14-15 Individual	**1**	730639	**2**	201499	**3**	22	**4**	232°	**5**	1016064
	6	32	**7**	126	**8**	1	**9**	13	**10**	$\frac{36}{55}$
14-15 Group	**1**	$\frac{1}{120900}$	**2**	$\frac{3\sqrt{3}}{2}$	**3**	4	**4**	18	**5**	–15
	6	140	**7**	$\frac{6}{23}$	**8**	454	**9**	$2\sqrt{14}$	**10**	$\frac{2014}{2015}$

Individual Events

I1 How many pairs of distinct integers between 1 and 2015 inclusively have their products as multiple of 5?

Multiples of 5 are 5, 10, 15, 20, 25, 30, … , 2015. Number = 403

Numbers which are not multiples of 5 = 2015 – 403 = 1612

Let the first number be x, the second number be y.

Number of pairs = No. of ways of choosing any two numbers from 1 to 2015 – no. of ways of choosing such that both x, y are not multiples of 5.

$$= C_2^{2015} - C_2^{1612} = \frac{2015\times2014}{2} - \frac{1612\times1611}{2} = 403\times\left(\frac{5\times2014}{2} - \frac{4\times1611}{2}\right)$$

$$= 403\times(5\times1007 - 2\times1611) = 403\times(5035 - 3222) = 403\times1813 = 730639$$

I2 Given that $\left(10^{2015}\right)^{-10^2} = 0.\underbrace{000\cdots0}_{n\text{ times}}1$. Find the value of n.

$10^{-201500} = 0.\underbrace{000\cdots0}_{n\text{ times}}1$

$n = 201500 - 1 = 201499$

I3 Let $x°$ be the measure of an interior angle of an n-sided regular polygon, where x is an integer, how many possible values of n are there?

If $x°$ is an integer, then each exterior angle, $360° - x°$, is also an integer.

Using the fact that the sum of exterior angle of a convex polygon is 360°.

Each exterior angle $= \frac{360°}{n}$, which is an integer.

$\therefore$ n must be an positive integral factor of 360.

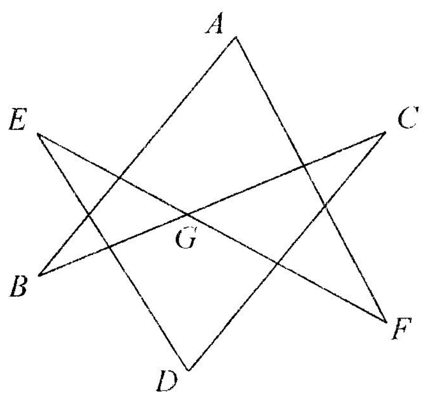
n = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360

However, $n = 1$ and $n = 2$ must be rejected because the least number of sides is 3.

$\therefore$ There are 22 possible value of n.

I4 As shown in the figure, $\angle EGB = 64°$,

$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F$ = ?



reflex $\angle BGF$ = reflex $\angle CGE$ = 180° + 64° = 244°

Consider quadrilateral $ABGF$,

$\angle A + \angle B$ +reflex $\angle BGF + \angle F$ = 360° ($\angle$ sum of polygon)

Consider quadrilateral $CDEG$,

$\angle C + \angle D + \angle E$+ reflex $\angle CGE$ = 360° ($\angle$ sum of polygon)

Add these two equations,

$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F$ = 720° – 2(244°) =232°

I5 It is given that $a_1, a_2, \dots, a_n, \dots$ is a sequence of positive real numbers such that $a_1 = 1$ and $a_{n+1} = a_n + \sqrt{a_n} + \frac{1}{4}$. Find the value of a_{2015}.

$a_2 = 2 + \frac{1}{4} = \frac{9}{4}$

$a_3 = \frac{9}{4} + \frac{3}{2} + \frac{1}{4} = \frac{16}{4}$

Claim: $a_n = \frac{(n+1)^2}{4}$ for $n \ge 1$

Pf: By M.I. $n = 1, 2, 3$, proved already.

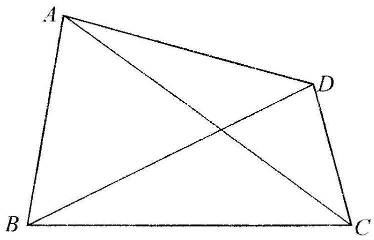
Suppose $a_k = \frac{(k+1)^2}{4}$ for some positive integer k.

$a_{k+1} = a_k + \sqrt{a_k} + \frac{1}{4} = \frac{(k+1)^2}{4} + \frac{k+1}{2} + \frac{1}{4} = \frac{(k+1)^2 + 2(k+1) + 1}{4} = \frac{(k+1+1)^2}{4}$

By M.I., the statement is true for $n \ge 1$

$a_{2015} = \frac{2016^2}{4} = 1008^2 = 1016064$

I6 As shown in the figure, $ABCD$ is a convex quadrilateral and $AB + BD + CD = 16$. Find the maximum area of $ABCD$.



Let $AB = a$, $BD = b$, $CD = c$, $\angle ABD = \alpha$, $\angle BDC = \beta$

Area of $ABCD$ = area of ΔABD + area of ΔBCD

$= \frac{1}{2} ab \sin\alpha + \frac{1}{2} bc \sin\beta$

$\le \frac{1}{2} ab + \frac{1}{2} bc$, equality holds when $\alpha = 90°$, $\beta = 90°$

$= \frac{1}{2} b(a+c) = \frac{1}{2} b(16-b)$

$\le \frac{1}{2}\left(\frac{b+16-b}{2}\right)^2$ (A.M. ≥ G.M., equality holds when $b = 8$, $a + c = 8$)

$= 32$

$\therefore$ The maximum area of $ABCD = 32$

I7 Let $x, y, z > 1$, $p > 0$, $\log_x p = 18$, $\log_y p = 21$ and $\log_{xyz} p = 9$. Find the value of $\log_z p$.

Reference: 1999 FG1.4, 2001 FG1.4

$\frac{\log p}{\log x} = 18$, $\frac{\log p}{\log y} = 21$, $\frac{\log p}{\log xyz} = 9$

$\frac{\log x}{\log p} = \frac{1}{18}$, $\frac{\log y}{\log p} = \frac{1}{21}$, $\frac{\log x + \log y + \log z}{\log p} = \frac{1}{9}$

$\frac{\log x}{\log p} + \frac{\log y}{\log p} + \frac{\log z}{\log p} = \frac{1}{18} + \frac{1}{21} + \frac{\log z}{\log p} = \frac{1}{9}$

$\frac{\log z}{\log p} = \frac{1}{126}$

$\log_z p = \frac{\log p}{\log z} = 126$

I8 Find the value of $\frac{1}{4029}+\frac{2\times 2014}{2014^2+2015^2}+\frac{4\times 2014^3}{2014^4+2015^4}-\frac{8\times 2014^7}{2014^8-2015^8}$.

$$\frac{1}{2014+2015}+\frac{2\times 2014}{2014^2+2015^2}+\frac{4\times 2014^3}{2014^4+2015^4}-\frac{8\times 2014^7}{2014^8-2015^8}$$

$$=-\frac{1}{2015-2014}+\frac{1}{2014+2015}+\frac{2\times 2014}{2014^2+2015^2}+\frac{4\times 2014^3}{2014^4+2015^4}-\frac{8\times 2014^7}{2014^8-2015^8}+1$$

$$=-\frac{2\times 2014}{2015^2-2014^2}+\frac{2\times 2014}{2014^2+2015^2}+\frac{4\times 2014^3}{2014^4+2015^4}-\frac{8\times 2014^7}{2014^8-2015^8}+1$$

$$=-\frac{4\times 2014^3}{2015^4-2014^4}+\frac{4\times 2014^3}{2014^4+2015^4}-\frac{8\times 2014^7}{2014^8-2015^8}+1$$

$$=-\frac{8\times 2014^7}{2015^8-2014^8}+\frac{8\times 2014^7}{2015^8-2014^8}+1=1$$

I9 Let x be a real number. Find the minimum value of $\sqrt{x^2-4x+13}+\sqrt{x^2-14x+130}$.

