Example on Method of Bisection

Created by Mr. Francis Hung Last updated: July 1, 2023

A goat is tied up by a rope of length x which is posted at a point C on the circumference of a circular grassland with centre A, radius R. (There is no grass outside the circle.) What is the ratio of $x : R$ so that the goat can eat just half of the area of the circular grassland?

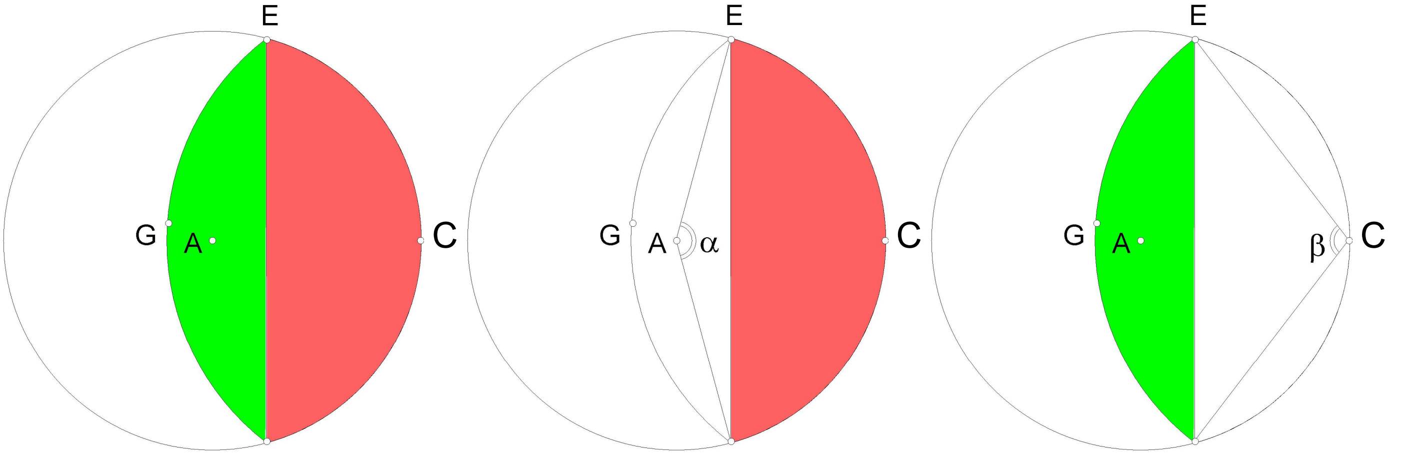

Figure 1 Figure 2 Figure 3

As shown in figure 1, the red and the green shaded area is just half the area of the circle. It is the sum of areas of the segment in figure 2 and the segment in figure 3.

Suppose the angle subtended by the segment ECF at the centre A is α, and the angle subtended by the segment EGF at the centre C is β. (α, β are in radians.)

Reflex $\angle$EAF $= 2\pi - \alpha$ ($\angle$s at a point)

Reflex $\angle$EAF $= 2\beta$ ($\angle$ at centre twice $\angle$ at circumference)

$2\pi - \alpha = 2\beta$

$\alpha = 2\pi - 2\beta$ …… (1)

$$\text{EF} = 2\text{AE} \sin\frac{\alpha}{2} = 2R\sin\frac{\alpha}{2}$$

$$\text{EF} = 2\text{CE} \sin\frac{\beta}{2} = 2x\sin\frac{\beta}{2}$$

$$\therefore R \sin\frac{\alpha}{2} = x\sin\frac{\beta}{2} \quad \text{…… (2)}$$

sub (1) into (2)

$$R \sin(\pi - \beta) = x\sin\frac{\beta}{2}$$

$$R\sin\beta = x\sin\frac{\beta}{2}$$

$$2R \sin\frac{\beta}{2}\cos\frac{\beta}{2} = x\sin\frac{\beta}{2}$$

$$2R \cos\frac{\beta}{2} = x$$

$$\frac{x}{R} = 2\cos\frac{\beta}{2} \quad \text{…… (3)}$$

Area of segment ECF = area of sector ECF – area of ΔEAF

$$= \frac{1}{2}R^2\alpha - \frac{1}{2}R^2 \sin\alpha \cdots\cdots (4)$$

Area of segment EGF = area of sector EGF – area of ΔECF

$$= \frac{1}{2}x^2\beta - \frac{1}{2}x^2 \sin\beta \cdots\cdots (5)$$

(4) + (5) = half of the area of the circle

$$\frac{1}{2}R^2\alpha - \frac{1}{2}R^2 \sin\alpha + \frac{1}{2}x^2\beta - \frac{1}{2}x^2 \sin\beta = \frac{1}{2}\pi R^2 \cdots\cdots (6)$$

sub (1) into (6), multiply by 2 and divide both sides by R^2 :

$$2\pi - 2\beta - \sin(2\pi - 2\beta) + \frac{x^2}{R^2}\beta - \frac{x^2}{R^2}\sin\beta = \pi \cdots\cdots (7)$$

Simplifying and sub (3) into (7)

$$\pi - 2\beta + \sin 2\beta + 4\beta\cos^2\frac{\beta}{2} - 4\sin\beta\cos^2\frac{\beta}{2} = 0$$

$$\pi + 2\beta\left(2\cos^2\frac{\beta}{2} - 1\right) + 2\sin\beta\cos\beta - 4\sin\beta\cos^2\frac{\beta}{2} = 0$$

$$\pi + 2\beta\cos\beta + 2\sin\beta\left(\cos\beta - 2\cos^2\frac{\beta}{2}\right) = 0$$

$$\pi + 2\beta\cos\beta - 2\sin\beta = 0 \cdots\cdots (8)$$

Let $f(\beta) = \pi + 2\beta\cos\beta - 2\sin\beta$

Clearly $f(\beta)$ is a continuous function.

$f(1) > 0$, $f(2) < 0 \Rightarrow$ there is a root between 1 and 2 .

x_n	$f(x_n)$
1.5	+
1.75	+
1.875	+
1.938	–
1.907	–
1.891	+
1.899	+
1.903	+
1.905	+

$\therefore$ 1.905 < root < 1.907

root ≈ 1.91 correct to 2 decimal places.

sub β = 1.91 into (3)

$$\frac{x}{R} = 2\cos\frac{1.91}{2} = 1.16 \text{ (correct to 2 decimal places)}$$