10-11 Individual	**1**	11	**2**	$\frac{26}{5}$ (= 5.2)	**3**	$\frac{1}{2}\sqrt{2010}$	**4**	20	**5**	237
	6	10	**7**	$\sqrt{6}-\sqrt{2}$	**8**	7	**9**	36	**10**	$24\sqrt{3}$
10-11 Group	**1**	a = 1006, b = 1002, c = 998, d = 996, e = 994	**2**	15	**3**	5026	**4**	4	**5**	1005
	6	$6\sqrt{2}$	**7**	125751	**8**	2012	**9**	99°	**10**	$2\sqrt{3}$

Individual Events

I1 Find the remainder when 2^{2011} is divided by 13.

Reference: 1972 American High School Mathematics Examination Q31, 2015 FI4.1

$2^6 = 64 = 13\times5 - 1 \equiv -1 \bmod 13$; $2^{12} \equiv 1 \bmod 13$

$2011 = 12\times167 + 7$

$2^{2011} = 2^{12\times167+7} = (2^{12})^{167}\times2^7 \equiv 2^7 \equiv 2^6\times2 \equiv -1\times2 \equiv -2 \equiv 11 \bmod 13$

I2 Given that $x^2 + y^2 = 1$, find the maximum value of $2x + 5y^2$.

Reference: 2003 FG1.2, 2009 HI5

$2x + 5y^2 = 2x + 5(1 - x^2) = -5x^2 + 2x + 5$

The maximum value $=\dfrac{4ac-b^2}{4a}=\dfrac{4(-5)\cdot5-2^2}{4(-5)}=\dfrac{26}{5}$ (= 5.2)

I3 Given that $a+b=\sqrt{\sqrt{2011}+\sqrt{2010}}$ and $a-b=\sqrt{\sqrt{2011}-\sqrt{2010}}$, find the value of ab. (Give your answer in surd form)

$(a+b)^2-(a-b)^2=\sqrt{2011}+\sqrt{2010}-\sqrt{2011}+\sqrt{2010}$

$4ab=2\sqrt{2010}$

$ab=\frac{1}{2}\sqrt{2010}$

I4 In ΔABC, the ratio of the altitudes perpendicular to the three sides AB, BC and CA is 3 : 4 : 5. If the lengths of the three sides are integers, find the minimum value of AB.

Let $AB = c$, $BC = a$, $CA = b$. By calculating the areas in 3 different ways:

$3a = 4b = 5c$

$a : b : c = 20 : 15 : 12$

The minimum value of AB = 20

I5 An integer x minus 12 is the square of an integer. x plus 19 is the square of another integer. Find the value of x.

$x - 12 = n^2$ ⋯⋯ (1); $x + 19 = m^2$ ⋯⋯ (2), where m, n are integers.

(2) – (1): $(m + n)(m - n) = 31$

$\because$ 31 is a prime number

$\therefore$ $m + n = 31$ and $m - n = 1$

$m = 16$, $n = 15$

$x = 15^2 + 12 = 237$

I6 A, B and C pass a ball among themselves. A is the first one to pass the ball to other one. In how many ways will the ball be passed back to A after 5 passes?

Construct the following table:

Number of passes	1	2	3	4	5
A	0	1+1=2	1+1=2	3+3=6	5+5=10
B	1	0+1=1	1+2=3	3+2=5	5+6=11
C	1	0+1=1	1+2=3	3+2=5	5+6=11

There will be 10 ways for the ball to pass back to A.

I7 Find the value of $\sqrt{7-\sqrt{12}-\sqrt{13-2\sqrt{12}}}$.

Reference: 1993 FI1.4, 1999 HG3, 2001 FG2.1, 2015 FI4.2, 2015 FG3.1

$$\sqrt{13-2\sqrt{12}}=\sqrt{12-2\sqrt{1\times 12}+1}$$
$$=\sqrt{\left(\sqrt{12}-1\right)^2}$$
$$=\sqrt{12}-1$$

$$\sqrt{7-\sqrt{12}-\sqrt{13-2\sqrt{12}}}=\sqrt{7-\sqrt{12}-\sqrt{12}+1}$$
$$=\sqrt{8-2\sqrt{12}}$$
$$=\sqrt{6-2\sqrt{2\times 6}+2}$$
$$=\sqrt{\left(\sqrt{6}-\sqrt{2}\right)^2}$$
$$=\sqrt{6}-\sqrt{2}$$

I8 A school issues 4 types of raffle tickets with face values \$10, \$15, \$25 and \$40. Class A uses several one-hundred dollar notes to buy 30 raffle tickets, including 5 tickets each for two of the types and 10 tickets each for the other two types. How many one-hundred dollars notes Class A use to buy the raffle tickets?

Suppose x tickets of \$10, y tickets of \$15, z tickets of \$25, t tickets of \$40 are bought.

100 is an even number, the face values \$15 and \$25 are odd numbers.

Then either $(x, y, z, t) = (5, 10, 10, 5)$ or $(x, y, z, t) = (10, 5, 5, 10)$ will make an even sum.

