Individual Events

SI	***a***	16	**I1**	***a***	72	**I2**	***A***	2	**I3**	***a***	3	**I4**	***a***	$*\frac{2}{3}$ see the remark	**IS**	***a***	20
	b	$\frac{3}{2}$		***b***	9		***B***	2		***b***	18		***b***	2		***b***	$-\frac{1}{3}$
	c	36		***c***	*31 see the remark		***C***	200		***c***	21		***c***	1		***c***	–4
	d	42		***d***	1984		***D***	100		***d***	$-\frac{1}{9}$		***d***	$\frac{3}{4}$		***d***	–24

Group Events

SG	***A***	$-\frac{17}{13}$	**G1**		44.5	**G2**	***p***	6	**G3**		2	**G4**	***m***	*3 see the remark	**GS**	***n***	4
	B	13		$\frac{x}{y}$	*2 see the remark		***m***	1		***n***	66		**Minimum *y***	$4\sqrt{2}$			86975
	C	46			$\frac{7}{2}$		***k***	15		***x***	$\frac{8}{\sqrt{3}}=\frac{8}{3}\sqrt{3}$		***A*+*B*+*C***	6			$*\frac{1}{3}$ see the remark
	D	30		***n***	10			478			8		***EC***	4			4

Sample Individual Event (2009 Final Individual Event 2)

SI.1 Let $[x]$ be the largest integer not greater than x. If $a=\left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right]+16$, find the value of a.

$$0<\sqrt{3}-\sqrt{2}=\frac{1}{\sqrt{3}+\sqrt{2}}<1$$

$$0<\left(\sqrt{3}-\sqrt{2}\right)^{2009}<1$$

$$a=\left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right]+16=0+16=16$$

SI.2 In the coordinate plane, if the area of the triangle formed by the x-axis, y-axis and the line $3x + ay = 12$ is b square units, find the value of b.

$3x + 16y = 12$

x-intercept = 4, y-intercept $=\frac{3}{4}$

Area $= b = \frac{1}{2}\cdot 4\cdot\frac{3}{4}=\frac{3}{2}$

SI.3 Given that $x-\frac{1}{x}=2b$ and $x^3-\frac{1}{x^3}=c$, find the value of c.

$$x-\frac{1}{x}=3\Rightarrow x^2-2+\frac{1}{x^2}=9\Rightarrow x^2+\frac{1}{x^2}=11$$

$$c=x^3-\frac{1}{x^3}=\left(x-\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right)=3\times(11+1)=36$$

SI.4 In Figure 1, $\alpha = c$, $\beta = 43$, $\gamma = 59$ and $\omega = d$, find the value of d.

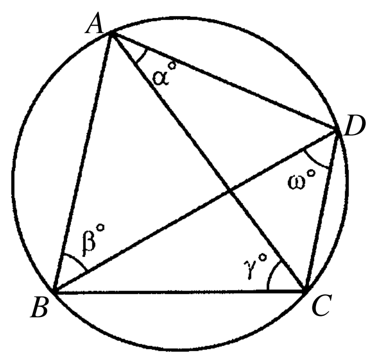
$\angle BAC = \omega°$ ($\angle$s in the same seg.)

$\angle ACD = \beta°$ ($\angle$s in the same seg.)

$\angle BAD + \angle BCD = 180°$ (opp. $\angle$s cyclic quad.)

$c + d + 43 + 59 = 180$

$d = 180 - 43 - 59 - 36 = 42$ ($\because c = 36$)



Individual Event 1

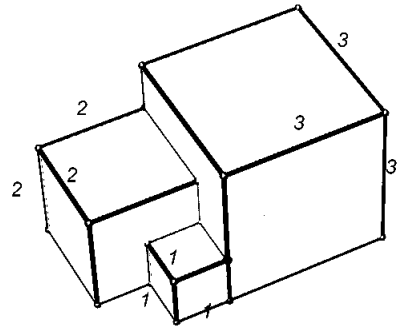
I1.1 Three cubes with volumes 1, 8, 27 are glued together at their faces. If a is the smallest possible surface area of the resulting polyhedron, find the value of a.

The lengths of the 3 cubes are 1, 2 and 3 with surface areas 6, 24 and 54 respectively.

As shown in the figure, if the three cubes are glued together, the faces stuck together are 2×2, 2×2, 1×1, 1×1, 1×1 and 1×1.

The smallest possible surface area is 6 + 24 + 54 – 4 – 4 – 4.

$a = 72$



I1.2 Given that $f(x) = -x^2 + 10x + 9$, and $2 \le x \le \frac{a}{9}$. If b is the difference of the maximum and minimum values of f , find the value of b.

$f(x) = -x^2 + 10x + 9 = -(x - 5)^2 + 34$ for $2 \le x \le 8$

Maximum = f(5) = 34; minimum = f(2) = f(8) = 25

$b = 34 - 25 = 9$

I1.3 Given that p and q are real numbers with $pq = b$ and $p^2q + q^2p + p + q = 70$.

If $c = p^2 + q^2$, find the value of c.

$pq = 9$ … (1), and $pq(p + q) + (p + q) = 70 \Rightarrow (pq + 1)(p + q) = 70$ … (2)

Sub. (1) into (2): $10(p + q) = 70 \Rightarrow p + q = 7$ … (3)

$c = p^2 + q^2 = (p + q)^2 - 2pq = 7^2 - 2\times 9 = 31$

Remark: The original question is

Given that p and q are **integers** with $pq = b$ and $p^2q + q^2p + p + q = 70$.

However, $pq = 9$, $p + q = 7$, which give no integral solution.

I1.4 There are c rows in a concert hall and each succeeding row has two more seats than the previous row. If the middle row has 64 seats, how many seats (d) does the concert have ?

There are altogether 31 rows. The 16th row is the middle row, which has 64 seats.

The 15th row has 64 – 2 = 62 seats.

The 14th row has 64 – 2×2 = 60 seats.

....

The 1st row has 64 – 2×15 = 34 seats.

Total number of seats $= \frac{n}{2}[2a + (n-1)d] = \frac{31}{2}[2\cdot 34 + (31-1)\cdot 2] = 1984$

Method 2 Total number of seats = (1st row + 31st row) + (2nd row + 30th row) + … + 16th row

= (64 + 64) + (64 + 64) + … + 64 (31 terms) = 1984

Individual Event 2

I2.1 If a, p, q are primes with $a < p$ and $a + p = q$, find the value of a.

'2' is the only prime number which is even. All other primes are odd numbers.

If both a and p are odd, then q must be even, which means that either q is not a prime or $q = 2$. Both cases lead to contradiction.

$\therefore a = 2$

I2.2 If b and h are positive integers with $b < h$ and $b^2 + h^2 = b(a + h) + ah$, find the value of b.

