88-89 Individual	**1**	7	**2**	2	**3**	24	**4**	$\frac{1}{2}$	**5**	12 km/h
	6	45	**7**	45	**8**	–4	**9**	2	**10**	12
	11	$\frac{5}{12}$	**12**	147	**13**	3	**14**	2	**15**	24
	16	12	**17**	6 : 25	**18**	24	**19**	230	**20**	12

88-89 Group	**1**	$\frac{9}{20}$	**2**	7	**3**	6585	**4**	(12, 20)	**5**	$\frac{3}{2}$
	6	43	**7**	480	**8**	$\frac{12}{5}$	**9**	5	**10**	110768

Individual Events

I1 Given that $x+\frac{1}{x}=3$, find $x^2+\frac{1}{x^2}$.

Reference: 1983 FG7.3, 1984 FG10.2, 1985 FI1.2, 1987 FG8.2, 1990 HI12, 1997 HG7

$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2$

$= 3^2 - 2 = 7$

I2 If $x \# y = xy - 2x$, find the value of $2 \# 3$.

$2 \# 3 = 2\times3 - 2\times2 = 2$

I3 Find the number of sides of a regular polygon if an interior angle exceeds an exterior angle by 150°. **(Reference 1997 HG6)**

Let x be the size of each interior angle, y be the size of each exterior angle, n be the number of sides.

$x=\frac{180^\circ(n-2)}{n}, y=\frac{360^\circ}{n}$

$x = y + 150°$

$\frac{180^\circ(n-2)}{n}=\frac{360^\circ}{n}+150^\circ$

$180(n - 2) = 360 + 150n$

$18n - 36 = 36 + 15n$

$\Rightarrow n = 24$

I4 Find the value of b such that $10^{\log_{10}9}=8b+5$.

$9 = 8b + 5$

$\Rightarrow b=\frac{1}{2}$

I5 A man cycles from P to Q with a uniform speed of 15 km/h and then back from Q to P with a uniform speed of 10 km/h. Find the average speed for the whole journey.

Let the distance between P and Q be x km.

Total distance travelled = $2x$ km. Total time $=\frac{x}{15}+\frac{x}{10}$ hour.

Average speed $=\frac{2x}{\frac{x}{15}+\frac{x}{10}}$ km/h

$=\frac{2}{\frac{2+3}{30}}$ = 12 km/h

I6 $[x]$ denotes the greatest integer less than or equal to x. For example, $[3] = 3$, $[5.7] = 5$.

If $\left[\sqrt[5]{1}\right]+\left[\sqrt[5]{2}\right]+\cdots+\left[\sqrt[5]{n}\right]=n+14$, find n.

Reference 1991 HI13

$\left\lfloor\sqrt[5]{1}\right\rfloor=1$, $\left\lfloor\sqrt[5]{2}\right\rfloor=1$, … , $\left\lfloor\sqrt[5]{31}\right\rfloor=1$; $\left\lfloor\sqrt[5]{32}\right\rfloor=2$, … , $\left\lfloor\sqrt[5]{242}\right\rfloor=2$; $\left\lfloor\sqrt[5]{243}\right\rfloor=3$

If $n \le 31$, $\left\lfloor\sqrt[5]{1}\right\rfloor+\left\lfloor\sqrt[5]{2}\right\rfloor+\cdots+\left\lfloor\sqrt[5]{n}\right\rfloor=n$

If $32 \le n \le 242$, $\left\lfloor\sqrt[5]{1}\right\rfloor+\left\lfloor\sqrt[5]{2}\right\rfloor+\cdots+\left\lfloor\sqrt[5]{n}\right\rfloor=31+2(n-31)=2n-31$

$2n - 31 = n + 14$

$\Rightarrow n = 45$

I7 A boy tries to find the area of a parallelogram by multiplying together the lengths of two adjacent sides. His answer is $\sqrt{2}$ times the correct area. If the acute angle of the parallelogram is $x°$, find x. **(Reference: 1991 FSG.3-4)**

Let the lengths of two adjacent sides be a and b, where the angle between a and b is $x°$.

$ab=\sqrt{2}ab\sin x°$

$\sin x°=\dfrac{1}{\sqrt{2}}$

$x = 45$

I8 If the points A (–8, 6), B (–2, 1) and C (4, c) are collinear, find c.

Reference: 1984 FSG.4, 1984 FG7.3, 1986 FG6.2, 1987 FG7.4

$m_{CB} = m_{BA}$

$\dfrac{c-1}{4+2}=\dfrac{1-6}{-2+8}$

$c = -4$

I9 The graphs of $x^2 + y = 8$ and $x + y = 8$ meet at two points. If the distance between these two points is $\sqrt{d}$, find d.

