Individual Events

I1	a	15	**I2**	a	3	**I3**	a	5	**I4**	a	0
	b	30		b	$\frac{1}{2}$		b	0		b	*4 see the remark
	c	11		c	4		c	–1		c	*2 see the remark
	d	979		d	24		d	4		d	0

Group Events

G1	P	12	**G2**	A	9	**G3**	K	4	**G4**	d	*72 see the remark
	Q	60		B	5		L	3		u	6
	n	11		c	5		x	52		c	$\frac{7}{13}$
	T	$*\frac{1}{2}\left(3^{2048}-1\right)$ see the remark		d	30		y	$\frac{13}{3}$		x	$\frac{1+\sqrt{5}}{2}$

Individual Event 1

I1.1 解方程 $\log_5 a + \log_3 a = \log_5 a \cdot \log_3 a$，其中 $a > 1$ 為實數。

Solve the equation $\log_5 a + \log_3 a = \log_5 a \cdot \log_3 a$ for real number $a > 1$.

$$\frac{\log a}{\log 5} + \frac{\log a}{\log 3} = \frac{\log a}{\log 5} \cdot \frac{\log a}{\log 3}$$

Multiple both sides by log3·log 5 and divide both sides by log a ($\neq 0$).

$\log 3 + \log 5 = \log a$

$a = 15$

I1.2 若 $\sqrt{b} = \sqrt{8+\sqrt{a}} + \sqrt{8-\sqrt{a}}$，求 b 的實數值。

If $\sqrt{b} = \sqrt{8+\sqrt{a}} + \sqrt{8-\sqrt{a}}$, determine the real value of b.

Reference: 2007 FI1.1

$$b = \left(\sqrt{8+\sqrt{15}} + \sqrt{8-\sqrt{15}}\right)^2$$
$$= 8+\sqrt{15} + 2\left(\sqrt{8+\sqrt{15}} \cdot \sqrt{8-\sqrt{15}}\right) + 8 - \sqrt{15}$$
$$= 16 + 2\sqrt{64-15}$$
$$= 16 + 2\sqrt{49}$$
$$= 16 + 2\times 7 = 30$$

I1.3 若方程 $x^2 - cx + b = 0$ 有兩個實數根及兩根之差為 1，求兩根之和的最大可能值 c。

If the equation $x^2 - cx + b = 0$ has two distinct real roots and their difference is 1, determine the greatest possible value of the sum of the roots, c.

Reference: 2008 FIS.3

Let the roots be α, β.

$\alpha + \beta = c$, $\alpha\beta = b = 30$, $\alpha - \beta = 1$

$(\alpha - \beta)^2 = 1$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$

$c^2 - 4\times 30 = 1$

$\Rightarrow c = 11$ or -11

The greatest possible value of $c = 11$

I1.4 設 $d=\overline{xyz}$ 為一不能被 10 整除的三位數。若 $\overline{xyz}$ 與 $\overline{zyx}$ 之和可被 c 整除，求此整數的最大可能值 d。

Let $d=\overline{xyz}$ be a three-digit integer that is **not** divisible by 10. If the sum of integers $\overline{xyz}$ and $\overline{zyx}$ is divisible by c, determine the greatest possible value of such an integer d.

$100x + 10y + z + 100z + 10y + x$
$= 100(x + z) + 20y + x + z$
$= 101(x + z) + 20y$
$= 99(x + z) + 22y + 2(x + z - y)$, which is divisible by 11
$x + z - y$ is a multiple of 11
$x + z - y = 0$ or 11
To maximize d, x should be as large as possible
$x + z = 11 + y$
$x = 9$, $z = 9$, $y = 7$
The greatest possible value of $d = 979$
Check: $\overline{xyz}+\overline{zyx} = 979 + 979 = 1958 = 11\times178$, which is divisible by 11.

Individual Event 2

I2.1 一個等邊三角形及一個正六邊形的周長比率為 1 : 1。若三角形與六邊形的面積比率為 2 : a，求 a 的值。

Let the ratio of perimeter of an equilateral triangle to the perimeter of a regular hexagon be 1 : 1. If the ratio of the area of the triangle to the area of the hexagon is 2 : a, determine the value of a.

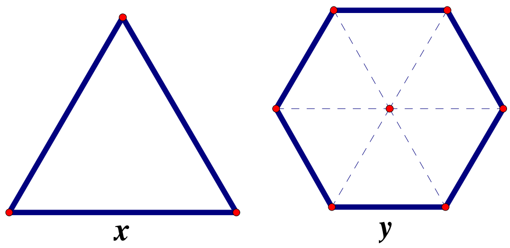
Reference: 1996 FI1.1, 2014 FI4.3

Let the length of the equilateral triangle be x, and that of the regular hexagon be y.

Since they have equal perimeter, $3x = 6y$

$\therefore x = 2y$

The hexagon can be divided into 6 identical equilateral triangles.



