13-14 Individual	**1**	$\frac{19}{121}$	**2**	12	**3**	120°	**4**	193	**5**	90
	6	*107 See the remark	**7**	±1	**8**	8	**9**	*14 See the remark	**10**	15
13-14 Group	**1**	5	**2**	23	**3**	$\frac{-65}{144}$	**4**	$15-4\sqrt{2}$	**5**	49
	6	$\frac{29}{4}$ (=7.25)	**7**	−1	**8**	1584	**9**	$3^{-\frac{16}{3}}$	**10**	$\frac{3+\sqrt{5}}{2}$

Individual Events

I1 Given that a, b, $c > 0$ and $\begin{cases}\frac{\sqrt{ab}}{\sqrt{a}+\sqrt{b}}=2\\ \frac{\sqrt{bc}}{\sqrt{b}+\sqrt{c}}=3\\ \frac{\sqrt{ca}}{\sqrt{c}+\sqrt{a}}=5\end{cases}$. Find the value of $\frac{a}{\sqrt{bc}}$.

$$\begin{cases}\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}=\frac{1}{2}\\ \frac{\sqrt{b}+\sqrt{c}}{\sqrt{bc}}=\frac{1}{3}\\ \frac{\sqrt{c}+\sqrt{a}}{\sqrt{ca}}=\frac{1}{5}\end{cases}\Rightarrow\begin{cases}\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}=\frac{1}{2}\cdots\cdots(1)\\ \frac{1}{\sqrt{c}}+\frac{1}{\sqrt{b}}=\frac{1}{3}\cdots\cdots(2)\\ \frac{1}{\sqrt{a}}+\frac{1}{\sqrt{c}}=\frac{1}{5}\cdots\cdots(3)\end{cases}$$

(1) + (2) − (3): $\frac{2}{\sqrt{b}}=\frac{19}{30}\Rightarrow b=\frac{3600}{361}$

(1) + (3) − (2): $\frac{2}{\sqrt{a}}=\frac{11}{30}\Rightarrow a=\frac{3600}{121}$

(2) + (3) − (1): $\frac{2}{\sqrt{c}}=\frac{1}{30}\Rightarrow c=3600$

$$\frac{a}{\sqrt{bc}}=\frac{3600}{121}\times\sqrt{\frac{361}{3600^2}}=\frac{19}{121}$$

I2 Given that $a = 2014x + 2011$, $b = 2014x + 2013$ and $c = 2014x + 2015$.
Find the value of $a^2 + b^2 + c^2 - ab - bc - ca$.

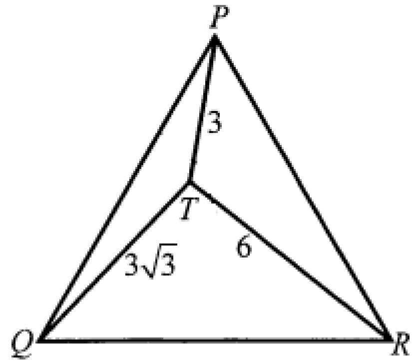
$$a^2 + b^2 + c^2 - ab - bc - ca$$

$$=\frac{1}{2}\left[(a-c)^2+(c-b)^2+(b-a)^2\right]$$

$$=\frac{1}{2}\left[(2014x+2011-2014x-2015)^2+(2014x+2015-2014x-2013)^2+(2014x+2013-2014x-2011)^2\right]$$

$$=\frac{1}{2}\left[(-4)^2+2^2+2^2\right]=12$$

I3 As shown in Figure 1, a point T lies in an equilateral triangle PQR such that $TP = 3$, $TQ = 3\sqrt{3}$ and $TR = 6$. Find the value of $\angle PTR$.



Reference: 2019 HG10

Rotate ΔPTR anticlockwise by 60° to ΔQSR.

Then $\Delta PTR \cong \Delta QSR$, $SR = 6$ and $\angle SRT = 60°$

Consider ΔTRS,

$SR = 6 = TR$

$\therefore$ ΔTRS is isosceles.

$\angle SRT = 60°$

$\therefore$ $\angle RTS = \angle RST = 60°$ ($\angle$s sum of isos. Δ)

$\therefore$ ΔTRS is an equilateral triangle

$TS = 6$

Consider ΔTQS,

$QS^2 + QT^2 = 3^2 + (3\sqrt{3})^2 = 9 + 27 = 36 = 6^2 = TS^2$

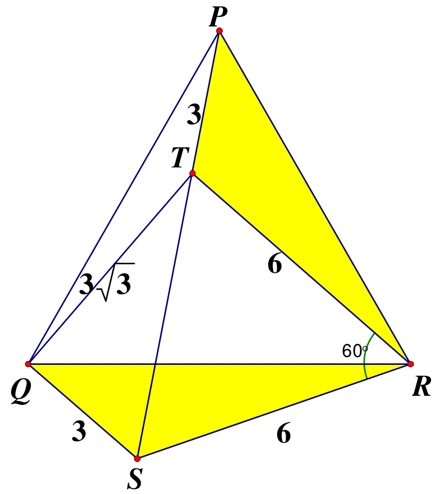
$\therefore$ $\angle TQS = 90°$ (converse, Pythagoras' theorem)

$\tan \angle TSQ = \dfrac{3\sqrt{3}}{3} = \sqrt{3}$

$\angle TSQ = 60°$

$\angle QSR = \angle TSQ + \angle RST = 60° + 60° = 120°$

$\angle PTR = \angle QSR = 120°$ (corr. $\angle$s, $\Delta PTR \cong \Delta QSR$)



Reference: C:/Users/孔德偉/Dropbox/Data/My%20Web/Home_Page/Geometry/7%20Construction%20by%20ruler%20and%20compasses/others/345.pdf

I4 Let α and β be the roots of the quadratic equation $x^2 - 14x + 1 = 0$.

