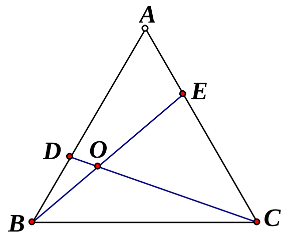
18-19 Individual	**1**	60	**2**	$24\sqrt{3}$	**3**	–4	**4**	1987	**5**	516
	6	516	**7**	$\frac{7\sqrt{5}}{3}$	**8**	8	**9**	32	**10**	4
	11	9	**12**	$5\sqrt{13}$	**13**	9	**14**	3	**15**	7
18-19 Group	**1**	1010	**2**	25	**3**	30	**4**	2	**5**	–1
	6	64	**7**	120	**8**	4	**9**	12	**10**	$25\sqrt{3}+37.5$

Individual Events

I1 在圖一中，ABC 是一個等邊三角形。D 和 E 分別是 AB 和 AC 上的點，使得 $AE = BD$。若 CD 和 BE 相交於 O 及 $\angle COE = y°$，求 y 的值。

In Figure 1, ABC is an equilateral triangle. D and E are points on AB and AC respectively such that $AE = BD$. Id CD and BE intersect at O and $\angle COE = y°$, find the value of y .



Reference: 2000 HG6

$AE = BD$ (已知)	$AE = BD$ (given)
$\angle BAE = \angle CBD = 60°$ (等邊三角形的性質)	$\angle BAE = \angle CBD = 60°$ (prop. of equilateral Δ)
$AB = CB$ (等邊三角形的性質)	$AB = CB$ (prop. of equilateral Δ)
$\therefore \Delta EAB \cong \Delta DBC$ (S.A.S.)	$\therefore \Delta EAB \cong \Delta DBC$ (S.A.S.)
$\angle ABE = \angle BCD = \theta$ (全等三角形對應邊)	$\angle ABE = \angle BCD = \theta$ (cor. sides $\cong \Delta$s)
$\angle CBE = 60° - \theta$ (等邊三角形的性質)	$\angle CBE = 60° - \theta$ (prop. of equilateral Δ)
$\angle COE = \angle CBE + \angle BCD$ (ΔBCO 的外角)	$\angle COE = \angle CBE + \angle BCD$ (ext. $\angle$ of ΔBCO)
$= 60° - \theta + \theta = 60°$	$= 60° - \theta + \theta = 60°$
$y = 60$	$y = 60$

I2 設 O 為極座標系統的極點。若 $P(6, 240°)$ 向右平移 16 單位至 Q 而ΔOPQ 的面積為 T 平方單位，求 T 的值。

Let O be the pole of the polar coordinate system. If $P(6, 240°)$. If P is translated to the right by 16 units to Q and the area of ΔOPQ is T square units, find the value of T .

Reference: 2016 HI9

P 的直角座標為	The rectangular coordinates of P is
$(6 \cos 240°, 6 \sin 240°) = (-3, -3\sqrt{3})$。	$(6 \cos 240°, 6 \sin 240°) = (-3, -3\sqrt{3})$.
Q 的直角座標為$(13, -3\sqrt{3})$。	The rectangular coordinates of Q is $(13, -3\sqrt{3})$.
$T=\frac{1}{2}\left\|\begin{vmatrix}-3 & -3\sqrt{3}\\ 13 & -3\sqrt{3}\end{vmatrix}\right\|=\frac{1}{2}\left(9\sqrt{3}+39\sqrt{3}\right)=24\sqrt{3}$	$T=\frac{1}{2}\left\|\begin{vmatrix}-3 & -3\sqrt{3}\\ 13 & -3\sqrt{3}\end{vmatrix}\right\|=\frac{1}{2}\left(9\sqrt{3}+39\sqrt{3}\right)=24\sqrt{3}$

I3 已知 x 及 y 均為實數，若 $y^2 - 4xy + 5x^2 - 8x + 16 = 0$ 及 $F = x - y$，求 F 的值。

Given that x and y are real numbers.

If $y^2 - 4xy + 5x^2 - 8x + 16 = 0$ and $F = x - y$, find the value of F .

Reference: 2015 HG4

$y^2 - 4xy + 4x^2 + x^2 - 8x + 16 = 0$	$y^2 - 4xy + 4x^2 + x^2 - 8x + 16 = 0$
$(y - 2x)^2 + (x - 4)^2 = 0$	$(y - 2x)^2 + (x - 4)^2 = 0$
兩個平方之和 $= 0$	sum of two squares $= 0$
$\Rightarrow$ 每一項$= 0$	$\Rightarrow$ Each term $= 0$
$y - 2x = 0$ 及 $x = 4 \Rightarrow y = 8$	$y - 2x = 0$ and $x = 4 \Rightarrow y = 8$
$F = x - y = 4 - 8 = -4$	$F = x - y = 4 - 8 = -4$

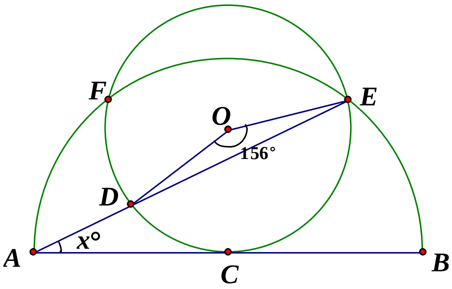
I4 設 n 為正整數。若 $a_n = 1 + 2 + \cdots + 2^n$ 及 $b = a_{10} - a_5 + a_1$，求 b 的值。

Let n be a positive integer. If $a_n = 1 + 2 + \cdots + 2^n$ and $b = a_{10} - a_5 + a_1$, find the value of b.

利用等比級數 n 項之和公式： $a_n = 2^{n+1} - 1$ 由 $n = 1, 2, 3, \cdots$ $b = a_{10} - a_5 + a_1$ $= (2^{11} - 1) - (2^6 - 1) + (1 + 2)$ $= 2048 - 64 + 3 = 1987$	By the sum to n terms of a geometric series formula, $a_n = 2^{n+1} - 1$ for $n = 1, 2, 3, \cdots$ $b = a_{10} - a_5 + a_1$ $= (2^{11} - 1) - (2^6 - 1) + (1 + 2)$ $= 2048 - 64 + 3 = 1987$

I5 在圖二中，AB 為半圓的直徑，C 為半圓的圓心。有一圓形，圓心 O 切 AB 於 C 及交半圓於 E 和 F。若 AE 交此圓形於 D、$\angle DOE = 156°$及$\angle BAE = x°$，求 x 的值。

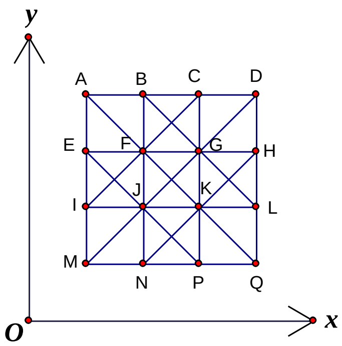
In Figure 2, AB is the diameter of the semi-circle, C is the centre of the semi-circle. A circle with centre at O, touching the semi-circle at C and cutting it at E and F. If AE cuts the circle at D, $\angle DOE = 156°$ and $\angle BAE = x°$, find the value of x.



反角 $\angle DOE = 360° - 156°$ (同頂角) $= 204°$ $\angle DCE = \frac{1}{2}$ 反角 $\angle DOE$ (圓心角兩倍於圓周角) $= 102°$ $\angle ACD = \angle AEC$ (交錯弓形的角) $\angle AEC = x°$ (等腰三角形底角) $\angle BCE = \angle CAE + \angle AEC$ (三角形外角) $= 2x°$ $\angle ACD + \angle DCE + \angle BCE = 180°$ (直綫上的鄰角) $x° + 102° + 2x° = 180°$ $x = 26$	Reflex $\angle DOE = 360° - 156°$ ($\angle$s at a pt.) $= 204°$ $\angle DCE = \frac{1}{2}$ reflex $\angle DOE$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$) $= 102°$ $\angle ACD = \angle AEC$ ($\angle$ in alt. segment) $\angle AEC = x°$ (base $\angle$s isos. Δ) $\angle BCE = \angle CAE + \angle AEC$ (ext. $\angle$ of Δ) $= 2x°$ $\angle ACD + \angle DCE + \angle BCE = 180°$ (adj. $\angle$s on st. line) $x° + 102° + 2x° = 180°$ $x = 26$

I6 在圖三中，直角座標平面上一個正方形的四個頂點的座標分別為 $(1, 1)$、$(1, 4)$、$(4, 1)$及$(4, 4)$。若在該正方形中(包括邊界)選擇任何三個座標均為整數的點，問可組成多少個三角形？

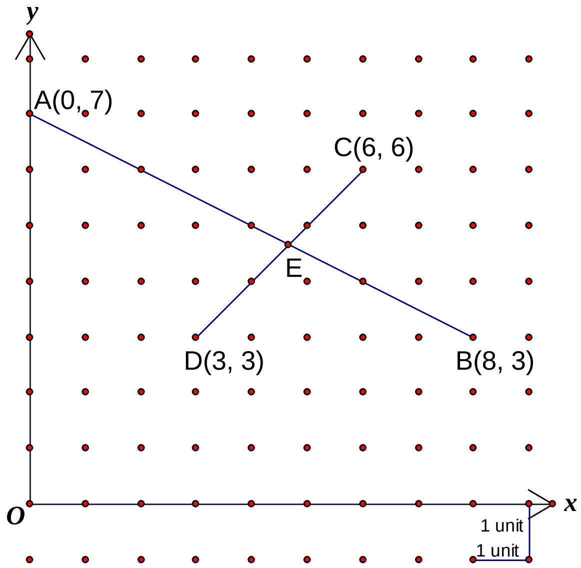
In Figure 3, the vertices of a square in the rectangular coordinate plane are $(1, 1)$, $(1, 4)$, $(4, 1)$ and $(4, 4)$. How many triangles can be formed by selecting any three points in the square (including the boundaries) with integer coordinates?