Ratio of areas $= \frac{1}{2}x^2\sin 60^\circ : 6\times\frac{1}{2}y^2\sin 60^\circ = 2 : a$

$x^2 : 6y^2 = 2 : a$

$(2y)^2 : 6y^2 = 2 : a$

$\Rightarrow a = 3$

I2.2 求 $b = \left[\log_2\left(a^2\right)+\log_4\left(\frac{1}{a^2}\right)\right]\times\left[\log_a 2+\log_{a^2}\left(\frac{1}{2}\right)\right]$ 的值。

Determine the value of $b = \left[\log_2\left(a^2\right)+\log_4\left(\frac{1}{a^2}\right)\right]\times\left[\log_a 2+\log_{a^2}\left(\frac{1}{2}\right)\right]$.

$$b = \left[\log_2\left(3^2\right)+\log_4\left(\frac{1}{3^2}\right)\right]\times\left[\log_3 2+\log_{3^2}\left(\frac{1}{2}\right)\right]$$

$$= \left[\log_2 9+\frac{\log_2\left(\frac{1}{9}\right)}{\log_2 4}\right]\times\left[\log_3 2+\frac{\log_3\left(\frac{1}{2}\right)}{\log_3 9}\right]$$

$$= \left[\log_2 9+\frac{\log_2\left(\frac{1}{9}\right)}{2}\right]\times\left[\log_3 2+\frac{\log_3\left(\frac{1}{2}\right)}{2}\right]$$

$$= \left[\log_2 9+\log_2\left(\frac{1}{9}\right)^{\frac{1}{2}}\right]\times\left[\log_3 2+\log_3\left(\frac{1}{2}\right)^{\frac{1}{2}}\right]$$

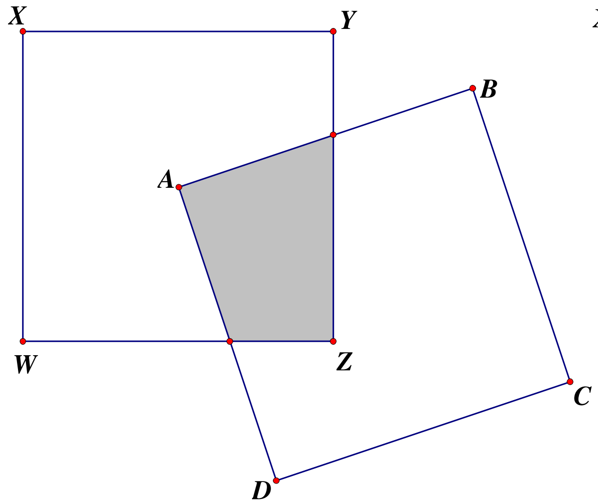
$$= \left[\log_2\left(9\times\frac{1}{\sqrt{9}}\right)\right]\times\left[\log_3\left(2\times\frac{1}{\sqrt{2}}\right)\right]$$

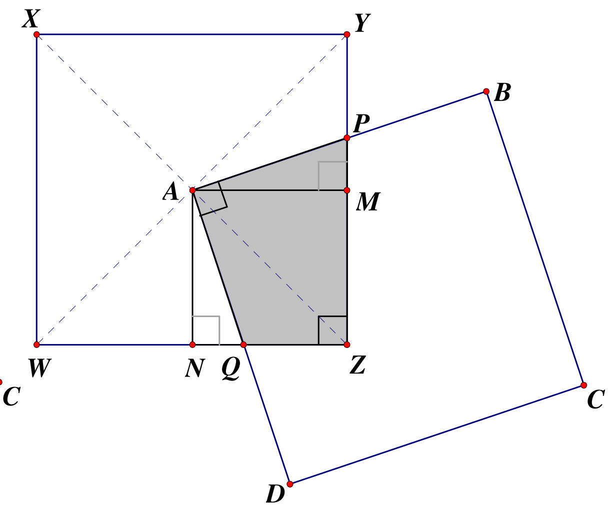
$$= [\log_2 3]\times\left[\log_3\left(2^{\frac{1}{2}}\right)\right]$$

$$= \frac{\log 3}{\log 2}\times\frac{\frac{1}{2}\log 2}{\log 3} = \frac{1}{2}$$

I2.3 在下圖中，正方形 $ABCD$ 及 $XYZW$ 相等而且互相交疊使得頂點 A 位在 $XYZW$ 的中心及綫段 AB 將綫段 YZ 邊分為 $1:2$。若 $XYZW$ 的面積與交疊部分的面積比率為 $c:1$，求 c 的值。

In the figure below, identical squares $ABCD$ and $XYZW$ overlap each other in such a way that the vertex A is at the centre of $XYZW$ and the line segment AB cuts line segment YZ into $1:2$. If the ratio of the area of $XYZW$ to the overlapped region is $c:1$, determine the value of c.





Reference: 2009 HI7

A is the intersection of the diagonals XZ and YW. Suppose AB cuts YZ at P and AD cuts WZ at Q. By the property of squares, $\angle PAQ = \angle PZQ = 90°$

$\therefore \angle PAQ + \angle PZQ = 90° + 90° = 180°$

$\Rightarrow A, P, Z, Q$ are concyclic (opp. $\angle$s supp.)

$\angle APM = \angle AQN$ (ext. $\angle$s cyclic quad.)

Let M, N be the mid-points of YZ and WZ respectively.

It is easy to show that $AM = AN$ and $\angle AMP = \angle ANQ = 90°$

$\therefore \Delta APM \cong \Delta ANQ$ (A.A.S.)

