

<b>89-90 Individual</b>	<b>1</b>	5	<b>2</b>	-2	<b>3</b>	7	<b>4</b>	6	<b>5</b>	(2, 4)
	<b>6</b>	4	<b>7</b>	120	<b>8</b>	-7	<b>9</b>	100	<b>10</b>	109
	<b>11</b>	9	<b>12</b>	0	<b>13</b>	2519	<b>14</b>	2	<b>15</b>	10 days
	<b>16</b>	5	<b>17</b>	$2\sqrt{13}$	<b>18</b>	1 : 2	<b>19</b>	$\frac{9}{20}$	<b>20</b>	58.5

<b>89-90 Group</b>	<b>1</b>	275	<b>2</b>	73	<b>3</b>	2	<b>4</b>	0	<b>5</b>	1783
	<b>6</b>	26	<b>7</b>	$\frac{125}{8} = 15\frac{5}{8}$	<b>8</b>	4 : 1	<b>9</b>	7	<b>10</b>	$2\pi$

**Individual Events**

- I1** Find the value of  $\frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2}$ .

$$\begin{aligned} & \frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2} \\ &= 3 + \sqrt{8} - (\sqrt{8} + \sqrt{7}) + (\sqrt{7} + \sqrt{6}) - (\sqrt{6} + \sqrt{5}) + (\sqrt{5} + 2) \\ &= 5 \end{aligned}$$

- I2** If  $b < 0$  and  $2^{2b+4} - 20 \times 2^b + 4 = 0$ , find  $b$ .

Let  $y = 2^b$ , then  $y^2 = 2^{2b}$ , the equation becomes  $16y^2 - 20y + 4 = 0$

$$4y^2 - 5y + 1 = 0$$

$$(4y - 1)(y - 1) = 0$$

$$y = 2^b = \frac{1}{4} \quad \text{or} \quad y = 1$$

$$b = -2 \quad \text{or} \quad 0$$

$$\because b < 0 \therefore b = -2 \text{ only}$$

- I3** If  $f(a) = a - 2$  and  $F(a, b) = a + b^2$ , find  $F(3, f(4))$ .

**Reference: 1985 FI3.3, 2013 FI3.2, 2015 FI4.3**

$$f(4) = 4 - 2 = 2$$

$$F(3, f(4)) = F(3, 2) = 3 + 2^2 = 7$$

- I4** For positive integers  $a$  and  $b$ , define  $a \# b = a^b + b^a$ . If  $2 \# w = 100$ , find the value of  $w$ .

**Reference: 1999 FI3.1**

$$2^w + w^2 = 100 \text{ for positive integer } w.$$

$$\text{By trial and error, } 64 + 36 = 100$$

$$w = 6.$$

- I5**  $a$  and  $b$  are constants. The straight line  $2ax + 3by = 4a + 12b$  passes through a fixed point  $P$  whose coordinates do not depend on  $a$  and  $b$ . Find the coordinates of  $P$ .

**Reference: 1991 HI6, 1996 HI6**

$$2ax + 3by = 4a + 12b \Rightarrow 2a(x - 2) + 3b(y - 4) = 0$$

$$\text{Put } b = 0 \Rightarrow x = 2,$$

$$\text{Put } a = 0 \Rightarrow y = 4$$

$$P(2, 4)$$

- I6** The sines of the angles of a triangle are in the ratio 3 : 4 : 5. If  $A$  is the smallest interior angle of the triangle and  $\cos A = \frac{x}{5}$ , find the value of  $x$ .

**Reference: 1989 HI10**

By Sine rule,  $a : b : c = \sin A : \sin B : \sin C = 3 : 4 : 5$

Let  $a = 3k$ ,  $b = 4k$ ,  $c = 5k$ .

$$a^2 + b^2 = (3k)^2 + (4k)^2 = (5k)^2 = c^2$$

$\therefore \angle C = 90^\circ$  (converse, Pythagoras' theorem)

$$\cos A = \frac{b}{c} = \frac{4}{5}$$

$$\Rightarrow x = 4$$

- I7** If  $x + y = 9$ ,  $y + z = 11$  and  $z + x = 10$ , find the value of  $xyz$ .