From (1), $y = 8 - x^2$ …… (3)

From (2), $y = 8 - x$ …… (4)

(3) = (4): $8 - x = 8 - x^2$

$x = 0$ or 1

When $x = 0$, $y = 8$; when $x = 1$, $y = 7$

Distance between the points (0, 8) and (1, 7) $=\sqrt{1^2+(7-8)^2}=\sqrt{2}$

$d = 2$

I10 The sines of the three angles of a triangle are in the ratio 3 : 4 : 5. If A is the smallest interior angle of the triangle and $\tan A=\dfrac{x}{16}$, find x.

Reference: 1990 HI6

By Sine rule, $a : b : c = \sin A : \sin B : \sin C = 3 : 4 : 5$

Let $a = 3k$, $b = 4k$, $c = 5k$.

$a^2 + b^2 = (3k)^2 + (4k)^2 = (5k)^2 = c^2$

$\therefore \angle C = 90°$ (converse, Pythagoras' theorem)

$\tan A=\dfrac{a}{b}=\dfrac{3}{4}=\dfrac{12}{16}$

$\Rightarrow x = 12$

I11 Two dice are thrown.

Find the probability that the sum of the two numbers shown is greater than 7.

Reference: 2002 HG7

$P(\text{ sum} > 7) = \text{P(sum} = 8 \text{ or } 9 \text{ or } 10 \text{ or } 11 \text{ or } 12)$

$$= \frac{5}{36}+\frac{4}{36}+\frac{3}{36}+\frac{2}{36}+\frac{1}{36}$$

$$= \frac{15}{36}=\frac{5}{12}$$

Method 2 $P(7)=\frac{6}{36}=\frac{1}{6}$

$P(2, 3, 4, 5, 6) = P(8, 9, 10, 11, 12)$

$P(2, 3, 4, 5, 6) + P(7) + P(8, 9, 10, 11, 12) = 1$

$2P(8, 9, 10, 11, 12) = 1 - P(7) = \frac{5}{6}$

$P(8, 9, 10, 11, 12) = \frac{5}{12}$

I12 F is a function defined by $F(x)=\begin{cases}2x+1, & \text{if } x\le 3\\ 3x^2, & \text{if } x>3\end{cases}$. Find $F(F(3))$.

$F(3) = 2\times3 + 1 = 7$

$F(F(3)) = F(7) = 3\times7^2 = 147$

I13 If $\begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=ax+by+cz$ and $\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}14\\ y\\ 2\end{pmatrix}=26$, find y. **(Reference: 1986 FI3.4)**

$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}14\\ y\\ 2\end{pmatrix} = 14 + 2y + 6 = 26 \Rightarrow y = 3$

I14 If $\frac{1}{B}=\frac{\sin 37^\circ \sin 45^\circ \cos 60^\circ \sin 60^\circ}{\cos 30^\circ \cos 45^\circ \cos 53^\circ}$, find B.

Reference: 1990 HI14

$$\frac{1}{B}=\frac{\sin 37^\circ \sin 45^\circ \cos 60^\circ \sin 60^\circ}{\cos 30^\circ \cos 45^\circ \cos 53^\circ}$$

$$=\frac{\sin 37^\circ \sin 45^\circ \cos 60^\circ \sin 60^\circ}{\sin 60^\circ \sin 45^\circ \sin 37^\circ} = \cos 60^\circ = \frac{1}{2}$$

$B = 2$

I15 If $x + y = -4$, $y + z = 5$ and $z + x = 7$, find the value of xyz.

Reference: 1986 FG10.1, 1990 HI7

$(1) + (2) - (3)$: $2y = -6 \Rightarrow y = -3$

$(1) + (3) - (2)$: $2x = -2 \Rightarrow x = -1$

$(2) + (3) - (1)$: $2z = 16 \Rightarrow z = 8$

$xyz = 24$

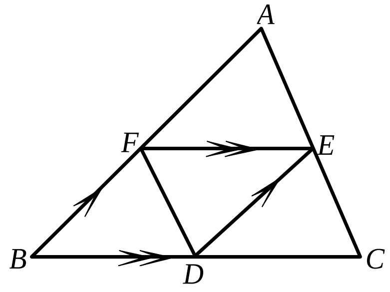
I16 α, β are the roots of the equation $x^2 - 10x + c = 0$. If $\alpha\beta = -11$ and $\alpha > \beta$, find the value of $\alpha - \beta$.

