

It is given that a triangle with one angle is  $\theta$ , where  $\cos \theta = \frac{3}{5}$ . If all sides are integers, find all solutions.

$$c^2 = a^2 + b^2 - 2ab \cos \theta \Rightarrow c^2 = a^2 + b^2 - 2ab \times \frac{3}{5}$$

$$5c^2 = 5a^2 - 10ab + 5b^2 + 4ab$$

$$5[c^2 - (a - b)^2] = 4ab$$

$$\frac{5(c + a - b)}{b} = \frac{4a}{c - a + b} = k$$

$$c + a - b = \frac{bk}{5} \dots\dots(1) \text{ and } c - a + b = \frac{4a}{k} \dots\dots(2)$$

$$[(1) - (2)] \div 2: a - b = \frac{bk}{10} - \frac{2a}{k} = \frac{bk^2 - 20a}{10k}$$

$$10ka - 10kb = bk^2 - 20a$$

$$(10k + 20)a = (k^2 + 10k)b$$

$$\frac{a}{b} = \frac{k^2 + 10k}{10k + 20}$$

$$\text{Let } a = (k^2 + 10k)p, b = (10k + 20)p$$

$$c = b - a + \frac{bk}{5} = [10k + 20 - k^2 - 10k + (2k + 4)k]p = (k^2 + 4k + 20)p. \text{ Let } p = 1.$$

$k$	$a = k^2 + 10k$	$b = 10k + 20$	$c = k^2 + 4k + 20$	Remark
1	11	30	25	
2	24	40	32	3-4-5
3	39	50	41	
4	56	60	52	13-14-15

It is given that a triangle with one angle is  $\theta$ , where  $\cos \theta = \frac{4}{5}$ . If all sides are integers, find all solutions.

$$c^2 = a^2 + b^2 - 2ab \cos \theta \Rightarrow c^2 = a^2 + b^2 - 2ab \times \frac{4}{5}$$

$$5c^2 = 5a^2 - 10ab + 5b^2 + 2ab$$

$$5[c^2 - (a - b)^2] = 2ab$$

$$\frac{5(c + a - b)}{b} = \frac{2a}{c - a + b} = k$$

$$c + a - b = \frac{bk}{5} \dots\dots(1) \text{ and } c - a + b = \frac{2a}{k} \dots\dots(2)$$

$$[(1) - (2)] \div 2: a - b = \frac{bk}{10} - \frac{a}{k} = \frac{bk^2 - 10a}{10k}$$

$$10ka - 10kb = k^2b - 10a$$

$$(10k + 10)a = (k^2 + 10k)b$$

$$\frac{a}{b} = \frac{k^2 + 10k}{10k + 10}$$

$$\text{Let } a = (k^2 + 10k)p, b = (10k + 10)p$$

$$c = b - a + \frac{bk}{5} = [10k + 10 - k^2 - 10k + (2k + 2)k]p = (k^2 + 2k + 10)p. \text{ Let } p = 1.$$

$k$	$a = k^2 + 10k$	$b = 10k + 10$	$c = k^2 + 2k + 10$	Remark
1	11	20	13	
2	24	30	18	3-4-5
3	39	40	25	
4	56	50	34	17-25-28
5	75	60	45	3-4-5
6	96	70	58	29-35-48

It is given that a triangle with one angle is  $\theta$ , where  $\cos \theta = \frac{7}{25}$ . If all sides are integers, find all solutions.

$$c^2 = a^2 + b^2 - 2ab \cos \theta \Rightarrow c^2 = a^2 + b^2 - 2ab \times \frac{7}{25}$$

$$25c^2 = 25a^2 - 50ab + 25b^2 + 36ab$$

$$25[c^2 - (a - b)^2] = 36ab$$

$$\frac{25(c + a - b)}{b} = \frac{36a}{c - a + b} = k$$

$$c + a - b = \frac{bk}{25} \dots\dots(1) \text{ and } c - a + b = \frac{36a}{k} \dots\dots(2)$$

$$[(1) - (2)] \div 2: a - b = \frac{bk}{50} - \frac{18a}{k} = \frac{bk^2 - 900a}{50k}$$

$$50ka - 50kb = k^2b - 900a$$

$$(50k + 900)a = (k^2 + 50k)b$$

$$\frac{a}{b} = \frac{k^2 + 50k}{50k + 900}$$

$$\text{Let } a = (k^2 + 50k)p, b = (50k + 900)p$$

$$c = b - a + \frac{bk}{25} = [50k + 900 - k^2 - 50k + (2k + 36)k]p = (k^2 + 36k + 900)p. \text{ Let } p = 1.$$

$k$	$a = k^2 + 50k$	$b = 50k + 900$	$c = k^2 + 36k + 900$	Remark
1	51	950	937	
2	104	1000	976	13-122-125
3	159	1050	1017	
4	216	1100	1060	54-275-265
5	275	1150	1105	55-221-230
6	336	1200	1152	7-24-25
7	399	1250	1201	
8	464	1300	1252	116-313-325
9	531	1350	1305	59-145-150
10	600	1400	1360	15-34-35
11	671	1450	1417	
12	744	1500	1476	62-123-125
13	819	1550	1537	
14	896	1600	1600	14-25-25
15	975	1650	1665	65-110-111
16	1056	1700	1732	264-425-433
17	1139	1750	1801	
18	1224	1800	1872	17-25-26
19	1311	1850	1945	
20	1400	1900	2020	70-95-101

It is given that the three sides of a triangle are positive integers.

Let  $\theta$  be an acute angle such that  $\sin \theta = \frac{p}{r}$ ,  $\cos \theta = \frac{q}{r}$  and  $\tan \theta = \frac{p}{q}$  are rational numbers

in the simplest forms.

Then  $(p, q, r)$  is a primitive Pythagorean triple.  $(p, q, r) = (2uv, u^2 - v^2, u^2 + v^2)$  or  $(u^2 - v^2, 2uv, u^2 + v^2)$ .

Let  $ABC$  be a triangle ( $BC = a$ ,  $AC = b$ ,  $AB = c$ ).  $\angle ACB = \theta$ .

Area of  $\triangle ABC = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B$ , which must be rational.

Denote the area of  $\triangle ABC$  by  $S$ , then  $\sin A = \frac{2S}{bc}$ ,  $\sin B = \frac{2S}{ca}$  and  $\sin C = \frac{2S}{ab}$  which are also rational.

By cosine formula,  $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ ,  $\cos B = \frac{c^2 + a^2 - b^2}{2ac}$ ,  $\cos C = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow \cos A, \cos B, \cos C \in \mathbb{Q}$ .

$\tan A = \frac{\sin A}{\cos A}$ ,  $\tan B = \frac{\sin B}{\cos B}$ ,  $\tan C = \frac{\sin C}{\cos C}$  which are still rational. Find all integral solutions for  $a, b$  and  $c$ .

$$c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow c^2 = a^2 + b^2 - 2ab \times \frac{q}{r}$$

**Case 1**  $q = 2uv$

$$(u^2 + v^2)c^2 = (u^2 + v^2)a^2 + (u^2 + v^2)b^2 - 4abuv$$

$$(u^2 + v^2)[c^2 - (a^2 - 2ab + b^2)] = -4abuv + 2ab(u^2 + v^2)$$

$$(u^2 + v^2)(c + a - b)(c - a + b) = 2ab(u - v)^2$$

$$\frac{(u^2 + v^2)(c + a - b)}{b} = \frac{2(u - v)^2 a}{c - a + b} = k$$

$$c + a - b = \frac{bk}{u^2 + v^2} \dots\dots(1) \text{ and } c - a + b = \frac{2(u - v)^2 a}{k} \dots\dots(2)$$

$$(1) - (2): 2(a - b) = \frac{bk}{u^2 + v^2} - \frac{2(u - v)^2 a}{k}$$

$$2(a - b)(u^2 + v^2)k = bk^2 - 2a(u - v)^2(u^2 + v^2)$$

$$2(u^2 + v^2)ka + 2(u - v)^2(u^2 + v^2)a = 2(u^2 + v^2)kb + k^2b$$

$$\frac{a}{b} = \frac{(2u^2 + 2v^2 + k)k}{2(u^2 + v^2)[(u - v)^2 + k]} \Rightarrow a = (2u^2 + 2v^2 + k)k, b = 2(u^2 + v^2)[(u - v)^2 + k].$$

$$\begin{aligned} \text{Sub. into (2): } c &= (2u^2 + 2v^2 + k)k - 2(u^2 + v^2)[(u - v)^2 + k] + 2(u - v)^2(2u^2 + 2v^2 + k) \\ &= k^2 + (2u^2 + 2v^2)k - 2(u^2 + v^2)(u - v)^2 - 2(u^2 + v^2)k + 4(u - v)^2(u^2 + v^2) + 2(u - v)^2k \\ &= k^2 + 2(u - v)^2k + 2(u - v)^2(u^2 + v^2) = k^2 + 2(u - v)^2(u^2 + v^2 + k) \end{aligned}$$

