Examples on Mathematical Induction: divisibility 16 & 36

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1. (a) Prove that $7^n - 6n - 1$ is divisible by 36 for all non-negative integers n.

   (b) Prove that $5^n - 4n - 1$ is divisible by 16 for all non-negative integers n.

   (c) If $2\cdot7^n - 3\cdot5^n + 1$ is divisible by p, find the greatest value of p.

   (a) $n = 0$, $7^n - 6n - 1 = 1 - 0 - 1 = 0$, which is divisible by 36.

   Suppose $7^k - 6k - 1 = 36r$, where r is an integer.

   $7^k = 6k + 1 + 36r \cdots\cdots (1)$

   $$\begin{aligned}7^{k+1} - 6(k+1) - 1 &= 7\cdot7^k - 6k - 7\\ &= 7\cdot(6k + 1 + 36r) - 6k - 7 \text{ by (1)}\\ &= 42k + 7 + 7\cdot36r - 6k - 7\\ &= 36k + 7\cdot36r\\ &= 36(k + 7r)\end{aligned}$$

   $\because$ $k + 7r$ is an integer

   $\therefore$ $7^{k+1} - 6(k+1) - 1$ is divisible by 36.

   If it is true for $n = k$, then it is also true for $n = k + 1$

   By the principle of mathematical induction, it is true for all non-negative integers n.

   (b) $n = 0$, $5^n - 4n - 1 = 1 - 0 - 1 = 0$, which is divisible by 16.

   Suppose $5^k - 4k - 1 = 16s$, where s is an integer.

   $5^k = 4k + 1 + 16s \cdots\cdots (2)$

   $$\begin{aligned}5^{k+1} - 4(k+1) - 1 &= 5\cdot5^k - 4k - 5\\ &= 5\cdot(4k + 1 + 16s) - 4k - 5 \text{ by (2)}\\ &= 20k + 5 + 5\cdot16s - 4k - 5\\ &= 16k + 5\cdot16s\\ &= 16(k + 5s)\end{aligned}$$

   $\because$ $k + 5s$ is an integer

   $\therefore$ $5^{k+1} - 4(k+1) - 1$ is divisible by 16.

   If it is true for $n = k$, then it is also true for $n = k + 1$

   By the principle of mathematical induction, it is true for all non-negative integers n.

   (c) By (a), $7^n - 6n - 1 = 36r \cdots\cdots (1)$, where r is an integer.

   $5^n - 4n - 1 = 16s \cdots\cdots (2)$, where s is an integer.

   $2(1) - 3(2)$: $2\cdot7^n - 3\cdot5^n + 1 = 72r - 48s = 24(3r - 2s)$

   The greatest value of $p = 24$.