Individual Events

I1	a	1	**I2**	R	$*\frac{9}{25}$ see the remark	**I3**	S	10	**I4**	k	1
	b	$\frac{1}{2}$		S	1		R	30		r	2
	C	10		T	1		T	6		s	$\frac{1}{\sqrt{2}}\left(=\frac{\sqrt{2}}{2}\right)$
	D	-2		W	$\sqrt{5}$		P	$\sqrt{7}+2$		w	9

Group Events

G1	k	1	**G2**	w	45	**G3**	r	2006	**G4**	R	$12\sqrt{3}$
	B	$\frac{16}{15}$		z	-13		x	$\frac{7}{4}$ (=1.75)		S	*8 see the remark
	C	$\frac{1}{4}$		s	$\frac{1}{4}$		z	30		T	$\frac{1}{2}$
	a	1		t	14		R	$\frac{15}{4}$ (= 3.75)		W	2013021

Individual Event 1

I1.1 If a is a real number satisfying $\log_2(x+3) - \log_2(x+1) = 1$, find the value of a.

$$\log_2\frac{x+3}{x+1} = \log_2 2$$

$x + 3 = 2x + 2$

$x = 1 \Rightarrow a = 1$

I1.2 In Figure 1, O is the centre of the circle with radius 1 cm. If the length of the arc AB is equal to a cm and the area of the shaded sector OAB is equal to b cm^2, find the value of b. (Take $\pi = 3$)

$$b = \frac{1}{2}rs = \frac{1}{2}1\cdot 1 = \frac{1}{2}$$

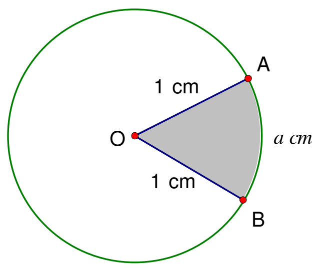

Figure 1

I1.3 An interior angle of a regular C-sided polygon is $288b°$, find the value of C.

Each interior angle = $288b° = 144°$

Each exterior angle = $36° = \frac{360°}{C}$

$C = 10$

I1.4 Given that C is a root of the equation $kx^2 + 2x + 5 = 0$, where k is a constant. If D is another root, find the value of D.

$100k + 20 + 5 = 0 \Rightarrow k = -\frac{1}{4}$

$C + D$ = sum of roots $= -\frac{2}{k}$

$10 + D = 8 \Rightarrow D = -2$

Individual Event 2

I2.1 Given that $a : b : c = 6 : 3 : 1$. If $R = \dfrac{3b^2}{2a^2 + bc}$, find the value of R.

Let $a = 6k$, $b = 3k$, $c = k$, then $R = \dfrac{3(3k)^2}{2(6k)^2 + (3k)k} = \dfrac{9}{25}$

Remark: original question is Given that a, b and c are three numbers not equal to 0 and $a : b : c = 6 : 3 : 1$ ……, the condition $a : b : c = 6 : 3 : 1$ already implies that a, b and c are not zero.

I2.2 Given that $\dfrac{|k+R|}{|R|} = 0$. If $S = \dfrac{|k+2R|}{|2k+R|}$, find the value of S.

$k + \dfrac{9}{25} = 0 \Rightarrow k = -\dfrac{9}{25}$

$S = \dfrac{|k+2R|}{|2k+R|} = \dfrac{\left|-\frac{9}{25} + \frac{18}{25}\right|}{\left|-\frac{18}{25} + \frac{9}{25}\right|} = 1$

I2.3 Given that $T = \sin 50°\times(S + \sqrt{3}\times\tan 10°)$, find the value of T.

$T = \sin 50°\times(1 + \sqrt{3}\cdot\dfrac{\sin 10^\circ}{\cos 10^\circ})$

$= \dfrac{\sin 50^\circ}{\cos 10^\circ}\cdot\left(\cos 10^\circ + \sqrt{3}\sin 10^\circ\right)$

$= \dfrac{2\sin 50^\circ}{\cos 10^\circ}\left(\dfrac{1}{2}\cos 10^\circ + \dfrac{\sqrt{3}}{2}\sin 10^\circ\right)$

$= \dfrac{2\sin 50^\circ}{\cos 10^\circ}\left(\cos 60^\circ\cdot\cos 10^\circ + \sin 60^\circ\cdot\sin 10^\circ\right)$

$= \dfrac{2\sin 50^\circ\cos 50^\circ}{\cos 10^\circ} = \dfrac{\sin 100^\circ}{\cos 10^\circ} = 1$

I2.4 Given that x_0 and y_0 are real numbers satisfying the system of equations $\begin{cases} y = \dfrac{T}{x} \\ y = |x| + T \end{cases}$.

