Given a regular pentagon *ABCDE*. To construct an inscribed square *PQRS* in *ABCDE* so that *RS* // *CD*.

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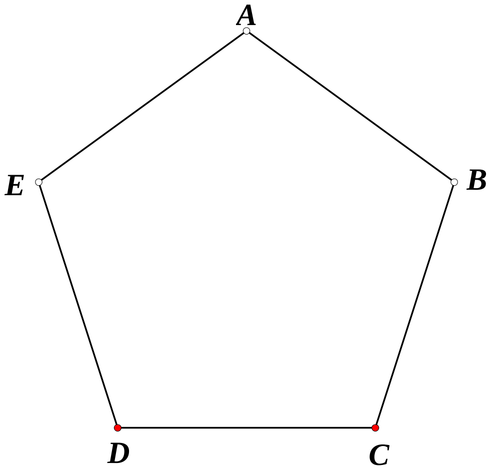

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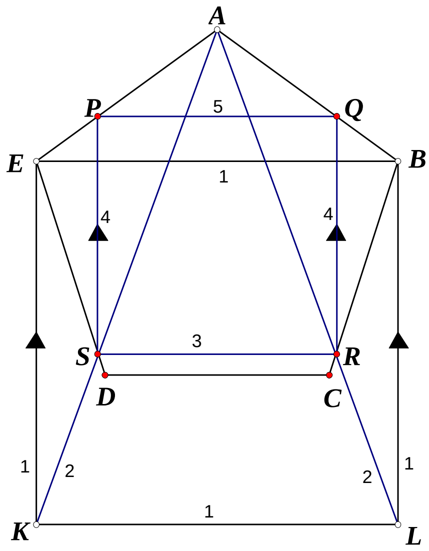

Construction steps:

(1) Join BE. Construct a square $BEKL$. (KL and DC lie on the same side of EB.)

(2) Join AK, cutting DE at S. Join AL, cutting BC at R.

(3) Join SR. Draw SP // KE, cutting AE at P. Draw RQ // LB, cutting AB at Q.

(4) Join PQ. Then $PQRS$ is the required inscribed square.

Proof: $AE = AB$	(sides of the regular pentagon)
$EK = BL$	(opp. sides of a square)
$\angle BAE = 108°$	($\angle$ sum of polygon)
$\angle AEB = \angle ABE = (180° - 108°)\div 2 = 36°$	($\angle$ sum of Δ, base $\angle$s isos. Δ)
$\angle KEB = \angle LBE = 90°$	(property of a square)
$\angle AEK = 36° + 90° = 126° = \angle ABL$	
$\Delta AEK \cong \Delta ABL$	(S.A.S.)
$AK = AL$	(corr. sides $\cong \Delta$s)
$AE = AB$	(proved)
$\angle EAS = \angle BAR$	(corr. $\angle$s $\cong \Delta$s)
$\angle AES = \angle ABR = 108°$	($\angle$ sum of polygon)
$\Delta AES \cong \Delta ABR$	(A.A.S.)
$AS = AR$	(corr. sides $\cong \Delta$s)
$\therefore \dfrac{AS}{AK} = \dfrac{AR}{AL}$	
$\angle SAR = \angle KAL$	(common $\angle$s)
$\Delta ASR \sim \Delta AKL$	(2 sides proportional, included $\angle$)
$\angle ASR = \angle AKL$	(corr. $\angle$s $\sim \Delta$s)
SR // KL	(corr. $\angle$s eq.)
$\angle PSR = \angle PSA + \angle ASR$	
$= \angle EKA + \angle AKL$	(corr. $\angle$s, KE // SB, KL // SR)
$= \angle EKL = 90°$	(property of a square)
PS // EK // BL // QR	(transitive property of // lines)
$\angle QRS = 180° - 90° = 90°$	(int. $\angle$s, BD // QR)
$\Delta APS \sim \Delta AEK$ and $\Delta AQR \sim \Delta ABL$	(equiangular)
$\dfrac{PS}{EK} = \dfrac{AS}{AK}$	(corr. sides $\sim \Delta$s)
$= \dfrac{AR}{AL} = \dfrac{SR}{KL}$	(corr. sides $\sim \Delta$s)

$= \dfrac{QR}{BL}$ (corr. sides ~ Δs)

$\because$ $EK = BL$ and $EK = KL$ (property of a square)

$\therefore$ $PS = SR = QR$

$PQRS$ is a //-gram (opp. sides are eq. and //)

$PQRS$ is a square (adj. sides are eq.)

