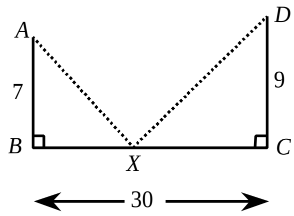
92-93 Individual	**1**	34	**2**	121	**3**	2	**4**	$\frac{3}{5}$	**5**	720
	6	11260	**7**	11	**8**	80	**9**	13	**10**	9

92-93 Group	**1**	2 km	**2**	45	**3**	6	**4**	211	**5**	9
	6	−7	**7**	26	**8**	$2+\sqrt{3}$	**9**	70	**10**	$\frac{\sqrt{13}}{3}$

Individual Events

I1 X is a point on the line segment BC as shown in figure 1.

If $AB = 7$, $CD = 9$ and $BC = 30$, find the minimum value of $AX + XD$.



(Figure 1)

Reference: 1983 FG8.1, 1991 HG9, 1996 HG9

Reflect point A along BC to A'.

By the property of reflection.

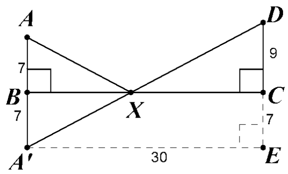
$A'B \perp BC$ and $A'B = 7$

Join $A'D$, which cuts BC at X.

$\Delta ABX \cong \Delta A'BX$ (S.A.S.)

$AX + XD = A'X + XD$

This is the minimum when A', X, D are collinear.



Draw AE // BC which intersects DC produced at E.

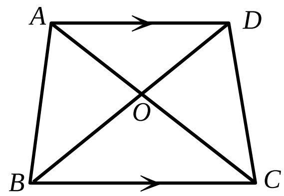
Then $A'E \perp DE$ (corr. $\angle$s, BC // $A'E$)

$A'E = 30$ and $CE = 7$ (opp. sides, rectangle)

$A'D^2 = 30^2 + (7 + 9)^2 = 1156 \Rightarrow A'D = 34$

The minimum value of $AX + XD = 34$

I2 In quadrilateral $ABCD$, AD // BC, and AC, BD intersect at O (as shown in figure 2). Given that area of ΔBOC = 36, area of ΔAOD = 25, determine the area of the quadrilateral $ABCD$.



(Figure 2)

Reference: 1997 HG3, 2000 FI2.2, 2002 FI1.3, 2004 HG7, 2010HG4, 2013 HG2

$\Delta AOD \sim \Delta COB$ (equiangular)

$$\frac{AO^2}{OC^2} = \frac{\text{area of } \Delta AOD}{\text{area of } \Delta BOC} = \frac{25}{36}$$

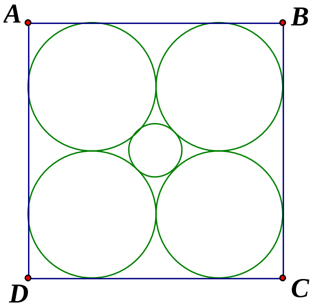
$$\frac{AO}{OC} = \frac{5}{6}$$

Area of $\Delta AOB = \frac{5}{6} \times$area of $\Delta BOC = \frac{5}{6} \times 36 = 30$

Area of $\Delta COD = \frac{6}{5} \times$area of $\Delta AOD = \frac{6}{5} \times 25 = 30$

Area of quadrilateral $ABCD$ = 25 + 30 + 36 + 30 = 121

I3 In figure 3, *ABCD* is a square of side $8(\sqrt{2}+1)$. Find the radius of the small circle at the centre of the square.



(Figure 3)

Let *AC* and *BD* intersect at *O*. $AC \perp BD$.

Let *H*, *K* be the centres of two adjacent circles touch each other at *E*.

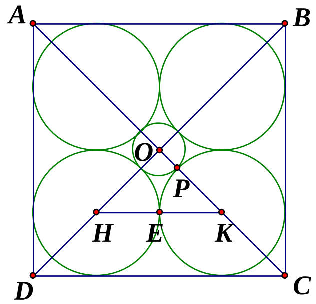
The small circle touches one of the other circles at *P*.