If $W = x_0 + y_0$, find the value of W.

$\begin{cases} y = \dfrac{1}{x} \\ y = |x| + 1 \end{cases}$

$\dfrac{1}{x} = |x| + 1$ ....(*)

$\dfrac{1}{x} = x + 1$ or $\dfrac{1}{x} = -x + 1$

$1 = x^2 + x$ or $1 = -x^2 + x$

$x^2 + x - 1 = 0$ or $x^2 - x + 1 = 0$

$x = \dfrac{-1\pm\sqrt{5}}{2}$ or no solution

Check: sub. $x = \dfrac{-1-\sqrt{5}}{2}$ into (*)

LHS $= \dfrac{2}{-1-\sqrt{5}} < 0$, RHS > 0 (rejected)

When $x = \dfrac{-1+\sqrt{5}}{2}$; LHS $= \dfrac{2}{-1+\sqrt{5}} = \dfrac{\sqrt{5}+1}{2}$; RHS $= \dfrac{-1+\sqrt{5}}{2} + 1 = \dfrac{1+\sqrt{5}}{2}$ (accepted)

$y = \dfrac{1+\sqrt{5}}{2} \Rightarrow W = x_0 + y_0 = \dfrac{-1+\sqrt{5}}{2} + \dfrac{1+\sqrt{5}}{2} = \sqrt{5}$

Individual Event 3

I3.1 Given that $\frac{2x-3}{x^2-x}=\frac{A}{x-1}+\frac{B}{x}$,where A and B are constants. If $S=A^2+B^2$, find the value of S

$\frac{2x-3}{x^2-x}=\frac{Ax+B(x-1)}{(x-1)x}$

$A+B=2$, $-B=-3$

$A=-1$, $B=3$

$S=(-1)^2+3^2=10$

I3.2 In Figure 1, $ABCD$ is an inscribed rectangle, $AB=(S-2)$ cm and $AD=(S-4)$ cm. If the circumference of the circle is R cm, find the value of R. (Take $\pi=3$)

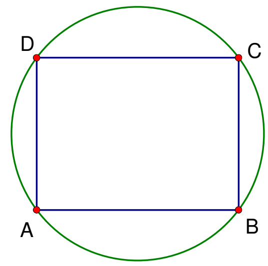

Figure 1

$AB=8$ cm, $CD=6$ cm

$AC=10$ cm (Pythagoras' theorem)

$R=10\pi=30$

I3.3 Given that x and y are integers satisfying the equation $\frac{R}{2}xy=21x+20y-13$.

If $T=xy$, find the value of T.

$15xy=21x+20y-13$

$(3x-4)(5y-7)=15$

$\begin{cases}3x-4=1\\5y-7=15\end{cases}$ or $\begin{cases}3x-4=3\\5y-7=5\end{cases}$ or $\begin{cases}3x-4=5\\5y-7=3\end{cases}$ or $\begin{cases}3x-4=15\\5y-7=1\end{cases}$ or $\begin{cases}3x-4=-1\\5y-7=-15\end{cases}$ or $\begin{cases}3x-4=-3\\5y-7=-5\end{cases}$ or $\begin{cases}3x-4=-5\\5y-7=-3\end{cases}$ or $\begin{cases}3x-4=-15\\5y-7=-1\end{cases}$

For integral solution: $3x-4=5$, $5y-7=3$

$x=3$, $y=2\Rightarrow T=3\times 2=6$

I3.4 Let a be the positive root of the equation $x^2-2x-T=0$.

If $P=3+\cfrac{T}{2+\cfrac{T}{2+\cfrac{T}{2+\cfrac{T}{a}}}}$, find the value of P.