Area of shaded region = area of APM + area of $AMZQ$

= area of ANQ + area of $AMZQ$

= area of $AMZN$

$= \frac{1}{4} \times$ area of $XYZW$

$\therefore c = 4$

I2.4 若 76 與 d 的最小公倍數(L.C.M.)為 456 及 76 與 d 的最大公因數(H.C.F.)為 c，求正整數 d 的值。

If the least common multiples (L.C.M.) of 76 and d is 456 and the highest common factor (H.C.F.) of 76 and d is c, determine the value of the positive integer d.

Reference: 2005 FI1.2

$76 \times d$ = L.C.M. $\times$ H.C.F. $= 456 \times 4$

$d = 24$

Individual Event 3

I3.1 若 $f(x) = x^4 + x^3 + x^2 + x + 1$，求 $f(x^5)$ 除以 $f(x)$ 的餘值 a。

If $f(x) = x^4 + x^3 + x^2 + x + 1$, determine the remainder a of $f(x^5)$ divided by $f(x)$.

Reference: 1996 FG10.2

Clearly $f(1) = 5$.

By division algorithm, $f(x) = (x - 1)Q(x) + 5$, where $Q(x)$ is a polynomial

$f(x^5) = (x^5 - 1)Q(x^5) + 5$

$= (x - 1)(x^4 + x^3 + x^2 + x + 1)Q(x^5) + 5$

$= f(x)(x - 1)Q(x^{10}) + 5$

The remainder is $a = 5$.

I3.2 設 n 為整數。求 $n^a - n$ 除以 30 的餘值 b。

Let n be an integer. Determine the remainder b of $n^a - n$ divided by 30.

$n^5 - n = n(n^4 - 1) = n(n^2 + 1)(n^2 - 1) = (n - 1)n(n + 1)(n^2 + 1)$

$n - 1$, n and $n + 1$ are three consecutive integers, the product of which must be divisible by 6.

If any one of $n - 1$, n or $n + 1$ is divisible by 5, then the product is divisible by 30.

Otherwise, let $n - 1 = 5k + 1$, $n = 5k + 2$, $n + 1 = 5k + 3$, $n^2 + 1 = (5k + 2)^2 + 1 = 25k^2 + 20k + 5$ which is a multiple of 5, the product is divisible by 30.

If $n - 1 = 5k + 2$, $n = 5k + 3$, $n + 1 = 5k + 4$, $n^2 + 1 = (5k + 3)^2 + 1 = 25k^2 + 30k + 10$ which is a multiple of 5, the product is divisible by 30.

In all cases, $n^5 - n$ is divisible by 30. The remainder when $n^5 - n$ divided by 30 is 0.

I3.3 若 $0 < x < 1$，求 $c=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\times\left(\sqrt{\frac{1}{x^2-b^2}-1}-\frac{1}{x-b}\right)$ 的值。

If $0 < x < 1$, determine the value of

$$c=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\times\left(\sqrt{\frac{1}{x^2-b^2}-1}-\frac{1}{x-b}\right).$$

Reference: 2017 FG3.2

$$c=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\times\left(\sqrt{\frac{1}{x^2}-1}-\frac{1}{x}\right)$$

$$=\left\{\frac{\sqrt{1+x}\cdot\left(\sqrt{1+x}+\sqrt{1-x}\right)}{(1+x)-(1-x)}+\frac{(1-x)\cdot\left[\sqrt{1-x^2}-(x-1)\right]}{\left(1-x^2\right)-(x-1)^2}\right\}\times\left(\sqrt{\frac{1-x^2}{x^2}}-\frac{1}{x}\right)$$

$$=\left\{\frac{1+x+\sqrt{1-x^2}}{2x}+\frac{(1-x)\cdot\left[\sqrt{1-x^2}+(1-x)\right]}{\left(1-x^2\right)-\left(1-2x+x^2\right)}\right\}\times\left(\frac{\sqrt{1-x^2}}{x}-\frac{1}{x}\right)$$

$$=\left\{\frac{1+x+\sqrt{1-x^2}}{2x}+\frac{(1-x)\cdot\left[\sqrt{1-x^2}+(1-x)\right]}{2x(1-x)}\right\}\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)$$

$$=\left[\frac{1+x+\sqrt{1-x^2}}{2x}+\frac{\sqrt{1-x^2}+(1-x)}{2x}\right]\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)$$

$$=\left(\frac{2+2\sqrt{1-x^2}}{2x}\right)\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)=\left(\frac{1+\sqrt{1-x^2}}{x}\right)\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)$$

$$=\frac{\left(1-x^2\right)-1}{x^2}=-1$$

Remark: You may substitute $x = 0.5$ directly to find the value of c.

I3.4 若實數 x 及 y 滿足方程 $2\log_{10}(x+2cy)=\log_{10}x+\log_{10}y$，求 $d=\dfrac{x}{y}$ 的值。

If real numbers x and y satisfy the equation $2\log_{10}(x+2cy)=\log_{10}x+\log_{10}y$, determine the value of $d=\dfrac{x}{y}$.

