120° triangle

Given a triangle with one angle is 120°. If all sides are integers, find all possible solution.

This document was created by Mr. Francis Hung, last updated: July 1, 2023.

$c^2 = a^2 + b^2 - 2ab\cos 120°$

$c^2 = a^2 + b^2 + ab$

$c^2 = (a+b)^2 - ab$

$ab = (a+b)^2 - c^2$

$ab = (a+b+c)(a+b-c)$

$\dfrac{a+b+c}{a} = \dfrac{b}{a+b-c} = k$, where k is a positive constant.

$a + b + c = ak$; $b = (a + b - c)k$

$$\Rightarrow \begin{cases} a(1-k)+b+c=0\cdots\cdots(1) \\ ak+b(k-1)-ck=0\cdots\cdots(2) \end{cases}$$

From (1): $c = a(k-1) - b$ ....(3)

Sub. (3) into (2): $ak + b(k-1) - a(k^2-k) + bk = 0$

$b(2k-1) = a(k^2-2k)$

Let $a = (2k-1)p$, $b = (k^2-2k)p$, then $c = (k^2-k+1)p$; where p is a positive integer.

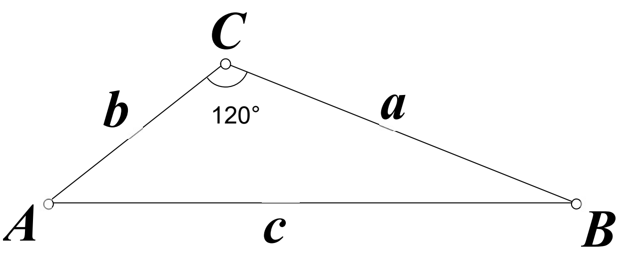
$a : b : c = (2k-1) : (k^2-2k) : (k^2-k+1)$

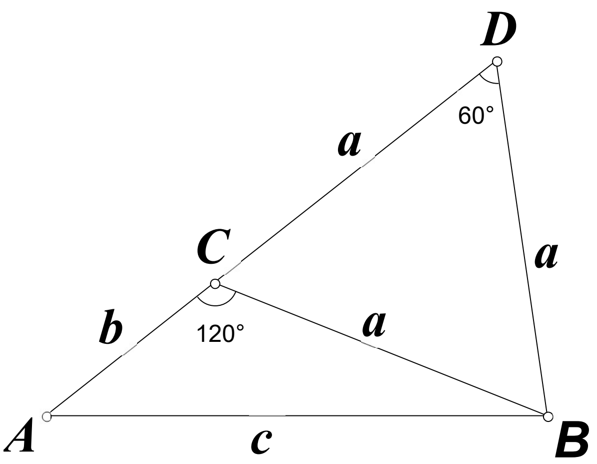
Let $a = (2k-1)p$, $b = k(k-2)p$, $c = (k^2-k+1)p$; where p is a positive integer.

k	p	a	b	c
3	1	5	3	7
4	1	7	8	13
5	1	9	15	21
6	1	11	24	31

Given a triangle with one angle is 60°. If all sides are integers, find all possible solution.

Given the above triangle with $\angle C = 120°$, we can construct another triangle ABD with $\angle D = 60°$





So, if (a, b, c) is a solution to a 120° triangle, then $(a, a+b, c)$ or $(a+b, b, c)$ is a solution to a 60° Δ.

The general solution are: $((2k-1)p, (k^2-1)p, (k^2-k+1)p)$ or $((k^2-1)p, (k^2-2k)p, (k^2-k+1)p)$

k	p	a	$a+b$	c	$a+b$	b	c
2	1	3	3	3			
3	1	5	8	7	8	3	7
4	1	7	15	13	15	8	13
5	1	9	24	21	24	15	21
6	1	11	35	31	35	24	31

"A cyclic quadrilateral with all 4 sides and 2 diagonals are integers."

(A) A cyclic quadrilateral with an equilateral triangle and another triangle.

If ΔABC is equilateral of side length a, ΔBCD with $\angle C = 120°$, $BC = c$, $CD = b$.