Reference: 2000 FI5.2, 2001 FI2.1, 2011 FI3.1, 2013 HG1

$b < h$ and $b^2 + h^2 = b(2 + h) + 2h$

$b^2 + h^2 = 2b + bh + 2h$

$(b + h)^2 - 2(b + h) = 3bh < 3\left(\dfrac{b+h}{2}\right)^2$ (G.M. < A.M., given that $b < h$)

Let $t = b + h$, $t^2 - 2t < \dfrac{3t^2}{4} \Rightarrow t^2 - 8t < 0$, where t is a positive integer

$t - 8 < 0 \Rightarrow t < 8 \Rightarrow b + h < 8 \Rightarrow 2b < b + h < 8 \Rightarrow 2b < 8 \Rightarrow b < 4$

$b = 1, 2$ or 3

When $b = 1$, $1 + h^2 = 2 + h + 2h \Rightarrow h^2 - 3h - 1 = 0 \Rightarrow h$ is not an integer, rejected

When $b = 2$, $4 + h^2 = 4 + 2h + 2h \Rightarrow h^2 - 4h = 0 \Rightarrow h = 4$

When $b = 3$, $9 + h^2 = 6 + 3h + 2h \Rightarrow h^2 - 5h + 3 = 0 \Rightarrow h$ is not an integer, rejected

$\therefore b = 2$

Method 2 $h^2 - (b + 2)h + b^2 - 2b = 0$

$\Delta = (b + 2)^2 - 4(b^2 - 2b) = m^2$, where m is an integer

$-3b^2 + 12b + 4 = m^2$

$-3(b - 2)^2 + 16 = m^2$

$m^2 + 3(b - 2)^2 = 16$, both b and m are integers

$m = 0$, no integral solution for b

$m = 1$, no integral solution for b

$m = 2$, $b = 4$, $h^2 - 6h + 8 = 0 \Rightarrow h = 2$ or $h = 4$, contradicting $b < h$, reject

$m = 3$, no integral solution for b

$m = 4$, $b = 2$, $h = 4$ (accept)

I2.3 In a $(2b + 1)\times(2b + 1)$ checkerboard, two squares not lying in the same row are randomly chosen. If c is the number of combinations of different pairs of squares chosen, find the value of c.

There are 25 squares. First we count the number of ways of choosing two squares lying in the same column or the same row: ${}_5C_2\times5 + {}_5C_2\times5 = 100$

$\therefore c = {}_{25}C_2 - 100 = 200$

Method 2 Label the two squares as A, B.

For each chosen square A (out of 25 squares), B has 16 possible positions.

$\therefore$ There are $25\times16 = 400$ combinations.

However, A, B may be inter-changed. $\therefore$ We have double counted. $c = 200$.

I2.4 Given that $f(x) = c\left|\dfrac{1}{x} - \left\lfloor \dfrac{1}{x} + \dfrac{1}{2} \right\rfloor\right|$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to the real number x. If d is the maximum value of $f(x)$, find the value of d.

Let $\dfrac{1}{x} + \dfrac{1}{2} = a + b$, where a is an integer and $0 \le b < 1$.

$\left\lfloor \dfrac{1}{x} + \dfrac{1}{2} \right\rfloor = a \Rightarrow -\left\lfloor \dfrac{1}{x} + \dfrac{1}{2} \right\rfloor = -a \Rightarrow \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} + \dfrac{1}{2} \right\rfloor = a + b - \dfrac{1}{2} - a = b - \dfrac{1}{2}$

$0 \le b < 1 \Rightarrow -\dfrac{1}{2} \le b - \dfrac{1}{2} < \dfrac{1}{2} \Rightarrow \left|b - \dfrac{1}{2}\right| \le \dfrac{1}{2}$ (equality holds when $b = 0$)

$f(x) = 200\times\left|\dfrac{1}{x} - \left\lfloor \dfrac{1}{x} + \dfrac{1}{2} \right\rfloor\right| = 200\times\left|b - \dfrac{1}{2}\right| \le 200\times\dfrac{1}{2} = 100$

$d = 100$ (You may verify the result by putting $x = 2$.)

Individual Event 3

I3.1 If a is the number of distinct prime factors of 15147, find the value of a.

$15147 = 3^4 \times 11 \times 17$

$a = 3$

I3.2 If $x+\frac{1}{x}=a$ and $x^3+\frac{1}{x^3}=b$, find the value of b.

Reference: 1983 FG7.3, 1996 FI1.2, 1998 FG5.2

$x+\frac{1}{x}=3 \Rightarrow x^2+2+\frac{1}{x^2}=9 \Rightarrow x^2+\frac{1}{x^2}=7$

$b=x^3+\frac{1}{x^3}$

$=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$

$= 3\times(7-1)$

$= 18$

I3.3 Let $f(x)=\begin{cases} x+5 & \text{if } x \text{ is an odd integer} \\ \frac{x}{2} & \text{if } x \text{ is an even integer}\end{cases}$.

If c is an odd integer and $f(f(f(c))) = b$, find the least value of c.

$f(c) = c + 5$, which is even

$f(f(c)) = \frac{c+5}{2}$

If $\frac{c+5}{2}$ is odd, $f(f(f(c))) = 18$

$\Rightarrow \frac{c+5}{2} + 5 = 18$

$\Rightarrow c + 5 = 26$

$\Rightarrow c = 21$

If $\frac{c+5}{2}$ is even, $f(f(f(c))) = 18$

$\Rightarrow \frac{c+5}{4} = 18$

$\Rightarrow c + 5 = 72$

$\Rightarrow c = 67$

The least value of $c = 21$.

I3.4 Let $f\left(\frac{x}{3}\right)=x^2+x+1$. If d is the sum of all x for which $f(3x) = c$, find the value of d.

$f(x) = (3x)^2 + 3x + 1$

$= 9x^2 + 3x + 1$

$f(3x) = 81x^2 + 9x + 1$

$f(3x) = 21$

$\Rightarrow 81x^2 + 9x + 1 = 21$

$\Rightarrow 81x^2 + 9x - 20 = 0$

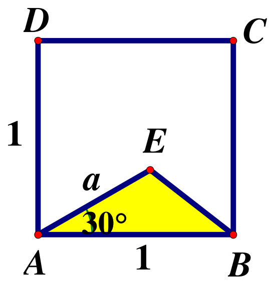
$\Rightarrow d =$ sum of roots

$=-\frac{9}{81}$

$=-\frac{1}{9}$

Individual Event 4

I4.1 In Figure 1, $ABCD$ is a square, E is a point and $\angle EAB = 30°$. If the area of $ABCD$ is six times that of ΔABE, then the ratio of $AE : AB = a : 1$. Find the value of a.



Let $AB = AD = 1$, $AE = a$, let the altitude of ΔABE from E to AB be h.

area of $ABCD$ is six times that of $\Delta ABE \Leftrightarrow 1^2 = 6\times\frac{1}{2}\cdot 1\cdot h$

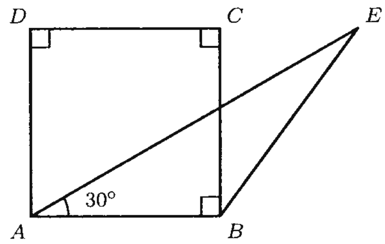
$h = \frac{1}{3}$, $a = \frac{h}{\sin 30^\circ} = 2h = \frac{2}{3}$

Remark: The original questions is

在圖一中，$ABCD$ 為一正方形，E 為此正方形以外的一點及 $\angle EAB = 30°$。若 $ABCD$ 的面積是 ΔABE 的面積的六倍，則 $AE : AB = a : 1$。求 a 的值。

In Figure 1, $ABCD$ is a square, E is a point outside the square and $\angle EAB = 30°$. If the area of $ABCD$ is six times that of ΔABE, then the ratio of $AE : AB = a : 1$. Find the value of a.

In fact, E must lie **inside** the square.



I4.2 Given that $b = \frac{\log 8^a + \log 27^a + \log 125^a}{\log 9 + \log 25 + \log 2 - \log 15}$, find the value of b.

$$b = \frac{a\log 8 + a\log 27 + a\log 125}{2\log 3 + 2\log 5 + \log 2 - \log 3 - \log 5} = \frac{a(3\log 2 + 3\log 3 + 3\log 5)}{\log 3 + \log 5 + \log 2} = \frac{2}{3}\times 3 = 2$$

I4.3 Let c be the remainder of $1^3 + 2^3 + \ldots + 2009^3 + 2010^3$ divided by b^2, find the value of c.