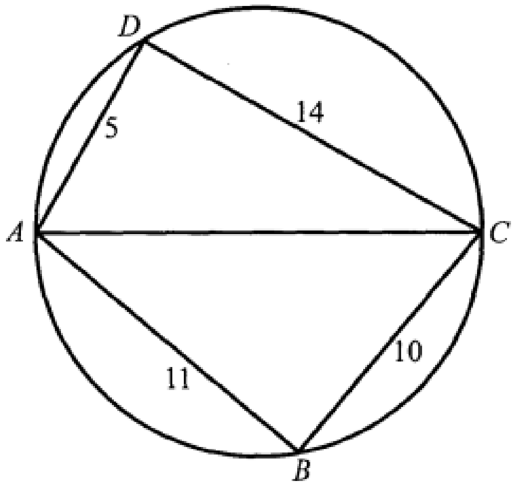
Find the value of $\dfrac{\alpha^2}{\beta^2+1} + \dfrac{\beta^2}{\alpha^2+1}$.

$\alpha^2 + 1 = 14\alpha$; $\beta^2 + 1 = 14\beta$; $\alpha + \beta = 14$ and $\alpha\beta = 1$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 196 - 2 = 194$

$$\frac{\alpha^2}{\beta^2+1} + \frac{\beta^2}{\alpha^2+1} = \frac{14\alpha-1}{14\beta} + \frac{14\beta-1}{14\alpha} = \frac{196\alpha^2 - 14\alpha + 196\beta^2 - 14\beta}{196\alpha\beta} = \frac{196(\alpha^2+\beta^2) - 14^2}{196} = 193$$

I5 As shown in Figure 2, $ABCD$ is a cyclic quadrilateral, where $AD = 5$, $DC = 14$, $BC = 10$ and $AB = 11$. Find the area of quadrilateral $ABCD$.



Reference: 2002 HI6

$AC^2 = 10^2 + 11^2 - 2\times 11\times 10 \cos \angle B$ ……(1)

$AC^2 = 5^2 + 14^2 - 2\times 5\times 14 \cos \angle D$ ……… (2)

(1) = (2): $221 - 220 \cos\angle B = 221 - 140 \cos\angle D$ …(3)

$\angle B + \angle D = 180°$ (opp. $\angle$s, cyclic quad.)

$\therefore \cos \angle D = -\cos \angle B$

(3): $(220 + 140) \cos \angle B = 0 \Rightarrow \angle B = 90° = \angle D$

Area of the cyclic quadrilateral

= area of ΔABC + area of ΔACD

$= \dfrac{1}{2}\cdot 11\cdot 10 + \dfrac{1}{2}\cdot 5\cdot 14 = 90$

I6 Let n be a positive integer and $n < 1000$.
If $(n^{2014} - 1)$ is divisible by $(n - 1)^2$, find the maximum value of n .
Let $p = 2014$.

$$\frac{n^p - 1}{(n-1)^2} = \frac{(n-1)(n^{p-1} + n^{p-2} + \cdots + n + 1)}{(n-1)^2} = \frac{n^{p-1} + n^{p-2} + \cdots + n + 1}{n-1}$$
$$= \frac{(n^{p-1} - 1) + (n^{p-2} - 1) + \cdots + (n-1) + p}{n-1}$$
$$= \frac{n^{p-1} - 1}{n-1} + \frac{n^{p-2} - 1}{n-1} + \cdots + 1 + \frac{p}{n-1}$$

Clearly $n - 1$ are factors of $n^{p-1} - 1$, $n^{p-2} - 1$, … , $n - 1$.

$\therefore \frac{n^{p-1} - 1}{n-1} + \frac{n^{p-2} - 1}{n-1} + \cdots + 1$ is an integer.

$\therefore \frac{p}{n-1} = \frac{2014}{n-1} = \frac{2\times 19\times 53}{n-1}$ is an integer

The largest value of $n - 1$ is $2\times 53 = 106$.
i.e. The maximum value of $n = 107$.
Remark: The original question is Let n be a positive **number** and $n < 1000$. If $(n^{2014} - 1)$ is divisible by $(n - 1)^2$, find the maximum value of n. 設 n 為正數，且 $n < 1000$。…
Note that n must be an integer for divisibility question.

I7 If $x^3 + x^2 + x + 1 = 0$, find the value of $x^{-2014} + x^{-2013} + x^{-2012} + \cdots + x^{-1} + 1 + x + x^2 + \cdots + x^{2013} + x^{2014}$.
Reference: 1997 FG4.2
The given equation can be factorised as $(1 + x)(1 + x^2) = 0 \Rightarrow x = -1$ or $\pm i$

$x^{-2014} + x^{-2013} + x^{-2012} + \cdots + x^{-1} + 1 + x + x^2 + \cdots + x^{2013} + x^{2014}$

$= x^{-2014}\cdot(1 + x + x^2 + x^3) + \cdots + x^{-6}\cdot(1 + x + x^2 + x^3) + x^{-2} + x^{-1} + 1 + x + x^2 + x^3\cdot(1 + x + x^2 + x^3)$
$+ \cdots + x^{2011}\cdot(1 + x + x^2 + x^3)$

$= x^{-2} + x^{-1} + 1 + x + x^2 = x^{-2}\cdot(1 + x + x^2 + x^3) + x^2 = x^2$
When $x = -1$, $x^2 = 1$
When $x = \pm i$, $x^2 = -1$

I8 Let $\overline{xy} = 10x + y$. If $\overline{xy} + \overline{yx}$ is a square number, how many numbers of this kind exist ?
$\overline{xy} + \overline{yx} = 10x + y + 10y + x = 10(x + y) + x + y = 11(x + y)$
Clearly x and y are integers ranging from 1 to 9.
$\therefore 2 \le x + y \le 18$.
In order that $\overline{xy} + \overline{yx} = 11(x + y)$ is a square number, $x + y = 11$
$(x, y) = (2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3)$ or $(9, 2)$.
There are 8 possible numbers.