$k$	$u$	$v$	$a = (2u^2 + 2v^2 + k)k$	$b = 2(u^2 + v^2)[(u - v)^2 + k]$	$c = k^2 + 2(u - v)^2(u^2 + v^2 + k)$	$\cos C = \frac{2uv}{u^2 + v^2}$
1	2	1	11	20	13	$\frac{4}{5}$
1	3	1	21	100	89	$\frac{6}{10} = \frac{3}{5}$
1	3	2	27	52	29	$\frac{12}{13}$
1	4	1	35	340	325	$\frac{8}{17}$
1	4	3	51	100	53	$\frac{24}{25}$
3	2	1	39	40	25	$\frac{4}{5}$

Note that  $u > v$  are relatively prime positive integers.  $\cos C = \frac{2uv}{u^2 + v^2} > 0$  and so  $\angle C$  must be acute.

$$\text{Area of } \triangle ABC = \frac{1}{2}(2u^2 + 2v^2 + k)k \cdot 2(u^2 + v^2) \left[ (u-v)^2 + k \right] \cdot \frac{u^2 - v^2}{u^2 + v^2} = k(u^2 - v^2)(2u^2 + 2v^2 + k)[(u-v)^2 + k]$$

$$\text{Case 2 } p = 2uv, \cos \theta = \frac{q}{r} = \frac{u^2 - v^2}{u^2 + v^2}$$

$$c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow c^2 = a^2 + b^2 - 2ab \cdot \frac{u^2 - v^2}{u^2 + v^2}$$

$$(u^2 + v^2)c^2 = (u^2 + v^2)(a^2 - 2ab + b^2) + 2ab(u^2 + v^2) - 2ab(u^2 - v^2)$$

$$(u^2 + v^2)(c + a - b)(c - a + b) = 4v^2ab$$

$$\frac{(u^2 + v^2)(c + a - b)}{b} = \frac{4av^2}{c - a + b} = k$$

$$c + a - b = \frac{bk}{u^2 + v^2} \dots\dots(1) \text{ and } c - a + b = \frac{4av^2}{k} \dots\dots(2)$$

$$(1) - (2): 2(a - b) = \frac{bk}{u^2 + v^2} - \frac{4av^2}{k}$$

$$2k(u^2 + v^2)a - 2k(u^2 + v^2)b = k^2b - 4v^2(u^2 + v^2)a$$

$$2(u^2 + v^2)(k + 2v^2)a = [k^2 + 2(u^2 + v^2)k]b$$

$$\frac{a}{b} = \frac{k^2 + 2(u^2 + v^2)k}{2(u^2 + v^2)(k + 2v^2)} \Rightarrow a = k^2 + 2(u^2 + v^2)k, b = 2(u^2 + v^2)(k + 2v^2)$$

$$\text{Sub. into (2): } c = \frac{k^2 + 2(u^2 + v^2)k - 2(u^2 + v^2)(k + 2v^2) + 4v^2[k + 2(u^2 + v^2)]}{k} \\ = k^2 + 4v^2(u^2 + v^2) + 4v^2k = k^2 + 4v^2(u^2 + v^2 + k)$$

$k$	$u$	$v$	$a = (2u^2 + 2v^2 + k)k$	$b = 2(u^2 + v^2)(2v^2 + k)$	$c = k^2 + 4v^2(u^2 + v^2 + k)$	$\cos C = \frac{u^2 - v^2}{u^2 + v^2}$
1	1	2	11	90	97	$-\frac{3}{5}$
1	2	1	11	30	25	$\frac{3}{5}$
1	2	3	27	494	505	$-\frac{5}{13}$
1	3	2	27	234	225	$\frac{5}{13}$
1	4	1	35	102	73	$\frac{15}{17}$
1	4	3	51	950	937	$\frac{7}{25}$
3	2	1	39	50	41	$\frac{3}{5}$

Note that  $u, v$  are distinct relatively prime positive integers.

If  $u > v$ , then  $\cos C = \frac{u^2 - v^2}{u^2 + v^2} > 0$  and so  $C$  is acute. If  $u < v$ , then  $\cos C = \frac{u^2 - v^2}{u^2 + v^2} < 0$  and so  $C$  is obtuse.

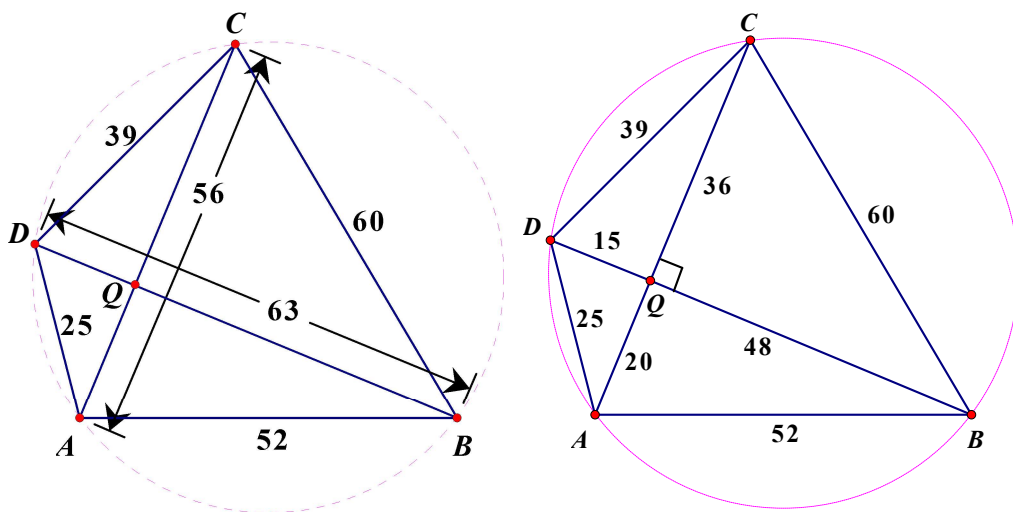
$$\text{Area of } \triangle ABC = \frac{1}{2}ab \sin C = \frac{1}{2}(2u^2 + 2v^2 + k)k \cdot 2(u^2 + v^2)(2v^2 + k) \cdot \frac{2uv}{u^2 + v^2}$$

$$= 2kuv(2u^2 + 2v^2 + k)(2v^2 + k), \text{ which is an integer}$$

Given that the three sides of a triangle  $ACB$  are positive integers.  $\angle ACB = \theta$  such that  $\sin \theta = \frac{p}{r}$ ,  $\cos \theta = \frac{q}{r}$  and  $\tan \theta = \frac{p}{q}$  are all rational numbers. If  $(p, q, r)$  is a Pythagorean triple, from the above analysis,  $(a, b, c) = ((2u^2 + 2v^2 + k)k, 2(u^2 + v^2)[(u-v)^2 + k], k^2 + 2(u-v)^2(u^2 + v^2 + k))$  or  $((2u^2 + 2v^2 + k)k, 2(u^2 + v^2)(2v^2 + k), k^2 + 4v^2(u^2 + v^2 + k))$  where  $k \in \mathbb{Z}^+$ ,  $u, v$  are distinct relatively prime positive integers.

In this section, we are going to find a cyclic quadrilateral for which all sides and all diagonals are integers.

Idea: In February 2008, I asked Dr. Man Keung Siu from the University of Hong Kong about how to find a solution to the above question. He quoted a paper “Normal Trigrade and cyclic quadrilateral with integral sides and diagonals” from April, 1951 American Mathematical Monthly. I didn’t understand the content of the paper. However, the author gave an example of integral cyclic quadrilateral  $ABCD$ ,  $AB = 52$ ,  $BC = 60$ ,  $CD = 39$ ,  $DA = 25$ ,  $AC = 56$ ,  $BD = 63$ .



It takes me years of time to investigate how to find another integral cyclic quadrilateral. Still, I failed to find any other solutions. After retirement, I use cosine formula to find  $\cos \angle ACB$ .

$$\cos \angle ACB = \frac{56^2 + 60^2 - 52^2}{2 \times 56 \times 60} = \frac{3}{5} \quad \text{and so } \sin \angle ACB = \frac{4}{5}, \tan \angle ACB = \frac{4}{3} \quad \text{which are all rational.}$$

I started to investigate a triangle with integral sides and rational sines of each angle.

