# Exercises on Binomial Theorem

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1. Let $n$ be a positive integer. Consider $(1 + x)^n = C_0^n + C_1^n \cdot x + \cdots + C_n^n \cdot x^n$.
   Find the values of
   (a) $C_0^n + C_1^n + \cdots + C_n^n$,
   (b) $C_0^n + C_2^n + C_4^n + \cdots$ and $C_1^n + C_3^n + C_5^n + \cdots$,
   (c) $C_1^n + 2 \cdot C_2^n + 3 \cdot C_3^n + \cdots + n \cdot C_n^n$ for $n \geq 1$,
   (d) $C_0^n + 2C_1^n + 3C_2^n + \cdots + (n+1)C_n^n$,
   (e) $C_0^n + \frac{C_1^n}{2} + \frac{C_2^n}{3} + \cdots + \frac{C_n^n}{n+1}$,
   (f) $\frac{C_0^n}{2} + \frac{C_1^n}{3} + \frac{C_2^n}{4} + \cdots + \frac{C_n^n}{n+2}$,
   (Note: It is impossible to find $C_1^n + \frac{C_1^n}{2} + \frac{C_3^n}{3} + \cdots + \frac{C_n^n}{n}$.)
   (g) $1 \times 2 \cdot C_2^n + 2 \times 3 \cdot C_3^n + 3 \times 4 \cdot C_4^n + \cdots + (n-1)n \cdot C_n^n$ for $n \geq 2$;
   (h) $1^2 \cdot C_1^n + 2^2 \cdot C_2^n + 3^2 \cdot C_3^n + \cdots + n^2 \cdot C_n^n$;
   (i) $C_0^n + C_3^n + C_6^n + \cdots$.
2. Let $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, where $n$ is a positive integer.
   Prove that $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \cdots + \frac{nC_n}{C_{n-1}} = \frac{n(n+1)}{2}$.
3. (a) Show that for $n$, $k$ are positive integers,
   $$x\left[1 + (x+1) + (x+1)^2 + \cdots + (x+1)^{n+k-1}\right] \equiv (x+1)^{n+k} - 1.$$
   (b) Hence, or otherwise, show that
   $$C_0^{n-1} + C_1^n + C_2^{n+1} + \cdots + C_k^{n+k-1} = C_k^{n+k}, \text{ where } C_r^n = \frac{n!}{r!(n-r)!}$$
4. Let $n$ be a positive integer. By applying binomial theorem to $11^n - 9^n = (10 + 1)^n - (10 - 1)^n$, prove that $9^n + 10^n < 11^n$ if $n \geq 5$.
5. (a) Find the first three terms in the expansion of $(1 + ax+bx^2)^4$ in the ascending powers of $x$.
   (b) Given $y = 1 + a\lambda + b\lambda^2$.
   (i) Express $y^4 - \lambda y - 1$ in ascending powers of $\lambda$ up to $\lambda^2$.
   (ii) Let $E$: $y^4 - \lambda y - 1 = 0$ and $\lambda$ be so small that the terms involving $\lambda^3$ and higher powers of $\lambda$ can be neglected. Using the result in (i) and equating coefficients of $\lambda$ and $\lambda^2$ to zero, find an approximate root of $E$.
6. Express $\frac{11x-2}{(x-2)^2(x^2+1)}$ in terms of partial fractions.
   Find the coefficients of $x^{2n}$ and $x^{2n+1}$ in the expression.
7. Prove that $C_2^n + C_5^n + C_8^n + \cdots = \frac{1}{3}\left[2^n + 2\cos\frac{(n-4)\pi}{3}\right]$.
8. Prove that $C_0^n \cdot C_p^m + C_1^n \cdot C_{p-1}^m + \cdots + C_p^n \cdot C_0^m = C_p^{m+n}$.

9. Prove that $C_0^n \cdot C_r^n + C_1^n \cdot C_{r+1}^n + \cdots + C_{n-r}^n \cdot C_n^n = \dfrac{(2n)!}{(n-r)!(n+r)!}$.

10. If $(1 + x + x^2)^{3n} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ , prove that $a_0 - a_1x + a_2 - a_3 + a_4 - \cdots = 1$ .

