Individual Events

I1	A	80	**I2**	A	2	**I3**	s	$\frac{25}{4}$	**I4**	α	$\frac{5}{8}$
	B	7		B	3		t	6		β	4
	C	15		C	$\frac{11}{50}$		u	1		γ	45
	D	$\frac{125}{36}=3\frac{17}{36}$		D	3		v	4		δ	16

Group Events

G1	a	8	**G2**	least A	7056	**G3**	least a	14	**G4**	α	24
	b	6		B	6		e	27		β	7333
	c	24		least Q	1		c	45		γ	$\frac{9}{5}=1.8$
	d	419		W	5		d	3		δ	0.01

Individual Event 1

I1.1 若 A 是 $(x^2+2)^5$ 展開式中 x^4 的係數，求 A 的值。

If A is the coefficient of x^4 in the expansion of $(x^2+2)^5$, determine the value of A .

$A = C_2^5\left(2\right)^3 = 80$

I1.2 若 x 和 y 為正整數，並且滿足以下等式 $\log_{10} x + \log_{10} y = \log_{10}\left(2x - Ay\right) + 1$，而 B 是 (x, y) 所有可能組合的數量，求 B 的值。

If x and y are positive integers that satisfy $\log_{10} x + \log_{10} y = \log_{10}\left(2x - Ay\right) + 1$, and B is the number of possible pairs of (x, y), determine the value of B .

$\log_{10} x + \log_{10} y = \log_{10}\left(2x - 80y\right) + 1$ $\log_{10}\left(xy\right) = \log_{10} 10\left(2x - 80y\right)$ $xy = 20x - 800y$ $20x - xy - 800y + 16000 = 16000$ $x(20 - y) + 800(20 - y) = 16000$ $(x + 800)(20 - y) = 16000$ x 和 y 為正整數，$(x + 800, 20 - y)$ $= (16000, 1), (8000, 2), (4000, 4), (3200, 5), (2000, 8), (1600, 10), (1000, 16)$ 一共有7 組答案 (x, y)。$B = 7$	$\log_{10} x + \log_{10} y = \log_{10}\left(2x - 80y\right) + 1$ $\log_{10}\left(xy\right) = \log_{10} 10\left(2x - 80y\right)$ $xy = 20x - 800y$ $20x - xy - 800y + 16000 = 16000$ $x(20 - y) + 800(20 - y) = 16000$ $(x + 800)(20 - y) = 16000$ For positive integral values $(x + 800, 20 - y)$ $= (16000, 1), (8000, 2), (4000, 4), (3200, 5), (2000, 8), (1600, 10), (1000, 16)$ There are 7 pairs of (x, y). $B = 7$

I1.3 若 $Y = 2^{3(B-1)}$ 並且 C 是 Y 中每個數字之和，求 C 的值。

If $Y = 2^{3(B-1)}$ and C is the sum of the digits of Y , determine the value of C .

$Y = 2^{3(7-1)} = 2^{18} = 262144$

$C = 2 + 6 + 2 + 1 + 4 + 4 = 19$

I1.4 在 ΔXYZ 中，已知 $XY \perp YZ$，$\angle XZY = \theta$，綫段 YZ 和 XZ 的長度分別為 $C - 3$ 和 $C + 5$。若 $D = (\sin\theta + \tan\theta)^2$，求 D 的值。

In ΔXYZ , $XY \perp YZ$, $\angle XZY = \theta$, and the length of YZ and XZ are $C - 3$ and $C + 5$ respectively. If $D = (\sin\theta + \tan\theta)^2$, determine the value of D .

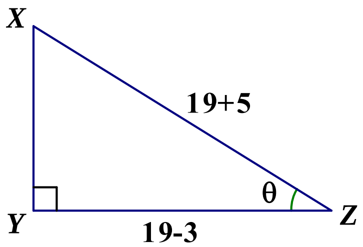
$YZ = 19 - 3 = 16$

$XZ = 19 + 5 = 24$

$XY^2 + 16^2 = 24^2$

$XY = \sqrt{320} = 8\sqrt{5}$

$\sin\theta = \frac{8\sqrt{5}}{24} = \frac{\sqrt{5}}{3}$; $\tan\theta = \frac{8\sqrt{5}}{16} = \frac{\sqrt{5}}{2}$



$$D = \left(\frac{\sqrt{5}}{3} + \frac{\sqrt{5}}{2}\right)^2 = \frac{125}{36} = 3\frac{17}{36}$$

Individual Event 2

I2.1 若 $\sqrt{A}=\sqrt{11+\sqrt{21}}-\sqrt{11-\sqrt{21}}$，求 A 的值。

If $\sqrt{A}=\sqrt{11+\sqrt{21}}-\sqrt{11-\sqrt{21}}$, determine the value of A.

Reference: 2007 FI1.1, 2016 FI1.2

$$A=\left(\sqrt{11+\sqrt{21}}-\sqrt{11-\sqrt{21}}\right)^2$$

$$=11+21-2\left(\sqrt{11+\sqrt{21}}\cdot\sqrt{11-\sqrt{21}}\right)+11-21$$

$$=22-2\sqrt{121-21}=22-2\times 10=2$$

I2.2 若直綫 $y=mx+B$ 經過兩點 $(4, 5)$ 和 $(-A, A)$，求 B 的值。

If the straight line $y=mx+B$ passes through the two points $(4, 5)$ and $(-A, A)$, determine the value of B.

$$m=\frac{5-2}{4+2}=\frac{1}{2}$$

$$B=y-mx=5-\frac{1}{2}\times 4=3$$

I2.3 若 $\cos x+\sin x=\frac{2B}{5}$ 及 $C=(\tan x+\cot x)^{-1}$，求 C 的值。

If $\cos x+\sin x=\frac{2B}{5}$ and $C=(\tan x+\cot x)^{-1}$, determine the value of C.