Suppose the diagonals intersect at  $Q$ . Let  $\angle BQC = \theta$ .

$$\text{It can be proved that } \tan \theta = \frac{4\sqrt{(s-a)(s-b)(s-c)(s-d)}}{a^2 + c^2 - b^2 - d^2},$$

$$\text{where } a = 52, b = 60, c = 39, d = 25, s = \frac{1}{2}(a + b + c + d).$$

$$a^2 + c^2 - b^2 - d^2 = 52^2 + 39^2 - 60^2 - 25^2 = 0 \Rightarrow \text{denominator} = 0 \Rightarrow \theta = 90^\circ$$

$$CQ = BC \cos \angle BCQ = 60 \times \frac{3}{5} = 36, BQ = BC \sin \angle BCQ = 48. \triangle BCQ \text{ is a 3-4-5 } \Delta.$$

$$\therefore DQ = BD - BQ = 63 - 48 = 15, AQ = AC - CQ = 56 - 36 = 20.$$

$$\triangle ABQ \text{ is a 5-12-13 } \Delta. \triangle ADQ \text{ is a 3-4-5 } \Delta. \triangle CDQ \text{ is a 5-12-13 } \Delta.$$

This is a **special case** of integral cyclic quadrilateral.

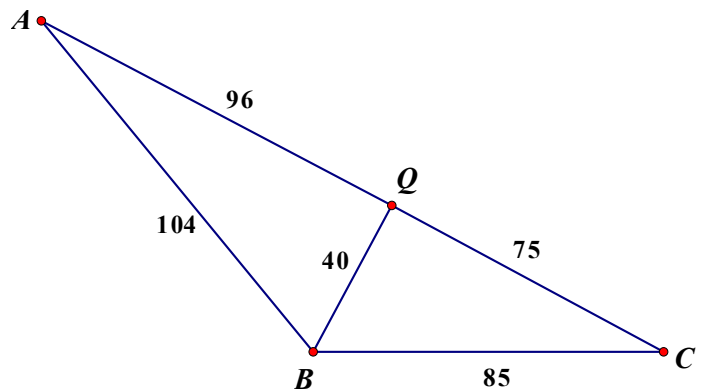
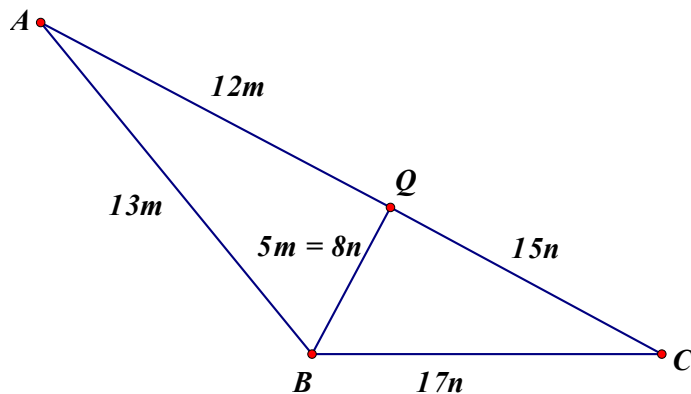
Will there be any other example(s) of integral cyclic quadrilateral  $ABCD$  with perpendicular diagonals?

$\angle ACB = \angle ADB$ ,  $\angle CAD = \angle CBD$ ,  $\angle ACD = \angle ABD$ ,  $\angle BDC = \angle BAC$  ( $\angle$ s in the same segment)

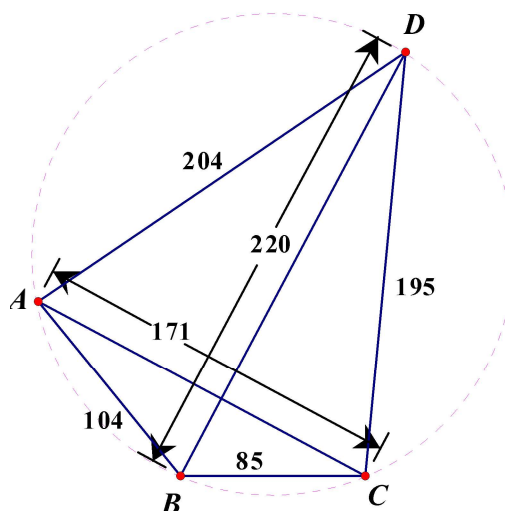
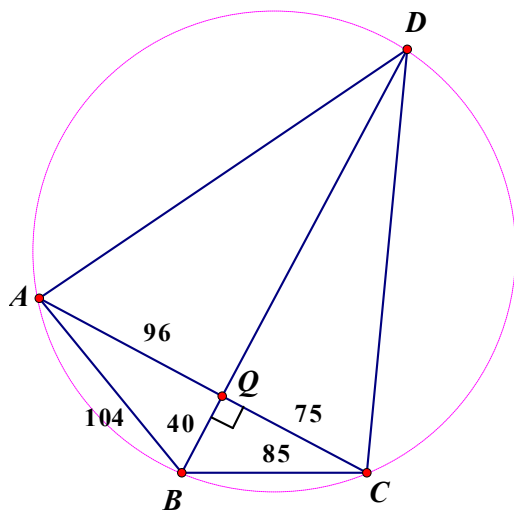
$\therefore \triangle ABQ \sim \triangle DCQ$ ,  $\triangle BCQ \sim \triangle ADQ$  (equiangular)

We try to find **two pairs of right-angled triangles with common sides**.

If  $\triangle ABQ$  is a 5-12-13  $\Delta$ ,  $\triangle BCQ$  is a 8-15-17  $\Delta$ .  $AQ = 12m$ ,  $BQ = 5m$ ,  $AB = 13m$ ,  $BQ = 8n$ ,  $CQ = 15n$ ,  $BC = 17n$ .



$BC = 5m = 8n$ , let  $m = 8$ ,  $n = 5$ , then  $AB = 13 \times 8 = 104$ ,  $AQ = 12 \times 8 = 96$ ,  $BC = 17 \times 5 = 85$ ,  $CQ = 15 \times 5 = 75$



Construct the circumscribed circle  $ABC$ . Extend  $BQ$  to cut the circumscribed circle at  $D$ .

$\triangle DCQ \sim \triangle ABQ$  which are 5-12-13  $\Delta$ s.  $DQ = \frac{75}{5} \times 12 = 180$ ,  $CD = \frac{75}{5} \times 13 = 195$ .

$\triangle ADQ \sim \triangle BCQ$  which are 8-15-17  $\Delta$ s.  $DQ = \frac{96}{8} \times 15 = 180$ ,  $AD = \frac{96}{8} \times 17 = 204$ .

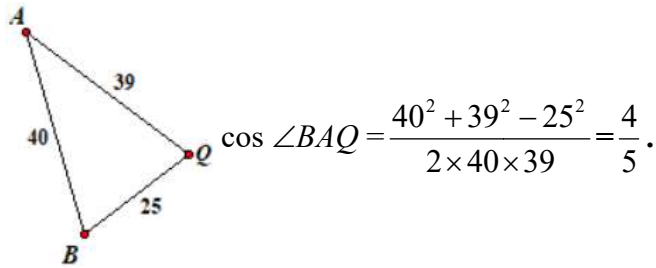
$\therefore ABCD$  is another integral cyclic quadrilateral with

$AB = 104$ ,  $BC = 85$ ,  $CD = 195$ ,  $DA = 204$ ,  $AC = 96 + 75 = 171$ ,  $BD = 40 + 180 = 220$ .

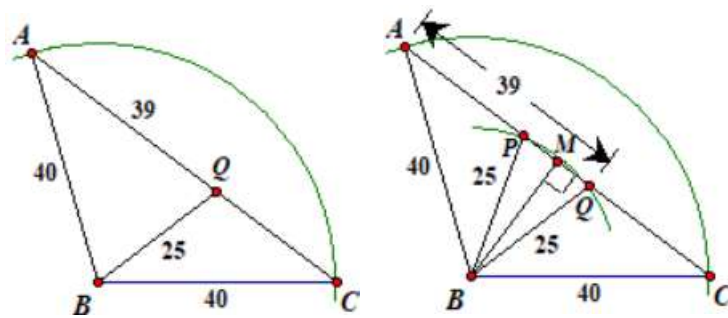
**Question:**

Can we find an integral cyclic quadrilateral  $ABCD$  so that the diagonals are not necessarily perpendicular?