11. By equating the coefficients of $x^r$ on both sides of the identity $(1 + x)^n = (1 + x)^2(1 + x)^{n-2}$ for $n \geq 2$, prove that $C_r^n = C_r^{n-2} + 2C_{r-1}^{n-2} + C_{r-2}^{n-2}$ .

12. (a) Find the coefficients of $x$ and $x^2$ in $(1 - x)^n (1 + x)^{2n}$ where $n$ is a positive integer.
    (b) If the coefficients of $x$ and $x^2$ in $(1 - x)^n (1 + x)^{2n}$ is equal, find $n$.

13. If $n$ and $r$ are positive integers such that $n > r > 1$, show that
    (a) $\dfrac{n+1-r}{n-r} > \dfrac{n+1}{n}$,
    (b) $\left(1+\dfrac{1}{n+1}\right)^{n+1} > \left(1+\dfrac{1}{n}\right)^n$.

14. Given the coefficient of $x$ in the expansion of the expression
    $(1 + x)^m + (1 + x)^n$ is 19, where $m$, $n$ are integers.
    (a) Find the minimum value of the coefficient of $x^2$ in the expansion.
    (b) Find the coefficient of $x^7$ in the expansion when the coefficient of $x^2$ attains its minimum value.

15. Write down the first three terms in the binomial expansion of $\left(2-\dfrac{1}{2x^2}\right)^{10}$ in ascending powers of $\dfrac{1}{x^2}$. Hence, find the value of $(1.995)^{10}$ correct to five significant figures without using calculators.

**End of Exercise**

1. (a) Put $x = 1$, $C_0^n + C_1^n + \cdots + C_n^n = (1 + 1)^n = 2^n$

(b) Put $x = -1$, $C_0^n - C_1^n + C_2^n - C_3^n + \cdots + (-1)^n C_n^n = 0$

$\Rightarrow C_0^n + C_2^n + C_4^n + \cdots = C_1^n + C_3^n + C_5^n + \cdots$

By (a), $2^n = C_0^n + C_1^n + \cdots + C_n^n$

$= C_0^n + C_2^n + C_4^n + \cdots + (C_1^n + C_3^n + C_5^n + \cdots)$

$= 2(C_0^n + C_2^n + C_4^n + \cdots\ )$

$\Rightarrow\ C_0^n + C_2^n + C_4^n + \cdots\ =\ C_1^n + C_3^n + C_5^n + \cdots\ = 2^{n-1}$

(c) Differentiating $(1 + x)^n$ with respect to $x$.

$n(1 + x)^{n-1} = C_1^n + 2\cdot C_2^n \cdot x + 3\cdot C_3^n \cdot x^2 + \cdots + n\cdot C_n^n \cdot x^{n-1}$

Put $x = 1$, $C_1^n + 2\cdot C_2^n + 3\cdot C_3^n + \cdots + n\cdot C_n^n = n\cdot 2^{n-1}$

(d) Multiply by $x$, $x(1 + x)^n = C_0^n \cdot x + C_1^n \cdot x^2 + C_2^n \cdot x^3 + \cdots + C_n^n \cdot x^{n+1}$

Differentiate with respect to $x$:

$(1 + x)^n + nx(1 + x)^{n-1} = C_0^n + 2C_1^n \cdot x + 3C_2^n \cdot x^2 + \cdots + (n+1)C_n^n \cdot x^{n-1}$

Put $x = 1$, $C_0^n + 2C_1^n + 3C_2^n + \cdots + (n+1)C_n^n = 2^n + n\cdot 2^{n-1}$

$C_0^n + 2C_1^n + 3C_2^n + \cdots + (n+1)C_n^n = (n + 2)\cdot 2^{n-1}$

(e) Integrating the expression from 0 to 1 .

$$\int_0^1 (1+x)^n \,\mathrm{d}x = \int_0^1 \left(C_0^n + C_1^n \cdot x + C_2^n \cdot x^2 + \cdots + C_n^n \cdot x^n\right)\mathrm{d}x$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right|_0^1 = \left.\left(C_0^n \cdot x + \frac{C_1^n \cdot x^2}{2} + \frac{C_2^n \cdot x^3}{3} + \cdots + \frac{C_n^n \cdot x^{n+1}}{n+1}\right)\right|_0^1$$

$$C_0^n + \frac{C_1^n}{2} + \frac{C_2^n}{3} + \cdots + \frac{C_n^n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

(f) Multiply by $x$, $x(1 + x)^n = C_0^n \cdot x + C_1^n \cdot x^2 + C_2^n \cdot x^3 + \cdots + C_n^n \cdot x^{n+1}$.