$2\log_{10}(x-2y)=\log_{10}x+\log_{10}y$

$(x-2y)^2=xy$

$x^2-5xy+4y^2=0$

$(x-y)(x-4y)=0$

$d=\dfrac{x}{y}=1$ or 4

Check: When $x=y$, L.H.S. $=2\log_{10}(y-2y)=2\log_{10}(-y)$, R.H.S. $=\log_{10}y+\log_{10}y$

When $y>0$. L.H.S. is undefined and R.H.S. is well defined, rejected

When $y=0$, L.H.S. is undefined and R.H.S. is undefined, rejected

When $y<0$, L.H.S. is well defined, R.H.S. is undefined, rejected

When $x=4y$, L.H.S. $=2\log_{10}(4y-2y)=\log_{10}4+2\log_{10}y=\log_{10}4y+\log_{10}y=$ R.H.S.

When $y>0$. L.H.S. is well defined and R.H.S. is well defined, accepted

$\therefore d=4$ only

Individual Event 4

I4.1 若 m 和 n 為正整數及 $a=\log_2\left[\left(\frac{m^4n^{-4}}{m^{-1}n}\right)^{-3}\div\left(\frac{m^{-2}n^2}{mn^{-1}}\right)^5\right]$，求 a 的值。

If m and n are positive integers and $a=\log_2\left[\left(\frac{m^4n^{-4}}{m^{-1}n}\right)^{-3}\div\left(\frac{m^{-2}n^2}{mn^{-1}}\right)^5\right]$, determine the value of a.

$$a=\log_2\left[\left(\frac{m^5}{n^5}\right)^{-3}\div\left(\frac{n^3}{m^3}\right)^5\right]$$

$$=\log_2\left[\left(\frac{n^{15}}{m^{15}}\right)\times\left(\frac{m^{15}}{n^{15}}\right)\right]=\log_2 1=0$$

I4.2 當整數 $1108+a$、1453、$1844+2a$ 及 2281 除以正整數 n (>1)都得相同餘數 b，求 b 的值。
When the integers $1108+a$, 1453, $1844+2a$ and 2281 divided by some positive integer n (> 1), they all get the same remainder b. Determine the value of b.
Reference: 2000 FG1.1

$1108=pn+b$ ······ (1)

$1453=qn+b$ ······ (2)

$1844=rn+b$ ······ (3)

$2281=sn+b$ ······ (4)

p, q, r, s are non-negative integers and $0\le b<n$.

(2) – (1): $345=(q-p)n$ ······ (5)

(3) – (2): $391=(r-q)n$ ······ (6)

(4) – (3): $437=(s-r)n$ ······ (7)

$\therefore$ n is the common factor of 345, 391 and 437.
$345=3\times5\times23$, $391=17\times23$, $437=19\times23$
$\therefore$ $n=1$ or 23
When $n=1$, $b=0$. (rejected)
When $n=23$, sub. $n=23$ into (1): $1108=23\times48+4$
$b=4$

Remark: original question: ······除以正整數n都得相同餘數b, ······divided by some positive integer n, they all get the same remainder b.
There are two possible answers for b: 0 or 4.

I4.3 若 $\frac{6}{b}<x<\frac{10}{b}$，求 $c=\sqrt{x^2-2x+1}+\sqrt{x^2-6x+9}$ 的值。

If $\frac{6}{b}<x<\frac{10}{b}$, determine the value of $c=\sqrt{x^2-2x+1}+\sqrt{x^2-6x+9}$.
Reference: 2005 HI2, 2018 FI1.2

$1.5<x<2.5$, $c=\sqrt{(x-1)^2}+\sqrt{(x-3)^2}=x-1+3-x=2$

Remark: original question: ······最大可能值 ······ the greatest possible value of c.
The value of c is a constant.

I4.4 求 $1 + 3^c + (3^c)^2 + (3^c)^3 + (3^c)^4$ 除以 $1 + 3 + 3^2 + 3^3 + 3^4$ 的餘值 d。

Determine the remainder d when $1 + 3^c + (3^c)^2 + (3^c)^3 + (3^c)^4$ is divided by $1 + 3 + 3^2 + 3^3 + 3^4$.

$1 + 3 + 3^2 + 3^3 + 3^4 = \frac{3^5-1}{3-1}$

$1 + 3^2 + (3^2)^2 + (3^2)^3 + (3^2)^4 = \frac{\left(3^2\right)^5-1}{3^2-1} = \frac{3^5+1}{3+1}\cdot\frac{3^5-1}{3-1}$

$$\frac{1+3^2+\left(3^2\right)^2+\left(3^2\right)^3+\left(3^2\right)^4}{1+3+3^2+3^3+3^4} = \frac{3^5+1}{3+1}$$

$$= \frac{(2+1)^5+1}{4}$$

$$= \frac{2^5+C_1^5 2^4+C_2^5 2^3+C_3^5 2^2+C_4^5 2+1+1}{4}$$

$$= 2^3+C_1^5 2^2+C_2^5 2+C_3^5+3, \text{ which is an integer}$$

The remainder $d = 0$

Method 2

$1 + 3 + 3^2 + 3^3 + 3^4 = \frac{3^5-1}{3-1}$

$1 + 3^2 + (3^2)^2 + (3^2)^3 + (3^2)^4 = \frac{\left(3^2\right)^5-1}{3^2-1} = \frac{3^5+1}{3+1}\cdot\frac{3^5-1}{3-1}$

$$\frac{1+3^2+\left(3^2\right)^2+\left(3^2\right)^3+\left(3^2\right)^4}{1+3+3^2+3^3+3^4} = \frac{3^5+1}{3+1} = \frac{244}{4} = 61, \text{ which is an integer}$$

The remainder $d = 0$

Method 3

$3 \equiv -1 \pmod 4$

$3^5 \equiv (-1)^5 \equiv -1 \pmod 4$

$3^5 + 1 \equiv 0 \pmod 4$

$\frac{3^5+1}{3+1}$ is an integer

$\frac{3^5+1}{3+1}\cdot\frac{3^5-1}{3-1}$ is an integral multiple of $\frac{3^5-1}{3-1}$.

