VII Cubic Equation: $ax^3 + bx^2 + cx + d = 0$

Transform $x \to y - \dfrac{b}{3a}$.

$$\Rightarrow a\left(y-\frac{b}{3a}\right)^3 + b\left(y-\frac{b}{3a}\right)^2 + c\left(y-\frac{b}{3a}\right) + d = 0$$

$$\Rightarrow y^3 - 3\cdot\frac{b}{3a}y^2 + 3\left(\frac{b}{3a}\right)^2 y - \left(\frac{b}{3a}\right)^3 + \frac{b}{a}\left[y^2 - 2\cdot\frac{b}{3a}y + \left(\frac{b}{3a}\right)^2\right] + \frac{c}{a}\left(y-\frac{b}{3a}\right) + \frac{d}{a} = 0$$

$$y^3 + \left(\frac{c}{a} - \frac{b^2}{3a^2}\right)y + \left[2\left(\frac{b}{3a}\right)^3 - \frac{c}{a}\left(\frac{b}{3a}\right) + \frac{d}{a}\right] = 0$$

For simplicity, let the new equation be $y^3 + 3py + q = 0$. $p = \dfrac{c}{3a} - \left(\dfrac{b}{3a}\right)^2$, $q = 2\left(\dfrac{b}{3a}\right)^3 - \dfrac{c}{a}\left(\dfrac{b}{3a}\right) + \dfrac{d}{a}$

This is called the **standard cubic equation.**

Theorem: $y^3 - 3uvy - (u^3 + v^3) \equiv (y - u - v)(y - \omega u - \omega^2 v)(y - \omega^2 u - \omega v)$

where $\omega = \dfrac{-1+\sqrt{3}i}{2}$ or $\dfrac{-1-\sqrt{3}i}{2}$

Proof: The above is an identity.

$$\begin{aligned}(y - \omega u - \omega^2 v)(y - \omega^2 u - \omega v) &= y^2 + u^2 + v^2 - (\omega + \omega^2)uy - (\omega + \omega^2)vy + (\omega^2 + \omega^4)uv \\ &= y^2 + u^2 + v^2 - (-1)uy - (-1)vy + (-1)uv \\ &= y^2 + u^2 + v^2 + uy + vy - uv\end{aligned}$$

$$\begin{aligned}\text{RHS} &= (y - u - v)(y - \omega u - \omega^2 v)(y - \omega^2 u - \omega v) \\ &= (y - u - v)(y^2 + u^2 + v^2 + uy + vy - uv) \\ &= y^3 + (u + v)y^2 - (u + v)y^2 + (u^2 + v^2 - uv)y - (u + v)^2 y - (u^2 + v^2 - uv)(u + v) \\ &= y^3 + [u^2 + v^2 - uv - (u + v)^2]y - (u + v)(u^2 - uv + v^2) \\ &= y^3 + [-uv - 2uv]y - (u^3 + v^3) \\ &= y^3 - 3uvy - (u^3 + v^3) = \text{LHS}\end{aligned}$$

Let $p = -uv$, $q = -(u^3 + v^3)$

Then u^3, v^3 are the roots of $t^2 + qt - p^3 = 0$

$$u^3 = \frac{-q-\sqrt{q^2+4p^3}}{2},\ v^3 = \frac{-q+\sqrt{q^2+4p^3}}{2} \Rightarrow u = \left(\frac{-q-\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}},\ v = \left(\frac{-q+\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}$$

From the cubic equation: $y^3 + 3py + q = 0$

$\Rightarrow y^3 - 3uvy - (u^3 + v^3) = 0$

$\Rightarrow (y - u - v)(y - \omega u - \omega^2 v)(y - \omega^2 u - \omega v) = 0$

$\Rightarrow y = u + v$, $\omega u + \omega^2 v$, or $\omega^2 u + \omega v$

$$y = \left(\frac{-q-\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}} + \left(\frac{-q+\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}},\ \left(\frac{-q-\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}\omega + \left(\frac{-q+\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}\omega^2$$

$$\text{or } \left(\frac{-q-\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}\omega^2 + \left(\frac{-q+\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}\omega,\ \text{where } \omega = \frac{-1+\sqrt{3}i}{2}$$

Suppose $\Delta = q^2 + 4p^3 > 0$

Then the second and the third roots of y are complex conjugates.

y has one real root and two complex roots.