By the notes on 120°-triangle,

$c = (2k - 1)p$, $b = (k^2 - 2k)p$, $a = (k^2 - k + 1)p$

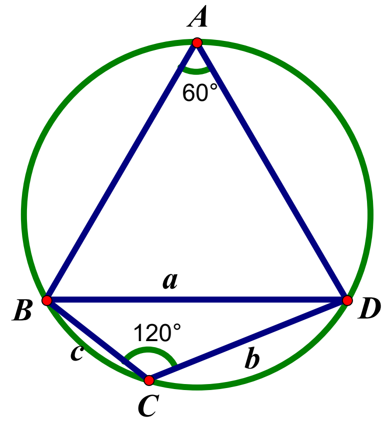
or $b = (2k - 1)p$, $c = (k^2 - 2k)p$, $a = (k^2 - k + 1)p$

Apply Ptolemy's theorem, $AC \times BD = ab + ac$

$\Rightarrow AC = b + c = (k^2 - 1)p$, which is a positive integer.

Hence, all 4 sides and the 2 diagonals are integers.

e.g. $k = 3$, $p = 1$, $AB = 7 = AD = BD$, $BC = 5$, $CD = 3$, $AC = 8$

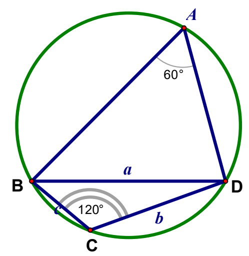


(B) A cyclic quadrilateral with $\angle A = 60°$, $\angle C = 120°$

By the notes on 60°-triangle,

$a = (k^2 - k + 1)p$, $AB = (2k - 1)p$, $AD = (k^2 - 1)p$

or $a = (k^2 - k + 1)p$, $AB = (k^2 - 1)p$, $AD = (k^2 - 2k)p$



Case 1 $a = (k^2-k+1)p$, $AB = (2k-1)p$, $AD = (k^2-1)p$, $b = (2k-1)p$, $c = (k^2-2k)p$

$\angle ADB = \angle CBD$ (eq. chords eq. $\angle$s)

$\therefore AD \parallel BC$ (alt. $\angle$s eq.)

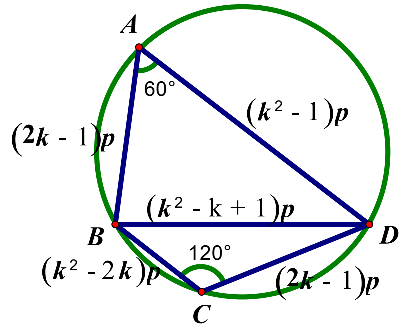
$\angle ABD = \angle ACD$, $\angle ABD = \angle CAD$ (eq. chords eq. $\angle$s)

$\Delta ABD \cong \Delta DCA$ (AAS)

$AC = BD$ (corr. sides $\cong \Delta$s)

Hence, all 4 sides and the 2 diagonals are integers.

e.g. $k = 3$, $p = 1$, $AB = 5$, $AD = 8$, $BD = 7 = AC$, $BC = 3$, $CD = 5$.



Case 2 $a = (k^2-k+1)p$, $AB = (2k-1)p$, $AD = (k^2-1)p$, $c = (2k-1)p$, $b = (k^2-2k)p$

Apply Ptolemy's Theorem,

$AC(k^2 - k + 1)p = (2k - 1)p \times (k^2 - 2k)p + (2k - 1)p \times (k^2 - 1)p$

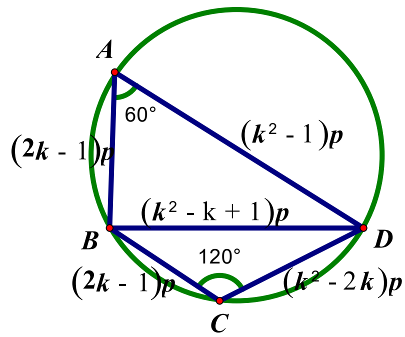
$AC = \dfrac{(2k-1)(2k^2-2k-1)}{k^2-k+1}p$, let $p = k^2 - k + 1$

$AB = (2k - 1)(k^2 - k + 1) = BC$, $CD = (k^2 - 2k)(k^2 - k + 1)$,

$AD = (k^2 - 1)(k^2 - k + 1)$, $AC = (2k - 1)(2k^2 - 2k - 1)$, $BD = (k^2-k+1)^2$

all 4 sides and the 2 diagonals are integers.

e.g. $k = 3$, $AB = 35 = BC$, $AD = 56$, $BD = 49$, $AC = 55$, $CD = 21$.