$\alpha + \beta = 10$

$\alpha - \beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= \sqrt{10^2 - 4(-11)} = 12$

I17 In figure 1, FE // BC and ED // AB. If $AF : FB = 3 : 2$, find the ratio area of ΔDEF : area of ΔABC. **Reference: 1990 HG8**



(Figure 1)

$BDEF$ is a parallelogram formed by 2 pairs of parallel lines

$\Delta DEF \cong \Delta FBD$ (A.S.A.)

Let $S_{\Delta DEF} = x = S_{\Delta FBD}$ (where S stands for the area)

$\Delta AEF \sim \Delta ACB$ ($\because$ FE // BC, equiangular)

$$\frac{S_{\Delta AEF}}{S_{\Delta ACB}} = \left(\frac{3}{2+3}\right)^2 = \frac{9}{25} \quad \ldots\ldots (1)$$

$\therefore$ $AE : EC = AF : FB = 3 : 2$ (theorem of equal ratio)

$\because$ DE // AB

$\therefore$ $AE : EC = BD : DC = 3 : 2$ (theorem of equal ratio)

$\Delta CDE \sim \Delta CBA$ ($\because$ DE // BA, equiangular)

$$\frac{S_{\Delta CDE}}{S_{\Delta CBA}} = \left(\frac{2}{2+3}\right)^2 = \frac{4}{25} \quad \ldots\ldots (2)$$

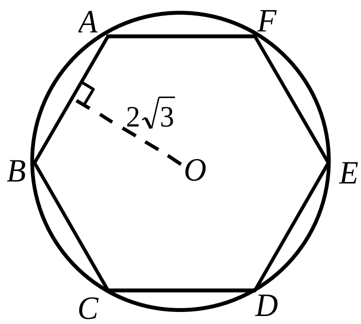
Compare (1) and (2) $S_{\Delta AEF} = 9k$, $S_{\Delta CDE} = 4k$, $S_{\Delta ABC} = 25k$

$9k + 4k + x + x = 25k$

$x = 6k$

$\Rightarrow$ area of ΔDEF : area of $\Delta ABC = 6 : 25$

I18 In figure 2, a regular hexagon $ABCDEF$ is inscribed in a circle centred at O. If the distance of O from AB is $2\sqrt{3}$ and p is the perimeter of the hexagon, find p.



(Figure 2)

Let H be the foot of perpendiculars drawn from O onto AB.

$\angle AOB = 360° \div 6 = 60°$ ($\angle$s at a point)

$\angle AOH = 30°$

$AH = OH \tan 30° = 2\sqrt{3} \times \frac{1}{\sqrt{3}} = 2$

$\Rightarrow AB = 4$

Perimeter = $6\times 4 = 24$

I19 In figure 3, $ABCD$ and $ACDE$ are cyclic quadrilaterals. Find the value of $x + y$.

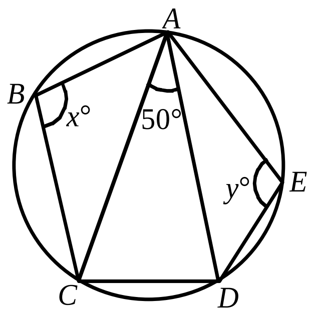
Reference: 1992 FI2.3

$\angle ADC = 180° - x°$ (opp. $\angle$s cyclic quad.)

$\angle ACD = 180° - y°$ (opp. $\angle$s cyclic quad.)

$180° - y° + 180° - x° + 50° = 180°$ ($\angle$s sum of Δ)

$x + y = 230$



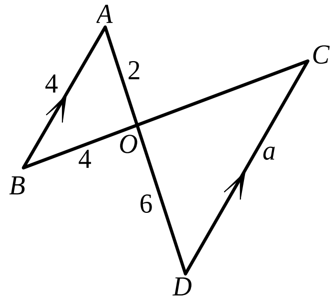
(Figure 3)

I20 Find the value of a in figure 4.

$\Delta AOB \sim \Delta DOC$ (equiangular)

$$\frac{a}{4} = \frac{6}{2}$$

$a = 12$



(Figure 4)

Group Events

G1 Given a and b are distinct real numbers satisfying $a^2 = 5a + 10$ and $b^2 = 5b + 10$.

Find the value of $\frac{1}{a^2}+\frac{1}{b^2}$. (**Reference: 1991 HI14**)

a and b are the roots of $x^2 = 5x + 10$; i.e. $x^2 - 5x - 10 = 0$

$a + b = 5$; $ab = -10$

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{a^2+b^2}{a^2b^2}=\frac{(a+b)^2-2ab}{(ab)^2}=\frac{5^2-2(-10)}{(-10)^2}=\frac{9}{20}$$

G2 An interior angle of an n-sided convex polygon is $x°$ while the sum of other interior angles is $800°$. Find the value of n. **(1990 FG10.3-4, 1992 HG3, 2002 FI3.4, 2013 HI6)**

$800 = 180\times4 + 80$

$800 + x = 180(n - 2)$ $\angle$s sum of polygon

$\because 0 < x < 180$

$\therefore 800 + x = 180\times5 = 180(n - 2)$

$n = 7$

G3 It is known that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n.