Case 1 $(x, y, z, t) = (5, 10, 10, 5)$

Total cost = 10(5) + 15(10) + 25(10) + 40(5) = 650, which is not a multiple of 100, rejected.

Case 2 $(x, y, z, t) = (10, 5, 5, 10)$

Total coast = 10(10) + 15(5) + 25(5) + 40(10) = 700

$\therefore$ Class A uses 7 \$100 notes.

I9 The length and the width of a rectangle are integers. If its area is larger than its perimeter by 9, find the perimeter.

Let the width be x and the length be y.

$xy - 2(x + y) = 9$

$x(y - 2) - 2y + 4 = 13$

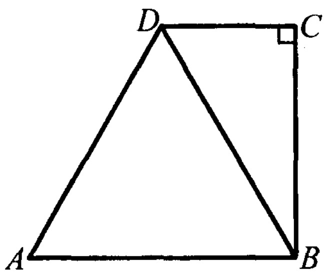
$(x - 2)(y - 2) = 13$

$x - 2 = 1, y - 2 = 13$

$x = 3, y = 15$

The perimeter = 2(3 + 15) = 36

I10 In the figure, $ABCD$ is a trapezium with $\angle C = 90°$. If the area of the equilateral triangle ABD is $16\sqrt{3}$, find the area of trapezium $ABCD$.



Area of $\Delta BCD = \frac{1}{2}$ area of ΔABD

Area of trapezium = $16\sqrt{3} + 8\sqrt{3} = 24\sqrt{3}$

Group Events

G1 If $(1000 - a)(1000 - b)(1000 - c)(1000 - d)(1000 - e) = 24^2$, where a, b, c, d and e are even numbers and $a > b > c > d > e$, find the values of a, b, c, d and e.

$24^2 = 2^6\times 3^2 = -6\times(-2)\times 2\times 4\times 6$

$= (1000 - 1006)(1000 - 1002)(1000 - 998)(1000 - 996)(1000 - 994)$

$a = 1006$, $b = 1002$, $c = 998$, $d = 996$, $e = 994$

G2 $\overline{ab}$ denotes a two digit number with a as tens digit and b as the unit digit. $R_{\overline{ab}}$ is the remainder when $\overline{ab}$ is divided by $a + b$. Find the maximum value of $R_{\overline{ab}}$.

$\overline{ab} = 10a + b = (a + b)m + R_{\overline{ab}}$, where $0 \le R_{\overline{ab}} < a + b$

To find the maximum value of $R_{\overline{ab}}$.

We start searching $R_{\overline{ab}}$ from $\overline{ab} = 99$ downwards to 10.

$\overline{ab} = 99$, $99 \div (9 + 9) = 5 \cdots\cdots 9 = R_{\overline{ab}}$

$\overline{ab} = 98$, $98 \div (9 + 8) = 5 \cdots\cdots 13 = R_{\overline{ab}}$

$\overline{ab} = 89$, $89 \div (9 + 8) = 5 \cdots\cdots 4 = R_{\overline{ab}}$

$\overline{ab} = 97$, $97 \div (9 + 7) = 6 \cdots\cdots 1 = R_{\overline{ab}}$

$\overline{ab} = 88$, $88 \div (8 + 8) = 5 \cdots\cdots 8 = R_{\overline{ab}}$

$\overline{ab} = 79$, $79 \div (7 + 9) = 4 \cdots\cdots 15 = R_{\overline{ab}}$

The maximum value of the divisor is 16, which gives the maximum value of remainder = 15
For other divisors, which must be less than 16, the remainders must be less than 15.

$\therefore$ The maximum value of $R_{\overline{ab}}$ is 15.

G3 Given that a, b and c are integers, and $a + b = 2011$, $c - a = 2010$, $a < b$. Find the greatest possible value of $a + b + c$.

$a + b + c = 2011 + c$

$c = a + 2010$

Maximum $a = 1005$, $b = 1006$, $c = 1005 + 2010 = 3015$

Maximum $a + b + c = 2011 + 3015 = 5026$

G4 Given that n is a positive integer and $n^4 - 18n^2 + 49$ is a prime number, find the value of n.

$n^4 - 18n^2 + 49 = (n^2 - 7)^2 - 4n^2 = (n^2 + 2n - 7)(n^2 - 2n - 7)$

$\because n^2 + 2n - 7 > n^2 - 2n - 7$

$n^2 - 2n - 7 = 1$ and $n^2 + 2n - 7$ is a prime

$n^2 - 2n - 8 = 0$

$(n - 4)(n + 2) = 0$

$n = 4$ only (verification: $n^2 + 2n - 7 = 17$ which is a prime)

G5 Given that $\mathrm{f}(x) = \dfrac{4^x}{4^x + 2}$, where x is a real number, find the value of

$$\mathrm{f}\left(\frac{1}{2011}\right)+\mathrm{f}\left(\frac{2}{2011}\right)+\mathrm{f}\left(\frac{3}{2011}\right)+\cdots+\mathrm{f}\left(\frac{2009}{2011}\right)+\mathrm{f}\left(\frac{2010}{2011}\right).$$

