Examples on Mathematical Induction: Sum of variables

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1. **1970 AM Paper 1 Q10**

Prove that $a + ar + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$, where $r \neq 1$, for all positive integer n

Let $P(n) \equiv$ '$a + ar + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$, where $r \neq 1$, for all positive integer n.'

$n = 1$, L.H.S. $= a$, R.H.S. $= \dfrac{a(1-r^1)}{1-r} = a$

L.H.S. = R.H.S. it is true for $n = 1$

Suppose $P(k)$ is true.

i.e. $a + ar + \cdots + ar^{k-1} = \dfrac{a(1-r^k)}{1-r}$ for some positive integer k.

When $n = k + 1$,

L.H.S. $= a + ar + \cdots + ar^{k-1} + ar^k$

$$= \frac{a(1-r^k)}{1-r} + ar^k$$

$$= \frac{a(1-r^k+r^k-r^{k+1})}{1-r}$$

$$= \frac{a(1-r^{k+1})}{1-r} = \text{R.H.S.}$$

If $P(k)$ is true then $P(k + 1)$ is also true

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

2. Prove that $a + (a + d) + (a+2d) + \cdots + a + (n-1)d = \dfrac{n}{2}[2a+(n-1)d]$ for all positive integer n.

3. Prove that $a + 2(a + b) + 3(a + 2b) + \cdots + n[a + (n-1)b] = \dfrac{1}{6}n(n+1)[2bn+(3a-2b)]$

Let $P(n) \equiv$ "$a+2(a+b)+3(a+2b)+\cdots+n[a+(n-1)b]=\dfrac{1}{6}n(n+1)[2bn+(3a-2b)]$ for $n = 1,2,3,\cdots$"

$n = 1$, L.H.S. $= a$, R.H.S. $= \dfrac{1}{6}\cdot 1\cdot 2[2b+(3a-2b)] = a$

L.H.S. = R.H.S., $P(1)$ is true.

Suppose $a + 2(a + b) + 3(a + 2b) + \cdots + k[a + (k-1)b] = \dfrac{1}{6}k(k+1)[2bk+(3a-2b)]$

When $n = k + 1$, L.H.S. $= a + 2(a + b) + 3(a + 2b) + \cdots + k[a + (k-1)b] + (k + 1)(a + kb)$

$$= (k+1)(a+kb) + \frac{1}{6}k(k+1)[2bk+(3a-2b)]$$

$$= \frac{1}{6}(k+1)\left[6(a+kb)+2bk^2+(3a-2b)k\right]$$

$$= \frac{1}{6}(k+1)\left[2bk^2+(3a+4b)k+6a\right]$$

$$= \frac{1}{6}(k+1)(k+2)(2bk+3a)$$

$$= \frac{1}{6}(k+1)(k+2)[2b(k+1)+(3a-2b)]$$

$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true. By induction, $P(n)$ is true for all positive n .

4. **General Mathematics (CUHK) 1975 Q4(b)**
Let $S(n)$ denote the following statement
$2 + 2(2 + 3a) + 3(2 + 6a) + \cdots + n[2 + 3(n - 1)a] = n(n + 1)[1 + (n - 1)a]$.
Prove the following statement:
(*) If we <u>assume</u> that $S(n)$ is true, then $S(n + 1)$ is also true.
Can your conclude, on the basis of statement (*) <u>alone</u>, that $S(n)$ is true for every positive integer n? Why?
Is the statement $S(n)$ in fact true for every positive integer n? Why?
Suppose $2 + 2(2+3a) + 3(2 + 6a) + \cdots + n[2+3(n-1)a] = n(n + 1)[1 + (n - 1)a]$ is true.
$S(n+1)$: $2 + 2(2+3a) + 3(2 + 6a) + \cdots + (n + 1)(2 + 3na) = (n + 1)(n + 2)(1 + na)$
L.H.S. $= 2 + 2(2 + 3a) + 3(2 + 6a) + \cdots + n[2 + 3(n - 1)a] + (n + 1)(2 + 3na)$
$= n(n + 1)[1 + (n - 1)a] + (n + 1)(2 + 3na)$ (by assumption)
$= (n + 1)[n + n(n - 1)a + 2 + 3na]$
$= (n + 1)[n +2 + n(n + 2)a]$
$= (n + 1)(n + 2)[1 + (n + 1 - 1)a] =$ R.H.S.
$\therefore$ $S(n + 1)$ is also true if $S(n)$ is true.
No, $S(n)$ may not be true for every positive integer n. We have to check whether it is true for
$S(1)$: $2 = 1\times2\times(1 + 0)$
L.H.S. = 2, R.H.S. = 2
$\therefore$ $S(1)$ is true.
By the principle of mathematical induction $S(n)$ is true for all positive integer n .

