

98-99	1	1	2	8	3	56	4	405	5	100000
Individual	6	2401	7	9	8	36	9	11	10	9

98-99	1	3	2	-24	3	$\frac{1}{2}$	4	$\frac{1}{2}$	5	6
Group	6	12	7	4	8	7	9	12	10	135

Individual Events

- I1** The circumference of a circle is 14π cm. Let X cm be the length of an arc of the circle, which subtends an angle of $\frac{1}{7}$ radian at the centre. Find the value of X .

Let r be the radius of the circle.

$$2\pi r = 14\pi$$

$$\Rightarrow r = 7$$

$$X = r\theta = 7 \times \frac{1}{7} = 1$$

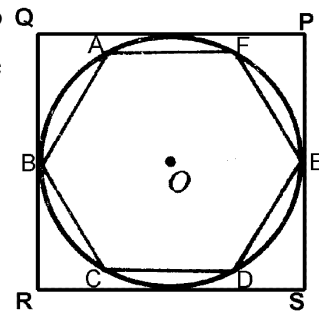
- I2** In Figure 1, $ABCDEF$ is a regular hexagon with area equal to $3\sqrt{3}$ cm². Let X cm² be the area of the square $PQRS$, find the value of X .

Area of the hexagon = 6 × areas of $\triangle AOB$

$$3\sqrt{3} = 6 \cdot \frac{1}{2} \cdot OB^2 \sin 60^\circ = \frac{3\sqrt{3}}{2} \cdot OB^2$$

$$OB^2 = 2$$

$$\text{Area of the square} = (2OB)^2 = 4 \times 2 = 8$$



- I3** 8 points are given and no three of them are collinear. Find the number of triangles formed by using any 3 of the given points as vertices.

The number of triangles formed

$$= {}_8C_3 = \frac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56$$

- I4** In Figure 2, there is a 3×3 square.

Let $\angle a + \angle b + \dots + \angle i = X^\circ$, find the value of X .

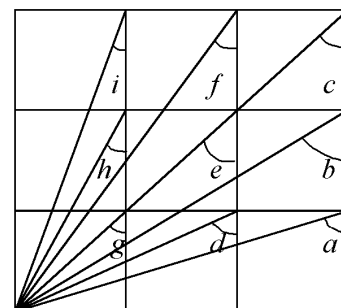
Reference: 廣州、武漢、福州、重慶、洛陽 初中數學聯賽

$$\angle c = \angle e = \angle g = 45^\circ$$

$$\angle a + \angle i = 90^\circ, \angle b + \angle f = 90^\circ, \angle d + \angle h = 90^\circ$$

$$\angle a + \angle b + \dots + \angle i = 45^\circ \times 3 + 90^\circ \times 3 = 405^\circ$$

$$X = 405$$



- I5** How many integers n are there between 0 and 10^6 , such that the unit digit of n^3 is 1?

$1^3 = 1$, the unit digit of n must be 1

There are $10^6 \div 10 = 100000$ possible integers.

- 16** Given that a, b, c are positive integers and $a < b < c = 100$, find the number of triangles formed with sides equal a cm, b cm and c cm.

By triangle inequality: $a + b > c = 100$

Possible pairs of (a, b) : $(2, 99), (3, 98), (3, 99), (4, 97), (4, 98), (4, 99), \dots, (50, 51), (50, 52), \dots, (50, 99), \dots, (98, 99)$

Total number of triangles = $1 + 2 + \dots + 48 + 49 + 48 + \dots + 2 + 1$

$$= \frac{1+49}{2} \times 49 \times 2 - 49 = 2401$$

- 17** A group of youngsters went for a picnic. They agreed to share all expenses. The total amount used was \$288. One youngster had no money to pay his share, and each of the others had to pay \$4 more to cover the expenses. How many youngsters were there in the group?

Let the number of youngsters be n .

$$\frac{288}{n-1} - \frac{288}{n} = 4$$

$$72 = n^2 - n$$

$$n = 9$$

- 18** A two-digit number is equal to 4 times the sum of the digits, and the number formed by reversing the digits exceeds 5 times the sum of the digits by 18. What is the number?

Let the unit digits of the original number be x and the tens digit by y .

$$10y + x = 4(x + y) \dots\dots\dots(1)$$

$$10x + y - 5(x + y) = 18 \dots\dots(2)$$

$$\text{From (1), } 6y = 3x \Rightarrow x = 2y \dots\dots\dots(3)$$

$$\text{Sub. (3) into (2): } 20y + y - 5(2y + y) = 18$$

$$\Rightarrow y = 3, x = 6$$

The number is 36.