Find the value of $21^2 + 22^2 + \ldots + 30^2$.

Reference: 1993 HI6

$21^2 + 22^2 + \ldots + 30^2 = 1^2 + 2^2 + \ldots + 30^2 - (1^2 + 2^2 + \ldots + 20^2)$

$$=\frac{1}{6}\cdot30\cdot31\cdot61-\frac{1}{6}\cdot20\cdot21\cdot41 = 9455 - 2870 = 6585$$

G4 One of the positive integral solutions of the equation $19x + 88y = 1988$ is given by $(100, 1)$. Find another positive integral solution. **(Reference: 1991 HG8)**

The line has a slope of $-\frac{19}{88}=\frac{y_2-y_1}{x_2-x_1}$

Given that $(100, 1)$ is a solution.

$$-\frac{19}{88}=\frac{y_2-1}{x_2-100}$$

Let $y_2 - 1 = -19t$; $x_2 - 100 = 88t$, where t is an integer.

$y_2 = 1 - 19t$, $x_2 = 100 + 88t$

For positive integral solution of (x_2, y_2), $1 - 19t > 0$ and $100 + 88t > 0$

$$-\frac{25}{22}<t<\frac{1}{19}$$

$\because$ t is an integer $\therefore$ $t = 0$ or -1

When $t = -1$, $x_2 = 12$, $y_2 = 20 \Rightarrow$ another positive integral solution is $(12, 20)$.

G5 The line joining $A(2, 3)$ and $B(17, 23)$ meets the line $2x - y = 7$ at P. Find the value of $\frac{AP}{PB}$.

Reference: 1990 HG3

Equation of AB: $\frac{y-3}{x-2}=\frac{23-3}{17-2}\Rightarrow 3y = 4x + 1$ …… (2)

From (1): $y = 2x - 7$ …… (3)

Sub. (3) into (2): $3(2x - 7) = 4x + 1 \Rightarrow x = 11$

Sub. $x = 11$ into (3): $y = 2(11) - 7 = 15$

The point of intersection is $P(11, 15)$.

$$\frac{AP}{PB}=\frac{11-2}{17-11}=\frac{3}{2}$$

G6 Find the remainder when 7^{2047} is divided by 100.

Reference: 2002 HG4

The question is equivalent to find the last 2 digits of 7^{2047}.

$7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$

The last 2 digits repeats for every multiples of 4.

$7^{2047} = 7^{4\times 511+3}$

The last 2 digits is 43.

G7 If $\log_2[\log_3(\log_7 x)] = \log_3[\log_7(\log_2 y)] = \log_7[\log_2(\log_3 z)] = 0$, find the value of $x + y + z$.

$\log_2[\log_3(\log_7 x)] = 0$

$\Rightarrow \log_3(\log_7 x) = 1$

$\Rightarrow \log_7 x = 3$

$\Rightarrow x = 7^3 = 343$

$\log_3[\log_7(\log_2 y)] = 0$

$\Rightarrow \log_7(\log_2 y) = 1$

$\Rightarrow \log_2 y = 7$

$\Rightarrow y = 2^7 = 128$

$\log_7[\log_2(\log_3 z)] = 0$

$\Rightarrow \log_2(\log_3 z) = 1$

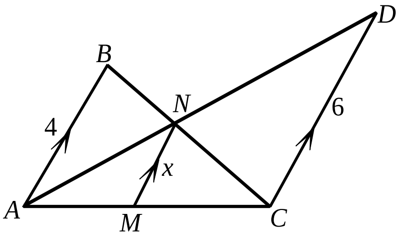
$\Rightarrow \log_3 z = 2$

$\Rightarrow z = 3^2 = 9$

$x + y + z = 343 + 128 + 9 = 480$

G8 In figure 1, $AB \parallel MN \parallel CD$. If $AB = 4$, $CD = 6$ and $MN = x$, find the value of x.