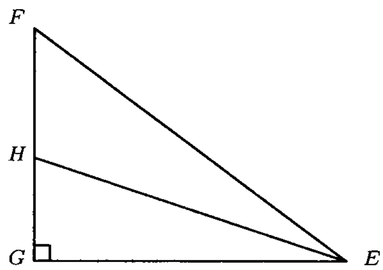
Use the formula $1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$,

$1^3 + 2^3 + \ldots + 2009^3 + 2010^3 = \frac{1}{4}\cdot 2010^2\cdot 2011^2 = 1005^2\cdot 2011^2 = (4\times 251 + 1)^2\cdot(4\times 502 + 3)^2$

$= (4p + 1)\cdot(4q + 1) = 4r + 1$, where p, q, r are positive integers.

$\therefore$ When it is divided by 2^2, the remainder is 1, $c = 1$.

I4.4 In Figure 2, EFG is a right-angled triangle. Given that H is a point on FG, such that $GH : HF = 4 : 5$ and $\angle GEH = \angle FEH$. If $EG = c$ and $FG = d$, find the value of d.



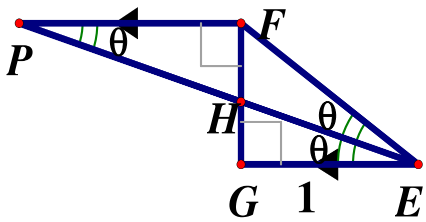
Let $\angle FEH = \theta = \angle FEH$, $GH = 4k$, $FH = 5k$, $EG = 1$

In ΔEGH, $\tan\theta = 4k$ …… (1)

In ΔEFG, $\tan 2\theta = 9k$ …… (2)

Sub. (1) into (2): $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \Leftrightarrow 9k = \frac{2\cdot 4k}{1-(4k)^2}$

$9(1 - 16k^2) = 8 \Leftrightarrow k = \frac{1}{12} \Leftrightarrow d = FG = 5k + 4k = 9k = \frac{3}{4}$



Method 2 Let $\angle GEH = \theta = \angle FEH$

From P, draw a line segment FP parallel to GE, which intersects with EH produced at P.

$\angle FPH = \theta$	(alt. $\angle$s, PF // GE)
$\Delta FPH \sim \Delta GEH$	(equiangular)
$\frac{GH}{HF} = \frac{GE}{PF} \Rightarrow \frac{4}{5} = \frac{1}{PF} \Rightarrow PF = 1.25$	(ratio of sides, ~Δ's)
$FE = PF = 1.25$	(sides opp. eq. $\angle$s)
$d = FG = \sqrt{1.25^2 - 1} = 0.75$	(Pythagoras' Theorem)

Individual Spare

IS.1 Given that $a=\sqrt{(19.19)^2+(39.19)^2-(38.38)(39.19)}$. Find the value of m.

Let $x = 19.19$, $y = 39.19$ then $x < y$ and $38.38 = 2x$

$a=\sqrt{x^2+y^2-2xy}=\sqrt{(y-x)^2}=y-x=39.19-19.19=20$

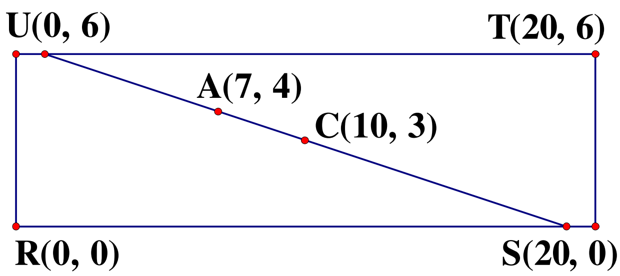
IS.2 Given four points $R(0, 0)$, $S(a, 0)$, $T(a, 6)$ and $U(0, 6)$. If the line $y = b(x - 7) + 4$ cuts the quadrilateral $RSTU$ into two halves of equal area, find the value of b.

$y = b(x - 7) + 4$ represents a family of straight lines with slope b which always pass through a fixed point $A(7, 4)$.

$R(0, 0)$, $S(20, 0)$, $T(20, 6)$ and $U(0, 6)$.

$RSTU$ is a rectangle whose base is parallel to x-axis with centre at $C(10, 3)$.

The line joining AC bisect the area of the rectangle. $b=\frac{3-4}{10-7}=-\frac{1}{3}$



IS.3 Given that c is the minimum value of $f(x)=\frac{x^2-2x+\frac{1}{b}}{2x^2+2x+1}$. Find the value of c.

Let $y=f(x)=\frac{x^2-2x-3}{2x^2+2x+1}$

$2yx^2 + 2yx + y = x^2 - 2x - 3$

$(2y - 1)x^2 + 2(y + 1)x + (y + 3) = 0$

For any values of x, the above quadratic equation has real solution.

$\therefore \Delta \geq 0$

$(y + 1)^2 - (2y - 1)(y + 3) \geq 0$

$y^2 + 2y + 1 - (2y^2 + 5y - 3) \geq 0$

$-y^2 - 3y + 4 \geq 0$

$y^2 + 3y - 4 \leq 0$

$(y + 4)(y - 1) \leq 0$

$-4 \leq y \leq 1$

c = the minimum of $y = -4$

IS.4 Given that $f(x) = px^6 + qx^4 + 3x - \sqrt{2}$, and p, q are non-zero real numbers.

If $d = f(c) - f(-c)$, find the value of d.

$d = (pc^6 + qc^4 + 3c - \sqrt{2}) - (pc^6 + qc^4 - 3c - \sqrt{2}) = 6c = 6(-4) = -24$

Sample Group Event (2009 Final Group Event 2)

SG.1 Given $\tan\theta = \frac{5}{12}$, where $180° \le \theta \le 270°$. If $A = \cos\theta + \sin\theta$, find the value of A.

$\cos\theta = -\frac{12}{13}$, $\sin\theta = -\frac{5}{13}$

$A = -\frac{12}{13} - \frac{5}{13} = -\frac{17}{13}$

SG.2 Let $[x]$ be the largest integer not greater than x. If $B = \left[10+\sqrt{10+\sqrt{10+\sqrt{10+\cdots}}}\right]$, find the value of B.

Reference 2007 FG2.2 … $x \ge 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}}$ …

Let $y = \sqrt{10+\sqrt{10+\sqrt{10+\cdots}}}$

$y^2 = 10 + \sqrt{10+\sqrt{10+\sqrt{10+\cdots}}} = 10 + y$

$y^2 - y - 10 = 0$

$y = \frac{1+\sqrt{41}}{2}$ or $\frac{1-\sqrt{41}}{2}$ (rejected)