Case 3 When $BD = (k^2 - k + 1)p$, $AB = (k^2 - 1)p$, $AD = (k^2 - 2k)p$, $BC = (2k - 1)p$, $CD = (k^2 - 2k)p$

$AC = \dfrac{(k^2-2k)(k^2+2k-2)}{k^2-k+1}p$, let $p = k^2 - k + 1$, then all sides are integers.

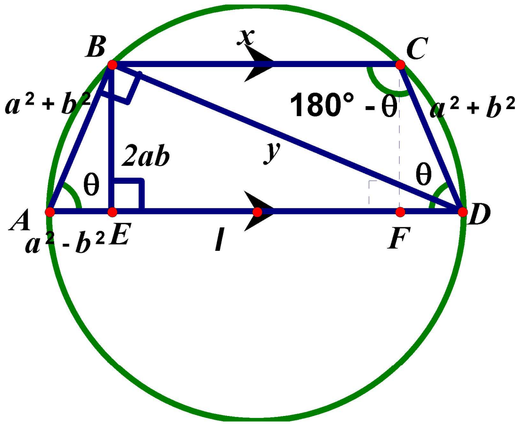
e.g. $k = 3$, $AB = 56$, $BC = 35$, $AD = 21$, $BD = 49$, $AC = 55$, $CD = 21$.

Case 4 When $BD = (k^2 - k + 1)p$, $AD = (k^2 - 1)p$, $AB = (k^2 - 2k)p$, $BC = (2k - 1)p$, $CD = (k^2 - 2k)p$

$AC = (k^2 - k + 1)p = BD$ (similar to Case 1), then all sides are integers.

e.g. $k = 3$, $p = 1$, $AB = 3$, $AD = 8$, $BD = 7 = AC$, $BC = 5$, $CD = 3$.

(C) A cyclic quadrilateral $ABCD$ with one pair of equal opposite sides $AB = CD$ and AD is the diameter.



$\angle ABD = 90°$ ($\angle$ in semi-circle)

Let $BC = x$, $BD = y = AC$, $AD = \ell$, $BE \perp AD$ (E lies on AD.)

$\angle BAD = \theta = \angle CDA$ (given $AB = CD$)

case 1 In ΔABE, $\angle AEB = 90°$, it can be proved that[*]

$BE = 2ab$, $AE = a^2 - b^2$, $AB = a^2 + b^2$

$\Delta ABE \sim \Delta ADB$ (equiangular)

$y : (a^2 + b^2) : \ell = 2ab : (a^2 - b^2) : (a^2 + b^2)$ (ratio of sides, ~Δ's)

$$y = \frac{2ab(a^2+b^2)}{(a^2-b^2)};\ \ell = \frac{(a^2+b^2)^2}{(a^2-b^2)}.$$

$\ell = AE + EF + FD = x + 2(a^2 - b^2) \Rightarrow x = \ell - 2(a^2 - b^2)$

$$x = \frac{(a^2+b^2)^2}{(a^2-b^2)} - 2(a^2-b^2) = \frac{(a^2+b^2)^2 - 2(a^2-b^2)^2}{(a^2-b^2)} = \frac{a^4+2a^2b^2+b^4-2(a^4-2a^2b^2+b^2)}{(a^2-b^2)} = \frac{6a^2b^2-a^4-b^4}{(a^2-b^2)}$$

$$x = \frac{4a^2b^2-(a^4-2a^2b^2+b^4)}{(a^2-b^2)} = \frac{(2ab)^2-(a^2-b^2)^2}{(a^2-b^2)} = \frac{(2ab+a^2-b^2)(2ab-a^2+b^2)}{(a^2-b^2)} = -\frac{(a^2+2ab-b^2)(a^2-2ab-b^2)}{(a^2-b^2)}$$