Reference: 1992 HI20, 1993 HG10, 1995 HI5, 2007 HI7, 2007 FI1.4, 2014 HG3

$$\cos x+\sin x=\frac{2\times 3}{5}=\frac{6}{5}$$

$$(\cos x+\sin x)^2=\frac{36}{25}$$

$$1+2\sin x\cos x=\frac{36}{25}\Rightarrow \sin x\cos x=\frac{11}{50}$$

$$C=(\tan x+\cot x)^{-1}=\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)^{-1}=\left(\frac{\sin^2 x+\cos^2 x}{\cos x\sin x}\right)^{-1}=\sin x\cos x=\frac{11}{50}$$

I2.4 假設 D、x、y 和 z 均為整數，其中 $D>x>y>z$。

若 D、x、y 和 z 滿足等式 $3^D-3^x+3^y-3^z=\frac{1000C+2}{9}$，求 D 的值。

Suppose that D, x, y and z are integers with $D>x>y>z$. If D, x, y and z satisfy the equation $3^D-3^x+3^y-3^z=\frac{1000C+2}{9}$, determine the value of D.

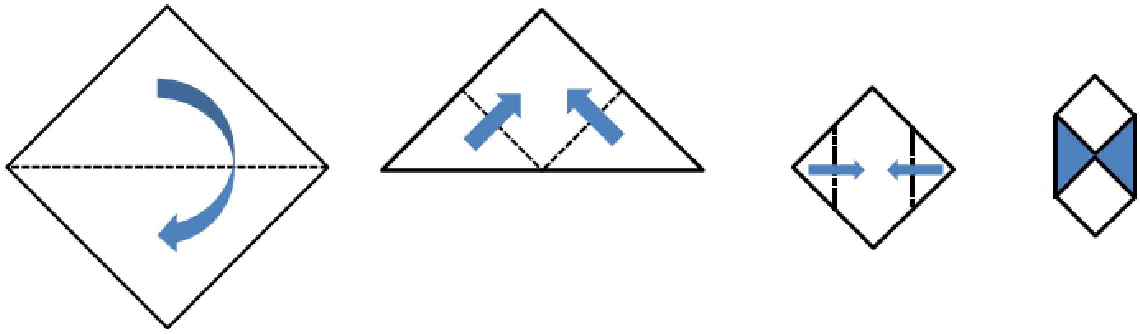
Reference: 2003 FI4.3

$$\frac{1000C+2}{9}=\frac{1000\times\frac{11}{50}+2}{9}=24\frac{2}{3}=27-3+1-\frac{1}{3}$$

$D=3$, $x=1$, $y=0$, $z=-1$

Individual Event 3

I3.1 一張正方形紙的面積為 100 cm^2，按照圖中的虛綫和箭咀的方向對摺。
若圖中的陰影部份為 s cm^2，求 s 的值。
A piece of square paper of area 100 cm^2, is folded in half along the dotted line as shown below. If the area of the shaded region in the last figure is s cm^2, determine the value of s .



由左至右，第二幅圖的面積為 50 cm^2，第三幅圖的面積為 25 cm^2。 陰影面積$=25\times\frac{2}{8}$ cm^2 $=\frac{25}{4}$ cm^2 $s=\frac{25}{4}$	Starting from left to right, the second diagram has an area 50 cm^2, the third diagram has an area 25 cm^2. The shaded area $=25\times\frac{2}{8}$ cm^2 $=\frac{25}{4}$ cm^2 $s=\frac{25}{4}$

I3.2 假設 $s=\frac{A}{B}$ 為最簡分數。若 c 和 d 分別為 A 和 B 的正因數的數量，
求 $t=c+d$ 的值。
Suppose that $s=\frac{A}{B}$ is in simplest form. If c and d are the number of positive factors of A and B respectively, determine the value of $t=c+d$.

$s=\frac{25}{4}$, $A=25$, $B=4$ 25 的正因數是 1、5、25。 4 的正因數是 1、2、4。 $t=3+3=6$	$s=\frac{25}{4}$, $A=25$, $B=4$ Positive factors of 25 are 1, 5, 25 . Positive factors of 4 are 1, 2, 4 . $t=3+3=6$

I3.3 若今天是某一週的第 2 日，以及已知 t^{2019} 日後的當天為該週的第 u 日，求 u 的值。
If today is the 2$^{\text{nd}}$ day of a week, and it is known that t^{2019} days later is the u^{th} day of the week, determine the value of u .

$6^{2019}=(7-1)^{2019}$

$=7^{2019}-2019\times7^{2018}+\cdots+(-1)^{2019}$ (binomial expansion 二項式展開)

$=7m-1$

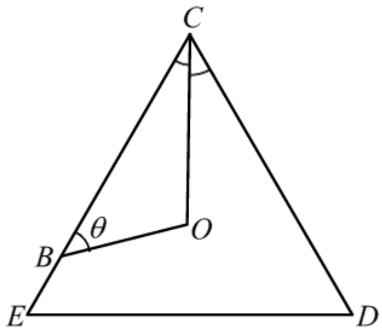
$u=-1+2=1$

I3.4 若 $v_{(u+5)}$ 為 $1231234_{(u+5)}\div123_{(u+5)}$ 的餘數，求 v 的值。
If $v_{(u+5)}$ is the remainder of $1231234_{(u+5)}\div123_{(u+5)}$, determine the value of v .
Reference: 2018 FG1.4
$1231234_6=1230000_6+1230_6+4_6$
$v=4$

Individual Event 4

I4.1 ΔCDE 為一個等邊三角形。點 O 在 ΔCDE 內。若點 B 在 CE 上，$\theta = \angle CBO$，OC 為 $\angle DCE$ 的角平分綫，以及 $OC : OB = 5 : 4$，求 $\alpha = \sin\theta$ 的值。

ΔCDE is an equilateral triangle. Point O is inside ΔCDE. If point B lies on CE, $\theta = \angle CBO$, $\angle DCE$ is bisected by OC, and $OC : OB = 5 : 4$, determine the value of $\alpha = \sin\theta$.