將這 16 個整數點命名如圖。 其中有 10 條綫段穿過 4 點。 另外有 4 條綫段穿過 3 點。 三角形的數目 $= C_3^{16}$ − 選中三點在同一直綫的數目 $= \frac{16 \times 15 \times 14}{1 \times 2 \times 3} - 10 \times C_3^4 - 4 \times C_3^3$ $= 560 - 40 - 4 = 516$	Label the 16 integral points as shown. There are 10 line segments passing through 4 integral points. There are 4 line segments passing through 3 integral points. Number of triangles $= C_3^{16}$ – number of choices of 3 collinear points $= \frac{16 \times 15 \times 14}{1 \times 2 \times 3} - 10 \times C_3^4 - 4 \times C_3^3$ $= 560 - 40 - 4 = 516$

I7 在圖四中，AB 與 CD 相交於 E。設 AE 的長度為 q 單位，求 q 的值。

In Figure 4, AB and CE intersect at E. Let the length of AE be q units. Find the value of q.

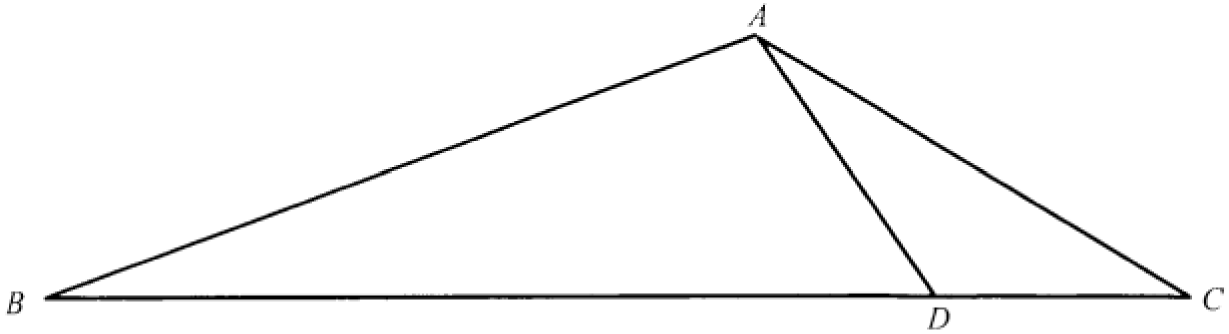


定義一個直角座標系統如圖。 A、B、C 和 D 的座標分別為(0, 7)、(8, 3)、(6, 6)及(3, 3)。 AB 的方程為：$y-7=\frac{7-3}{0-8}\cdot(x-0)$ $y=-\frac{1}{2}x+7\cdots(1)$ CD 的方程為：$y=x\cdots(2)$ 代 (2) 入 (1): $x=-\frac{1}{2}x+7\Rightarrow x=\frac{14}{3}=y$ $q=AE=\sqrt{\left(\frac{14}{3}-0\right)^2+\left(\frac{14}{3}-7\right)^2}=\frac{7\sqrt{5}}{3}$	Define a rectangular co-ordinates system as shown. The coordinates of A, B, C and D are (0, 7), (8, 3), (6, 6) and (3, 3) respectively. Equation of AB: $y-7=\frac{7-3}{0-8}\cdot(x-0)$ $y=-\frac{1}{2}x+7\cdots(1)$ Equation of CD: $y=x\cdots(2)$ Sub. (2) into (1): $x=-\frac{1}{2}x+7\Rightarrow x=\frac{14}{3}=y$ $q=AE=\sqrt{\left(\frac{14}{3}-0\right)^2+\left(\frac{14}{3}-7\right)^2}=\frac{7\sqrt{5}}{3}$

I8 在圖五中，D 是在 BC 上的一點使得$\angle ABD = \angle CAD$ 及 $\frac{BD}{AC} = \frac{8}{3}$。若 $\frac{\Delta ABD \text{ 的面積}}{\Delta ADC \text{ 的面積}} = k$，求 k 的值。

In Figure 5, D is a point on BC such that $\angle ABD = \angle CAD$ and $\frac{BD}{AC} = \frac{8}{3}$.

If $\frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ADC} = k$, find the value of k.



$\Delta ACD \sim \Delta BCA$ (A.A.A.) $\frac{AC}{CD} = \frac{BD+DC}{AC}$ (相似三角形對應邊) $\frac{AC}{CD} = \frac{BD}{AC} + \frac{DC}{AC}$ 設 $t = \frac{AC}{CD}$，則 $\frac{1}{t} = \frac{DC}{AC}$ $t = \frac{8}{3} + \frac{1}{t}$ $3t^2 - 8t - 3 = 0$ $(3t+1)(t-3) = 0$ $t = -\frac{1}{3}$ (捨去) 或 $t = 3$ $CD = \frac{1}{3}AC$ $BD = \frac{8}{3}AC$ $k = \frac{\Delta ABD \text{ 的面積}}{\Delta ADC \text{ 的面積}} = \frac{BD}{CD} = 8$	$\Delta ACD \sim \Delta BCA$ (A.A.A.) $\frac{AC}{CD} = \frac{BD+DC}{AC}$ (corr. sides, ~Δs) $\frac{AC}{CD} = \frac{BD}{AC} + \frac{DC}{AC}$ Let $t = \frac{AC}{CD}$, then $\frac{1}{t} = \frac{DC}{AC}$ $t = \frac{8}{3} + \frac{1}{t}$ $3t^2 - 8t - 3 = 0$ $(3t+1)(t-3) = 0$ $t = -\frac{1}{3}$ (rejected) or $t = 3$ $CD = \frac{1}{3}AC$ $BD = \frac{8}{3}AC$ $k = \frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ADC} = \frac{BD}{CD} = 8$

I9 已知 α 及 β 為方程 $x^2 + 32x - 1 = 0$ 的兩個根。

若 $P = (\alpha^2 + 31\alpha - 2)(\beta^2 + 33\beta)$，求 P 的值。

Given that α and β are the two roots of the equation $x^2 + 32x - 1 = 0$.

If $P = (\alpha^2 + 31\alpha - 2)(\beta^2 + 33\beta)$, find the value of P.

Reference: 2013 HG4

$\alpha^2 + 32\alpha - 1 = 0 \Rightarrow \alpha^2 + 31\alpha - 2 = -\alpha - 1$

$\beta^2 + 32\beta - 1 = 0 \Rightarrow \beta^2 + 33\beta = \beta + 1$

$$\begin{aligned}(\alpha^2 + 31\alpha - 2)(\beta^2 + 33\beta) &= (-\alpha - 1)(\beta + 1)\\ &= -(\alpha + 1)(\beta + 1)\\ &= -(\alpha\beta + \alpha + \beta + 1)\\ &= -(-1 - 32 + 1)\\ &= 32\end{aligned}$$

$P = 32$

I10 設 $c=\sqrt[3]{7+5\sqrt{2}}+\sqrt[3]{7-5\sqrt{2}}$。若 $w=c^2$，求 w 的值。

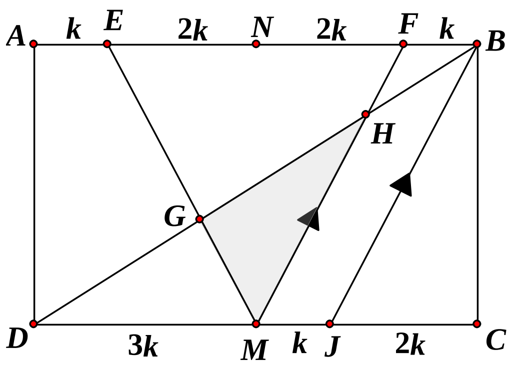
Let $c=\sqrt[3]{7+5\sqrt{2}}+\sqrt[3]{7-5\sqrt{2}}$. If $w=c^2$, find the value of w.