Let $AB = BC = CD = DE = 2a$ and let $\theta = 36°$

$5\theta = 180° \Rightarrow 3\theta = 180° - 2\theta$

$\sin 3\theta = \sin(180° - 2\theta) = \sin 2\theta$

$3 \sin\theta - 4\sin^3\theta = 2\sin\theta\cos\theta$

$3 - 4\sin^2\theta = 2\cos\theta$

$3 - 4(1 - \cos^2\theta) = 2\cos\theta$

$4\cos\theta - 2\cos\theta - 1 = 0$

$\cos\theta = \dfrac{1+\sqrt{5}}{4}$ or $\dfrac{1-\sqrt{5}}{4}$ (rejected)

In ΔABE, $AB = AE = 2a$, $\angle BAE = 108°$ ($\angle$ sum of polygon)

$\angle ABE = \angle AEB = (180° - 108°)\div 2 = 36°$ ($\angle$ sum of Δ, base $\angle$s isos. Δ)

$BE = 2\,AE\cos\angle AEB = 4a\cos 36° = \left(1+\sqrt{5}\right)a = BL = KL = EK$

$\angle AES = 108°$ ($\angle$ sum of polygon)

$\angle BES = 108° - 36° = 72°$

$\angle KES = 90° - 72° = 18°$

Apply sine rule on ΔAES and ΔKES. Let $\angle ASE = \alpha$, then $\angle KSE = 180° - \alpha$ (adj. $\angle$s on st. line)

$\dfrac{AS}{\sin 108^\circ} = \dfrac{2a}{\sin\alpha}$ …… (1) $\quad \dfrac{SK}{\sin 18^\circ} = \dfrac{\left(1+\sqrt{5}\right)a}{\sin\left(180^\circ - \alpha\right)}$ …… (2)

Using the fact that $\sin(180° - \alpha) = \sin\alpha$, $\sin 108° = \cos 18°$ and (2)÷(1):

$$\frac{SK}{AS}\cdot\frac{\cos 18^\circ}{\sin 18^\circ} = \frac{\left(1+\sqrt{5}\right)a}{2a}$$

$$\frac{SK}{AS} = \frac{1+\sqrt{5}}{2}\cdot\tan 18^\circ$$

$$\frac{AK}{AS} = \frac{AS+SK}{AS} = 1+\frac{SK}{AS} = 1+\frac{1+\sqrt{5}}{2}\cdot\tan 18^\circ = \frac{2+\left(1+\sqrt{5}\right)\tan 18^\circ}{2}$$

$$\frac{AS}{AK} = \frac{2}{2+\left(1+\sqrt{5}\right)\tan 18^\circ}$$

$$\frac{PS}{EK} = \frac{AS}{AK} = \frac{2}{2+\left(1+\sqrt{5}\right)\tan 18^\circ}$$

PS = the length of a side of the square $= \dfrac{2\left(1+\sqrt{5}\right)a}{2+\left(1+\sqrt{5}\right)\tan 18^\circ}$

Let $\beta = 18°$, then $5\beta = 90°$, $3\beta = 90° - 2\beta$, let $\tan\beta = t$

$\tan 3\beta = \tan(90° - 2\beta) = \cot 2\beta$

$$\frac{3t-t^3}{1-3t^2} = \frac{1-t^2}{2t}$$

$6t^2 - 2t^4 = 1 - t^2 - 3t^2 + 3t^4$

$5t^4 - 10t^2 + 1 = 0$

$t^2 = \dfrac{5+\sqrt{20}}{5}$ (rejected) or $\dfrac{5-\sqrt{20}}{5} = \dfrac{5-2\sqrt{5}}{5}$

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$$\tan 18^\circ = \sqrt{\frac{5-2\sqrt{5}}{5}}$$

$$\begin{aligned}
PS &= \frac{2\left(1+\sqrt{5}\right)a}{2+\left(1+\sqrt{5}\right)\tan 18^\circ} \\
&= \frac{2a}{\dfrac{2}{1+\sqrt{5}}+\sqrt{\dfrac{5-2\sqrt{5}}{5}}} \\
&= \frac{2a}{\dfrac{\sqrt{5}-1}{2}+\sqrt{\dfrac{5-2\sqrt{5}}{5}}} \\
&= \frac{4\sqrt{5}a}{5-\sqrt{5}+2\sqrt{5-2\sqrt{5}}} \\
&\approx 2.12a
\end{aligned}$$