97-98 Individual	1	2	2	40	3	–12	4	3	5	$-\frac{1}{2}$
	6	466	7	19	8	3	9	2	10	744

97-98 Group	1	2	2	12	3	27	4	64	5	14
	6	14	7	$-\frac{1}{2}$	8	1	9	20	10	19

Individual Events

I1 Given that $x^3 - 5x^2 + 2x + 8$ is divisible by $(x - a)$ and $(x - 2a)$, where a is an integer, find the value of a.

Let $f(x) = x^3 - 5x^2 + 2x + 8$

$f(-1) = -1 - 5 - 2 + 8 = 0 \Rightarrow x + 1$ is a factor

$f(2) = 8 - 20 + 4 + 8 = 0 \Rightarrow x - 2$ is a factor

$f(x) = (x + 1)(x - 2)(x - 4)$

$a = 2$

I2 Given that 8, a, b form an A.P. and a, b, 36 form a G.P. If a and b are both positive numbers, find the sum of a and b.

$a = \frac{8+b}{2}$ … (1); $b^2 = 36a$ … (2)

Sub. (1) into (2): $b^2 = 18(8 + b)$

$b^2 - 18b - 144 = 0$

$(b + 6)(b - 24) = 0$

$b = -6$ (rejected) or $b = 24$

$a = \frac{8+24}{2} = 16$

$a + b = 40$

I3 Find the smallest real root of the following equation: $\frac{x}{(x-4)(x+3)} = \frac{x}{(x+4)(x-6)}$.

Reference: 1995 FI2.1

$x(x + 4)(x - 6) = x(x - 4)(x + 3)$

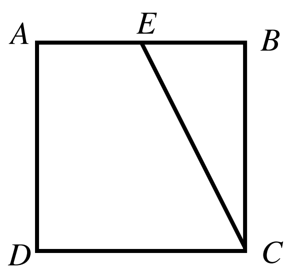
$x(x^2 - 2x - 24) = x(x^2 - x - 12)$

$0 = x(x + 12)$

$x = 0$ or -12

The smallest root $= -12$

I4 In figure 1, $ABCD$ is a square. E is a point on AB such that $BE = 1$ and $CE = 2$. Find the area of the square $ABCD$.



$BC^2 = 2^2 - 1^2$ (Pythagoras' theorem on ΔBCE)

Area of the square $= BC^2 = 3$

I5 If $2x+3=\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$, find the value of x.

$(2x+3)^2=2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}=2+(2x+3)$

$4x^2+12x+9=2x+5$

$4x^2+10x+4=0$

$2x^2+5x+2=0$

$(2x+1)(x+2)=0$

$x=-\frac{1}{2}$ or -2

$\because 2x+3=\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}>0$

$\therefore x\neq -2, x=-\frac{1}{2}$ only

I6 Given that n is a positive integer which is less than 1000. If n is divisible by 3 or 5, find the number of possible values of n. **(Reference: 1993 FG8.3-4, 1994 FG8.1-2, 2015 FI3.1)**

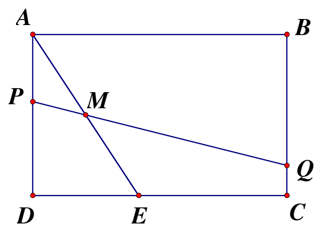
Number of multiples of 3 = 333

Number of multiples of 5 = 199

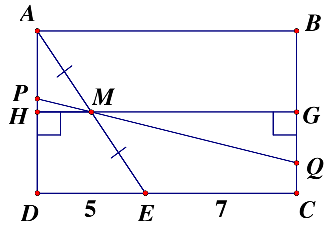
Number of multiples of 15 = 66

Number of possible n = 333 + 199 – 66 = 466

I7 In figure 2, $ABCD$ is a rectangle with $CD = 12$. E is a point on CD such that $DE = 5$. M is the mid-point of AE and P, Q are points on AD and BC respectively such that PMQ is a straight line. If $PM : MQ = 5 : k$, find the value of k.



Draw a straight line HMG // CD (H lies on AD, G lies on BC)



$AH = HD$ (Intercept theorem)

$\Delta PHM \sim \Delta QGM$ (equiangular)

$\Delta AHM \sim \Delta ADE$ (equiangular)

$PM : MQ = HM : MG$ (ratio of sides, ~Δ's)

$=\frac{1}{2}DE : (HG - HM)$ (ratio of sides, ~Δ's)

$= 2.5 : (12 - 2.5) = 5 : 19$ (opp. sides, rectangle)

$k = 19$

I8 Find the last digit of the value of $6^{20} - 5^{12} - 8$.

$6^1 = 6$, $6^2 = 36$, …, the last digit of 6^{20} is 6, the last digit of 5^{12} is 5.