Reference: 1999 FI3.2, 2005 FI2.2, 2016 FG3.3

<table>
<tr>
<td>設 $\left(a+\sqrt{b}\right)^3=7+5\sqrt{2}$
$a^3+3a^2\sqrt{b}+3ab+b\sqrt{b}=7+5\sqrt{2}$
$b=2$, $a^3+3ab=7$, $3a^2+b=5\Rightarrow a=1$
$\left(1+\sqrt{2}\right)^3=7+5\sqrt{2}$ 及 $\left(1-\sqrt{2}\right)^3=7-5\sqrt{2}$
$c=1+\sqrt{2}+\left(1-\sqrt{2}\right)=2$
$w=c^2=4$</td>
<td>Let $\left(a+\sqrt{b}\right)^3=7+5\sqrt{2}$
$a^3+3a^2\sqrt{b}+3ab+b\sqrt{b}=7+5\sqrt{2}$
$b=2$, $a^3+3ab=7$, $3a^2+b=5\Rightarrow a=1$
$\left(1+\sqrt{2}\right)^3=7+5\sqrt{2}$ and $\left(1-\sqrt{2}\right)^3=7-5\sqrt{2}$
$c=1+\sqrt{2}+\left(1-\sqrt{2}\right)=2$
$w=c^2=4$</td>
</tr>
<tr>
<td>方法二
$c^3=7+5\sqrt{2}+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)^2\left(7-5\sqrt{2}\right)}$
$+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)\left(7-5\sqrt{2}\right)^2}+7-5\sqrt{2}$
$=14+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)(49-50)}$
$+3\times\sqrt[3]{(49-50)\left(7-5\sqrt{2}\right)}$
$=14-3c$
$c^3+3c-14=0$
$(c-2)(c^2+2c+7)=0$
$c=2$ 或 沒有實數解
$w=c^2=4$</td>
<td>Method 2
$c^3=7+5\sqrt{2}+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)^2\left(7-5\sqrt{2}\right)}$
$+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)\left(7-5\sqrt{2}\right)^2}+7-5\sqrt{2}$
$=14+3\times\sqrt[3]{\left(7+5\sqrt{2}\right)(49-50)}$
$+3\times\sqrt[3]{(49-50)\left(7-5\sqrt{2}\right)}$
$=14-3c$
$c^3+3c-14=0$
$(c-2)(c^2+2c+7)=0$
$c=2$ or no real solution
$w=c^2=4$</td>
</tr>
</table>

I11 在圖六中，$ABCD$ 為一個長方形。M 和 N 分別是 DC 和 AB 的中點且 $AE : EN = BF : FN = 1 : 2$。$DB$ 分別交 EM 和 FM 於 G 及 H。若長方形 $ABCD$ 及三角形 GHM 的面積分別是 96 和 S，求 S 的值。

In Figure 6, $ABCD$ is rectangle M and N are the mid-points of DC and AB respectively and $AE : EN = BF : FN = 1 : 2$. DB intersects EM and FM at G and H respectively. If the areas of the rectangle $ABCD$ and the triangle GHM are 96 and S respectively, find the value of S.



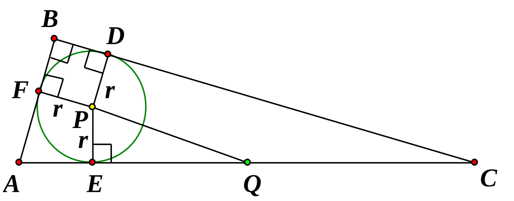
圖六 Figure 6

Reference 1998 HG5, 2016 HI14, 2018 FG3.1

設 $AE = BF = k$, $EN = NF = 2k$, $DM = MC = 3k$	Let $AE = BF = k$, $EN = NF = 2k$, $DM = MC = 3k$
$\Delta BHF \sim \Delta DHM$ (A.A.A.)	$\Delta BHF \sim \Delta DHM$ (A.A.A.)
$\Delta BGE \sim \Delta DGM$ (A.A.A.)	$\Delta BGE \sim \Delta DGM$ (A.A.A.)
$\frac{DH}{HB}=\frac{DM}{BF}=\frac{3k}{k}=3$ (相似三角形對應邊)	$\frac{DH}{HB}=\frac{DM}{BF}=\frac{3k}{k}=3$ (corr. side ~Δs)
$\frac{DG}{GB}=\frac{DM}{BE}=\frac{3k}{5k}=\frac{3}{5}$ (相似三角形對應邊)	$\frac{DG}{GB}=\frac{DM}{BE}=\frac{3k}{5k}=\frac{3}{5}$ (corr. side ~Δs)
$BH=\frac{1}{4}DB$, $DG=\frac{3}{8}DB$	$BH=\frac{1}{4}DB$, $DG=\frac{3}{8}DB$
$GH=DB-DG-BH=\left(1-\frac{3}{8}-\frac{1}{4}\right)DB=\frac{3}{8}DB$	$GH=DB-DG-BH=\left(1-\frac{3}{8}-\frac{1}{4}\right)DB=\frac{3}{8}DB$
$DG : GH=\frac{3}{8}DB:\frac{3}{8}DB \;=1:1$ …… (1)	$DG : GH=\frac{3}{8}DB:\frac{3}{8}DB \;=1:1$ …… (1)
過 B 作 $BJ // MF$，交 CD 於 J。	Draw $BJ // MF$, cutting CD at J.
$\frac{DM}{MJ}=\frac{DH}{HB}=3$ (等比定理)	$\frac{DM}{MJ}=\frac{DH}{HB}=3$ (theorem of equal ratios)
$MJ = k$, $JC = 2k$	$MJ = k$, $JC = 2k$
$\Delta BCD \cong \Delta DAB$ (S.S.S.)	$\Delta BCD \cong \Delta DAB$ (S.S.S.)
$S_{\Delta BCD}=S_{\Delta DAB}=\frac{1}{2}\times 96=48$	$S_{\Delta BCD}=S_{\Delta DAB}=\frac{1}{2}\times 96=48$
$\frac{S_{\Delta BDJ}}{S_{\Delta BCJ}}=\frac{DJ}{CJ}=\frac{4k}{2k}=2\Rightarrow S_{\Delta BDJ}=\frac{2}{3}\times 48=32$	$\frac{S_{\Delta BDJ}}{S_{\Delta BCJ}}=\frac{DJ}{CJ}=\frac{4k}{2k}=2\Rightarrow S_{\Delta BDJ}=\frac{2}{3}\times 48=32$
$\Delta DMH \sim \Delta DJB$ (A.A.A.)	$\Delta DMH \sim \Delta DJB$ (A.A.A.)
$\frac{S_{\Delta DMH}}{S_{\Delta DJB}}=\left(\frac{DM}{DJ}\right)^2=\left(\frac{3}{4}\right)^2\Rightarrow S_{\Delta DMH}=\frac{9}{16}\times 32=18$	$\frac{S_{\Delta DMH}}{S_{\Delta DJB}}=\left(\frac{DM}{DJ}\right)^2=\left(\frac{3}{4}\right)^2\Rightarrow S_{\Delta DMH}=\frac{9}{16}\times 32=18$
由(1), $S_{\Delta GHM}=S_{\Delta GDM}=\frac{1}{2}\times S_{\Delta DMH}=9$	By (1), $S_{\Delta GHM}=S_{\Delta GDM}=\frac{1}{2}\times S_{\Delta DMH}=9$

I12 在三角形 ABC 中，$AB = 14$、$BC = 48$ 及 $AC = 50$。將 P 及 Q 分別記為ΔABC 的內心及外心。設 PQ 的長度為 d 單位。求 d 的值。

In triangle ABC, $AB = 14$, $BC = 48$ and $AC = 50$. Denote the in-centre and circumcentre of ΔABC by P and Q respectively. Let the length of PQ be d units. Find the value of d.



$AB^2 + BC^2 = 14^2 + 48^2 = 196+2304 = 2500 = AC^2$ $\angle ABC = 90°$ (畢氏定理的逆定理) AC 是外接圓 ABC 的直徑(半圓上的圓周角的定理) $Q = AC$ 的中點 (外接圓的圓心) $AQ = 25 \cdots (1)$ 假設內切圓分別切 BC、AC 及 AB 於 D、E 及 F。 $PD\perp BC$，$PE\perp AC$，$PF\perp AB$ (切綫⊥半徑) $PDBF$ 是一個長方形 (它有 3 隻直角) 設內切圓的半徑為 r。 $PD = PE = PF = r$ $PDBF$ 是一個正方形 ($PD = PF$) $BF = BD = r$ $AF = 14 - r$, $CD = 48 - r$ $AE = 14 - r$, $CE = 48 - r$ (由外點引切綫) $AE + EC = AC$ $14 - r + 48 - r = 50$ $r = 6$, $AE = 14 - 6 = 8$ $EQ = AQ - AE = 25 - 8 = 17$ 在ΔPEQ 中，$PE^2 + EQ^2 = PQ^2$ (畢氏定理) $6^2 + 17^2 = PQ^2$ $d = \sqrt{325} = 5\sqrt{13}$	$AB^2 + BC^2 = 14^2 + 48^2 = 196+2304 = 2500 = AC^2$ $\angle ABC = 90°$ (converse, Pyth. thm.) AC is the diameter of the circumcircle ABC (converse, $\angle$ in semi-circle) Q = mid-point of AC (centre of circumcircle) $AQ = 25 \cdots (1)$ Suppose the in-circle touches BC, AC and AB at D, E and F respectively. $PD\perp BC$, $PE\perp AC$, $PF\perp AB$ (tangent ⊥ radius) $PDBF$ is a rectangle (it has 3 ⊥ ∠s) Let r be the radius of the inscribed circle. $PD = PE = PF = r$ $PDBF$ is a square ($PD = PF$) $BF = BD = r$ $AF = 14 - r$, $CD = 48 - r$ $AE = 14 - r$, $CE = 48 - r$ (tangent from ext. pt.) $AE + EC = AC$ $14 - r + 48 - r = 50$ $r = 6$, $AE = 14 - 6 = 8$ $EQ = AQ - AE = 25 - 8 = 17$ In ΔPEQ, $PE^2 + EQ^2 = PQ^2$ (Pythagoras' thm.) $6^2 + 17^2 = PQ^2$ $d = \sqrt{325} = 5\sqrt{13}$

I13 已知正整數 a、b 及 c 滿足下列條件：

(i) $a > b > c$，

(ii) $(a - b)(b - c)(a - c) = 84$，

(iii) $abc < 100$。

設 M 為 a 的最大值。求 M 的值。

Given that a, b and c are positive integers satisfying the following conditions:

(i) $a > b > c$,

(ii) $(a - b)(b - c)(a - c) = 84$,

(iii) $abc < 100$.

