdot \frac{BP}{PD} \cdot \frac{18b+b}{18b+2b} = 1$$

$$\frac{DP}{BP} = \frac{19}{12}$$

G6 設 x 、 y 及 z 為實數，且滿足方程
$$\begin{cases} x + yz = 6 & \dots\dots(1) \\ y + zx = 6 & \dots\dots(2) \\ z + xy = 6 & \dots\dots(3) \end{cases}$$
，求 xyz 的最大值。

If x, y and z are real numbers that satisfy the system of equations
$$\begin{cases} x + yz = 6 & \dots\dots(1) \\ y + zx = 6 & \dots\dots(2), \\ z + xy = 6 & \dots\dots(3) \end{cases}$$

find the largest possible value of xyz .

From (1): $x = 6 - yz \dots\dots (*)$

Sub. (*) into (2): $y + z(6 - yz) = 6 \Rightarrow y + 6z - yz^2 = 6 \Rightarrow y(1 - z^2) = 6(1 - z) \dots\dots (4)$

If $z = 1$, then (1) becomes $x + y = 6$ and (3) becomes $xy = 5 \Rightarrow (x, y) = (1, 5)$ or $(5, 1) \Rightarrow xyz = 5$

If $z \neq 1$, then (4) becomes $y(1 + z) = 6 \Rightarrow y = \frac{6}{1+z} \dots\dots (5)$

Sub. (5) into (*) $x = 6 - \frac{6z}{1+z} = \frac{6}{1+z} \dots\dots (6)$

Sub. (5), (6) into (3): $z + \frac{6}{1+z} \cdot \frac{6}{1+z} = 6$

$$z(z^2 + 2z + 1) + 36 = 6(z^2 + 2z + 1)$$

$$z^3 - 4z^2 - 11z + 30 = 0$$

$$(z - 5)(z^2 + z - 6) = 0$$

$$(z - 5)(z + 3)(z - 2) = 0$$

$$z = 5, -3 \text{ or } 2$$

$$\text{When } z = 5, x = y = 1, xyz = 5$$

$$\text{When } z = -3, x = y = -3, xyz = -27$$

$$\text{When } z = 2, x = y = 2, xyz = 8$$

The largest possible value of xyz is 8.

- G7** 整數數列 $\{a_n\}$ 定義為 $a_n = 100 + n^2$, 其中 n 為正整數。設 d_n 為 a_n 和 a_{n+1} 的最大公因數。求 d_n 的最大值。

A sequence of integers $\{a_n\}$ is defined by $a_n = 100 + n^2$, where n is a positive integer.

Let d_n be the greatest common divisor of a_n and a_{n+1} . Find the greatest possible value of d_n .

The solution is provided by Mr. Tam Chi Leung.

$$\begin{cases} n^2 + 100 \equiv 0 \pmod{d_n} & \dots\dots(1) \\ n^2 + 2n + 101 \equiv 0 \pmod{d_n} & \dots\dots(2) \end{cases}$$

$$(2) - (1): 2n + 1 \equiv 0 \pmod{d_n}$$

$$2n \equiv -1 \pmod{d_n}$$

$$4n^2 \equiv 1 \pmod{d_n} \dots\dots (3)$$

$$\text{From (1) } n^2 \equiv -100 \pmod{d_n}$$

$$4n^2 \equiv -400 \pmod{d_n} \dots\dots (4)$$

$$(3) - (4): 0 \equiv 401 \pmod{d_n}$$

As 401 is a prime number, $d_n = 401$ or 1

\therefore The largest possible value of d_n is 401.

- G8** 已知 x 及 y 為正實數且滿足 $x^2 - y^2 = 4 \dots (1)$ 及 $xy = 2 \dots (2)$ 。

若 $x + y$ 可寫成 $a\sqrt{b} + \sqrt{c}$, 其中 a 、 b 及 c 均為正整數, 求 $100a + 10b + c$ 的最小值。

Given that x and y are positive real numbers satisfying $x^2 - y^2 = 4 \dots (1)$ and $xy = 2 \dots (2)$.

