Individual Events

SI			I1			I2			I3			I4		
	A	15		***R***	30		***a***	16		***m***	3		***m***	3
	B	3		***S***	120		***b***	$\frac{3}{2}$		***n***	9		***n***	$\frac{9}{4}$
	C	4		***T***	11		***c***	36		***p***	2		***p***	9
	D	8		***U***	72		***d***	42		***q***	1141		***q***	8

Group Events

SG			G1			G2			G3			G4		
	z	540		***q***	3		***A***	$-\frac{17}{13}$		***A***	5		***P***	$\frac{3}{8}$
	R	6		***k***	1		***B***	13		***R***	4018		***R***	$\frac{1}{2}$
	k	5		***w***	25		***C***	46		***Q***	$\frac{4\sqrt{5}}{5}$		***S***	320
	xyz	1		***p***	$\frac{3}{2}$		***D***	30		***T***	$5-2\sqrt{3}$		***Q***	–1

Sample Individual Event (2008 Final Individual Event 1)

SI.1 Let $A = 15\times\tan 44°\times\tan45°\times\tan46°$, find the value of A.

Similar question 2012 G2.1

$$A = 15\times\tan 44^\circ\times 1\times\frac{1}{\tan 44^\circ} = 15$$

SI.2 Let n be a positive integer and $\overbrace{20082008\cdots2008}^{n\text{ 2008's}}15$ is divisible by A. If the least possible value of n is B, find the value of B.

The given number is divisible by 15. Therefore it is divisible by 3 and 5.

The last 2 digits of the given number is 15, which is divisible by 15.

The necessary condition is: $\overbrace{20082008\cdots2008}^{n\text{ 2008's}}$ must be divisible by 3.

2 + 0 + 0 + 8 = 10 which is not divisible by 3.

The least possible n is 3: 2+0+0+8+2+0+0+8+2+0+0+8 = 30 which is divisible by 3.

SI.3 Given that there are C integers that satisfy the equation $|x – 2| + |x + 1| = B$, find the value of C

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2012 FG2.3

$|x – 2| + |x + 1| = 3$

If $x < –1$, $2 – x – x – 1 = 3 \Rightarrow x = –1$ (rejected)

If $–1 \le x \le 2$, $2 – x + x + 1 = 3 \Rightarrow 3 = 3$, always true $\quad –1 \le x \le 2$

If $2 < x$, $x – 2 + x + 1 = 3 \Rightarrow x = 2$ (reject)

$–1 \le x \le 2$ only

$\because$ x is an integer, $\quad x = –1, 0, 1, 2$; $C = 4$

SI.4 In the coordinate plane, the distance from the point $(–C, 0)$ to the straight line $y = x$ is $\sqrt{D}$, find the value of D.

The distance from $P(x_0, y_0)$ to the straight line $Ax + By + C = 0$ is $\left|\frac{Ax_0+By_0+C}{\sqrt{A^2+B^2}}\right|$.

The distance from $(–4, 0)$ to $x – y = 0$ is $\sqrt{D} = \left|\frac{-4-0+0}{\sqrt{1^2+(-1)^2}}\right| = \frac{4}{\sqrt{2}} = 2\sqrt{2} = \sqrt{8}$; $D = 8$

Individual Event 1

I1.1 Let a, b, c and d be the distinct roots of the equation $x^4 - 15x^2 + 56 = 0$.

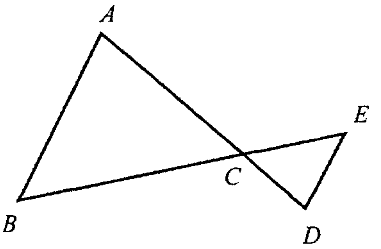
If $R = a^2 + b^2 + c^2 + d^2$, find the value of R.

$x^4 - 15x^2 + 56 = 0 \Rightarrow (x^2 - 7)(x^2 - 8) = 0$

$a = \sqrt{7}$, $b = -\sqrt{7}$, $c = \sqrt{8}$, $d = -\sqrt{8}$

$R = a^2 + b^2 + c^2 + d^2 = 7 + 7 + 8 + 8 = 30$

I1.2 In Figure 1, AD and BE are straight lines with $AB = AC$ and AB // ED. If $\angle ABC = R°$ and $\angle ADE = S°$, find the value of S.



$\angle ABC = 30° = \angle ACB$ (base $\angle$ isos. Δ)

$\angle BAC = 120°$ ($\angle$s sum of Δ)

$\angle ADE = 120°$ (alt. $\angle$s AB // ED)

$S = 120$

I1.3 Let $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^S$ and $T = \sqrt{\dfrac{\log(1+F)}{\log 2}}$, find the value of T.