Reference 2010 FG4.2, 2021 P1Q12

Consider the following problem:

Let $P(2, 3)$ and $Q(7, 9)$ be two points. $R(x, 0)$ is a variable point on x-axis.

To find the minimum sum of distances $PR + RQ$.

Let y = sum of distances $=\sqrt{(x-2)^2+9}+\sqrt{(x-7)^2+81}$

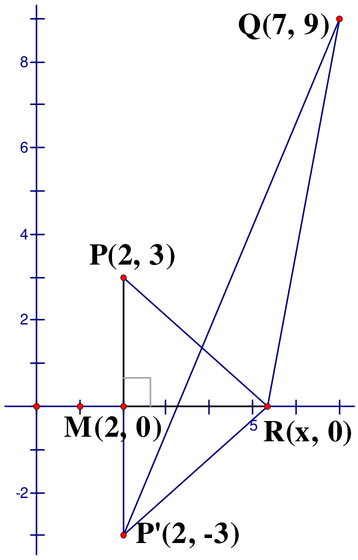
If we reflect $P(2, 3)$ along x-axis to $P'(2, -3)$, $M(2, 0)$ is the foot of perpendicular,

then $\Delta PMR \cong \Delta P'MR$ (S.A.S.)

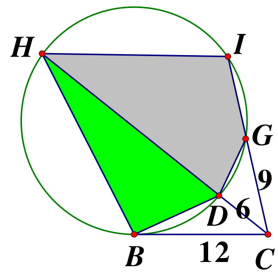
$y = PR + RQ = P'R + RQ \ge P'Q$ (triangle inequality)

$y \ge \sqrt{(7-2)^2+(9+3)^2} = 13$

The minimum value of $\sqrt{x^2-4x+13}+\sqrt{x^2-14x+130}$ is 13.



I10 B, H and I are points on the circle. C is a point outside the circle. BC is tangent to the circle at B. HC and IC cut the circle at D and G respectively. It is given that HDC is the angle bisector of $\angle BCI$, $BC = 12$, $DC = 6$ and $GC = 9$. Find the value of $\dfrac{\text{area of } \Delta BDH}{\text{area of } DHIG}$.



Reference 2018 HG7

By intersecting chords theorem,

$CH \times CD = BC^2$

$6 \times CH = 12^2$

$CH = 24$

$DH = 24 - 6 = 18$

Let $\angle BCD = \theta = \angle GCD$ ($\because$ HDC is the angle bisector)

$$\frac{S_{\Delta BCD}}{S_{\Delta CDG}} = \frac{\frac{1}{2} BC \cdot CD \sin\theta}{\frac{1}{2} GC \cdot CD \sin\theta} = \frac{12}{9} = \frac{4}{3} \quad \ldots\ldots (1)$$

Consider ΔBCD and ΔBDH

They have the same height but different bases.

$$\frac{S_{\Delta BDH}}{S_{\Delta BCD}} = \frac{DH}{CD} = \frac{18}{6} = 3 \quad \ldots\ldots (2)$$

Consider ΔCDG and ΔCIH

$\angle DCG = \angle ICH$ (common $\angle$s)

$\angle CDG = \angle CIH$ (ext. $\angle$, cyclic quad.)

$\angle CGD = \angle CHI$ (ext. $\angle$, cyclic quad.)

$\therefore$ $\Delta CDG \sim \Delta CIH$ (equiangular)

$$\frac{S_{\Delta CIH}}{S_{\Delta CDG}} = \left(\frac{CH}{CG}\right)^2 = \left(\frac{24}{9}\right)^2 = \frac{64}{9}$$

$$\Rightarrow \frac{S_{DHIG}}{S_{\Delta CDG}} = \frac{64-9}{9} = \frac{55}{9}$$

$$\Rightarrow \frac{S_{\Delta CDG}}{S_{DHIG}} = \frac{9}{55} \quad \ldots\ldots (3)$$

(1)×(2)×(3): $$\frac{\text{area of } \Delta BDH}{\text{area of } DHIG} = \frac{S_{\Delta BCD}}{S_{\Delta CDG}} \times \frac{S_{\Delta BDH}}{S_{\Delta BCD}} \times \frac{S_{\Delta CDG}}{S_{DHIG}} = \frac{4}{3} \times 3 \times \frac{9}{55} = \frac{36}{55}$$

Group Events

G1 Find the value of $\frac{1}{1860\times1865}+\frac{1}{1865\times1870}+\frac{1}{1870\times1875}+\cdots+\frac{1}{2010\times2015}$.

Reference: 2010 HI3

$$\frac{1}{1860\times1865}+\frac{1}{1865\times1870}+\frac{1}{1870\times1875}+\cdots+\frac{1}{2010\times2015}$$

$$=\frac{1}{25}\cdot\left(\frac{1}{372\times373}+\frac{1}{373\times374}+\frac{1}{374\times375}+\cdots+\frac{1}{402\times403}\right)$$

$$=\frac{1}{25}\left[\left(\frac{1}{372}-\frac{1}{373}\right)+\left(\frac{1}{373}-\frac{1}{374}\right)+\left(\frac{1}{374}-\frac{1}{375}\right)+\cdots+\left(\frac{1}{402}-\frac{1}{403}\right)\right]$$

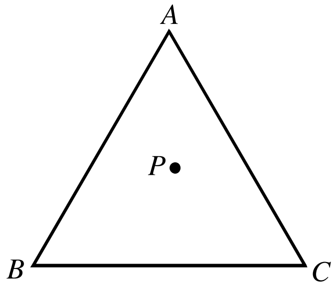
$$=\frac{1}{25}\left(\frac{1}{372}-\frac{1}{403}\right)$$

$$=\frac{31}{3747900}$$

$$=\frac{1}{120900}$$

G2 Given an equilateral triangle ABC with each side of length 3 and P is an interior point of the triangle. Let PX, PY and PZ be the feet of perpendiculars from P to AB, BC and CA respectively, find the value of $PX + PY + PZ$. (**Reference 1992 HG8, 2005 HG9, 2021 P1Q6**)

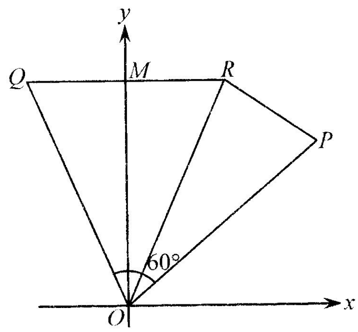
Let the distance from P to AB, BC, CA be h_1, h_2, h_3 respectively.



$$\frac{1}{2}\cdot3h_1+\frac{1}{2}\cdot3h_2+\frac{1}{2}\cdot3h_3=\text{area of }\Delta ABC=\frac{1}{2}\cdot3^2\sin60^\circ=\frac{9\sqrt{3}}{4}$$

$$PX+PY+PZ=h_1+h_2+h_3=\frac{3\sqrt{3}}{2}$$

G3 The coordinates of P are ($\sqrt{3}+1$, $\sqrt{3}+1$). P is rotated 60° anticlockwise about the origin to Q. Q is then reflected along the y-axis to R. Find the value of PR^2. **Reference: 2007 HI10**