$(1 + x)^{n+1} - (1 + x)^n = C_0^n \cdot x + C_1^n \cdot x^2 + C_2^n \cdot x^3 + \cdots + C_n^n \cdot x^{n+1}$

Integrating the expression from 0 to 1.

$$\int_0^1 (1+x)^{n+1} - (1+x)^n \,dx = \int_0^1 \left(C_0^n \cdot x + C_1^n \cdot x^2 + C_2^n \cdot x^3 + \cdots + C_n^n \cdot x^{n+1}\right)dx$$

$$\left[\frac{(1+x)^{n+2}}{n+2} - \frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left.\left(\frac{C_0^n \cdot x^2}{2} + \frac{C_1^n \cdot x^3}{3} + \frac{C_2^n \cdot x^4}{4} + \cdots + \frac{C_n^n \cdot x^{n+2}}{n+2}\right)\right|_0^1$$

$$\frac{C_0^n}{2} + \frac{C_1^n}{3} + \frac{C_2^n}{4} + \cdots + \frac{C_n^n}{n+2} = \left(\frac{2^{n+2}-1}{n+2}\right) - \left(\frac{2^{n+1}-1}{n+1}\right) = \frac{n\cdot 2^{n+1}+1}{(n+1)(n+2)}$$

(g) Differentiate the expression twice:

$n(n - 1)(1 + x)^{n-2} = 1\times 2C_2^n + 2\times 3C_3^n \cdot x + \cdots + (n-1)nC_n^n \cdot x^{n-2}$

Put $x = 1$, $1\times 2\cdot C_2^n + 2\times 3\cdot C_3^n + 3\times 4\cdot C_4^n + \cdots + (n-1)n\cdot C_n^n = n(n - 1)2^{n-2}$

(h) From (g), $1\times 2\cdot C_2^n + 2\times 3\cdot C_3^n + 3\times 4\cdot C_4^n + \cdots + (n-1)n\cdot C_n^n = n(n - 1)2^{n-2}$

$(2-1)\times 2\cdot C_2^n + (3-1)\times 3\cdot C_3^n + (4-1)\times 4\cdot C_4^n + \cdots + (n-1)n\cdot C_n^n = n(n - 1)\cdot 2^{n-2}$

$1^2\cdot C_1^n + 2^2\cdot C_2^n + 3^2\cdot C_3^n + \cdots + n^2\cdot C_n^n - \left(C_1^n + 2\cdot C_2^n + 3\cdot C_3^n + \cdots + n\cdot C_n^n\right) = n(n-1)\cdot 2^{n-2}$

$1^2\cdot C_1^n + 2^2\cdot C_2^n + 3^2\cdot C_3^n + \cdots + n^2\cdot C_n^n = n(n-1)\cdot 2^{n-2} + n\cdot 2^{n-1} = n(n + 1)\cdot 2^{n-2}$

(i) If $\omega = \dfrac{-1+\sqrt{3}i}{2} = \text{cis}\dfrac{2\pi}{3}$, then $\omega \neq 1$, $1 + \omega + \omega^2 = 0$ …… (1) and $\omega^3 = 1$ …… (2)

$1 + \omega = \omega^{\frac{1}{2}}\left(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}}\right)$ …… (3), $1 + \omega^2 = 1 + \omega^{-1} = \omega^{-\frac{1}{2}}\left(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}}\right)$ …… (4)

Put $x = 1$: $(1 + 1)^n = C_0^n + C_1^n + \cdots + C_n^n$

Put $x = \omega$: $(1 + \omega)^n = C_0^n + C_1^n \cdot \omega + \cdots + C_n^n \cdot \omega^n$

Put $x = \omega^2$: $(1 + \omega^2)^n = C_0^n + C_1^n \cdot \omega^2 + \cdots + C_n^n \cdot \omega^{2n}$