5. Prove that $[a + (n - 1)b] +2[a + (n - 2)b] + \cdots + (n - 1)(a + b) + na = \frac{1}{6}n(n+1)[bn+(3a-b)]$.

6. Prove that $a+3(a+b) +6(a+ 2b) + \cdots + \frac{1}{2}n(n-1)[a+(n-1)b] = \frac{1}{24}n(n+1)(n+2)[3bn+(4a-3b)]$

7. Prove that $[a+(n-1)b] + 3[a+(n-2)b] +\cdots+ \frac{1}{2}n(n-1)(a+b) + \frac{1}{2}n(n+1)a = \frac{1}{24}n(n+1)(n+2)[bn+(4a-b)]$.

8. **2007 Q5** Let $a \neq 0$ and $a \neq 1$. Prove by mathematical induction that
$$\frac{1}{a-1}-\frac{1}{a}-\frac{1}{a^2}-\cdots-\frac{1}{a^n}=\frac{1}{a^n(a-1)}$$ for all positive integers n.
Let $P(n)$ be the statement "$\frac{1}{a-1}-\frac{1}{a}-\frac{1}{a^2}-\cdots-\frac{1}{a^n}=\frac{1}{a^n(a-1)}$ for all positive integers n."
$n = 1$, L.H.S. $= \frac{1}{a-1}-\frac{1}{a}=\frac{a-(a-1)}{a(a-1)}=\frac{1}{a(a-1)}$ = R.H.S., $P(1)$ is true.
Suppose $P(k)$ is true for some positive integer k.
i.e. $\frac{1}{a-1}-\frac{1}{a}-\frac{1}{a^2}-\cdots-\frac{1}{a^k}=\frac{1}{a^k(a-1)}$
When $n = k + 1$, L.H.S. $=\frac{1}{a-1}-\frac{1}{a}-\frac{1}{a^2}-\cdots-\frac{1}{a^k}-\frac{1}{a^{k+1}}$
$=\frac{1}{a^k(a-1)}-\frac{1}{a^{k+1}}$, by induction assumption
$=\frac{a-(a-1)}{a^{k+1}(a-1)}=\frac{1}{a^{k+1}(a-1)}$ = R.H.S., $P(k + 1)$ is also true.
If $P(k)$ is true, then $P(k + 1)$ is also true.
By mathematical induction, $P(n)$ is true for all positive integer n.

9. Prove, by mathematical induction, that for all positive integers n,

$$\frac{2}{x-2}-\left(\frac{2}{x}+\frac{2^2}{x^2}+\cdots+\frac{2^n}{x^n}\right)=\frac{2^{n+1}}{x^n(x-2)}, \text{ where } x \neq 0 \text{ and } x \neq 2.$$

Let $P(n) \equiv$ "$\frac{2}{x-2}-\left(\frac{2}{x}+\frac{2^2}{x^2}+\cdots+\frac{2^n}{x^n}\right)=\frac{2^{n+1}}{x^n(x-2)}$ for all positive integers n ."

$n = 1$, L.H.S. $=\frac{2}{x-2}-\frac{2}{x}=\frac{2(x-x+2)}{x(x-2)}=\frac{2^{1+1}}{x^1(x-2)}=$ R.H.S., $P(1)$ is true.