$\angle BCO = \angle DCO = 30°$ 設 $OB = 4k$, $OC = 5k$ 於 ΔBOC 應用正弦定理， $\frac{\sin\theta}{5k} = \frac{\sin 30^\circ}{4k}$ $\alpha = \sin\theta = \frac{5}{8}$	$\angle BCO = \angle DCO = 30°$ Let $OB = 4k$, $OC = 5k$ Apply sine rule on ΔBOC, $\frac{\sin\theta}{5k} = \frac{\sin 30^\circ}{4k}$ $\alpha = \sin\theta = \frac{5}{8}$

I4.2 假設有一函數 $\mathrm{f}(x)$，對於任何整數 x 及任何整數 $y \neq 0$，均滿足 $\mathrm{f}\left(\frac{x}{y}\right) = \mathrm{f}(x) - \mathrm{f}(y)$ 和 $\mathrm{f}(2) = -1$。若 $\beta = \mathrm{f}\left(\frac{\alpha}{10}\right)$，求 β 的值。

Suppose that there exists a function $\mathrm{f}(x)$, defined for all integers x and for all integers $y \neq 0$, such that $\mathrm{f}\left(\frac{x}{y}\right) = \mathrm{f}(x) - \mathrm{f}(y)$ and $\mathrm{f}(2) = -1$. If $\beta = \mathrm{f}\left(\frac{\alpha}{10}\right)$, determine the value of β.

$$\mathrm{f}(2) = \mathrm{f}\left(\frac{2}{1}\right) = \mathrm{f}(2) - \mathrm{f}(1) \Rightarrow \mathrm{f}(1) = 0$$

$$-1 = \mathrm{f}(2) = \mathrm{f}\left(\frac{4}{2}\right) = \mathrm{f}(4) - \mathrm{f}(2) = \mathrm{f}(4) + 1 \Rightarrow \mathrm{f}(4) = -2$$

$$-2 = \mathrm{f}(4) = \mathrm{f}\left(\frac{8}{2}\right) = \mathrm{f}(8) - \mathrm{f}(2) = \mathrm{f}(8) + 1 \Rightarrow \mathrm{f}(8) = -3$$

$$-3 = \mathrm{f}(8) = \mathrm{f}\left(\frac{16}{2}\right) = \mathrm{f}(16) - \mathrm{f}(2) = \mathrm{f}(16) + 1 \Rightarrow \mathrm{f}(16) = -4$$

$$\beta = \mathrm{f}\left(\frac{\alpha}{10}\right) = \mathrm{f}\left(\frac{5}{80}\right) = \mathrm{f}\left(\frac{1}{16}\right) = \mathrm{f}(1) - \mathrm{f}(16) = -\mathrm{f}(16) = 4$$

I4.3 若 $B = \gamma p + 2\gamma(1 - p)$ 以及 $p = \frac{\beta(\gamma - 40)}{100}$，當 B 取最大值時，求 γ 的值。

If $B = \gamma p + 2\gamma(1 - p)$ and $p = \frac{\beta(\gamma - 40)}{100}$, determine the value of γ such that B attains the maximum.

$p = \frac{4(\gamma - 40)}{100} = \frac{\gamma - 40}{25}$

$1 - p = 1 - \frac{\gamma - 40}{25} = \frac{65 - \gamma}{25}$

$B = \gamma \cdot \frac{\gamma - 40}{25} + 2\gamma \cdot \frac{65 - \gamma}{25}$

$= -\frac{1}{25}(\gamma^2 - 90\gamma)$

$= -\frac{1}{25}(\gamma - 45)^2 + 81$

When $\gamma = 45$, B attains the maximum.

I4.4 已知 a、b 和 c 是 γ 的正因數，而且 $a < b < c < \gamma$ 及 $ab = c$。

若 $x + y = a$，$x + 2y + z = b$，$y + 2z + t = c$ 及 $\delta = x + y + z + t$，求 δ 的值。

Given that a , b and c are positive factors of γ with $a < b < c < \gamma$ and $ab = c$.

If $x + y = a$, $x + 2y + z = b$, $y + 2z + t = c$ and $\delta = x + y + z + t$, determine the value of δ .

$a = 3$, $b = 5$, $c = 15$ $x + y = 3$ …… (1) $x + 2y + z = 5$ …… (2) $y + 2z + t = 15$ …… (3) 由(1)式，$y = 3 - x$ …… (4) 代(4)式入(2)式, $x + 2(3 - x) + z = 5$ $z = -1 + x$ …… (5) 代(4)式及(5)式入(3)式， $3 - x + 2(-1 + x) + t = 15$ $1 + x + t = 15$ $t = 14 - x$ …… (6) 代(4)式、(5)式及(6)式入 $\delta = x + y + z + t$ $\delta = x + (3 - x) + (-1 + x) + (14 - x) = 16$	$a = 3$, $b = 5$, $c = 15$ $x + y = 3$ …… (1) $x + 2y + z = 5$ …… (2) $y + 2z + t = 15$ …… (3) From (1), $y = 3 - x$ …… (4) Sub. (4) into (2), $x + 2(3 - x) + z = 5$ $z = -1 + x$ …… (5) Sub. (4) and (5) into (3), $3 - x + 2(-1 + x) + t = 15$ $1 + x + t = 15$ $t = 14 - x$ …… (6) Sub. (4), (5) and (6) into $\delta = x + y + z + t$ $\delta = x + (3 - x) + (-1 + x) + (14 - x) = 16$

Group Event 1

G1.1 已知 $x+y=32$，其中 x、$y\geq 0$。若 a 為 $\sqrt{x}+\sqrt{y}$ 的最大值，求 a 的值。

Let $x+y=32$ with $x, y\geq 0$. If a is the maximum value of $\sqrt{x}+\sqrt{y}$, determine the value of a.