$HE = EK = \frac{CD}{4} = 2\left(\sqrt{2}+1\right) = KP$, $HK = 4\left(\sqrt{2}+1\right)$

$OH = OK = HK \cos 45° = 2\left(2+\sqrt{2}\right)$

$OP = OK - KP = 2\left(2+\sqrt{2}\right) - 2\left(\sqrt{2}+1\right) = 2$

$\therefore$ The radius = 2



I4 Thirty cards are marked from 1 to 30 and one is drawn at random. Find the probability of getting a multiple of 2 or a multiple of 5.

Let *A* be the event that the number drawn is a multiple of 2.

B be the event that the number drawn is a multiple of 5.

$A \cap B$ is the event that the number drawn is a multiple of 10.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{15}{30} + \frac{6}{30} - \frac{3}{30}$

$= \frac{18}{30} = \frac{3}{5}$

I5 The areas of three different faces of a rectangular box are 120, 72 and 60 respectively. Find its volume.

Let the lengths of sides of the box be *a*, *b*, *c*, where $a > b > c$.

$ab = 120$ …… (1)

$bc = 60$ …… (2)

$ca = 72$ …… (3)

(1)×(2) ×(3): $(abc)^2 = (60\times6\times2)^2$

$abc = 720$

The volume is 720 .

I6 For any positive integer n , it is known that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. Find the value of $12^2 + 14^2 + 16^2 + \ldots + 40^2$. **(Reference: 1989 HG3)**

$12^2 + 14^2 + 16^2 + \ldots + 40^2 = 4\times\left(6^2+7^2+8^2+\cdots+20^2\right)$

$= 4\times\left[1^2+\cdots+20^2-\left(1^2+\cdots+5^2\right)\right]$

$= 4\times\left[\frac{20(21)(41)}{6} - \frac{5(6)(11)}{6}\right]$

$= 4(2870 - 55) = 11260$

I7 If x and y are prime numbers such that $x^2 - y^2 = 117$, find the value of x.

Reference: 1995 HG4, 1997 HI1

$(x + y)(x - y) = 117 = 3^2 \times 13$

Without loss of generality, assume $x \ge y$.

$x + y = 117, x - y = 1 \cdots\cdots (1)$

or $x + y = 39, x - y = 3 \cdots\cdots (2)$

or $x + y = 13, x - y = 9 \cdots\cdots (3)$

From (1), $x = 59, y = 58$, not a prime, rejected

From (2), $x = 21, y = 18$, not a prime, rejected

From (3), $x = 11, y = 2 \Rightarrow x = 11$

I8 If m is the total number of positive divisors of 54000, find the value of m.

Reference 1994 FI3.2, 1997 HI3, 1998 HI10, 1998 FI1.4, 2002 FG4.1, 2005 FI4.4

$54000 = 2^4 \times 3^3 \times 5^3$

Positive divisors are in the form $2^x \times 3^y \times 5^z$ where x, y, z are integers and $0 \le x \le 4$, $0 \le y \le 3$, $0 \le z \le 3$

Total number of positive factors = $5 \times 4 \times 4 = 80$

I9 If a is a real number such that $a^2 - a - 1 = 0$, find the value of $a^4 - 2a^3 + 3a^2 - 2a + 10$.

Reference: 2000 HG1, 2001 FG2.1, 2007 HG3, 2009 HG2

By division algorithm,

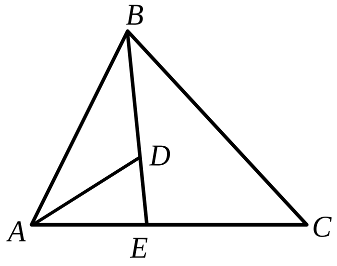
$a^4 - 2a^3 + 3a^2 - 2a + 10$

$= (a^2 - a - 1)(a^2 - a + 3) + 13$

$= 13$

$$\begin{array}{r}
a^2 - a + 3 \\
a^2 - a - 1 \overline{\smash{)}\, a^4 - 2a^3 + 3a^2 - 2a + 10} \\
\underline{a^4 - a^3 - a^2 \qquad\qquad} \\
-a^3 + 4a^2 - 2a \qquad \\
\underline{-a^3 + a^2 + a \qquad} \\
3a^2 - 3a + 10 \\
\underline{3a^2 - 3a - 3} \\
13
\end{array}$$

I10 In figure 4, BDE and AEC are straight lines, $AB = 2$, $BC = 3$, $\angle ABC = 60°$, $AE : EC = 1 : 2$. If $BD : DE = 9 : 1$ and area of $\Delta DBA = \frac{a\sqrt{3}}{20}$, find the value of a.