Add up these three equations:

$3\left(C_0^n + C_3^n + C_6^n + \cdots\right) = 2^n + (1 + \omega)^n + (1 + \omega^2)^n$

$$C_0^n + C_3^n + C_6^n + \cdots = \frac{1}{3}\left[2^n + \omega^{\frac{n}{2}}\left(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}}\right)^n + \omega^{-\frac{n}{2}}\left(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}}\right)^n\right] \text{ by (3) and (4)}$$

$$= \frac{1}{3}\left[2^n + \left(\omega^{\frac{n}{2}} + \omega^{-\frac{n}{2}}\right)\left(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}}\right)^n\right]$$

$$= \frac{1}{3}\left[2^n + 2\cos\left(\frac{n}{2}\cdot\frac{2\pi}{3}\right)\left(2\cos\frac{1}{2}\cdot\frac{2\pi}{3}\right)^n\right] = \frac{1}{3}\left[2^n + 2\cos\left(\frac{n\pi}{3}\right)\right]$$

Using a similar technique, we can find $C_1^n + C_4^n + C_7^n + \cdots$; and $C_2^n + C_5^n + C_8^n + \cdots$.

2. $C_r = \dfrac{n!}{r!(n-r)!}$

$$\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \cdots + \frac{nC_n}{C_{n-1}} = \frac{n}{1} + \frac{n(n-1)}{n} + \frac{\frac{n(n-1)(n-2)}{2}}{\frac{n(n-1)}{2}} + \cdots + \frac{n\times 1}{n}$$

$= n + (n - 1) + (n - 2) + \ldots + 1$

$= \dfrac{n(n+1)}{2}$ (sum of $n$ terms of an A.P.)

3. (a) $x\left[1 + (x+1) + (x+1)^2 + \cdots + (x+1)^{n+k-1}\right]$

$= x\cdot\dfrac{a\left(r^{n+k} - 1\right)}{r-1}$, (sum of $n + k$ terms of a G.P., $r = x + 1$ for $r \neq 1$.)

$= x\dfrac{\left[(x+1)^{n+k} - 1\right]}{x+1-1} = (x + 1)^{n+k} - 1$

(b) Compare coefficient of $x^n$

LHS = coefficient in $x(x+1)^{n-1}$ + coefficient in $x(x+1)^n$ + … + coefficient in $x(x + 1)^{n+k-1}$

$= C_{n-1}^{n-1} + C_{n-1}^{n} + C_{n-1}^{n+1} + \cdots + C_{n-1}^{n+k-1} = C_0^{n-1} + C_1^n + C_2^{n+1} + \cdots + C_k^{n+k-1}$

RHS $= C_k^{n+k}$ $\therefore$ $C_0^{n-1} + C_1^n + C_2^{n+1} + \cdots + C_k^{n+k-1} = C_k^{n+k}$

4. $11^n = (10 + 1)^n = 10^n + C_1^n\cdot 10^{n-1} + C_2^n\cdot 10^{n-2} + \cdots$

$9^n = (10 - 1)^n = 10^n - C_1^n\cdot 10^{n-1} + C_2^n\cdot 10^{n-2} - \cdots$

$11^n - 9^n = 2\left(C_1^n\cdot 10^{n-1} + C_3^n\cdot 10^{n-3} + \cdots\right) > 2\cdot n\cdot 10^{n-1}$

$\geq 2\times 5\times 10^{n-1}$ for $n \geq 5$

$= 10\times 10^{n-1} = 10^n$

$\therefore$ $11^n > 9^n + 10^n$ for $n \geq 5$ .