$1+3^2+(3^2)^2+(3^2)^3+(3^2)^4 = \frac{\left(3^2\right)^5-1}{3^2-1} = \frac{3^5+1}{3+1}\cdot\frac{3^5-1}{3-1}$ is an integral multiple of $1 + 3 + 3^2 + 3^3 + 3^4$

The remainder $d = 0$

Group Event 1

G1.1 一項工程包括三個項目：A、B 和 C。若項目 A 開始三天後，項目 B 才可開始進行。項目 C 亦必須在項目 B 開始四天後才可開始進行。若完成項目 A、B 和 C 分別需要四天、六天和五天，求最少天數 (P) 完成全項工程。

A project comprises of three tasks, A, B and C. Suppose task B must begin 3 days later than task A begins, and task C must begin 4 days later than task B begins. If the numbers of days to complete tasks A, B and C are 4, 6 and 5, respectively, determine the least number of days (P) to complete the project.

$P = 3 + 4 + 5 = 12$

G1.2 指示牌牌上掛有紅、黃、綠閃燈。紅、黃、綠閃燈分別每隔 3 秒、4 秒、8 秒閃爍一次。當 0 秒時，紅、黃、綠閃燈同時閃爍。若當 Q 秒時，第三次出現只有紅及黃閃燈同時閃爍，求 Q 的值。

There are 3 blinking lights, red, yellow and green, on a panel. Red, yellow and green lights blink at every 3, 4 and 8 seconds, respectively. Suppose each light blinks at the time $t = 0$. At time Q (in seconds), there is the third time at which only red and yellow lights blink, determine the value of Q.

The L.C.M. of 3, 4 and 8 is 24. i.e. The lights blink patterns repeat for every 24 seconds.

If only red and yellow lights blink, but not the green light, then the first time it happens is 12 s, the second time it happens is 12 + 24 = 36 s, in the third time it happens is 36 + 24 = 60 s.

$Q = 60$

G1.3 設 $f_{n+1}=\begin{cases} f_n+3 & \text{若 } n \text{ 是雙數} \\ f_n-2 & \text{若 } n \text{ 是單數} \end{cases}$。

若 $f_1 = 60$，求 n 的最少可能值，令當 $m \ge n$ 時，滿足 $f_m \ge 63$。

Let $f_{n+1}=\begin{cases} f_n+3 & \text{if } n \text{ is even} \\ f_n-2 & \text{if } n \text{ is odd} \end{cases}$.

If $f_1 = 60$, determine the smallest possible value of n satisfying $f_m \ge 63$ for all $m \ge n$.

$f_2 = 58, f_3 = 61, f_4 = 59, f_5 = 62, f_6 = 60, f_7 = 63, f_8 = 61, f_9 = 64, f_{10} = 62, f_{11} = 65, f_{12} = 63\cdots\cdots$

Now $f_m \ge 63$

The smallest possible value of n is 11.

G1.4 求 $T=\left(3^{2^0}+1\right)\times\left(3^{2^1}+1\right)\times\left(3^{2^2}+1\right)\times\cdots\times\left(3^{2^{10}}+1\right)$ 的值。(答案以指數表示。)

Determine the value of $T=\left(3^{2^0}+1\right)\times\left(3^{2^1}+1\right)\times\left(3^{2^2}+1\right)\times\cdots\times\left(3^{2^{10}}+1\right)$. (Leave your answer in index form.)

Reference: 1994 FG6.2

$T = (3 - 1)(3 + 1)(3^2 + 1)(3^4 + 1)(3^8 + 1)\times\cdots\times\left(3^{2^{10}}+1\right)\div 2$

$= (3^2 - 1)(3^2 + 1)(3^4 + 1)(3^8 + 1)\times\cdots\times\left(3^{2^{10}}+1\right)\div 2$

$= (3^4 - 1)(3^4 + 1)(3^8 + 1)\times\cdots\times\left(3^{2^{10}}+1\right)\div 2$

$= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$=\frac{1}{2}\left(3^{2^{11}}-1\right)$

$=\frac{1}{2}\left(3^{2048}-1\right)$

Remark: The original question: 求 $T=\left(3^{2^0}+1\right)\times\left(3^{2^1}+1\right)\times\left(3^{2^2}+1\right)\times\cdots\times\left(3^{2^{10}}+1\right)$ 的值。

Determine the value of $T=\left(3^{2^0}+1\right)\times\left(3^{2^1}+1\right)\times\left(3^{2^2}+1\right)\times\cdots\times\left(3^{2^{10}}+1\right)$.

It is difficult to find the exact value of the expression.

Group Event 2

G2.1 一個盒子有五個球，球面上分別印上號碼 3、4、6、9 或 10。由盒中同時隨機取出 2 個球，並得出其號碼的總和。若 A 為不同總和的數量，求 A 的值。

A box contains five distinctly marked balls with number markings being 3, 4, 6, 9 or 10. Two balls are randomly drawn without replacement from the box. If A is the number of possible distinct sums of the selected numbers, determine the value of A.