Reference: 2015 FI1.4, 2017 FI3.4

$F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{120}$

$= \dfrac{2^{121}-1}{2-1} = 2^{121} - 1$

$T = \sqrt{\dfrac{\log(1+F)}{\log 2}}$

$= \sqrt{\dfrac{\log 2^{121}}{\log 2}} = 11$

I1.4 Let f(x) be a function such that f(n) = (n – 1) f(n – 1) and f(n) ≠ 0 hold for all integers $n \geq 6$.

If $U = \dfrac{\mathrm{f}(T)}{(T-1)\,\mathrm{f}(T-3)}$, find the value of U.

Reference: 1999 FI5.4

$\mathrm{f}(n) = (n-1)\,\mathrm{f}(n-1) = (n-1)(n-2)\mathrm{f}(n-2) = \ldots\ldots$

$U = \dfrac{\mathrm{f}(11)}{(11-1)\,\mathrm{f}(11-3)}$

$= \dfrac{10\times9\times8\times f(8)}{10\times f(8)}$

$= 8\times9 = 72$

Individual Event 2

I2.1 Let $[x]$ be the largest integer not greater than x. If $a=\left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right]+16$, find the value of a.

$$0<\sqrt{3}-\sqrt{2}=\frac{1}{\sqrt{3}+\sqrt{2}}<1$$

$$0<\left(\sqrt{3}-\sqrt{2}\right)^{2009}<1$$

$$\Rightarrow a=\left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right]+16$$

$= 0 + 16 = 16$

I2.2 In the coordinate plane, if the area of the triangle formed by the x-axis, y-axis and the line $3x + ay = 12$ is b square units, find the value of b.

$3x + 16y = 12$

x-intercept = 4

y-intercept $=\frac{3}{4}$

Area $= b=\frac{1}{2}\cdot 4\cdot\frac{3}{4}=\frac{3}{2}$

I2.3 Given that $x-\frac{1}{x}=2b$ and $x^3-\frac{1}{x^3}=c$, find the value of c.

Reference: 1990 FI2.2

$$x-\frac{1}{x}=3$$

$$\Rightarrow x^2-2+\frac{1}{x^2}=9$$

$$\Rightarrow x^2+\frac{1}{x^2}=11$$

$$c=x^3-\frac{1}{x^3}$$

$$=\left(x-\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right)$$

$= 3\times(11 + 1) = 36$

I2.4 In Figure 1, $\alpha = c$, $\beta = 43$, $\gamma = 59$ and $\omega = d$, find the value of d.

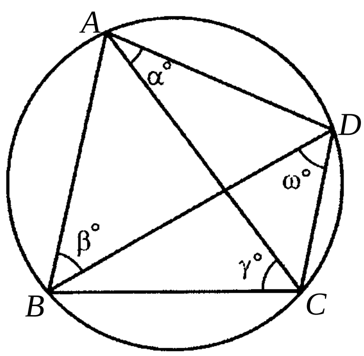
$\angle BAC = \omega°$ ($\angle$s in the same seg.)

$\angle ACD = \beta°$ ($\angle$s in the same seg.)

$\angle BAD + \angle BCD = 180°$ (opp. $\angle$s cyclic quad.)

$c + d + 43 + 59 = 180$

$d = 180 - 43 - 59 - 36 = 42$ ($\because$ $c = 36$)



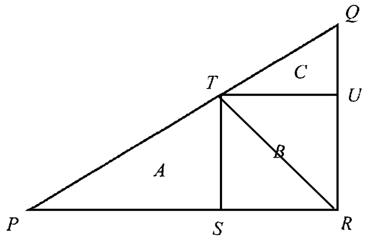
Individual Event 3

I3.1 Given that $\frac{4}{\sqrt{6}+\sqrt{2}}-\frac{1}{\sqrt{3}+\sqrt{2}}=\sqrt{a}-\sqrt{b}$. If $m = a - b$, find the value of m.

$$\frac{4}{\sqrt{6}+\sqrt{2}}-\frac{1}{\sqrt{3}+\sqrt{2}}=\frac{4}{\sqrt{6}+\sqrt{2}}\cdot\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}-\frac{1}{\sqrt{3}+\sqrt{2}}\cdot\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$
$$=\frac{4\left(\sqrt{6}-\sqrt{2}\right)}{6-2}-\frac{\sqrt{3}-\sqrt{2}}{3-2}$$
$$=\sqrt{6}-\sqrt{2}-\left(\sqrt{3}-\sqrt{2}\right)$$
$$=\sqrt{6}-\sqrt{3}$$

$a = 6$, $b = 3$; $m = 6 - 3 = 3$

I3.2 In figure 1, PQR is a right-angled triangle and $RSTU$ is a rectangle. Let A, B and C be the areas of the corresponding regions. If $A : B = m : 2$ and $A : C = n : 1$, find the value of n.