If the value of $x + y$ can be expressed in the form of $a\sqrt{b} + \sqrt{c}$, where a , b and c are positive integers, find the least value of $100a + 10b + c$.

$$\text{From (2), } y = \frac{2}{x} \dots (3)$$

$$\text{Sub. (3) into (1): } x^2 - \frac{4}{x^2} = 4$$

$$x^4 - 4x^2 - 4 = 0$$

$$x^2 = 2 + 2\sqrt{2} \text{ or } 2 - 2\sqrt{2} \text{ } (< 0, \text{ rejected})$$

$$x = \sqrt{2 + 2\sqrt{2}} \text{ or } -\sqrt{2 + 2\sqrt{2}} \text{ } (< 0, \text{ rejected})$$

$$= \sqrt{2(1 + \sqrt{2})} = \sqrt{2(\sqrt{2} + 1)}$$

$$y = \frac{2}{\sqrt{2(\sqrt{2} + 1)}}$$

$$= \frac{\sqrt{2}}{\sqrt{\sqrt{2} + 1}} \cdot \frac{\sqrt{\sqrt{2} - 1}}{\sqrt{\sqrt{2} - 1}} = \sqrt{2(\sqrt{2} - 1)}$$

$$x + y = \sqrt{2(\sqrt{2} + 1)} + \sqrt{2(\sqrt{2} - 1)}$$

$$(x + y)^2 = 2(\sqrt{2} + 1) + 2\sqrt{4\sqrt{2} - 1} + 2(\sqrt{2} - 1)$$

$$= 4 + 4\sqrt{2}$$

$$x + y = \sqrt{4 + 4\sqrt{2}} = 2\sqrt{1 + \sqrt{2}}$$

$$(a, b, c) = (1, 4, 32), \text{ or } (2, 1, 2).$$

$$100a + 10b + c = 140 + 32 = 172 \text{ or } 200 + 10 + 2 = 212$$

The least value of $100a + 10b + c$ is 172.

G9 定義 $f(z) = z^2 + 4z$ ，其中 z 是一個複數，設 $z = x + 2i$ ，當中 x 為非零實數。

若 $\frac{f(f(z)) - f(z)}{z - f(z)}$ 是一個純虛數，求 x 的值。

Define $f(z) = z^2 + 4z$, where z is a complex number. let $z = x + 2i$, where x is a non-zero real number. If $\frac{f(f(z)) - f(z)}{z - f(z)}$ is a purely imaginary number, find the value of x .

$$\begin{aligned} f(f(z)) &= (z^2 + 4z)^2 + 4(z^2 + 4z) \\ &= z^4 + 8z^3 + 16z^2 + 4z^2 + 16z \\ &= z^4 + 8z^3 + 20z^2 + 16z \\ f(f(z)) - f(z) &= z^4 + 8z^3 + 20z^2 + 16z - (z^2 + 4z) \\ &= z^4 + 8z^3 + 19z^2 + 12z \\ z - f(z) &= z - (z^2 + 4z) = -(z^2 + 3z) \\ \frac{f(f(z)) - f(z)}{z - f(z)} &= -\frac{z^4 + 8z^3 + 20z^2 + 16z}{z^2 + 3z} \\ &= -\frac{z^3 + 8z^2 + 20z + 16}{z + 3} \\ &= -(z^2 + 5z + 4) \\ &= -[(x + 2i)^2 + 5(x + 2i) + 4] \\ &= -(x^2 + 4xi - 4 + 5x + 10i + 4) \\ &= -[(x^2 + 5x) + (4x + 10)i] \end{aligned}$$

It is a purely imaginary number \Rightarrow Real part = 0

$$x^2 + 5x = 0$$

$$x = 0 \text{ or } -5$$

$\because x$ is non-zero $\therefore x = -5$ only

G10 下列方程有一個實數解： $\begin{cases} 3\log_a(\sqrt{x}\log_a x) = 26 \dots\dots(1) \\ \log_{\log_a x} x = 24 \dots\dots(2) \end{cases}$ ，求 a 的值。

The following system of equations has one real number solution $\begin{cases} 3\log_a(\sqrt{x}\log_a x) = 26 \dots\dots(1) \\ \log_{\log_a x} x = 24 \dots\dots(2) \end{cases}$,

find the value of a .

$$\text{From (1), } \sqrt{x}\log_a x = a^{\frac{26}{3}} \dots (3)$$

$$\text{From (3), } \log_a x = x^{\frac{1}{24}} \dots\dots (4)$$

$$\text{Sub. (4) into (3): } x^{\frac{1}{2}} \cdot x^{\frac{1}{24}} = a^{\frac{26}{3}}$$

$$x^{\frac{13}{24}} = a^{\frac{26}{3}}$$

$$x = a^{16} \dots\dots (5)$$

$$\text{Sub. (5) into (4): } \log_a a^{16} = \left(a^{16}\right)^{\frac{1}{24}}$$

$$16 = a^{\frac{2}{3}}$$

$$a = 64$$