$\because x > 0;\ a > b\ \therefore -(a^2 + 2ab - b^2)(a^2 - 2ab - b^2) > 0$

$[(a + b)^2 - 2b^2][(a - b)^2 - 2b^2] < 0$

$(a+b+\sqrt{2}b)(a+b-\sqrt{2}b)(a-b+\sqrt{2}b)(a-b-\sqrt{2}b) < 0$

$(a+b-\sqrt{2}b)(a-b-\sqrt{2}b) < 0 \Rightarrow (\sqrt{2}-1)b < a < (\sqrt{2}+1)b$

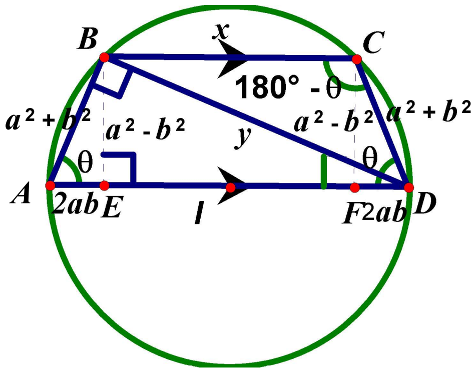
$a > b \Rightarrow b < a < (\sqrt{2}+1)b$ to order to ensure that $x > 0$

Multiply every side by $(a^2 - b^2)$,

$AB = a^4 - b^4 = CD$, $AD = (a^2+b^2)^2$, $BC = 6a^2b^2 - a^4 - b^4$, $AC = 2ab(a^2 + b^2) = BD$; then all sides are integers.

e.g. $a = 2$, $b = 1$, then $1 < 2 < (\sqrt{2}+1)$; $AB = 15 = CD$, $AD = 25$, $BC = 7$, $AC = 20 = BD$

case 2 $AE = 2ab$, $BE = a^2 - b^2$, $AB = a^2 + b^2$



$\Delta ABE \sim \Delta ADB$ (equiangular)

$y : (a^2 + b^2) : \ell = (a^2 - b^2) : 2ab : (a^2 + b^2)$ (ratio of sides, ~Δ's)

$$y = \frac{(a^2-b^2)(a^2+b^2)}{2ab};\ \ell = \frac{(a^2+b^2)^2}{2ab}.$$

$\ell = AE + EF + FD = x + 2(2ab) \Rightarrow x = \ell - 4ab$

$$x = \frac{(a^2+b^2)^2}{2ab} - 4ab = \frac{(a^2+b^2)^2 - 8a^2b^2}{2ab} = \frac{a^4+b^4-6a^2b^2}{2ab}$$

$(\sqrt{2}+1)b < a$ to order to ensure that $x > 0$

Multiply every side by $2ab$,

$AB = 2ab(a^2 + b^2) = CD$, $AD = (a^2+b^2)^2$, $BC = a^4+b^4 - 6a^2b^2$, $AC = a^4 - b^4 = BD$; then all sides are integers.

e.g. $a = 4$, $b = 1$, then $(\sqrt{2}+1) < 4$; $AB = 136 = CD$, $AD = 289$, $BC = 161$, $AC = 255 = BD$

[*] http://scicomp.sinaman.com/Number_Theory/Pythagorean_triple.pdf

(D) A cyclic quadrilateral with one pair of adjacent equal sides.

Follow the same steps in (C), $\angle ADB = 90° - \theta$ ($\angle$ sum of Δ)

$\angle BDC = \angle ADB - \angle BDC = \theta - (90° - \theta) = 2\theta - 90°$

Let G be a point on the circle so that $BD \perp GD$

$\angle AGD = 90°$ ($\angle$ in semi-circle)

$ABDG$ is a rectangle (there are 3 right angles)

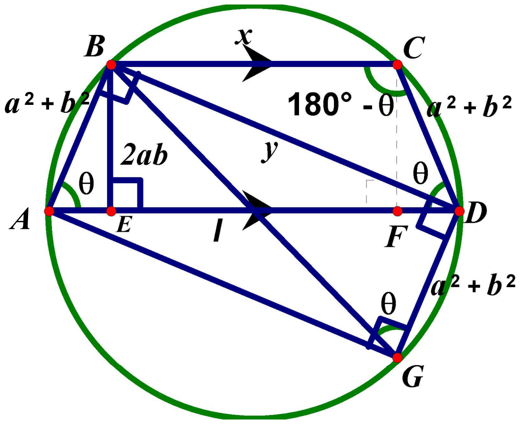
$BG = AD = \ell$ (diagonal of the rectangle)