Let the inclination of OP be θ.

$$\tan\theta=\frac{\sqrt{3}+1}{\sqrt{3}+1}=1\Rightarrow\theta=45^\circ$$

Inclination of OQ = 45° + 60° = 105°

Angle between OQ and positive y-axis = 105° – 90° = 15°

∴ Inclination of OR = 90° – 15° = 75°

$\angle POR$ = 75° – 45° = 30°

$$OP=OR=\left(\sqrt{3}+1\right)\sqrt{1^2+1^2}=\sqrt{6}+\sqrt{2}$$

Apply cosine rule on ΔPOR

$$PR^2=\left(\sqrt{6}+\sqrt{2}\right)^2+\left(\sqrt{6}+\sqrt{2}\right)^2-2\left(\sqrt{6}+\sqrt{2}\right)\left(\sqrt{6}+\sqrt{2}\right)\cos30^\circ$$

$$=\left(6+2+2\sqrt{12}\right)\left(2-2\times\frac{\sqrt{3}}{2}\right)=\left(8+4\sqrt{3}\right)\left(2-\sqrt{3}\right)$$

$$=4$$

G4 Given that $a^2+\dfrac{b^2}{2}+9\le ab-3b$, where a and b are real numbers. Find the value of ab.

Reference: 2005 FI4.1, 2006 FI4.2, 2009 FG1.4, 2011 FI4.3, 2013 FI1.4, 2015 FI1.1

$4a^2 + 2b^2 + 36 \le 4ab - 12b$

$(4a^2 - 4ab + b^2) + (b^2 + 12b + 36) \le 0$

$(2a - b)^2 + (b + 6)^2 \le 0$

$\Rightarrow 2a - b = 0$ and $b + 6 = 0$

$\Rightarrow b = -6$ and $a = -3$

$ab = 18$

G5 Given that the equation $x^2+15x+58=2\sqrt{x^2+15x+66}$ has two real roots. Find the sum of the roots.

Let $y = x^2 + 15x$

$(y + 58)^2 = 4(y + 66)$

$y^2 + 116y + 3364 = 4y + 264$

$y^2 + 112y + 3100 = 0$

$(y + 62)(y + 50) = 0$

$x^2 + 15x + 62 = 0$ or $x^2 + 15x + 50 = 0$

$\Delta = 225 - 248 < 0$ or $\Delta = 225 - 200 > 0$

$\therefore$ The first equation has no real roots and the second equation has two real roots

$\therefore$ Sum of the two real roots $= -15$

G6 Given that the sum of two interior angles of a triangle is $n°$, and the largest interior angle is 30° greater than the smallest one. Find the largest possible value of n .

Let the 3 angles of the triangle be $x°$, $y°$ and $x° - 30°$, where $x \ge y \ge x - 30 \cdots\cdots (1)$

$x + y + x - 30 = 180$ ($\angle$ sum of Δ)

$\Rightarrow y = 210 - 2x \cdots\cdots (2)$

Sub. (2) into (1): $x \ge 210 - 2x \ge x - 30$

$x \ge 70$ and $80 \ge x$

$\therefore 80 \ge x \ge 70 \cdots\cdots (3)$

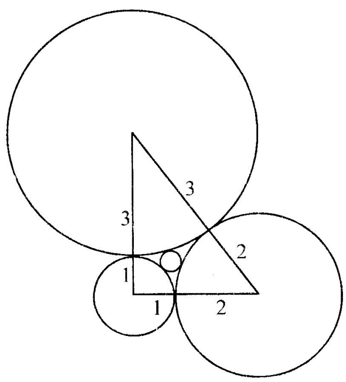
$n = x + y = x + 210 - 2x$ by (2)

$\therefore x = 210 - n \cdots\cdots (4)$

Sub. (4) into (3): $130 \le n \le 140$

$\therefore$ The largest possible value of $n = 140$

G7 Four circles with radii 1 unit, 2 units, 3 units and r units are touching one another as shown in the figure. Find the value of r.



Let the centre of the smallest circle be O and the radius be r. Let the centres of the circles with radii 2, 3, 1 be A, B and C respectively. $AB = 3 + 2 = 5$, $AC = 2 + 1 = 3$, $BC = 3 + 1 = 4$

$AC^2 + BC^2 = 3^2 + 4^2 = 25 = AC^2$

ΔABC is a $\perp\angle\Delta$ with $\angle C = 90°$ (converse, Pythagoras' theorem)

$OA = r + 2$, $OB = r + 3$, $OC = r + 1$

Let the feet of $\perp$ drawn from O to BC and AC respectively.

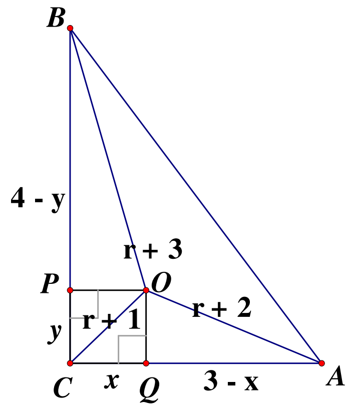
Let $CQ = x$, $CP = y$; then $AQ = 3 - x$, $BP = 4 - y$.

In ΔOCQ, $x^2 + y^2 = (r + 1)^2$ …… (1) (Pythagoras' theorem)

In ΔOAQ, $(3 - x)^2 + y^2 = (r + 2)^2$ …… (2) (Pythagoras' theorem)

In ΔOBP, $x^2 + (4 - y)^2 = (r + 3)^2$ …… (3) (Pythagoras' theorem)

(2) – (1): $9 - 6x = 2r + 3 \Rightarrow x = 1-\dfrac{1}{3}r$ …… (4)



(3) – (1): $16 - 8y = 4r + 8 \Rightarrow y = 1 - \frac{1}{2}r$ …… (5)

Sub. (4), (5) into (1): $\left(1 - \frac{1}{3}r\right)^2 + \left(1 - \frac{1}{2}r\right)^2 = (r+1)^2$

$1 - \frac{2}{3}r + \frac{1}{9}r^2 + 1 - r + \frac{1}{4}r^2 = 1 + 2r + r^2$

$\frac{23}{36}r^2 + \frac{11}{3}r - 1 = 0$

$23r^2 + 132r - 36 = 0$

$(23r - 6)(r + 6) = 0$

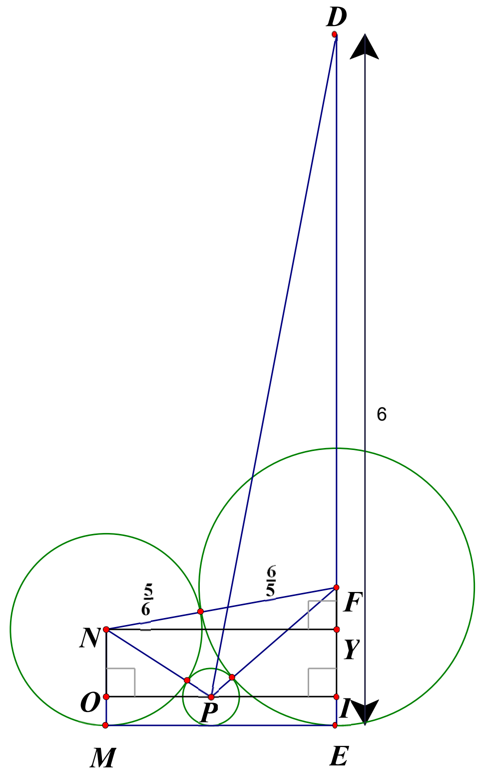
$r = \frac{6}{23}$ or –6 (rejected)

Method 2 We shall use the method of circle inversion to solve this problem.

