

Series Example (HKAL 1992 Paper 1 Q10)

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Let $\{a_1, a_2, \dots\}$, $\{b_1, b_2, \dots\}$ be two sequence of real numbers, and $b_0 = 0$.

(a) Show that $\sum_{i=1}^k a_i (b_i - b_{i-1}) = a_k b_k + \sum_{i=1}^{k-1} (a_i - a_{i+1}) b_i$, $k = 2, 3, \dots$.

(b) Suppose $\{a_i\}$ is decreasing and $|b_i| \leq K$ for all i , where K is a constant.

Show that $\left| \sum_{i=1}^k a_i (b_i - b_{i-1}) \right| \leq K \{ |a_1| + 2|a_k| \}$, $k = 1, 2, \dots$.

(c) Using (b), or otherwise, show that for any positive integers n and p ,

$$\left| \sum_{i=n}^{n+p} \frac{(-1)^i}{i} \right| \leq \frac{3}{2n}$$

(Remark: the marking scheme in part (c) is wrong, thank for Mr. Ng Ka Lok in Wah Yan College, Kowloon for pointing out this.)

(a)
$$\begin{aligned} \sum_{i=1}^k a_i (b_i - b_{i-1}) &= a_1 (b_1 - b_0) + a_2 (b_2 - b_1) + \dots + a_k (b_k - b_{k-1}) \\ &= a_1 b_1 + a_2 b_2 - a_2 b_1 + \dots + a_k b_k - a_k b_{k-1}, \because b_0 = 0 \\ &= (a_1 - a_2) b_1 + (a_2 - a_3) b_2 + \dots + (a_{k-1} - a_k) b_{k-1} + a_k b_k \\ &= a_k b_k + \sum_{i=1}^{k-1} (a_i - a_{i+1}) b_i \end{aligned}$$

(b) Given $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$ and $|b_i| \leq K$ for $i = 0, 1, 2, \dots$

$$\begin{aligned} \left| \sum_{i=1}^k a_i (b_i - b_{i-1}) \right| &= \left| a_k b_k + \sum_{i=1}^{k-1} (a_i - a_{i+1}) b_i \right| \\ &\leq |a_k b_k| + \left| \sum_{i=1}^{k-1} (a_i - a_{i+1}) b_i \right| \\ &\leq K |a_k| + \sum_{i=1}^{k-1} |(a_i - a_{i+1})| |b_i| \\ &\leq K |a_k| + K \sum_{i=1}^{k-1} (a_i - a_{i+1}) \\ &\leq K \left\{ |a_k| + \sum_{i=1}^{k-1} (a_i - a_{i+1}) \right\}, \because a_i - a_{i+1} \geq 0 \\ &\leq K \{ |a_k| + a_1 - a_k \} \\ &\leq K \{ |a_1| + 2|a_k| \} \end{aligned}$$

(c) Case 1 $p = 2m + 1$,

$$\begin{aligned} \sum_{i=n}^{n+p} \frac{(-1)^i}{i} &= \left[\frac{(-1)^n}{n} + \frac{(-1)^{n+1}}{n+1} + \dots + \frac{(-1)^{n+p}}{n+p} \right] \\ &= (-1)^n \left[\frac{1}{n} - \frac{1}{n+1} + \dots + \frac{(-1)^{2m+1}}{n+2m+1} \right] \\ &= (-1)^n \left[\left(\frac{1}{n} - \frac{1}{n+1} \right) + \left(\frac{1}{n+2} - \frac{1}{n+3} \right) + \dots + \left(\frac{1}{n+2m} - \frac{1}{n+2m+1} \right) \right] \end{aligned}$$

On the other hand,

$$\sum_{i=n}^{n+p} \frac{(-1)^i}{i} = (-1)^n \left[\frac{1}{n} - \left(\frac{1}{n+1} - \frac{1}{n+2} \right) - \left(\frac{1}{n+3} - \frac{1}{n+4} \right) - \dots - \left(\frac{1}{n+2m-1} - \frac{1}{n+2m} \right) - \frac{1}{n+2m+1} \right]$$

$$\begin{aligned} \therefore 0 &< \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \cdots + \left(\frac{1}{n+2m} - \frac{1}{n+2m+1}\right) \text{ and} \\ \frac{1}{n} - \left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \left(\frac{1}{n+3} - \frac{1}{n+4}\right) - \cdots - \left(\frac{1}{n+2m-1} - \frac{1}{n+2m}\right) - \frac{1}{n+2m+1} &< \frac{1}{n} \\ \therefore 0 &< \left| \sum_{i=n}^{n+p} \frac{(-1)^i}{i} \right| < \frac{1}{n} = \frac{2}{2n} < \frac{3}{2n} \end{aligned}$$

Case 2 $p = 2m$

$$\begin{aligned} \sum_{i=n}^{n+p} \frac{(-1)^i}{i} &= \left[\frac{(-1)^n}{n} + \frac{(-1)^{n+1}}{n+1} + \cdots + \frac{(-1)^{n+p}}{n+p} \right] \\ &= (-1)^n \left[\frac{1}{n} - \frac{1}{n+1} + \cdots + \frac{(-1)^{2m}}{n+2m} \right] \\ &= (-1)^n \left[\left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \cdots + \left(\frac{1}{n+2m-2} - \frac{1}{n+2m-1}\right) + \frac{1}{n+2m} \right] \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{i=n}^{n+p} \frac{(-1)^i}{i} &= (-1)^n \left[\frac{1}{n} - \left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \left(\frac{1}{n+3} - \frac{1}{n+4}\right) - \cdots - \left(\frac{1}{n+2m-1} - \frac{1}{n+2m}\right) \right] \\ \therefore 0 &< \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \cdots + \left(\frac{1}{n+2m-2} - \frac{1}{n+2m-1}\right) + \frac{1}{n+2m} \text{ and} \\ \frac{1}{n} - \left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \left(\frac{1}{n+3} - \frac{1}{n+4}\right) - \cdots - \left(\frac{1}{n+2m-1} - \frac{1}{n+2m}\right) &< \frac{1}{n} \\ \therefore 0 &< \left| \sum_{i=n}^{n+p} \frac{(-1)^i}{i} \right| < \frac{1}{n} = \frac{2}{2n} < \frac{3}{2n} \end{aligned}$$

The proof is completed.

In fact, a smaller upper bound for $\left| \sum_{i=n}^{n+p} \frac{(-1)^i}{i} \right|$ is $\frac{1}{n}$.