The last digit of the number is $6 - 5 - 8 \pmod{10} = -7 = 3 \pmod{10}$

I9 Let a be the positive root of the equation $\sqrt{\frac{x+2}{x-1}}+\sqrt{\frac{x-1}{x+2}}=\frac{5}{2}$, find the value of a.

Cross multiplying: $2(x+2+x-1)=5\sqrt{(x-1)(x+2)}$

$4(4x^2+4x+1)=25(x^2+x-2)$

$\Rightarrow 9x^2+9x-54=0$

$\Rightarrow 9(x-2)(x+3)=0$

$\Rightarrow a=x=2$

I10 Find the sum of all positive factors of 240.

Reference 1993 HI8, 1994 FI3.2, 1997 HI3, 1998 FI1.4, 2002 FG4.1, 2005 FI4.4

$240 = 2^4\times3\times5$

Positive factors are in the form $2^a3^b5^c$, $0 \le a \le 4$, $0 \le b, c \le 1$, a, b, c are integers.

Sum of positive factors = $(1+2+2^2+2^3+2^4)(1+3)(1+5) = 31\times4\times6 = 744$

Group Events

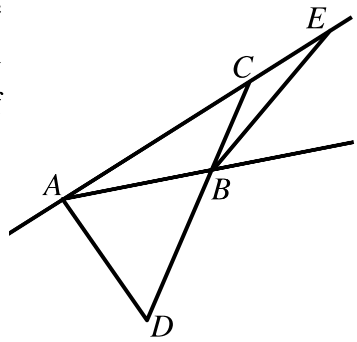
G1 If $x+\frac{1}{x}=2$, find the value of $x^3+\frac{1}{x^3}$.

Reference: 1984 FG10.2

$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2=2$$

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)=2\times(2-1)=2$$

G2 In Figure 1, ABC is a triangle. AD and BE are the bisectors of the exterior angles A and B respectively meeting CB and AC produced at D and E. Let $AD = BE = AB$ and $\angle BAC = a°$. Find the value of a. **Reference: 1986 上海市初中數學競賽**



$\angle BAD=\frac{180^\circ-a^\circ}{2}=90^\circ-\frac{a^\circ}{2}$ (adj. $\angle$s on st. line, $\angle$ bisector)

$\angle ABD=\frac{180^\circ-\angle BAD}{2}=45^\circ+\frac{a^\circ}{4}$ ($\angle$ sum of ΔABD, base $\angle$s isos. Δ)

$\angle CBE=\frac{\angle ABD}{2}=22.5^\circ+\frac{a^\circ}{8}$ (vert. opp. $\angle$s, bisector)

$\angle ABE = \angle ABC + \angle CBE = 180° - \angle ABD + \angle CBE$

$=135^\circ-\frac{a^\circ}{4}+22.5^\circ+\frac{a^\circ}{8}=157.5^\circ-\frac{a^\circ}{8}$

$\angle AEB = a°$ (base $\angle$s isosceles Δ)

$a°+a°+157.5^\circ-\frac{a^\circ}{8}=180°$ ($\angle$ sum of ΔABE)

$a = 12$

G3 If $-6 \le a \le 4$ and $3 \le b \le 6$, find the greatest value of $a^2 - b^2$.

$0 \le a^2 \le 36$ and $9 \le b^2 \le 36$

$-36 \le a^2 - b^2 \le 27$

$\Rightarrow$ The greatest value = 27.

G4 Let a, b, c be integers such that $a^2 = b^3 = c$. If $c > 1$, find the smallest value of c.

Reference: 1999 FG3.1, 2021 P2Q4

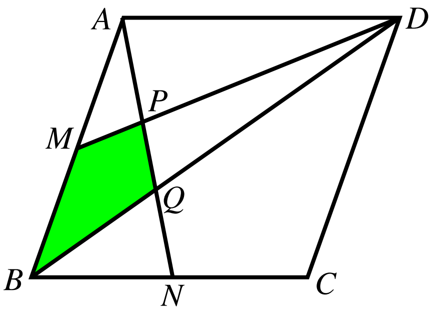
Let $a = k^3$, $b = k^2$, $c = k^6$

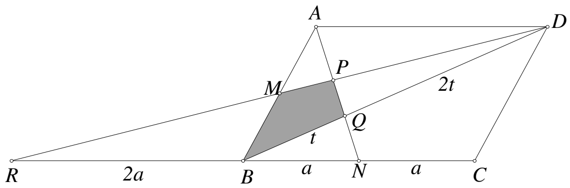
$c > 1 \Rightarrow k > 1$

The smallest $k = 2$

$\Rightarrow$ The smallest $c = 2^6 = 64$

G5 In figure 2, the area of the parallelogram $ABCD$ is 120. M and N are the mid-points of AB and BC respectively. AN intersects MD and BD at points P and Q respectively. Find the area of $BQPM$. (**Reference: 2016HI14, 2019 HI11**)





Produce DM and CB to meet at R.