I9 Given that x, y and z are positive real numbers such that $xyz = 64$.
If $S = x + y + z$, find the value of S when $4x^2 + 2xy + y^2 + 6z$ is a minimum.
$4x^2 + 2xy + y^2 + 6z = 4x^2 - 4xy + y^2 + 6xy + 6z$
$= (2x - y)^2 + 6(xy + z) \ge 0 + 6\times 2\sqrt{xyz} = 96$ (A.M. ≥ G.M.)
When $4x^2 + 2xy + y^2 + 6z$ is a minimum, $2x - y = 0$ and $xy = z$
$\therefore y = 2x, z = 2x^2$
$\because xyz = 64 \therefore x(2x)(2x^2) = 64 \Rightarrow x^4 = 16$
$x = 2, y = 4, z = 8 \Rightarrow S = 2 + 4 + 8 = 14$
Remark: The original question is: Given that x, y and z are real numbers such that $xyz = 64$
Note that the steps in inequality fails if $xy < 0$ and $z < 0$.

I10 Given that ΔABC is an acute triangle, where $\angle A > \angle B > \angle C$.
If $x°$ is the minimum of $\angle A - \angle B$, $\angle B - \angle C$ and $90° - \angle A$, find the maximum value of x .
In order to attain the maximum value of x, the values of $\angle A - \angle B$, $\angle B - \angle C$ and $90° - \angle A$ must be equal.
$\angle A - \angle B = \angle B - \angle C = 90° - \angle A$
$2\angle B = \angle A + \angle C$ ⋯⋯ (1)
$2\angle A = 90° + \angle B$ ⋯⋯ (2)
$\angle A + \angle B + \angle C = 180°$ ⋯⋯ (3) ($\angle$ sum of Δ)
Sub. (1) into (3), $3\angle B = 180°$
$\angle B = 60°$ ⋯⋯ (4)
Sub. (4) into (2): $2\angle A = 90° + 60°$
$\angle A = 75°$ ⋯⋯ (5)
Sub. (4) and (5) into (1): $2(60°) = 75° + \angle C$
$\angle C = 45°$
The maximum value of $x = 75 - 60 = 15$

Method 2

$90° - \angle A \ge x°$
$\Rightarrow 90° - \angle A + \angle A + \angle B + \angle C \ge 180° + x°$ ($\angle$ sum of Δ)
$\Rightarrow \angle B + \angle C \ge 90° + x°$ …… (1)
$\because \angle B - \angle C \ge x°$ ⋯⋯ (2)
((1) + (2))÷2: $\angle B \ge 45° + x°$ ⋯⋯ (3)
$\because \angle A - \angle B \ge x°$ ⋯⋯ (4)
(3) + (4): $\angle A \ge 45° + 2x°$ ⋯⋯ (5)
$90° - \angle A \ge x°$
$\Rightarrow 90° - x° \ge \angle A$
$\Rightarrow 90° - x° \ge \angle A \ge 45° + 2x°$ by (5)
$\Rightarrow 90° - x° \ge 45° + 2x°$
$\Rightarrow 45° \ge 3x°$
$\Rightarrow 15° \ge x°$
$\therefore$ The maximum value of x is 15 .

Group Events

G1 Given that $\sqrt{2014-x^2}-\sqrt{2004-x^2}=2$, find the value of $\sqrt{2014-x^2}+\sqrt{2004-x^2}$.

Reference: 1992 FI5.4

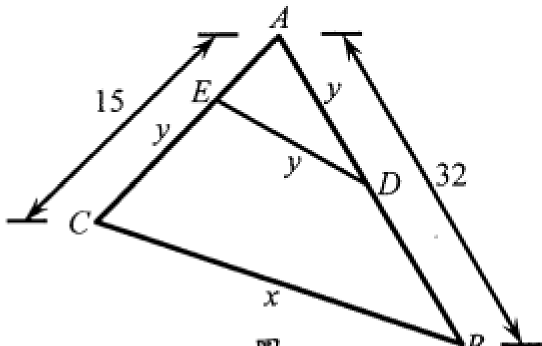
$$\frac{\left(\sqrt{2014-x^2}-\sqrt{2004-x^2}\right)\cdot\left(\sqrt{2014-x^2}+\sqrt{2004-x^2}\right)}{\sqrt{2014-x^2}+\sqrt{2004-x^2}}=2$$

$$\frac{\left(2014-x^2\right)-\left(2004-x^2\right)}{\sqrt{2014-x^2}+\sqrt{2004-x^2}}=2$$

$10=2(\sqrt{2014-x^2}+\sqrt{2004-x^2})$

$\sqrt{2014-x^2}+\sqrt{2004-x^2}=5$

G2 Figure 1 shows a ΔABC, $AB = 32$, $AC = 15$ and $BC = x$, where x is a positive integer. If there are points D and E lying on AB and AC respectively such that $AD = DE = EC = y$, where y is a positive integer. Find the value of x.



圖一

Let $\angle BAC = \theta$, $AE = 15 - y$, $y = 1, 2, \cdots, 14$.

Apply triangle inequality on ΔADE, $y + y > 15 - y$

$\Rightarrow y > 5$ …… (1)

$\angle AED = \theta$ (base $\angle$s, isos. Δ)

By drawing a perpendicular bisector of AE,

$\cos\theta = \frac{15-y}{2y}$ …… (2)

Apply cosine formula on ΔABC,

$x^2 = 15^2 + 32^2 - 2(15)(32)\cos\theta$

$x^2 = 1249 - 480\times\frac{15-y}{y}$ by (2)

$x^2 = 1729 - \frac{7200}{y}$ …… (3)

$\because$ x is a positive integer

$\therefore$ x^2 is a positive integer

$\Rightarrow \frac{7200}{y}$ is a positive integer.

$\Rightarrow$ y is a positive factor of 7200 and $y = 6, 7, 8, \cdots, 14$ by (1) and (3)

$\Rightarrow y = 6, 8, 9, 10$ or 12.

When $y = 6$, $x^2 = 1729 - 1200 = 529 \Rightarrow x = 23$, accepted.

When $y = 8$, $x^2 = 1729 - 900 = 829$, which is not a perfect square, rejected.

When $y = 9$, $x^2 = 1729 - 800 = 929$, which is not a perfect square, rejected.

When $y = 10$, $x^2 = 1729 - 720 = 1009$, which is not a perfect square, rejected.

When $y = 12$, $x^2 = 1729 - 600 = 1129$, which is not a perfect square, rejected.