Reference: 1999 FG3.4

$$\left(\sqrt{x}+\sqrt{y}\right)^2=x+2\sqrt{x}\cdot\sqrt{y}+y$$
$$=32+2\sqrt{xy}$$
$$\leq 32+2\left(\frac{x+y}{2}\right) \quad \text{(A.M.} \geq \text{G.M.)}$$
$$= 32 + 32 = 64$$

$a = 8$

G1.2 一個盒中只有 x 個一元硬幣，$x+2$ 個二元硬幣及 $x+4$ 個五元硬幣。已知隨機從盒中拿出一元硬幣的概率小於 0.1。若盒中有 b 個硬幣，求 b 的值。

A box contains only x-one-dollar coins, $x+2$ two-dollar coins and $x+4$ five-dollar coins. Given that the probability of drawing a one-dollar coin randomly from the box is less than 0.1 . If the box contains b coins, determine the value of b.

$b = x + x + 2 + x + 4 = 3x + 6$

$\frac{x}{3x+6}<0.1$

$10x < 3x + 6$

$7x < 6$

$x = 0$

$b = 3\times 0 + 6 = 6$

G1.3 若 c 是以下數的最大公因數，3^3-3、5^5-5、7^7-7、9^9-9、⋯、$2019^{2019}-2019$，求 c 的值。

If c is the greatest common factor of the following numbers

$3^3-3, 5^5-5, 7^7-7, 9^9-9, \cdots, 2019^{2019}-2019$, determine the value of c.

Reference: Mathematics Excalibur Volume 6, Number 4 Problem Corner Problem 131

$3^3-3=24$, H.C.F. 最大是 24。 設 $n=2k+1$，其中 k 為正整數。 $n^n-n=n(n^{n-1}-1)$ $=n(n^{2k}-1)$ $=n(n^2-1)[(n^2)^{k-1}+(n^2)^{k-2}+\cdots+1]$ $=(n-1)n(n+1)(n^{2k-2}+n^{2k-4}+\cdots+1)$ $(n-1)n(n+1)$為三個連續整數之積，因此必然被 3 整除。 另外，$(n-1)(n+1)=2k(2k+2)=4k(k+1)$ 由於 $k(k+1)$為兩個連續整數之積，因此必然被 2 整除。 $\therefore$ $4k(k+1)$能被 8 整除。 即是說，$(n-1)(n+1)$ 能被 8 整除。 由於 3 和 8 互質，n^n-n 能被 $3\times 8=24$ 整除。 H.C.F. $= c = 24$	$3^3-3=24$, the H.C.F. is at most 24. Let $n = 2k + 1$, where k is a positive integer. $n^n-n=n(n^{n-1}-1)$ $=n(n^{2k}-1)$ $=n(n^2-1)[(n^2)^{k-1}+(n^2)^{k-2}+\cdots+1]$ $=(n-1)n(n+1)(n^{2k-2}+n^{2k-4}+\cdots+1)$ $(n-1)n(n+1)$ is the product of 3 consecutive integers, which is divisible by 3. Also, $(n-1)(n+1)=2k(2k+2)=4k(k+1)$ As $k(k+1)$ is the product of 2 consecutive integers, which must be divisible by 2. $\therefore$ $4k(k+1)$ is divisible by 8. i.e. $(n-1)(n+1)$ is divsible by 8. As 3 and 8 are relatively prime, n^n-n is divisble by $3\times 8=24$. H.C.F. $= c = 24$

G1.4 設 $x=\dfrac{\sqrt{5}+\sqrt{7}}{\sqrt{5}-\sqrt{7}}$ 和 $y=\dfrac{\sqrt{5}-\sqrt{7}}{\sqrt{5}+\sqrt{7}}$。若 $d=3x^2-7xy+3y^2$，求 d 的值。

Let $x=\dfrac{\sqrt{5}+\sqrt{7}}{\sqrt{5}-\sqrt{7}}$ and $y=\dfrac{\sqrt{5}-\sqrt{7}}{\sqrt{5}+\sqrt{7}}$. If $d=3x^2-7xy+3y^2$, determine the value of d.

Reference: 2000 FG1.2

$$x=\frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}-\sqrt{7}}\times\frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{7}}=-\frac{12+2\sqrt{35}}{2}=-\left(6+\sqrt{35}\right)$$

$$y=\frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}+\sqrt{7}}\times\frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}-\sqrt{7}}=-\frac{12-2\sqrt{35}}{2}=-\left(6-\sqrt{35}\right)$$

$x^2=\left[-\left(6+\sqrt{35}\right)\right]^2=71+12\sqrt{35}$; $y^2=71-12\sqrt{35}$; $xy=\left[-\left(6+\sqrt{35}\right)\right]\left[-\left(6-\sqrt{35}\right)\right]=1$

$d=3x^2-7xy+3y^2=3\left(71+12\sqrt{35}\right)-7+3\left(71-12\sqrt{35}\right)=419$

Group Event 2

G2.1 設 $X=\sqrt{2020-\sqrt{A}}$ 為正整數，求 A 的最小值。

Let $X=\sqrt{2020-\sqrt{A}}$ be a positive integer. Determine the least value of A.