(Figure 4)

Area of $\Delta ABC = \frac{1}{2} AB \cdot BC \cdot \sin 60° = \frac{2}{2} \cdot 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

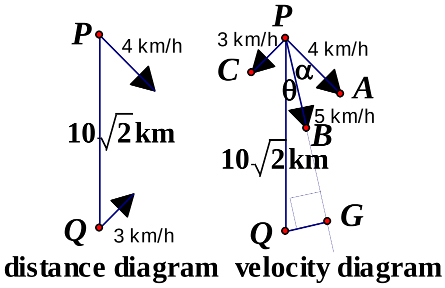
Area of $\Delta ABE = \frac{1}{3} \cdot$area of $\Delta ABC = \frac{\sqrt{3}}{2}$

Area of $\Delta ABD = \frac{9}{10} \cdot$area of $\Delta ABE = \frac{9\sqrt{3}}{20}$

$\Rightarrow a = 9$

Group Events

G1 A car P is $10\sqrt{2}$ km north of another car Q. The two cars start to move at the same time with P moving south-east at 4 km/h and Q moving north-east at 3 km/h. Find their smallest distance of separation in km.



Consider the relative velocity.

Keep Q fixed, the velocity of P relative to Q is 5 km/h in the direction of PB, where $\angle BPQ = \theta$.

Let $\angle APB = \alpha$, $\angle APQ = 45°$

$\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$

$\sin\theta = \sin(45° - \alpha) = \sin 45° \cos\alpha - \cos 45° \sin\alpha$

$= \frac{1}{\sqrt{2}}\cdot\frac{4}{5} - \frac{1}{\sqrt{2}}\cdot\frac{3}{5} = \frac{1}{5\sqrt{2}}$

When the course of PB is nearest to Q (i.e at G),

The shortest distance is $GQ = PQ \sin\theta = 10\sqrt{2} \times \frac{1}{5\sqrt{2}} = 2$ km

G2 If α, β are the roots of the equation $x^2 - 3x - 3 = 0$, find $\alpha^3 + 12\beta$.

Reference: 2010 HI2, 2013 HG4

$\alpha^2 - 3\alpha - 3 = 0$

$\Rightarrow \alpha^3 = 3\alpha^2 + 3\alpha = 3(3\alpha + 3) + 3\alpha = 12\alpha + 9$

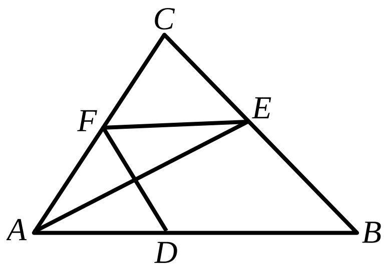
$\alpha + \beta = 3$, $\alpha\beta = -3$

$\alpha^3 + 12\beta = 12\alpha + 9 + 12\beta$

$= 12\times 3 + 9 = 45$

G3 As shown in figure 1, the area of ΔABC is 10. D, E, F are points on AB, BC and CA respectively such that $AD : DB = 2 : 3$, and area of ΔABE = area of quadrilateral $BEFD$.

Find the area of ΔABE.



(Figure 1)

Join DE. Area of ΔADE = area of ΔDEF

$\because$ ΔADE and ΔDEF have the same base and the same height

$\therefore$ $DE \parallel AC$

$BE : EC = BD : DA = 3 : 2$ (theorem of equal ratio)

Area of ΔABE = Area of $\Delta ABC \times \frac{BE}{BC} = 10 \times \frac{3}{3+2} = 6$

G4 What is the maximum number of regions produced by drawing 20 straight lines on a plane?