Conclusion, $x = 23$

G3 If $0° \le \theta \le 180°$ and $\cos\theta + \sin\theta = \frac{7}{13}$, find the value of $\cos\theta + \cos^3\theta + \cos^5\theta + \cdots$.

Reference: 1992 HI20, 1993 HG10, 1995 HI5, 2007 HI7, 2007 FI1.4

Similar question: 2006 HG3

$\cos\theta + \sin\theta = \frac{7}{13} \quad \cdots (1)$

$(\cos\theta + \sin\theta)^2 = \frac{49}{169}$

$\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = \frac{49}{169}$

$1 + 2\sin\theta\cos\theta = \frac{49}{169}$

$2\sin\theta\cos\theta = -\frac{120}{169} \quad \cdots (*)$

$-2\sin\theta\cos\theta = \frac{120}{169}$

$1 - 2\sin\theta\cos\theta = \frac{289}{169}$

$\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta = \frac{289}{169}$

$(\cos\theta - \sin\theta)^2 = \frac{289}{169}$

$\cos\theta - \sin\theta = \frac{17}{13}$ or $-\frac{17}{13}$

From (1), $\sin\theta\cos\theta < 0$ and $0° \le \theta \le 180°$

$\therefore \cos\theta < 0$ and $\sin\theta > 0$

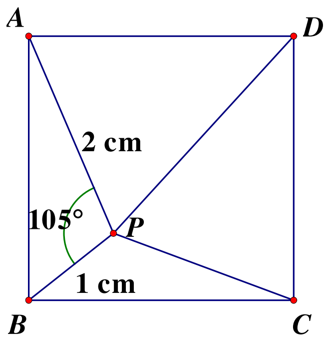
$\therefore \cos\theta - \sin\theta = -\frac{17}{13} \quad \cdots (2)$

(1) + (2): $2\cos\theta = -\frac{10}{13}$

$\cos\theta = -\frac{5}{13}$

$$\cos\theta + \cos^3\theta + \cos^5\theta + \cdots = \frac{\cos\theta}{1-\cos^2\theta} = \frac{-\frac{5}{13}}{1-\frac{25}{169}} = \frac{-65}{144}$$

G4 As shown in Figure 2, $ABCD$ is a square. P is a point lies in $ABCD$ such that $AP = 2$ cm, $BP = 1$ cm and $\angle APB = 105°$.
If $CP^2 + DP^2 = x$ cm^2, find the value of x .



Reference: 1999 HG10

Let $CP = c$ cm, $DP = d$ cm

Rotate ΔAPB about A anticlockwise through 90° to ΔAQD.

Rotate ΔAPB about B clockwise through 90° to ΔCRB .

Join PQ, PR .

$AQ = AP = 2$cm, $\angle PAQ = 90°$, $BR = BP = 1$cm, $\angle PBR = 90°$

ΔAPQ and ΔBPR are right angled isosceles triangles.

$\angle AQP = 45°$, $\angle BRP = 45°$

$PQ = 2\sqrt{2}$ cm, $PR = \sqrt{2}$ cm (Pythagoras' theorem)

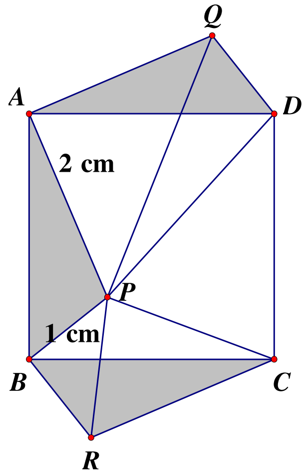
$\angle DQP = 105° - 45° = 60°$, $\angle CRP = 105° - 45° = 60°$

Apply cosine formula on ΔDQP and ΔCRP.

$DP^2 = \left(2\sqrt{2}\right)^2 + 1^2 - 2(1)\left(2\sqrt{2}\right)\cos 60° = 9 - 2\sqrt{2}$ cm^2

$CP^2 = \left(\sqrt{2}\right)^2 + 2^2 - 2(2)\left(\sqrt{2}\right)\cos 60° = 6 - 2\sqrt{2}$ cm^2

$\therefore x = 6 - 2\sqrt{2} + 9 - 2\sqrt{2} = 15 - 4\sqrt{2}$



G5 If x, y are real numbers and $x^2 + 3y^2 = 6x + 7$, find the maximum value of $x^2 + y^2$.

$x^2 + 3y^2 = 6x + 7 \Rightarrow (x - 3)^2 + 3y^2 = 16$ ⋯⋯ (1) and $y^2 = \frac{1}{3}\left(-x^2 + 6x + 7\right)$ ⋯⋯ (2)

Sub. (2) into $x^2 + y^2$:

$$x^2 + y^2 = \frac{1}{3}\left(3x^2 - x^2 + 6x + 7\right)$$
$$= \frac{1}{3}\left(2x^2 + 6x + 7\right)$$
$$= \frac{1}{3}\left[2\left(x^2 + 3x\right) + 7\right]$$
$$= \frac{1}{3}\left[2\left(x^2 + 3x + 1.5^2\right) - 2\times 1.5^2 + 7\right]$$
$$= \frac{1}{3}\left[2(x + 1.5)^2 + 2.5\right]$$
$$= \frac{2}{3}(x + 1.5)^2 + \frac{5}{6}$$

From (1), $3y^2 = 16 - (x - 3)^2 \geq 0$

$\Rightarrow -4 \leq x - 3 \leq 4$

$\Rightarrow -1 \leq x \leq 7$

$0.5 \leq x + 1.5 \leq 8.5$

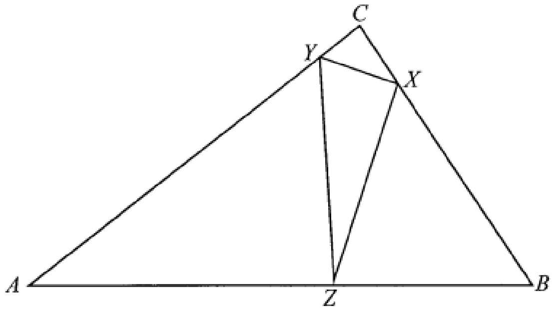
$0.25 \leq (x + 1.5)^2 \leq 72.25$

$\frac{1}{6} \leq \frac{2}{3}(x + 1.5)^2 \leq \frac{289}{6}$

$1 \leq \frac{2}{3}(x + 1.5)^2 + \frac{5}{6} \leq \frac{289}{6} + \frac{5}{6} = 49$

The maximum value of $x^2 + y^2$ is 49.