$x^2-2x-6=0$

$a=1+\sqrt{7}$

$a^2-2a-6=0\Rightarrow a^2=2a+6\Rightarrow a=2+\frac{6}{a}$

$2+\frac{T}{a}=2+\frac{6}{a}=a\Rightarrow 2+\cfrac{T}{2+\cfrac{T}{a}}=2+\frac{6}{a}=a\Rightarrow 2+\cfrac{T}{2+\cfrac{T}{2+\cfrac{T}{a}}}=2+\frac{6}{a}=a$

$P=3+\frac{6}{a}=1+2+\frac{6}{a}=1+a=1+1+\sqrt{7}=2+\sqrt{7}$

Individual Event 4

I4.1 Let $\frac{k}{4}=\left(1+\frac{1}{2}+\frac{1}{3}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\times\left(\frac{1}{2}+\frac{1}{3}\right)$, find the value of k.

Reference: 1995 FG6.2

Let $x=1+\frac{1}{2}+\frac{1}{3}$, $y=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, then $\frac{k}{4}=x(y-1)-y(x-1)=-x+y=\frac{1}{4}\Rightarrow k=1$

I4.2 Let x and y be real numbers satisfying the equation $y^2+4y+4+\sqrt{x+y+k}=0$.

If $r=|xy|$, find the value of r.

Reference: 2005 FI4.1, 2009 FG1.4, 2011 FI4.3, 2013 FI1.4, 2015 HG4, 2015 FI1.1

The equation is equivalent to $(y+2)^2+\sqrt{x+y+k}=0$

which is a sum of two non-negative numbers.

$\Rightarrow y+2=0$ and $x+y+1=0$

$y=-2$ and $x-2+1=0$, $x=1$

$r=|-2\times1|=2$

I4.3 In Figure 1, there are eight positive numbers in series. Starting from the 3rd number, each number is the product of the previous two numbers. Given that the 5th number is $\frac{1}{r}$ and the 8th number is $\frac{1}{r^4}$. If the first number is s, find the value of s.

s				$\frac{1}{r}$			$\frac{1}{r^4}$

Let the 2nd number be a, then other numbers are as follows:

s	a	sa	sa^2	$s^2a^3=\frac{1}{r}$	s^3a^5	s^5a^8	$s^8a^{13}=\frac{1}{r^4}$

$s^2a^3=\frac{1}{r}$ .... (1)

$s^8a^{13}=\frac{1}{r^4}$ .... (2)

(2)÷(1)^4: $a=1$

Sub. $a=1$ into (1): $s^2=\frac{1}{r}=\frac{1}{2}$; $s>0\Rightarrow s=\frac{1}{\sqrt{2}}$

I4.4 Let [x] be the largest integer not greater than x. For example, [2.5] = 2.

Let $w=1+[10\times s^2]+[10\times s^4]+[10\times s^6]+\ldots+[10\times s^{2n}]+\ldots$, find the value of w.

$w=1+[10\times\frac{1}{2}]+[10\times\frac{1}{4}]+[10\times\frac{1}{8}]+\ldots+[10\times\frac{1}{2^n}]+\ldots$

$=1+5+2+1+0+0+\ldots=9$

Group Event 1

G1.1 Given that k is a real number. If $x^2 + 2kx - 3k^2$ can be divisible by $x - 1$, find the greatest value of k.

By factor theorem, $1^2 + 2k - 3k^2 = 0$

$3k^2 - 2k - 1 = 0$

$(3k + 1)(k - 1) = 0$

$k = -\frac{1}{3}$ or 1

Greatest value of $k = 1$

G1.2 Given that $x = x_0$ and $y = y_0$ satisfy the system of equations $\begin{cases} \frac{x}{3} + \frac{y}{5} = 1 \\ \frac{x}{5} + \frac{y}{3} = 1 \end{cases}$. If $B = \frac{1}{x_0} + \frac{1}{y_0}$, find the value of B.

$\frac{x}{3} + \frac{y}{5} = \frac{x}{5} + \frac{y}{3} = 1 \Rightarrow \frac{x}{3} - \frac{x}{5} = \frac{y}{3} - \frac{y}{5} \Rightarrow x = y$

Sub. $x = y$ into the first equation: $\frac{x}{3} + \frac{x}{5} = 1 \Rightarrow x = y = \frac{15}{8} \Rightarrow B = \frac{16}{15}$

G1.3 Given that $x = 2 + \sqrt{3}$ is a root of the equation $x^2 - (\tan\alpha + \cot\alpha)x + 1 = 0$.

If $C = \sin\alpha \times \cos\alpha$, find the value of C.