Suppose $P(k)$ is true. i.e. $\frac{2}{x-2}-\left(\frac{2}{x}+\frac{2^2}{x^2}+\cdots+\frac{2^k}{x^k}\right)=\frac{2^{k+1}}{x^k(x-2)}$

When $n = k + 1$,

$$\begin{aligned}\text{L.H.S.} &= \frac{2}{x-2}-\left(\frac{2}{x}+\frac{2^2}{x^2}+\cdots+\frac{2^k}{x^k}+\frac{2^{k+1}}{x^{k+1}}\right)\\ &= \frac{2^{k+1}}{x^k(x-2)}-\frac{2^{k+1}}{x^{k+1}}\\ &= \frac{2^{k+1}x}{x^{k+1}(x-2)}-\frac{2^{k+1}(x-2)}{x^{k+1}(x-2)}\\ &= \frac{2^{k+1}(x-x+2)}{x^{k+1}(x-2)}=\frac{2^{k+2}}{x^{k+1}(x-2)}=\text{R.H.S.}\end{aligned}$$

It is also true for $n = k + 1$ if it is true for $n = k$. By induction, it is true for all +ve integers n .

10. Prove that $a^2 + (a + d)^2 + (a + 2d)^2 + \cdots + (a + nd)^2 = \frac{(n+1)}{6}\left[6a(a+nd)+d^2n(2n+1)\right]$.

Hence prove that $2^2 + 4^2 + 6^2 + \cdots + p^2 = \frac{1}{6}p(p+1)(p+2)$ for any even positive integer p .

$n = 1$, L.H.S. $= a^2 + (a + d)^2 = 2a^2 + 2ad + d^2$

R.H.S. $= \frac{2}{6}\left[6a(a+d)+d^2(2+1)\right] = 2a(a + d) + d^2$

$\therefore$ L.H.S. = R.H.S., it is true for $n = 1$

Suppose $a^2 + (a + d)^2 + (a + 2d)^2 + \cdots + (a + kd)^2 = \frac{(k+1)}{6}\left[6a(a+kd)+d^2k(2k+1)\right]$, for $k > 0$.

$a^2 + (a + d)^2 + (a + 2d)^2 + \cdots + (a + kd)^2 + [a + (k + 1)d]^2$

$= \frac{(k+1)}{6}\left[6a(a+kd)+d^2k(2k+1)\right] + [a + (k + 1)d]^2$

$= \frac{1}{6}\left[6a^2(k+1)+6adk(k+1)+d^2k(2k+1)(k+1)+6a^2+12(k+1)ad+6(k+1)^2d^2\right]$

$= \frac{1}{6}\left[6a^2(k+2)+6ad(k+1)(k+2)+d^2(k+1)(2k^2+k+6k+6)\right]$

$= \frac{1}{6}\left[6a^2(k+2)+6ad(k+1)(k+2)+d^2(k+1)(2k+7k+6)\right]$

$= \frac{1}{6}\left[6a^2(k+2)+6ad(k+1)(k+2)+d^2(k+1)(k+2)(2k+3)\right]$

$= \frac{k+2}{6}\left[6a^2+6ad(k+1)+d^2(k+1)(2k+3)\right]$

R.H.S. $= \frac{(k+2)}{6}\left\{6a[a+(k+1)d]+d^2(k+1)[2(k+1)+1]\right\}$

$= \frac{(k+2)}{6}\left[6a^2+6ad(k+1)+d^2(k+1)(2k+3)\right]$

L.H.S. = R.H.S., it is also true for $n = k + 1$ if it is true for $n = k$.

By the principle of mathematical induction, it is true for all positive integers n .

11. If a and d are positive, decide which n starts and prove that

$$\frac{1}{\sqrt{a}+\sqrt{a+d}}+\frac{1}{\sqrt{a+d}+\sqrt{a+2d}}+\cdots+\frac{1}{\sqrt{a+(n-2)d}+\sqrt{a+(n-1)d}}=\frac{n-1}{\sqrt{a}+\sqrt{a+(n-1)d}}$$

$n = 1$, L.H.S. $\neq$ R.H.S.

$n = 2$, L.H.S. $=\frac{1}{\sqrt{a}+\sqrt{a+d}}=$ R.H.S.

Suppose $\frac{1}{\sqrt{a}+\sqrt{a+d}}+\frac{1}{\sqrt{a+d}+\sqrt{a+2d}}+\cdots+\frac{1}{\sqrt{a+(k-2)d}+\sqrt{a+(k-1)d}}=\frac{k-1}{\sqrt{a}+\sqrt{a+(k-1)d}}$ for some $k>1$.