G6 As shown in Figure 3, X, Y and Z are points on BC, CA and AB of ΔABC respectively such that $\angle AZY = \angle BZX$, $\angle BXZ = \angle CXY$ and $\angle CYX = \angle AYZ$. If $AB = 10$, $BC = 6$ and $CA = 9$, find the length of AZ.



Let $\angle AZY = \gamma$, $\angle BXZ = \alpha$ and $\angle CYX = \beta$.

$\angle ZXY = 180° - 2\alpha$ (adj. $\angle$s on st. line)

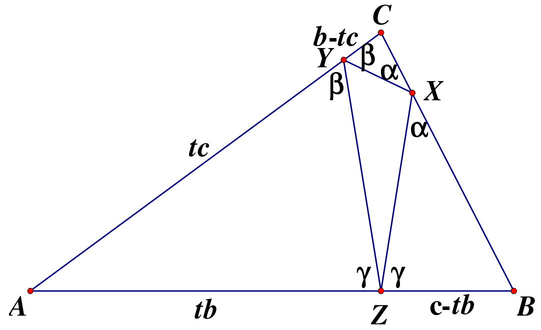
$\angle XYZ = 180° - 2\beta$ (adj. $\angle$s on st. line)

$\angle YZX = 180° - 2\gamma$ (adj. $\angle$s on st. line)

$\angle ZXY + \angle XYZ + \angle YZX = 180°$ ($\angle$s sum of Δ)

$180° - 2\alpha + 180° - 2\beta + 180° - 2\gamma = 180°$

$\Rightarrow \alpha + \beta + \gamma = 180°$ …… (1)



In ΔCXY, $\angle C + \alpha + \beta = 180°$ ($\angle$s sum of Δ)

$\angle C = 180° - (\alpha + \beta) = \gamma$ by (1)

Similarly, $\angle B = \beta$, $\angle A = \alpha$

$\therefore \Delta AYZ \sim \Delta ABC$, $\Delta BXZ \sim \Delta BAC$, $\Delta CXY \sim \Delta CAB$ (equiangular)

Let $BC = a$, $CA = b$, $AB = c$.

$\frac{AZ}{AC} = \frac{AY}{AB} = t$ (corr. sides, $\sim\Delta$'s), where t is the proportional constant

$\frac{AZ}{b} = \frac{AY}{c} = t \Rightarrow AZ = bt, AY = ct$

$BZ = AB - AZ = c - tb$; $CY = AC - AY = b - tc$

$\frac{BZ}{BC} = \frac{BX}{AB}$ (corr. sides, $\sim\Delta$'s)

$\frac{c - tb}{a} = \frac{BX}{c} \Rightarrow BX = \frac{c^2 - bct}{a}$ …… (1)

$\frac{CY}{BC} = \frac{CX}{AC}$ (corr. sides, $\sim\Delta$'s)

$\frac{b - tc}{a} = \frac{CX}{b} \Rightarrow CX = \frac{b^2 - bct}{a}$ …… (2)

$BX + CX = BC$

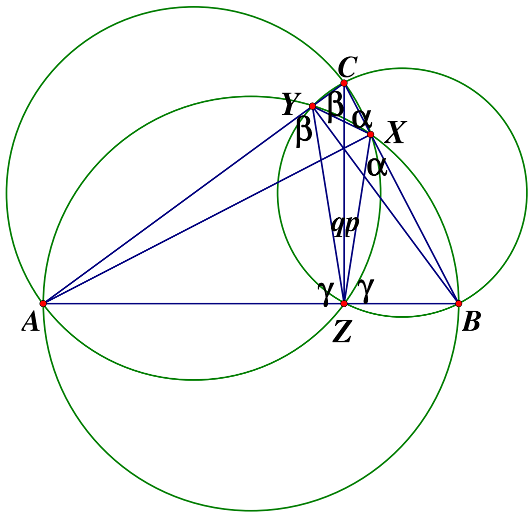
$\frac{c^2 - bct}{a} + \frac{b^2 - bct}{a} = a$ by (1) and (2)

$b^2 + c^2 - 2bct = a^2$

$AZ = tb = \frac{b^2 + c^2 - a^2}{2c} = \frac{9^2 + 10^2 - 6^2}{2\times 10} = \frac{145}{20} = \frac{29}{4}$ (= 7.25)

Method 2

Join AX, BY, CZ.



Let $\angle AZY = \gamma$, $\angle BXZ = \alpha$ and $\angle CYX = \beta$.

$\angle ZXY = 180° - 2\alpha$ (adj. $\angle$s on st. line)

$\angle XYZ = 180° - 2\beta$ (adj. $\angle$s on st. line)

$\angle YZX = 180° - 2\gamma$ (adj. $\angle$s on st. line)

$\angle ZXY + \angle XYZ + \angle YZX = 180°$ ($\angle$ sum of Δ)

$180° - 2\alpha + 180° - 2\beta + 180° - 2\gamma = 180°$

$\Rightarrow \alpha + \beta + \gamma = 180° \cdots\cdots (1)$

In ΔCXY, $\angle C + \alpha + \beta = 180°$ ($\angle$ sum of Δ)

$\angle C = 180° - (\alpha + \beta) = \gamma$ by (1)

Similarly, $\angle B = \beta$, $\angle A = \alpha$

$\therefore$ $ABXY$, $BCYZ$, $CAZX$ are cyclic quadrilaterals (ext. $\angle$ = int. opp. $\angle$)

Let $\angle XZC = p$, $\angle YZC = q$.