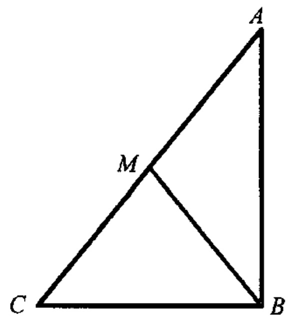
Reference: 2004 FG4.1, 2012 FI2.2

$$\mathrm{f}(x)+\mathrm{f}(1-x)=\frac{4^x}{4^x+2}+\frac{4^{1-x}}{4^{1-x}+2}=\frac{4+4+2\cdot 4^x+2\cdot 4^{1-x}}{4+4+2\cdot 4^x+2\cdot 4^{1-x}}=1$$

$$\mathrm{f}\left(\frac{1}{2011}\right)+\mathrm{f}\left(\frac{2}{2011}\right)+\mathrm{f}\left(\frac{3}{2011}\right)+\cdots+\mathrm{f}\left(\frac{2009}{2011}\right)+\mathrm{f}\left(\frac{2010}{2011}\right)$$

$$=\mathrm{f}\left(\frac{1}{2011}\right)+\mathrm{f}\left(\frac{2010}{2011}\right)+\mathrm{f}\left(\frac{2}{2011}\right)+\mathrm{f}\left(\frac{2009}{2011}\right)+\cdots+\mathrm{f}\left(\frac{1005}{2011}\right)+\mathrm{f}\left(\frac{1006}{2011}\right)=1005$$

G6 In the figure below, M is a point on AC, $AM = MC = BM = 3$. Find the maximum value of $AB + BC$. **Reference: 2007 HG8**



M is the centre of the circle ABC with AC as diameter.

$\angle ABC = 90°$ (∠ in semi-circle)

Let $\angle ACB = x$, $AB = 6 \sin \theta$, $BC = 6 \cos \theta$

$AB + BC = 6 \sin \theta + 6 \cos \theta$

$= 6\sqrt{2}\sin(\theta + 45^\circ) \le 6\sqrt{2}$

∴ The maximum value is $6\sqrt{2}$.

G7 Given that $n! = n\times(n-1)\times(n-2)\times \ldots \times 3\times 2\times 1$ and $\frac{2011!}{10^k}$ is an integer, where k is a positive integer. If S is the sum of all possible values of k, find the value of S.

Reference: 1990 HG6, 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2012 FI1.4, 2012 FG1.3

When each factor of 5 is multiplied by 2, a trailing zero will appear in $n!$.

The number of factors of 2 is clearly more than the number of factors of 5 in 2011!

It is sufficient to find the number of factors of 5.

5, 10, 15, … , 2010; altogether 402 numbers, each have at least one factor of 5.

25, 50, 75, … , 2000; altogether 80 numbers, each have at least two factors of 5.

125, 250, 375, … , 2000; altogether 16 numbers, each have at least three factors of 5.

625, 1250, 1875; altogether 3 numbers, each have at least four factors of 5.

∴ Total number of factors of 5 is 402 + 80 + 16 + 3 = 501

There are 501 trailing zeros of 2011! ⇒ If $\frac{2011!}{10^k}$ is an integer, then $k = 1, 2, \ldots, 501$.

$S = 1 + 2 + \ldots + 501 = \frac{1+501}{2}\times 501 = 125751$

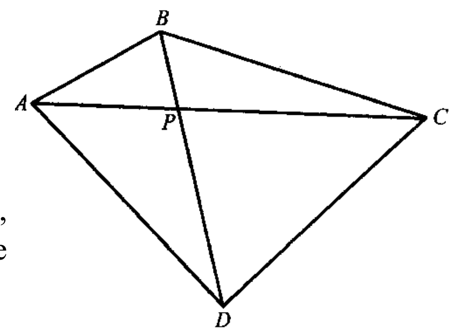
G8 Given that a, b, c and d are non-negative integers and $ac + bd + ad + bc = 2011$. Find the value of $a + b + c + d$.

$(a + b)(c + d) = 2011$, which is a prime number

$a + b = 2011$ and $c + d = 1$ or vice versa.

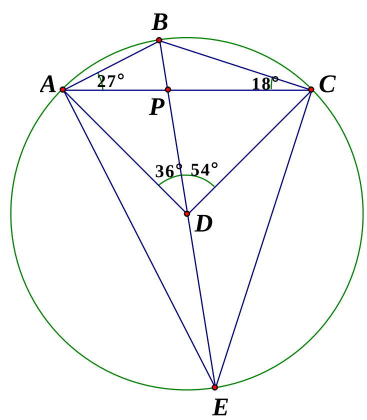
$a + b + c + d = 2012$

G9 As shown in the figure, $ABCD$ is a convex quadrilateral, $\angle BAC = 27°$, $\angle BCA = 18°$, $\angle BDC = 54°$, $\angle BDA = 36°$. The diagonals AC and BD intersect at P. Find $\angle CPB$.