$6 < \sqrt{41} < 7 \Rightarrow \frac{7}{2} < \frac{1+\sqrt{41}}{2} < 4$

$13.5 < 10+\sqrt{10+\sqrt{10+\sqrt{10+\cdots}}} < 14$

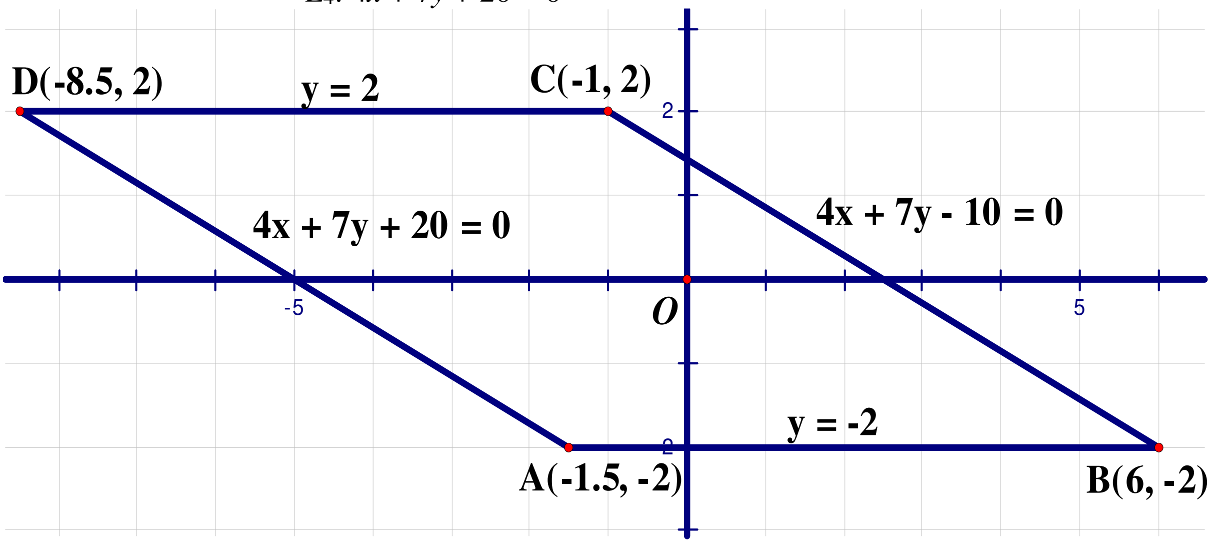
$B = 13$

SG.3 Let $a \oplus b = ab + 10$. If $C = (1\oplus 2) \oplus 3$, find the value of C.

$1\oplus 2 = 2 + 10 = 12$; $C = 12\oplus 3 = 36 + 10 = 46$

SG.4 In the coordinate plane, the area of the region bounded by the following lines is D square units, find the value of D.

L_1: $y - 2 = 0$
L_2: $y + 2 = 0$
L_3: $4x + 7y - 10 = 0$
L_4: $4x + 7y + 20 = 0$



It is easy to show that the bounded region is a parallelogram $ABCD$ with vertices $A(-1.5, -2)$, $B(6, -2)$, $C(-1, 2)$, $C(-8.5, 2)$.

The area $D = |6 - (-1.5)|\times|2 - (-2)| = 7.5\times 4 = 30$

Group Event 1

G1.1 Find the value of $\sin^2 1° + \sin^2 2° + \dots + \sin^2 89°$.

Reference 2012 HG9

$\sin^2 1° + \sin^2 89° = \sin^2 1° + \cos^2 1° = 1$

$\sin^2 2° + \sin^2 88° = \sin^2 2° + \cos^2 2° = 1$

....

$\sin^2 44° + \sin^2 46° = \sin^2 44° + \cos^2 44° = 1$

$\sin^2 1° + \sin^2 2° + \dots + \sin^2 89° = (\sin^2 1° + \sin^2 89°) + \dots + (\sin^2 44° + \sin^2 46°) + \sin^2 45°$

$= 44.5$

G1.2 Let x, y and z be positive numbers. Given that $\frac{x+z}{2z-x} = \frac{z+2y}{2x-z} = \frac{x}{y}$. Find the value of $\frac{x}{y}$.

Reference 1998 FG1.2

It is equivalent to $\frac{x+z}{2z-x} = \frac{x}{y}$ … (1) and $\frac{z+2y}{2x-z} = \frac{x}{y}$ … (2)

From (2), $yz + 2y^2 = 2x^2 - xz \Rightarrow (x + y)z = 2(x^2 - y^2)$

$\Rightarrow x + y = 0$ (rejected, $x > 0$ and $y > 0$) or $z = 2(x - y)$ … (3)

From (1): $xy + yz = 2xz - x^2 \Rightarrow (2x - y)z = x^2 + xy$ … (4)

Sub. (4) into (5): $2(x - y)(2x - y) = x^2 + xy$

$2(2x^2 - 3xy + y^2) = x^2 + xy$

$3x^2 - 7xy + 2y^2 = 0$

$(3x - y)(x - 2y) = 0 \Rightarrow \frac{x}{y} = \frac{1}{3}$ or 2

When $y = 3x$, sub. into (3): $z = 2(x - 3x) = -4x$ (rejected, $x > 0$ and $z > 0$)

$\therefore \frac{x}{y} = 2$

Method 2 $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k \Rightarrow a = bk,\ c = dk,\ e = fk \Rightarrow \frac{a+c+e}{b+d+f} = \frac{bk+dk+fk}{b+d+f} = k$

$k = \frac{x}{y} = \frac{a+c+e}{b+d+f} = \frac{(x+z)+(z+2y)+x}{(2z-x)+(2x-z)+y} = \frac{2x+2y+2z}{x+y+z} = 2$ ($\because x + y + z > 0$)

Remark: The original question is: Given that $\frac{x+z}{2z-x} = \frac{z+2y}{2x-z} = \frac{x}{y}$. Find the value of $\frac{x}{y}$.

The question has **more than one solution**.

G1.3 Find the sum of all real roots x of the equation $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$.

Let $a = 2^x - 4$, $b = 4^x - 2$, $a + b = 4^x + 2^x - 6$, the equation is equivalent to $a^3 + b^3 = (a + b)^3$

$(a + b)(a^2 - ab + b^2) = (a + b)^3$

$a^2 - ab + b^2 = a^2 + 2ab + b^2$ or $a + b = 0$

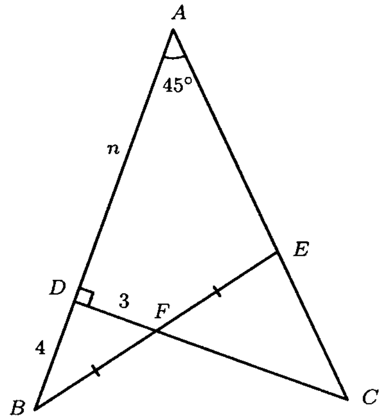
$3ab = 0$ or $4^x + 2^x - 6 = 0$

$2^x = 4$ or $4^x = 2$ or $(2^x - 2)(2^x + 3) = 0$

$x = 2$, $\frac{1}{2}$ or 1

Sum of all real roots = 3.5

G1.4 In Figure 1, if $AB \perp CD$，F is the midpoint of BE, $\angle A = 45°$, $DF = 3$, $BD = 4$ and $AD = n$, find the value of n.



Let G be the foot of perpendicular drawn from E onto CF.

$\angle BFD = \angle CFE$	(vert. opp. $\angle$s)
$BF = 5$	(Pythagoras' Theorem)
$EF = 5$	(Given F is the midpoint)
$\angle BDF = 90° = \angle EGF$	(by construction)
$\Delta BDF \cong \Delta EGF$	(A.A.S.)
$\therefore FG = DF = 3$	(corr. sides, $\cong \Delta$'s)
$EG = 4$	(corr. sides, $\cong \Delta$'s)
$\angle ACD = 45°$	($\angle$s sum of ΔACD)

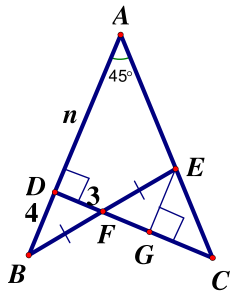
$\therefore \Delta ACD$ is a right-angled isosceles triangle.