We see from page 3 the last line that 25-39-40 is an integral triangle  $\triangle ABQ$ .



Using  $B$  as centre,  $BA$  as radius, draw a circular arc  $AC$ . Extend  $AQ$ , cutting the circular arc at  $C$ . Join  $BC$ .



Using  $B$  as centre,  $BQ$  as radius, draw a circular arc, cutting  $AQ$  at  $P$ .  $BP = BQ = 25$  (radii)

Let  $M$  be the foot of perpendicular from  $B$  to  $PQ$ .  $\triangle BPM \cong \triangle BQM$  (R.H.S.). Join  $BM$ .

$$AM = AB \cos \angle BAQ = 40 \times \frac{4}{5} = 32, \quad QM = AQ - AM = 39 - 32 = 7 = PM \text{ (corr. sides, } \cong \Delta \text{s)}$$

$$AP = AQ - QM - PM = 39 - 7 - 7 = 25$$

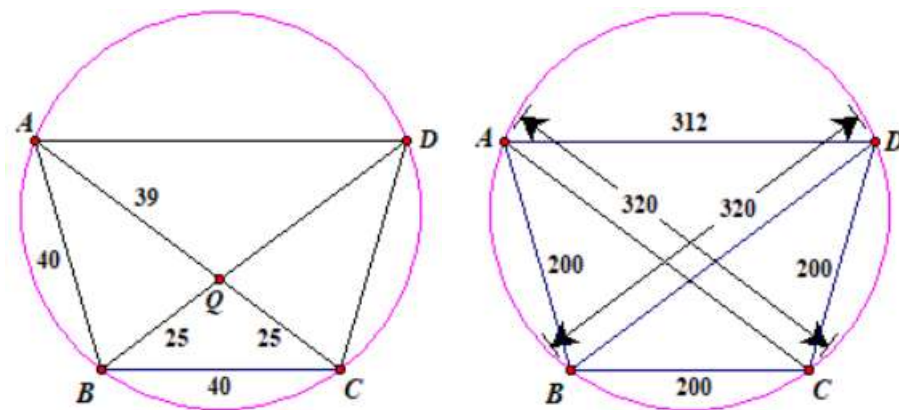
$$\angle BAC = \angle BCA, \quad \angle BPQ = \angle BQP \text{ (base } \angle \text{s, isos. } \Delta \text{)}$$

$$\angle APB = 180^\circ - \angle BPM = 180^\circ - \angle BQM = \angle BQC \text{ (adj. } \angle \text{s on st. line)}$$

$$\triangle ABP \cong \triangle CBQ \text{ (A.A.S.)}$$

$$CQ = AP = 25 \text{ (corr. sides, } \cong \Delta \text{s)}$$

Construct a circumscribed circle through  $A$ ,  $B$  and  $C$ . Extend  $BQ$  to cut the circle again at  $D$ . Join  $AD$  and  $CD$ .



It is easy to show that  $\triangle ABQ \cong \triangle DCQ$  (A.A.S.)

$$DQ = AQ = 39, \quad DC = AB = 40 \text{ (corr. sides, } \cong \Delta \text{s)}$$

$\triangle ADQ \sim \triangle BCQ$  (equiangular)

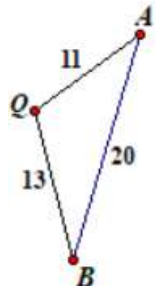
$$\frac{AD}{BC} = \frac{AQ}{BQ} \text{ (corr. sides, } \sim \Delta \text{s)} \Rightarrow AD = 40 \times \frac{39}{25} = \frac{312}{5}$$

Multiply every side by 5 to give integral sides.  $AB = BC = CD = 200$ ,  $AD = 312$ ,  $BD = AC = 320$ .

Again, this is a **special case** for **three equal adjacent sides of integral cyclic quadrilateral**.

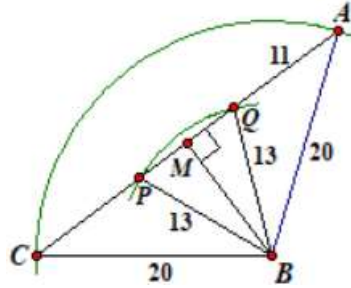
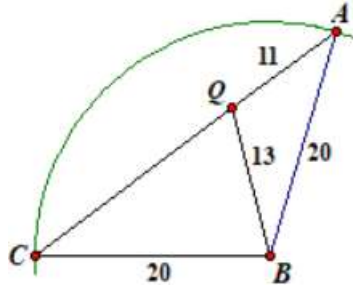
**Second example:**

We see from page 3 the first line in the table that 11-13-20 is an integral triangle  $\triangle ABQ$ .



$$\cos \angle BAQ = \frac{11^2 + 20^2 - 13^2}{2 \times 11 \times 20} = \frac{4}{5}$$

Using  $B$  as centre,  $BA$  as radius, draw a circular arc  $AC$ . Extend  $AQ$ , cutting the circular arc at  $C$ . Join  $BC$ .



Using  $B$  as centre,  $BQ$  as radius, draw a circular arc, cutting  $AQ$  at  $P$ .  $BP = BQ = 13$  (radii)

Let  $M$  be the foot of perpendicular from  $B$  to  $PQ$ .  $\triangle BPM \cong \triangle BQM$  (R.H.S.). Join  $BM$ .

$$AM = AB \cos \angle BAQ = 20 \times \frac{4}{5} = 16, \quad QM = AM - AQ = 16 - 11 = 5 = PM \text{ (corr. sides, } \cong \Delta \text{s)}$$

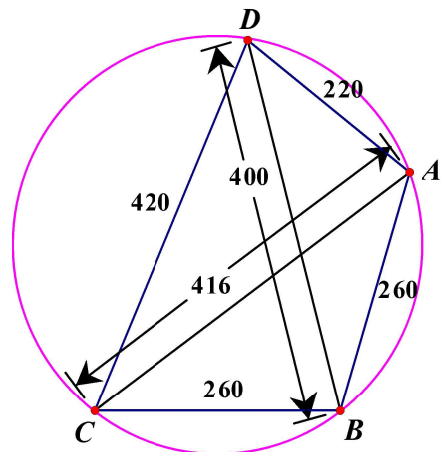
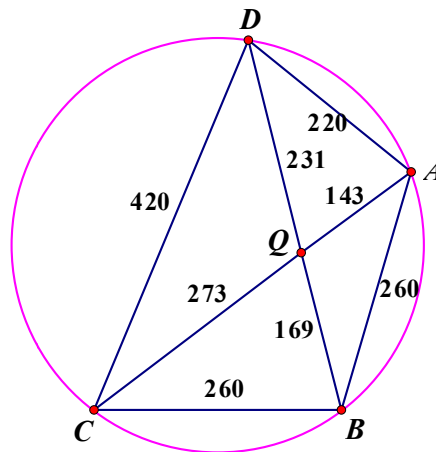
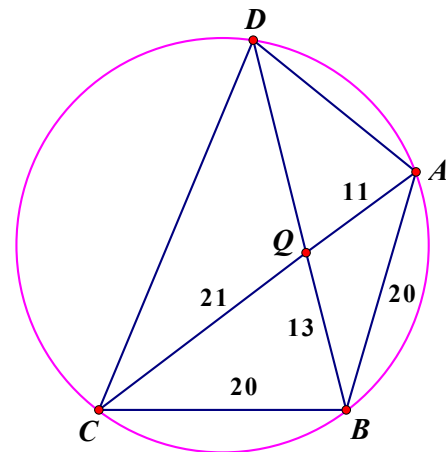
$$AP = AM + PM = 16 + 5 = 21$$

$$\angle BAC = \angle BCA, \quad \angle BPQ = \angle BQP \text{ (base } \angle \text{s, isos. } \Delta \text{)}$$

$$\triangle ABP \cong \triangle CBQ \text{ (A.A.S.)}$$

$$CQ = AP = 21 \text{ (corr. sides, } \cong \Delta \text{s)}$$

Construct a circumscribed circle through  $A$ ,  $B$  and  $C$ . Extend  $BQ$  to cut the circle again at  $D$ . Join  $AD$  and  $CD$ .



