Integration Example - symmetry

Cambridge University Mathematics Entrance Examination 2015 Q6

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(i) Show that $\sec^2\left(\frac{1}{4}\pi - \frac{1}{2}x\right) = \frac{2}{1+\sin x}$.

Hence integrate $\frac{2}{1+\sin x}$ with respect to x.

(ii) By means of substitution $y = \pi - x$, show that $\int_0^\pi x\mathrm{f}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi \mathrm{f}(\sin x)\,\mathrm{d}x$,

where f is any function for which these integrals exist.

Hence evaluate $\int_0^\pi \frac{x}{1+\sin x}\,\mathrm{d}x$.

(iii) Evaluate $\int_0^\pi \frac{2x^3 - 3\pi x^2}{(1+\sin x)^2}\,\mathrm{d}x$.

(i) $\sec^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)=\dfrac{1}{\cos^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)}$

$=\dfrac{1}{\frac{1}{2}\left[1+\cos 2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\right]}$ (double angle formula cos 2θ = 2 cos² θ – 1)

$=\dfrac{2}{1+\cos\left(\frac{1}{2}\pi-x\right)}=\dfrac{2}{1+\sin x}$

$\int\frac{1}{1+\sin x}\,\mathrm{d}x=\frac{1}{2}\int\sec^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\mathrm{d}x$

$=-\int\sec^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\mathrm{d}\left(\frac{1}{4}\pi-\frac{1}{2}x\right)$

$=-\tan\left(\frac{1}{4}\pi-\frac{1}{2}x\right)+C$, where C is a constant

(ii) Let $y=\pi-x$, then $x=\pi-y$; $\mathrm{d}x=-\mathrm{d}y$; when $x=0$, $y=\pi$; when $x=\pi$, $y=0$.

$\int_0^\pi x\mathrm{f}(\sin x)\,\mathrm{d}x=\int_\pi^0(\pi-y)\mathrm{f}(\sin(\pi-y))(-\mathrm{d}y)$

$=\int_0^\pi(\pi-y)\mathrm{f}(\sin y)\,\mathrm{d}y$

$=\int_0^\pi \pi\mathrm{f}(\sin y)\,\mathrm{d}y-\int_0^\pi y\mathrm{f}(\sin y)\,\mathrm{d}y$

$=\int_0^\pi \pi\mathrm{f}(\sin x)\,\mathrm{d}x-\int_0^\pi x\mathrm{f}(\sin x)\,\mathrm{d}x$

$2\int_0^\pi x\mathrm{f}(\sin x)\,\mathrm{d}x=\pi\int_0^\pi \mathrm{f}(\sin x)\,\mathrm{d}x$

$\int_0^\pi x\mathrm{f}(\sin x)\,\mathrm{d}x=\frac{\pi}{2}\int_0^\pi \mathrm{f}(\sin x)\,\mathrm{d}x$

Let $\mathrm{f}(x)=\dfrac{1}{1+\sin x}$, then by the above result,

$\int_0^\pi\frac{x}{1+\sin x}\,\mathrm{d}x=\frac{\pi}{2}\int_0^\pi\frac{1}{1+\sin x}\,\mathrm{d}x$

$=\frac{\pi}{2}\left(-\tan\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\right)\Bigg|_0^\pi$ by the result of (a)

$=-\frac{\pi}{2}\left(\tan\left(-\frac{\pi}{4}\right)-\tan\frac{\pi}{4}\right)=\pi$

(iii) Consider $I=\int_0^{\pi}\frac{x^3}{(1+\sin x)^2}\,\mathrm{d}x$.

Let $y = \pi - x$, then $x = \pi - y$; $\mathrm{d}x = -\mathrm{d}y$; when $x = 0$, $y = \pi$; when $x = \pi$, $y = 0$.

$$I=\int_{\pi}^{0}\frac{(\pi-y)^3}{\left[1+\sin(\pi-y)\right]^2}(-\mathrm{d}y)=\int_0^{\pi}\frac{\pi^3-3\pi^2y+3\pi y^2-y^3}{(1+\sin y)^2}\mathrm{d}y=\int_0^{\pi}\frac{\pi^3-3\pi^2x+3\pi x^2-x^3}{(1+\sin x)^2}\mathrm{d}x$$

$$\int_0^{\pi}\frac{x^3}{(1+\sin x)^2}\mathrm{d}x+\int_0^{\pi}\frac{x^3}{(1+\sin x)^2}\mathrm{d}x-\int_0^{\pi}\frac{3\pi x^2}{(1+\sin x)^2}\mathrm{d}x=\int_0^{\pi}\frac{\pi^3-3\pi^2x}{(1+\sin x)^2}\mathrm{d}x$$

$$\int_0^{\pi}\frac{2x^3-3\pi x^2}{(1+\sin x)^2}\mathrm{d}x=\pi^3\int_0^{\pi}\frac{1}{(1+\sin x)^2}\mathrm{d}x-3\pi^2\int_0^{\pi}\frac{x}{(1+\sin x)^2}\mathrm{d}x$$

$$=\pi^3\int_0^{\pi}\frac{1}{(1+\sin x)^2}\mathrm{d}x-3\pi^2\cdot\frac{\pi}{2}\int_0^{\pi}\frac{1}{(1+\sin x)^2}\mathrm{d}x \quad \text{by the result of (ii)}$$

$$=-\frac{\pi^3}{2}\int_0^{\pi}\frac{1}{(1+\sin x)^2}\mathrm{d}x$$

$$=-\frac{\pi^3}{8}\int_0^{\pi}\sec^4\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\mathrm{d}x \quad \text{by the result of (i)}$$

$$=-\frac{\pi^3}{8}\int_0^{\pi}\sec^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\left(1+\tan^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\right)\mathrm{d}x$$

$$=\frac{\pi^3}{4}\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\sec^2 u\left(1+\tan^2 u\right)\mathrm{d}u,\ u=\frac{\pi}{4}-\frac{x}{2};\ x=0,\ u=\frac{\pi}{4};\ x=\pi,\ u=-\frac{\pi}{4};\ \mathrm{d}x=-2\mathrm{d}u$$

$$=-\frac{\pi^3}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(1+\tan^2 u\right)\mathrm{d}(\tan u)=-\frac{\pi^3}{4}\left(\tan u+\frac{1}{3}\tan^3 u\right)_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$=-\frac{\pi^3}{4}\left[\left(1+\frac{1}{3}\right)-\left(-1-\frac{1}{3}\right)\right]$$

$$=-\frac{2\pi^3}{3}$$