5. (a) $(1 + ax+bx^2)^4 = [1 + (ax+bx^2)]^4$
$= 1 + 4(ax + bx^2) + 6(ax + bx^2)^2$ + terms involving $x^3$ or higher powers
$= 1 + 4ax + 4bx^2 + 6a^2x^2$ + terms involving $x^3$ and or powers
$= 1 + 4ax + (6a^2 + 4b)x^2$ + terms involving $x^3$ and or powers

(b) (i) $y = 1 + a\lambda + b\lambda^2$
$y^4 - \lambda y - 1 = (1 + a\lambda + b\lambda^2)^4 - \lambda(1 + a\lambda + b\lambda^2) - 1$
$= 1 + 4a\lambda + (6a^2 + 4b)\lambda^2 - \lambda(1 + a\lambda + b\lambda^2) - 1 + \cdots$
$= (4a - 1)\lambda + (6a^2 + 4b - a)\lambda^2 + \cdots$

(ii) $4a - 1 = 0 \Rightarrow a = \dfrac{1}{4}$
$6a^2 + 4b - a = 0$
$\dfrac{3}{8} + 4b - \dfrac{1}{4} = 0$
$b = -\dfrac{1}{32}$
$y \approx 1 + a\lambda + b\lambda^2 = 1 + \dfrac{\lambda}{4} - \dfrac{\lambda^2}{32}$

6. $$\frac{11x-2}{(x-2)^2(x^2+1)} \equiv \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+1}$$
$11x - 2 \equiv A(x - 2)(x^2 + 1) + B(x^2 + 1) + (Cx + D)(x - 2)^2$
Put $x = 2$, $20 = 5B \Rightarrow B = 4$
Put $x = i$, $-2 + 11i = (Ci + D)(-2 + i)(-2 + i)$
$= (Ci + D)(4 - 1 - 4i)$
$= (D + Ci)(3 - 4i)$
$= 3D + 4C + (3C - 4D)i$
Compare coefficients: $3D + 4C = -2$ ⋯⋯ (1), $3C - 4D = 11$ ⋯ ⋯ (2)
(1)×3: $12C + 9D = -6$ ⋯⋯ (3)
(2)×4: $12C - 16D = 44$ ⋯ ⋯ (4)
(3) – (4): $25D = -50$
$D = -2$
Sub. $D = -2$ into (2): $3C - 4(-2) = 11$
$3C = 3$
$C = 1$
Compare coefficients of $x^3$: $0 = A + C \Rightarrow A = -1$
$$\therefore \frac{11x-2}{(x-2)^2(x^2+1)}$$
$$\equiv -\frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{x-2}{x^2+1}$$
$$\equiv -\frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{x-2}{x^2+1}$$
$$\equiv \frac{1}{2}\cdot\frac{1}{\left(1-\frac{x}{2}\right)} + \frac{1}{\left(1-\frac{x}{2}\right)^2} + (x-2)\cdot\frac{1}{1+x^2}$$
$$\equiv \frac{1}{2}\cdot\left[1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\cdots+\left(\frac{x}{2}\right)^r+\cdots\right]+\left[1+2\left(\frac{x}{2}\right)+3\left(\frac{x}{2}\right)^2+\cdots+(r+1)\left(\frac{x}{2}\right)^r+\cdots\right]$$

$+(x-2)\cdot\left[1-x^2+x^4-x^6+\cdots+(-1)^r x^{2r}+\cdots\right]$, valid for $|x| < 1$

Coefficient of $x^{2n}$ is $\frac{1}{2}\cdot\left(\frac{1}{2}\right)^{2n}+(2n+1)\left(\frac{1}{2}\right)^{2n}-2(-1)^n=\left(\frac{1}{2}\right)^{2n}\left(2n+\frac{3}{2}\right)-2(-1)^n$

$$=\frac{4n+3}{2^{2n+1}}+2(-1)^{n+1}$$

Coefficient of $x^{2n+1}$ is $\frac{1}{2}\cdot\left(\frac{1}{2}\right)^{2n+1}+(2n+2)\left(\frac{1}{2}\right)^{2n+1}+(-1)^n=\frac{4n+5}{2^{2n+2}}+(-1)^n$

7. $(1+x)^n=C_0^n+C_1^n\cdot x+\cdots+C_n^n\cdot x^n$ ……(*)

Using $\omega=\frac{-1+\sqrt{3}i}{2}=\text{cis}\frac{2\pi}{3}$

then $\omega\neq 1$, $1+\omega+\omega^2=0$ …… (1) and $\omega^3=1$ …… (2)

Put $x = 1$: $2^n = C_0^n+C_1^n+C_2^n+C_3^n+C_4^n+C_5^n+\cdots$ …… (3)