$CD = AD = n$ (sides opp. equal angle)

$EG \perp CD \Rightarrow EG \parallel AD \Rightarrow \Delta CEG \sim \Delta CAD$ (equiangular)

$\Rightarrow CG = EG = 4$ (ratio of sides, ~Δ's)

$n = AD = CD = 3 + 3 + 4 = 10$



Method 2

Draw $EG \parallel CD$, which intersects AB at G.

$GD = BD = 4$ ($BF = FE$ and $FD \parallel EG$, intercept theorem)

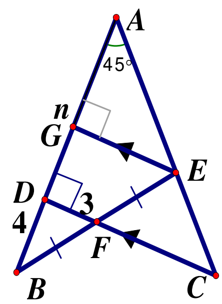
$GE = 2DF = 6$ (mid-points theorem)

$\angle AGE = 90°$ (corr. $\angle$s, $EG \parallel CD$)

ΔAGE is a right-angled isosceles triangle.

$\therefore AG = GE = 6$ (sides opp. eq. $\angle$s)

$n = AG + GD = 6 + 4 = 10$



Group Event 2

G2.1 If $p = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^9 - 2^{10} + 2^{11}$, find the value of p.

$$p = 2 + 2^{11} - (2^2 + 2^3 + 2^4 + \ldots + 2^9 + 2^{10}) = 2 + 2^{11} - \frac{2^2(2^9-1)}{2-1} = 2 + 2^{11} - 2^{11} + 2^2 = 6$$

G2.2 Given that x, y, z are three distinct real numbers.

Reference: 2008 FG2.4, 2017 FG2.1

If $x+\frac{1}{y} = y+\frac{1}{z} = z+\frac{1}{x}$ and $m = x^2y^2z^2$, find the value of m.

Let $x+\frac{1}{y} = k$ … (1), $y+\frac{1}{z} = k$ … (2), $z+\frac{1}{x} = k$ … (3)

From (1), $x = k - \frac{1}{y} = \frac{ky-1}{y} \Rightarrow \frac{1}{x} = \frac{y}{ky-1}$

Sub. into (3): $z + \frac{y}{ky-1} = k \Rightarrow z = \frac{(k^2-1)y-k}{ky-1}$

Sub. into (2): $y + \frac{ky-1}{(k^2-1)y-k} = k \Rightarrow (k^2-1)y^2 - ky + ky - 1 = k(k^2-1)y - k^2$

$\Rightarrow (k^2-1)y^2 - k(k^2-1)y + (k^2-1) = 0$

$\Rightarrow k^2 - 1 = 0$ or $y^2 - ky + 1 = 0$

If $k^2 - 1 \neq 0$, then $y^2 - ky + 1 = 0 \Rightarrow y = \frac{k \pm \sqrt{k^2-4}}{2}$

$\because$ The system is symmetric for x, y, z

$\therefore$ x or $z = \frac{k \pm \sqrt{k^2-4}}{2}$, this contradict to the fact that x, y, z are distinct.

$\therefore y^2 - ky + 1 \neq 0$

$\Rightarrow k^2 = 1 \Rightarrow k = 1$ or -1

When $k = 1$, $x = k - \frac{1}{y} = 1 - \frac{1}{y} = \frac{y-1}{y}$; $z = \frac{(k^2-1)y-k}{ky-1} = \frac{-1}{y-1}$

$xyz = \frac{y-1}{y} \cdot y \cdot \frac{-1}{y-1} = -1$

When $k = -1$, $x = k - \frac{1}{y} = -1 - \frac{1}{y} = -\frac{y+1}{y}$; $z = \frac{(k^2-1)y-k}{ky-1} = \frac{-1}{y+1}$

$xyz = -\frac{y+1}{y} \cdot y \cdot \frac{-1}{y+1} = 1$

$\therefore m = x^2y^2z^2 = 1$

Method 2 $x+\frac{1}{y} = y+\frac{1}{z} \Leftrightarrow x - y = \frac{1}{z} - \frac{1}{y} \Leftrightarrow x - y = \frac{y-z}{yz}$ …… (1)

$y+\frac{1}{z} = z+\frac{1}{x} \Leftrightarrow y - z = \frac{1}{x} - \frac{1}{z} \Leftrightarrow y - z = \frac{z-x}{xz}$ …… (2)

$x+\frac{1}{y} = z+\frac{1}{x} \Leftrightarrow z - x = \frac{1}{y} - \frac{1}{x} \Leftrightarrow z - x = \frac{x-y}{xy}$ …… (3)

(1)×(2)×(3): $(x-y)(y-z)(z-x) = \frac{y-z}{yz} \cdot \frac{z-x}{xz} \cdot \frac{x-y}{xy}$

$\Leftrightarrow 1 = \frac{1}{x^2y^2z^2}$

$\Leftrightarrow m = x^2y^2z^2 = 1$

G2.3 Given that x is a positive real number and $x \cdot 3^x = 3^{18}$. If k is a positive integer and $k < x < k + 1$, find the value of k.

The equation is equivalent to $3^{18-x} - x = 0$.

Let $f(x) = 3^{18-x} - x$.

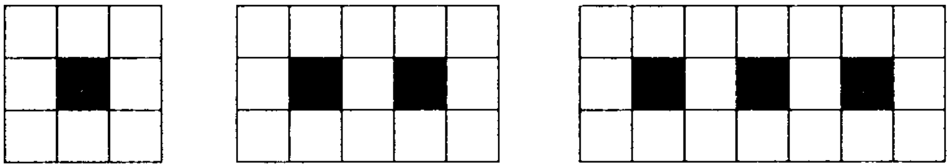
Clearly $f(x)$ is a continuous function.

$f(15) = 3^3 - 15 = 12 > 0$, $f(16) = 3^2 - 16 = -7 < 0$

By intermediate value theorem (or Bolzano's theorem), we can find a real root $15 < x < 16$.

$k = 15$

G2.4 Figure 1 shows the sequence of figures that are made of squares of white and black. Find the number of white squares in the 95th figure.



1st figure 2nd figure 3rd figure

1st figure = 8 white squares; 2nd figure = 13 squares; 3rd figure = 18 squares

$T(1) = 8$, $T(2) = 8 + 5$, $T(3) = 8 + 5\times2$, … , $T(95) = 8 + 5\times(95 - 1) = 478$

Group Event 3

G3.1 Find the smallest prime factor of $101^{303} + 301^{101}$.

Both 101^{303} and 301^{101} are odd integers

$\therefore$ $101^{303} + 301^{101}$ is even

The smallest prime factor is 2.

G3.2 Let n be the integral part of $\dfrac{1}{\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{2009}}$. Find the value of n.

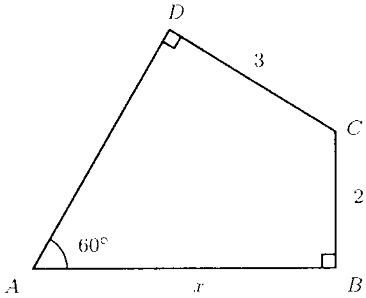
$$\frac{1}{2009}+\cdots+\frac{1}{2009}<\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{2009}<\frac{1}{1980}+\cdots+\frac{1}{1980}\quad \text{(30 terms)}$$

$$\frac{30}{2009}<\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{2009}<\frac{30}{1980}$$

$$66=\frac{1980}{30}<\frac{1}{\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{2009}}<\frac{2009}{30}<\frac{2010}{30}=67$$

n = integral part = 66

G3.3 In Figure 1, $\angle A = 60°$, $\angle B = \angle D = 90°$. $BC = 2$, $CD = 3$ and $AB = x$, find the value of x.