- 19** Given that the denominator of the 1001^{th} term of the following sequence is 46, find the numerator of this term. $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots$

Suppose the numerator of the 1001^{th} term is n .

$$1 + 2 + 3 + \dots + 44 + n = 1001, n \leq 45$$

$$\frac{1}{2} (45)(44) + n = 1001$$

$$n = 1001 - 990 = 11$$

- 110** In the following addition, if the letter 'S' represents 4, what digit does the letter 'A' represent? **SEE**

$$3E + 4 = 10a + Y \dots\dots\dots(1), \text{ where } a \text{ is the carry digit in the tens digit.} \quad \text{4EE} \quad \text{SEE}$$

$$4E + a = 10b + 4 \dots\dots(2), \text{ where } b \text{ is the carry digit in the hundreds digit.} \quad \text{4EE} \quad + \quad \text{YES}$$

$$4 \times 3 + Y + b = 10E + A \dots\dots\dots(3) \quad \text{4EE} \quad \text{EASY}$$

$$\text{From (3), } E = 1 \text{ or } 2 \quad \text{+ YE4}$$

$$\text{When } E = 1, (1) \Rightarrow Y = 7, a = 0, (2) \Rightarrow b = 0, (3) \Rightarrow A = 9 \quad \text{EA4Y}$$

When $E = 2, (2) \Rightarrow a = 1, Y = 0$ reject because $YE4$ is a 3-digit number.

$$\therefore A = 9$$

Group Events

- G1** If a is a prime number and $a^2 - 2a - 15 < 0$, find the greatest value of a .

$$(a+3)(a-5) < 0$$

$$\Rightarrow a < 5$$

The greatest prime number is 3.

- G2** If $a : b : c = 3 : 4 : 5$ and $a + b + c = 48$, find the value of $a - b - c$.

$$a = 3k, b = 4k, c = 5k; \text{ sub. into } a + b + c = 48$$

$$\Rightarrow 3k + 4k + 5k = 48$$

$$\Rightarrow k = 4$$

$$a = 12, b = 16, c = 20$$

$$a - b - c = 12 - 16 - 20 = -24$$

- G3** Find the value of $\log(\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}})$.

Reference: 1993 FI1.4, 2001 FG2.1, 2011 HI7, 2015 FI4.2, 2015 FG3.1

$$\begin{aligned} \log(\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}}) &= \log\left(\sqrt{\frac{6+2\sqrt{5}}{2}} + \sqrt{\frac{6-2\sqrt{5}}{2}}\right) \\ &= \log\left(\frac{\sqrt{(1+\sqrt{5})^2} + \sqrt{(\sqrt{5}-1)^2}}{\sqrt{2}}\right) \\ &= \log\left(\frac{1+\sqrt{5}+\sqrt{5}-1}{\sqrt{2}}\right) = \log\left(\frac{2\sqrt{5}}{\sqrt{2}}\right) = \log(\sqrt{2}\sqrt{5}) = \log\sqrt{10} = \frac{1}{2} \end{aligned}$$

- G4** Find the area enclosed by the straight line $x + 4y - 2 = 0$ and the two coordinate axes.

$$x\text{-intercept} = 2, y\text{-intercept} = \frac{1}{2}; \text{ the area} = \frac{1}{2} \times \frac{1}{2} \times 2 = \frac{1}{2}$$

- G5** Natural numbers are written in order starting from 1 until 198th digit as shown 123456789101112...... If the number obtained is divided by 9, find the remainder.

198 digits

123456789 has 9 digits

10111213...9899 has $90 \times 2 = 180$ digits

\therefore 1234567891011...9899100101102 has 198 digits.

$1 + 2 + 3 + \dots + 9 = 45$, $11 + 12 + \dots + 19$ is also divisible by 9, \dots ,

$91 + 92 + \dots + 99$ is divisible by 9.

$10 + 20 + \dots + 90$ is divisible by 9

\therefore the remainder is the same as 100101102 divided by 9.

$1 + 1 + 1 + 1 + 2 = 6$, the remainder is 6.

- G6** The average of 2, a , 5, b , 8 is 6. If n is the average of a , $2a+1$, 11, b , $2b+3$, find the value of n .

$$2 + a + 5 + b + 8 = 30 \dots\dots (1), a + 2a + 1 + 11 + b + 2b + 3 = 5n \dots\dots (2)$$

From (1): $a + b = 15$

$$(2) 5n = 3a + 3b + 15 = 3(a + b) + 15 = 3 \times 15 + 15 = 60$$

$$\Rightarrow n = 12$$

- G7** If $p = 2x^2 - 4xy + 5y^2 - 12y + 16$, where x and y are real numbers, find the least value of p .