$A : B = 3 : 2$, $A : C = n : 1 \Rightarrow A : B : C = 3n : 2n : 3$

Let $TS = UR = x$, $QU = y$

$\Delta PTS \sim \Delta TQU \sim \Delta PQR$ (equiangular)

$S_{\Delta PTS} : S_{\Delta TQU} : S_{\Delta PQR} = A : C : (A + B + C) = 3n : 3 : (5n + 3)$

$x^2 : y^2 : (x + y)^2 = 3n : 3 : (5n + 3)$

$\frac{y}{x}=\frac{1}{\sqrt{n}}$ … (1), $\frac{x+y}{y}=\frac{\sqrt{5n+3}}{\sqrt{3}}$ … (2)

From (2): $\frac{x}{y}+1=\frac{\sqrt{5n+3}}{\sqrt{3}}$ … (3)

Sub. (1) into (3): $\sqrt{n}+1=\frac{\sqrt{5n+3}}{\sqrt{3}}$

$$\sqrt{3n}+\sqrt{3}=\sqrt{5n+3}$$
$$\left(\sqrt{3n}+\sqrt{3}\right)^2=5n+3$$
$$3n+6\sqrt{n}+3=5n+3$$
$$6\sqrt{n}=2n\Rightarrow n=9$$

Method 2

Let $SR = x$, $PS = z$

Join TR which bisects the area of the rectangle.

$\frac{A}{B}=\frac{3}{2}\Rightarrow\frac{A}{\frac{B}{2}}=\frac{3}{1}$

$\frac{S_{\Delta TPS}}{S_{\Delta TSR}}=\frac{3}{1}\Rightarrow\frac{z}{x}=\frac{3}{1}$ … (4)

$\because$ $\Delta PTS \sim \Delta TQU$ (equiangular)

$\therefore$ $\frac{A}{C}=\frac{n}{1}\Rightarrow\left(\frac{z}{x}\right)^2=\frac{n}{1}$

$n = 3^2 = 9$ (by (4))

I3.3 Let x_1, x_2, x_3, x_4 be real numbers and $x_1 \neq x_2$. If $(x_1 + x_3)(x_1 + x_4) = (x_2 + x_3)(x_2 + x_4) = n - 10$ and $p = (x_1 + x_3)(x_2 + x_3) + (x_1 + x_4)(x_2 + x_4)$, find the value of p.

Reference: 2002 HI7, 2006HG6, 2014 HG7

$$x_1^2+x_1x_3+x_1x_4+x_3x_4=x_2^2+x_2x_3+x_2x_4+x_3x_4=-1$$
$$x_1^2-x_2^2+x_1x_3-x_2x_3+x_1x_4-x_2x_4=0$$

$(x_1 - x_2)(x_1 + x_2 + x_3 + x_4) = 0 \Rightarrow x_1 + x_2 + x_3 + x_4 = 0$ … (1)

$p = (x_1 + x_3)(x_2 + x_3) + (x_1 + x_4)(x_2 + x_4)$

$= -(x_2 + x_3)(x_2 + x_4) - (x_1 + x_3)(x_1 + x_4)$ (by (1), $x_1 + x_3 = -(x_2 + x_4)$, $x_2 + x_4 = -(x_1 + x_3)$)

$= 1 + 1 = 2$ (given $(x_1 + x_3)(x_1 + x_4) = (x_2 + x_3)(x_2 + x_4) = -1$)

I3.4 The total number of students in a school is a multiple of 7 and not less than 1000. Given that the same remainder 1 will be obtained when the number of students is divided by $p + 1$, $p + 2$ and $p + 3$. Let q be the least of the possible numbers of students in the school, find the value of q.

$p + 1 = 3$, $p + 2 = 4$, $p + 3 = 5$; HCF = 1, LCM = 60

$q = 60m + 1$, where m is an integer.

$\because$ $q \geq 1000$ and $q = 7k$, k is an integer.

$60m + 1 = 7k$

$7k - 60m = 1$

$k = 43$, $m = 5$ satisfies the equation

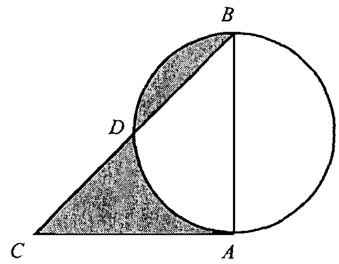
$k = 43 + 60t$; $7k \geq 1000 \Rightarrow 7(43 + 60t) \geq 1000 \Rightarrow t \geq 2 \Rightarrow$ Least $q = 7\times(43 + 60\times2) = 1141$

Individual Event 4

I4.1 Given that $x_0^2 + x_0 - 1 = 0$. If $m = x_0^3 + 2x_0^2 + 2$, find the value of m.

$m = x_0^3 + 2x_0^2 + 2 = x_0^3 + x_0^2 - x_0 + x_0^2 + x_0 - 1 + 3 = 3$

I4.2 In Figure 1, ΔBAC is a right-angled triangle, AB = AC = m cm. Suppose that the circle with diameter AB intersects the line BC at D, and the total area of the shaded region is n cm^2. Find the value of n.