$\angle BGD = \angle BAD = \theta$ ($\angle$s in the same segment)

$\angle CDG = \angle BCD + \angle BDG = 2\theta - 90° + 90° = 2\theta$

$GD = AB = a^2 + b^2$ (opp. sides of rectangle)

ΔCDG is isosceles



$CG = 2\ CD \sin\frac{1}{2}\angle CDG = 2(a^2 + b^2) \sin\theta = 2(a^2+b^2)\cdot\frac{2ab}{a^2+b^2} = 4ab$ (note that in ΔABE, $\sin\theta = \frac{2ab}{a^2+b^2}$)

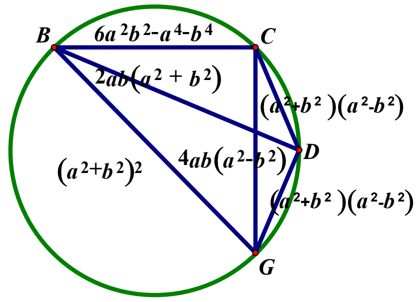
Multiply every side by $(a^2 - b^2)$, then

$CD = (a^2 + b^2)(a^2 - b^2) = DG$, $BC = 6a^2b^2 - a^4 - b^4$,

$BG = (a^2 + b^2)^2$, $BD = 2ab(a^2 + b)$, $CG = 4ab(a^2 - b^2)$.

All sides are positive integers.

e.g. $a = 2$, $b = 1$, $CD = 15 = DG$, $BC = 7$, $BG = 25$, $BD = 20$, $CG = 24$



(E) A cyclic quadrilateral $ABCD$ with $AD = BC$.

$\angle ACD = \theta = \angle BAC$ (eq. chords eq. $\angle$s)

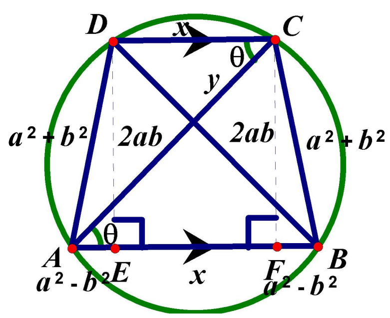
$AB \parallel DC$ (alt. $\angle$s eq.)

Let $DE \perp AB$, $CF \perp AB$, $CD = x = EF$, $AC = y = BD$.

case 1 Use the method in (C), $DE = CF = 2ab$, $AE = a^2 - b^2$, $AD = a^2 + b^2 = BC$

$AB = x + 2(a^2 - b^2)$

Apply cosine rule on ΔABC and ΔACD.



$$\cos\theta = \frac{\left[x + 2\left(a^2 - b^2\right)\right]^2 + y^2 - \left(a^2 + b^2\right)^2}{2\left[x + 2\left(a^2 - b^2\right)\right]y} = \frac{x^2 + y^2 - \left(a^2 + b^2\right)^2}{2xy}$$

$x[x^2 + 4(a^2 - b^2)x + 4(a^2 - b^2)^2] + xy^2 - (a^2 + b^2)^2x = x^2[x + 2(a^2 - b^2)] + y^2[x + 2(a^2 - b^2)] - (a^2 + b^2)^2[x + 2(a^2 - b^2)]$

$x^3 + 4(a^2 - b^2)x^2 + 4(a^2 - b^2)^2x - (a^2 + b^2)^2x = x^3 + 2(a^2 - b^2)x^2 + 2(a^2 - b^2)y^2 - (a^2 + b^2)^2x - 2(a^2 + b^2)^2(a^2 - b^2)$

$2(a^2 - b^2)x^2 + 4(a^2 - b^2)^2x + 2(a^2 + b^2)^2(a^2 - b^2) = 2(a^2 - b^2)y^2$

$x^2 + 2(a^2 - b^2)x + (a^2 + b^2)^2 = y^2$

$[x + (a^2 - b^2)]^2 + (a^2 + b^2)^2 - (a^2 - b^2)^2 = y^2$

$[x + (a^2 - b^2)]^2 = y^2 - 4a^2b^2 = (y + 2ab)(y - 2ab)$

$y^2 = (2ab)^2 + m^2$, where m is an integer. $2ab$, m, y forms a Pythagorean Triple, by the notes [*]

$2ab = 2rs$, $m = r^2 - s^2$, $y = r^2 + s^2$; $r > s > 0$ are integers.