It is easy to show that  $\triangle ABQ \sim \triangle DCQ$  (equiangular)

$$\frac{DQ}{AQ} = \frac{CQ}{BQ} = \frac{CD}{AB} \text{ (corr. sides, } \sim \Delta \text{s)}$$

$$DQ = 11 \times \frac{21}{13} = \frac{231}{13}, \quad CQ = 20 \times \frac{21}{13} = \frac{420}{13}, \quad AC = 11 + 21 = 32, \quad BD = 13 + \frac{231}{13} = \frac{400}{13}$$

$\triangle ADQ \sim \triangle BCQ$  (equiangular)

$$\frac{AD}{BC} = \frac{AQ}{BQ} \text{ (corr. sides, } \sim \Delta \text{s)}$$

$$AD = 20 \times \frac{11}{13} = \frac{220}{13}$$

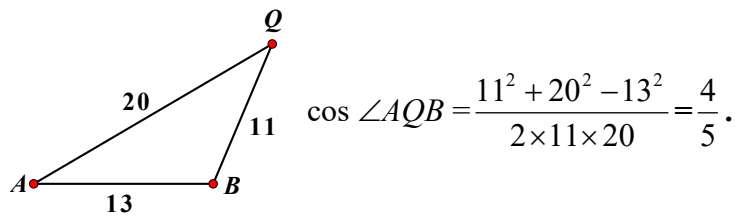
Multiply every side by 13 to give integral sides.  $AB = BC = 260$ ,  $CD = 420$ ,  $AD = 220$ ,  $AC = 416$ ,  $BD = 400$ .

Again, this is a **special case for two equal adjacent sides of integral cyclic quadrilateral**.

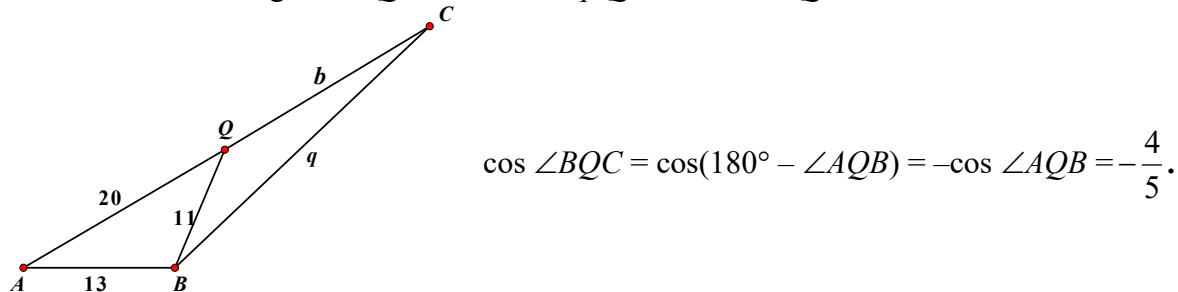


**Question:** Can find an integral cyclic quadrilateral  $ABCD$  so that all adjacent sides are unequal?

We see from page 3 the first line in the table that 11-13-20 is an integral triangle  $\triangle ABQ$ .



We find another triangle  $\triangle BCQ$  so that  $BC = q$ ,  $QC = b$  and  $A, Q, C$  are collinear.



Apply cosine rule on  $\triangle BCQ$ :  $q^2 = b^2 + 11^2 + 22b \times \frac{4}{5}$

$$5q^2 = 5(b^2 + 22b + 11^2) - 110b + 88b$$

$$22b = 5[(b + 11)^2 - q^2] = 5(b + q + 11)(b - q + 11)$$

$$\frac{b + q + 11}{b} = \frac{22}{5(b - q + 11)} = t$$

$$b + q + 11 = bt \dots\dots (1), \quad b - q + 11 = \frac{22}{5t} \dots\dots (2)$$

$$(1) + (2): 2b + 22 = bt + \frac{22}{5t}$$

$$10tb + 110t = 5t^2b + 22$$

$$110t - 22 = (5t^2 - 10t)b$$

$$b = \frac{22(5t - 1)}{5t(t - 2)}$$

$$(1) - (2): 2q = bt - \frac{22}{5t} = \frac{22t(5t - 1)}{5t(t - 2)} - \frac{22}{5t}$$

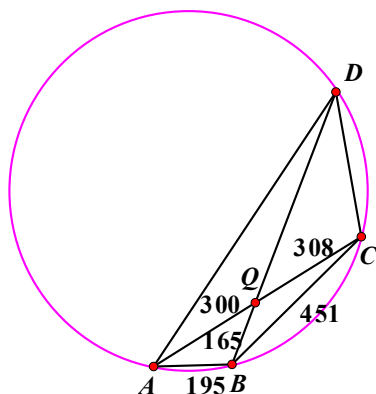
$$q = 11 \cdot \frac{t(5t - 1) - (t - 2)}{5t(t - 2)} = 11 \cdot \frac{5t^2 - 2t + 2}{5t(t - 2)}$$

$$\text{Put } t = 3, \quad b = \frac{22 \times 14}{15} = \frac{308}{15}, \quad q = 11 \times \frac{41}{15} = \frac{451}{15}$$

Multiply every side by 15 to give integral sides.

$$AB = 13 \times 15 = 195, \quad BQ = 11 \times 15 = 165, \quad AQ = 20 \times 15 = 300.$$

Construct a circumscribed circle through  $A$ ,  $B$  and  $C$ . Extend  $BQ$  to cut the circle again at  $D$ . Join  $AD$  and  $CD$ .



It is easy to show that  $\triangle ABQ \sim \triangle DCQ$  (equiangular)

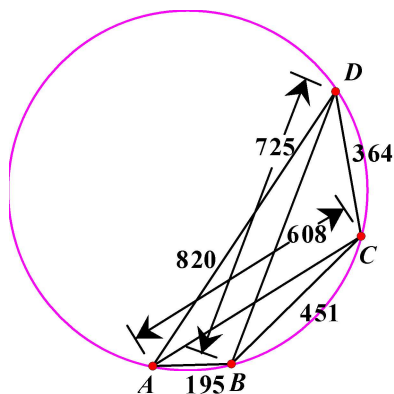
$$\frac{DQ}{AQ} = \frac{CQ}{BQ} = \frac{CD}{AB} \quad (\text{corr. sides, } \sim \Delta s)$$

$$DQ = 300 \times \frac{308}{165} = 560, \quad CD = 195 \times \frac{308}{165} = 364, \quad AC = 300 + 308 = 608, \quad BD = 165 + 560 = 725$$

$\triangle ADQ \sim \triangle BCQ$  (equiangular)

$$\frac{AD}{BC} = \frac{AQ}{BQ} \quad (\text{corr. sides, } \sim \Delta s)$$

$$AD = 451 \times \frac{300}{165} = 820$$



**Question:** Can we find a general formula for integral cyclic quadrilateral for which the diagonals are not necessarily perpendicular and the adjacent sides are not necessarily equal?

Let the cyclic quadrilateral be  $ABCD$ . The diagonals  $AC$  and  $BD$  intersect at  $Q$ .

Let  $AQ = a$ ,  $BQ = b$ ,  $AB = c$ ,  $DQ = d$ ,  $CQ = e$ ,  $CD = f$ ,  $AD = g$ ,  $BC = h$

as shown in the figure.

$\triangle ABQ \sim \triangle DCQ$  (equiangular)

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f} = m, \text{ where } m \text{ is a constant (corr. sides, } \sim \Delta s)$$

$$a = dm, b = em, c = fm$$

$\triangle ADQ \sim \triangle BCQ$  (equiangular)

$$\frac{a}{b} = \frac{d}{e} = \frac{g}{h} = n, \text{ where } n \text{ is a constant. (corr. sides, } \sim \Delta s)$$

$$a = bn = dmn, d = en, g = hn$$

There are five variables  $e, f, h, m, n$  in the figure.

Let  $\triangle BCQ$  be an obtuse-angled triangle with  $\angle BQC > 90^\circ$ .

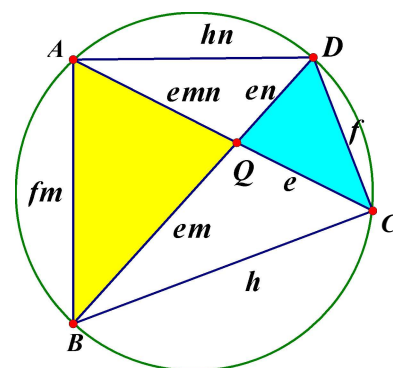
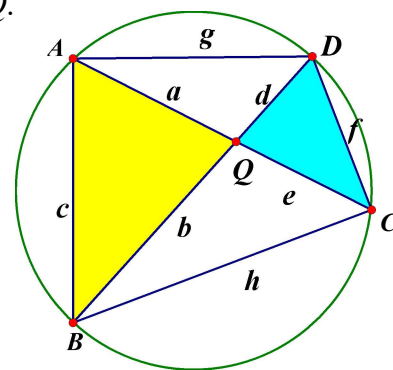
Then  $\triangle CDQ$  is an acute-angled triangle.