$AB = AC = 3$ cm; $\angle ADB = 90°$ ($\angle$ in semi-circle)

Shaded area = area of $\Delta ACD = \frac{1}{2}$ area of $\Delta ABC = \frac{1}{2} \cdot \frac{1}{2} \cdot 3 \cdot 3\,\text{cm}^2$

$n = \frac{9}{4}$

I4.3 Given that $p = 4n\left(\frac{1}{2^{2009}}\right)^{\log(1)}$, find the value of p.

$p = 4n\left(\frac{1}{2^{2009}}\right)^{0} = 4 \cdot \frac{9}{4} = 9$

I4.4 Let x and y be real numbers satisfying the equation $\left(x - \sqrt{p}\right)^2 + \left(y - \sqrt{p}\right)^2 = 1$.

If $k = \frac{y}{x-3}$ and q is the least possible values of k^2, find the value of q.

$(x - 3)^2 + (y - 3)^2 = 1$ … (1)

Sub. $y = k(x - 3)$ into (1): $(x - 3)^2 + [k(x - 3) - 3]^2 = 1$

$(x - 3)^2 + k^2(x - 3)^2 - 6k(x - 3) + 9 = 1$

$(1 + k^2)(x - 3)^2 - 6k(x - 3) + 8 = 0 \Rightarrow (1 + k^2)t^2 - 6kt + 8 = 0$; where $t = x - 3$

For real values of t: $\Delta = 4[3^2k^2 - 8(1 + k^2)] \geq 0$

$k^2 \geq 8$

The least possible value of $k^2 = q = 8$.

Method 2

The circle $(x - 3)^2 + (y - 3)^2 = 1$ intersects with the variable line $y = k(x - 3)$ which passes through a fixed point (3, 0), where k is the slope of the line.

From the graph, the extreme points when the variable line touches the circle at B and C.

H(3, 3) is the centre of the circle.

In ΔABH, $\angle ABH = 90°$ ($\angle$ in semi-circle)

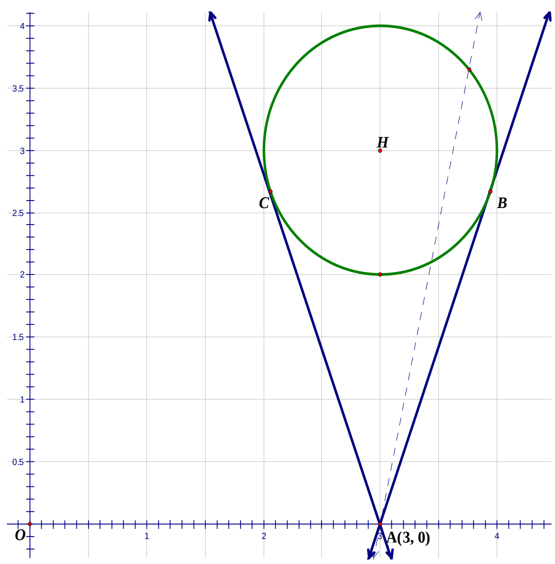
Let $\angle HAB = \theta$, $AH = 3$, $HB = 1$,

$AB = \sqrt{8} = 2\sqrt{2}$ (Pythagoras' theorem)

$\tan\theta = \frac{1}{\sqrt{8}}$

Slope of $AB = \tan(90° - \theta) = \sqrt{8}$

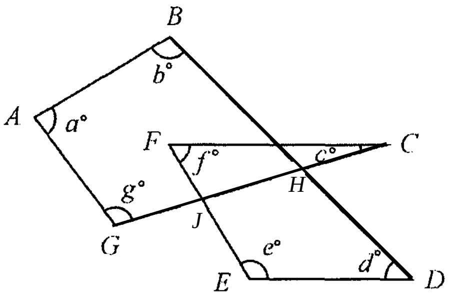
Least possible $k^2 = q = 8$



Sample Group Event (2008 Final Group Event 2)

SG.1 In Figure 1, BD, FC, GC and FE are straight lines.

If $z = a + b + c + d + e + f + g$, find the value of z.



$a° + b° + g° + \angle BHG = 360°$ ($\angle$s sum of polygon $ABHG$)

$c° + f° = \angle CJE$ (ext. $\angle$ of ΔCFJ)

$c° + f° + e° + d° + \angle JHD = 360°$ ($\angle$s sum of polygon $JHDE$)

$a° + b° + g° + \angle BHG + c° + f° + e° + d° + \angle JHD = 720°$

$a° + b° + c° + d° + e° + f° + g° + 180° = 720°$

$z = 540$

SG.2 If R is the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7, find the value of R.

