VI Reciprocal Equation.

Theory Suppose $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, has roots $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Transform $x \to \frac{1}{y}$

$a_ny^n + a_{n-1}y^{n-1} + \cdots + a_1y + a_0 = 0$, has roots $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \cdots, \frac{1}{\alpha_n}$.

<u>If</u> the two equations have **<u>the same roots</u>**, then

$$\frac{a_0}{a_n} = \frac{a_1}{a_{n-1}} = \cdots = \frac{a_{n-1}}{a_1} = \frac{a_n}{a_0}$$

Property 1 $\frac{a_0}{a_n} = \frac{a_n}{a_0} \Rightarrow \left(\frac{a_0}{a_n}\right)^2 = 1 \Rightarrow \frac{a_0}{a_n} = \pm 1$. The ratio is equal to 1 or -1.

Property 2 If $n = 2m - 1$, the reciprocal equation becomes

$$a_0 x^{2m-1} + a_1x^{2m-2} + \cdots + a_{2m-2}x + a_{2m-1} = 0$$

If the ratio is 1, then $a_0 = a_{2m-1}$, $a_1 = a_{2m-2}$, $\cdots$, $a_{m-1} = a_m$, the equation becomes:

$$a_0 x^{2m-1} + a_1x^{2m-2} + \cdots + a_{m-1}x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0 = 0$$

Put $x = -1$ into the equation:

L.H.S. $= -a_0 + a_1 - \cdots + (-1)^m a_{m-1} + (-1)^{m-1} a_{m-1} + \cdots - a_1 + a_0 = 0 =$ R.H.S.

$\therefore$ $x = -1$ is a root.
Consequently the equation becomes $(x + 1)Q(x) = 0$,
where $Q(x) = 0$ is another reciprocal equation of even degree.

If the ratio is -1, then $a_0 = -a_{2m-1}$, $a_1 = -a_{2m-2}$, $\cdots$, $a_{m-1} = -a_m$, the equation becomes:

$$a_0 x^{2m-1} + a_1x^{2m-2} + \cdots + a_{m-1}x^m - a_{m-1}x^{m-1} - \cdots - a_1x - a_0 = 0$$

Put $x = 1$ into the equation:

LHS $= a_0 + a_1 + \cdots + a_{m-1} - a_{m-1} - \cdots - a_1 - a_0 = 0 =$ RHS

$\therefore$ $x = 1$ is a root.
Consequently the equation becomes $(x - 1)Q(x) = 0$,
where $Q(x) = 0$ is another reciprocal equation of even degree.

Property 3 If $n = 2m$, the reciprocal equation becomes

$$a_0 x^{2m} + a_1x^{2m-1} + \cdots + a_{2m-1}x + a_{2m} = 0$$

If the ratio is –1, then $a_0 = -a_{2m}$, $a_1 = -a_{2m-1}$, $\cdots$, $a_{m-1} = -a_{m+1}$, $a_m = -a_m$,

$\Rightarrow a_m = 0$

the equation becomes: $a_0 x^{2m} + a_1 x^{2m-1} + \cdots + a_{m-1}x^{m+1} - a_{m-1}x^{m-1} - \cdots - a_1 x - a_0 = 0$

Put $x = 1$ into the equation:

LHS $= a_0 + a_1 + \cdots + a_{m-1} - a_{m-1} - \cdots - a_1 - a_0 = 0 =$ RHS

$\therefore$ $x = 1$ is a root.

Put $x = -1$ into the equation:

LHS $= a_0 - a_1 + \cdots + (-1)^{m+1}a_{m-1} - (-1)^{m-1}a_{m-1} - \cdots + a_1 - a_0 = 0 =$ RHS

$\therefore$ $x = -1$ is a root.

Hence $x = 1$ or $x = -1$ are the roots of the equation: $(x^2 - 1)Q(x) = 0$,

where $Q(x) = 0$ is another **reciprocal equation** of degree $2m - 2$.

If the ratio is 1, then $a_0 = a_{2m}$, $a_1 = a_{2m-1}$, $\cdots$, $a_{m-1} = a_{m+1}$, the equation becomes:

$$a_0 x^{2m} + a_1 x^{2m-1} + \cdots + a_{m-1}x^{m+1} + a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 = 0$$

The equation is called **<u>standard type</u>**. It can be solved using the substitution:

$y = x + \frac{1}{x}$, $y^2 - 2 = x^2 + \frac{1}{x^2}$, $\cdots$, etc.

Exercise 1

1. By inspection, show that $x = -1$ is a root of $x^5 - 4x^4 + 3x^3 + 3x^2 - 4x + 1 = 0$.

Hence solve it completely. [Ans: 1, 1, –1, $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$]

2. By inspection, show that $x = 1$ is a root of $2x^3 - 7x^2 + 7x - 2 = 0$.

Hence solve it completely. [Ans: 1, 2, $\frac{1}{2}$]

3. Show that $(x^2 - 1)$ is a factor of $x^6 + x^5 - 5x^4 + 5x^2 - x - 1 = 0$.

Hence solve it completely. [Ans: 1, 1, 1, –1, $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$]

4. Solve the standard type of the reciprocal equation:

$3x^4 - 20x^3 - 94x^2 - 20x + 3 = 0$. [Ans. –3, $-\frac{1}{3}$, $5 \pm 2\sqrt{6}$]

5. The equation $6x^4 + 7x^3 - 36x^2 - 7x + 6 = 0$ is **<u>NOT</u>** an reciprocal equation (why?)

Use $y = x - \frac{1}{x}$ to solve it . [Ans. –3, $\frac{1}{3}$, 2, $-\frac{1}{2}$]