Suppose $\Delta = q^2 + 4p^3 = 0$

$$y = 2\left(\frac{-q}{2}\right)^{\frac{1}{3}},\ \left(\frac{-q}{2}\right)^{\frac{1}{3}}\omega + \left(\frac{-q}{2}\right)^{\frac{1}{3}}\omega^2 \text{ or } \left(\frac{-q}{2}\right)^{\frac{1}{3}}\omega^2 + \left(\frac{-q}{2}\right)^{\frac{1}{3}}\omega$$

$$= -(4q)^{\frac{1}{3}},\ \left(\frac{q}{2}\right)^{\frac{1}{3}} \text{ or } \left(\frac{q}{2}\right)^{\frac{1}{3}}$$

y has three real roots, amongst them two are equal.

Suppose $\Delta = q^2 + 4p^3 < 0$

$\left(\frac{-q-\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}$, $\left(\frac{-q+\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}$ are complex conjugates $\Rightarrow$ 1st root is real

$\left(\frac{-q-\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}\omega$, $\left(\frac{-q+\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}\omega^2$ are complex conjugates $\Rightarrow$ 2nd root is real

$\left(\frac{-q-\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}\omega^2$, $\left(\frac{-q+\sqrt{q^2+4p^3}}{2}\right)^{\frac{1}{3}}\omega$ are complex conjugates $\Rightarrow$ 3rd root is real

y has three real roots.

In this case, $q^2 + 4p^3 < 0 \Rightarrow p < 0$

let $y = 2\sqrt{-p}\cos\theta$

$y^3 + 3py + q = 0 \Rightarrow (2\sqrt{-p}\cos\theta)^3 + 3p(2\sqrt{-p}\cos\theta) + q = 0$

$-8p\sqrt{-p}\ \cos^3\theta + 6p\ \sqrt{-p}\cos\theta + q = 0$

$$4\cos^3\theta - 3\cos\theta = -\frac{q}{2(-p)^{\frac{3}{2}}}$$

$$\cos 3\theta = -\frac{q}{2(-p)^{\frac{3}{2}}}$$

$$\theta = \frac{1}{3}\cos^{-1}\left[-\frac{q}{2(-p)^{\frac{3}{2}}}\right],\ \frac{1}{3}\cos^{-1}\left[-\frac{q}{2(-p)^{\frac{3}{2}}}\right] + \frac{2\pi}{3},\ \frac{1}{3}\cos^{-1}\left[-\frac{q}{2(-p)^{\frac{3}{2}}}\right] + \frac{4\pi}{3}$$

$$y = 2\sqrt{-p}\cos\left\{\frac{1}{3}\cos^{-1}\left[-\frac{q}{2(-p)^{\frac{3}{2}}}\right]\right\}, 2\sqrt{-p}\cos\left\{\frac{1}{3}\cos^{-1}\left[-\frac{q}{2(-p)^{\frac{3}{2}}}\right] + \frac{2\pi}{3}\right\},$$

$$2\sqrt{-p}\cos\left\{\frac{1}{3}\cos^{-1}\left[-\frac{q}{2(-p)^{\frac{3}{2}}}\right] + \frac{4\pi}{3}\right\}$$