$x^6 + y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5) + 2y^6$

$6^6 + 1^6 = 7Q_1 + 2$; $5^6 + 2^6 = 7Q_2 + 2\times 2^6$; $4^6 + 3^6 = 7Q_3 + 2\times 3^6$

$2 + 2\times 2^6 + 2\times 3^6 = 2(1 + 64 + 729) = 1588 = 7\times 226 + 6$; $R = 6$

Method 2 $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6 \equiv 1^6 + 2^6 + 3^6 + (-3)^6 + (-2)^6 + (-1)^6 \bmod 7$

$\equiv 2(1^6 + 2^6 + 3^6) \equiv 2(1 + 64 + 729) \bmod 7$

$\equiv 2(1 + 1 + 1) \bmod 7 \equiv 6 \bmod 7$

SG.3 If 14! is divisible by 6^k, where k is an integer, find the largest possible value of k.

We count the number of factors of 3 in 14!. They are 3, 6, 9, 12. So there are 5 factors of 3.

$k = 5$

SG.4 Let x, y and z be real numbers that satisfy $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = \frac{7}{3}$.

Find the value of xyz. (**Reference 2010 FG2.2**)

Method 1

From (1), $x = 4 - \frac{1}{y} = \frac{4y-1}{y}$

$\Rightarrow \frac{1}{x} = \frac{y}{4y-1}$ …… (4)

Sub. (4) into (3): $z + \frac{y}{4y-1} = \frac{7}{3}$

$z = \frac{7}{3} - \frac{y}{4y-1}$ …… (5)

From (2): $\frac{1}{z} = 1 - y$

$z = \frac{1}{1-y}$ …… (6)

(5) = (6): $\frac{1}{1-y} = \frac{7}{3} - \frac{y}{4y-1}$

$\frac{1}{1-y} = \frac{28y-7-3y}{3(4y-1)}$

$3(4y - 1) = (1 - y)(25y - 7)$

$12y - 3 = -25y^2 - 7 + 32y$

$25y^2 - 20y + 4 = 0$

$(5y - 2)^2 = 0$

$y = \frac{2}{5}$

Sub. $y = \frac{2}{5}$ into (6): $z = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

Sub. $y = \frac{2}{5}$ into (1): $x + \frac{5}{2} = 4 \Rightarrow x = \frac{3}{2}$

$xyz = \frac{2}{5}\times\frac{5}{3}\times\frac{3}{2} = 1$

Method 2

$$\begin{cases} x + \frac{1}{y} = 4 \cdots\cdots (1) \\ y + \frac{1}{z} = 1 \cdots\cdots (2) \\ z + \frac{1}{x} = \frac{7}{3} \cdots\cdots (3) \end{cases}$$

(1)×(2): $xy + 1 + \frac{x}{z} + \frac{1}{yz} = 4$

$x\left(y + \frac{1}{z}\right) + \frac{1}{yz} = 3$

Sub. (2) into the eqt.: $x + \frac{x}{xyz} = 3$

Let $a = xyz$, then $x + \frac{x}{a} = 3 \cdots\cdots (4)$

(2)×(3): $y\left(\frac{7}{3}\right) + \frac{y}{a} = \frac{4}{3} \Rightarrow y\left(\frac{7}{3} + \frac{1}{a}\right) = \frac{4}{3} \cdots\cdots (5)$

(1)×(3): $z(4) + \frac{z}{a} = \frac{25}{3} \Rightarrow z\left(4 + \frac{1}{a}\right) = \frac{25}{3} \cdots\cdots (6)$

(4)×(5)×(6): $a\left(1 + \frac{1}{a}\right)\left(\frac{7}{3} + \frac{1}{a}\right)\left(4 + \frac{1}{a}\right) = \frac{100}{3}$

$\frac{(a+1)(7a+3)(4a+1)}{3a^2} = \frac{100}{3}$

which reduces to $28a^3 - 53a^2 + 22a + 3 = 0$

$\Rightarrow (a - 1)^2(28a + 3) = 0$

$\therefore a = 1$

Method 3 (1)×(2)×(3) – (1) – (2) – (3):

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} - \left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{28}{3} - \frac{22}{3} \Rightarrow xyz + \frac{1}{xyz} = 2$$

$xyz = 1$

Group Event 1

G1.1 Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

When $a = 1$, possible $b = 2$

When $a = 2$, possible $b = 2$ or 3

$\therefore q = 3$

G1.2 Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

When $x > 0$: $x^2 - 4 = 3x \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x + 1)(x - 4) = 0 \Rightarrow x = 4$

When $x < 0$: $-x^2 - 4 = -3x \Rightarrow x^2 - 3x + 4 = 0$; $D = 9 - 16 < 0 \Rightarrow$ no real roots.

$k = 1$ (There is only one real root.)

G1.3 Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x - y = 7$. If $w = x + y$, find the value of w.