Then $\angle XBY = \angle CBY = \angle CZY = q$ ($\angle$s in the same segment)

$\angle XAY = \angle XAC = \angle XZC = p$ ($\angle$s in the same segment)

But $\angle XAY = \angle XBY$ ($\angle$s in the same segment)

$\therefore p = q$

On the straight line AZB, $\gamma + q + p + \gamma = 180°$ (adj. $\angle$s on st. line)

$\therefore \angle AZC = \angle BZC = 90°$

i.e. CZ is an altitude of ΔABC.

By cosine formula, $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{9^2+10^2-6^2}{2\times 9\times 10} = \dfrac{145}{180} = \dfrac{29}{36}$

$AZ = AC\cos A = 9\times\dfrac{29}{36} = \dfrac{29}{4}$

G7 Given that a, b, c and d are four distinct numbers, where $(a+c)(a+d) = 1$ and $(b+c)(b+d) = 1$. Find the value of $(a+c)(b+c)$. (**Reference: 2002 HI7, 2006 HG6, 2009 FI3.3**)

$$\begin{cases} a^2+ac+ad+cd=1 & \cdots\cdots(1) \\ b^2+bc+bd+cd=1 & \cdots\cdots(2) \end{cases}$$

(1) – (2): $a^2 - b^2 + (a-b)c + (a-b)d = 0$

$(a-b)(a+b+c+d) = 0$

$\because a - b \neq 0 \therefore a + b + c + d = 0$

$\Rightarrow b + c = -(a + d)$

$(a+c)(b+c) = -(a+c)(a+d) = -1$

G8 Let $a_1 = 215$, $a_2 = 2014$ and $a_{n+2} = 3a_{n+1} - 2a_n$, where n is a positive integer . Find the value of $a_{2014} - 2a_{2013}$.

$a_{n+2} = 3a_{n+1} - 2a_n$

$\Rightarrow a_{n+2} - 2a_{n+1} = a_{n+1} - 2a_n$

$a_{2014} - 2a_{2013} = a_{2013} - 2a_{2012} = a_{2012} - 2a_{2011}$

$= \cdots\cdots = a_2 - 2a_1$

$= 2014 - 2(215) = 1584$

G9 Given that the minimum value of the function $y = \sin^2 x - 4\sin x + m$ is $\frac{-8}{3}$.

Find the minimum value of m^y.

$y = \sin^2 x - 4\sin x + m = (\sin x - 2)^2 + m - 4$

$m - 3 \le (\sin x - 2)^2 + m - 4 \le m + 5$

$m - 3 = \frac{-8}{3}$

$m = \frac{1}{3}$

$\frac{-8}{3} \le y \le \frac{16}{3}$

$3^{\frac{8}{3}} = \left(\frac{1}{3}\right)^{-\frac{8}{3}} \ge m^y \ge \left(\frac{1}{3}\right)^{\frac{16}{3}}$

$\therefore$ The minimum value of m^y is $\left(\frac{1}{3}\right)^{\frac{16}{3}} = 3^{-\frac{16}{3}}$.

G10 Given that $\tan\left(\frac{90^\circ}{\tan x}\right) \times \tan(90^\circ \tan x) = 1$ and $1 < \tan x < 3$. Find the value of $\tan x$.

$\frac{90^\circ}{\tan x} + 90^\circ \tan x = 90^\circ$ or $\frac{90^\circ}{\tan x} + 90^\circ \tan x = 270^\circ$ or $\frac{90^\circ}{\tan x} + 90^\circ \tan x = 90^\circ \cdot (2m+1)$, $m \in \mathbf{Z}$

$\frac{1}{\tan x} + \tan x = 1$ or $\frac{1}{\tan x} + \tan x = 3$ or $\frac{1}{\tan x} + \tan x = 2m+1$

$\tan^2 x - \tan x + 1 = 0$ or $\tan^2 x - 3\tan x + 1 = 0$ or $\tan^2 x - (2m + 1)\tan x + 1 = 0$

$\Delta = -3 < 0$, no solution or $\tan x = \frac{3 \pm \sqrt{5}}{2}$ or $\frac{2m+1 \pm \sqrt{(2m+1)^2 - 4}}{2}$

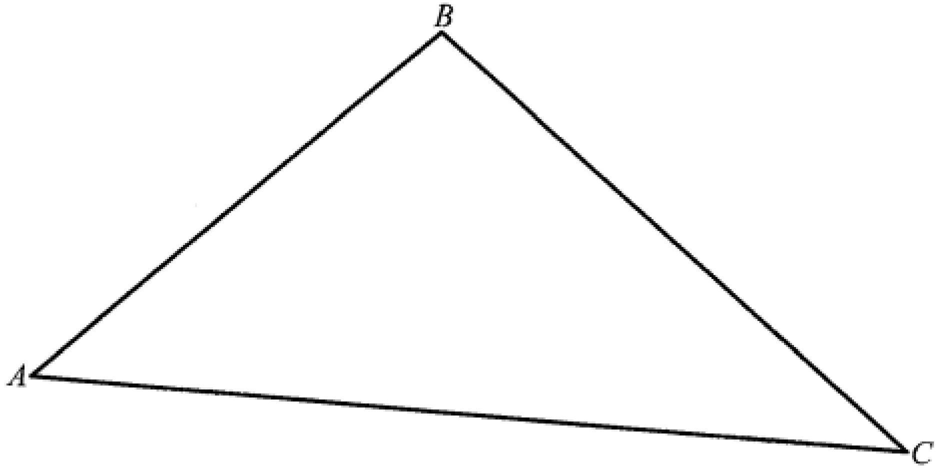
$\because$ $1 < \tan x < 3$ and $\frac{3+\sqrt{5}}{2} \approx 2.6$, $\frac{3-\sqrt{5}}{2} \approx 0.4$

$\therefore$ $\tan x = \frac{3+\sqrt{5}}{2}$ only

Geometrical Construction

1. Figure 1 shows a ΔABC. Construct a circle with centre O inside the triangle such that the three sides of the triangle are tangents to the circle.

Reference: 2009 HSC1, 2012 HC2, 2019 HC3



The steps are as follows: (The question is the same as 2009 construction sample paper Q1)

(1) Draw the bisector of $\angle BAC$.