$\frac{a}{r} = \frac{s}{b} = k$, $a = kr$, $b = \frac{s}{k}$

$x = -(a^2 - b^2) \pm \sqrt{(y + 2ab)(y - 2ab)}$, $\because x > 0$, $\therefore x = -(a^2 - b^2) + \sqrt{(y + 2ab)(y - 2ab)}$

$x = -(a^2 - b^2) + (r^2 - s^2) = (r^2 - a^2) + (b^2 - s^2) = \left(r^2 - k^2r^2\right) + \left(\frac{s^2}{k^2} - s^2\right) = \left(1 - k^2\right)\left(r^2 + \frac{s^2}{k^2}\right) > 0 \Rightarrow 0 < k < 1$

$s < b < a < r$; $ab = rs$; $AD = a^2 + b^2 = BC$, $AB = (r^2 - s^2) + (a^2 - b^2)$, $CD = (r^2 - a^2) + (b^2 - s^2)$, $AC = r^2 + s^2 = BD$

e.g. $a = 3$, $b = 2$, $s = 1$, $r = 6$, $AD = 13 = BC$, $CD = 30$, $AB = 40$, $AC = 37 = BD$

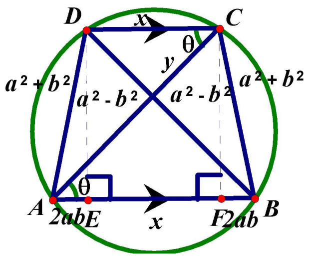
e.g. $a = 4$, $b = 3$, $s = 2$, $r = 6$, $AD = 25 = BC$, $CD = 25$, $AB = 39$, $AC = 40 = BD$

e.g. $a = 4$, $b = 2$, $s = 1$, $r = 8$, $AD = 20 = BC$, $CD = 51$, $AB = 75$, $AC = 65 = BD$

case 2 $DE = a^2 - b^2 = CF$, $AE = 2ab = BF$, $AD = a^2 + b^2 = BC$

$CD = x = EF$, $AC = y = BD$, $AB = x + 4ab$

Apply cosine rule on ΔABC and ΔACD.



$$\cos\theta = \frac{(x+4ab)^2 + y^2 - (a^2+b^2)^2}{2(x+4ab)y} = \frac{x^2 + y^2 - (a^2+b^2)^2}{2xy}$$

$x^3+8abx^2+16a^2b^2x+xy^2 - (a^2+b^2)^2x = x^3+4abx^2+xy^2+4aby^2 - (a^2+b^2)^2x - 4ab(a^2+b^2)^2$

$4abx^2 + 16a^2b^2x + 4ab(a^2 + b^2)^2 = 4aby^2$

$x^2 + 4abx + (a^2 + b^2)^2 = y^2$

$(x + 2ab)^2 = y^2 - (a^2 - b^2)^2 = m^2$, where m is an integer.

by the notes [*]: $y = r^2 + s^2$, $a^2 - b^2 = r^2 - s^2$, $m = 2rs$, where $r > s > 0$ are integers