The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$

Sub. $y = \frac{144}{x}$ into $x - y = 7$: $x - \frac{144}{x} = 7 \Rightarrow x^2 - 7x - 144 = 0 \Rightarrow (x + 9)(x - 16) = 0$

$x = -9$ or 16; when $x = -9$, $y = -16$ (rejected $\because \sqrt{x}$ is undefined); when $x = 16$; $y = 9$

$w = 16 + 9 = 25$

Method 2 The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$ …… (1)

$\because x - y = 7$ and $x + y = w$

$\therefore x = \frac{w+7}{2}, y = \frac{w-7}{2}$

Sub. these equations into (1): $\left(\frac{w+7}{2}\right)\left(\frac{w-7}{2}\right) = 144$

$w^2 - 49 = 576 \Rightarrow w = \pm 25$

$\because$ From the given equation $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$, we know that both $x > 0$ and $y > 0$

$\therefore w = x + y = 25$ only

G1.4 Given that x and y are real numbers and $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$.

Let $p = |x| + |y|$, find the value of p.

Reference: 2005 FI4.1, 2006 FI4.2, 2011 FI4.3, 2013 FI1.4, 2015 HG4, 2015 FI1.1

Both $\left|x - \frac{1}{2}\right|$ and $\sqrt{y^2 - 1}$ are non-negative numbers.

The sum of two non-negative numbers = 0 means each of them is zero

$x = \frac{1}{2}$, $y = \pm 1$; $p = \frac{1}{2} + 1 = \frac{3}{2}$

Group Event 2

G2.1 Given $\tan\theta = \frac{5}{12}$, where $180° \le \theta \le 270°$. If $A = \cos\theta + \sin\theta$, find the value of A.

$\cos\theta = -\frac{12}{13}$, $\sin\theta = -\frac{5}{13}$

$A = -\frac{12}{13} - \frac{5}{13} = -\frac{17}{13}$

G2.2 Let $[x]$ be the largest integer not greater than x.

If $B = \left[10 + \sqrt{10 + \sqrt{10 + \sqrt{10 + \cdots}}}\right]$, find the value of B.

Reference: 2007 FG2.2 … $x \ge 3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3}}}}}$ …

Let $y = \sqrt{10 + \sqrt{10 + \sqrt{10 + \cdots}}}$

$y^2 = 10 + \sqrt{10 + \sqrt{10 + \sqrt{10 + \cdots}}} = 10 + y$

$y^2 - y - 10 = 0$

$y = \frac{1+\sqrt{41}}{2}$ or $\frac{1-\sqrt{41}}{2}$ (rejected)

$6 < \sqrt{41} < 7 \Rightarrow \frac{7}{2} < \frac{1+\sqrt{41}}{2} < 4$

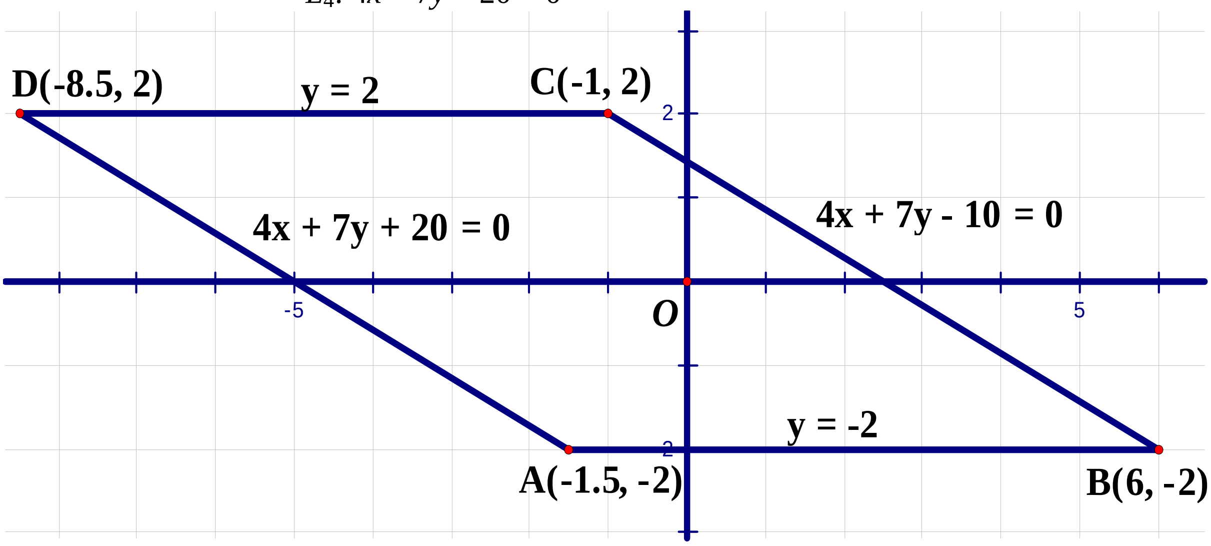
$13.5 < 10 + \sqrt{10 + \sqrt{10 + \sqrt{10 + \cdots}}} < 14$; $B = 13$

G2.3 Let $a \oplus b = ab + 10$. If $C = (1\oplus 2) \oplus 3$, find the value of C.

