Supplementary Exercise - Conic Section

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1. The tangents at points P, Q on a parabola intersect at T and the line through T perpendicular to PQ meets the axis in M. Prove that the projection of TM on the axis is of constant length $2a$.(Assume that $a > 0$.)

2. The tangent to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ cuts the x-axis at L and any line through L meets the parabola at points with parameters t_1, t_2. Prove that t_1, t, t_2 are in geometrical progression. (Hint: 3 numbers d, e, f are in geometric progression if they satisfy $e^2 = df$.)

3. The normal to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ meets the axis of the parabola at G and GP is produced beyond P to Q so that $GP = PQ$. Show that the locus of Q is the parabola $y^2=16a(x+2a)$.

4. Find the locus of the mid-points of chords of the parabola $y^2 = 4ax$, the extremities of which subtend a right angle at the vertex of the parabola.

5. $P(h,k)$ is a fixed point. From P variable lines are drawn, intersecting the parabola $y^2 = 4ax$ in different chords. Find the locus of the mid-point of such chords.

6. Show that the line $y = mx + \frac{a}{m}$ touches the circle $x^2 + y^2 = r^2$ if $m^4 + m^2 - \frac{a^2}{r^2} = 0$. Hence find the tangents common to the parabola $y^2 = 4ax$ and the circle $20x^2 + 20y^2 = a^2$.

7. (a) The straight line $y = mx + c$ and the parabola $y^2 = 4ax$ intersect at two points P_1 and P_2. Prove that

   (i) the mid-point of the chord P_1P_2 is $\left(\frac{2a - mc}{m^2}, \frac{2a}{m}\right)$.

   (ii) the length of the chord P_1P_2 is $\frac{4}{m^2}\sqrt{a(a - mc)(1 + m^2)}$

   (b) Given $L_1 : 3x - y - 1 = 0$, $L_2 : 2x - y = 0$. Determine the equations of the two straight lines which pass through the point of intersection of L_1 and L_2 , and the sum of x and y intercepts is 6 .

8. P is the point (4,1.8) on the ellipse $9x^2 + 25y^2 = 225$. S is (4,0) and S' is (–4,0). SK, $S'K'$ are drawn perpendicular to the tangent at P to meet it at K, K'. Prove that

   (i) $SP + S'P = 10$,

   (ii) $SK \cdot S'K' = 9$,

   (iii) $\angle SPK = \angle S'PK' = \tan^{-1} 1.25$

9. P, Q are points on an ellipse such that CQ is parallel to the tangent at P, where C is the centre. Show that the eccentric angles of P and Q differ by a right angle.

   Show also that the locus of the intersection of perpendiculars from the foci S and S' to CP and CQ respectively is a concentric ellipse $a^2x^2 + b^2y^2 = a^2(a^2 - b^2)$.

10. Let $P(h,k)$ be a fixed point on the ellipse E: $b^2x^2 + a^2y^2 = a^2b^2$. Find the locus of the mid-points of chords drawn from P.

11. Using eccentric angles, show that the equation of chord joining the points θ, φ is

$$\frac{x}{a}\cos\frac{\theta+\varphi}{2} + \frac{y}{b}\sin\frac{\theta+\varphi}{2} = \cos\frac{\theta-\varphi}{2}$$

   and the equation of tangent at θ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

1. Let R be the projection of T on x-axis.

The equation of tangent through P is $x - t_1 y + a t_1^2 = 0$ ⋯⋯ (1)

The equation of tangent through Q is $x - t_2 y + a t_2^2 = 0$ ⋯ ⋯ (2)

(1) − (2) $(t_2 - t_1)y = a(t_2^2 - t_1^2)$

$y = a(t_1 + t_2)$ ∵ P≠Q, ∴ $t_1 \neq t_2$

$\therefore x = t_1 y - a t_1^2$

$= a t_1 (t_1 + t_2) - a t_1^2$

$= a\, t_1 t_2$

∴ $T(a\, t_1 t_2, a(t_1 + t_2))$

$R(a\, t_1 t_2, 0)$

$TR = a|t_1 + t_2|$

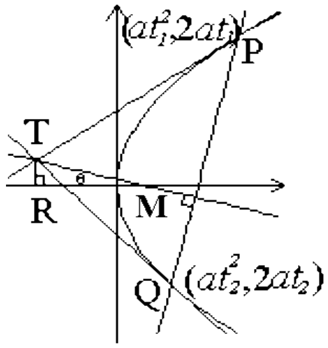
Slope of $PQ = \dfrac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \dfrac{2}{t_1 + t_2}$

∵ $TM \perp PQ$, ∴ Slope of $TM = -\dfrac{t_1 + t_2}{2}$

$\tan\theta = \dfrac{TR}{RM}$

$\dfrac{t_1 + t_2}{2} = \dfrac{a(t_1 + t_2)}{RM}$

$RM = 2a$.