Reference: 2001 HI3, 2012 HG5, 2018 HI1

$$p = 2x^2 - 4xy + 2y^2 + 3y^2 - 12y + 16 = 2(x - y)^2 + 3(y^2 - 4y + 4) + 4 = 2(x - y)^2 + 3(y - 2)^2 + 4$$

$p \geq 4$, the least value of p is 4.

- G8** Find the units digit of 333^{335} .

$3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$, the units digit of 3^{4m} is 1, where m is any positive integer.

$333^{335} = 333^{4 \times 83 + 3} = (333^4)^{83} \times 333^3 = (\dots 1)^{83} \times (\dots 3^3) = \dots 7$, the units digit is 7.

- G9** In Figure 1, $\angle MON = 20^\circ$, A is a point on OM , $OA = 4\sqrt{3}$, D is a point on ON , $OD = 8\sqrt{3}$, C is any point on AM , B is any point on OD . If $\ell = AB + BC + CD$, find the least value of ℓ .

(Reference: 2016 HG5)

Reflect the figure along the line OM , then reflect the figure between $\angle MON_1$ along the line ON_1 .

$$\angle NOM_2 = 3 \times 20^\circ = 60^\circ$$

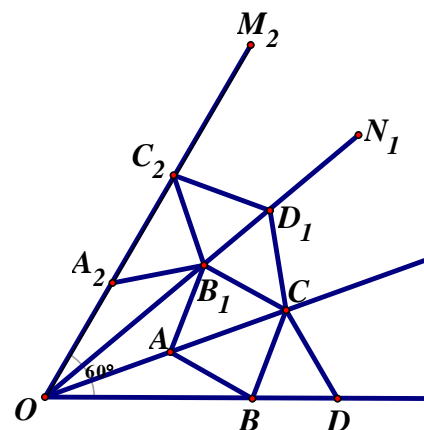
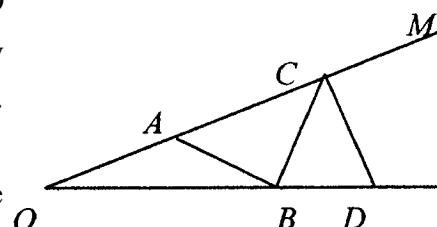
$$\ell = AB + BC + CD = AB_1 + B_1C + CD$$

$$\ell = A_2B_1 + B_1C + CD$$

ℓ is the shortest when A_2, B_1, C, D are collinear.

By cosine formula on $\triangle OA_2D$,

$$\begin{aligned} \text{Shortest } \ell = A_2D &= \sqrt{(4\sqrt{3})^2 + (8\sqrt{3})^2 - 2(4\sqrt{3})(8\sqrt{3})\cos 60^\circ} \\ &= \sqrt{48 + 192 - 96} = 12 \end{aligned}$$



G10 In figure 2, P is a point inside the square $ABCD$, $PA = a$, $PB = 2a$, $PC = 3a$ ($a > 0$). If $\angle APB = x^\circ$, find the value of x .

Reference: 2014 HG4

Rotate $\triangle APB$ by 90° in anti-clockwise direction about B .

Let P rotate to Q , A rotate to E .

$\triangle APB \cong \triangle EQB$ (by construction)

$EQ = a$, $BQ = 2a = PB$. Join AQ .

$\angle PBQ = 90^\circ$ (Rotation)

$\angle ABQ = 90^\circ - \angle ABP = \angle PBC$

$AB = BC$ (sides of a square)

$\triangle ABQ \cong \triangle CBP$ (S.A.S.)

$AQ = CP = 3a$ (corr. sides \cong \triangle s)

$\therefore \angle PBQ = 90^\circ$ (Rotation)

$\therefore PQ^2 = PB^2 + QB^2$ (Pythagoras' Theorem)
 $= (2a)^2 + (2a)^2 = 8a^2$

$AP^2 + PQ^2 = a^2 + 8a^2 = 9a^2$

$AQ^2 = (3a)^2$

$\therefore AP^2 + PQ^2 = AQ^2$

$\angle APQ = 90^\circ$

$\therefore \angle PBQ = 90^\circ$ and $PB = QB$

$\therefore \angle BPQ = 45^\circ$

$\angle APB = 45^\circ + 90^\circ = 135^\circ$