Reference: 1985 FI2.4, 1990 FG6.4



(Figure 1)

$\Delta AMN \sim \Delta ACD$ (equiangular)

$\frac{x}{6} = \frac{AM}{AC}$ … (1) (ratio of sides, ~Δ's)

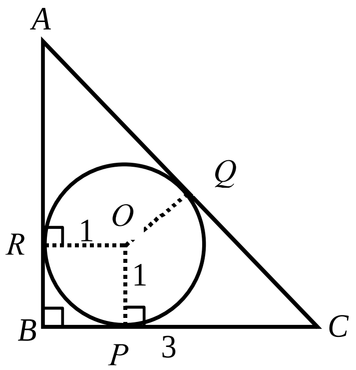
$\Delta CMN \sim \Delta CAB$ (equiangular)

$\frac{x}{4} = \frac{MC}{AC}$ … (2) (ratio of sides, ~Δ's)

(1) + (2): $\frac{x}{6} + \frac{x}{4} = \frac{AM + MC}{AC} = 1 \Rightarrow x = \frac{12}{5}$

G9 In figure 2, $\angle B = 90°$, $BC = 3$ and the radius of the inscribed circle of ΔABC is 1. Find the length of AC.

Reference: 1985 FG9.2



Let the centre be O. Suppose the circle touches BC at P, AC at Q and AB at R respectively. Let $AB = c$ and $AC = b$.

$OP \perp BC$, $OQ \perp AC$, $OR \perp AB$ (tangent ⊥ radius)

$OPBR$ is a rectangle $\Rightarrow$ $OPBR$ is a square.

$BP = BR = 1$ (opp. sides of rectangle)

$CP = 3 - 1 = 2$

$CQ = CP = 2$ (tangent from ext. point)

Let $AR = AQ = t$ (tangent from ext. point)

$3^2 + (1 + t)^2 = (2 + t)^2$ (Pythagoras' theorem)

$9 + 1 + 2t + t^2 = 4 + 4t + t^2$

$6 = 2t$

$t = 3 \Rightarrow AC = 2 + 3 = 5$

G10 In the attached division (see figure 3), the dividend in (*a*) is divisible by the divisor in line (*b*). Find the dividend in line (*a*). (Each asterisk * is an integer from 0 to 9.)

$$\begin{array}{r}
* \; 8 \; * \\
(b) \text{ .... } * * * \,\overline{)\, * * * * * *} \text{ .... } (a) \\
\underline{* * * *} \quad\; \\
* * * \;\; \\
\underline{* * *} \;\; \\
* * * * \\
\underline{* * * *}
\end{array}$$

Relabel the '*' as shown.

Let $a = (a_1 8 a_3)_x$,

$b = (b_1 b_2 b_3)_x$

$c = (c_1 c_2 c_3 c_4 c_5 c_6)_x$

$d = (d_1 d_2 d_3 d_4)_x$

$e = (e_1 e_2 e_3)_x$

$f = (f_1 f_2 f_3)_x$

$g = (g_1 g_2 g_3 g_4)_x$

$$\begin{array}{r}
a_1 \; 8 \; a_3 \\
(b) \text{ .... } b_1 b_2 b_3 \,\overline{)\, c_1 c_2 c_3 c_4 c_5 c_6} \text{ .... } (a) \\
\underline{d_1 d_2 d_3 d_4} \quad\; \\
e_1 e_2 e_3 \;\; \\
\underline{f_1 f_2 f_3} \;\; \\
g_1 g_2 g_3 g_4 \\
\underline{g_1 g_2 g_3 g_4}
\end{array}$$

$\because$ $8b = e$ and $a_1 \times b = d$, $a_3 \times b = g$ 4-digit numbers

$\therefore$ $a_1 = 9$ and $a_3 = 9$ and $d = g$

$d = 9b > 1000$ and $f = 8b < 999$

$112 \le b \le 124$ …… (1)

$b_1 = 1$, $d_1 = 1$

$b_2 = 1$ or 2, $f_1 = 8$ or 9, $d_2 = 0$ or 1

$c_1 = 1$

$e_1 - f_1 = 1$

$\therefore$ $f_1 = 8$ and $e_1 = 9$

$8b = (8 f_2 f_3)_x < 900$

$b < 112.5$ …… (2)

Combine (1) and (2)

$b = 112$

$c = 989 \times 112 = 110768$

$$\begin{array}{r}
9 \; 8 \; 9 \\
(b) \text{ .... } 1\, b_2 b_3 \,\overline{)\, 1\, c_2 c_3 c_4 c_5 c_6} \text{ .... } (a) \\
\underline{1\, d_2 d_3 d_4} \quad\; \\
e_1 e_2 e_3 \;\; \\
\underline{f_1 f_2 f_3} \;\; \\
1\, d_2 d_3 d_4 \\
\underline{1\, d_2 d_3 d_4}
\end{array}$$