Let M be the maximum value of a. Find the value of M.

84 的正因子包括 1、2、3、4、6、7、12、14、21、28、42 及 84。 $(a - b) + (b - c) = a - c$ $(a - b, b - c, a - c)$的可能值$= (3, 4, 7)$或$(4, 3, 7)$ $(a, b, c) = (a, a - 3, a - 7)$或$(a, a - 4, a - 7)$ 為了使得 a 為最大，b 和 c 必須盡量小 $\therefore (a, b, c) = (a, a - 4, a - 7)$ $9\times5\times2 = 90$，$10\times6\times3 = 180$ $M = 9$	Positive factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42 and 84. $(a - b) + (b - c) = a - c$ Possible $(a - b, b - c, a - c) = (3, 4, 7)$ or $(4, 3, 7)$ $(a, b, c) = (a, a - 3, a - 7)$ or $(a, a - 4, a - 7)$ For largest a, b and c must be as small as possible $\therefore (a, b, c) = (a, a - 4, a - 7)$ $9\times5\times2 = 90$, $10\times6\times3 = 180$ $M = 9$

I14 已知 $3\sin x + 2\sin y = 4$。設 N 為 $3\cos x + 2\cos y$ 的最大值。求 N 的值。

Given that $3\sin x + 2\sin y = 4$. Let N be the maximum value of $3\cos x + 2\cos y$.
Find the value of N.

The following method is provided by Ms. Wong Ka Man from St. Mark's College.

$(3\cos x + 2\cos y)^2$

$= 9\cos^2 x + 12\cos x\cos y + 4\cos^2 y$

$= 9(1 - \sin^2 x) + 12(\cos x\cos y + \sin x\sin y) + 4(1 - \sin^2 y) - 12\sin x\sin y$

$= 13 + 12\cos(x - y) - (3\sin x + 2\sin y)^2$

$= 13 + 12\cos(x - y) - 4^2 = 12\cos(x - y) - 3$

$\le 12 - 3 = 9$

$\therefore 3\cos x + 2\cos y \le 3$

$N = 3$

I15 已知 x、y 及 z 為正實數且滿足 $\begin{cases} x^2+xy+y^2=7 \\ y^2+yz+z^2=21 \\ x^2+xz+z^2=28 \end{cases}$。若 $a=x+y+z$，求 a 的值。

Given that x, y and z are positive real numbers satisfying $\begin{cases} x^2+xy+y^2=7 & \cdots(1) \\ y^2+yz+z^2=21 & \cdots(2) \\ x^2+xz+z^2=28 & \cdots(3) \end{cases}$.

If $a = x + y + z$, find the value of a.

$\begin{cases} (x-y)(x^2+xy+y^2)=7(x-y) \\ (y-z)(y^2+yz+z^2)=21(y-z) \\ (z-x)(x^2+xz+z^2)=28(z-x) \end{cases}$ $\begin{cases} x^3-y^3=7x-7y \\ y^3-z^3=21y-21z \\ z^3-x^2=28z-28x \end{cases}$ 將以上三題方程相加: $0 = -21x + 14y + 7z$ $z = 3x - 2y \cdots (4)$ $(1) + (2) - (3)$：$2y^2 + (x + z)y - xz = 0 \cdots (5)$ 代(4)入(5)：$2y^2 + (x + 3x - 2y)y - x(3x - 2y) = 0$ $2y^2 + (4x - 2y)y - (3x^2 - 2xy) = 0$ $3x^2 - 6xy = 0$ $x = 0$ (x 為正實數，捨去) 或 $x = 2y \cdots (6)$ 代(6)入(1)：$4y^2 + 2y^2 + y^2 = 7$ $y = 1$ 或 -1 (y 為正實數，捨去) $x = 2$ $z = 3x - 2y = 4$ $a = x + y + z = 7$	$\begin{cases} (x-y)(x^2+xy+y^2)=7(x-y) \\ (y-z)(y^2+yz+z^2)=21(y-z) \\ (z-x)(x^2+xz+z^2)=28(z-x) \end{cases}$ $\begin{cases} x^3-y^3=7x-7y \\ y^3-z^3=21y-21z \\ z^3-x^2=28z-28x \end{cases}$ Add up these equations: $0 = -21x + 14y + 7z$ $z = 3x - 2y \cdots (4)$ $(1) + (2) - (3)$: $2y^2 + (x + z)y - xz = 0 \cdots (5)$ Sub. (4) into (5): $2y^2 + (x + 3x - 2y)y - x(3x - 2y) = 0$ $2y^2 + (4x - 2y)y - (3x^2 - 2xy) = 0$ $3x^2 - 6xy = 0$ $x = 0$ (x is real positive, rejected) or $x = 2y \cdots (6)$ Sub. (6) into (1): $4y^2 + 2y^2 + y^2 = 7$ $y = 1$ or -1 (y is real positive, rejected) $x = 2$ $z = 3x - 2y = 4$ $a = x + y + z = 7$

Group Events

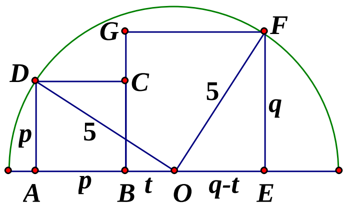
G1 對所有正實數 x，定義 $f(x)=\log_{2019} x^{2020}$。若 $D=f\left(\sqrt{3}\right)+f\left(\sqrt{673}\right)$，求 D 的值。

For all positive value real numbers x, define $f(x)=\log_{2019} x^{2020}$. If $D=f\left(\sqrt{3}\right)+f\left(\sqrt{673}\right)$, find the value of D.

$$D=\log_{2019}\left(\sqrt{3}\right)^{2020}+\log_{2019}\left(\sqrt{673}\right)^{2020}$$
$$=\log_{2019}\left(\sqrt{3}\times\sqrt{673}\right)^{2020}$$
$$=\log_{2019}(2019)^{1010}$$
$$=1010$$

G2 圖一所示，$ABCD$ 和 $BEFG$ 是兩個緊貼的正方形，躺臥在一個以 O 為圓心，半徑為 5 cm 的半圓上。其中 A、B 和 E 在半圓的直徑，D 和 F 在半圓的弧上。設 $ABCD$ 與 $BEFG$ 的面積之和為 S cm^2，求 S 的值。

Figure 1 shows two adjacent squares $ABCD$ and $BEFG$ lying on a semi-circle with centre O and radius 5 cm. A, B and E lie on the diameter of the semi-circle, D and F lie on the semi-circular arc. Let the sum of areas of $ABCD$ and $BEFG$ be S cm^2, find the value of S.

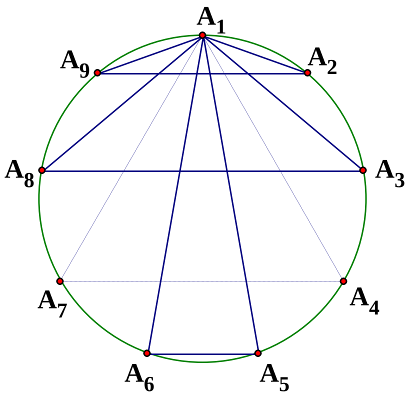


圖一 Figure 1

$OD=OE=5$ cm。設 $AD=p$、$EF=q$。 不妨假設 $q>p$。 設 $OB=t$，則 $OE=q-t$. $AD\perp AB$，$FE\perp BE$ $AD^2+AO^2=OE^2+EF^2=OF^2$ (畢氏定理) $p^2+(p+t)^2=(q-t)^2+q^2=5^2$ $p^2+p^2+2pt+t^2=q^2-2qt+t^2+q^2$ $2p^2+2pt=2q^2-2qt$ $p^2+pt=q^2-qt$ $(q+p)t=q^2-p^2$ $t=q-p$ $OA=p+t=p+q-p=q$ $AD^2+AO^2=OD^2$ (畢氏定理) $p^2+q^2=5^2$ $S=p^2+q^2=25$	$OD=OE=5$ cm. Let $AD=p$, $EF=q$. Without loss of generality, assume $q>p$. Let $OB=t$, then $OE=q-t$. $AD\perp AB$, $FE\perp BE$ $AD^2+AO^2=OE^2+EF^2=OF^2$ (Pythagoras' theorem) $p^2+(p+t)^2=(q-t)^2+q^2=5^2$ $p^2+p^2+2pt+t^2=q^2-2qt+t^2+q^2$ $2p^2+2pt=2q^2-2qt$ $p^2+pt=q^2-qt$ $(q+p)t=q^2-p^2$ $t=q-p$ $OA=p+t=p+q-p=q$ $AD^2+AO^2=OD^2$ (Pythagoras' theorem) $p^2+q^2=5^2$ $S=p^2+q^2=25$

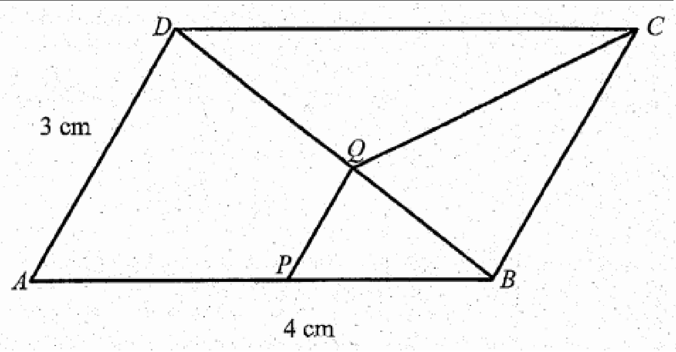
G3 若從一個正九邊形的 9 個頂點中選 3 點，共可組成多少個等腰三角形？

If three vertices are chosen from the nine vertices of a regular nonagon, how many possible isosceles triangles are there ?