Put $x = \omega$ in (*) and multiply by $\omega$:

$\omega(1+\omega)^n = \omega C_0^n+C_1^n\cdot\omega^2+C_2^n\cdot\omega^3+C_3^n\cdot\omega^4+C_4^n\cdot\omega^5+C_5^n\cdot\omega^6+\cdots$ …… (4)

Put $x = \omega^2$ in (*) and multiply by $\omega^2$:

$\omega^2(1+\omega^2)^n=\omega^2C_0^n+C_1^n\cdot\omega^4+C_2^n\cdot\omega^6+C_3^n\cdot\omega^8+C_4^n\cdot\omega^{10}+C_5^n\cdot\omega^{12}+\cdots$ …… (5)

Add up these three equations (3) + (4) + (5):

$2^n+\omega(1+\omega)^n+\omega^2(1+\omega^2)^n=(1+\omega+\omega^2)+(1+\omega^2+\omega)C_1^n+3C_2^n+(1+\omega+\omega^2)C_3^n+(1+\omega^2+\omega)C_4^n+3C_5^n+\cdots$

$3(C_2^n+C_5^n+C_8^n+\cdots)=\left[2^n+\omega(-\omega^2)^n+\omega^2(-\omega)^n\right]$ by (1)

$$\begin{aligned}C_2^n+C_5^n+C_8^n+\cdots &=\frac{1}{3}\left[2^n+(-1)^n\left(\omega^{2n+1}+\omega^{n+2}\right)\right]\\
&=\frac{1}{3}\left[2^n+(-1)^n\left(\omega^{\frac{n-1}{2}}+\omega^{-\frac{n-1}{2}}\right)\omega^{\frac{3(n+1)}{2}}\right]\\
&=\frac{1}{3}\left\{2^n+(-1)^n\cdot 2\cos\left(\frac{n-1}{2}\cdot\frac{2\pi}{3}\right)\left[\text{cis}\frac{3(n+1)}{2}\cdot\frac{2\pi}{3}\right]\right\}\\
&=\frac{1}{3}\left\{2^n+(-1)^n\cdot 2\cos\frac{(n-1)\pi}{3}\left[\cos(n+1)\pi+i\sin(n+1)\pi\right]\right\}\\
&=\frac{1}{3}\left[2^n+(-1)^n\cdot 2\cos\frac{(n-1)\pi}{3}\cdot(-1)^{n+1}\right]\\
&=\frac{1}{3}\left[2^n-\cdot 2\cos\frac{(n-1)\pi}{3}\right]\\
&=\frac{1}{3}\left[2^n+2\cos\frac{(n-4)\pi}{3}\right]\end{aligned}$$

8. $(1+x)^n=C_0^n+C_1^n\cdot x+C_2^n\cdot x^2+\cdots+C_p^n\cdot x^p+\cdots+C_n^n\cdot x^n$ ……(1)

$(1+x)^m=C_0^m+C_1^m\cdot x+C_2^m\cdot x^2+\cdots+C_p^m\cdot x^p+\cdots+C_m^m\cdot x^m$ ……(2)

(1)×(2): $(1+x)^{n+m}=\left(C_0^n+C_1^nx+C_2^nx^2+\cdots+C_p^nx^p+\cdots+C_n^nx^n\right)\left(C_0^m+C_1^mx+C_2^mx^2+\cdots+C_p^mx^p+\cdots+C_m^mx^m\right)$

Compare coefficient of $x^p$, $0\le p\le\min(m, n)$

$C_0^n\cdot C_p^m+C_1^n\cdot C_{p-1}^m+\cdots+C_p^n\cdot C_0^m=C_p^{m+n}$

9. $\left(1+\frac{1}{x}\right)^n = C_0^n + \frac{C_1^n}{x} + \cdots + \frac{C_{n-r}^n}{x^{n-r}} + \cdots + \frac{C_n^n}{x^n}$ …… (1)

$(1+x)^n = C_0^n + C_1^n \cdot x + C_2^n \cdot x^2 + \cdots + C_r^n \cdot x^r + \cdots + C_n^n \cdot x^n$ …… (2)