Reference: 2003 HG8, 2014 FG2.4

Draw a circumscribed circle ABC.



Produce BD to cut the circle at E.

$\angle BEA = 18° = \angle BCA$ (∠s in the same segment)

$\angle BEC = 27° = \angle BAC$ (∠s in the same segment)

$\angle DAE = 36° - 18° = 18°$ (ext. ∠ of ΔADE)

$\angle DCE = 54° - 27° = 27°$ (ext. ∠ of ΔADE)

∴ $DA = DE$ and $DC = DE$ (sides opp. eq. ∠s)

∴ D is the centre of the circle

∴ $DA = DB = DC$ (radii)

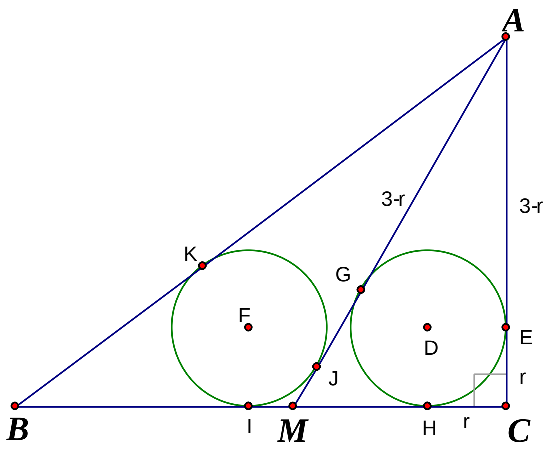
$\angle DCB = \angle DBC = (180° - 54°) \div 2 = 63°$

(∠s sum of isos. ΔBCD)

$\angle DCP = 63° - 18° = 45°$

$\angle CPB = 45° + 54° = 99°$ (ext. ∠ of ΔCDP)

G10 As shown in the figure, $AC = 3$, $BC = 4$ and $\angle C = 90°$. M is a point on BC such that the radii of the incircles in ΔABM and ΔACM are equal. Find the length of AM.



Let D, F be the centres of the circles, with radii r, the points of contact be E, G, H, I, J, K as shown. Let $AM = x$, $AB = 5$, $CH = CE = r$.

$AE = 3 - r = AG$ (tangent from ext. point)

$S_{\Delta ABF}+S_{\Delta AMF}+S_{\Delta BMF}+S_{\Delta AMD}+S_{\Delta ACD}+S_{\Delta CMD}=S_{\Delta ABC}$

$$\frac{3+4+5}{2}\cdot r+\frac{2x\cdot r}{2}=\frac{3\times 4}{2}$$

$r(6 + x) = 6$

$$r=\frac{6}{6+x} \quad \cdots\cdots (1)$$

In ΔACM, $MG = x + r - 3 = MH$ (tangent from ext. point)

$$MC = x + 2r - 3 = x+2\times\frac{6}{6+x}-3 \text{ by (1)}$$

$$=\frac{(x-3)(6+x)+12}{6+x}=\frac{x^2+3x-6}{6+x}$$

$AM^2 - AC^2 = MC^2$ (Pythagoras' Theorem)

$$x^2-9=\left(\frac{x^2+3x-6}{6+x}\right)^2$$

$(x^2 - 9)(x + 6)^2 = (x^2 + 3x - 6)^2$

$(x^2 - 9)(x^2 + 12x +36) = x^4 + 6x^3 - 3x^2 - 36x + 36$

$6x^3 + 30x^2 - 72x - 360 = 0$

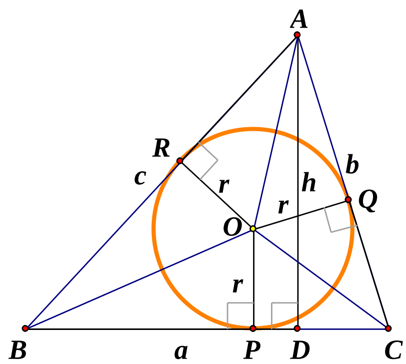
$x^3 + 5x^2 - 12x - 60 = 0$

$x(x^2 - 12) + 5(x^2 - 12) = 0$

$(x + 5)(x^2 - 12) = 0$

$AM = x = 2\sqrt{3}$

Method 2 Lemma In the figure, given a triangle ABC. $AD\perp BC$, $AD = h$, O is the centre of the inscribed circle with radius r. $s=\frac{1}{2}(a+b+c)$



Then $r=\frac{\text{Area of }\Delta ABC}{s}=\frac{ah}{a+b+c}$

Proof: Area ΔOBC + area ΔOCA + area ΔOAB = Area of ΔABC

$$\frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr=\frac{1}{2}ah$$

$$\therefore\ r=\frac{\text{Area of }\Delta ABC}{s}=\frac{ah}{a+b+c}$$

Let $CM = t$, then $BM = 4 - t$. Now apply the above lemma to ΔACM and ΔABM respectively.