將 9 個頂點依次序命名為 A_1、A_2、…、A_9。 其中有 4 個等腰三角形通過 $A_iA_1A_j$ 及 $A_1A_i = A_1A_j$。當中 $A_4A_1A_7$ 是一個等邊三角形。 若果不計算等邊三角形，所有等腰三角形的總數為 $3\times9 = 27$。 若果包括了所有等邊三角形，所有等腰三角形的總數為 $27 + 3 = 30$。	Label the 9 vertices as $A_1, A_2, \cdots, A_9$ in order. There are 4 isosceles triangles in the form $A_iA_1A_j$ such that $A_1A_i = A_1A_j$. Amongst these 4 isosceles triangles, $A_4A_1A_7$ is an equilateral triangle. If we do not count these equilateral triangles, the total number of isosceles triangles are $3\times9 = 27$ If we include these equilateral triangles, the total number of isosceles triangles $= 27 + 3 = 30$

G4 在圖二中，$ABCD$ 為一個平行四邊形，其中 $AB = 4$ cm、$AD = 3$ cm 及 $\sin A = \frac{2}{3}$。P 和 Q 分別是 AB 和 BD 上的點使得 $PQ // AD$，且四邊形 $PBCQ$ 的面積為 3 cm^2。設 AP 的長度為 q cm，求 q 的值。



圖二 Figure 2

In Figure 2, $ABCD$ is a parallelogram, where $AB = 4$ cm, $AD = 3$ cm and $\sin A = \frac{2}{3}$. P and Q are points on AB and BD respectively such that $PQ // AD$, and the area of the quadrilateral $PBCQ$ is 3 cm^2. Let the length of AP be q cm, find the value of q.

設 S 表示面積。 $S_{\Delta ABD} = S_{\Delta CDB} = \frac{1}{2}AB\cdot AD\sin A = \frac{1}{2}\cdot3\cdot4\cdot\frac{2}{3} = 4$ $\Delta BPQ \sim \Delta BAD$ (A.A.A.) 設 $BQ : QD = k : (1-k)$ $\frac{S_{\Delta BPQ}}{S_{\Delta BAD}} = k^2 \Rightarrow S_{\Delta BPQ} = 4k^2$ ΔBCQ 及 ΔBCD 有相同高度 $\frac{S_{\Delta BCQ}}{S_{\Delta BCD}} = k \Rightarrow S_{\Delta BCQ} = 4k$ $S_{PBCQ} = 3 \Rightarrow 4k^2 + 4k = 3$ $(2k+3)(2k-1) = 0$ $k = -1.5$ (捨去) 或 0.5 $BP : PA = BQ : QD = 0.5 : (1-0.5) = 1 : 1$ $\Rightarrow q = 2$	Let S denote the area. $S_{\Delta ABD} = S_{\Delta CDB} = \frac{1}{2}AB\cdot AD\sin A = \frac{1}{2}\cdot3\cdot4\cdot\frac{2}{3} = 4$ $\Delta BPQ \sim \Delta BAD$ (A.A.A.) Let $BQ : QD = k : (1-k)$ $\frac{S_{\Delta BPQ}}{S_{\Delta BAD}} = k^2 \Rightarrow S_{\Delta BPQ} = 4k^2$ ΔBCQ and ΔBCD have the same height $\frac{S_{\Delta BCQ}}{S_{\Delta BCD}} = k \Rightarrow S_{\Delta BCQ} = 4k$ $S_{PBCQ} = 3 \Rightarrow 4k^2 + 4k = 3$ $(2k+3)(2k-1) = 0$ $k = -1.5$ (rejected) or 0.5 $BP : PA = BQ : QD = 0.5 : (1-0.5) = 1 : 1$ $\Rightarrow q = 2$

G5 已知 $f(x)-2f\left(\frac{1}{x}\right)=x$，其中 $x\neq 0$。設 y 為滿足方程 $f(x)=1$ 的 x 的最大值。求 y 的值。

Given that $f(x)-2f\left(\frac{1}{x}\right)=x$, where $x\neq 0$. Let y be the maximum value of x that satisfies the equation $f(x)=1$. Find the value of y. **Reference: 2018 HG4**

$f(x)-2f\left(\frac{1}{x}\right)=x\ \cdots(1)$ $f\left(\frac{1}{x}\right)-2f(x)=\frac{1}{x}\ \cdots(2)$ (1) + 2(2): $-3f(x)=x+\frac{2}{x}$ $\Rightarrow f(x)=-\frac{1}{3}\left(x+\frac{2}{x}\right)$ $f(x)=1\Rightarrow -\frac{1}{3}\left(x+\frac{2}{x}\right)=1$ $x^2+2=-3x$ $x^2+3x+2=0$ $x=-1$ 或 -2 $y=-1$	$f(x)-2f\left(\frac{1}{x}\right)=x\ \cdots(1)$ $f\left(\frac{1}{x}\right)-2f(x)=\frac{1}{x}\ \cdots(2)$ (1) + 2(2): $-3f(x)=x+\frac{2}{x}$ $\Rightarrow f(x)=-\frac{1}{3}\left(x+\frac{2}{x}\right)$ $f(x)=1\Rightarrow -\frac{1}{3}\left(x+\frac{2}{x}\right)=1$ $x^2+2=-3x$ $x^2+3x+2=0$ $x=-1$ or -2 $y=-1$

G6 設 a_k 為多項式 $(2x-2)^3(2x+2)^3(2x+1)^3$ 中 x^k 的係數。
若 $Q=a_2+a_4+a_6+a_8$，求 Q 的值。

Let a_k be the coefficient of x^k in the polynomial $(2x-2)^3(2x+2)^3(2x+1)^3$.
If $Q=a_2+a_4+a_6+a_8$, find the value of Q.

$(2x-2)^3(2x+2)^3(2x+1)^3=64(x^2-1)^3(2x+1)^3=64(x^6-3x^4+3x^2-1)(8x^3+12x^2+6x+1)$

$a_2=64(3-12)=64\times(-9)$

$a_4=64(-3+3\times 12)=64\times 33$

$a_6=64(1-3\times 12)=64\times(-35)$

$a_8=64\times 12$

$Q=a_2+a_4+a_6+a_8=64\times(-9)+64\times 33+64\times(-35)+64\times 12=64\times(-9+33-35+12)=64$

G7 設 $f(x)=-6x^2+4x\cos\theta+\sin\theta$，其中 $0°\leq\theta\leq 360°$。已知對所有實數 x，$f(x)\leq 0$。若 θ 的最大值與最小值之差為 $d°$，求 d 的值。

Let $f(x)=-6x^2+4x\cos\theta+\sin\theta$, where $0°\leq\theta\leq 360°$. If is given that $f(x)\leq 0$ for all real numbers x. If $d°$ is the difference between the greatest and the least values of θ, find the value of d.

設 $a=-6$，$b=4\cos\theta$，$c=\sin\theta$ $f(x)$ 的最大值 $=\frac{4ac-b^2}{4a}\leq 0$ $\frac{4(-6)\sin\theta-(4\cos\theta)^2}{4(-6)}\leq 0$ $24\sin\theta+16\cos^2\theta\leq 0$ $3\sin\theta+2(1-\sin^2\theta)\leq 0$ $2\sin^2\theta-3\sin\theta-2\geq 0$ $(2\sin\theta+1)(\sin\theta-2)\geq 0$ $\sin\theta\leq -0.5$ 或 $\sin\theta\geq 2$ (捨去) $210°\leq\theta\leq 330°\Rightarrow d=330-210=120$	Let $a=-6$, $b=4\cos\theta$, $c=\sin\theta$ Maximum value of $f(x)=\frac{4ac-b^2}{4a}\leq 0$ $\frac{4(-6)\sin\theta-(4\cos\theta)^2}{4(-6)}\leq 0$ $24\sin\theta+16\cos^2\theta\leq 0$ $3\sin\theta+2(1-\sin^2\theta)\leq 0$ $2\sin^2\theta-3\sin\theta-2\geq 0$ $(2\sin\theta+1)(\sin\theta-2)\geq 0$ $\sin\theta\leq -0.5$ or $\sin\theta\geq 2$ (rejected) $210°\leq\theta\leq 330°\Rightarrow d=330-210=120$

G8 設 $\{a_n\}$ 為一個正實數序列使當 $n>1$ 時，$a_n=a_{n-1}a_{n+1}-1$。
已知 2018 在序列中及 $a_2=2019$。若 a_1 的所有可取的數目為 s，求 s 的值。
Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n=a_{n-1}a_{n+1}-1$ for $n>1$.
It is given that 2018 is in the sequence and $a_2=2019$. If the number of all possible values of a_1 is s, find the value of s.