Reference: 2016 HI3, 2018FG4.1

$45=\sqrt{2025}>\sqrt{2020-\sqrt{A}}$ $X=\sqrt{2020-\sqrt{A}}=1$、2、…、43 或 44 $2020-\sqrt{A}=1^2$、2^2、… 、43^2 或 44^2 $\sqrt{A}=2020-1^2$、$2020-2^2$、…、或$2020-44^2$ $A=(2020-1^2)^2$、$(2020-2^2)^2$、…、或$(2020-44^2)^2$ A 的最小值$=(2020-44^2)^2=84^2=7056$	$45=\sqrt{2025}>\sqrt{2020-\sqrt{A}}$ $X=\sqrt{2020-\sqrt{A}}=1, 2, \cdots, 43$ or 44 $2020-\sqrt{A}=1^2, 2^2, \cdots, 43^2$ or 44^2 $\sqrt{A}=2020-1^2, 2020-2^2, \cdots$, or $2020-44^2$ $A=(2020-1^2)^2, (2020-2^2)^2, \cdots$, or $(2020-44^2)^2$ The least value of $A=(2020-44^2)^2=84^2=7056$

G2.2 假設 $\begin{cases} x+y=5 \\ 4x^2+y^2=80 \end{cases}$，及 $P=(x_1,y_1)$ 和 $Q=(x_2,y_2)$ 為兩個不同的點，同時滿足這兩個等式。若 $B=y_1-x_1+y_2-x_2$，求 B 的值。

Suppose that $\begin{cases} x+y=5 \\ 4x^2+y^2=80 \end{cases}$, and $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ are two different points, simultaneously satisfy these two equations. If $B=y_1-x_1+y_2-x_2$, determine the value of B.

由(1)式，$y=5-x\cdots(3)$ 代(3)式入(2)式：$4x^2+(5-x)^2=80$ $4x^2+25-10x+x^2=80$ $5x^2-10x-55=0$ $x^2-2x-11=0$，其根為 x_1 及 x_2。 $x_1+x_2=$兩根之和$=2$ (x_1, y_1) 及 (x_2, y_2) 兩皆滿足 $y=5-x$ $y_1+y_2=5-(x_1+x_2)=5-2=3$ $B=y_1-x_1+y_2-x_2=3-2=1$	From (1), $y=5-x\cdots(3)$ Sub. (3) into (2): $4x^2+(5-x)^2=80$ $4x^2+25-10x+x^2=80$ $5x^2-10x-55=0$ $x^2-2x-11=0$, the roots are x_1 and x_2. $x_1+x_2=$ sum of roots $=2$ (x_1, y_1) and (x_2, y_2) both satisfy $y=5-x$ $y_1+y_2=10-(x_1+x_2)=10-2=8$ $B=y_1-x_1+y_2-x_2=8-2=6$

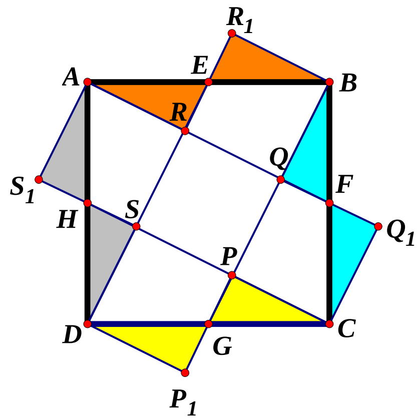
G2.3 若 $Q=a^b-b^a$ 為正整數，求 Q 的最小值。

If $Q=a^b-b^a$ is a positive integer, determine the least value of Q.

設 $a=2$，$b=1$ $Q=2^1-1^2=1$ Q 的最小值為 1。	Let $a=2$, $b=1$ $Q=2^1-1^2=1$ The least value of Q is 1.

G2.4 在正方形 $ABCD$ 中，E、F、G 和 H 分別是 AB、BC、CD 和 AD 的中點。DE 分別與 AF 和 CH 相交於點 R 和 S。BG 分別與 AF 和 CH 相交於點 Q 和 P。若 U 是正方形 $ABCD$ 的面積，而 V 是四邊形 $PQRS$ 的面積，求 $W=\frac{U}{V}$ 的值。

In square $ABCD$, E, F, G, H are the mid-points of AB, BC, CD and AD respectively. DE intersects with AF and CH at R and S respectively. Moreover, BG intersects with AF and CH at Q and P respectively. If U is the area of square $ABCD$ and V is the area of the quadrilateral $PQRS$, determine the value of $W=\frac{U}{V}$.