Lemma 1 In the figure, a circle centre at N, with radius $\frac{5}{6}$ touches another circle centre at F, with radius $\frac{6}{5}$ externally. ME is the common tangent of the two circles. A third circle with centre at P touches the given two circles externally and also the line ME.

EF is produced to D so that $DE = 6$. Join DP.

O lies on NM, Y, I lies on FE so that $NM \perp OP$, $PI \perp FE$, $NY \perp FE$. Prove that



(a) the radius of the smallest circle is $\frac{30}{121}$;

(b) $ME = 2$;

(c) $DP = \frac{\sqrt{501840}}{121}$

Proof: Let the radius of the smallest circle be a.

Then $PN = \frac{5}{6} + a$, $PF = \frac{6}{5} + a$, $NM = \frac{5}{6}$, $FE = \frac{6}{5}$

$NO = \frac{5}{6} - a$, $FI = \frac{6}{5} - a$, $FY = \frac{6}{5} - \frac{5}{6} = \frac{11}{30}$

In ΔPNO, $OP^2 = (\frac{5}{6} + a)^2 - (\frac{5}{6} - a)^2 = \frac{10a}{3}$ (Pythagoras' theorem)

In ΔPIF, $PI^2 = (\frac{6}{5} + a)^2 - (\frac{6}{5} - a)^2 = \frac{24a}{5}$ (Pythagoras' theorem)

In ΔNYF, $NY^2 + FY^2 = NF^2$ (Pythagoras' theorem) $\Rightarrow (OP + PI)^2 + FY^2 = NF^2$

$$\left(\sqrt{\frac{10a}{3}} + \sqrt{\frac{24a}{5}}\right)^2 + \left(\frac{11}{30}\right)^2 = \left(\frac{5}{6} + \frac{6}{5}\right)^2$$

$$\left(\sqrt{\frac{10}{3}} + \sqrt{\frac{24}{5}}\right)^2 a = 4 \Rightarrow \left(\frac{\sqrt{50} + \sqrt{72}}{\sqrt{15}}\right)^2 a = 4 \Rightarrow \left(\frac{5\sqrt{2} + 6\sqrt{2}}{\sqrt{15}}\right)^2 a = 4 \Rightarrow \left(\frac{11\sqrt{2}}{\sqrt{15}}\right)^2 a = 4$$

$$\Rightarrow \frac{242}{15}a = 4 \Rightarrow a = \frac{30}{121}$$

$$ME = OP + PI = \sqrt{\frac{10a}{3}} + \sqrt{\frac{24a}{5}} = \frac{11\sqrt{2}}{\sqrt{15}} \cdot \sqrt{a} = \frac{11\sqrt{2}}{\sqrt{15}} \cdot \frac{\sqrt{30}}{11} = 2$$

$$DI = DE - IE = 6 - \frac{30}{121} = \frac{696}{121}$$

$$PI = \sqrt{\frac{24a}{5}} = \sqrt{\frac{24}{5} \times \frac{30}{121}} = \frac{12}{11}$$

In ΔDPI, $DI^2 + PI^2 = DP^2$ (Pythagoras' theorem)

$$\Rightarrow \left(\frac{696}{121}\right)^2 + \left(\frac{12}{11}\right)^2 = DP^2$$

$$DP^2 = \frac{501840}{121^2} \Rightarrow DP = \frac{\sqrt{501840}}{121}$$

Lemma 2 Given a circle C with centre at O and radius r.

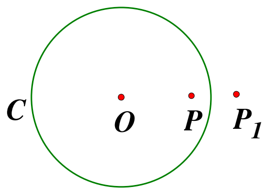


Figure 1

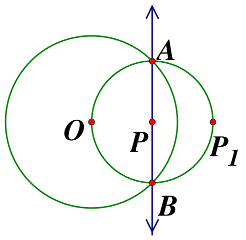


Figure 2

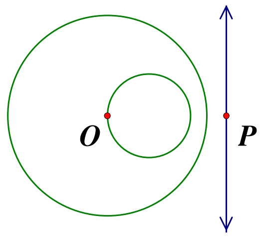


Figure 3

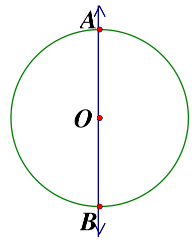


Figure 4

P and P_1 are points such that O, P, P_1 are collinear.
If $OP \times OP_1 = r^2$, then P_1 is the point of inversion of P respect to the circle C. (Figure 1)
P is also the point of inversion of P_1. O is called the centre of inversion.
If P lies on the circumference of the circle, then $OP = r$, $OP_1 = r$, P and P_1 coincide.
If $OP < r$, then $OP_1 > r$; if $OP > r$, then $OP_1 < r$; if $OP = 0$, then $OP_1 = \infty$; $OP = \infty$, $OP_1 = 0$.
If $OP < r$ and APB is a chord, then the inversion of APB is the arc AP_1B; the inversion of the straight line AB is the circle AP_1B which has a common chord AB. (Figure 2)
If $OP > r$, the inversion of a line (outside the given circle) is another smaller circle inside the given circle passing through the centre O. (Figure 3)
If $OP = 0$, the inversion of a line through the centre is itself, the line AB. (Figure 4)

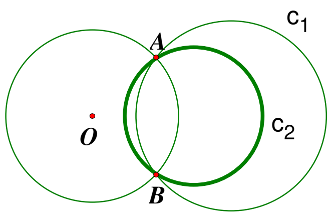


Figure 5

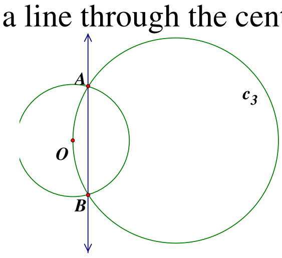


Figure 6

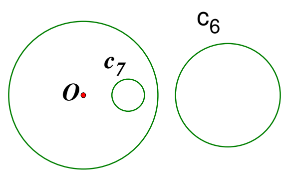


Figure 7

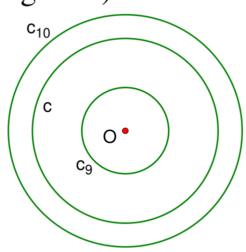


Figure 8

Given another circle C_1 which intersects the original circle at A and B, but does not pass through O. Then the inversion of C_1 with respect to the given circle is another circle C_2 passing through A and B but does not pass through O. (Figure 5)
Given another circle C_3 which intersects the original circle at A and B, and passes through O. Then the inversion of C_3 with respect to the given circle is the straight line through A and B. (Figure 6)
Given a circle C_6 outside but does not intersect the original circle. The inversion of C_6 respect to the given circle is another circle C_7 inside but does not pass through O. Conversely, the inversion of C_7 is C_6. (Figure 7)
Given a concentric circle C_9 with the common centre O inside the given circle C. Then the inversion of C_9 is another concentric circle C_{10} outside C. Conversely, the inversion of C_{10} is C_9. (Figure 8)