**Reference: 1986 FG10.1, 1989 HI15**

$$(1) + (2) - (3): 2y = 10 \Rightarrow y = 5$$

$$(1) + (3) - (2): 2x = 8 \Rightarrow x = 4$$

$$(2) + (3) - (1): 2z = 12 \Rightarrow z = 6$$

$$\Rightarrow xyz = 120$$

- I8** If  $\alpha, \beta$  are the roots of the equation  $2x^2 + 4x - 3 = 0$  and  $\alpha^2, \beta^2$  are the roots of the equation  $x^2 + px + q = 0$ , find the value of  $p$ .

$$\alpha + \beta = -2$$

$$\alpha\beta = -\frac{3}{2}$$

$$\begin{aligned} p &= -(\alpha^2 + \beta^2) = -(\alpha + \beta)^2 + 2\alpha\beta \\ &= -(-2)^2 - 3 = -7 \end{aligned}$$

- I9** If  $x^{\log_{10} x} = \frac{x^3}{100}$  and  $x > 10$ , find the value of  $x$ .

Take log on both sides,  $\log x \cdot \log x = 3 \log x - \log 100$

$$(\log x)^2 - 3 \log x + 2 = 0$$

$$(\log x - 1)(\log x - 2) = 0$$

$$\log x = 1 \text{ or } \log x = 2$$

$$x = 10 \text{ or } 100$$

$$\because x > 10 \therefore x = 100 \text{ only}$$

- I10** Given that  $a_0 = 1$ ,  $a_1 = 3$  and  $a_n^2 - a_{n-1}a_{n+1} = (-1)^n$  for positive integers  $n$ . Find  $a_4$ .

$$\text{Put } n = 1, \quad a_1^2 - a_0a_2 = (-1)^1 \Rightarrow 3^2 - a_2 = -1 \Rightarrow a_2 = 10$$

$$\text{Put } n = 2, \quad a_2^2 - a_1a_3 = (-1)^2 \Rightarrow 10^2 - 3a_3 = 1 \Rightarrow a_3 = 33$$

$$\text{Put } n = 3, \quad a_3^2 - a_2a_4 = (-1)^3 \Rightarrow 33^2 - 10a_4 = -1 \Rightarrow a_4 = 109$$

- I11** Find the units digit of  $2137^{754}$ .

**Reference 1991 HG1**

$$7^1 = 7, 7^2 = 49, 7^3 = 343, 7^4 = 2401$$

The pattern of units digit repeats for every multiples of 4.

$$2137^{754} \equiv (7^4)^{188} \cdot 7^2 \equiv 9 \pmod{10}$$

The units digit is 9.

**I12** If  $\left(r + \frac{1}{r}\right)^2 = 3$ , find  $r^3 + \frac{1}{r^3}$ .

**Reference: 1985 FI1.2, 2017 FI1.4**

$$r + \frac{1}{r} = \pm\sqrt{3}$$

$$r^2 + \frac{1}{r^2} = \left(r + \frac{1}{r}\right)^2 - 2 = 3 - 2 = 1$$

$$\begin{aligned} r^3 + \frac{1}{r^3} &= \left(r + \frac{1}{r}\right)\left(r^2 - 1 + \frac{1}{r^2}\right) \\ &= \pm\sqrt{3}(1-1) = 0 \end{aligned}$$

- I13** A positive integer  $N$ , when divided by 10, 9, 8, 7, 6, 5, 4, 3 and 2, leaves remainders 9, 8, 7, 6, 5, 4, 3, 2 and 1 respectively. Find the least value of  $N$ .

**Reference: 1985 FG7.2, 2013FG4.3**

$N + 1$  is divisible by 10, 9, 8, 7, 6, 5, 4, 3, 2.

The L.C.M. of 2, 3, 4, 5, 6, 7, 8, 9, 10 is 2520.

$\therefore N = 2520k - 1$ , where  $k$  is an integer.

The least positive integral of  $N = 2520 - 1 = 2519$

**I14** If  $\frac{1}{A} = \frac{\cos 45^\circ \sin 70^\circ \cos 60^\circ \tan 40^\circ}{\cos 340^\circ \sin 135^\circ \tan 220^\circ}$ , find the value of  $A$ .