Calculator Programme for Casio fx-50FH II MODE MODE 6 3 3 AC 3 (CMPLX) 2

(1)	?	(2)	→	(3)	A	(4)	:	(5)	?
(6)	→	(7)	B	(8)	:	(9)	?	(10)	→
(11)	C	(12)	:	(13)	?	(14)	→	(15)	D
(16)	:	(17)	B	(18)	x^3	(19)	–	(20)	9
(21)	┘	(22)	2	(23)	A	(24)	(	(25)	B
(26)	C	(27)	–	(28)	3	(29)	D	(30)	A
(31)	→	(32)	D	(33)	:	(34)	B	(35)	x^2
(36)	–	(37)	3	(38)	A	(39)	C	(40)	→
(41)	C	(42)	:	(43)	$\sqrt{}$(	(44)	D	(45)	x^2
(46)	–	(47)	C	(48)	x^3	(49)	:	(50)	Ans
(51)	–	(52)	D	(53)	–	(54)	2	(55)	Ans
(56)	(	(57)	Ans	(58)	=	(59)	D	(60)	→
(61)	D	(62)	:	(63)	$\sqrt[3]{}$(	(64)	Abs(	(65)	Ans
(66)	⇒	(67)	Ans	(68)	∠	(69)	(	(70)	3
(71)	x^{-1}	(72)	arg(	(73)	D	(74)	→	(75)	D
(76)	:	(77)	While	(78)	1	(79)	:	(80)	Abs(
(81)	D	(82)	⇒	(83)	D	(84)	+	(85)	C
(86)	┘	(87)	D	(88)	:	(89)	(	(90)	Ans
(91)	–	(92)	B	(93)	)	(94)	┘	(95)	(
(96)	3	(97)	A	(98)	◢	(99)	D	(100)	×
(101)	1	(102)	∠	(103)	5	(104)	!	(105)	°
(106)	→	(107)	D	(108)	:	(109)	WhileEnd		

Press MODE 1 (COMP) to exit the progamme mode.
Remark: to press the degree symbol ° : Press Shift Ans 1 .
Programme demonstration To solve $x^3 - 6x - 9 = 0$

Key sequences	Display	Explanation
AC Prog P3	A? 0.	Enter into P3 CMPLX mode
1 EXE 0 EXE -6 EXE -9 EXE	3. Disp	A = 1, B = 0, C = -6, D = -9, 1st ans.= 3
EXE	-1.5 Disp R⇔I	
SHIFT EXE	0.866025403*i* Disp R⇔I	2nd answer = -1.5 + 0.866025403*i*
EXE	-1.5 Disp R⇔I	
SHIFT EXE	-0.866025403*i* Disp R⇔I	3rd answer M = -1.5 – 0.866025403*i*

Press AC and then MODE 1 to exit the programme mode and the CMPLX mode.
To solve $x^2 + 2x + 3 = 0$. Multiply the equation by X to give $x^3 + 2x^2 + 3x = 0$.
Remaining steps are the same, discard the first answer $X = 0$. Press MODE 1 to exit CMPLX mode.

Exercise 2

1. Solve $x^3 - 6x - 9 = 0$ [Ans. 3, $\frac{-3 \pm \sqrt{3}i}{2}$]
2. Solve $x^3 - 12x - 16 = 0$ [Ans. 4, –2, –2]
3. Solve $4x^3 - 43x + 21 = 0$ [Ans. $\frac{1}{2}$, 3, $-\frac{7}{2}$]
4. Remove the coefficient of x^2 in $x^3 - 8x^2 + 20x - 16 = 0$. Hence solve for x. [Ans. 4, 2, 2]
5. If $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ is rewritten as $A(x + p)^3 + B(x + q)^3 = 0$, show that p and q are the roots of the equation $(a_0 a_2 - a_1^2)t^2 - (a_0a_3 - a_1a_2)t + (a_1a_3 - a_2^2) = 0$.
   Hence find the solution to x.

VII Cubic Equation

1. $x^3 - 6x - 9 = 0$

$p = -2$, $q = -9$

$\Delta = q^2 + 4p^3 = (-9)^2 + 4(-2)^3 = 81 - 32 > 0$

$\therefore$ The equation has one real root and 2 complex conjugate roots.