(2) Draw the bisector of $\angle ACB$.

O is the intersection of the two angle bisectors.

(3) Join BO.

(4) Let R, S, T be the feet of perpendiculars from O onto AC, BC and AB respectively.

$\Delta AOT \cong \Delta AOR$ (A.A.S.)

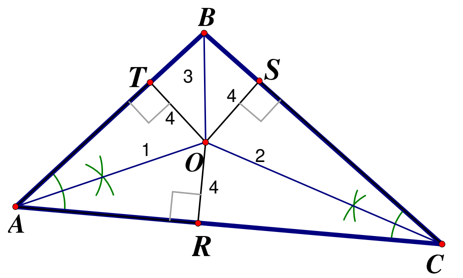
$\Delta COS \cong \Delta COR$ (A.A.S.)

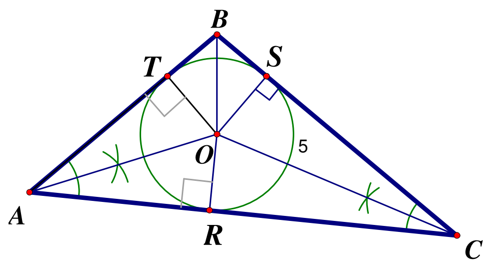
$OT = OR = OS$ (Corr. sides, $\cong \Delta$'s)

$\Delta BOT \cong \Delta BOS$ (R.H.S.)

$\angle OBT = \angle OBS$ (Corr. $\angle$s, $\cong \Delta$'s)

$\therefore$ BO is the angle bisector of $\angle ABC$.

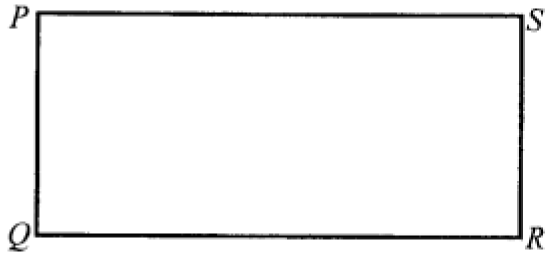




The three angle bisectors are concurrent at one point.

(5) Using O as centre, OR as radius to draw a circle. This circle touches ΔABC internally at R, S, and T. It is called the **inscribed circle**.

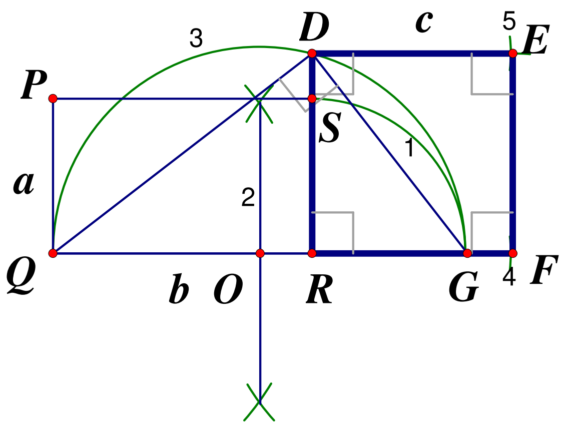
2. Figure 2 shows a rectangle $PQRS$. Construct a square of area equal to that of a rectangle.



Reference: C:\Users\孔德偉\Dropbox\Data\My Web\Home_Page\Geometry\7 Construction by ruler and compasses\others\rectangle_into_rectangle.pdf

作圖方法如下：

假設該長方形為 $PQRS$，其中 $PQ = a$，$QR = b$。



(1) 以 R 為圓心，RS 為半徑作一弧，交 QR 的延長線於 G。

(2) 作 QG 的垂直平分線，O 為 QG 的中點。

(3) 以 O 為圓心，OQ 為半徑作一半圓，交 RS 的延長線於 D，連接 QD、DG。

(4) 以 R 為圓心，RD 為半徑作一弧，交 QR 的延長線於 F。

(5) 以 F 為圓心，FR 為半徑作一弧，以 D 為圓心，DR 為半徑作一弧，兩弧相交於 E。

(6) 連接 DE、FE。

作圖完畢，證明如下：

$\angle GDQ = 90°$ (半圓上的圓周角)

$RG = RS = a$

$\Delta DRG \sim \Delta QRD$ (等角)

$\frac{RG}{DR} = \frac{DR}{QR}$ (相似三角形三邊成比例)

$DR^2 = ab \cdots\cdots (1)$

$RF = DR = DE = EF$ (半徑相等)

$\angle DRF = 90°$ (直線上的鄰角)

$\therefore$ $DEFR$ 便是該正方形，其面積與長方形 $PQRS$ 相等。(由(1)式得知)