(1)×(2): $\left(1+\frac{1}{x}\right)^n (1+x)^n = \left(C_0^n + \frac{C_1^n}{x} + \cdots + \frac{C_{n-r}^n}{x^{n-r}} + \cdots + \frac{C_n^n}{x^n}\right)\left(C_0^n + C_1^n \cdot x + C_2^n \cdot x^2 + \cdots + C_r^n \cdot x^r + \cdots + C_n^n \cdot x^n\right)$

$\frac{1}{x^n}(1+x)^{2n} = \left(C_0^n + \frac{C_1^n}{x} + \cdots + \frac{C_{n-r}^n}{x^{n-r}} + \cdots + \frac{C_n^n}{x^n}\right)\left(C_0^n + C_1^n \cdot x + C_2^n \cdot x^2 + \cdots + C_r^n \cdot x^r + \cdots + C_n^n \cdot x^n\right)$

Compare coefficient of $x^r$: $C_0^n \cdot C_r^n + C_1^n \cdot C_{r+1}^n + \cdots + C_{n-r}^n \cdot C_n^n = C_{n+r}^{2n} = \frac{(2n)!}{(n-r)!(n+r)!}$

10. Put $x = -1$, $a_0 - a_1 x + a_2 - a_3 + a_4 - \cdots = 1$

11. $(1+x)^n = (1+2x+x^2)(C_0^{n-2} + C_1^{n-2}x + C_2^{n-2}x^2 + \cdots + C_{r-2}^{n-2}x^{r-2} + C_{r-1}^{n-2}x^{r-1} + C_r^{n-2}x^r + \cdots + C_n^n x^n)$

For $2 \le r \le n-2$.

Compare coefficient of $x^r$: $C_r^n = C_r^{n-2} + 2C_{r-1}^{n-2} + C_{r-2}^{n-2}$

12. (a) $(1-x)^n(1+x)^{2n} = [1 - nx + \frac{n(n-1)}{2}x^2 - \ldots][1 + 2nx + n(2n-1)x^2 + \ldots]$

Coefficient of $x = 2n - n = n$

Coefficient of $x^2 = \frac{n(n-1)}{2} - 2n^2 + n(2n-1) = \frac{n^2-3n}{2}$

(b) $\frac{n^2-3n}{2} = n$

$n^2 - 3n = 2n$

$n = 5$

13. (a) $n(n+1-r) - (n-r)(n+1) = n^2 + n - nr - n^2 + nr - n + r = r > 0$

$\therefore n(n+1-r) > (n-r)(n+1)$

$\frac{n+1-r}{n+1} > \frac{n-r}{n}$

(b) $\left(1+\frac{1}{n+1}\right)^{n+1} = C_0^{n+1} + \frac{C_1^{n+1}}{n+1} + \cdots + \frac{C_{k+1}^{n+1}}{(n+1)^{k+1}} + \cdots + \frac{C_{n+1}^{n+1}}{(n+1)^{n+1}}$, where $1 \le k \le n$.

$\left(1+\frac{1}{n+1}\right)^{n+1} = 1 + \frac{n+1}{n+1} + \frac{\frac{(n+1)n}{2}}{(n+1)^2} \cdots + \frac{\frac{(n+1)n\cdots(n-k+1)}{(k+1)!}}{(n+1)^{k+1}} + \cdots + \frac{1}{(n+1)^{n+1}}$

$\left(1+\frac{1}{n+1}\right)^{n+1} = 1 + 1 + \frac{n}{2(n+1)} + \frac{n(n-1)}{6(n+1)^2} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{(k+1)!(n+1)^k} + \cdots + \frac{1}{(n+1)^{n+1}}$ …(1)

$\left(1+\frac{1}{n}\right)^n = C_0^n + \frac{C_1^n}{n} + \cdots + \frac{C_{k+1}^n}{n^{k+1}} + \cdots + \frac{C_n^n}{n^n}$, where $1 \le k < n$.