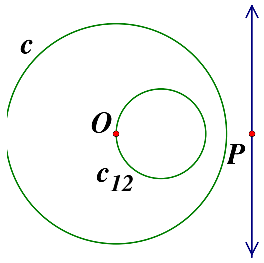


Figure 9

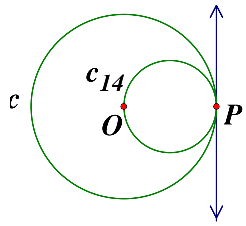


Figure 10

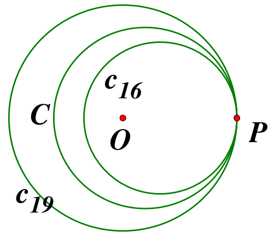


Figure 11

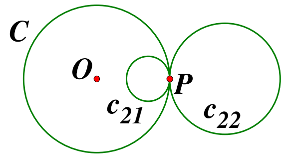


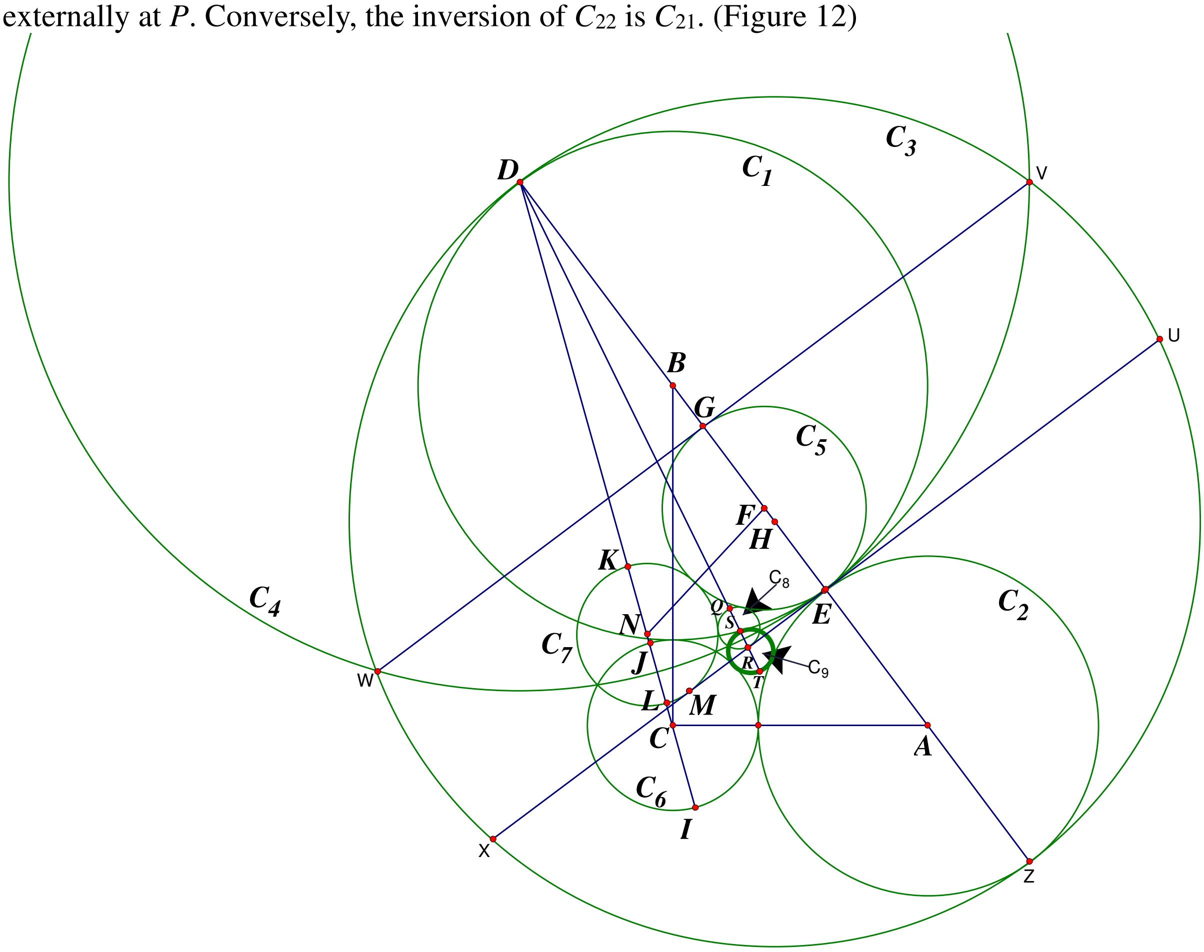
Figure 12

Given a circle C_{12} inside the given circle C but does not intersect the original circle, and passes through O. Then the inversion of C_{12} with respect to the given circle is the straight line outside C. (Figure 9)

Given a circle C_{14} inside the given circle C passes through O and touches C internally at P. Then the inversion of C_{14} with respect to the given circle is the tangent at P. Conversely, the inversion of the tangent at P is C_{14}. (Figure 10)

Given a circle C_{16} inside the given circle C which encloses O but touches C internally at P. Then the inversion of C_{16} with respect to the given circle is a circle C_{19} encloses C and touches C at P. Conversely, the inversion of C_{19} is C_{16}. (Figure 11)

Given a circle C_{21} inside the given circle C which does not enclose O but touches C internally at P. Then the inversion of C_{21} with respect to the given circle is a circle C_{22} which touches C externally at P. Conversely, the inversion of C_{22} is C_{21}. (Figure 12)



Suppose the circle C_1 with centre at B and the circle C_2 with centre at A touch each other at E. Draw a common tangent XEU. Let EZ and ED be the diameters of these two circles.
Let H be the mid-point of DZ. Use H as centre HD as radius to draw a circle C_3.
Use D as centre, DE as radius to draw a circle C_4. C_4 and C_3 intersect at W and V.
Join VW. VW intersects DZ at G. Let F be the mid-point of EG.
Use F as centre, FE as radius to draw a circle C_5.
$BE = 3$, $AE = 2$ (given), $DE = 6$, $EZ = 4$, $DZ = 6 + 4 = 10$.

$HD = HZ = HW = 5$

Let the diameter of C_5 be x, i.e. $GE = x$, $DG = 6 - x$.

$HG = HD - DG = 5 - (6 - x) = x - 1$

In ΔDGW, $WG^2 = 6^2 - (6 - x)^2 = 12x + x^2$ …… (1) (Pythagoras' theorem)

In ΔHGW, $WG^2 = 5^2 - (x - 1)^2 = 24 + 2x - x^2$ …… (2) (Pythagoras' theorem)

(1) = (2): $24 + 2x - x^2 = 12x + x^2 \Rightarrow x = 2.4 \Rightarrow$ The radius of C_5 is $1.2 = \frac{6}{5}$

$DG \times DZ = (6 - 2.4) \times 10 = 36 = DE^2 \Rightarrow G$ is the point of inversion of Z w.r.t. C_4.

Clearly E is the point of inversion of E w.r.t. C_4.

$\Rightarrow$ The inversion of C_2 w.r.t. C_4 is C_5.

The inversion of C_1 w.r.t. C_4 is the tangent UEX (see Figure 10).

Let the circle, with centre at C and radius 1 be C_6.

Join DC. DC cuts C_6 at J. DC is produced to cut C_6 again I. Then IJ = diameter of $C_6 = 2$.

In ΔABC, let $\angle ABC = \theta$, $\cos\theta = \frac{4}{5}$, $\angle CBD = 180° - \theta$ (adj. $\angle$s on st. line)

In ΔBCD, $CD^2 = 3^2 + 4^2 - 2\times3\times4\cos(180° - \theta) = 25 + 24\times\frac{4}{5} = \frac{221}{5} \Rightarrow CD = \sqrt{\frac{221}{5}}$

$DJ = DC - CJ = \sqrt{\frac{221}{5}} - 1$; $DI = DC + CI = \sqrt{\frac{221}{5}} + 1$

Invert C_6 w.r.t. C_4 to C_7 centre at N. Suppose DI intersects C_7 at K and L in the figure.