將 ΔCPG 繞 G 旋轉 $180°$ 至 ΔDP_1G。 將 ΔBQF 繞 F 旋轉 $180°$ 至 ΔCQ_1F。 將 ΔARE 繞 E 旋轉 $180°$ 至 ΔBR_1E。 將 ΔDSH 繞 G 旋轉 $180°$ 至 ΔAS_1H。 $AE = EB = BF = FC = CG = GD = DH = HA$ $\angle ABF = \angle BCG = \angle CDH = \angle DAE = 90°$ $\Delta ABF \cong \Delta BCG \cong \Delta CDH \cong \Delta DAE$ (S.A.S.) $\angle PCG = \angle P_1DG = \angle QBF = \angle Q_1CF$ $= \angle RAE = \angle R_1BE = \angle SDH = \angle S_1AH$ $\angle CGP = \angle DGP_1 = \angle BFQ = CFQ_1$ $= \angle AER = \angle BER_1 = \angle DHS = \angle AHS_1$ $\Delta ARE \cong \Delta BR_1E \cong \Delta BQF \cong \Delta CQ_1F$ $\cong \Delta CPG \cong \Delta DP_1G \cong \Delta DSH \cong \Delta AS_1H$ (A.S.A.) $AS_1 = DS = DP_1 = CP = CQ_1 = BQ = BR_1 = AR$ 在ΔARE中，$\angle ARE = 180° - \angle RAE - \angle REA$ $= 180° - \angle RAE - \angle AFB$ $= 90°$ 相似地，$\angle BR_1E = \angle BQF = \angle CQ_1F$ $= \angle CPG = \angle DP_1G = \angle DSH = \angle AS_1H = 90°$ $\therefore$ $PQRS$、$ARSS_1$、$BQRR_1$、$CPQQ_1$ 及$DSPP_1$ 為全等的正方形而這五個正方形的面積之和為大正方形 $ABCD$ 的面積。 $W=\frac{U}{V}=5$	Rotate ΔCPG about G through $180°$ to ΔDP_1G. Rotate ΔBQF about F through $180°$ to ΔCQ_1F. Rotate ΔARE about E through $180°$ to ΔBR_1E. Rotate ΔDSH about G through $180°$ to ΔAS_1H. $AE = EB = BF = FC = CG = GD = DH = HA$ $\angle ABF = \angle BCG = \angle CDH = \angle DAE = 90°$ $\Delta ABF \cong \Delta BCG \cong \Delta CDH \cong \Delta DAE$ (S.A.S.) $\angle PCG = \angle P_1DG = \angle QBF = \angle Q_1CF$ $= \angle RAE = \angle R_1BE = \angle SDH = \angle S_1AH$ $\angle CGP = \angle DGP_1 = \angle BFQ = CFQ_1$ $= \angle AER = \angle BER_1 = \angle DHS = \angle AHS_1$ $\Delta ARE \cong \Delta BR_1E \cong \Delta BQF \cong \Delta CQ_1F$ $\cong \Delta CPG \cong \Delta DP_1G \cong \Delta DSH \cong \Delta AS_1H$ (A.S.A.) $AS_1 = DS = DP_1 = CP = CQ_1 = BQ = BR_1 = AR$ In ΔARE, $\angle ARE = 180° - \angle RAE - \angle REA$ $= 180° - \angle RAE - \angle AFB$ $= 90°$ Simiarly, $\angle BR_1E = \angle BQF = \angle CQ_1F$ $= \angle CPG = \angle DP_1G = \angle DSH = \angle AS_1H = 90°$ $\therefore$ $PQRS$, $ARSS_1$, $BQRR_1$, $CPQQ_1$ and $DSPP_1$ are identical squares and the sum of areas of these five squares is equal to that of the larger square $ABCD$. $W=\frac{U}{V}=5$

Group Event 3

G3.1 若 $\sqrt{32\times81\times343}=b\sqrt{a}$，其中 a 和 b 是正整數，求 a 的最小值。

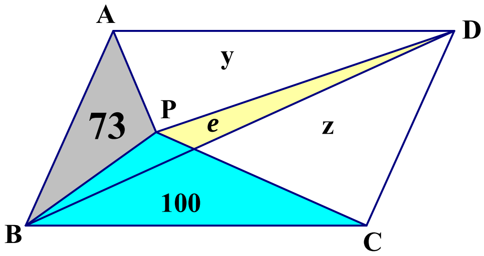
If $\sqrt{32\times81\times343}=b\sqrt{a}$, where a and b are positive integers, determine the least value of a.

Reference: 2014 FI2.3

$\sqrt{32\times81\times343}=b\sqrt{a}$ $\sqrt{16\times2\times81\times49\times7}=b\sqrt{a}$ $4\times9\times7\sqrt{2\times7}=b\sqrt{a}$ a 的最小值 $=14$。	$\sqrt{32\times81\times343}=b\sqrt{a}$ $\sqrt{16\times2\times81\times49\times7}=b\sqrt{a}$ $4\times9\times7\sqrt{2\times7}=b\sqrt{a}$ The least value of $a=14$.

G3.2 下圖中，P 點在平行四邊形 $ABCD$ 內。若 ΔABP、ΔBPC 和 ΔBPD 的面積分別為 73 cm^2、100 cm^2 和 e cm^2，求 e 的值。

In the diagram below, point P is inside parallelogram $ABCD$. If areas of ΔABP, ΔBPC and ΔBPD are 73 cm^2, 100 cm^2 and e cm^2 respectively, determine the value of e.



Reference: 2011 FG4.4, 2014 FG3.2

設ΔAPD及ΔCDP的面積分別為y cm^{2}及 z cm^2。 $100+y=73+z=73+y+e=100+z-e$ $e=27$	Let the areas of ΔAPD and ΔCDP be y cm^2 and z cm^2 respectively. $100+y=73+z=73+y+e=100+z-e$ $e=27$

G3.3 在以下的 3×3 幻方中，每行、列和兩斜行(對角綫)的和相等。如下圖所示，部份數值已經填上。求 c 的值。

A 3×3 magic square is filled with a number in each square such that the sum of the three numbers in each row, column and the two main diagonals are equal. The partially completed grid is shown below. Determine the value of c.

c	16	20
2		

c	16	20
2	x	y
z	t	v

設其他未填的空格為 x、y、z、t 及v 如圖所示。設每一行、列和斜行之和為 s。 $s=c+16+20\cdots(1)$，$s=2+x+y\cdots(2)$ $s=c+2+z\cdots(3)$，$s=c+x+v\cdots(4)$ $s=z+t+v\cdots(5)$，$s=16+x+t\cdots(6)$ $s=20+y+v\cdots(7)$，$s=20+x+z\cdots(8)$ $(1)=(4)$：$x+v=36\cdots(9)$ $(2)=(7)$：$x=18+v\cdots(10)$ 代(10)入(9)：$18+v+v=36\Rightarrow v=9\cdots(11)$ $(1)=(3)$：$z+2=16+20$ $z=34\cdots(12)$ $(2)=(8)$ 及代 (12)：$2+x+y=20+x+34$ $y=52\cdots(13)$ $(2)=(6)$ 及代 (13): $2+x+52=16+x+t$ $t=38\cdots(14)$ $(4)=(8)$及代(11)，(12): $c+x+9=20+x+34$ $c=45$	Let the other empty cells be x, y, z, t and v as shown. Let the sum be s. $s=c+16+20\cdots(1)$, $s=2+x+y\cdots(2)$ $s=c+2+z\cdots(3)$, $s=c+x+v\cdots(4)$ $s=z+t+v\cdots(5)$, $s=16+x+t\cdots(6)$ $s=20+y+v\cdots(7)$, $s=20+x+z\cdots(8)$ $(1)=(4)$ $x+v=36\cdots(9)$ $(2)=(7)$: $x=18+v\cdots(10)$ Sub. (10) into (9): $18+v+v=36\Rightarrow v=9\cdots(11)$ $(1)=(3)$: $z+2=16+20$ $z=34\cdots(12)$ $(2)=(8)$ and sub. (12): $2+x+y=20+x+34$ $y=52\cdots(13)$ $(2)=(6)$ and sub. (13): $2+x+52=16+x+t$ $t=38\cdots(14)$ $(4)=(8)$, sub. (11),(12): $c+x+9=20+x+34$ $c=45$