Let $BC = 2a$. Then $BN = NC = a$ (mid-point)

$\Delta AQD \sim \Delta BQN$ (equiangular)

$\frac{BQ}{QD} = \frac{BN}{AD}$ (ratio of sides, ~Δ's)

$= \frac{1}{2}$ (N = mid-point, opp. sides of //-gram)

Area of $\Delta ABD = \frac{1}{2} \times 120 = 60$

Area of $\Delta AQD = \frac{2}{3} \times \Delta \text{ABD} = 40$

$\frac{\text{Area of } \Delta\text{BQN}}{\text{Area of } \Delta\text{AQD}} = \left(\frac{BN}{AD}\right)^2 = \frac{1}{4}$

$\therefore$ Area of $\Delta BQN = \frac{1}{4} \times 40 = 10$ ……..(1)

As M is the mid-point, $\Delta AMD \cong \Delta BMR$ (ASA)

$\Rightarrow RM = MD$ (corr. sides ≅ Δ's) ……………(2)

Also $\Delta APD \sim \Delta NPR$ (equiangular)

$\frac{DP}{PR} = \frac{AD}{NR}$ (ratio of sides, ~Δ's)

$= \frac{2a}{3a} = \frac{2}{3}$ (opp. sides of //-gram, corr. sides ≅ Δ's) ………..(3)

Combine (2) and (3)

$PD = \frac{2}{5} RD$; $MD = \frac{1}{2} RD$

$MP = MD - PD = \frac{1}{2} RD - \frac{2}{5} RD = \frac{1}{10} RD$

$\Rightarrow \frac{MP}{PD} = \frac{\frac{1}{10} RD}{\frac{2}{5} RD} = \frac{1}{4}$ ……………..(4)

Area of $\Delta AMD = \frac{1}{4} \times 120 = 30$

By (4): Area of $\Delta AMP = \frac{1}{5} \times \text{Area of } \Delta\text{AMD} = \frac{1}{5} \times 30 = 6$ …………(5)

Area of $\Delta ABN = \frac{1}{4} \times 120 = 30$

$\therefore$ Area of $BQPM$ = Area of ΔABN – Area of ΔAMP – Area of ΔBQN

$= 30 - 6 - 10 = 14$ (by (1) and (5))

G6 In figure 3, find the number of possible paths from point A to point B following the direction of arrow heads.

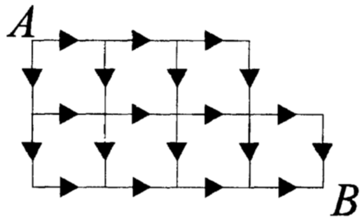
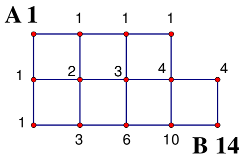


Figure 3

Reference 1983 FI4.1, 2000 HI4, 2007 HG5

The numbers at each of the vertices of in the following figure show the number of possible ways.



So the total number of ways = 14

G7 Find the smallest real root of the equation $(x-2)(2x-1)=5$.

$2x^2-5x-3=0$

$(2x+1)(x-3)=0$

$x=-\frac{1}{2}$ or 3

The smallest real root is $-\frac{1}{2}$.

G8 In figure 4, four circles with radius 1 touch each other inside a square. Find the shaded area. (Correct your answer to the nearest integer.)

The line segments joining the four centres form a square of sides = 2

Shaded area = $2^2-\pi\cdot 1^2\approx 1$

G9 In figure 5, $ABCD$ is a square and points E, F, G, H are the mid-points of sides AB, BC, CD, DA respectively, find the number of right-angled triangles in the figure. **(Reference: 1995 HG9)**

Let the shortest side of the smallest right-angled triangle be 1.

Then $AE=\sqrt{2}$, $EH=2$, $AB=2\sqrt{2}$, $AC=4$

We count the number of right-angled triangles with different hypotenuses.

Hypotenuse	Number of triangles
$\sqrt{2}$	8
2	4
$2\sqrt{2}$	4
4	4

Total number of triangles = 20

A H D

E G

B F C

G10 A test is composed of 25 multiple-choice questions. 4 marks will be awarded for each correct answer and 1 mark will be deducted for each incorrect answer. A pupil answered all questions and got 70 marks. How many questions did the pupil answer correctly?

Reference: 1994 FI1.2

Suppose he answer x questions correctly and $25-x$ question wrongly.

$4\cdot x-(25-x)=70$

$x=19$