**Reference: 1989 HI14**

$$\begin{aligned} \frac{1}{A} &= \frac{\cos 45^\circ \cos 20^\circ \cos 60^\circ \tan 40^\circ}{\cos 20^\circ \cos 45^\circ \tan 40^\circ} \\ &= \cos 60^\circ = \frac{1}{2} \end{aligned}$$

$$A = 2$$

- I15** If 10 men can make 20 tables in 5 days, how many days are required to make 60 tables by 15 men?

$$1 \text{ man can make } \frac{20}{10 \times 5} = \frac{2}{5} \text{ table in 1 day.}$$

$$15 \text{ men can make } \frac{2}{5} \times 15 = 6 \text{ tables in one day.}$$

They can make 60 tables in 10 days

- I16** In figure 1, the exterior angles of the triangle are in the ratio  $x^\circ : y^\circ : z^\circ = 4 : 5 : 6$  and the interior angles are in the ratio  $x : y : z = a : b : 3$ . Find the value of  $b$ .

$$\text{Let } x^\circ = 4k, y^\circ = 5k, z^\circ = 6k$$

$$4k + 5k + 6k = 360^\circ \text{ (sum of ext. } \angle \text{ of polygon)}$$

$$15k = 360^\circ$$

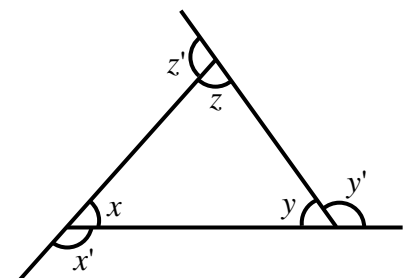
$$\Rightarrow k = 24$$

$$x^\circ = 96^\circ, y^\circ = 120^\circ, z^\circ = 144^\circ$$

$$x = 84^\circ, y = 60^\circ, z = 36^\circ \text{ (adj. } \angle \text{s on st. line)}$$

$$x : y : z = 7 : 5 : 3$$

$$\Rightarrow b = 5$$



(Figure 1)

- I17** In  $\triangle ABC$ ,  $\angle C = 90^\circ$  and  $D, E$  are the mid-points of  $BC$  and  $CA$  respectively. If  $AD = 7$  and  $BE = 4$ , find the length of  $AB$ .  
(See figure 2.)

Let  $BD = x = DC$ ,  $AE = y = EC$

$$x^2 + (2y)^2 = 7^2 \dots\dots (1)$$

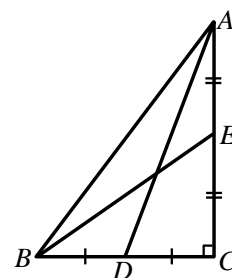
$$(2x)^2 + y^2 = 4^2 \dots\dots (2)$$

$$4(1) - (2): 15y^2 = 180 \Rightarrow y^2 = 12$$

$$4(2) - (1): 15x^2 = 15 \Rightarrow x^2 = 1$$

$$AB^2 = (2x)^2 + (2y)^2 = 4 + 48$$

$$\Rightarrow AB = \sqrt{52} = 2\sqrt{13}$$



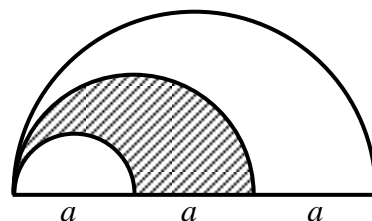
(Figure 2)

- I18** Figure 3 shows 3 semi-circles of diameters  $a$ ,  $2a$  and  $3a$  respectively. Find the ratio of the area of the shaded part to that of the unshaded part.

$$\text{Area of the shaded part} = \frac{\pi}{2} \cdot a^2 - \frac{\pi}{2} \cdot \left(\frac{a}{2}\right)^2 = \frac{3\pi}{8} a^2$$

$$\text{Area of the unshaded part} = \frac{\pi}{2} \cdot \left(\frac{3a}{2}\right)^2 - \frac{3\pi}{8} \cdot a^2 = \frac{6\pi}{8} \cdot a^2$$

$$\text{The ratio} = 3 : 6 = 1 : 2$$



(Figure 3)

- I19** Find the value of  $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{19 \times 20}$ .

$$\begin{aligned} & \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{19 \times 20} \\ &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{19} - \frac{1}{20}\right) \\ &= \frac{1}{2} - \frac{1}{20} = \frac{9}{20} \end{aligned}$$

- I20** In figure 4,  $\angle C = 90^\circ$ ,  $AD = DB$  and  $DE$  is perpendicular to  $AB$ . If  $AB = 20$  and  $AC = 12$ , find the area of the quadrilateral  $ADEC$ .