G3.4 若 $X = 2^{2018} + 3^{2018}$ 及 d 是其個位數字，求 d 的值。

If $X = 2^{2018} + 3^{2018}$ and d is the units digit, determine the value of d.

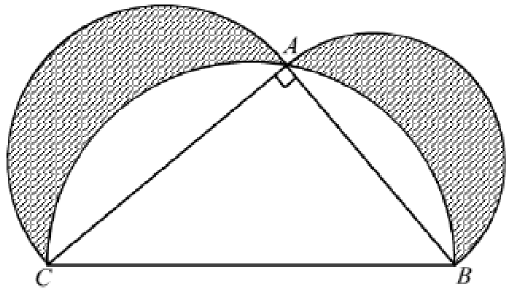
Reference: 1996 FG9.1

$2 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32$ 2^n 的個位數字跟以下規律重覆：2、4、8、6。 $2018 = 4\times504 + 2$ 2^{2018} 的個位數字是4。 $3 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$ 3^n 的個位數字跟以下規律重覆：3、9、7、1。 3^{2018} 的個位數字是9。 $d = 4 + 9 \pmod{10} = 3$	$2 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32$ The units digit of 2^n repeats in the pattern 2,4,8,6 $2018 = 4\times504 + 2$ The units digit of 2^{2018} is 4. $3 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$ The units digit of 3^n repeats in the pattern 3,9,7,1 The units digit of 3^{2018} is 9. $d = 4 + 9 \pmod{10} = 3$

Group Event 4

G4.1 如圖所示，ΔABC 是一直角三角形，其中 $AC = 8$，$BC = 10$。以 AB、AC 和 BC 為直徑分別畫了三個半圓。若陰影部分的總面積是 α，求 α 的值。

As shown in the figure, ΔABC is a right-angled triangle with $AC = 8$ and $BC = 10$. Semi-circles are drawn with AB, AC and BC as diameters. If the total shaded area is α, determine the value of α.



Reference: 1994 HI9, 2009 FG4.2

$AB^2 + 8^2 = 10^2$ (畢氏定理) $AB = 6$ $\alpha = 6\times8\div2 + \pi(3)^2 + \pi(4)^2 - \pi(5)^2 = 24$	$AB^2 + 8^2 = 10^2$ (Pythagoras' theorem) $AB = 6$ $\alpha = 6\times8\div2 + \pi(3)^2 + \pi(4)^2 - \pi(5)^2 = 24$

G4.2 對所有的正整數 n，設某一個函數 $F(n)$ 有如下定義：

$F(1) = 0$，

當 $n \geq 2$，

如果 n 只能被 2 整除而不能被 3 整除，則 $F(n) = F(n-1) + 2$；

如果 n 只能被 3 整除而不能被 2 整除，則 $F(n) = F(n-1) + 3$；

如果 n 既能被 2 整除而又能被 3 整除，則 $F(n) = F(n-1) + 4$；

如果 n 既不能被 2 整除而又不能被 3 整除，則 $F(n) = F(n-1)$。

若 $\beta = F(4000)$，求 β 的值。

For all positive integers n, suppose there exists a function $F(n)$ defined as follows:

$F(1) = 0$,

for all $n \geq 2$,

$F(n) = F(n-1) + 2$ if 2 divides n but 3 does not divide n;

$F(n) = F(n-1) + 3$ if 3 divides n but 2 does not divide n;

$F(n) = F(n-1) + 4$ if 2 and 3 both divide n;

$F(n) = F(n-1)$ if neither 2 nor 3 divides n.

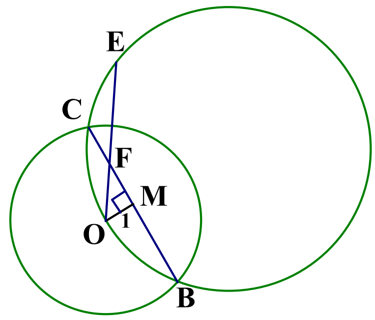
If $\beta = F(4000)$, determine the value of β.