$x = u + v$

$(u + v)^3 - 6(u + v) - 9 = 0$

$u^3 + 3u^2v + 3uv^2 + v^3 - 6(u + v) - 9 = 0$

$u^3 + v^3 + (u + v)(3uv - 6) - 9 = 0$

Let $uv = 2$, then

$$\begin{cases} u^3v^3 = 8 \\ u^3 + v^3 = 9 \end{cases}$$

u^3, v^3 are roots of $t^2 - 9t + 8 = 0$

$(t - 1)(t - 8) = 0$

$t = 1$ or $8 \Rightarrow u^3 = 1, v^3 = 8 \Rightarrow u = 1, v = 2$

$x = u + v = 1 + 2 = 3$

$x - 3$ is a factor of $x^3 - 6x - 9$

By division, $x^3 - 6x - 9 = (x - 3)(x^2 + 3x - 3) = 0$

$x = 3$ or $\dfrac{-3 \pm \sqrt{3}i}{2}$

2. $x^3 - 12x - 16 = 0$

$p = -4$, $q = -16$

$\Delta = q^2 + 4p^3 = (-16)^2 + 4(-4)^3 = 256 - 256 = 0$

$\therefore$ The equation has three real roots, amongst them two are equal.

$x = u + v$

$(u + v)^3 - 12(u + v) - 16 = 0$

$u^3 + 3u^2v + 3uv^2 + v^3 - 12(u + v) - 16 = 0$

$u^3 + v^3 + (u + v)(3uv - 12) - 16 = 0$

Let $uv = 4$, then

$$\begin{cases} u^3v^3 = 64 \\ u^3 + v^3 = 16 \end{cases}$$

u^3, v^3 are roots of $t^2 - 16t + 64 = 0$

$(t - 8)^2 = 0$

$t = 8 \Rightarrow u^3 = 8, v^3 = 8 \Rightarrow u = 2, v = 2$

$x = u + v = 2 + 2 = 4$

$x - 4$ is a factor of $x^3 - 12x - 16$

By division, $x^3 - 12x - 16 = (x - 4)(x^2 + 4x + 4) = 0$

$x = 4, -2$ or -2

3. $4x^3 - 43x + 21 = 0$ [Ans. $\frac{1}{2}$, 3, $-\frac{7}{2}$]

$$x^3 - 3\left(\frac{43}{12}\right)x + \frac{21}{4} = 0$$

$p = -\dfrac{43}{12}$, $q = \dfrac{21}{4}$

$$\Delta = q^2 + 4p^3 = \left(\frac{21}{4}\right)^2 + 4\left(-\frac{43}{12}\right)^3 = -\frac{4225}{27} < 0$$

$\therefore$ The equation has three distinct real roots

$x = 2\sqrt{-p}\cos\theta = 2\sqrt{\frac{43}{12}}\cos\theta = \sqrt{\frac{43}{3}}\cos\theta$

$4\left(\sqrt{\frac{43}{3}}\cos\theta\right)^3 - 43\left(\sqrt{\frac{43}{3}}\cos\theta\right) + 21 = 0$

$4\cdot\frac{43}{3}\cdot\sqrt{\frac{43}{3}}\cos^3\theta - 43\left(\sqrt{\frac{43}{3}}\cos\theta\right) + 21 = 0$

$\frac{43}{3}\cdot\sqrt{\frac{43}{3}}\left(4\cos^3\theta - 3\cos\theta\right) = -21$

$\cos 3\theta = -\frac{63\sqrt{3}}{43\sqrt{43}}$

$3\theta = 112.7672684°$

$\theta = 37.58908947°, 157.5890895°, 277.5890895°$

$x = \sqrt{\frac{43}{3}}\cos 37.58908947°,\ \sqrt{\frac{43}{3}}\cos 157.5890895°,\ \sqrt{\frac{43}{3}}\cos 277.5890895°$