$BD = 10$ ,  $BC = 16$  (Pythagoras' theorem)

$\triangle BDE \sim \triangle BCA$  (equiangular)

$BD : DE : BE = 16 : 12 : 20$  (ratio of sides,  $\sim \Delta$ 's)

$DE = 7.5$ ,  $BE = 12.5$

$CE = 16 - 12.5 = 3.5$

$$S_{ADEC} = \frac{1}{2} \cdot 10 \cdot 7.5 + \frac{1}{2} \cdot 12 \cdot 3.5 = 58.5$$

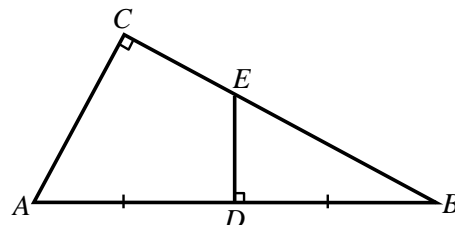
### Method 2

$BD = 10$ ,  $BC = 16$  (Pythagoras' theorem)

$\triangle BDE \sim \triangle BCA$  (equiangular)

$$S_{\triangle BDE} = \left(\frac{BD}{BC}\right)^2 \cdot S_{\triangle ABC} = \left(\frac{10}{16}\right)^2 \cdot \frac{1}{2} \cdot 12 \cdot 16 = 37.5$$

$$S_{ADEC} = \frac{1}{2} \cdot 12 \cdot 16 - 37.5 = 58.5$$



(Figure 4)

**Group Events**

**G1** If  $\frac{1}{a} + \frac{1}{b} = 5$  and  $\frac{1}{a^2} + \frac{1}{b^2} = 13$ , find the value of  $\frac{1}{a^5} + \frac{1}{b^5}$ .

$$(1)^2 - (2): \frac{2}{ab} = 12$$

$$\Rightarrow ab = \frac{1}{6} \dots\dots (4)$$

$$\text{From (1): } (a+b) \cdot \frac{1}{ab} = 5 \dots\dots (5)$$

$$\text{Sub. (4) into (5): } 6(a+b) = 5$$

$$\Rightarrow a+b = \frac{5}{6} \dots\dots (6)$$

From (4) and (6),  $a$  and  $b$  are roots of  $6t^2 - 5t + 1 = 0$

$$(2t-1)(3t-1) = 0$$

$$\Rightarrow t = \frac{1}{2} \text{ or } \frac{1}{3}$$

$$\frac{1}{a^5} + \frac{1}{b^5} = 2^5 + 3^5$$

$$= 32 + 243 = 275$$

**G2** There are  $N$  pupils in a class.

When they are divided into groups of 4, 1 pupil is left behind.

When they are divided into groups of 5, 3 pupils are left behind.

When they are divided into groups of 7, 3 pupils are left behind.

Find the least value of  $N$ .

**Reference: 1992 HG4**

$$N = 4p + 1 \dots\dots (1), p \text{ is an integer}$$

$$N = 5q + 3 \dots\dots (2), q \text{ is an integer}$$

$$N = 7r + 3 \dots\dots (3), r \text{ is an integer}$$

$$(3) - (2): 7r = 5q$$

$$r = 5k, q = 7k, \text{ where } k \text{ is an integer}$$

$$N = 35k + 3 = 4p + 1$$

$$4p - 35k = 2$$

By trial and error,

$$p = 18, k = 2 \text{ is a solution}$$

$$N = 73$$

- G3** The coordinates of  $A$ ,  $B$ ,  $C$  and  $D$  are  $(10, 1)$ ,  $(1, 7)$ ,  $(-2, 1)$  and  $(1, 3)$  respectively.  $AB$  and  $CD$  meet at  $P$ . Find the value of  $\frac{AP}{PB}$ .

**Reference: 1989 HG5**

$$\text{Equation of } AB: \frac{y-1}{x-10} = \frac{1-7}{10-1}$$

$$\Rightarrow 2x + 3y - 23 = 0 \dots\dots (1)$$

$$\text{Equation of } CD: \frac{y-1}{x+2} = \frac{3-1}{1+2}$$

$$\Rightarrow 2x - 3y + 7 = 0 \dots\dots (2)$$

$$(1) + (2): 4x - 16 = 0$$

$$\Rightarrow x = 4$$

$$(1) - (2): 6y - 30 = 0$$

$$\Rightarrow y = 5$$

$$\text{Let } \frac{AP}{PB} = r$$

$$4 = \frac{10+r}{1+r}$$

$$\Rightarrow 4 + 4r = 10 + r$$

$$\Rightarrow r = 2$$

- G4** Find the remainder when  $2^{1989} + 1$  is divided by 3.

$$2^{1989} + 1 = (3 - 1)^{1989} + 1 = 3m - 1 + 1, \text{ binomial theorem, } m \text{ is an integer.}$$

The remainder is 0.