$a_{n+1}=\frac{1+a_n}{a_{n-1}}=\frac{1+\frac{1+a_{n-1}}{a_{n-2}}}{a_{n-1}}=\frac{a_{n-2}+a_{n-1}+1}{a_{n-1}a_{n-2}}$ $=\frac{a_{n-2}+\frac{1+a_{n-2}}{a_{n-3}}+1}{\frac{1+a_{n-2}}{a_{n-3}}\cdot a_{n-2}}=\frac{a_{n-2}a_{n-3}+1+a_{n-2}+a_{n-3}}{(1+a_{n-2})\cdot a_{n-2}}$ $=\frac{(1+a_{n-2})(1+a_{n-3})}{(1+a_{n-2})\cdot a_{n-2}}=\frac{1+a_{n-3}}{a_{n-2}}=\frac{1+a_{n-3}}{\frac{1+a_{n-3}}{a_{n-4}}}$ $=a_{n-4}$ 對於 $n\ge 5$ $\therefore\ a_1=a_6=a_{11}=\cdots=a_{5n+1}$ $a_2=a_7=a_{12}=\cdots=a_{5n+2}=2019$ $\because\ a_k=2018$ 對某些正整數 $k\ne 5n+2$。 $\therefore\ a_n$ 的數值由 a_2 及 a_k 決定。 a_1 有 4 種不同的值，$s=4$	$a_{n+1}=\frac{1+a_n}{a_{n-1}}=\frac{1+\frac{1+a_{n-1}}{a_{n-2}}}{a_{n-1}}=\frac{a_{n-2}+a_{n-1}+1}{a_{n-1}a_{n-2}}$ $=\frac{a_{n-2}+\frac{1+a_{n-2}}{a_{n-3}}+1}{\frac{1+a_{n-2}}{a_{n-3}}\cdot a_{n-2}}=\frac{a_{n-2}a_{n-3}+1+a_{n-2}+a_{n-3}}{(1+a_{n-2})\cdot a_{n-2}}$ $=\frac{(1+a_{n-2})(1+a_{n-3})}{(1+a_{n-2})\cdot a_{n-2}}=\frac{1+a_{n-3}}{a_{n-2}}=\frac{1+a_{n-3}}{\frac{1+a_{n-3}}{a_{n-4}}}$ $=a_{n-4}$ for $n\ge 5$ $\therefore\ a_1=a_6=a_{11}=\cdots=a_{5n+1}$ $a_2=a_7=a_{12}=\cdots=a_{5n+2}=2019$ $\because\ a_k=2018$ for some positive integer $k\ne 5n+2$. $\therefore\ a_n$ is uniquely determined by a_2 and a_k. a_1 can have 4 different values, $s=4$

G9 有多少對正整數 x、y 可滿足 $xy=6\left(x+y+\sqrt{x^2+y^2}\right)$？

How many pairs of positive integers x, y are there satisfying $xy=6\left(x+y+\sqrt{x^2+y^2}\right)$?

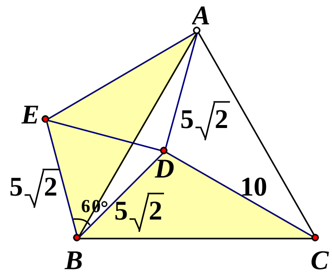
$(xy-6x-6y)^2=36(x^2+y^2)$ $x^2y^2-12x^2y-12xy^2+72xy=0$ $xy-12x-12y+72=0$ $xy-12x-12y+144=72$ $(x-12)(y-12)=72$ $(x-12, y-12)=(1, 72), (2, 36), (3, 24), (4, 18), (6, 12), (8, 9), (9, 8), (12, 6), (18, 4), (24, 3), (36, 2), (72, 1)$。 一共有 12 對正整數。 註：當 $(x-12, y-12)=(-8, -9)$ 或 $(-9, -8)$ $(x, y)=(4, 3)$ 或 $(3, 4)$。這兩組答案未能滿足原方程 $\therefore$ 捨去。	$(xy-6x-6y)^2=36(x^2+y^2)$ $x^2y^2-12x^2y-12xy^2+72xy=0$ $xy-12x-12y+72=0$ $xy-12x-12y+144=72$ $(x-12)(y-12)=72$ $(x-12, y-12)=(1, 72), (2, 36), (3, 24), (4, 18), (6, 12), (8, 9), (9, 8), (12, 6), (18, 4), (24, 3), (36, 2), (72, 1)$. There are 12 pairs of positive integers. Remark: When $(x-12, y-12)=(-8,-9)$ or $(-9,-8)$ $(x, y)=(4, 3)$ or $(3, 4)$. These two solutions do not satisfy the original equation $\therefore$ rejected.

G10 D 是等邊三角形 ABC 內的一點使得 $AD = BD = 5\sqrt{2}$ 及 $CD = 10$。設 ΔABC 的面積為 S，求 S 的值。

D is a point inside the equilateral triangle ABC such that $AD = BD = 5\sqrt{2}$ and $CD = 10$.

Let the area of ΔABC be S, find the value of S.

Reference: 2014 HI3



如圖所示，將 BD 繞 B 反時針方向轉 $60°$，得 BE。

由作圖所得，$BD = BE = 5\sqrt{2}$ 及 $\angle DBE = 60°$

$\angle BDE = \angle BED$ (等腰三角形底角)

$= \dfrac{180^\circ - 60^\circ}{2} = 60°$ (三角形內角和)

ΔBDE 是一個等邊三角形。

$DE = BD = 5\sqrt{2}$ (等邊三角形性質)

$AB = AC$ (等邊三角形性質)

$\angle ABC = 60°$ (等邊三角形性質)

$\angle ABE = \angle DBE - \angle ABD = 60° - \angle ABD$

$= \angle CBD$

$\Delta ABE \cong \Delta CBD$ (S.A.S.)

$AE = CD = 10$ (全等三角形對應邊)

$DE^2 + DA^2 = \left(5\sqrt{2}\right)^2 + \left(5\sqrt{2}\right)^2 = 100 = AE^2$

$\angle ADE = 90°$ (畢氏定理逆定理)

$\angle ADB = \angle ADE + \angle BDE = 90° + 60° = 150°$

設 $AB = x$。於ΔABD 中應用餘弦公式：

$x^2 = AD^2 + BD^2 - 2AD\cdot BD \cos \angle ADB$

$x^2 = \left(5\sqrt{2}\right)^2 + \left(5\sqrt{2}\right)^2 - 2\left(5\sqrt{2}\right)^2 \cos 150^\circ$

$x^2 = 100 - 100\left(-\dfrac{\sqrt{3}}{2}\right) = 100 + 50\sqrt{3}$

$S = \Delta ABC$ 的面積 $= \dfrac{1}{2}\cdot AB\cdot BC \sin 60^\circ$

$= \dfrac{1}{2}\cdot\left(100 + 50\sqrt{3}\right)\cdot\dfrac{\sqrt{3}}{2}$

$= \dfrac{25}{2}\cdot\left(2\sqrt{3} + 3\right) = 25\sqrt{3} + 37.5$

As shown in the figure, rotate BD about B anti-clockwise through $60°$ to BE.

By construction, $BD = BE = 5\sqrt{2}$ and $\angle DBE = 60°$

$\angle BDE = \angle BED$ (base $\angle$s isos. Δ)

$= \dfrac{180^\circ - 60^\circ}{2} = 60°$ ($\angle$ sum of Δ)

ΔBDE is an equilateral triangle.

$DE = BD = 5\sqrt{2}$ (prop. of equil. Δ)

$AB = AC$ (prop. of equil. Δ)

$\angle ABC = 60°$ (prop. of equil. Δ)

$\angle ABE = \angle DBE - \angle ABD = 60° - \angle ABD$

$= \angle CBD$

$\Delta ABE \cong \Delta CBD$ (S.A.S.)

$AE = CD = 10$ (corr. sides, $\cong \Delta$s)

$DE^2 + DA^2 = \left(5\sqrt{2}\right)^2 + \left(5\sqrt{2}\right)^2 = 100 = AE^2$

$\angle ADE = 90°$ (converse, Pyth. thm.)