$DI \times DK = 6^2$ and $DJ \times DL = 6^2 \Rightarrow \left(\sqrt{\frac{221}{5}} + 1\right)DK = 36$ and $\left(\sqrt{\frac{221}{5}} - 1\right)DL = 36$

$\Rightarrow DK = \frac{36}{\sqrt{\frac{221}{5}} + 1} = \frac{5}{6}\left(\sqrt{\frac{221}{5}} - 1\right)$ and $DL = \frac{36}{\sqrt{\frac{221}{5}} - 1} = \frac{5}{6}\left(\sqrt{\frac{221}{5}} + 1\right)$

$LK = DL - DK = \frac{5}{6}\left(\sqrt{\frac{221}{5}} + 1\right) - \frac{5}{6}\left(\sqrt{\frac{221}{5}} - 1\right) = \frac{5}{3} \Rightarrow$ The radius of C_7 is $\frac{5}{6}$.

Now construct a smaller circle C_8 centre P, touches C_5 and C_7 externally and also touches XU. P is not shown in the figure.

By the result of **Lemma 1**, the radius of C_8 is $\frac{30}{121}$; $ME = 2$ and $DP = \frac{\sqrt{501840}}{121}$

DP cuts C_8 at Q, DP is produced further to cut C_8 again at R.

$DQ = DP - PQ = \frac{\sqrt{501840}}{121} - \frac{30}{121}$; $DR = DP + PR = \frac{\sqrt{501840}}{121} + \frac{30}{121}$

Now invert C_8 w.r.t C_4 to give C_9. This circle will touch C_1, C_2 and C_6 externally.

DR intersects C_9 at S, produce DR further to meet C_9 again at T. Then

$DS \times DR = 6^2$ and $DT \times DQ = 6^2 \Rightarrow \left(\frac{\sqrt{501840} + 30}{121}\right)DS = 36$ and $\left(\frac{\sqrt{501840} - 30}{121}\right)DT = 36$

$\Rightarrow DS = \frac{36\times121}{\sqrt{501840} + 30} = \frac{\sqrt{501840} - 30}{115}$; $DT = \frac{36\times121}{\sqrt{501840} - 30} = \frac{\sqrt{501840} + 30}{115}$

$ST = DT - DS = \frac{\sqrt{501840} + 30}{115} - \frac{\sqrt{501840} - 30}{115} = \frac{12}{23}$ = diameter of C_9

$\therefore$ The radius of C_9 is $\frac{6}{23}$.

G8 Given that a, b, x and y are non-zero integers, where $ax + by = 4$, $ax^2 + by^2 = 22$, $ax^3 + by^3 = 46$ and $ax^4 + by^4 = 178$. Find the value of $ax^5 + by^5$.

$ax + by = 4$ …(1), $ax^2 + by^2 = 22$ …(2), $ax^3 + by^3 = 46$ … (3), $ax^4 + by^4 = 178$ … (4).

Let $ax^5 + by^5 = m$ … (5)

$(x + y)(2)$: $(x + y)(ax^2 + by^2) = 22(x + y)$

$ax^3 + by^3 + xy(ax + by) = 22(x + y)$

Sub. (1) and (3): $46 + 4xy = 22(x + y) \Rightarrow 23 + 2xy = 11(x + y)$ … (6)

$(x + y)(3)$: $(x + y)(ax^3 + by^3) = 46(x + y)$

$ax^4 + by^4 + xy(ax^2 + by^2) = 46(x + y)$

Sub. (2) and (4): $178 + 22xy = 46(x + y) \Rightarrow 89 + 11xy = 23(x + y)$ … (7)

$11(7) - 23(6)$: $450 + 75xy = 0 \Rightarrow xy = -6$ … (8)

$11(6) - 2(7)$: $75(x + y) = 75 \Rightarrow x + y = 1$ … (9)

$(x + y)(4)$: $(x + y)(ax^4 + by^4) = 178(x + y)$

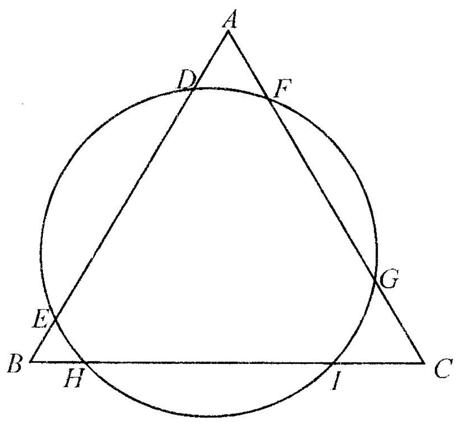
$ax^5 + by^5 + xy(ax^3 + by^3) = 178(x + y)$

Sub. (3) and (5): $m + 46xy = 178(x + y)$

Sub. (8) and (9): $m + 46\times(-6) = 178\times 1$

$\Rightarrow m = 454$

G9 Given that, in the figure, ABC is an equilateral triangle with $AF = 2$, $FG = 10$, $GC = 1$ and $DE = 5$. Find the value of HI. **(Reference: 2017 HG9)**



$AF + FG + GC = 2 + 10 + 1 = 13$

$\therefore AB = BC = CA = 13$ (property of equilateral triangle)

Let $AD = x$, then $BE = 13 - 5 - x = 8 - x$

Let $HI = y$, $BH = z$, then $IC = 13 - y - z$

By intersecting chords theorem,

$AD\times AE = AF\times AG$

$x(x + 5) = 2\times 12$

$x^2 + 5x - 24 = 0$

$(x - 3)(x + 8) = 0$

$x = 3$ or -8 (rejected)

$BE = 8 - x = 5$

$BH\times BI = BE\times BD$

$z(z + y) = 5\times 10 = 50$ …… (1)

$CI\times CH = CG\times CF$

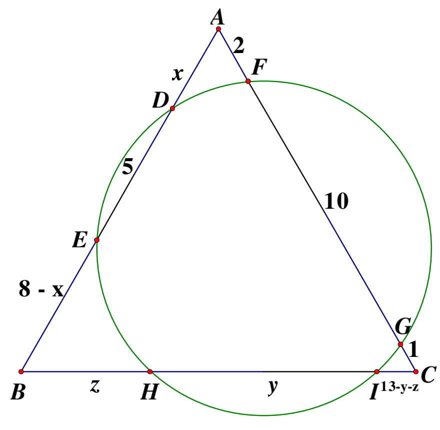
$(13 - y - z)(13 - z) = 11$

$169 - 13(y + 2z) + z(y + z) = 11$ …… (2)

Sub. (1) into (2): $169 - 13(y + 2z) + 50 = 11$

$y + 2z = 16$

$y = 16 - 2z$ …… (3)



Sub. (3) into (1): $z(z + 16 - 2z) = 50$

$z^2 - 16z + 50 = 0$

$z = 8 + \sqrt{14}$ or $8 - \sqrt{14}$

From (3), $2z \le 16 \Rightarrow z \le 8 \Rightarrow 8 + \sqrt{14}$ is rejected $\Rightarrow z = 8 - \sqrt{14}$ only

$HI = y = 16 - 2z = 16 - 2(8 - \sqrt{14}) = 2\sqrt{14}$

G10 Let a_n and b_n be the x-intercepts of the quadratic function $y = n(n-1)x^2 - (2n-1)x + 1$, where n is an integer greater than 1. Find the value of $a_2b_2 + a_3b_3 + \ldots + a_{2015}b_{2015}$.

Reference: 2005 HI6

The quadratic function can be written as $y = (nx - 1)[(n - 1)x - 1]$

$\therefore$ The x-intercepts are $\frac{1}{n}$ and $\frac{1}{n-1}$.