$1\oplus 2 = 2 + 10 = 12$; $C = 12\oplus 3 = 36 + 10 = 46$

G2.4 In the coordinate plane, the area of the region bounded by the following lines is D square units, find the value of D.

L_1: $y - 2 = 0$
L_2: $y + 2 = 0$
L_3: $4x + 7y - 10 = 0$
L_4: $4x + 7y + 20 = 0$



It is easy to show that the bounded region is a parallelogram $ABCD$ with vertices $A(-1.5, -2)$, $B(6, -2)$, $C(-1, 2)$, $C(-8.5, 2)$.

The area $D = |6 - (-1.5)|\times|2 - (-2)| = 7.5\times 4 = 30$

Group Event 3

G3.1 Let $[x]$ be the largest integer not greater than x.

If $A=\left[\dfrac{2008\times 80+2009\times 130+2010\times 180}{2008\times 15+2009\times 25+2010\times 35}\right]$, find the value of A.

Reference: 2008 FGS.4 Calculate the value of $\dfrac{2008^3+4015^3}{2007^3+4015^3}$.

Let a = 2009, b = 130, c = 25

$$\frac{2008\times 80+2009\times 130+2010\times 180}{2008\times 15+2009\times 25+2010\times 35}=\frac{(a-1)(b-50)+ab+(a+1)(b+50)}{(a-1)(c-10)+ac+(a+1)(c+10)}$$
$$=\frac{ab-b-50a+50+ab+ab+b+50a+50}{ac-c-10a+10+ac+ac+c+10a+10}$$
$$=\frac{3ab+100}{3ac+20}=\frac{3\cdot 2009\cdot 130+100}{3\cdot 2009\cdot 25+20}=\frac{783610}{150695}=5+d$$

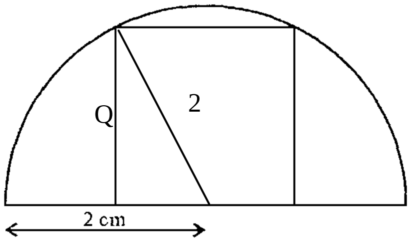
where $0 < d < 1$; $A = 5$

G3.2 There are R zeros at the end of $\underbrace{99\ldots9}_{2009\text{ of 9's}}\times\underbrace{99\ldots9}_{2009\text{ of 9's}}+1\underbrace{99\ldots9}_{2009\text{ of 9's}}$, find the value of R.

$$\underbrace{99\ldots9}_{2009\text{ of 9's}}\times\underbrace{99\ldots9}_{2009\text{ of 9's}}+1\underbrace{99\ldots9}_{2009\text{ of 9's}}=\left(1\underbrace{0\ldots0}_{2009\text{ of 0's}}-1\right)\times\left(1\underbrace{0\ldots0}_{2009\text{ of 0's}}-1\right)+\left(2\underbrace{0\ldots0}_{2009\text{ of 0's}}-1\right)$$
$$=(10^{2009}-1)(10^{2009}-1)+2\times 10^{2009}-1$$
$$=10^{4018}-2\times 10^{2009}+1+2\times 10^{2009}-1=10^{4018}$$

$R = 4018$

G3.3 In Figure 1, a square of side length Q cm is inscribed in a semi-circle of radius 2 cm. Find the value of Q.

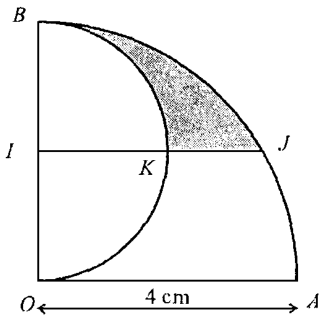


$Q^2+\left(\dfrac{Q}{2}\right)^2=4$ (Pythagoras' Theorem)

$5Q^2 = 16$

$Q=\dfrac{4}{\sqrt{5}}=\dfrac{4\sqrt{5}}{5}$

G3.4 In Figure 2, the sector OAB has radius 4 cm and $\angle AOB$ is a right angle. Let the semi-circle with diameter OB be centred at I with IJ // OA, and IJ intersects the semi-circle at K. If the area of the shaded region is T cm^2, find the value of T. (Take $\pi = 3$)



OI = 2 cm, OJ = 4 cm

$\cos\angle IOJ=\dfrac{OI}{OJ}=\dfrac{1}{2}$

$\angle IOJ = 60°$

$S_{BIJ}=S_{\text{sector } OBJ}-S_{\Delta OIJ}$

$=(\frac{1}{2}\cdot 4^2\cdot\frac{\pi}{3}-\frac{1}{2}\cdot 2\cdot 4\sin 60^\circ)\text{cm}^2$