$$r=\frac{3t}{t+3+\sqrt{9+t^2}} \quad \cdots\cdots (1)$$

$$r=\frac{3(4-t)}{4-t+5+\sqrt{9+t^2}} \quad \cdots\cdots (2)$$

(1) = (2) $\Rightarrow \dfrac{3t}{t+3+\sqrt{9+t^2}}=\dfrac{3(4-t)}{9-t+\sqrt{9+t^2}}$

$\dfrac{t}{4-t}=\dfrac{t+3+\sqrt{9+t^2}}{9-t+\sqrt{9+t^2}}$

$\dfrac{2t-4}{4}=\dfrac{2t-6}{12+2\sqrt{9+t^2}}$ ($\because \dfrac{A}{B}=\dfrac{C}{D}\Rightarrow\dfrac{A-B}{A+B}=\dfrac{C-D}{C+D}$)

$\dfrac{t-2}{2}=\dfrac{t-3}{6+\sqrt{9+t^2}}$

$(t-2)\left(6+\sqrt{9+t^2}\right)=2t-6$

$6t-12+(t-2)\sqrt{9+t^2}=2t-6$

$(t-2)\sqrt{9+t^2}=6-4t$

$(t^2-4t+4)(9+t^2)=36-48t+16t^2$

$t^4-4t^3+4t^2+9t^2-36t+36=36-48t+16t^2$

$t^4-4t^3-3t^2+12t=0$

$t[t^2(t-4)-3(t-4)]=0$

$t(t-4)(t^2-3)=0$

$\therefore$ $t = 0$ (rejected), 4 (rejected because $CM < BC = 4$) or $\sqrt{3}$.

$AM=\sqrt{AC^2+CM^2}=\sqrt{3^2+\sqrt{3}^2}=2\sqrt{3}$

Geometrical Construction

1. Given a straight line L, and two points P and Q lying on the same side of L. Mark a point T on L so that the sum of the lengths of PT and QT is minimal. (Hint: Consider the reflection image of P about the line L.)

Reference: 2008 Heat Sample construction Q2

Figure 1

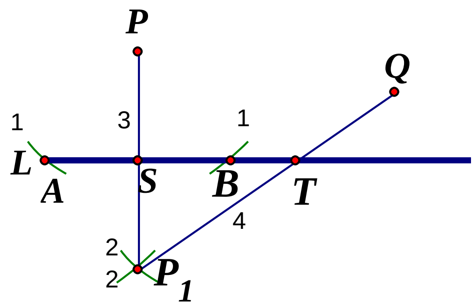


Figure 2

Let A be one of the end point of L nearer to P.

(1) Use P as centre, PA as radius to draw an arc, which intersects L at A and B.

(2) Use A as centre, AP as radius to draw an arc. Use B as centre, BP as radius to draw an arc. The two arcs intersect at P_1.

(3) Join PP_1 which intersects L at S.

(4) Join P_1Q which intersects L at T 。

$AP = AP_1$	(same radii)
$AB = AB$	(common side)
$BP = BP_1$	(same radii)
$\Delta APB \cong \Delta AP_1B$	(S.S.S.)
$\angle PBA = \angle P_1BA$	(corr. sides ≅Δ's)
$BS = BS$	(common side)
$\Delta PBS \cong \Delta P_1BS$	(S.A.S.)
$\angle BSP = \angle BSP_1$	(corr. sides ≅Δ's)
$= 90°$	(adj. ∠s on st. line)
$SP = SP_1$	(corr. sides ≅Δ's)
$ST = ST$	(common side)
$\Delta PST \cong \Delta P_1ST$	(S.A.S.)
$PT = P_1T$	(corr. sides ≅ Δ's)

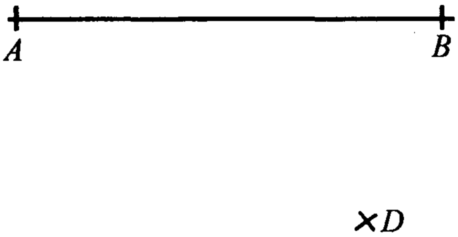
$PT + QT = P_1T + QT$

It is known that $P_1T + QT$ is a minimum when P_1, T, Q are collinear.

$\therefore$ T is the required point.

2. The figure shows a line segment AB. D is a fixed point such that A, B, D are not collinear. Construct a triangle ABC so that C, B and D are collinear and $AC - BC = BD$.

如圖所示為一綫段 AB。D 為一固定點，且 A、B、D 不共綫。試作ΔABC，使得 C、B 及 D 共綫，及 $AC - BC = BD$。



Remark: The wording in both versions are ambiguous, so I have changed it.

The original question is:

Figure 2 shows a line segment AB which is a ΔABC and D is any point not lying on AB. If the difference between the other two sides of ΔABC (i.e. $AC - BC$) is equal to BD and C, B and D are collinear, construct ΔABC.

