Examples on Mathematical Induction: Product

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1. Prove that $(2+1)(2^2+1)(2^4+1)\cdots(2^{2^n}+1)=2^{2^{n+1}}-1$ for all positive integers n.
2. Prove that $(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n})=\dfrac{1-x^{2^{n+1}}}{1-x}$ for all positive integers n.
3. Prove that $(n+1)(n+2)\cdots(n+n)=2^n\cdot 1\cdot 3\cdot 5\cdots(2n-1)$ for all positive integers n.

Let $P(n) \equiv$ "$(n+1)(n+2)\cdots(n+n)=2^n\cdot 1\cdot 3\cdot 5\cdots(2n-1)$ for all positive integers n."

$n = 1$, LHS = 2, RHS = $2^1\cdot 1 = 2$, $P(1)$ is true.

Suppose $(k+1)(k+2)\cdots(k+k)=2^k\cdot 1\cdot 3\cdot 5\cdots(2k-1)$ for some positive integer k.

When $n = k + 1$, LHS
$$\begin{aligned}&=(k+2)\cdots(2k)(2k+1)(2k+2)\\&=2(k+1)(k+2)\cdots(2k)(2k+1)\\&=2\cdot 2^k\cdot 1\cdot 3\cdot 5\cdots(2k-1)(2k+1)\\&=2^{k+1}\cdot 1\cdot 3\cdot 5\cdots(2k+1)\end{aligned}$$

$\therefore$ If it is true for $n = k$ then it is also true for $n = k + 1$.

By the principle of mathematical induction, it is true for all positive $n > 1$.

4. Prove that $\left(1+\frac{a}{1}\right)\left(1+\frac{a}{2}\right)\cdots\left(1+\frac{a}{n}\right)=1+a+\frac{a(a+1)}{2!}+\cdots+\frac{a(a+1)\cdots(a+n-1)}{n!}$ for all positive integers n.

Let $P(n)\equiv$"$\left(1+\frac{a}{1}\right)\left(1+\frac{a}{2}\right)\cdots\left(1+\frac{a}{n}\right)=1+a+\frac{a(a+1)}{2!}+\cdots+\frac{a(a+1)\cdots(a+n-1)}{n!}$ for all positive integers n."

$n = 1$, LHS = $1 + a$, RHS = $1 + a$, $P(1)$ is true.

Suppose $\left(1+\frac{a}{1}\right)\left(1+\frac{a}{2}\right)\cdots\left(1+\frac{a}{k}\right)=1+a+\frac{a(a+1)}{2!}+\cdots+\frac{a(a+1)\cdots(a+k-1)}{k!}$ for some positive integer k.

$n = k + 1$,

$$\begin{aligned}\text{LHS}&=\left(1+\frac{a}{1}\right)\left(1+\frac{a}{2}\right)\cdots\left(1+\frac{a}{k}\right)\left(1+\frac{a}{k+1}\right)\\&=\left[1+a+\frac{a(a+1)}{2!}+\cdots+\frac{a(a+1)\cdots(a+k-1)}{k!}\right]\left(1+\frac{a}{k+1}\right)\\&=1+a+\frac{a(a+1)}{2!}+\cdots+\frac{a(a+1)\cdots(a+k-1)}{k!}+\left(1+\frac{a}{1}\right)\left(1+\frac{a}{2}\right)\cdots\left(1+\frac{a}{k}\right)\frac{a}{k+1}\\&=1+a+\frac{a(a+1)}{2!}+\cdots+\frac{a(a+1)\cdots(a+k-1)}{k!}+\left(\frac{a+1}{1}\right)\left(\frac{a+2}{2}\right)\cdots\left(\frac{a+k}{k}\right)\frac{a}{k+1}\\&=1+a+\frac{a(a+1)}{2!}+\cdots+\frac{a(a+1)\cdots(a+k-1)}{k!}+\frac{a(a+1)\cdots(a+k-1)(a+k)}{(k+1)!}\\&=\text{RHS}\end{aligned}$$

$\therefore$ If it is true for $n = k$ then it is also true for $n = k + 1$.