$\left(1+\frac{1}{n}\right)^n = 1 + \frac{n}{n} + \frac{\frac{n(n-1)}{2}}{n^2} \cdots + \frac{\frac{n\cdots(n-k)}{(k+1)!}}{n^{k+1}} + \cdots + \frac{1}{n^n}$

$\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{n-1}{2n} + \frac{(n-1)(n-2)}{6n^2} + \cdots + \frac{(n-1)(n-2)\cdots(n-k)}{(k+1)!n^k} + \cdots + \frac{1}{n^n}$ …… (2)

Compare term by term in (1) and (2):

$\frac{n(n-1)\cdots(n-k+1)}{(k+1)!(n+1)^k} \div \frac{(n-1)\cdots(n-k)}{(k+1)!n^k} = \frac{\frac{n(n-1)\cdots(n-k+1)}{(n+1)^k}}{\frac{(n-1)(n-2)\cdots(n-k)}{n^k}}$, $1 \le k < n$.

$$=\frac{\left(\frac{n}{n+1}\right)\left(\frac{n-1}{n+1}\right)\cdots\left(\frac{n-k+1}{n+1}\right)}{\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k}{n}\right)}$$

$$=\frac{\left(\frac{n}{n+1}\right)\left(\frac{n-1}{n+1}\right)\cdots\left(\frac{n-k+1}{n+1}\right)}{\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k}{n}\right)} \quad \cdots\cdots (3)$$

By ($a$), $\frac{n+1-r}{n+1}>\frac{n-r}{n}$ for $n \ge r \ge 1$

Put $r = 1$, $\frac{n}{n+1}>\frac{n-1}{n}$

Put $r = 2$, $\frac{n-1}{n+1}>\frac{n-2}{n}$

…………………………………………………

Put $r = k$, $\frac{n+1-k}{n+1}>\frac{n-k}{n}$, where $1 \le k < n$.

Multiply these equations: $\frac{n}{n+1}\cdot\frac{n-1}{n+1}\cdots\cdots\frac{n+1-k}{n+1}>\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\cdots\frac{n-k}{n}$

$\therefore$ $\frac{\left(\frac{n}{n+1}\right)\left(\frac{n-1}{n+1}\right)\cdots\left(\frac{n-k+1}{n+1}\right)}{\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k}{n}\right)}>1$ for $1 \le k < n$.

(3): $\frac{n(n-1)\cdots(n-k+1)}{(k+1)!(n+1)^k}\div\frac{(n-1)\cdots(n-k)}{(k+1)!n^k}>1$

From the third term to the $(n+1)^{\text{th}}$ term in (1) > the third term to the $(n+1)^{\text{th}}$ term in (2)

$\therefore$ (1) > (2): $\left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^{n}$.

14. (a) $(1 + x)^m + (1 + x)^n = [1 + mx + \frac{m(m-1)}{2}x^2 + \cdots ] + [1 + nx + \frac{n(n-1)}{2}x^2 + \cdots ]$

$= 1 + (m + n)x + \frac{m^2-m+n^2-n}{2}x^2 + \cdots$

Coefficient of $x = m + n = 19$

Coefficient of $x^2 = \frac{m^2+n^2-(m+n)}{2} = \frac{(m+n)^2-2mn-(m+n)}{2}$

$= \frac{19^2-2mn-19}{2} = \frac{342-2mn}{2}$

$= 171 – mn = 171 – n(19 – n)$

$= 171 – 19n + n^2 = n^2 – 19n + 9.5^2 – 9.5^2 + 171$

$= (n – 9.5)^2 + 80.75$

When $n = 9$ or $10$, coefficient of $x^2 = 81$

(b) When $n = 9$, $m = 10$, coefficient of $x^7 = C_7^9 + C_7^{10} = 36 + 120 = 150$

15. $\left(2-\frac{1}{2x^2}\right)^{10} = 2^{10}-10\cdot 2^9\cdot\frac{1}{2x^2}+45\cdot 2^8\cdot\frac{1}{4x^4}+\cdots = 1024-\frac{2560}{x^2}+\frac{2880}{x^4}+\cdots$

Put $x = 10$, $\left(2-\frac{1}{2x^2}\right)^{10} = \left(2-\frac{1}{2\times 10^2}\right)^{10} = (1.995)^{10}$

$= 1024-\frac{2560}{10^2}+\frac{2880}{10^4}+\cdots$

$= 1024 – 25.6 + 0.288 + \cdots$

$\approx 998.688 \approx 998.69$ correct to 5 significant figures.