$AC^2 = x^2 + 4$ (Pythagoras' Theorem on ΔABC)

$AD^2 = AC^2 - 3^2$ (Pythagoras' Theorem on ΔACD)

$= x^2 - 5$

$BD^2 = x^2 + (x^2 - 5) - 2x\sqrt{x^2-5}\cos 60°$ (cosine rule on ΔABD)

$BD^2 = 2^2 + 3^2 - 2\cdot 2\cdot 3\cos 120°$ (cosine rule on ΔBCD)

$\therefore$ $2x^2 - 5 - x\sqrt{x^2-5} = 13 + 6$

$x\sqrt{x^2-5} = 2x^2 - 24$

$x^2(x^2 - 5) = 4x^4 - 96x^2 + 576$

$3x^4 - 91x^2 + 576 = 0$

$(x^2 - 9)(3x^2 - 64) = 0$

$x = 3$ or $\dfrac{8}{\sqrt{3}}$

When $x = 3$, $AD = \sqrt{x^2-5} = 2$

$\tan\angle BAC = \dfrac{2}{3}$, $\tan\angle CAD = \dfrac{3}{2} = \tan(90° - \angle BAC)$

$\angle BAD = 90° \neq 60°$ $\therefore$ reject $x = 3$

When $x = \dfrac{8}{\sqrt{3}}$, $AD = \sqrt{x^2-5} = \dfrac{7}{\sqrt{3}}$

$\tan\angle BAC = \dfrac{\sqrt{3}}{4}$, $\tan\angle CAD = \dfrac{3\sqrt{3}}{7}$

$$\tan(\angle BAC + \angle CAD) = \frac{\frac{\sqrt{3}}{4}+\frac{3\sqrt{3}}{7}}{1-\frac{\sqrt{3}}{4}\cdot\frac{3\sqrt{3}}{7}} = \frac{19\sqrt{3}}{19} = \sqrt{3} = \tan 60°$$

$\therefore$ $x = \dfrac{8}{\sqrt{3}} = \dfrac{8}{3}\sqrt{3}$

Method 2

$BD^2 = 2^2 + 3^2 - 2\cdot 2\cdot 3\cos 120°$ (cosine rule on ΔBCD)

$BD = \sqrt{19}$

$\angle ABC + \angle ADC = 180°$

A, B, C, D are concyclic (opp. $\angle$s supp.)

$AC = \sqrt{x^2+4}$ = diameter = $2R$

(converse, $\angle$ in semi-circle, R = radius)

$\dfrac{BD}{\sin 60°} = 2R$ (Sine rule on ΔABD)

$\dfrac{\sqrt{19}}{\frac{\sqrt{3}}{2}} = \sqrt{x^2+4}$

$76 = 3x^2 + 12$

$x = \dfrac{8}{\sqrt{3}} = \dfrac{8}{3}\sqrt{3}$

G3.4 Given that the function f satisfies $f(2 + x) = f(2 - x)$ for every real number x and that $f(x) = 0$ has exactly four distinct real roots. Find the sum of these four distinct real roots.

Reference: 1994 FI3.4

Let two of these distinct roots be $2 + \alpha$, $2 + \beta$, where $\alpha \neq \beta$ and $\alpha, \beta \geq 0$.

$f(2 + x) = f(2 - x) \Leftrightarrow f(2 + \alpha) = f(2 - \alpha) = 0$; $f(2 + \beta) = f(2 - \beta) = 0$

If $\alpha = 0$ and $\beta \neq 0 \Rightarrow$ there are only three real roots $2, 2 + \beta, 2 - \beta$ contradiction, rejected.

$\therefore$ $\alpha \neq 0$ and $\beta \neq 0 \Leftrightarrow$ The four roots are $2 + \alpha, 2 - \alpha, 2 + \beta, 2 - \beta$.

Sum of roots $= 2 + \alpha + 2 - \alpha + 2 + \beta + 2 - \beta = 8$

Group Event 4

G4.1 Let a be an integer and $a \neq 1$. Given that the equation $(a-1)x^2 - mx + a = 0$ has two roots which are positive integers. Find the value of m.

Let the 2 roots be α, β.

$$\alpha\beta = \frac{a}{a-1} = \frac{1}{a-1} + 1$$

α, β are positive integers $\Rightarrow \dfrac{1}{a-1}$ is a positive integer

$\Rightarrow a - 1 = 1$ or -1

$\Rightarrow a = 2$ or 0 (rejected)

Put $a = 2$ into the original equation: $x^2 - mx + 2 = 0$

$\alpha\beta = 2 \Rightarrow \alpha = 2, \beta = 1$ or $\alpha = 1, \beta = 2$

$m = \alpha + \beta = 3$

Remark: The original question is

Given that the equation $(a-1)x^2 - mx + a = 0$ has two roots which are positive integers. Find the value of m.

If $a = 1$, then it is not a quadratic equation, it cannot have 2 positive integral roots.

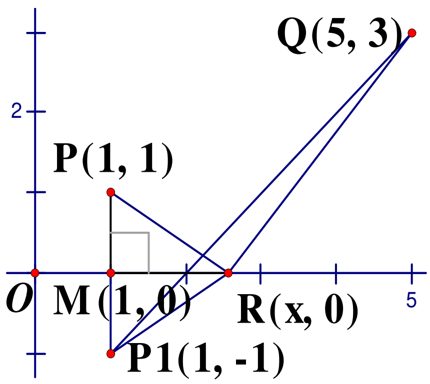
If a is any real number $\neq 1$, the equality $\alpha\beta = \dfrac{1}{a-1} + 1$ could not implies $a = 2$

e.g. $a = 1.5 \Rightarrow \alpha\beta = \dfrac{1}{1.5-1} + 1 = 3 \Rightarrow \alpha = 1, \beta = 3 \Rightarrow \alpha + \beta = 4 = \dfrac{m}{1.5-1} \Rightarrow m = 2$

e.g. $a = 1.1 \Rightarrow \alpha\beta = \dfrac{1}{1.1-1} + 1 = 11 \Rightarrow \alpha = 1, \beta = 11 \Rightarrow \alpha + \beta = 12 = \dfrac{m}{1.1-1} \Rightarrow m = 1.2$

There are infinitely many possible values of m !!!

G4.2 Given that x is a real number and $y = \sqrt{x^2 - 2x + 2} + \sqrt{x^2 - 10x + 34}$. Find the minimum value of y. **Reference: 2015 HI9, 2021 P1Q12**



Consider the following problem:

Let $P(1, 1)$ and $Q(5, 3)$ be two points. $R(x, 0)$ is a variable point on x-axis.

To find the minimum sum of distances $PR + RQ$.

Let y = sum of distances $= \sqrt{(x-1)^2 + 1} + \sqrt{(x-5)^2 + 9}$

If we reflect $P(1, 1)$ along x-axis to $P_1(1, -1)$, $M(1, 0)$ is the foot of perpendicular,

then $\Delta PMR \cong \Delta P_1MR$ (S.A.S.)

$y = PR + RQ = P_1R + RQ \geq P_1Q$ (triangle inequality)

$y \geq \sqrt{(5-1)^2 + (3+1)^2} = 4\sqrt{2}$

G4.3 Given that A, B, C are positive integers with their greatest common divisor equal to 1.

If A, B, C satisfy $A \log_{500} 5 + B \log_{500} 2 = C$, find the value of $A + B + C$.