Let the other root be y, $(2 + \sqrt{3})y$ = product of roots $= 1 \Rightarrow y = 2 - \sqrt{3}$

$\tan\alpha + \cot\alpha$ = sum of roots $= 2 + \sqrt{3} + 2 - \sqrt{3} = 4$

$\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha} = 4 \Rightarrow \frac{1}{\sin\alpha\cos\alpha} = 4 \Rightarrow C = \sin\alpha \times \cos\alpha = \frac{1}{4}$

G1.4 Let a be an integer. If the inequality $|x + 1| < a - 1.5$ has no integral solution, find the greatest value of a.

$\because$ $|x+ 1| \ge 0$, In order that the equation has no integral solution, it is sufficient that $a - 1.5 < 0$

$a < 1.5$

Greatest integral value of $a = 1$

Group Event 2

G2.1 In Figure 1, PRS is a straight line, $PQ = PR = QS$ and $\angle QPR = 30°$. If $\angle RQS = w°$, find the value of w.

$\angle QPR = \angle QSP = 30°$ (base $\angle$s isos. Δ)

$\angle PQS = 120°$ ($\angle$s sum of Δ)

$\angle PQR = \angle PRQ = (180° - 30°) \div 2 = 75°$ ($\angle$s sum of isos. Δ)

$\angle RQS = 120° - 75° = 45° \Rightarrow w = 45$

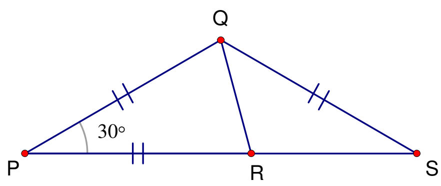

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Figure 1

G2.2 Let $f(x) = px^7 + qx^3 + rx - 5$, where p, q and r are real numbers.

If $f(-6) = 3$ and $z = f(6)$, find the value of z. **(Reference: 1995 FI1.3)**

$f(-6) = 3 \Rightarrow -p\times6^7 - q\times6^3 - 6r - 5 = 3$

$f(6) = p\times6^7 + q\times6^3 + 6r - 5 = -(-p\times6^7 - q\times6^3 - 6r - 5) - 10 = -3 - 10 = -13$

G2.3 If $n \neq 0$ and $s = \left(\frac{20}{2^{2n+4} + 2^{2n+2}}\right)^{\frac{1}{n}}$, find the value of s.

$$s = \left(\frac{20}{16\cdot2^{2n} + 4\cdot2^{2n}}\right)^{\frac{1}{n}} = \left(\frac{20}{20\cdot2^{2n}}\right)^{\frac{1}{n}} = \frac{1}{4}$$

G2.4 Given that x and y are positive integers and $x + y + xy = 54$. If $t = x + y$, find the value of t.

$1 + x + y + xy = 55$

$(1 + x)(1 + y) = 55$

$1 + x = 5$, $1 + y = 11$ or $1 + x = 11$, $1 + y = 5$

$x = 4$, $y = 10$ or $x = 10$, $y = 4$

$t = 14$

Group Event 3

G3.1 Given that $r = 2006 \times \frac{\sqrt{8} - \sqrt{2}}{\sqrt{2}}$, find the value of r.

It is easy to show that $r = 2006$.

G3.2 Given that $6^{x+y} = 36$ and $6^{x+5y} = 216$, find the value of x.

$x + y = 2$ .... (1)

$x + 5y = 3$ .... (2)

$5(1) - (2)$: $4x = 7 \Rightarrow x = \frac{7}{4}$

G3.3 Given that $\tan x + \tan y + 1 = \cot x + \cot y = 6$. If $z = \tan(x + y)$, find the value of z.

$\tan x + \tan y + 1 = \frac{\tan y + \tan x}{\tan x \tan y} = 6$

$\tan x + \tan y = 5$; $\tan x \tan y = \frac{5}{6}$

$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{5}{1 - \frac{5}{6}} = 30$

G3.4 In Figure 1, $ABCD$ is a rectangle, F is the midpoint of CD and $BE : EC = 1 : 3$. If the area of the rectangle $ABCD$ is 12 cm^2 and the area of $BEFD$ is R cm^2, find the value of R.

