Examples on Mathematical Induction: Inequality

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1. Prove by M.I. that $2^{n+4} > 2n + 9$ for all positive integers n .
2. Prove by M.I. that $2n + 7 < 2^{n+3}$ for all positive integers n .
3. Prove that $10^n > 6^n + 8^n$ for all positive integers $n > 2$.
4. Prove that $(n + 3)(n + 4) \geq 0$ for all positive integers $n \geq -4$.
5. Prove that $n! \geq 2^n > n^2$ for $n \geq 5$.
6. Prove that $2^n > n^3$ for $n \geq 10$.
7. Prove that $3^n > n^3$ for $n > 3$.
8. If $x > -1$ and $x \neq 0$, prove by mathematical induction that $(1 + x)^n > 1 + nx$ for all positive integers $n > 1$.
   $n = 2$, $(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x =$ R.H.S., it is true for $n = 2$
   Suppose $(1 + x)^k > 1 + kx$ for some positive integer $k > 1$.
   Multiply both sides by $(1 + x)$, $(1 + x)^{k+1} > (1 + x)(1 + kx) = 1 + (k + 1)x + kx^2$
   $> 1 + (k + 1)x =$ R.H.S.
   If it is true for $n = k$, then it is also true for $n = k + 1$.
   By the principle of mathematical induction, it is true for all positive integer $n > 1$.
9. Prove that $\dfrac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n)} < \dfrac{1}{\sqrt{2n+1}}$ for all $n \geq 1$.
10. If $a > 0$ and $a \neq 1$, prove that $\dfrac{a^{2n+2}-1}{a\left(a^{2n}-1\right)} > \dfrac{n+1}{n}$ for all $n \geq 1$. (using reciprocal)
11. If $p_0 = 1$, and p_1, p_2, $\cdots$, p_r, $\cdots$ are positive numbers such that $p_{r+1}^2 > p_r\, p_{r+2}$, then
    $p_1 > p_2^{1/2} > \cdots > p_2^{1/n}$.

11. $p_0 = 1$, $p_1^2 > p_0 p_2 = p_2$
    $\because$ $p_1 > 0$ $\therefore$ $p_1 > p_2^{1/2}$, it is true for $n = 1$.
    Suppose $p_1 > p_2^{1/2} > \cdots > p_k^{1/k}$ for some integer k
    $(p_0 p_2)(p_1 p_3)^2 (p_2 p_4)^3 \cdots (p_{k-1} p_{k+1})^k < p_1^2 p_2^4 p_3^6 \cdots p_k^{2k}$
    Cancel the common factors on both sides,
    $p_{k+1}^k < p_k^{k+1}$
    $\therefore$ $p_k^{1/k} > p_{k+1}^{1/(k+1)}$, it is also true for $n = k + 1$
    By M.I., it is true for all positive integer n .

12. Let $a_1 = 1$, $a_2 = 2$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$

    Prove by M.I. that $a_n < \left(\dfrac{1+\sqrt{5}}{2}\right)^n$ for all $n \geq 1$.

13. Let $a, b > 0$, prove that $\dfrac{a^n + b^n}{2} \geq \left(\dfrac{a+b}{2}\right)^n$ for all $n \geq 1$.

    (Hint: Using $(a^n - b^n)(a - b) > 0$ and consider $\dfrac{a^n + b^n}{2}\cdot\dfrac{a+b}{2}$.)