By the formula on Page 4, the only possible solution for  $\triangle CDQ$  is:

$$CD = k^2 + 4v^2(u^2 + v^2 + k), CQ = (2u^2 + 2v^2 + k)k, DQ = 2(u^2 + v^2)(2v^2 + k)$$

$$\cos \angle CQD = \frac{u^2 - v^2}{u^2 + v^2} > 0,$$

where  $u > v$  are distinct relatively prime positive integers.



Again,  $\triangle BCQ$  is another triangle adjacent to  $\triangle CDQ$  with  $\cos \angle BQC = \frac{v^2 - u^2}{u^2 + v^2} < 0$

The roles of  $u$  and  $v$  are interchanged.  $BC = h = k^2 + 4u^2(u^2 + v^2 + k)$ ,  $BQ = em = 2(u^2 + v^2)(2u^2 + k)$

$$n = \frac{en}{e} = \frac{DQ}{CQ} = \frac{2(u^2 + v^2)(2v^2 + k)}{(2u^2 + 2v^2 + k)k}$$

$$AQ = emn = BQ \times n$$

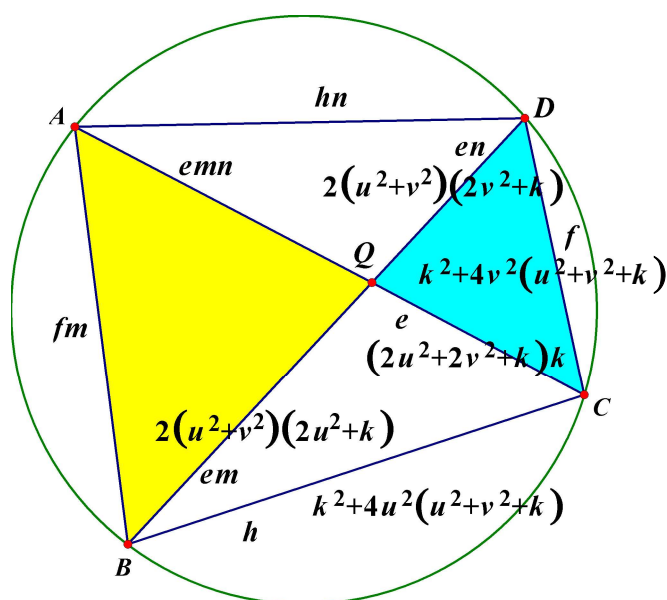
$$= 2(u^2 + v^2)(2u^2 + k) \cdot \frac{2(u^2 + v^2)(2v^2 + k)}{(2u^2 + 2v^2 + k)k}$$

$$= 4 \cdot \frac{(u^2 + v^2)^2(2u^2 + k)(2v^2 + k)}{(2u^2 + 2v^2 + k)k}$$

$$AD = hn = [k^2 + 4u^2(u^2 + v^2 + k)] \cdot \frac{2(u^2 + v^2)(2v^2 + k)}{(2u^2 + 2v^2 + k)k}$$

$$m = \frac{em}{e} = \frac{BQ}{CQ} = \frac{2(u^2 + v^2)(2u^2 + k)}{(2u^2 + 2v^2 + k)k}$$

$$AB = fm = [k^2 + 4v^2(u^2 + v^2 + k)] \cdot \frac{2(u^2 + v^2)(2v^2 + k)}{(2u^2 + 2v^2 + k)k}$$



Multiply every side by  $(2u^2 + 2v^2 + k)k$  to give integral sides:

$$CQ = (2u^2 + 2v^2 + k)^2 k^2$$

$$CD = (2u^2 + 2v^2 + k)k[k^2 + 4v^2(u^2 + v^2 + k)]$$

$$DQ = 2k(u^2 + v^2)(2v^2 + k)(2u^2 + 2v^2 + k)$$

$$BC = (2u^2 + 2v^2 + k)k[k^2 + 4u^2(u^2 + v^2 + k)]$$

$$BQ = 2(u^2 + v^2)(2u^2 + k)(2u^2 + 2v^2 + k)k$$

$$AQ = 4(u^2 + v^2)^2(2u^2 + k)(2v^2 + k)$$

$$AD = 2(u^2 + v^2)(2v^2 + k)[k^2 + 4u^2(u^2 + v^2 + k)]$$

$$AB = 2(u^2 + v^2)(2u^2 + k)[k^2 + 4v^2(u^2 + v^2 + k)]$$

For example, take  $k = 1$ ,  $u = 2$ ,  $v = 1$ :

$$CQ = 121$$

$$CD = 11 \times (1 + 4 \times 6) = 275$$

$$DQ = 2 \times 5 \times 3 \times 11 = 330$$

$$BC = 11 \times (1 + 16 \times 6) = 1067$$

$$BQ = 2 \times 5 \times 9 \times 11 = 990$$

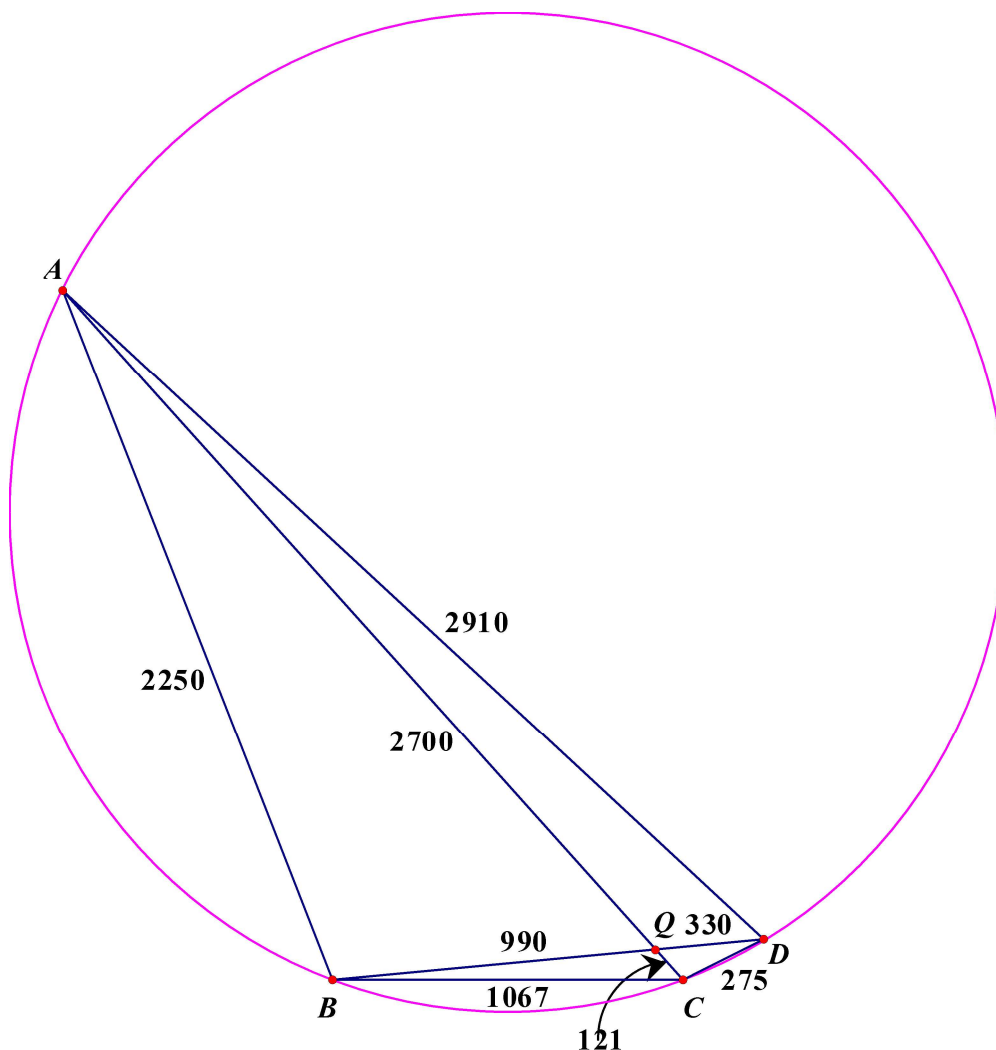
$$AQ = 4 \times 5^2 \times 9 \times 3 = 2700$$