$n = k + 1$,

$$\text{L.H.S.}=\frac{1}{\sqrt{a}+\sqrt{a+d}}+\frac{1}{\sqrt{a+d}+\sqrt{a+2d}}+\cdots+\frac{1}{\sqrt{a+(k-2)d}+\sqrt{a+(k-1)d}}+\frac{1}{\sqrt{a+(k-1)d}+\sqrt{a+kd}}$$

$$=\frac{k-1}{\sqrt{a}+\sqrt{a+(k-1)d}}+\frac{1}{\sqrt{a+(k-1)d}+\sqrt{a+kd}}$$

$$=\frac{(k-1)}{\left[\sqrt{a}+\sqrt{a+(k-1)d}\right]}\cdot\frac{\left[\sqrt{a+(k-1)d}-\sqrt{a}\right]}{\left[\sqrt{a+(k-1)d}-\sqrt{a}\right]}+\frac{1}{\left[\sqrt{a+(k-1)d}+\sqrt{a+kd}\right]}\cdot\frac{\left[\sqrt{a+kd}-\sqrt{a+(k-1)d}\right]}{\left[\sqrt{a+kd}-\sqrt{a+(k-1)d}\right]}$$

$$=\frac{(k-1)\left[\sqrt{a+(k-1)d}-\sqrt{a}\right]}{(k-1)d}+\frac{\sqrt{a+kd}-\sqrt{a+(k-1)d}}{d}$$

$$=\frac{\sqrt{a+(k-1)d}-\sqrt{a}}{d}+\frac{\sqrt{a+kd}-\sqrt{a+(k-1)d}}{d}$$

$$=\frac{\sqrt{a+(k-1)d}-\sqrt{a}}{d}+\frac{\sqrt{a+kd}-\sqrt{a+(k-1)d}}{d}$$

$$=\frac{\sqrt{a+kd}-\sqrt{a}}{d}$$

$$\text{R.H.S.}=\frac{k}{\sqrt{a}+\sqrt{a+kd}}=\frac{k}{\left(\sqrt{a}+\sqrt{a+kd}\right)}\cdot\frac{\left(\sqrt{a+kd}-\sqrt{a}\right)}{\left(\sqrt{a+kd}-\sqrt{a}\right)}$$

$$=\frac{k\left(\sqrt{a+kd}-\sqrt{a}\right)}{kd}=\text{L.H.S.}$$

$\therefore$ If it is true for $n = k$ then it is also true for $n = k + 1$.

By the principle of mathematical induction, it is true for all positive $n > 1$.

12. **1971 香港中文中學會考高級數學試卷一 Q6(a)**

某數列之第 r 項為 $r^3(1+3r^2)$。試利用數學歸納法證明其首 n 項之和為 $\frac{1}{2}n^3(1+n)^3$。

Let P(n) ≡ ‘$\sum_{r=1}^{n} r^3\left(1+3r^2\right)=\frac{1}{2}n^3\left(1+n\right)^3$ for all positive integer n .’

P(1): L.H.S. = $1^3(1+3)=4$, R.H.S. = $\frac{1}{2}\cdot 1^3\left(1+1\right)^3=4$

P(1) is true.
Suppose P(k) is true for some positive integer k .
P(k + 1):

$$\text{L.H.S.}=\sum_{r=1}^{k+1} r^3\left(1+3r^2\right)$$
$$=\sum_{r=1}^{k} r^3\left(1+3r^2\right)+\left(k+1\right)^3\left[1+3\left(k+1\right)^2\right]$$
$$=\frac{1}{2}k^3\left(1+k\right)^3+\left(k+1\right)^3\left[1+3\left(k+1\right)^2\right] \quad \text{(induction assumption)}$$
$$=\frac{1}{2}\left(1+k\right)^3\left[k^3+2+6\left(k+1\right)^2\right]$$
$$=\frac{1}{2}\left(1+k\right)^3\left(k^3+6k^2+12k+8\right)$$
$$\text{R.H.S.}=\frac{1}{2}\cdot\left(k+1\right)^3\left(k+2\right)^3$$
$$=\frac{1}{2}\cdot\left(k+1\right)^3\left(k^3+3\times 2k^2+3\times 4k+8\right)$$

L.H.S. = R.H.S.
If P(k) is true then P(k + 1) is also true
By the principle of mathematical induction, P(n) is true for all positive integer n .