圖 2 所示為ΔABC 的其中一條邊 AB 及 D 為一非綫段 AB 上的任意點。若ΔABC 的其餘兩條邊(即 $AC - BC$)的長度差距等如 BD，且 C、B 及 D 共綫，試構作ΔABC。

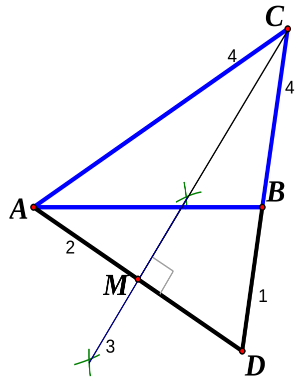
(1) Join DB and produce it further.

(2) Join AD.

(3) Construct the perpendicular bisector of AD, cutting DB produced at C. Let M be the mid-point of AD.

(4) Join AC, BC.

ΔABC is the required triangle.



Proof: $\Delta ACM \cong \Delta DCM$ (S.A.S.)

$AC = DC$ (corr. sides, $\cong \Delta$s)

$BC = DC - BD = AC - BD$

$\therefore BD = AC - BC$.

Discussion: There are some positions on the plane for which ΔABC is not constructible.

Use B as centre BA as radius to draw a circle.

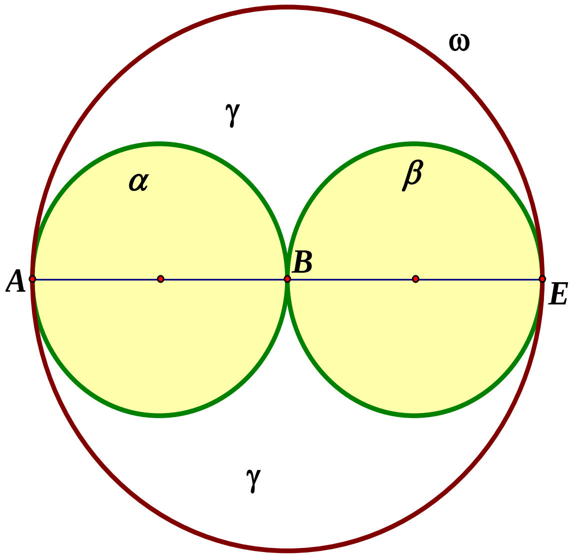
Let AE be the diameter of this circle. $AB = BE$.

Draw 2 circles with AB, BE as diameters.

Let the region (including the boundary) bounded by the circles with AB, BE as diameters be α and β respectively.

Let the region inside the great circle with AE as diameter but not in α and β be γ.

Let the region on or outside the great circle be ω.



When D lies on ω, then $BD \ge AB$

So $AC - BC \ge AB$

$AC \ge AB + BC$

This inequality violates the triangle law:

The sum of 2 sides of a triangle is larger than the third side.

i.e. $AC \ge AB + BC > AC$, which is false.

$\therefore$ **When D lies on or outside the circle, ΔABC is not constructible.**

When *D* lies on α,

$\Delta AC_1M \cong \Delta DC_1M$ (S.A.S.)

$\angle C_1AM = \angle C_1DM$ (corr. ∠s, ≅ Δs)

$\angle ADC_1 = \angle BAD + \angle ABD$ (ext. ∠ of ΔABD)

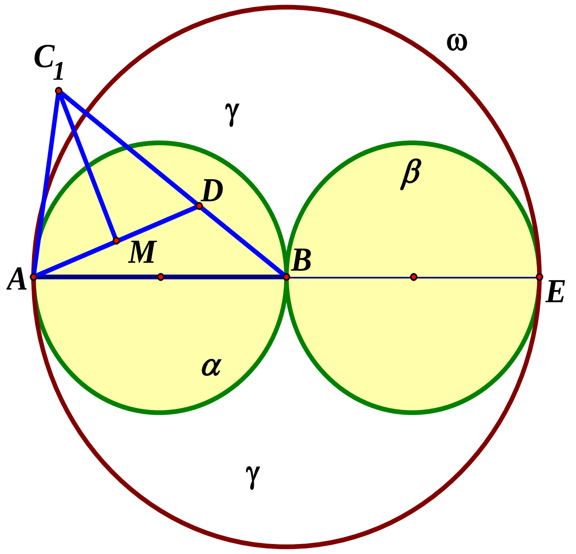
$> \angle ABD$

$\therefore \angle C_1AM = \angle ADC_1 > \angle ABD$

$\angle C_1AB > \angle ABD$

$AC_1 < BC_1$ (greater sides opp. greater ∠s)

The contradicts to the fact that $BD = AC_1 - BC_1$ because $AC_1 = BD + BC_1 > BC_1$



In this case the steps are as follows:

(1) Join DB and produce it further.

(2) Use B as centre, BD as radius to draw a circle, cutting DB produced at F.

(3) Join AF.

(4) Construct the perpendicular bisector of AF, cutting DB at C_2. Let M_2 be the mid-point of AF.

(5) Join AC_2, BC_2.

ΔABC_2 is the required triangle.