2. Equation of tangent at P: $y = \dfrac{1}{t}x + at$

When $y = 0$, $x = -at^2$

Straight line through L: $\dfrac{y - 0}{x + at^2} = m$

The chord through t_1 , t_2 is:

$\dfrac{y - 2at_1}{x - at_1^2} = \dfrac{2at_2 - 2at_1}{at_2^2 - at_1^2}$

$\dfrac{y - 2at_1}{x - at_1^2} = \dfrac{2}{t_1 + t_2}$

Its slope is equal to the slope through L:

$\dfrac{y}{x + at^2} = \dfrac{2}{t_1 + t_2}$

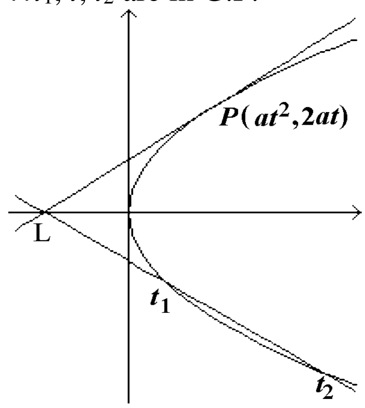
It also passes through $\left(at_1^2, 2at_1\right)$.

$\dfrac{2at_1}{at_1^2 + at^2} = \dfrac{2}{t_1 + t_2}$

$t_1(t_1 + t_2) = t_1^2 + t^2$

$t_1\, t_2 = t^2$

∴ t_1, t, t_2 are in G.P.



3. Let Q be (x_0, y_0).

Slope of tangent at $P = \dfrac{1}{t}$

Slope of normal $= -t$

Normal $\dfrac{y - 2at}{x - at^2} = -t$

To find G, $y = 0$,

$\dfrac{-2at}{x - at^2} = -t$

$2a = x - at^2$

$x = 2a + at^2$

∴ $G(2a + at^2, 0)$

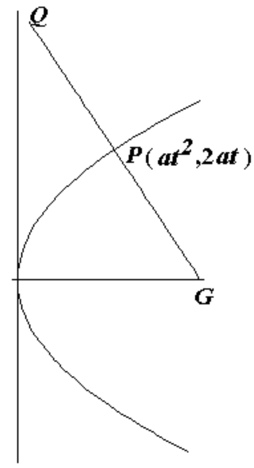
$$\begin{cases} 2at = \dfrac{1}{2}(y_0 + 0) \\ at^2 = \dfrac{1}{2}(x_0 + 2a + at^2) \end{cases}$$

$$\begin{cases} t = \dfrac{y_0}{4a} \\ at^2 = x_0 + 2a \end{cases}$$

$$a\left(\dfrac{y_0}{4a}\right)^2 = x_0 + 2a$$

$$\dfrac{y_0^2}{16a} = x_0 + 2a$$

$y^2 = 16a(x + 2a)$



4. Let $A(x_1, y_1)$. $B(x_2, y_2)$ lie on the parabola. $M(x_0, y_0)$ is the mid point of AB.

$\because OA\perp OB$, $\dfrac{y_1}{x_1}\times\dfrac{y_2}{x_2}=-1 \quad \ldots(1)$

$y_1^2=4ax_1$; $y_2^2=4ax_2 \quad \ldots(2)$

By (1): $y_1y_2=-x_1x_2$

Multiply (2) together $(y_1y_2)^2=16a^2x_1x_2$

Sub. (1) into it: $(-x_1x_2)^2=16a^2x_1x_2$

$x_1x_2=16a^2 \quad \ldots(3)$

Mid point $M(x_0, y_0)=\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2}\right)$