$\log_{500} 5^A + \log_{500} 2^B = \log_{500} 500^C \Rightarrow \log_{500} 5^A \cdot 2^B = \log_{500} 500^C \Rightarrow 5^A \cdot 2^B = 5^{3C} \cdot 2^{2C}$

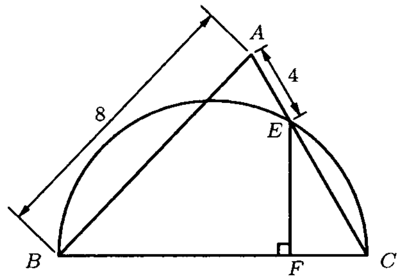
$A = 3C$, $B = 2C$ (unique factorization theorem)

$\because A, B, C$ are relatively prime.

$\therefore C = 1, A = 3, B = 2$

$A + B + C = 6$

G4.4 In figure 1, BEC is a semicircle and F is a point on the diameter BC. Given that $BF : FC = 3 : 1$, $AB = 8$ and $AE = 4$. Find the length of EC.



Join BE. It is easy to show that $\Delta BEF \sim \Delta ECF$ (equiangular)

Let $BF = 3k$, $CF = k$

$EF : 3k = k : EF$ (ratio of sides, ~Δ's)

$EF = \sqrt{3}\,k$

$BE^2 = BF^2 + EF^2 = 9k^2 + 3k^2$ (Pythagoras' Theorem on ΔBEF) $\Rightarrow BE = \sqrt{12}\,k$

$\angle BEC = 90°$ ($\angle$ in semi-circle)

$BE^2 + AE^2 = AB^2$ (Pythagoras' Theorem on ΔABE)

$12k^2 + 16 = 64 \Rightarrow k = 2$

$EC^2 = CF^2 + EF^2 = 2^2 + 3\times 2^2$ (Pythagoras' Theorem on ΔCEF)

$EC = 4$

Method 2

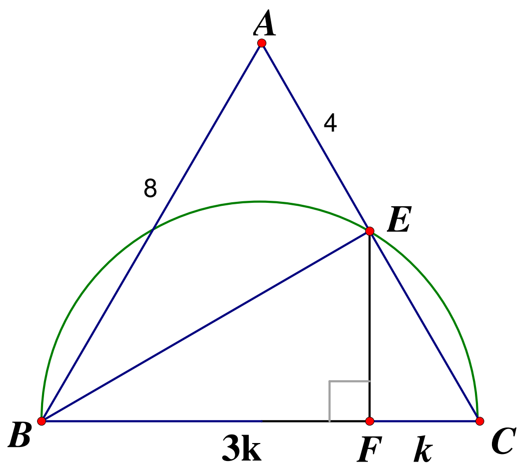
$\angle BEC = 90°$ ($\angle$ in semi-circle)

$\angle BEA = 90°$ (adj. $\angle$s on st. line)

$\cos\angle BAE = \frac{AE}{AB} = \frac{4}{8} = \frac{1}{2}$

$\angle BAE = 60°$

Let $BF = 3k$, $CF = k$, $\angle ECB = \theta$.



$\angle CEF = 90° - \theta$ ($\angle$s sum of ΔCEF)

$\angle CBE = 90° - \theta$ ($\angle$s sum of ΔBCE)

$\angle BEF = \theta$ ($\angle$s sum of ΔBEF)

$\Delta CEF \sim \Delta EBF$ (equiangular)

$\frac{CF}{EF} = \frac{EF}{BF}$ (corr. sides, ~Δs)

$EF^2 = k\cdot 3k$

$EF = \sqrt{3}\,k$

$\tan\angle ECF = \frac{EF}{CF} = \frac{\sqrt{3}k}{k} = \sqrt{3}$

$\angle ECF = 60°$

$\Delta ABE \cong \Delta CBE$ (A.A.S.)

$EC = AE = 4$ (corr. sides, $\cong$ Δs)

Group Spare

GS.1 Given that n is a positive integer. If $n^2 + 5n + 13$ is a perfect square, find the value of n.

$n^2 + 5n + 13 = n^2 + 5n + 2.5^2 - 2.5^2 + 13 = (n + 2.5)^2 + 6.75 = m^2$, where m is an integer

$m^2 - (n + 2.5)^2 = 6.75 \Rightarrow (m + n + 2.5)(m - n - 2.5) = 6.75 \Rightarrow (2m + 2n + 5)(2n - 2m - 5) = 27$

$$\begin{cases} 2m+2n+5=27 \\ 2m-2n-5=1 \end{cases} \text{ or } \begin{cases} 2m+2n+5=9 \\ 2m-2n-5=3 \end{cases}$$

$n = 4$ or $n = -1$ (rejected, $\because n > 0$)

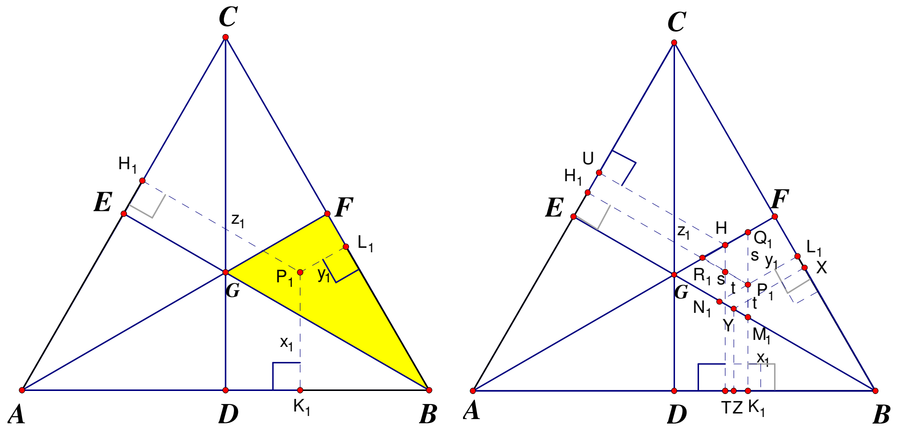
GS.2 Given that $1^3 + 2^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$. Find the value of $11^3 + 12^3 + \ldots + 24^3$.

$$11^3 + 12^3 + \ldots + 24^3 = 1^3 + 2^3 + \ldots + 24^3 - (1^3 + 2^3 + \ldots + 10^3)$$

$$= \frac{1}{4}\cdot 24^2\cdot 25^2 - \frac{1}{4}\cdot 10^2\cdot 11^2 = \frac{1}{4}\cdot 6^2\cdot 100^2 - \frac{1}{4}\cdot 10^2\cdot 121$$

$$= \frac{1}{4}\cdot(360000-12100) = \frac{1}{4}\cdot 347900 = 86975$$

GS.3 If P is an arbitrary point in the interior of the equilateral triangle ABC, find the probability that the area of ΔABP is greater than **each** of the areas of ΔACP and ΔBCP.



D, E, F be the mid-points of AB, AC and BC respectively.

The medians CD, BE and AF are concurrent at the centroid G.

It is easy to see that ΔCEG, ΔCFG, ΔAEG, ΔADG, ΔBDG, ΔBFG are congruent triangles having the same areas.

P is any point inside the triangle $\Rightarrow P$ lies on or inside one of these six congruent triangles.

As shown in the diagram, P_1 lies inside ΔBFG. Let the feet of perpendiculars from P_1 to AB, BC, CA be K_1, L_1, H_1 with lengths x_1, y_1 and z_1 respectively.