$\angle ABC_2 = \angle BAF + \angle AFB$ (ext. ∠ of ΔABF)

$> \angle AFB$

$= \angle FAC_2$ (corr. ∠s, ≅ Δs)

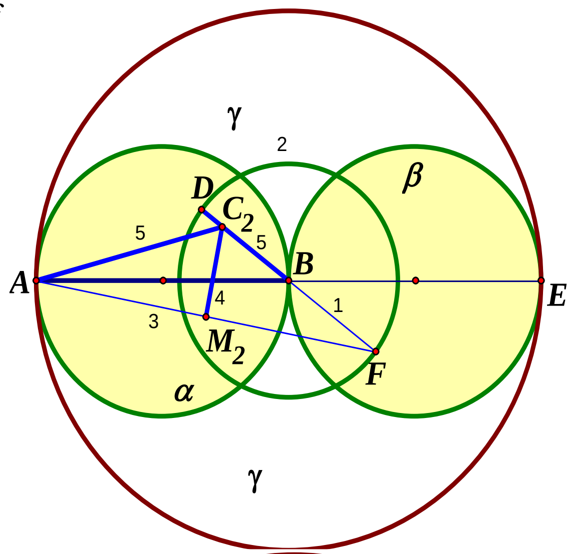
$> \angle BAC_2$

$\therefore AC_2 > BC_2$ (greater sides opp. greater ∠s)

The triangle ABC is constructible.

When *D* lies on β, relabel D as F and F as D, the construction steps are shown above.

There is only one possible triangle.



When *D* lies on the boundary of α,

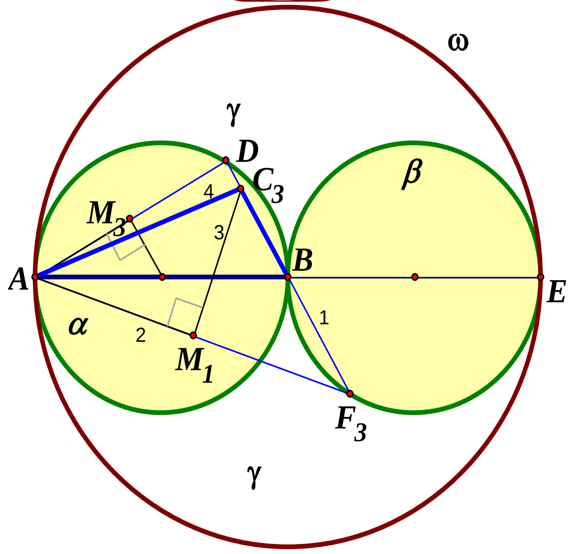
$\angle ADB = 90°$ (∠ in semi-circle)

The perpendicular bisector of AD is parallel to DB. ∴ It will not intersect DB produced.

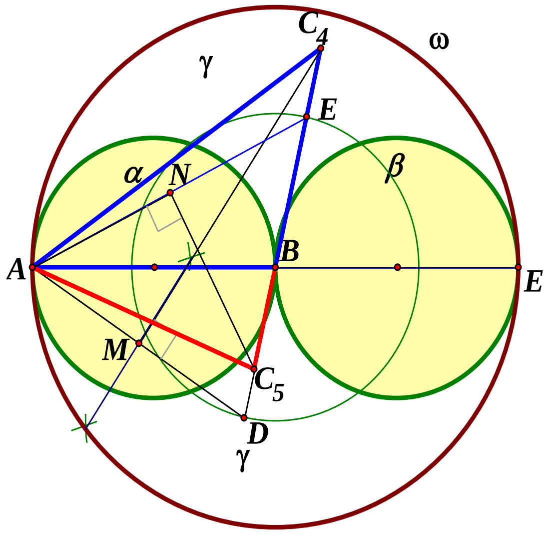
Suppose DB produced intersects the boundary of β at F_3. Then the perpendicular bisector of AF_3 intersects DB at C_3.

ΔABC_3 is the required triangle.

Proof: omitted.

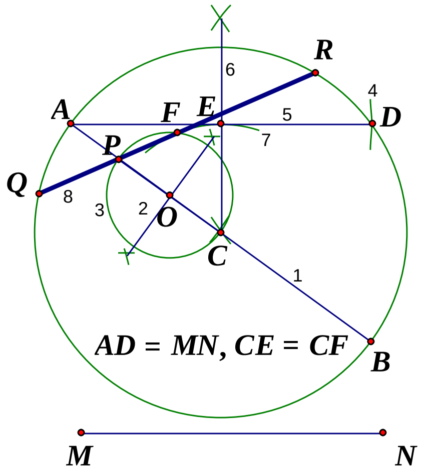


When *D* lies on γ,
Use B as centre, BD as radius to draw a circle, cutting DB produced at E on γ.
The perpendicular bisector of AD cuts DB produced at C_4. The perpendicular bisector of AE cut BD at C_5.
∴ **We can construct 2 possible triangles** ΔABC_4 and ΔABC_5 satisfying the given conditions.