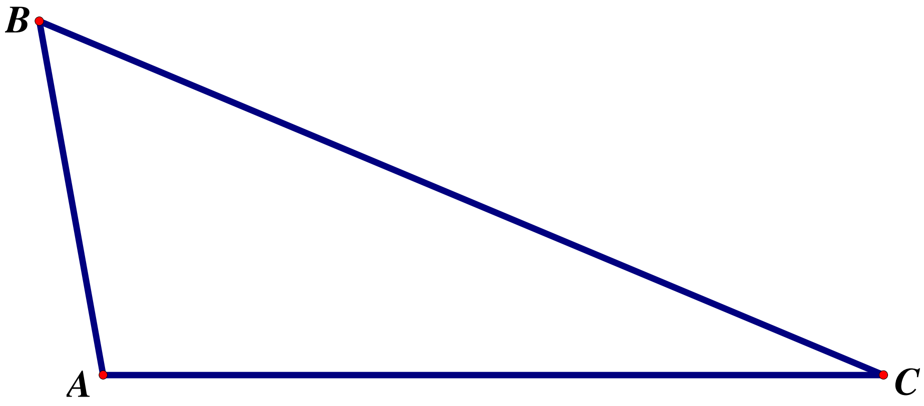
$a_n\, b_n = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$ for $n > 1$

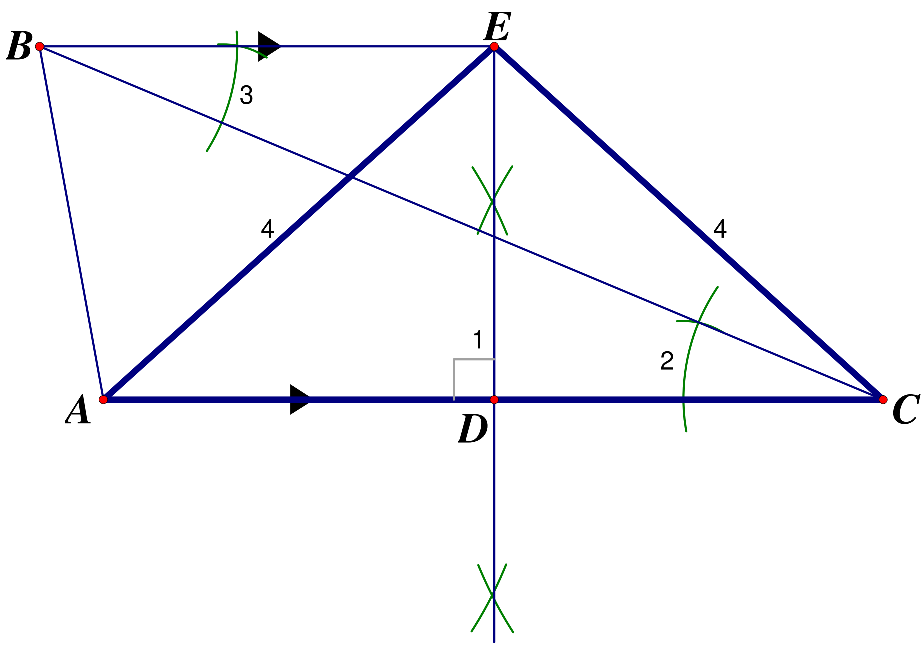
$$a_2b_2 + a_3b_3 + \ldots + a_{2015}b_{2015} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2014} - \frac{1}{2015}\right)$$

$$= 1 - \frac{1}{2015} = \frac{2014}{2015}$$

Geometrical Construction

1. Construct an isosceles triangle which has the same base and height to the following triangle.





Steps

(1) Construct the perpendicular bisector of AC, D is the mid point of AC.

(2) Copy $\angle ACB$.

(3) Draw $\angle CBE$ so that it is equal to $\angle ACB$, then $BE \parallel AE$ (alt. $\angle$s eq.)
BE and the $\perp$ bisector in step 1 intersect at E.

(4) Join AE, CE.

Then ΔAEC is the required isosceles triangle with $AE = CE$. Construction steps completed.

Proof: ΔABC and ΔAEC are two triangles with the same base and the same height

$\therefore$ ΔABC and ΔAEC have the same areas

$DE = DE$ (common sides)

$\angle ADE = 90° = \angle CDE$ (property of perpendicular bisector)

$AD = DC$ (property of perpendicular bisector)

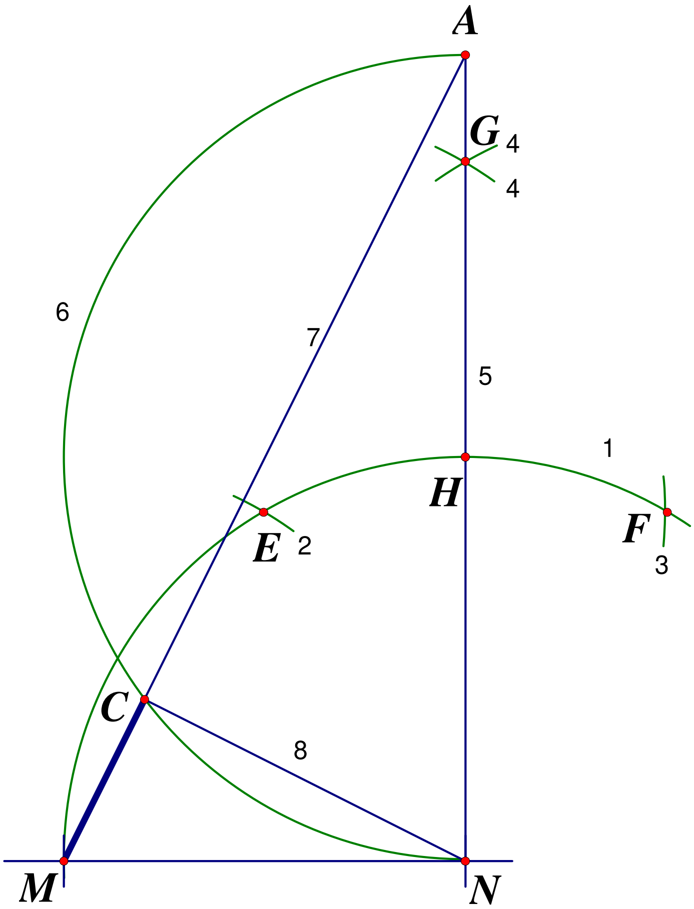
$\Delta ADE \cong \Delta CDE$ (S.A.S.)

$AE = CE$ (corr. sides $\cong\Delta$s)

Then ΔAEC is the required isosceles triangle with $AE = CE$.

The proof is completed.

2. Given the following line segment MN represent a unit length, construct a line segment of length $\sqrt{\frac{1}{5}}$.



Steps

(1) Use N as centre, MN as radius to draw an arc.

(2) Use M as centre, MN as radius to draw an arc, cutting the arc in step 1 at E.

(3) Use E as centre, MN as radius to draw an arc, cutting the arc in step 1 at F.

(4) Use E as centre, MN as radius to draw an arc. Use F as centre, MN as radius to draw an arc. The two arcs intersect at G. ΔEFG is an equilateral triangle.

(5) Join NG and produce it longer. NG intersects the arc in step 1 at H.

(6) Use H as centre, HN as radius to draw a semi-circle, cutting NG produced at A.

(7) Join AM, cutting the semicircle in step (6) at C.

(8) Join NC.

Then MC is the required length.