$\angle ADB = \angle ADE + \angle BDE = 90° + 60° = 150°$

Let $AB = x$. Apply cosine formula on ΔABD:

$x^2 = AD^2 + BD^2 - 2AD\cdot BD \cos \angle ADB$

$x^2 = \left(5\sqrt{2}\right)^2 + \left(5\sqrt{2}\right)^2 - 2\left(5\sqrt{2}\right)^2 \cos 150^\circ$

$x^2 = 100 - 100\left(-\dfrac{\sqrt{3}}{2}\right) = 100 + 50\sqrt{3}$

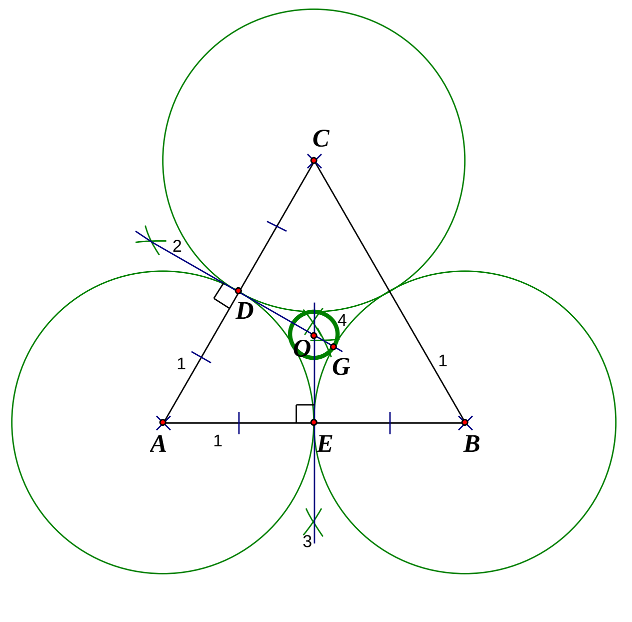
S = area of $\Delta ABC = \dfrac{1}{2}\cdot AB\cdot BC \sin 60^\circ$

$= \dfrac{1}{2}\cdot\left(100 + 50\sqrt{3}\right)\cdot\dfrac{\sqrt{3}}{2}$

$= \dfrac{25}{2}\cdot\left(2\sqrt{3} + 3\right) = 25\sqrt{3} + 37.5$

Geometrical Construction

1. 圖一所示為三個半徑相等且兩兩相切的圓。試作一圓使得它與圖中每一圓相切於一點。
Figure 1 shows three circles with equal radius which are pairwise tangents to each other. Construct a circle which will touch each circle in the figure at a point.

<table>
<tr>
<td>
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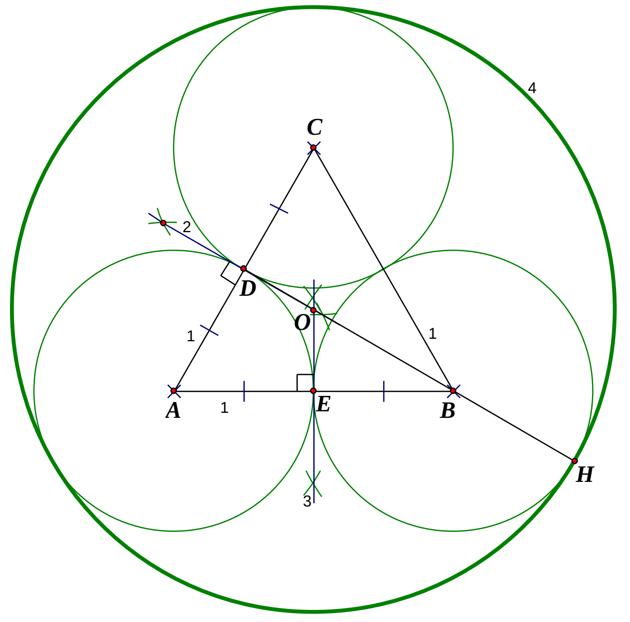
<td>
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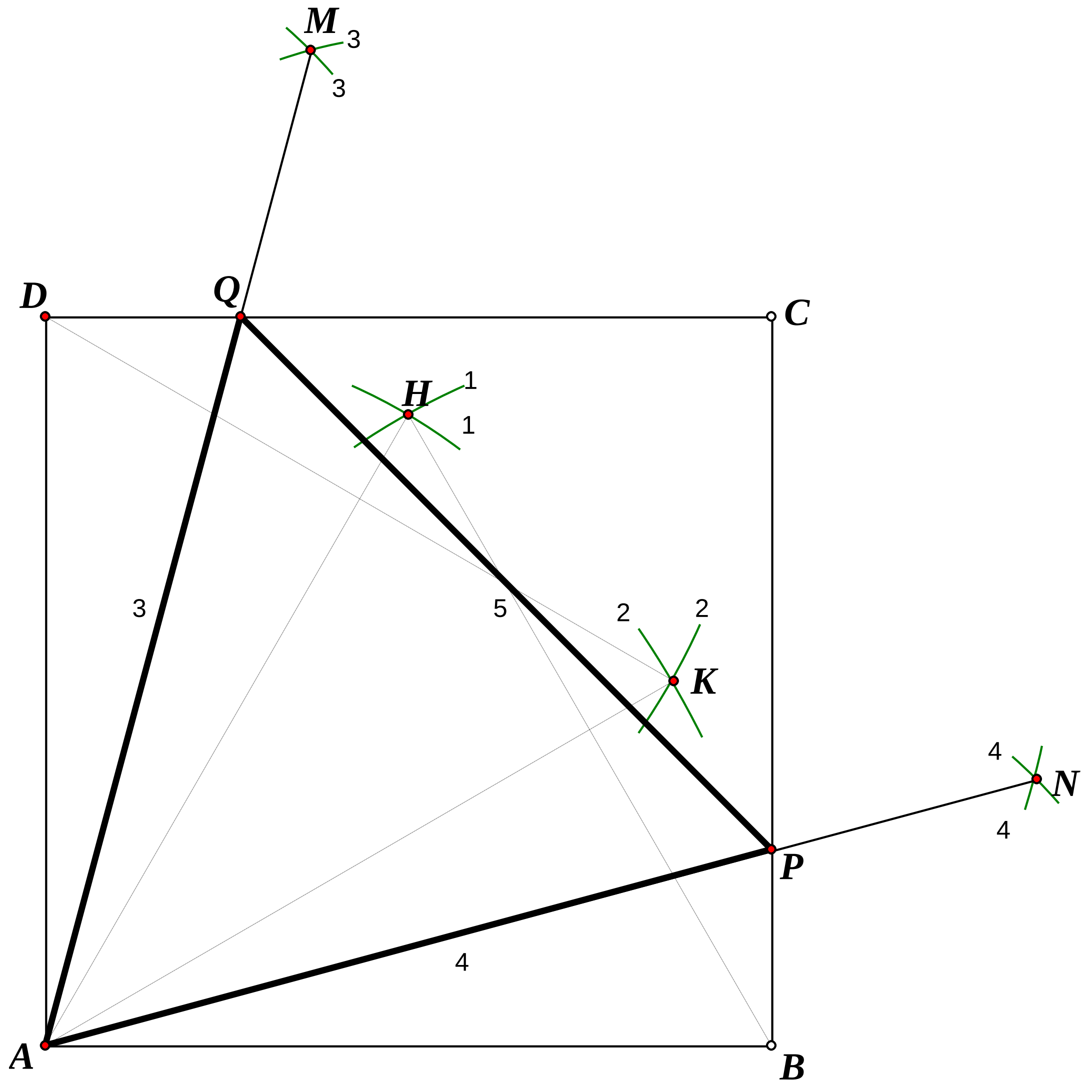
<tr>
<td>作圖步驟：
(1) 連接 AB、AC 及 BC。
(2) 作 AC 的垂直平分綫。D 為 AC 的中點。此中垂綫交以 B 為圓心的圓形於 G。($OG < AD$)
(3) 作 AB 的垂直平分綫。E 為 AB 的中點。兩中垂綫相交於 O。
(4) 作圓 $\odot(O, OG)$。
此圓滿足所求。</td>
<td>Steps:
(1) Join AB, AC and BC.
(2) Draw the perpendicular bisectors of AC. D is the mid-point of AC. It intersects the circle with centre B at G. ($OG < AD$)
(3) Draw the perpendicular bisectors of AB. E is the mid-point of AB. The 2 $\perp$ bisectors intersect at O.
(4) Draw a circle $\odot(O, OG)$.
This is the required circle.</td>
</tr>
<tr>
<td>方法二：
於步驟(2)中，中垂綫交以 B 為圓心的圓形於 H。
($OH > AD$)
(4) 作圓 $\odot(O, OH)$。
此圓亦滿足所求。</td>
<td>Method 2
In step (2), the perpendicular bisector intersects the circle with centre B at H. ($OH > AD$)
(4) Draw a circle $\odot(O, OH)$.
This is another solution.</td>
</tr>
</table>

2. 圖二所示為一個邊長為 1 單位的正方形 $ABCD$。試作一個三角形 APQ，其中 P、Q 分別位於綫段 BC、CD 上且 $\angle PAB = \angle QAD = 15^\circ$。寫出 APQ 是哪一類三角形。