$$AD = 2 \times 5 \times 3 \times (1 + 16 \times 6) = 2910$$

$$AB = 2 \times 5 \times 9 \times (1 + 4 \times 6) = 2250$$

$$AC = AQ + CQ = 2700 + 121 = 2821$$

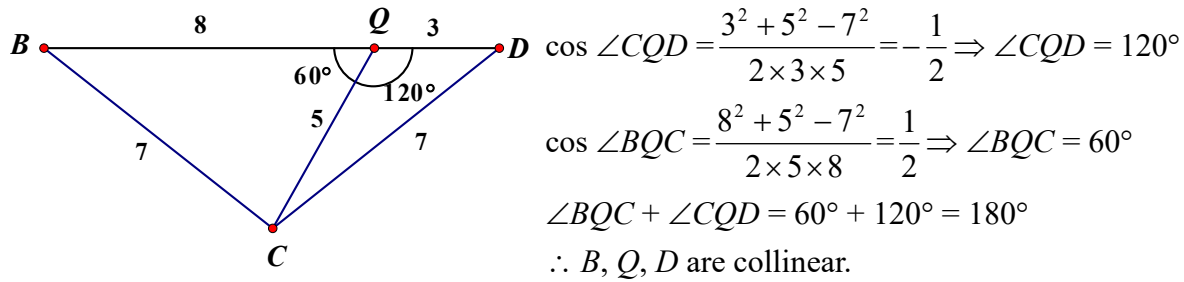
$$BD = BQ + DQ = 990 + 330 = 1320$$



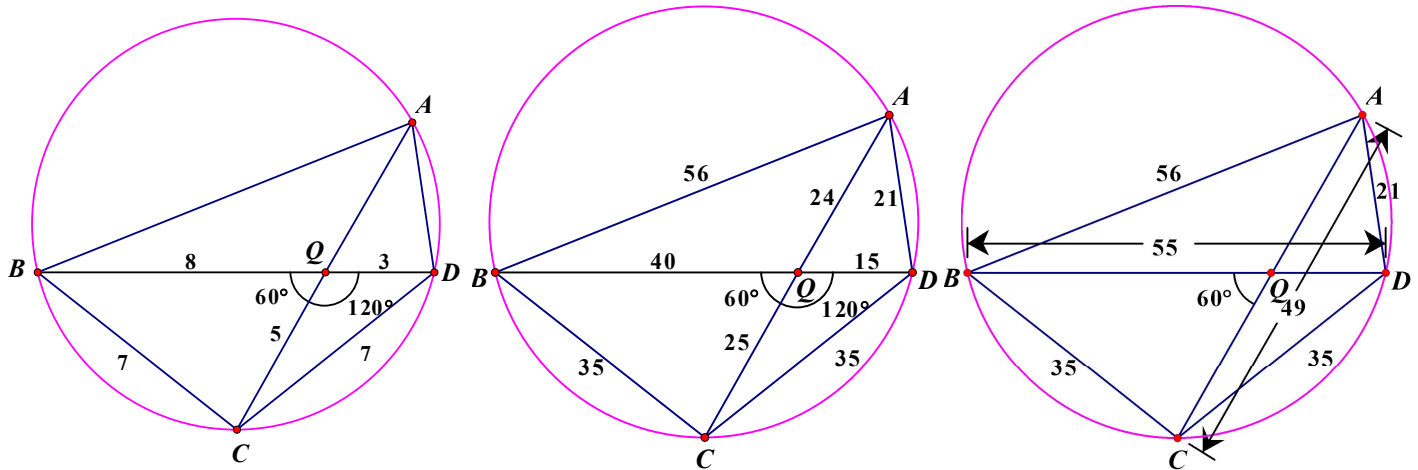
**Think further:** From the document: [https://twhung78.github.io/Number\\_Theory/120triangle.pdf](https://twhung78.github.io/Number_Theory/120triangle.pdf),

we know that 3-5-7 is a  $120^\circ$  triangle, whereas 5-7-8 is a  $60^\circ$  triangle.

Combine the common side '5' to give a bigger triangle as shown:



Construct a circumscribed circle through  $B, C$  and  $D$ . Extend  $CQ$  to cut the circle again at  $A$ . Join  $AB$  and  $AD$ .



It is easy to show that  $\triangle ABQ \sim \triangle DCQ$  (equiangular)

$$\frac{DQ}{AQ} = \frac{CQ}{BQ} = \frac{CD}{AB} \quad (\text{corr. sides, } \sim \Delta s)$$

$$AQ = 3 \times \frac{8}{5} = \frac{24}{5}, AB = 7 \times \frac{8}{5} = \frac{56}{5}, AC = \frac{24}{5} + 5 = \frac{49}{5}, BD = 8 + 3 = 11$$

$\triangle ADQ \sim \triangle BCQ$  (equiangular)

$$\frac{AD}{BC} = \frac{AQ}{BQ} \quad (\text{corr. sides, } \sim \Delta s)$$

$$AD = 7 \times \frac{3}{5} = \frac{21}{5}$$

Multiply every side by 5 to give integral sides.  $BC = CD = 35, AD = 21, AB = 56, AC = 49, BD = 55$ .

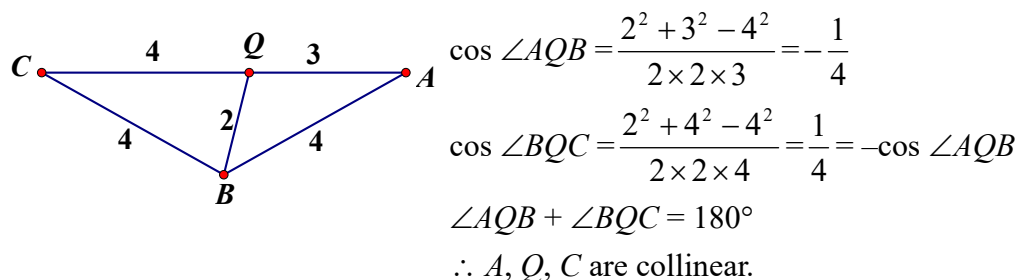
$\therefore$  We can construct another integral cyclic quadrilateral with a simpler formula, but the area of each smaller triangle inside (and hence the cyclic quadrilateral) are not integers.

Again, this is a **special case for two equal adjacent sides of integral cyclic quadrilateral and the angle between the two diagonals is  $60^\circ$ .**

**Question:** Given any triangle  $\triangle ABQ$  with integral sides, can we construct an integral cyclic quadrilateral, while the angle between the diagonals are not necessarily  $60^\circ$  using a similar method?

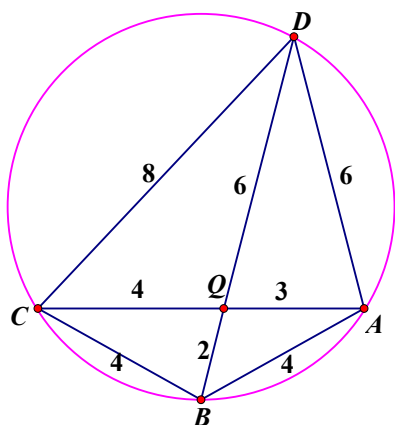
Let  $\triangle ABC$  be a 2-3-4 triangle with  $AB = 4$ ,  $BQ = 2$ ,  $CQ = 3$ .

We can construct (method on page 8) another triangle  $\triangle BCQ$  (with common sides  $BQ$ ) so that  $BC = 4$ ,  $CQ = 4$ .



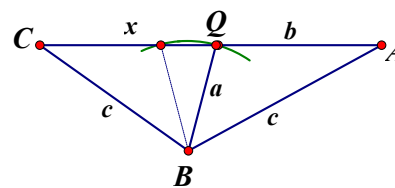
Construct a circumscribed circle through  $A, B$  and  $C$ . Extend  $BQ$  to cut the circle again at  $D$ . Join  $AD$  and  $CD$ .

Using a similar method, we can prove that  $DQ = 6$ ,  $AD = 6$ ,  $CD = 8$ .



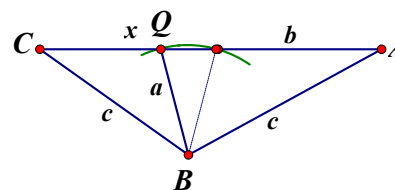
In this case,  $AB = BC = 4$

(two equal adjacent sides, whereas the angle between the diagonals  $\neq 60^\circ$ .)