$3 + 4 = 7$, $3 + 6 = 9$, $3 + 9 = 12$, $3 + 10 = 13$, $4 + 6 = 10$, $4 + 9 = 13$, $4 + 10 = 14$, $6 + 9 = 15$, $6 + 10 = 16$, $9 + 10 = 19$

The distinct sums are 7, 9, 10, 12, 13, 14, 15, 16, 19. $A = 9$

G2.2 設 $f_1 = 9$ 及 $f_n = \begin{cases} f_{n-1} + 3 & \text{若 } n \text{ 是3的倍數} \\ f_{n-1} - 1 & \text{若 } n \text{ 不是3的倍數} \end{cases}$。

若 B 為 k 的值的可能數量，使得 $f_k < 11$，求 B 的值。

Let $f_1 = 9$ and $f_n = \begin{cases} f_{n-1} + 3 & \text{if } n \text{ is a multiple of } 3 \\ f_{n-1} - 1 & \text{if } n \text{ is not a multiple of } 3 \end{cases}$.

If B is the number of possible values of k such that $f_k < 11$, determine the value of B.

$f_1 = 9, f_2 = 8, f_3 = 11, f_4 = 10, f_5 = 9, f_6 = 12, f_7 = 11, f_8 = 10, f_9 = 13, \cdots$

There are 5 values of k such that $f_k < 11$, $B = 5$

Remark: 中文版 …其中 k 滿足 $f_k < 11$… 改為 …使得 $f_k < 11$…

G2.3 設 a_1、a_2、a_3、a_4、a_5、a_6 為非負整數，並滿足 $\begin{cases} a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 = 26 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 5 \end{cases}$。

若 c 為方程系統的解的數量，求 c 的值。

Let a_1, a_2, a_3, a_4, a_5, a_6 be non-negative integers and satisfy

$$\begin{cases} a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 = 26 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 5 \end{cases}.$$

If c is the number of solutions to the system of equations, determine the value of c.

Let $(a_1, a_2, a_3, a_4, a_5, a_6)$ be a solution, then $0 \le a_1, a_2, a_3, a_4, a_5, a_6 \le 5$

The solutions are:

(0, 1, 0, 0, 0, 4), (0, 0, 1, 0, 1, 3), (0, 0, 0, 2, 0, 3), (0, 0, 0, 1, 2, 2), (0, 0, 0, 0, 4, 1).

$c = 5$

G2.4 設 d 及 f 為正整數及 $a_1 = 0.9$。若 $a_{i+1} = a_i^2$ 及 $\prod_{i=1}^{4} a_i = \frac{3^d}{f}$，求 d 的最小可能值。

Let d and f be positive integers and $a_1 = 0.9$. If $a_{i+1} = a_i^2$ and $\prod_{i=1}^{4} a_i = \frac{3^d}{f}$, determine the smallest possible value of d.

$0.9 \times 0.9^2 \times 0.9^4 \times 0.9^8 = 0.9^{15} = \frac{9^{15}}{10^{15}} = \frac{3^{30}}{10^{15}}$

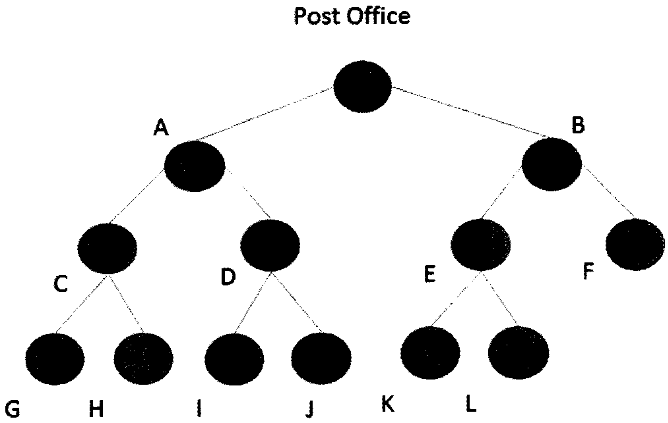
$d = 30$

Remark 最少可能值改為最小可能值

Group Event 3

G3.1 下圖是郵差的送信路線圖：從郵局開始，到達十二個地點送信，最後返回郵局。若郵差從一地點步行到另一地點需要十分鐘及 K 為郵差需要的時數來完成整天路線，求 K 的最小可能值。

The figure below represents routes of a postman. Starting at the post office, the postman walks through all the 12 points and finally returns to the post office. If he takes 10 minutes from a point to another adjacent point by walk and K is the number of hours required for the postman to finish the routes, find the smallest possible value of K.



Post Office →A→C→G→C→H→C→A→D→I→D→J→D→A→Post Office
→B→E→K→E→L→E→B→F→B→Post Office

Total time = 240 minutes = 4 hours

$K = 4$

Remark 最少可能值改為最小可能值

G3.2 若 n 為正整數，$a_1 = 0.8$ 及 $a_{n+1} = a_n^2$，求 L 的最小值，滿足 $a_1 \times a_2 \times \cdots \times a_L < 0.3$。

If $a_1 = 0.8$ and $a_{n+1} = a_n^2$ for positive integers n, determine the least value of L satisfying $a_1 \times a_2 \times \cdots \times a_L < 0.3$.