2 lines: maximum number of regions = 4 = 2 + 2

3 lines: maximum number of regions = 7 = 2 + 2 + 3

4 lines: maximum number of regions = 11 = 2 + 2 + 3 + 4

……………………………………………………………………………………

20 lines, maximum number of regions = $2 + 2 + 3 + \ldots + 20 = 1 + \frac{21}{2}\times 20 = 211$

G5 The product of 4 consecutive positive integers is 3024. Find the largest integer among the four.

Let the four integers be x, $x + 1$, $x + 2$, $x + 3$.

Reference 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3, 2013HI5

$x(x + 1)(x + 2)(x + 3) = 3024$

$(x^2 + 3x)(x^2 + 3x + 2) = 3024$

$(x^2 + 3x)^2 + 2(x^2 + 3x) + 1 = 3025$

$(x^2 + 3x + 1)^2 = 55^2$

$x^2 + 3x + 1 = 55$ or $x^2 + 3x + 1 = -55$

$x^2 + 3x - 54 = 0$ or $x^2 + 3x + 56 = 0$

$(x - 6)(x + 9) = 0$ or no real solution

$\because x > 0 \therefore x = 6$

The largest integer = 9

Method 2

$3024 + 1 = 3025 = 55^2$

$3024 = 55^2 - 1^2 = (55 - 1)(55 + 1)$

$3024 = 54\times 56 = 6\times 9\times 7\times 8$

The largest integer is 9.

G6 Find the sum of all real roots of the equation $(x + 2)(x + 3)(x + 4)(x + 5) = 3$.

Reference 1993 HG5, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3, 2013 HI5

Let $t = x + 3.5$

$(t - 1.5)(t - 0.5)(t + 0.5)(t + 1.5) = 3$

$t^4 - \frac{5}{2}t^2 + \frac{9}{16} - 3 = 0$

$\left(t^2 - \frac{5}{4}\right)^2 - 4 = 0$

$\left(t^2 - \frac{5}{4} + 2\right)\left(t^2 - \frac{5}{4} - 2\right) = 0$

$t^2 = \frac{13}{4} \Rightarrow t = \pm\frac{\sqrt{13}}{2}$

$x = t - 3.5 = \frac{-7 \pm \sqrt{13}}{2}$

Sum of real roots = -7

Method 2

$(x + 2)(x + 5)(x + 3)(x + 4) = 3$

$(x^2 + 7x + 10)(x^2 + 7x + 12) = 3$

Let $y = x^2 + 7x$

$(y + 10)(y + 12) = 3$

$y^2 + 22y + 117 = 0$

$(y + 9)(y + 13) = 0$

When $y = -9 = x^2 + 7x$

$x^2 + 7x + 9 = 0$

When $y = -13 = x^2 + 7x$

$x^2 + 7x + 13 = 0$

$\Delta = 49 - 52 < 0$, no solution

$\therefore$ Sum of roots = -7

G7 If a is an integer and $a^7 = 8031810176$, find the value of a.

$1280000000 = 20^7 < 8031810176 < 30^7 = 21870000000$

Clearly a is an even integer.

$2^7 \equiv 8, 4^7 \equiv 4, 6^7 \equiv 6, 8^7 \equiv 2 \pmod{10}$

$\therefore a = 26$

G8 If x and y are real numbers satisfying $\begin{cases} x^2 - xy + y^2 - 3x - 3y = 1 \\ xy = 1 \end{cases}$ and $x > y > 0$,

find the value of x. **Reference: 2010 FI1.3, 2013 FI4.4**

Let $t = x + y$, (1) becomes $(x + y)^2 - 3 - 3(x + y) = 1$

$t^2 - 3t - 4 = 0$

$(t + 1)(t - 4) = 0$

$t = -1$ (rejected) or $t = 4$

$x + y = 4$ and $xy = 1$

x and y are the roots of $u^2 - 4u + 1 = 0$

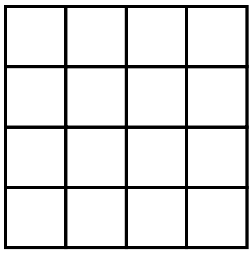
$x = 2 + \sqrt{3}$

G9 Each side of a square is divided into four equal parts and straight lines are joined as shown in figure 2. Find the number of rectangles which are not squares. **(Reference: 2013 FI1.1)**

Number of rectangles including squares $= C_2^5 \times C_2^5 = 100$

Number of squares = 16 + 9 + 4 + 1 = 30

Total number of rectangles which are not squares = 100 – 30 = 70



(Figure 2)

G10 If $0° \le \theta \le 90°$ and $\cos\theta - \sin\theta = \frac{\sqrt{5}}{3}$, find the value of $\cos\theta + \sin\theta$.