Given any triangle  $\triangle ABQ$  with integral sides ( $AB = c$ ,  $BQ = a$ ,  $AQ = b$ ). We can use the method on page 9 to find another triangle  $\triangle BCQ$  with a common side  $BQ$  and  $A, Q, C$  are collinear.  $BC = c$ ,  $CQ = x$ .

$$\cos \angle AQB = \frac{a^2 + b^2 - c^2}{2ab}, \quad \cos \angle BQC = -\frac{a^2 + b^2 - c^2}{2ab}$$



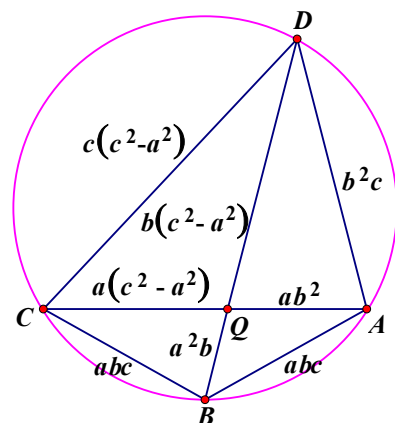
$$\text{If } \angle AQB \text{ is obtuse, then } x = b + 2a \cos \angle BQC = b - \frac{a^2 + b^2 - c^2}{2ab} = \frac{c^2 - a^2}{b}.$$

$$\text{If } \angle AQB \text{ is acute, then } x = b - 2a \cos \angle AQB = b - \frac{a^2 + b^2 - c^2}{2ab} = \frac{c^2 - a^2}{b}.$$

$$\text{If } \angle AQB = 90^\circ, \text{ then } x = b = \frac{c^2 - a^2}{b}.$$

Finally, a cyclic quadrilateral with integral sides is formed.

The only necessary condition is  $c > a$  and  $a, b, c$  obey triangle inequality.



In a more general case, let  $CQ = x$ ,  $BC = y$ .

$$y^2 = x^2 + a^2 - 2ax \cos \angle BQC = x^2 + a^2 + 2ax \cos \angle AQB = x^2 + a^2 + 2ax \cdot \frac{a^2 + b^2 - c^2}{2ab}$$

$$y^2 = x^2 + 2ax + a^2 + \frac{x}{b}(a^2 + b^2 - c^2) - 2ax$$

$$(x + y + a)(x - y + a) = \frac{x}{b}[c^2 - (a^2 - 2ab + b^2)] = \frac{x}{b}[c^2 - (a - b)^2]$$

$$\frac{x + y + a}{x} = \frac{c^2 - (a - b)^2}{b(x - y + a)} = k$$

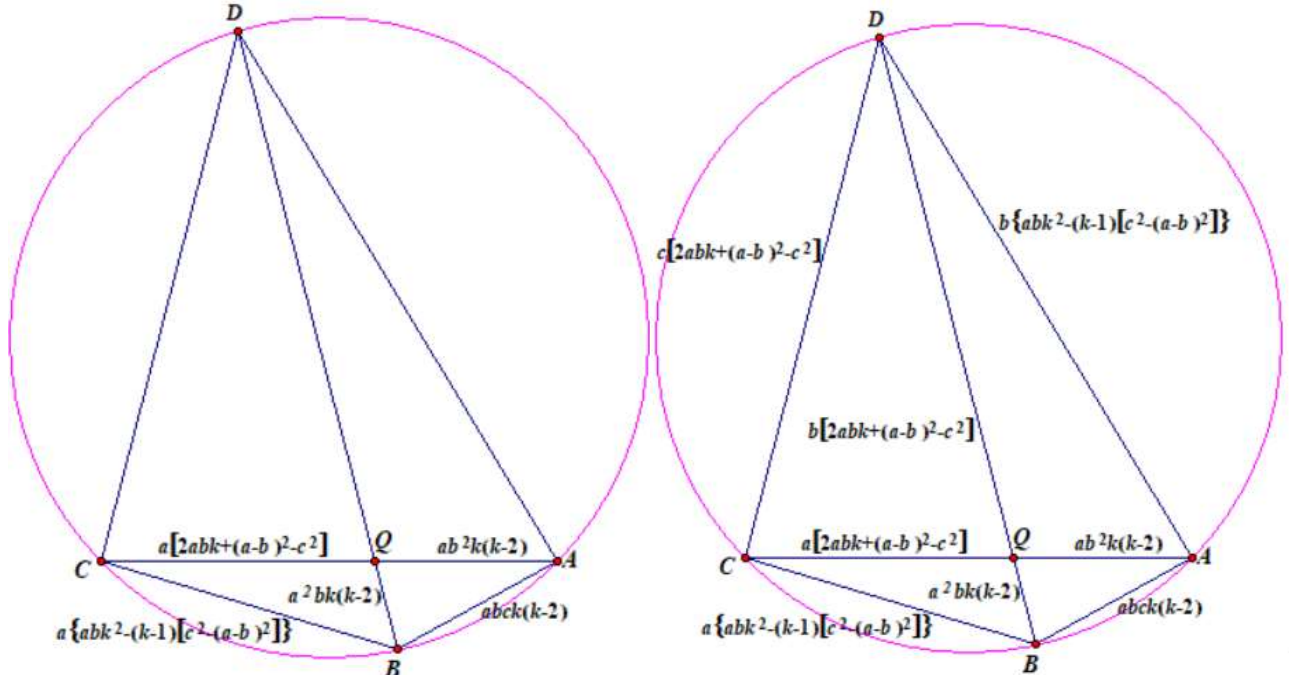
$$x + y + a = kx \dots \dots (1), \quad x - y + a = \frac{c^2 - (a - b)^2}{bk} \dots \dots (2)$$

$$(1) + (2): 2x + 2a = kx + \frac{c^2 - (a - b)^2}{bk} \Rightarrow x = \frac{2abk + (a - b)^2 - c^2}{bk(k - 2)}$$

$$(1) - (2): 2y = kx + \frac{(a - b)^2 - c^2}{bk} = \frac{2abk + (a - b)^2 - c^2}{b(k - 2)} + \frac{(a - b)^2 - c^2}{bk}$$

$$y = \frac{ak}{k - 2} - \frac{k - 1}{bk(k - 2)}[c^2 - (a - b)^2]$$

Multiply every side by  $abk(k - 2)$ , construct a circumscribed circle through  $ABC$ , extend  $BQ$  to the circle at  $D$ .



$$\frac{DQ}{a[2abk + (a - b)^2 - c^2]} = \frac{ab^2k(k - 2)}{a^2bk(k - 2)} = \frac{AD}{a\{abk^2 - (k - 1)[c^2 - (a - b)^2]\}} \quad (\text{corr. sides, } \triangle ADQ \sim \triangle BCQ)$$

$$DQ = b[2abk + (a - b)^2 - c^2], \quad AD = b\{abk^2 - (k - 1)[c^2 - (a - b)^2]\}$$

$$\frac{DQ}{ab^2k(k - 2)} = \frac{CD}{abck(k - 2)} \quad (\text{corr. sides, } \triangle CDQ \sim \triangle BAQ)$$

$$CD = c[2abk + (a - b)^2 - c^2]$$

Necessary condition:  $a, b, c$  obey triangle inequality and  $k > 2$ .

The following is the feedback from Doctor M. K. Siu, retired professor from the University of Hong Kong:

We know quite well how to compute the diagonals of a cyclic quadrilateral from the four sides. Further analysis follows. However, it is always good to try to solve a problem by oneself. The pleasure does not lie so much in the solution than in the process of solving it.

Author's email: [twhung78-hkame@yahoo.com](mailto:twhung78-hkame@yahoo.com)

**Reference:**

1. Pythagorean Triple: [https://twhung78.github.io/Number\\_Theory/Pythagorean\\_triple.pdf](https://twhung78.github.io/Number_Theory/Pythagorean_triple.pdf)
2. Angle between two diagonals in a cyclic quadrilateral:  
[https://twhung78.github.io/Geometry/6%20Circles/2%20Cyclic%20quadrilateral/Angle\\_diagonals\\_cyclic\\_quadrilateral.pdf](https://twhung78.github.io/Geometry/6%20Circles/2%20Cyclic%20quadrilateral/Angle_diagonals_cyclic_quadrilateral.pdf)
3. “Normal Trigrade and cyclic quadrilateral with integral sides and diagonals” from April, 1951 American Mathematical Monthly.
4.  $120^\circ$  triangle: [https://twhung78.github.io/Number\\_Theory/120triangle.pdf](https://twhung78.github.io/Number_Theory/120triangle.pdf)