證明完畢。

3. 圖三所示為兩綫段 AB 及 AC 相交於 A 點。試在它們之間構作兩個大小不同的圓使得

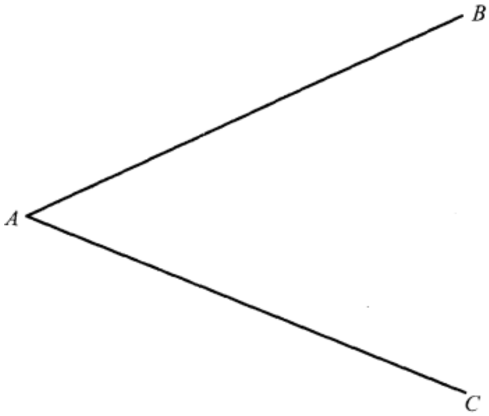
(i) 該兩圓相切於一點；及

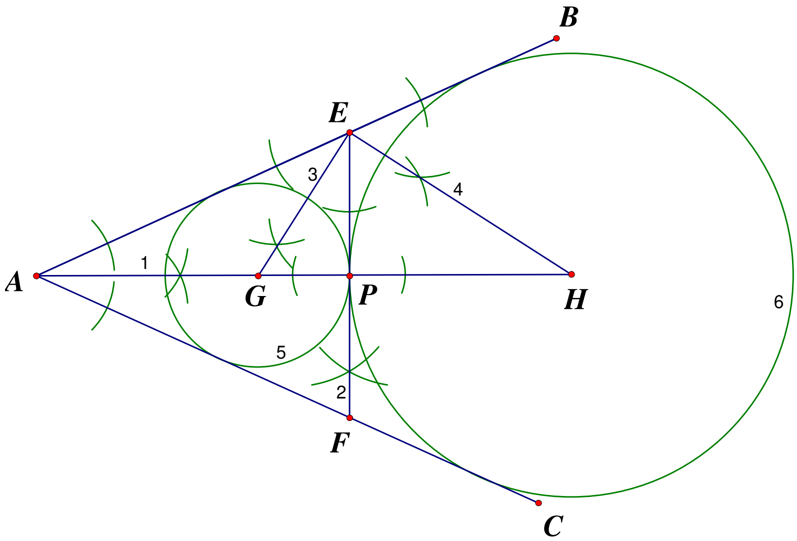
(ii) 綫段 AB 及 AC 均為該圓的切綫。

Figure 3 shows two line segments AB and AC intersecting at the point A. Construct two circles of different sizes between them such that

(i) They touch each other at a point; and

(ii) the line segments AB and AC are tangents to both circles.





Steps (Assume that $\angle BAC < 180°$, otherwise we cannot construct the circles touching $\angle BAC$.)

(1) Draw the angle bisector AH of $\angle BAC$.

(2) Choose any point P on AH. Construct a line through P and perpendicular to AH, intersecting AB and AC at E and F respectively.

(3) Draw the angle bisector EG of $\angle AEF$, intersecting AH at G.

(4) Draw the angle bisector EH of $\angle BEF$, intersecting AH at H.

(5) Use G as radius, GP as radius to draw a circle.

(6) Use H as radius, HP as radius to draw another circle.

The two circles in steps (5) and (6) are the required circles satisfying the conditions.

Proof: $\because$ G is the incentre of ΔAEF and H is the excentre of ΔAEF

$\therefore$ The two circles in steps (5) and (6) are the incircle and the excircle satisfying the conditions.

Remark: The question Chinese version and English version have different meaning, so I have changed it. The original question is:

圖三所示為兩相交於 A 點的綫段 AB 及 AC。試在它們之間構作兩個大小不同的圓使得

(i) 該兩圓相交於一點；及

(ii) 綫段 AB 及 AC 均為該圓的切綫。

A suggested solution to the Chinese version is given as follows:

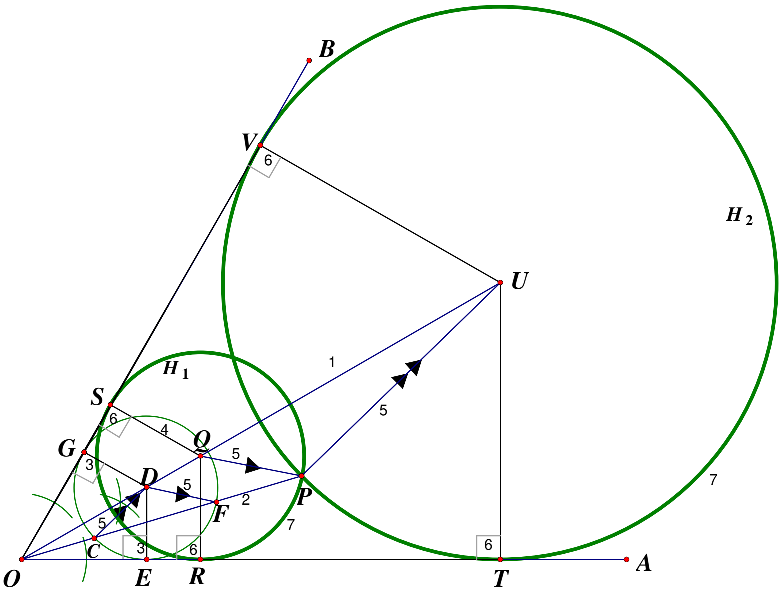
作圖方法如下：

(1) 作 AOB 的角平分綫 OU。

(2) 找一點 P 不在角平分綫上，連接 OP。

(3) 在角平分綫上任取一點 D。分別作過 D 且垂直於 OA 及 OB 之綫段，E 和 G 分別為兩垂足。

(4) 以 D 為圓心，DE 為半徑作一圓，交 OP 於 C 及 F，其中 $OC < OF$。



(5) 連接 DF，過 P 作一綫段與 DF 平行，交角平分綫於 Q。
連接 CD，過 P 作一綫段與 CD 平行，交角平分綫於 U。

(6) 分別作過 Q 且垂直於 OA 及 OB 之綫段，R 和 S 分別為兩垂足。
分別作過 U 且垂直於 OA 及 OB 之綫段，T 和 V 分別為兩垂足。

(7) 以 Q 為圓心，QR 為半徑作一圓 H_1。以 U 為圓心，UT 為半徑作另一圓 H_2。

作圖完畢。

證明如下：

一如上文分析，步驟 4 的圓分別切 OA 及 OB 於 E 及 G。

$\angle QOR = \angle QOS$ (角平分綫)

$OQ = OQ$ (公共邊)

$\angle QRO = 90° = \angle QSO$ (由作圖所得)

$\therefore \Delta QOR \cong \Delta QOS$ (A.A.S.)

$QR = QS$ (全等三角形的對應邊)

圓 H_1 分別切 OA 及 OB 於 R 及 S。 (切綫⊥半徑的逆定理)

$\Delta ODG \sim \Delta OQS$ 及 $\Delta ODF \sim \Delta OQP$ (等角)

$\dfrac{QS}{DG} = \dfrac{OQ}{OD}$ 及 $\dfrac{OQ}{OD} = \dfrac{QP}{DF}$ (相似三角形的對應邊)

$\therefore \dfrac{QS}{DG} = \dfrac{QP}{DF}$

$\because DG = DF$

$\therefore QS = QP$

$\therefore$ 圓 H_1 經過 P。

利用相同的方法，可證明圓 H_2 分別切 OA 及 OB 於 T 及 V，及經過 P。

證明完畢。