P_1H_1 and AF meet at R_1, P_1K_1 intersects BE at M_1, and AF at Q_1, L_1P_1 produced meet BE at N_1

By the properties on parallel lines, we can easily prove that $\Delta P_1M_1N_1$ and $\Delta P_1Q_1R_1$ are equilateral triangles. Let $P_1M_1 = P_1N_1 = N_1M_1 = t$, $P_1Q_1 = P_1R_1 = Q_1R_1 = s$

Let H and Y be the midpoints of Q_1R_1 and N_1M_1 respectively. $R_1H = 0.5s$, $YM_1 = 0.5t$

Let U and T be the feet of perpendiculars from H to AC and AB respectively.

Let X and Z be the feet of perpendiculars from Y to BC and AB respectively.

$UH = z_1 - s + 0.5s \cos 60° = z_1 - 0.75s$, $YZ = x_1 - t + 0.5t \cos 60° = x_1 - 0.75t$

$HT = x_1 + 0.75s$, $YX = y_1 + 0.75t$

It is easy to show that $\Delta AHU \cong \Delta AHT$, $\Delta BYX \cong \Delta BYZ$ (A.A.S.)

$UH = HT$ and $YZ = YX$ (corr. sides, $\cong$ Δ's) $\Rightarrow z_1 - 0.75s = x_1 + 0.75s$, $x_1 - 0.75t = y_1 + 0.75t$

$z_1 = x_1 + 1.5s$, $x_1 = y_1 + 1.5t \Rightarrow z_1 > x_1 > y_1$

$\therefore \frac{1}{2}AC\cdot z_1 > \frac{1}{2}AB\cdot x_1 > \frac{1}{2}BC\cdot y_1 \Rightarrow$ area of ΔACP_1 > area of ΔABP_1 > area of ΔBCP_1

If P_2 lies inside ΔBDG, using a similar method, we can easily prove that

area of ΔACP_2 > area of ΔBCP_2 > area of ΔABP_2.

If P_3 lies inside ΔADG, then area of ΔBCP_3 > area of ΔACP_3 > area of ΔABP_3.

If P_4 lies inside ΔAEG, then

area of ΔBCP_4 > area of ΔABP_4 > area of ΔACP_4.

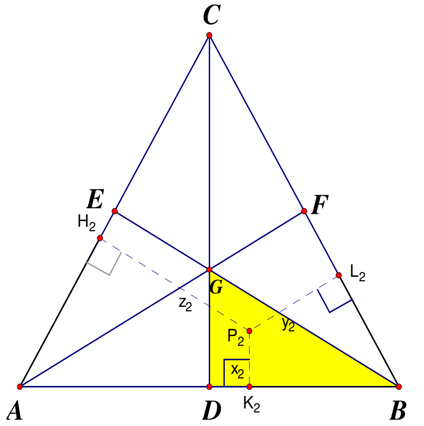
If P_5 lies inside ΔCEG, then

area of ΔABP_5 > area of ΔBCP_5 > area of ΔACP_5.

If P_6 lies inside ΔCFG, then

area of ΔABP_6 > area of ΔACP_6 > area of ΔBCP_6

In order that the area of ΔABP is greater than **each** of the areas of ΔACP and ΔBCP, P must lie inside ΔCEG or ΔCFG



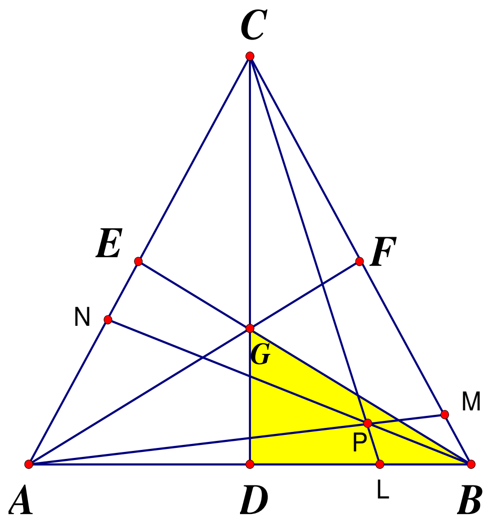
Required probability

$$=\frac{\text{Area of }\Delta CEG+\text{area of }\Delta CFG}{\text{Area of }\Delta ABC}=\frac{2}{6}=\frac{1}{3}$$

Method 2 Suppose P lies inside ΔBDG.

Produce AP, BP, CP to intersect BC, CA, AB at L, M, N respectively.

Let $S_{\Delta XYZ}$ denotes the area of ΔXYZ.



$$\frac{S_{\Delta APB}}{S_{\Delta APC}}=\frac{S_{\Delta ABM}-S_{\Delta BPM}}{S_{\Delta ACM}-S_{\Delta CPM}}$$

$$=\frac{\frac{1}{2}BM\cdot AM\sin AMB-\frac{1}{2}BM\cdot PM\sin AMB}{\frac{1}{2}CM\cdot AM\sin AMC-\frac{1}{2}CM\cdot PM\sin AMC}$$

$$=\frac{\frac{1}{2}BM\sin AMB\cdot(AM-PM)}{\frac{1}{2}CM\sin AMC\cdot(AM-PM)}=\frac{BM\sin AMB}{CM\sin AMC}$$

$$=\frac{BM}{MC}\quad(\because \sin AMB=\sin(180°-AMC)=\sin AMC)$$

$$\frac{S_{\Delta APB}}{S_{\Delta BPC}}=\frac{S_{\Delta ABN}-S_{\Delta APN}}{S_{\Delta BCN}-S_{\Delta CPN}}$$

$$=\frac{\frac{1}{2}BN\cdot AN\sin ANB-\frac{1}{2}AN\cdot PN\sin ANB}{\frac{1}{2}BN\cdot NC\sin BNC-\frac{1}{2}NC\cdot PN\sin BNC}$$

$$=\frac{\frac{1}{2}AN\sin ANB\cdot(BN-PN)}{\frac{1}{2}NC\sin BNC\cdot(BN-PN)}=\frac{AN\sin ANB}{NC\sin BNC}$$

$$=\frac{AN}{NC}\quad(\because \sin ANB=\sin(180°-BNC)=\sin BNC)$$

In order that the area of ΔABP is greater than **each** of the areas of ΔACP and ΔBCP,

$BM > MC$ and $AN > NC$

$\therefore$ P must lie inside ΔCEG or ΔCFG

$$\text{Required probability}=\frac{S_{\Delta CEG}+S_{\Delta CFG}}{S_{\Delta ABC}}=\frac{2}{6}=\frac{1}{3}$$

Remark: The original question is

若 P 是等邊三角形 ABC 內部的隨意一點，求ΔABP 的面積同時大於ΔACP 及ΔBCP 的面積的概率。

If P is an arbitrary point in the interior of the equilateral triangle ABC, find the probability that the area of ΔABP is greater than **both** of the areas of ΔACP and ΔBCP.

There is a slight difference between the Chinese version and the English version.

GS.4 How many positive integers m are there for which the straight line passing through points $A(-m, 0)$ and $B(0, 2)$ and also passes through the point $P(7, k)$, where k is a positive integer?

Let the slope of the variable straight line be a. Then its equation is: $y = ax + 2$

It passes through $A(-m, 0)$ and $P(7, k)$: $\begin{cases}-am+2=0\cdots(1)\\ 7a+2=k\cdots(2)\end{cases}$

$7(1) + m(2)$: $14 + 2m = km \Rightarrow m(k-2) = 14$

$m = 1, k = 16$ or $m = 2, k = 9$ or $m = 7, k = 4$ or $m = 14, k = 3$

Number of positive integral values of m is 4.