$0.8 \times 0.8^2 \times 0.8^4 \times \cdots \times 0.8^{\left(2^{L-1}\right)} < 0.3$

$0.8^{\left(1+2+4+\cdots+2^{L-1}\right)} < 0.3$

$0.8^{\left(2^L-1\right)} < 0.3$

$(2^L - 1) \log 0.8 < \log 0.3$

$2^L - 1 > \frac{\log 0.3}{\log 0.8} = \frac{1-\log 3}{1-\log 8} \approx \frac{1-0.48}{1-3\times 0.30} = \frac{0.52}{0.10} = 5.2$

$2^L > 6.2 \Rightarrow$ The least value of $L = 3$

Remark 最少值 改為 最小值

G3.3 若方程 $\sqrt[3]{5+\sqrt{x}}+\sqrt[3]{5-\sqrt{x}}=1$，求實數根 x。

Solve $\sqrt[3]{5+\sqrt{x}}+\sqrt[3]{5-\sqrt{x}}=1$ for real number x.

Reference: 1999 FI3.2, 2005 FI2.2, 2019 HI10

$\sqrt[3]{5+\sqrt{x}}+\sqrt[3]{5-\sqrt{x}}=1$

$\left(\sqrt[3]{5+\sqrt{x}}+\sqrt[3]{5-\sqrt{x}}\right)^3=1$

$5+\sqrt{x}+3\left(5+\sqrt{x}\right)^{2/3}\left(5-\sqrt{x}\right)^{1/3}+3\left(5+\sqrt{x}\right)^{1/3}\left(5-\sqrt{x}\right)^{2/3}+5-\sqrt{x}=1$

$10+3\left(25-x\right)^{1/3}\left(5+\sqrt{x}\right)^{1/3}+3\left(25-x\right)^{1/3}\left(5-\sqrt{x}\right)^{1/3}=1$

$9+3\left(25-x\right)^{1/3}\left[\left(5+\sqrt{x}\right)^{1/3}+\left(5-\sqrt{x}\right)^{1/3}\right]=0$

$3+\left(25-x\right)^{1/3}=0$

$\left(25-x\right)^{1/3}=-3$

$25-x=-27$

$x=52$

G3.4 若 a、b 及 y 為實數，並滿足 $\begin{cases}a+b+y=5\\ab+by+ay=3\end{cases}$，求 y 的最大值。

If a, b and y are real numbers and satisfy $\begin{cases}a+b+y=5\\ab+by+ay=3\end{cases}$,

determine the greatest possible value of y.

From (1): $a = 5 - (b + y)$ ⋯⋯ (3)

Sub. (3) into (2): $[5 - (b + y)]b + by + [5 - (b + y)]y = 3$

$b^2 + (y - 5)b + (y^2 - 5y + 3) = 0$, this is a quadratic equation in b.

For real values of b, $\Delta = (y - 5)^2 - 4(y^2 - 5y + 3) \ge 0$

$-3y^2 + 10y + 13 \ge 0$

$3y^2 - 10y - 13 \le 0$

$(3y - 13)(y + 1) \le 0$

$-1 \le y \le \frac{13}{3}$

The maximum value of $y = \frac{13}{3}$.

Group Event 4

G4.1 若 a 及 b 為整數，且 a^2 與 b^2 相差 144，求 $d = a + b$ 的最大值。

Let a and b are two integers and the difference between a^2 and b^2 is 144, determine the largest possible value of $d = a + b$.

$a^2 - b^2 = 144$

$(a + b)(a - b) = 144 = 144\times1 = 72\times2 = \cdots\cdots$

When $a + b = 144$, $a - b = 1$, then $a = 72.5$, $b = 71.5$, which are not integers.

When $a + b = 72$, $a - b = 2$, then $a = 37$, $b = 35$

$\therefore$ The largest possible value of $a + b = 72$.

Remark: The original question is

若 a^2 及 b^2 為整數，且相差 144，求 $d = a + b$ 的最大值。

Let a^2 and b^2 are two integers that differ by 144, determine the largest possible value of $d = a + b$.

The original question is wrong because d can be any positive number.

e.g. $a^2 = 100000144$, $b^2 = 100000000$, then $a^2 - b^2 = 144$ and $d = a + b \approx 200000$

e.g. $a^2 = 10^{100} + 144$, $b^2 = 10^{100}$, then $a^2 - b^2 = 144$ and $d = a + b \approx 2\times10^{50}$

G4.2 若 n 為整數，n^2 的個位及 10 位分別為 u 及 7，求 u 的值。

If n is an integer, and the units and tens digits of n^2 are u and 7, respectively, determine the value of u.

$0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$

Let $n = 10a + b$, $n^2 = (10a + b)^2 = 100a + 20ab + b^2 \equiv 70 + u \pmod{100}$

$u = 0, 1, 4, 5, 6$ or 9

$20ab + b^2 \equiv 70 + u \pmod{100}$

$b(20a + b) \equiv 70 + u \pmod{100}$

If $b = 0$, L.H.S. $\not\equiv$ R.H.S.