**Method 2**

$$2^1 + 1 = 3 \equiv 0 \pmod{3}, 2^2 + 1 = 5 \equiv 2 \pmod{3}, 2^3 + 1 \equiv 0 \pmod{3}, 2^4 + 1 \equiv 2 \pmod{3}$$

The pattern of the remainder repeats for every multiples of 2.

$$2^{1989} + 1 \equiv 2^1 + 1 \equiv 0 \pmod{3}$$

$$\Rightarrow \text{the remainder} = 0$$

- G5** Euler was born and died between 1700 A.D. and 1800 A.D. He was  $n + 9$  years old in  $n^3$  A.D. and died at the age of 76. Find the year in which Euler died.

Suppose he was born in  $x$  years after 1700 A.D.

$$1700 + x + n + 9 - 1 = n^3 \dots\dots (1)$$

$$11^3 = 1331, 12^3 = 1728, 13^3 > 1800$$

$$\therefore n = 12, x = 1728 - 1700 - 12 - 9 + 1 = 8$$

$$1700 + x + 76 - 1 = 1783$$

$$\Rightarrow \text{He was died in A.D. 1783.}$$

**G6** Let  $N!$  denotes the product of the first  $N$  natural numbers, i.e.  $N! = 1 \times 2 \times 3 \times \dots \times N$ .

If  $k$  is a positive integer such that  $30! = 2^k \times$  an odd integer, find  $k$ .

**Reference: 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FI1.4, 2012 FG1.3**

2, 4, 6, 8, ..., 30 each has at least one factor of 2. Subtotal = 15

4, 8, ..., 28 each has at least 2 factors of 2. Subtotal = 7

8, 16, 24 each has at least 3 factors of 2. Subtotal = 3

16 has 4 factors of 2. Subtotal = 1

Total number of factors of 2 =  $15 + 7 + 3 + 1 = 26$

**G7** The graph of the parabola  $y = x^2 - 4x - \frac{9}{4}$  cuts the  $x$ -axis at  $A$  and  $B$  (figure 1). If  $C$  is the vertex of the parabola, find the area of  $\triangle ABC$ .

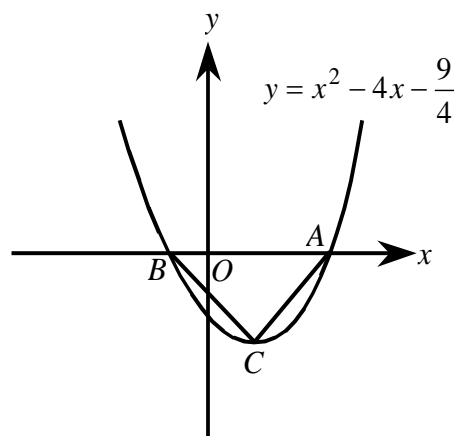
Let the roots be  $\alpha, \beta$ , where  $\alpha > \beta$ .

$$AB = \alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$= \sqrt{4^2 + 4 \cdot \frac{9}{4}} = 5$$

$$\text{Minimum} = \frac{4ac - b^2}{4a} = \frac{4\left(-\frac{9}{4}\right) - (-4)^2}{4} = -\frac{25}{4}$$

$$\text{Area of } \triangle ABC = \frac{1}{2} \times \frac{25}{4} \times 5 = \frac{125}{8}$$



(Figure 1)

**G8** In figure 2,  $FE \parallel BC$  and  $ED \parallel AB$ . If  $AF : FB = 1 : 4$ , find the ratio of area of  $\triangle EDC$  : area of  $\triangle DEF$ .

**Reference: 1989 HI17**

$BDEF$  is a parallelogram formed by 2 pairs of parallel lines

$\triangle DEF \cong \triangle FBD$  (A.S.A.)