$x = 3, -3.5, 0.5$

4. $x^3 - 8x^2 + 20x - 16 = 0$

Let $x = y - \frac{b}{3a} = y + \frac{8}{3}$

$\left(y + \frac{8}{3}\right)^3 - 8\left(y + \frac{8}{3}\right)^2 + 20\left(y + \frac{8}{3}\right) - 16 = 0$

$(3y + 8)^3 - 24(3y + 8)^2 + 180(3y + 8) - 432 = 0$

$27y^3 + 3(9y^2)(8) + 3(3y)(64) + 512 - 24(9y^2 + 48y + 64) + 540y + 1440 - 432 = 0$

$27y^3 - 36y - 16 = 0$

$y^3 - \frac{4}{3}y - \frac{16}{27} = 0$; $p = -\frac{4}{9}$, $q = -\frac{16}{27}$

$\Delta = q^2 + 4p^3 = \left(-\frac{16}{27}\right)^2 + 4\left(-\frac{4}{9}\right)^3 = 0$

The equation has 3 real roots, 2 of which are equal.

$y = u + v$

$27(u + v)^3 - 36(u + v) - 16 = 0$

$27(u^3 + 3u^2v + 3uv^2 + v^3) - 36(u + v) - 16 = 0$

$27(u^3 + v^3) + (u + v)(81uv - 36) - 16 = 0$

Let $uv = \frac{4}{9}$, then

$$\begin{cases} u^3v^3 = \frac{64}{729} \\ u^3 + v^3 = \frac{16}{27} \end{cases}$$

u^3, v^3 are roots of $729t^2 - 432t + 64 = 0$

$(27t - 8)^2 = 0$

$t=\dfrac{8}{27}\Rightarrow u^3=\dfrac{8}{27}$, $v^3=\dfrac{8}{27}\Rightarrow u=\dfrac{2}{3}$, $v=\dfrac{2}{3}$

$y=u+v=\dfrac{2}{3}+\dfrac{2}{3}=\dfrac{4}{3}$

$3y-4$ is a factor of $27y^3-36y-16$

By division, $27y^3-36y-16=(3y-4)(9x^2+12x+4)=(3y-4)(3y+2)^2$

$y=\dfrac{4}{3},-\dfrac{2}{3}$ or $-\dfrac{2}{3}$

$x=y+\dfrac{8}{3}=4$, 2 or 2

5. $A(x+p)^3+B(x+q)^3=0$

$A(x^3+3px^2+3p^2x+p^3)+B(x^3+3qx^2+3q^2x+q^3)=0$

Compare it with $a_0x^3+3a_1x^2+3a_2x+a_3=0$

$A+B=a_0$ …… (1)

$Ap+Bq=a_1$ …… (2)

$Ap^2+Bq^2=a_2$ …… (3)

$Ap^3+Bq^3=a_3$ …… (4)

$$\begin{aligned}a_0a_2-a_1^2&=(A+B)(Ap^2+Bq^2)-(Ap+Bq)^2\\&=A^2p^2+ABp^2+ABq^2+B^2q^2-(A^2p^2+2ABpq+B^2q^2)\\&=AB(p-q)^2\end{aligned}$$

$$\begin{aligned}a_0a_3-a_1a_2&=(A+B)(Ap^3+Bq^3)-(Ap+Bq)(Ap^2+Bq^2)\\&=A^2p^3+ABq^3+ABp^3+B^2q^3-(A^2p^3+ABp^2q+ABpq^2+B^2q^3)\\&=AB(p^3+q^3-p^2q-pq^2)\\&=AB(p-q)^2(p+q)\end{aligned}$$

$$\begin{aligned}a_1a_3-a_2^2&=(Ap+Bq)(Ap^3+Bq^3)-(Ap^2+Bq^2)^2\\&=A^2p^4+ABp^3q+ABpq^3+B^2q^4-(A^2p^4+2ABp^2q^2+B^2q^4)\\&=AB(p-q)^2\cdot pq\end{aligned}$$

$\therefore p, q$ are roots of the equation $AB(p-q)^2\cdot[x^2-(p+q)x+pq]=0$

i.e. $(a_0a_2-a_1^2)t^2-(a_0a_3-a_1a_2)t+(a_1a_3-a_2^2)=0$.