If $b = 1$, $20a + 1 \equiv 71 \Rightarrow 20a \equiv 70$. No solution

If $b = 2$, $40a + 4 \equiv 74 \Rightarrow 40a \equiv 70$. No solution

If $b = 3$, $60a + 9 \equiv 79 \Rightarrow 60a \equiv 70$. No solution

If $b = 4$, $80a + 16 \equiv 76 \pmod{100}$

$80a \equiv 60 \pmod{100}$

$8a \equiv 6 \pmod{10}$

$a = 2$ or 7 (e.g. $24^2 = 576$ and $74^2 = 5476$)

If $b = 5$, $100a + 25 \equiv 75 \Rightarrow 100a \equiv 50$. No solution

If $b = 6$, $120a + 36 \equiv 76 \pmod{100}$

$20a \equiv 40 \pmod{100}$

$a = 2$ or 7 (e.g. $26^2 = 676$ and $76^2 = 5776$)

If $b = 7$, $140a + 49 \equiv 79 \Rightarrow 140a \equiv 30$. No solution

If $b = 8$, $160a + 64 \equiv 74 \Rightarrow 160a \equiv 10$. No solution

If $b = 9$, $180a + 81 \equiv 71 \Rightarrow 180a \equiv 90$. No solution

Conclusion $u = 6$

G4.3 求實數 $c = \dfrac{\left(4+\sqrt{15}\right)^{\frac{3}{2}} + \left(4-\sqrt{15}\right)^{\frac{3}{2}}}{\left(6+\sqrt{35}\right)^{\frac{3}{2}} - \left(6-\sqrt{35}\right)^{\frac{3}{2}}}$ 的值。

Determine the value of real number $c = \dfrac{\left(4+\sqrt{15}\right)^{\frac{3}{2}} + \left(4-\sqrt{15}\right)^{\frac{3}{2}}}{\left(6+\sqrt{35}\right)^{\frac{3}{2}} - \left(6-\sqrt{35}\right)^{\frac{3}{2}}}$.

$\sqrt{8+2\sqrt{15}} = \sqrt{5+2\sqrt{3\times5}+3} = \sqrt{5}+\sqrt{3}$, $\sqrt{8-2\sqrt{15}} = \sqrt{5-2\sqrt{3\times5}+3} = \sqrt{5}-\sqrt{3}$

$\sqrt{12+2\sqrt{35}} = \sqrt{7+2\sqrt{7\times5}+5} = \sqrt{7}+\sqrt{5}$, $\sqrt{12-2\sqrt{35}} = \sqrt{7-2\sqrt{7\times5}+5} = \sqrt{7}-\sqrt{5}$

$$c=\frac{\left(4+\sqrt{15}\right)^{\frac{3}{2}}+\left(4-\sqrt{15}\right)^{\frac{3}{2}}}{\left(6+\sqrt{35}\right)^{\frac{3}{2}}-\left(6-\sqrt{35}\right)^{\frac{3}{2}}}$$

$$=\frac{2^{\frac{3}{2}}\left[\left(4+\sqrt{15}\right)^{\frac{3}{2}}+\left(4-\sqrt{15}\right)^{\frac{3}{2}}\right]}{2^{\frac{3}{2}}\left[\left(6+\sqrt{35}\right)^{\frac{3}{2}}-\left(6-\sqrt{35}\right)^{\frac{3}{2}}\right]}$$

$$=\frac{\left(8+2\sqrt{15}\right)^{\frac{3}{2}}+\left(8-2\sqrt{15}\right)^{\frac{3}{2}}}{\left(12+2\sqrt{35}\right)^{\frac{3}{2}}-\left(12-2\sqrt{35}\right)^{\frac{3}{2}}}$$

$$=\frac{\left(\sqrt{5}+\sqrt{3}\right)^3+\left(\sqrt{5}-\sqrt{3}\right)^3}{\left(\sqrt{7}+\sqrt{5}\right)^3-\left(\sqrt{7}-\sqrt{5}\right)^3}$$

$$=\frac{\left(\sqrt{5}\right)^3+3\left(\sqrt{5}\right)\left(\sqrt{3}\right)^2}{3\left(\sqrt{7}\right)^2\left(\sqrt{5}\right)+\left(\sqrt{5}\right)^3}=\frac{5\left(\sqrt{5}\right)+9\left(\sqrt{5}\right)}{21\left(\sqrt{5}\right)+5\left(\sqrt{5}\right)}=\frac{7}{13}$$

G4.4 求下列方程 $x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}}$ 的正實數解。

Determine the positive real root of the following equation: $x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}}$.

$$\frac{1}{x-1}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}\Rightarrow\frac{1}{x-1}-1=\cfrac{1}{1+\cfrac{1}{\frac{x+1}{x}}}$$

$$\frac{2-x}{x-1}=\cfrac{1}{1+\cfrac{x}{x+1}}$$

$$\frac{2-x}{x-1}=\frac{1}{\frac{2x+1}{x+1}}$$

$$\frac{2-x}{x-1}=\frac{x+1}{2x+1}$$

$(2-x)(2x+1)=(x-1)(x+1)$

$-2x^2+3x+2=x^2-1$

$3x^2-3x-3=0$

$x^2-x-1=0$

$x=\frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$ (< 0, rejected)

Remark: "求實數……的正數值。" is changed into "求下列方程……的正實數解。"

"Determine the positive value of the real number ……" is changed into "Determine the positive real root of the following equation ……"