Summation

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Prove that $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos[\alpha+(n-1)\beta] = \dfrac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$

Let $C = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos[\alpha+(n-1)\beta]$

$S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin[\alpha+(n-1)\beta]$

$z = \cos\beta + i\sin\beta$

$C + i\,S = \cos\alpha + \cos(\alpha+\beta) + \cdots + \cos[\alpha+(n-1)\beta] + i\{\sin\alpha + \sin(\alpha+\beta) + \cdots + \sin[\alpha+(n-1)\beta)]\}$

$= [\cos\alpha + i\sin\alpha] + [\cos(\alpha+\beta) + i\sin(\alpha+\beta)] + \cdots + \{\cos[\alpha+(n-1)\beta] + i\sin[\alpha+(n-1)\beta]\}$

$= \text{cis}\,\alpha + z\,\text{cis}\,\alpha + z^2\,\text{cis}\,\alpha + \cdots + z^{n-1}\,\text{cis}\,\alpha$

$= \text{cis}\,\alpha\,(1 + z + z^2 + \cdots + z^{n-1})$

$$= cis\,\alpha \cdot \frac{z^n - 1}{z - 1},\ z \neq 1$$

$$= cis\,\alpha \cdot \frac{z^{\frac{n}{2}}\left(z^{\frac{n}{2}} - z^{-\frac{n}{2}}\right)}{z^{\frac{1}{2}}\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)}$$

$$= z^{\frac{n-1}{2}}\,cis\,\alpha \cdot \frac{2i\sin\frac{n\beta}{2}}{2i\sin\frac{\beta}{2}}$$

$$= cis\left(\alpha + \frac{n-1}{2}\beta\right) \cdot \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

Compare real part: $\cos\alpha + \cos(\alpha+\beta) + \cdots + \cos[\alpha+(n-1)\beta] = \dfrac{\cos\left(\alpha + \frac{n-1}{2}\beta\right)\cdot\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$

Special Cases

(1) Put $\alpha = 5°$, $\beta = 72°$, $n = 5$

$\cos 5° + \cos 77° + \cos 149° + \cos 221° + \cos 293°$

$$= \frac{\cos(5^\circ + 2\times 72^\circ)\sin\frac{5\times 72^\circ}{2}}{\sin\frac{72^\circ}{2}}$$

$$= \frac{\cos 221^\circ \sin 180^\circ}{\sin 36^\circ} = 0$$

(2) Put $\alpha = 1°$, $\beta = 1°$, $n = 90$

$$\cos 1° + \cos 2° + \cos 3° + \cdots + \cos 90° = \frac{\cos 45.5^\circ \sin 45^\circ}{\sin \frac{1^\circ}{2}} = 56.794$$

(3) Put $\alpha = \frac{\pi}{2n-1}$, $\beta = \frac{2\pi}{2n-1}$ and replace n by $n - 1$

$$\cos\frac{\pi}{2n-1} + \cos\frac{3\pi}{2n-1} + \cdots + \cos\frac{(2n-3)\pi}{2n-1} = \frac{\cos\left(\frac{\pi}{2n-1} + \frac{(n-2)\pi}{2n-1}\right)\sin\frac{(n-1)\pi}{2n-1}}{\sin\frac{\pi}{2n-1}}$$

$$= \frac{\cos\frac{(n-1)\pi}{2n-1}\sin\frac{(n-1)\pi}{2n-1}}{\sin\frac{\pi}{2n-1}}$$

$$= \frac{\sin\frac{2(n-1)\pi}{2n-1}}{2\sin\frac{\pi}{2n-1}}\text{, using } \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{1}{2}\text{, } \because \sin\frac{2(n-1)\pi}{2n-1} = \sin\frac{\pi}{2n-1}$$

In particular, $n = 4$, $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$

(4) Put $\alpha = \frac{2\pi}{2n-1}$, $\beta = \frac{2\pi}{2n-1}$ and replace n by $n - 1$

$$\cos\frac{2\pi}{2n-1} + \cos\frac{4\pi}{2n-1} + \cdots + \cos\frac{2(n-1)\pi}{2n-1} = \frac{\cos\left[\frac{2\pi}{2n-1} + \frac{(n-2)\pi}{2n-1}\right]\sin\frac{(n-1)\pi}{2n-1}}{\sin\frac{\pi}{2n-1}}$$

$$= \frac{\cos\frac{n\pi}{2n-1}\sin\frac{(n-1)\pi}{2n-1}}{\sin\frac{\pi}{2n-1}}$$

$$= \frac{\sin\frac{(2n-1)\pi}{2n-1} - \sin\frac{\pi}{2n-1}}{2\sin\frac{\pi}{2n-1}}$$

$$= -\frac{1}{2}$$

Exercise: Prove that $\cos\frac{\pi}{2n-1} - \cos\frac{2\pi}{2n-1} + \cos\frac{3\pi}{2n-1} - \cdots + (-1)^{n-2}\cos\frac{(n-1)\pi}{2n-1} = \frac{1}{2}$

Note that $-\cos\frac{2\pi}{2n-1} = \cos\frac{(2n-3)\pi}{2n-1}$, $-\cos\frac{2\pi}{2n-1} = \cos\frac{(2n-5)\pi}{2n-1}$, $\cdots$

$$\cos\frac{\pi}{2n-1} - \cos\frac{2\pi}{2n-1} + \cos\frac{3\pi}{2n-1} - \cdots + (-1)^{n-2}\cos\frac{(n-1)\pi}{2n-1}$$

$$= \cos\frac{\pi}{2n-1} + \cos\frac{3\pi}{2n-1} + \cos\frac{5\pi}{2n-1} + \cdots + \cos\frac{(2n-5)\pi}{2n-1} + \cos\frac{(2n-3)\pi}{2n-1} = \frac{1}{2} \quad \text{(By Part 3)}$$