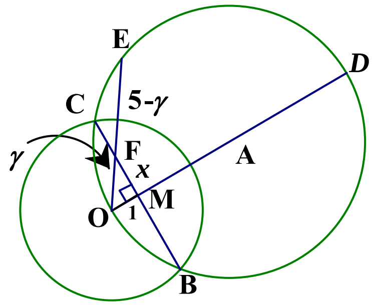
$F(n)=\begin{cases}F(n-1)+2 & 若\ n=6k+2\ 或\ 6k+4\\ F(n-1)+3 & 若\ n=6k+3\\ F(n-1)+4 & 若\ n=6k\\ F(n-1) & 若\ n=6k+1\ 或\ 6k+5\end{cases}$ $F(1) = 0$，$F(2) = 2$，$F(3) = F(2) + 3 = 5$ $F(4) = F(3) + 2 = 7$，$F(5) = F(4) = 7$ $F(6) = F(5) + 4 = 11$，$F(7) = F(6) = 11$ $F(8) = F(7) + 2 = 13$，$F(9) = F(8) + 3 = 16$ $F(10) = F(9) + 2 = 18$，$F(11) = F(10) = 18$ $F(12) = F(11) + 4 = 22$ 利用數學歸納法， $F(n)=\begin{cases}11k & 若\ n=6k+1\\ 11k+2 & 若\ n=6k+2\\ 11k+5 & 若\ n=6k+3\\ 11k+7 & 若\ n=6k+4或n=6k+5\\ 11k & 若\ n=6k\end{cases}$ $4000 = 6\times666 + 4$ $F(4000) = 11\times666 + 7 = 7333$	$F(n)=\begin{cases}F(n-1)+2 & \text{if } n=6k+2 \text{ or } 6k+4\\ F(n-1)+3 & \text{if } n=6k+3\\ F(n-1)+4 & \text{if } n=6k\\ F(n-1) & \text{if } n=6k+1 \text{ or } 6k+5\end{cases}$ $F(1) = 0$, $F(2) = 2$, $F(3) = F(2) + 3 = 5$ $F(4) = F(3) + 2 = 7$, $F(5) = F(4) = 7$ $F(6) = F(5) + 4 = 11$, $F(7) = F(6) = 11$ $F(8) = F(7) + 2 = 13$, $F(9) = F(8) + 3 = 16$ $F(10) = F(9) + 2 = 18$, $F(11) = F(10) = 18$ $F(12) = F(11) + 4 = 22$ Inductively, $F(n)=\begin{cases}11k & \text{if } n=6k+1\\ 11k+2 & \text{if } n=6k+2\\ 11k+5 & \text{if } n=6k+3\\ 11k+7 & \text{if } n=6k+4 \text{ or } n=6k+5\\ 11k & \text{if } n=6k\end{cases}$ $4000 = 6\times666 + 4$ $F(4000) = 11\times666 + 7 = 7333$

G4.3 如圖所示，兩圓相交於 B、C 兩點。M 是 BC 的中點。O 在大圓上，使得 $OM \perp BC$。$OM = 1$、$OC = 3$ 及 $OE = 5$。若 $\gamma = OF$，求 γ 的值。

Two circles intersect at B, C as shown in the figure. M is the mid-point of BC. O is a point on the larger circle, so that $OM \perp BC$. $OM = 1$, $OC = 3$ and $OE = 5$.

If $\gamma = OF$, determine the value of γ.





設 A 為大圓的圓心。
連接並延長 OA 使其再交大圓於 D。
$AO \perp BC$ (圓心至弦綫的中點垂直於弦)
A、M、O 共綫 (普萊費爾公理)
在 ΔCMO 中，$OC = 3$，$OM = 1$，$OM \perp BC$
$1^2 + CM^2 = 3^2$ (畢氏定理)
$CM = 2\sqrt{2} = BM$
在 ΔOFM 中，設 $MF = x$，$\gamma^2 - x^2 = 1$ (畢氏定理)
$OF = \gamma$, $EF = 5 - x$, $CF = CM - MF = 2\sqrt{2} - x$
$BF = BM + MF = 2\sqrt{2} + x$
由相交弦定理，$OF \times EF = BF \times CF$
$\gamma(5 - \gamma) = \left(2\sqrt{2} - x\right)\left(2\sqrt{2} + x\right)$
$5\gamma - \gamma^2 = 8 - x^2$
$5\gamma = 8 + (\gamma^2 - x^2)$
$5\gamma = 8 + 1 = 9$
$\gamma = \frac{9}{5} = 1.8$

Let A be the centre of the larger circle.
Join and produce OA to cut the circle again at D.
$AO \perp BC$ (line joining centre to mid-pt. of chord $\perp$ chord)
A, M, O are collinear (Playfair's axiom)
In ΔCMO, $OC = 3$, $OM = 1$, $OM \perp BC$
$1^2 + CM^2 = 3^2$ (Pythagoras' theorem)
$CM = 2\sqrt{2} = BM$
In ΔOFM, let $MF = x$, $\gamma^2 - x^2 = 1$ (Pythagoras' thm)
$OF = \gamma$, $EF = 5 - x$, $CF = CM - MF = 2\sqrt{2} - x$
$BF = BM + MF = 2\sqrt{2} + x$
By intersecting chords theorem, $OF \times EF = BF \times CF$
$\gamma(5 - \gamma) = \left(2\sqrt{2} - x\right)\left(2\sqrt{2} + x\right)$
$5\gamma - \gamma^2 = 8 - x^2$
$5\gamma = 8 + (\gamma^2 - x^2)$
$5\gamma = 8 + 1 = 9$
$\gamma = \frac{9}{5} = 1.8$

G4.4 若 $f(x) = \left(x + \frac{1}{2000}\right) \times \left(x + \frac{1}{2001}\right) \times \cdots \times \left(x + \frac{1}{2019}\right)$ 以及 $\delta = f(1) - 1$，求 δ 的值。

If $f(x) = \left(x + \frac{1}{2000}\right) \times \left(x + \frac{1}{2001}\right) \times \cdots \times \left(x + \frac{1}{2019}\right)$ and $\delta = f(1) - 1$,

determine the value of δ.

$$\delta = f(1) - 1 = \left(1 + \frac{1}{2000}\right) \times \left(1 + \frac{1}{2001}\right) \times \cdots \times \left(1 + \frac{1}{2019}\right) - 1$$
$$= \frac{2001}{2000} \times \frac{2002}{2001} \times \cdots \times \frac{2020}{2019} - 1 = \frac{2020}{2000} - 1$$
$$= 1.01 - 1 = 0.01$$