Proof: $\angle MNE = 60°$ (ΔMNE is an equilateral triangle)

$\angle FNE = 60°$ (ΔFNE is an equilateral triangle)

$\angle MNG = 60° + 30° = 90°$ (NG is the $\angle$ bisector of $\angle ENF$)

$\angle ACN = 90°$ ($\angle$ in semi-circle)

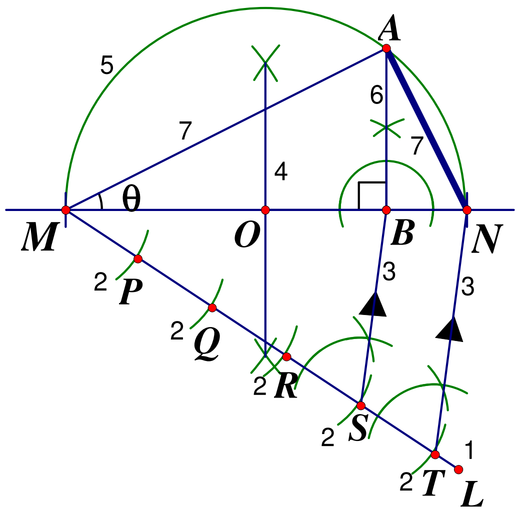
$AN = 2,\ AM = \sqrt{1^2 + 2^2} = \sqrt{5}$ (Pythagoras' theorem)

$\Delta CMN \sim \Delta NMA$ (equiangular)

$\frac{MC}{MN} = \frac{MN}{AM}$ (cor. sides, ~Δ's)

$MC = \sqrt{\frac{1}{5}}$

Method 2



Steps

(1) Draw a line segment ML in any direction which is not parallel to MN.

(2) Use any radius to mark points P, Q, R, S, T on ML so that $MP = PQ = QR = RS = ST$.

(3) Join TN, draw a line segment SB parallel to TN, cutting MN at B.

(4) Construct the perpendicular bisector of MN, O is the mid-point of MN.

(5) Use O as centre, OM as radius to draw a semi-circle with MN as diameter.

(6) Through B, draw a line segment $AB \perp MN$, cutting the semi-circle in step 5 at A.

(7) Join AM and AN.

Then $AN = \sqrt{\frac{1}{5}}$. The construction is completed.

Proof: By steps (2) and (3), $MS : ST = 4 : 1$ and $SB \parallel TN$

$\therefore MB : BN = MS : ST = 4 : 1$ (Theorem of equal ratios)

$MB = \frac{4}{5}$, $BN = \frac{1}{5}$

$\angle MAN = 90°$ ($\angle$ in semi-circle)

Let $\angle AMN = \theta$

$\angle AMN = 180° - 90° - \theta = 90° - \theta$ ($\angle$ sum of ΔAMN)

$\angle BAN = \theta$ (ext $\angle$ of ΔABN)

$\frac{AB}{MB} = \frac{BN}{AB} = \tan\theta$

$AB^2 = MB \times BN = \frac{4}{5} \times \frac{1}{5} = \frac{4}{25}$

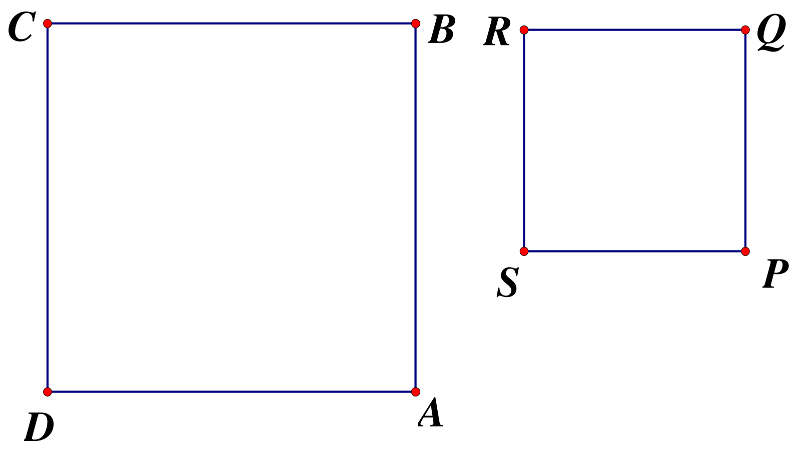
$AB^2 + BN^2 = AN^2$ (Pythagoras' theorem on ΔABN)

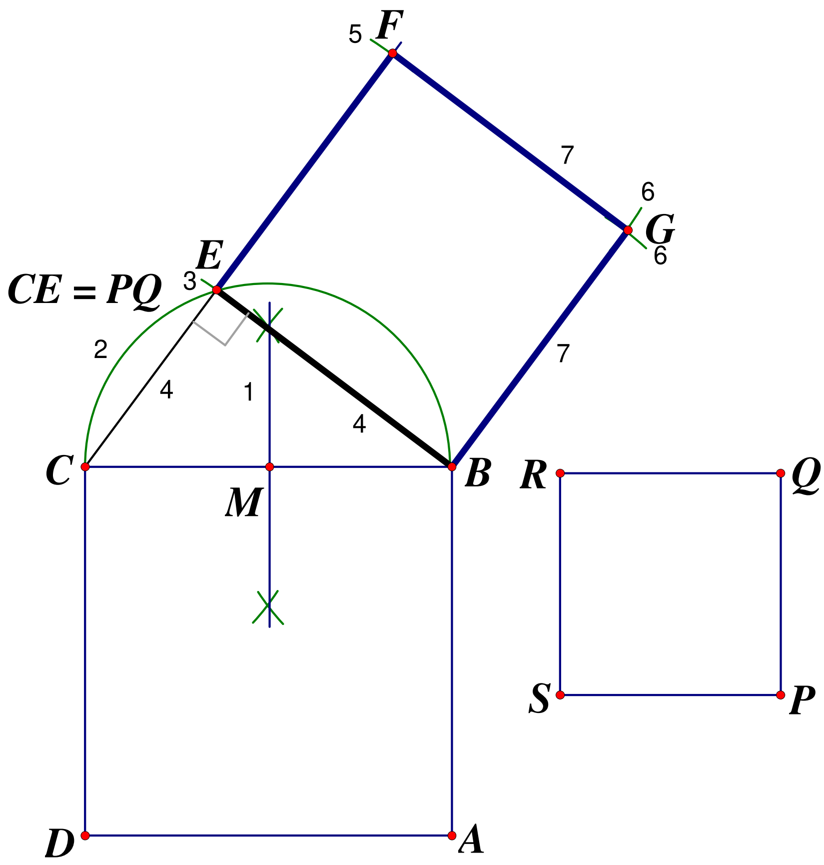
$AN^2 = \frac{4}{25} + \frac{1}{25} = \frac{1}{5}$

$AN = \sqrt{\frac{1}{5}}$

The proof is completed

3. Construct a square whose area is equal to the difference between the areas of the following two squares $ABCD$ and $PQRS$. (**Reference: 2018 HC1**)





Steps

(1) Draw the perpendicular bisector of BC , M is the mid-point of BC.

(2) Use M as centre, MB as radius to draw a semi-circle outside the square $ABCD$.

(3) Use C as centre, PQ as radius to draw an arc, cutting the semicircle in (2) at E.

(4) Join CE and produce it longer. Join BE.

(5) Use E as centre, BE as radius to draw an arc, cutting CE produced at F.

(6) Use B as centre, BE as radius to draw an arc. Use F as centre, FE as radius to draw an arc. The two arcs intersect at G.

(7) Join FG and BG.

$\angle BEC = 90°$ (∠ in semi-circle)

$BE^2 = BC^2 - CE^2 = BC^2 - PQ^2$ (Pythagoras' theorem)

$BEFG$ is the required square.