Reference: 1992 HI20, 1995 HI5, 2007 HI7, 2007 FI1.4, 2014 HG3

$(\cos\theta - \sin\theta)^2 = \frac{5}{9}$

$1 - 2\sin\theta\cos\theta = \frac{5}{9}$

$\frac{4}{9} - 2\sin\theta\cos\theta = 0$

$2 - 9\sin\theta\cos\theta = 0$

$2(\sin^2\theta + \cos^2\theta) - 9\sin\theta\cos\theta = 0$

$2\tan^2\theta - 9\tan\theta + 2 = 0$

$\tan\theta = \frac{9+\sqrt{65}}{4}$ or $\frac{9-\sqrt{65}}{4}$

When $\tan\theta = \frac{9+\sqrt{65}}{4}$, $\sin\theta = \frac{9+\sqrt{65}}{3(\sqrt{13}+\sqrt{5})}$, $\cos\theta = \frac{4}{3(\sqrt{13}+\sqrt{5})}$,

Original equation LHS $= \cos\theta - \sin\theta = -\frac{5+\sqrt{65}}{3(\sqrt{13}+\sqrt{5})} = -\frac{\sqrt{5}}{3}$ (reject)

When $\tan\theta = \frac{9-\sqrt{65}}{4}$, $\sin\theta = \frac{9-\sqrt{65}}{3(\sqrt{13}-\sqrt{5})}$, $\cos\theta = \frac{4}{3(\sqrt{13}-\sqrt{5})}$,

Original equation LHS $= \cos\theta - \sin\theta = \frac{\sqrt{5}}{3}$

$\therefore \cos\theta + \sin\theta = \frac{9-\sqrt{65}}{3(\sqrt{13}-\sqrt{5})} + \frac{4}{3(\sqrt{13}-\sqrt{5})} = \frac{\sqrt{13}}{3}$

Method 2

$\cos\theta > \sin\theta \Rightarrow \theta < 45° \Rightarrow 2\theta < 90°$

$(\cos\theta - \sin\theta)^2 = \frac{5}{9}$

$1 - 2\sin\theta\cos\theta = \frac{5}{9}$

$\sin 2\theta = \frac{4}{9} \Rightarrow \cos 2\theta = \frac{\sqrt{65}}{9}$ $\because 2\theta < 90°$

$(\cos\theta - \sin\theta)(\cos\theta + \sin\theta) = \frac{\sqrt{5}}{3}(\cos\theta + \sin\theta)$

$$\cos^2\theta - \sin^2\theta = \frac{\sqrt{5}}{3}(\cos\theta + \sin\theta)$$

$$\frac{\sqrt{65}}{9} = \cos 2\theta = \frac{\sqrt{5}}{3}(\cos\theta + \sin\theta)$$

$$\cos\theta + \sin\theta = \frac{\sqrt{65}}{9} \div \frac{\sqrt{5}}{3} = \frac{\sqrt{13}}{3}$$

Method 3

$$(\cos\theta - \sin\theta)^2 = \frac{5}{9}$$

$$1 - 2\sin\theta\cos\theta = \frac{5}{9}$$

$$2\sin\theta\cos\theta = \frac{4}{9}$$

$$1 + 2\sin\theta\cos\theta = \frac{13}{9}$$

$$\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = \frac{13}{9}$$

$$(\cos\theta + \sin\theta)^2 = \frac{13}{9}$$

$$\cos\theta + \sin\theta = \frac{\sqrt{13}}{3} \quad \text{or} \quad -\frac{\sqrt{13}}{3}$$

$\because$ $0° \le \theta \le 90°$

$\therefore$ $\cos\theta + \sin\theta > 0$

$$\cos\theta + \sin\theta = \frac{\sqrt{13}}{3}$$