$(a + b)(a - b) = (r + s)(r - s)$

$$\frac{a+b}{r+s} = \frac{r-s}{a-b} = k$$

$$a + b = k(r + s)\text{..........(1)},\ a - b = \frac{1}{k}(r-s)\text{..........(2)}$$

$$((1)+(2))\div 2:\ a = \frac{1}{2k}\left[(k^2+1)r + (k^2-1)s\right];\ ((1)-(2))\div 2:\ b = \frac{1}{2k}\left[(k^2-1)r + (k^2+1)s\right]$$

$$x = m - 2ab = 2rs - 2ab = 2\left\{rs - \frac{1}{2k}\left[(k^2+1)r + (k^2-1)s\right]\frac{1}{2k}\left[(k^2-1)r + (k^2+1)s\right]\right\}$$

$$x = 2\left\{rs - \frac{1}{4k^2}\left[k^2(r+s) + (r-s)\right]\left[k^2(r+s) - (r-s)\right]\right\} = \frac{2}{4k^2}\left\{4k^2rs - \left[k^4(r+s)^2 - (r-s)^2\right]\right\}$$

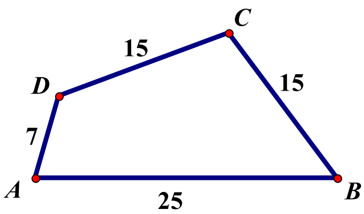
$$x = -\frac{1}{2k^2}\left[k^4(r+s)^2 - 4k^2rs - (r-s)^2\right] = -\frac{1}{2k^2}\left[k^2(r+s)^2 + (r-s)^2\right](k^2-1) > 0 \Rightarrow 0 < k < 1$$

$0 < \dfrac{a+b}{r+s} = \dfrac{r-s}{a-b} = k < 1 \Rightarrow$ $\boldsymbol{a + b < r + s}$, $\boldsymbol{r - s < a - b}$, $\boldsymbol{r > s}$, $\boldsymbol{a > b}$ and $\boldsymbol{(a + b)(a - b) = (r + s)(r - s)}$

$CD = 2(rs - ab)$, $AD = a^2 + b^2 = BC$, $AC = r^2 + s^2 = BD$, $AB = 2(ab + rs)$

e.g. $a = 7$, $b = 1$, $s = 4$, $r = 8$, $AD = 50 = BC$, $CD = 50$, $AB = 78$, $AC = 80 = BD$

Example 1 Given $ABCD$ is a quadrilateral such that $AB = 7$, $BC = CD = 15$, $DA = 25$. Find the maximum area of the quadrilateral.



By cosine law, $AC^2 = 7^2+15^2 - 2\cdot7\cdot15\cos D = 25^2+15^2 - 2\cdot25\cdot15\cos B$

$576 = 2(125\cos B - 105\cos D)$

$288 = 375\cos B - 105\cos D$ .... (1)

Let K = area of $ABCD$ = area of ΔABC + area of ΔACD

$$= \frac{1}{2}25\cdot15\sin B + \frac{1}{2}7\cdot15\sin D$$

$4K^2 = (375\sin B + 105\sin D)^2$ .... (2)

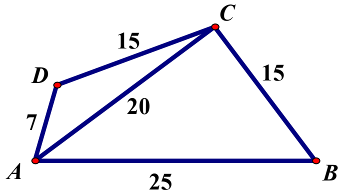
$(1)^2 + (2)$: $4K^2 + 288^2 = 375^2 + 105^2 - 2\times375\times105(\cos B\cos D - \sin B\sin D)$

$4K^2 = 68706 - 78750\cos(B + D)$

K is a maximum when $\cos(B + D)$ is a minimum

$-1 \le \cos(B + D) \le 1$, maximum area $= \dfrac{1}{2}\sqrt{68706+78750} = 192$

Example 2 Given $ABCD$ is a quadrilateral such that $AB = 7$, $BC = CD = 15$, $DA = 25$, $AC = 20$. Find BD.



$AC^2 + BC^2 = 20^2 + 15^2 = 25^2 = AB^2$, $\angle ACB = 90°$ (converse, Pyth. Thm)

In ΔACD, $\cos\angle ACD = \dfrac{15^2 + 20^2 - 7^2}{2\cdot15\cdot20} = \dfrac{24}{25}$

In ΔBCD, $BD^2 = 15^2 + 15^2 - 2\times15\times15\cos(90° + \angle ACD)$

$$BD^2 = 450 + 450\sin\angle ACD = 450\cdot\left[1 + \sqrt{1 - \left(\frac{24}{25}\right)^2}\right] = 450\cdot\left(1 + \frac{7}{25}\right) = 576$$

$BD = 24$