$2y_0=y_1+y_2$

$4y_0^2=y_1^2+y_2^2+2y_1y_2$

5. Let $M(x_0, y_0)$ = mid point

$\dfrac{y-y_0}{x-x_0}=m=\dfrac{k-y_0}{h-x_0}$

$y = mx + y_0 - mx_0$

$(mx + y_0 - mx_0)^2 - 4ax = 0$

$m^2x^2 + [2m(y_0 - mx_0) - 4a]x + (y_0 - mx_0)^2 = 0$

$x_0=\dfrac{x_1+x_2}{2}$

$=-\dfrac{2m(y_0-mx_0)-4a}{2m^2}$

$m^2x_0 = 2a - m(y_0 - mx_0)$

$2a = my_0$

6. $\begin{cases}x^2+y^2=r^2\\ y=mx+\dfrac{a}{m}\end{cases}$

$x^2+\left(mx+\dfrac{a}{m}\right)^2=r^2$

$(1+m^2)x^2+2ax+\dfrac{a^2}{m^2}-r^2=0$

$\Delta=4\left[a^2-(1+m^2)\left(\dfrac{a^2}{m^2}-r^2\right)\right]$

$=\dfrac{4}{m^2}\left[a^2m^2-(1+m^2)(a^2-r^2m^2)\right]$

$=\dfrac{4}{m^2}\left[-a^2+r^2m^2+r^2m^4\right]$

$=\dfrac{4r^2}{m^2}\left[m^4+m^2-\dfrac{a^2}{r^2}\right]$

Sub. (2) into it:

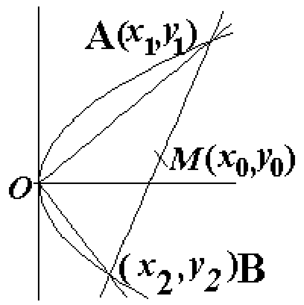
$4y_0^2=4ax_1+4ax_2+2y_1y_2$

Sub. (1) into it:

$4y_0^2=4a(x_1+x_2)-2x_1x_2$

Sub. (3) into it: $4y_0^2=8ax_0-2x_1x_2$

$\therefore$ The locus is $y^2=2a(x-4a)$

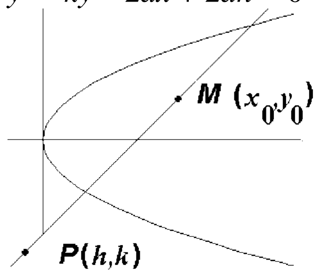


$2a=\dfrac{y_0-k}{x_0-h}\cdot y_0$

$2a(x_0 - h) = (y_0 - k)y_0$

$2ax_0 - 2ah = y_0^2 - ky_0$

$y^2 - ky - 2ax + 2ah = 0$



$\therefore$ If $m^4+m^2-\dfrac{a^2}{r^2}=0$ then $\Delta = 0$

$y=mx+\dfrac{a}{m}$ touches the circle $x^2+y^2=r^2$

$x-ty+at^2=0$ is a tangent to $y^2 = 4ax$

$y=\dfrac{1}{t}x+at$; let $t=\dfrac{1}{m}$;

$y=mx+\dfrac{a}{m}$ is also a tangent to $y^2 = 4ax$

$\therefore x^2+y^2=\left(\dfrac{a}{\sqrt{20}}\right)^2 \Rightarrow r^2=\dfrac{a^2}{20}\Rightarrow\dfrac{a^2}{r^2}=20$

by the condition: $m^4+m^2-\dfrac{a^2}{r^2}=0$

$m^4+m^2-20=0\Rightarrow m=\pm2$

$\Rightarrow y=2x+\dfrac{a}{2}$ or $y=-2x-\dfrac{a}{2}$

$4x - 2y + a = 0$ or $4x + 2y + a = 0$

7.(a) $\begin{cases} y^2 = 4ax \\ y = mx + c \end{cases}$

$(mx + c)^2 = 4ax$

$m^2 x^2 + 2mcx - 4ax + c^2 = 0$

$x_1 x_2 = \dfrac{c^2}{m^2}$; $\dfrac{x_1 + x_2}{2} = \dfrac{4a - 2mc}{2m^2} = \dfrac{2a - mc}{m^2}$

$y_1 y_2 = \sqrt{(4a)^2 x_1 x_2} = \sqrt{(4a)^2 \dfrac{c^2}{m^2}} = \dfrac{4ac}{m}$;