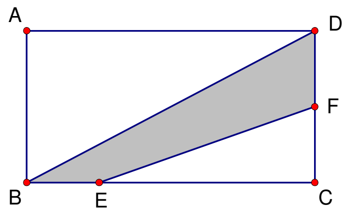

Figure 1

Area of ΔBCD = 6 cm^2

Area of $\Delta CEF = \frac{3}{4} \cdot \frac{1}{2} \cdot 6 \text{ cm}^2 = \frac{9}{4} \text{ cm}^2$

Area of $BEFD = (6 - \frac{9}{4}) \text{ cm}^2 = \frac{15}{4} \text{ cm}^2$

Group Event 4

G4.1 In Figure 1, $ABCD$ is a parallelogram, $BE \perp CD$, $BF \perp AD$, $CE = 2$ cm, $DF = 1$ cm and $\angle EBF = 60°$.
If the area of the parallelogram $ABCD$ is R cm^2, find the value of R.

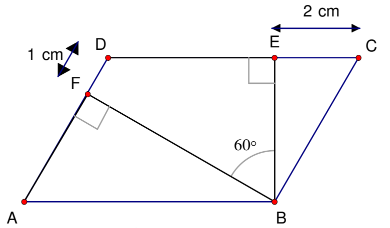

Figure 1

$\angle EDF = 360° - 90° - 90° - 60° = 120°$ ($\angle$s sum of polygon)
$\angle BAD = \angle BCD = 180° - 120° = 60°$ (int. $\angle$s //-lines)

$BC = \dfrac{2}{\cos 60^\circ}$ cm = 4 cm = AD

$BE = 2 \tan 60° = 2\sqrt{3}$ cm

$AF = (4 - 1)$ cm = 3 cm

$AB = \dfrac{3}{\cos 60^\circ}$ cm = 6 cm

Area of $ABCD = AB \times BE = 6 \times 2\sqrt{3}$ cm^2 = $12\sqrt{3}$ cm^2

G4.2 Given that a and b are positive numbers and $a + b = 2$. If $S = \left(a + \dfrac{1}{a}\right)^2 + \left(b + \dfrac{1}{b}\right)^2$, find the minimum value S. (**Reference: HKAL Pure Mathematics 1964 Paper 1 Q5 (b)**)

$\left(\sqrt{a} - \sqrt{b}\right)^2 \ge 0 \Rightarrow a + b - 2\sqrt{ab} \ge 0 \Rightarrow 1 \ge \sqrt{ab} \Rightarrow 1 \ge ab \cdots\cdots (1)$

$a^2 + b^2 = (a + b)^2 - 2ab = 4 - 2ab \ge 4 - 2 = 2 \cdots \cdots (2)$

$\dfrac{1}{ab} \ge 1 \Rightarrow \dfrac{1}{a^2b^2} \ge 1 \cdots\cdots (3)$

$$S = \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 = a^2 + b^2 + \frac{1}{a^2} + \frac{1}{b^2} + 4$$

$$= a^2 + b^2 + \frac{a^2 + b^2}{a^2b^2} + 4$$

$$= \left(a^2 + b^2\right)\left(1 + \frac{1}{a^2b^2}\right) + 4$$

$\ge 2 \times (1 + 1) + 4 = 8$ (by (2) and (3))

Remark: original question $\cdots$ a and b are positive real numbers $\cdots$
Positive numbers must be real, there is no need to emphasise the word 'real'.

G4.3 Let $2^x = 7^y = 196$. If $T = \dfrac{1}{x} + \dfrac{1}{y}$, find the value of T.

Reference: 2001 HI1, 2003 FG2.2, 2004 FG4.3, 2005 HI9

$x \log 2 = y \log 7 = \log 196$

$x = \dfrac{\log 196}{\log 2}$, $y = \dfrac{\log 196}{\log 7}$

$$T = \frac{1}{x} + \frac{1}{y} = \frac{\log 2 + \log 7}{\log 196} = \frac{\log 14}{\log 14^2} = \frac{1}{2}$$

Method 2 (provided by Denny)

$2 = 196^{\frac{1}{x}}$, $7 = 196^{\frac{1}{y}}$

$2 \times 7 = 14 = \sqrt{196} = 196^{\frac{1}{x}} \times 196^{\frac{1}{y}} = 196^{\frac{1}{x} + \frac{1}{y}}$

$\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2}$

G4.4 If $W = 2006^2 - 2005^2 + 2004^2 - 2003^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2$, find the value of W.

$W = (2006 + 2005)(2006-2005) + (2004 + 2003)(2004-2003) + \ldots + (4+3)(4-3) + (2+1)(2-1)$
$= 2006 + 2005 + 2004 + \ldots + 4 + 3 + 2 + 1$
$= \dfrac{2006}{2}(2006 + 1) = 1003 \times 2007 = 2013021$