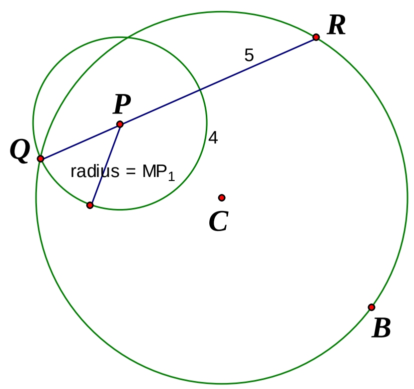
3. Figure 3 shows a circle of centre C and a line segment MN. P is a point lies inside the circle. Construct a chord QR with points Q and R on the circumference of the circle so that it passes through P and its length is equal to that of MN.

(1) Join CP and produce it to 2 ends A, B of the diameter of the circle as shown.
(2) Draw the perpendicular bisector of CP, O is the mid-point of CP.
(3) Use O as centre, OC as radius to draw a circle.
(4) Use A as centre, MN as radius to draw an arc, cutting the given circle at D.
(5) Join AD.
(6) Draw the perpendicular bisector of AD, E is the mid-point of AD.
(7) Use C as centre, CE as radius to draw an arc, cutting the circle in step (3) at F.
(8) Join PF and produce it to cut the circle at Q and R. Then QR is the required chord.



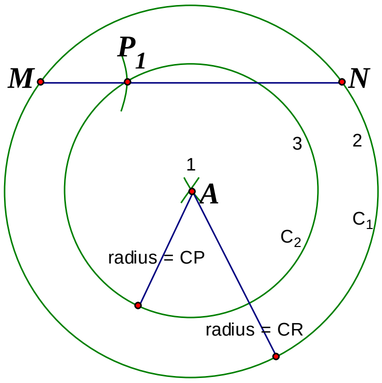
Proof: $\angle PFC = 90°$ ($\angle$ in semi-circle)
$CE = CF$ (by construction)
$QR = AD = MN$ (chords eq. distance from centre are equal)

Method 2: (Provided by Tsuen Wan Government Secondary School Tam Lok Him)
Let the radius of the given circle be R, the distance between CP be r.



(1) Use M as centre and R as radius to draw an arc, use N as centre and R as radius to draw an arc. The two arcs intersect at A.
(2) Use A as centre, R as radius to draw a circle C_1. The circle C_1 must pass through M, N.
(3) Use A as centre, r as radius to draw a circle C_2, which cuts MN at P_1.
(4) On the given circle, use P as centre, MP_1 as radius to draw a circle, which cuts the given circle at Q.
(5) Join QP and produce it to cut the given circle at R.

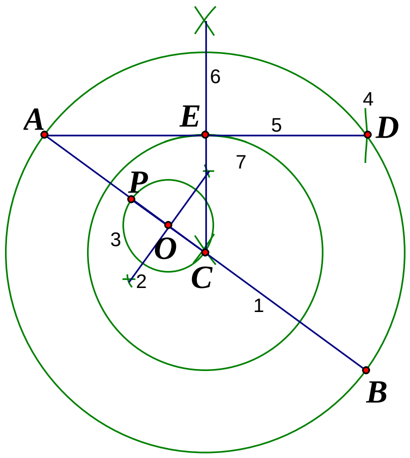
Then QPR is the required chord.

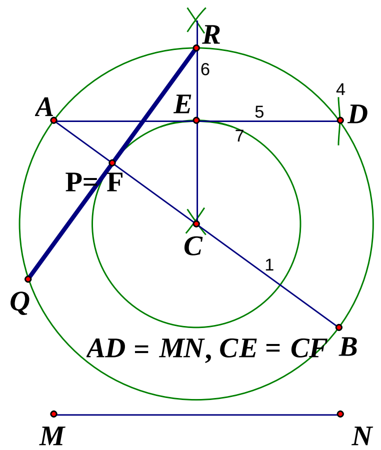


Proof: $QP = MP_1$ (by construction)
$CP = AP_1$ (by construction)
$CQ = AM$ (by construction)
$\Delta CPQ \cong \Delta AP_1M$ (S.S.S.)
height of ΔCPQ = height of ΔAP_1M
$QR = MN$ (chords eq. distance from centre are equal)

Remark There are some positions of P inside the circle for which the chord is not constructible. The circle in step 7 may not cut the circle in step 3. At the limiting position, $CQ^2 = PQ^2 + CP^2$

$$4r^2 = QR^2 + 4CP^2 \Rightarrow QR = 2\sqrt{r^2 - CP^2}$$





$\therefore$ For fixed position P, the length of chord must satisfy $2\sqrt{r^2 - CP^2} \leq MN \leq 2r$.

Percentage of correct questions

1	23.55%	**2**	26.6%	**3**	52.16%	**4**	25.87%	**5**	60.04%
6	31.34%	**7**	10.83%	**8**	70.35%	**9**	41.22%	**10**	62.88%
1	22.18%	**2**	24.69%	**3**	40.17%	**4**	43.1%	**5**	17.57%
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