$=(\frac{8\pi}{3}-2\sqrt{3})$ cm^2

Shaded area = $S_{BIJ}-S_{BIK}$

$=\left(\dfrac{8\pi}{3}-2\sqrt{3}-\dfrac{1}{4}\pi\cdot 2^2\right)\text{cm}^2$

$=\left(\dfrac{5\pi}{3}-2\sqrt{3}\right)\text{cm}^2$

$T=5-2\sqrt{3}$

Group Event 4

G4.1 Let P be a real number. If $\sqrt{3-2P}+\sqrt{1-2P}=2$, find the value of P.

$$\left(\sqrt{3-2P}\right)^2=\left(2-\sqrt{1-2P}\right)^2$$

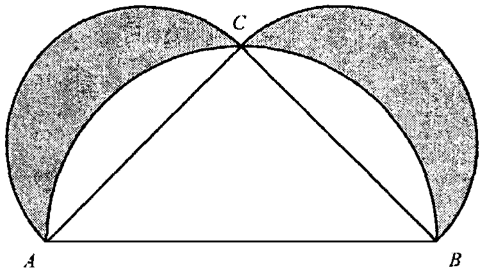
$$3-2P=4-4\sqrt{1-2P}+1-2P$$

$$4\sqrt{1-2P}=2$$

$$4(1-2P)=1$$

$$P=\frac{3}{8}$$

G4.2 In Figure 1, let AB, AC and BC be the diameters of the corresponding three semi-circles. If $AC = BC = 1$ cm and the area of the shaded region is R cm^2. Find the value of R.



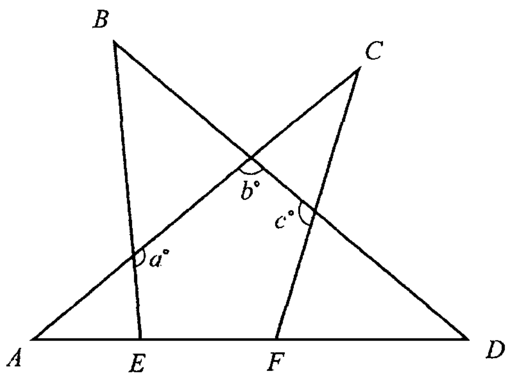
Reference: 1994 HI9

$AB=\sqrt{2}$

Shaded area = R cm^2 = $S_{\text{circle with diameter } AC} - 2\, S_{\text{segment } AC}$

$$R=\pi\left(\frac{1}{2}\right)^2-\left[\frac{1}{2}\pi\cdot\left(\frac{\sqrt{2}}{2}\right)^2-\frac{1}{2}\cdot 1^2\right]=\frac{1}{2}$$

G4.3 In Figure 2, AC, AD, BD, BE and CF are straight lines. If $\angle A+\angle B+\angle C+\angle D=140^\circ$ and $a+b+c=S$, find the value of S.



$\angle CFD=\angle A+\angle C$ (ext. $\angle$ of Δ)

$\angle AEB=\angle B+\angle D$ (ext. $\angle$ of Δ)

$a^\circ=\angle A+\angle AEB=\angle A+\angle B+\angle D$ (ext. $\angle$ of Δ)

$c^\circ=\angle D+\angle CFD=\angle D+\angle A+\angle C$ (ext. $\angle$ of Δ)

$b^\circ+\angle A+\angle D=180^\circ$ ($\angle$s sum of Δ)

$$\begin{aligned}a^\circ+b^\circ+c^\circ&=\angle A+\angle B+\angle D+180^\circ-(\angle A+\angle D)\\&\quad+\angle D+\angle A+\angle C\\&=\angle A+\angle B+\angle C+\angle D+180^\circ\end{aligned}$$

$S=a+b+c=140+180=320$

G4.4 Let $Q=\log_{2+\sqrt{2^2-1}}\left(2-\sqrt{2^2-1}\right)$, find the value of Q.

$$Q=\frac{\log\left(2-\sqrt{3}\right)}{\log\left(2+\sqrt{3}\right)}$$

$$=\frac{\log\left(2-\sqrt{3}\right)}{\log\left(2+\sqrt{3}\right)\cdot\frac{\left(2-\sqrt{3}\right)}{\left(2-\sqrt{3}\right)}}$$

$$=\frac{\log\left(2-\sqrt{3}\right)}{\log\frac{1}{2-\sqrt{3}}}$$

$$=\frac{\log\left(2-\sqrt{3}\right)}{-\log\left(2-\sqrt{3}\right)}=-1$$

Method 2 $Q=\log_{2+\sqrt{3}}\left(2-\sqrt{3}\right)\cdot\frac{\left(2+\sqrt{3}\right)}{\left(2+\sqrt{3}\right)}$

$$=\log_{2+\sqrt{3}}\frac{1}{\left(2+\sqrt{3}\right)}$$

$$=\log_{2+\sqrt{3}}\left(2+\sqrt{3}\right)^{-1}=-1$$