Let  $S_{\triangle DEF} = x = S_{\triangle FBD}$  (where  $S$  stands for the area)

$\triangle AEF \sim \triangle ACB$  ( $\because FE \parallel BC$ , equiangular)

$$\frac{S_{\triangle AEF}}{S_{\triangle ACB}} = \left(\frac{1}{1+4}\right)^2 = \frac{1}{25} \dots\dots (1)$$

$$\therefore AE : EC = AF : FB = 1 : 4 \text{ (theorem of equal ratio)}$$

$$\because DE \parallel AB$$

$$\therefore AE : EC = BD : DC = 1 : 4 \text{ (theorem of equal ratio)}$$

$$\triangle CDE \sim \triangle CBA \text{ } (\because DE \parallel BA, \text{ equiangular})$$

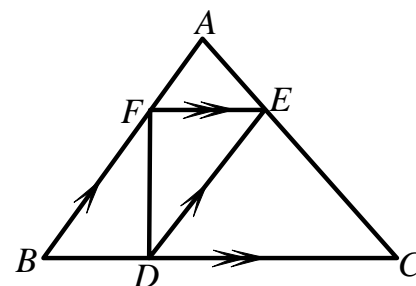
$$\frac{S_{\triangle CDE}}{S_{\triangle CBA}} = \left(\frac{4}{1+4}\right)^2 = \frac{16}{25} \dots\dots (2)$$

$$\text{Compare (1) and (2) } S_{\triangle AEF} = k, S_{\triangle CDE} = 16k, S_{\triangle ABC} = 25k$$

$$k + 16k + x + x = 25k$$

$$x = 4k$$

$$\Rightarrow \text{area of } \triangle DEF : \text{area of } \triangle ABC = 16 : 4 = 4 : 1$$



(Figure 2)

- G9** In the attached multiplication (figure 3), the letters  $O, L, Y, M, P, I, A$  and  $D$  represent different integers ranging from 1 to 9. Find the integer  $\times \frac{D}{O O O O O O O O O}$  represented by  $A$ .

$$1^2 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 16, 5^2 = 25, 6^2 = 36, 7^2 = 49, 8^2 = 64, 9^2 = 81$$

Possible  $(D, O) = (2, 4), (3, 9), (4, 6), (7, 9), (8, 4), (9, 1)$

When  $D = 2, O = 4, (OLYMPIAD) = 444444444 \div 2 = 222222222$  rejected

When  $D = 3, O = 9, (OLYMPIAD) = 999999999 \div 3 = 333333333$  rejected

When  $D = 4, O = 6, (OLYMPIAD) = 666666666 \div 4 = 166666666.5$  rejected

When  $D = 7, O = 9, (OLYMPIAD) = 999999999 \div 7 = 142857142.7$  rejected

When  $D = 8, O = 4, (OLYMPIAD) = 444444444 \div 8 = 55555555.5$  rejected

When  $D = 9, O = 1, (OLYMPIAD) = 111111111 \div 9 = 12345679$

$A = 7$

- G10** Three circles, with centres  $A, B$  and  $C$  respectively, touch one another as shown in figure 4. If  $A, B$  and  $C$  are collinear and  $PQ$  is a common tangent to the two smaller circles, where  $PQ = 4$ , find the area of the shaded part in terms of  $\pi$ .

**Reference: 2018FG1.2**

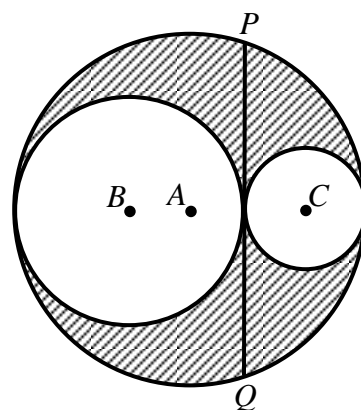
Let the radii of the 3 circles with centres  $A, B$  and  $C$  be  $a, b, c$ .

$$2b + 2c = 2a \Rightarrow a = b + c \dots\dots (1)$$

By intersecting chords theorem,  $2c \times 2b = 2^2$

$$bc = 1 \dots\dots (2)$$

$$\begin{aligned} \text{Shaded area} &= \pi a^2 - \pi b^2 - \pi c^2 \\ &= \pi[a^2 - (b^2 + c^2)] \\ &= \pi[a^2 - (b + c)^2 + 2bc] \\ &= \pi(a^2 - a^2 + 2) \text{ by (1) and (2)} \\ &= 2\pi \end{aligned}$$



(Figure 4)