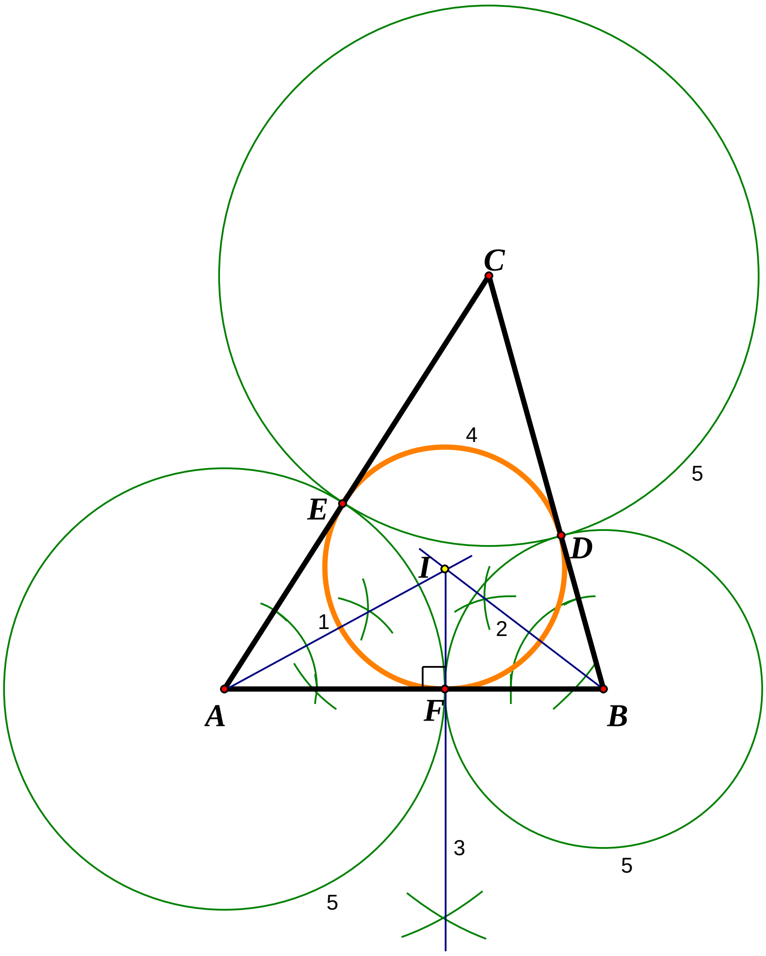
Figure 2 shows a square $ABCD$ with side 1 unit. Construct a triangle APQ, in which P, Q lie on the line segments BC and CD respectively, and $\angle PAB = \angle QAD = 15^\circ$. Write down the type of triangle that APQ is.



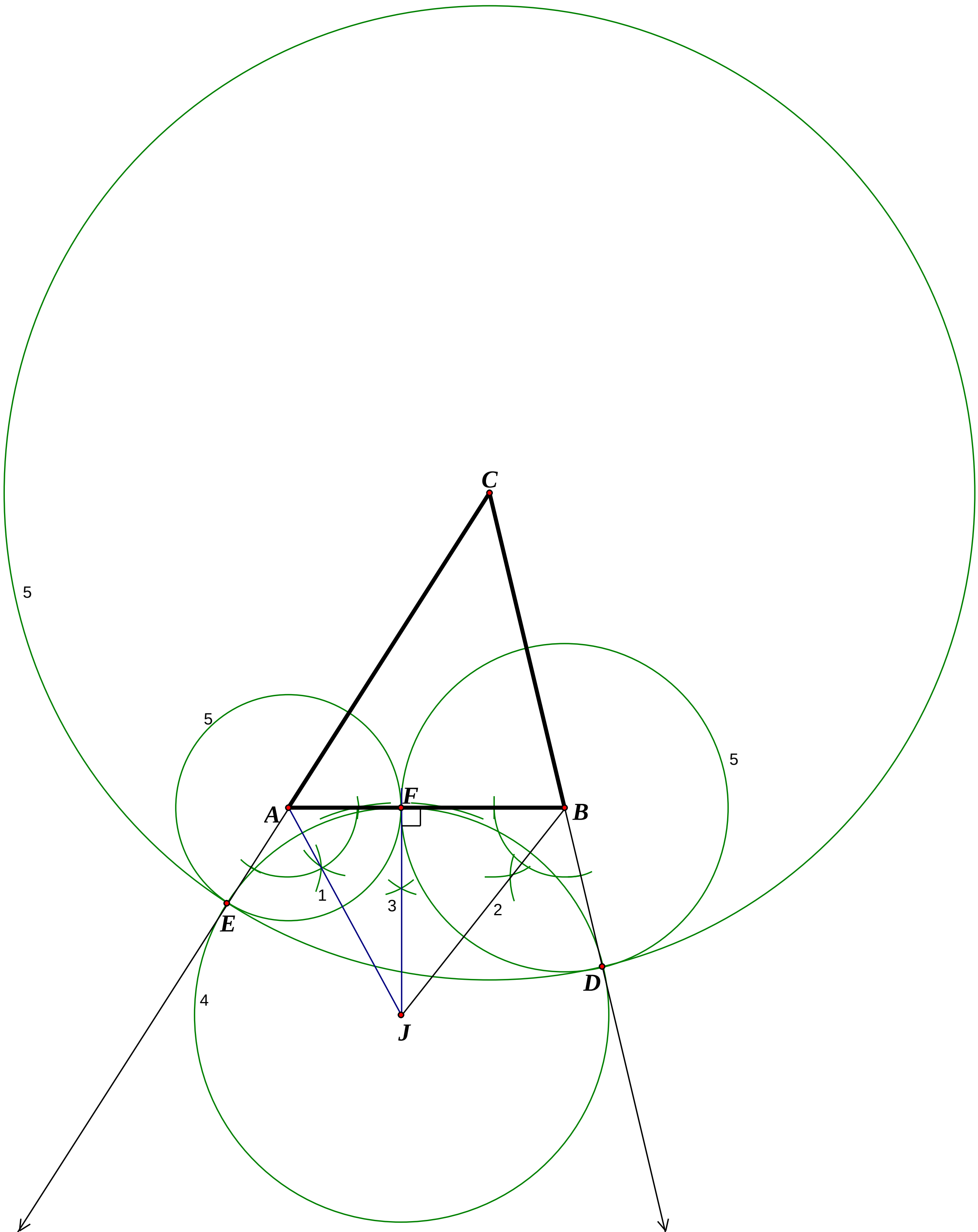
作圖步驟：	Steps:
(1) 作等邊三角形 AHB。 $\angle BAH = 60^\circ$，$\angle DAH = 30^\circ$。 (2) 作等邊三角形 AKD。 $\angle DAK = 60^\circ$，$\angle BAK = 30^\circ$。 (3) 作 $\angle DAH$ 的角平分綫 AM，交 CD 於 Q。 $\angle DAQ = 15^\circ$。 (4) 作 $\angle BAK$ 的角平分綫 AN，交 CB 於 P。 $\angle BAP = 15^\circ$。 (5) 連接 PQ。 ΔAPQ 是一個等邊三角形。	(1) Construct an equilateral triangle AHB. $\angle BAH = 60^\circ$, $\angle DAH = 30^\circ$. (2) Construct an equilateral triangle AKD. $\angle DAK = 60^\circ$, $\angle BAK = 30^\circ$. (3) Construct the angle bisector AM of $\angle DAH$, cutting CD at Q. $\angle DAQ = 15^\circ$. (4) Construct the angle bisector AN of $\angle BAK$, cutting CB at P. $\angle BAP = 15^\circ$. (5) Join PQ. ΔAPQ is an equilateral triangle.

3. 圖三所示為一個三角形 ABC。試以 A、B 及 C 為圓心分別構作三個圓，使得它們兩兩相切。

Figure 3 shows a triangle ABC. Use A, B and C as centres to construct three circles respectively that are pairwise tangent to each other. **Reference: 2009 HSC1, 2012HC2, 2014 HC1**



作圖步驟：	Steps:
(1) 作$\angle A$ 的角平分綫。	(1) Construct the angle bisector of $\angle A$.
(2) 作$\angle B$ 的角平分綫。 兩條角平分綫相交於內切圓心 I。	(2) Construct the angle bisector of $\angle B$. The two $\angle$ bisectors intersect at the incentre I.
(3) 作綫段 $IF \perp AB$。	(3) Construct a line $IF \perp AB$.
(4) 作內切圓$\odot(I, IF)$，分別切 BC 和 AC 於 D 和 E。 由切綫性質，$AE = AF$、$BD = BF$、$CD = CE$。	(4) Construct the incircle $\odot(I, IF)$, touching BC and AC at D and E respectively. By tangent property, $AE = AF$, $BD = BF$, $CD = CE$.
(5) 作三圓$\odot(A, AE)$、$\odot(B, BD)$、$\odot(C, CE)$。	(5) Draw 3 circles $\odot(A, AE)$, $\odot(B, BD)$, $\odot(C, CE)$.



方法二作圖步驟：	**Method 2** Steps:
(1) 作$\angle A$的外角平分綫。	(1) Construct the exterior angle bisector of $\angle A$.
(2) 作$\angle B$的外角平分綫。 兩條角平分綫相交於旁切圓心 J。	(2) Construct the exterior angle bisector of $\angle B$. The two $\angle$ bisectors intersect at the excentre J.
(3) 作綫段 $JF \perp AB$。	(3) Construct a line $JF \perp AB$.
(4) 作旁切圓$\odot(J, JF)$，分別切 CB 和 CA 的延綫於 D 和 E。	(4) Construct the excircle $\odot(J, JF)$, touching CB produced and CA produced at D and E respectively.
由切綫性質，$AE = AF$、$BD = BF$、$CD = CE$。	By tangent property, $AE = AF$, $BD = BF$, $CD = CE$.
(5) 作三圓$\odot(A, AE)$、$\odot(B, BD)$、$\odot(C, CE)$。	(5) Draw 3 circles $\odot(A, AE)$, $\odot(B, BD)$, $\odot(C, CE)$.