$\dfrac{y_1 + y_2}{2} = m\left(\dfrac{x_1 + x_2}{2}\right) + c = \dfrac{2a - mc}{m} + c = \dfrac{2a}{m}$

(i) mid point = $\left(\dfrac{2a - mc}{m^2}, \dfrac{2a}{m}\right)$

(ii) length of chord = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2 - 4x_1 x_2 - 4y_1 y_2}$

$= \sqrt{4\left(\dfrac{2a - mc}{m^2}\right)^2 + 4\left(\dfrac{2a}{m}\right)^2 - 4\dfrac{c^2}{m^2} - \dfrac{16ac}{m}}$

$= \dfrac{1}{m^2}\sqrt{16a^2 - 16acm + 4m^2c^2 + 16a^2m^2 - 4c^2m^2 - 16acm^3}$

$= \dfrac{1}{m^2}\sqrt{16a^2 - 16acm + 16a^2m^2 - 16acm^3}$

$= \dfrac{4}{m^2}\sqrt{a \cdot \left(a - cm + am^2 - cm^3\right)}$

8. Put (4,1.8) into $9x^2 + 25y^2 = 225$

$9(4)^2 + 25(1.8)^2 = 144 + 81 = 225$

$\left(\dfrac{x}{5}\right)^2 + \left(\dfrac{y}{3}\right)^2 = 1$

$a = 5$, $b = 3 \Rightarrow c = 4$

$\therefore$ the foci are S(4,0) and S'(–4,0)

$SP = \sqrt{(4-4)^2 + (1.8-0)^2} = 1.8$

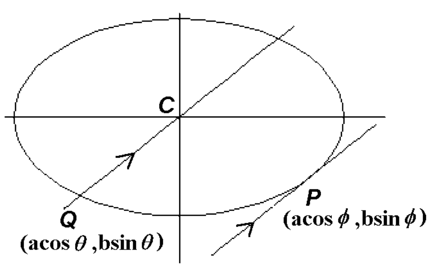
$S\ P = \sqrt{(4--4)^2 + (1.8-0)^2} = \sqrt{64 + 3.24} = 8.2$

$SP + S'P = 1.8 + 8.2 = 10$

tangent at (4,1.8) is $9\times4x + 25\times1.8y = 225$

$4x + 5y - 25 = 0$

9.



tangent at P: $\dfrac{x}{a}\cos\phi + \dfrac{y}{b}\sin\phi = 1$

$= \dfrac{4}{m^2}\sqrt{a \cdot (a - cm)\left(1 + m^2\right)}$

7(b) $3x - y - 1 + k(2x - y) = 0$

$(3 + 2k)x - (1 + k)y = 0$

$\dfrac{x}{\frac{1}{3+2k}} + \dfrac{y}{-\frac{1}{1+k}} = 1$

$\dfrac{1}{3+2k} - \dfrac{1}{1+k} = 6$

$1 + k - 3 - 2k = 6(3 + 2k)(1 + k)$

$-k - 2 = 6(3 + 5k + 2k^2)$

$12k^2 + 31k + 20 = 0$

$(3k + 4)(4k + 5) = 0$

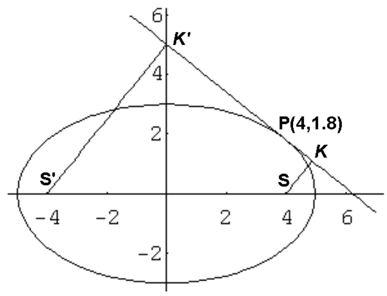
$k = -\dfrac{4}{3}$ or $-\dfrac{5}{4}$

$3(3x - y - 1) - 4(2x - y) = 0$

$x + y - 3 = 0$

$4(3x - y - 1) - 5(2x - y) = 0$

$2x + y - 4 = 0$



at (4,0) $d_1 = SK = \left|\dfrac{4\times4 + 0 - 25}{\sqrt{41}}\right| = \dfrac{9}{\sqrt{41}}$

at (–4,0) $d_2 = S'K' = \left|\dfrac{4\times(-4) + 0 - 25}{\sqrt{41}}\right| = \sqrt{41}$

$\therefore$ SK · S'K' = 9

$\sin\angle SPK = \dfrac{SK}{SP} = \dfrac{9}{\sqrt{41}} \times \dfrac{1}{1.8} = \dfrac{5}{\sqrt{41}}$