By the principle of mathematical induction, it is true for all positive integers n.

5. **1985 Paper 2 Q2**

Let $T_n = \dfrac{n(n+2)}{(n+1)^2}$, where n is a positive integer.

Prove by mathematical induction that $T_1 \times T_2 \times \cdots \times T_n = \dfrac{n+2}{2(n+1)}$ for all n .

Let $P(n) \equiv$ "$T_1 \times T_2 \times \cdots \times T_n = \dfrac{n+2}{2(n+1)}$ for all positive integer n ."

$n = 1$, $T_1 = \dfrac{1(1+2)}{(1+1)^2} = \dfrac{3}{4}$, R.H.S. $= \dfrac{1+2}{2(1+1)} = \dfrac{3}{4} =$ L.H.S.

$P(1)$ is true

Suppose $P(k)$ is true for some positive integer k .

i.e. $T_1 \times T_2 \times \cdots \times T_k = \dfrac{k+2}{2(k+1)}$

Multiply both sides by T_{k+1} :

$$T_1 \times T_2 \times \cdots \times T_k \times T_{k+1} = \frac{k+2}{2(k+1)} \times \frac{(k+1)(k+1+2)}{(k+1+1)^2}$$

$$= \frac{k+2}{2(k+1)} \times \frac{(k+1)(k+3)}{(k+2)^2}$$

$$= \frac{k+3}{2(k+2)} = \frac{k+1+2}{2(k+1+1)}$$

If $P(k)$ is true then $P(k + 1)$ is also true

By the principle of mathematical induction, $T_1 \times T_2 \times \cdots \times T_n = \dfrac{n+2}{2(n+1)}$ for all positive integer n .

6. (a) Prove that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for all positive integers $n \ge 2$.

(b) Hence, find the values of

(i) $\frac{3}{4}\cdot\frac{8}{9}\cdot\frac{15}{16}\cdots\frac{99}{100}$, and

(ii) $\frac{120}{121}\cdot\frac{143}{144}\cdot\frac{168}{169}\cdots\frac{399}{400}$.

(c) If $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\le\frac{22}{39}$, find the smallest possible value of n .

6. (a) Let $P(n) \equiv$ "$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for all positive integers $n \ge 2$."

When $n = 2$.

L.H.S. $=1-\frac{1}{2^2}=\frac{3}{4}$; R.H.S. $=\frac{2+1}{2\times 2}=\frac{3}{4}$

$\therefore$ P(1) is true.

Suppose P(k) is true for some positive integer $k \ge 2$.

i.e. $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$

When $n = k + 1$,

To prove $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left[1-\frac{1}{(k+1)^2}\right]=\frac{k+2}{2(k+1)}$.

$$\text{LHS}=\frac{k+1}{2k}\cdot\left[1-\frac{1}{(k+1)^2}\right]=\frac{k+1}{2k}\cdot\frac{k^2+2k+1-1}{(k+1)^2}=\frac{k^2+2k}{2k(k+1)}=\frac{k(k+2)}{2k(k+1)}=\frac{k+2}{2(k+1)}=\text{RHS}$$

P(k + 1) is true when P(k) is true.

By the principle of mathematical induction, P(n) is true for all positive integer $n \ge 2$

(b) (i) $\frac{3}{4}\cdot\frac{8}{9}\cdot\frac{15}{16}\cdots\frac{99}{100}$

$$=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{10^2}\right)=\frac{10+1}{2(10)}=\frac{11}{20}$$

(ii)
$$\frac{120}{121}\cdot\frac{143}{144}\cdot\frac{168}{169}\cdots\frac{399}{400}=\frac{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{20^2}\right)}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{10^2}\right)}$$

$$=\frac{\frac{20+1}{2\times 20}}{\frac{11}{20}}=\frac{21}{22}$$

(c) $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\le\frac{22}{39}$

$\frac{n+1}{2n}\le\frac{22}{39}$

$39n + 39 \le 44n$

$5n \ge 39$

$n \ge 7.8$

$\therefore$ The smallest $n = 8$.