We consider the following cases:

(i) $p=q=$ real, then $A(x+p)^3+B(x+q)^3=0$

$\Rightarrow (A+B)(x+p)^3=0$

$\Rightarrow x=-p$ (three equal real roots)

(ii) p, q are complex. Then $\overline{p}=q$ and $|p|=|q|$.

$\because$ complex roots occur in conjugate pairs, the equation $A(x+p)^3+B(x+q)^3=0$ must contain one real root, let's say x_1. Then

$A(x_1+p)^3=-B(x_1+q)^3$

$$\frac{A}{B}=-\frac{(x_1+q)^3}{(x_1+p)^3}$$

$$\Rightarrow\left|\frac{A}{B}\right|=\left|\frac{x_1+q}{x_1+p}\right|^3=\frac{|x_1+q|^3}{|x_1+p|^3}=\frac{\left|\overline{x_1+p}\right|^3}{|x_1+p|^3}=1 \ (\because x_1 \text{ is real, } \ \overline{p}=q)$$

Hence $\frac{B}{A}=z$, where $|z| = 1$

The equation $A(x + p)^3 + B(x + q)^3 = 0$

$$\Rightarrow\frac{(x+p)^3}{(x+q)^3}=-\frac{B}{A}=-z$$

$$\Rightarrow\frac{x+p}{x+q}=(-z)^{\frac{1}{3}}, \text{ let } (-z)^{\frac{1}{3}}=z_1, z_2, z_3$$

Clearly $|z_1| = |z_2| = |z_3| = \left|(-z)^{\frac{1}{3}}\right| = 1$,

$$\frac{x+p}{x+q}=z_1\Rightarrow x=\frac{qz_1-p}{1-z_1}=\frac{(qz_1-p)}{(1-z_1)}\cdot\frac{(1-\overline{z}_1)}{(1-\overline{z}_1)}=\frac{qz_1-p-q+p\overline{z}_1}{|1-z_1|^2}=\frac{\overline{p}z_1+p\overline{z}_1-(p+\overline{p})}{|1-z_1|^2}$$

Observe that x is real, by similar working, the other two values of x are also real.

Hence the three roots are all real.

Hence if $\Delta \le 0$, i.e. $(a_0a_3 - a_1a_2)^2 \le 4(a_0 a_2 - a_1^2)(a_1a_3 - a_2^2)$, the equation has 3 real roots.

(iii) If p, q are real and $p \ne q$, $A(x + p)^3 + B(x + q)^3 = 0$

$$\Rightarrow\left[\sqrt[3]{A}(x+p)+\sqrt[3]{B}(x+q)\right]\left[\sqrt[3]{A^2}(x+p)^2-\sqrt[3]{AB}(x+p)(x+q)+\sqrt[3]{B^2}(x+q)^2\right]=0$$

$x=-\frac{\sqrt[3]{A}p+\sqrt[3]{B}q}{\sqrt[3]{A}+\sqrt[3]{B}}$ one real root. (From equation (1), $A + B = a_0 \ne 0 \Leftrightarrow \sqrt[3]{A}+\sqrt[3]{B}\ne 0$)

The equation $\sqrt[3]{A^2}(x+p)^2-\sqrt[3]{AB}(x+p)(x+q)+\sqrt[3]{B^2}(x+q)^2 = 0$ has a negative discriminant and hence it has two complex conjugate roots.

$\therefore$ The equation has one real root and two complex conjugate roots.