$\tan\angle SPK = \dfrac{5}{4}$

Similarly $\tan\angle S'PK' = \dfrac{5}{4}$

$\angle$SPK = $\angle$S'PK' = $\tan^{-1}$ 1.25

slope $= -\dfrac{b}{a}\cot\phi$

$\therefore$ slope of CQ = $= -\dfrac{b}{a}\cot\phi$

$\therefore \dfrac{b\sin\theta - 0}{a\cos\theta - 0} = -\dfrac{b}{a}\cot\phi$

$\tan\theta = -\cot\phi$

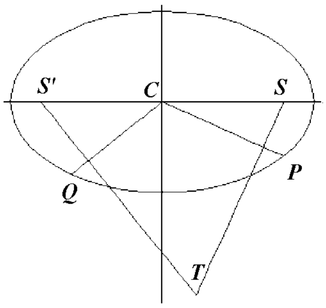
$|\theta - \phi| = \dfrac{\pi}{2}$

Suppose they intersect at T,

$P(a\cos(\theta + \dfrac{\pi}{2}), b\sin(\theta + \dfrac{\pi}{2}))$,

slope $CP = -\dfrac{b}{a}\cot\theta$

$Q(a\cos\theta, b\sin\theta)$, slope $CQ = \dfrac{b}{a}\tan\theta$



slope of $ST = \dfrac{a}{b}\tan\theta$

slope of $S'T = -\dfrac{a}{b}\cot\theta$

let T be (x,y),

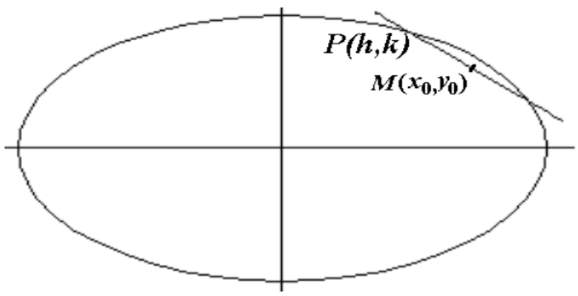
equation of ST: $\dfrac{y}{x-c} = \dfrac{a}{b}\tan\theta$ ....(1)

equation of $S'T$: $\dfrac{y}{x+c} = -\dfrac{a}{b}\cot\theta$ ....(2)

(1)×(2) $\dfrac{y^2}{x^2-c^2} = -\dfrac{a^2}{b^2}$

$\therefore b^2y^2 = -a^2(x^2-c^2)$

$a^2x^2 + b^2y^2 = a^2(a^2-b^2)$

10. 

Let the chord through P cuts the ellipse again at $Q(a\cos\theta, b\sin\theta)$ and $M(x_0, y_0)$ = mid point.

Then $\begin{cases} x_0 = \dfrac{1}{2}(h + a\cos\theta) \\ y_0 = \dfrac{1}{2}(k + b\sin\theta) \end{cases}$

$$\begin{cases} \cos\theta = \dfrac{2x_0 - h}{a} = \dfrac{x_0 - \frac{h}{2}}{\frac{a}{2}} \\ \sin\theta = \dfrac{2y_0 - k}{b} = \dfrac{y_0 - \frac{k}{2}}{\frac{b}{2}} \end{cases}$$

$\because \cos^2\theta + \sin^2\theta = 1$

$$\therefore \left(\frac{x_0 - \frac{h}{2}}{\frac{a}{2}}\right)^2 + \left(\frac{y_0 - \frac{k}{2}}{\frac{b}{2}}\right)^2 = 1$$

The locus is an ellipse with centre at $\left(\dfrac{h}{2}, \dfrac{k}{2}\right)$

$$b^2\left(x_0 - \frac{h}{2}\right)^2 + a^2\left(y_0 - \frac{k}{2}\right)^2 = \frac{a^2b^2}{4}$$

$$b^2x_0^2 + a^2y_0^2 - b^2hx_0 - a^2ky_0 + \frac{b^2h^2 + a^2k^2}{4} = \frac{a^2b^2}{4}$$

$b^2x_0^2 + a^2y_0^2 - b^2hx_0 - a^2ky_0 = 0$

(As (h,k) lies on the ellipse, $b^2h^2 + a^2k^2 = a^2b^2$)

The locus is $b^2x^2 + a^2y^2 - b^2hx - a^2ky = 0$

11. Routine work. Do it yourself.