Miscellaneous Examples

9.1 The use of "**Complementary Events**" at a suitable time may help to solve the problem.

Example 1 Assume that a single torpedo (魚雷) has a probability $\frac{1}{6}$ of sinking a ship, a probability $\frac{1}{3}$ of damaging it, and probability $\frac{1}{2}$ of missing. Assume further that 2 damaging shots are sufficient to sink a ship. What is the probability that 4 torpedoes will succeed in sinking a ship?

Solution: There are many cases in which 4 torpedoes may succeed in sinking the ship. It is easier to find the probability that 4 torpedoes fail to do so.

Cases in which 4 torpedoes fail to sink a ship:

(i) all shots missing : P_1 = _________

(ii) 3 shots missing and 1 shot damaging.

$P_2 = 4 \times$ ___________

$\therefore$ Required Probability = $1 - P_1 - P_2$ = ____________________. [Ans: $\frac{37}{48}$]

Exercise 1 If the probability of a plane being shot down during a mission (飛行任務) is $\frac{1}{10}$. Find the probability that a pilot will be shot down within 10 missions. (The answer is not 1.)

9.2 Estimates

Example 3 A man fires at a target with a probability of 0.2 hitting it. Find the minimum number of fires in order that the probability of hitting the target is at least 0.8.

Solution: H = event of hitting, P(H) = 0.2

L = event of losing. P(L) = 0.8

n = number of fires

P(at least one shot hitting) ≥ 0.8

1 – P(no shots hit the target) ≥ 0.8

$1 - 0.8 \geq ($ ______ $)^n$

$0.2 \geq 0.8^n$

$\frac{\log 0.2}{\log 0.8} \leq n$ ($\because \log 0.8 < 0$)

$\therefore$ minimum $n = 8$

Exercise 2 How many times should a die be thrown so that the probability that the number "1" appears at least once is greater than $\frac{1}{2}$? Answer: 4

Class Activity 1 How to estimate the number of fish in a lake ?

You may wonder how we can know the number of fish in a lake. It seems that it is impossible to catch all the fish for counting. Instead, an estimation algorithm called " Capture-Recapture " is more preferable.

Let N be the total number of fish in the lake.

Assumption: The probability that a fish being caught is equal.
The number of fish in the lake is a constant.

Steps:
1. A sample of n fish are caught. Mark them with the same symbols and then return them to the lake.
2. Catch another sample of M fish. Among these M fish, some of them are those which have been marked in step 1, say m of them.

From the second capture, we can obtain a sample probability of getting a marked fish (i.e. $\frac{m}{M}$). This probability is close to the population probability of getting a marked fish (i.e. $\frac{n}{N}$) when M, N are large enough.

9.3 Examination Type Questions

Very often, an examination-type question consists of the following parts

(a) Simple probability calculation.
(b) Mixed mode of addition law and multiplication law.
(c) The use of complementary events.

Example 3 When throwing a die, we denote the probability of getting the number X by P(X). If, given that $P(1) = \frac{1}{12}$, $P(2) = P(3) = P(4) = P(5) = \frac{1}{6}$, find

(a) the probability of getting an odd number in one throw,
(b) P(6),
(c) the probability that an odd number is obtained in the first throw and an even number is obtained in the second throw, if the die is thrown twice.

Solution (a) P(odd) = P(1) + P(3) + P(5) (they are mutually exclusive events)

$$= \frac{1}{12} + \frac{1}{6} + \frac{1}{6} = \frac{5}{12}$$

(b) P(6) = 1 − P(1) − P(2) − P(3) − P(4) − P(5)

$$= 1 - \frac{1}{12} - 4\times\frac{1}{6} = \frac{1}{4} \quad \text{(complementary events)}$$

(c) P(first odd and second even)
= P(first odd) × P(second even) (Independent Events)

= _________ × ___________

= _________

Example 4 Two men A and B play a "hammer, scissor and cloth" game. (Scissor defeats cloth, hammer defeats scissor, cloth defeats hammer). In each round, they give "hammer, scissor or cloth" at random. If they give the same gesture, the game will be a tie. They will continue to play until one of them wins that round.

(a) Find the probability that A wins a single round.
(b) Find the probability that A wins in three consecutive rounds.
(c) If they play in the following manner: the one who wins three rounds first will be the winner. Find the probability that
 (i) A wins just after the fourth round;
 (ii) the game will be ended in exactly five rounds.

Solution: (a) $P(A \text{ wins 1 round}) = \frac{1}{2}$

(b) $P(A \text{ wins 3 consecutive rounds}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$

(c) (i) P(A wins just after the fourth round)
= P(B wins exactly one round in the 1st, 2nd or 3rd round, A wins last)

$$= 3\times\frac{1}{2^4} = \frac{3}{16}$$

(ii) P(The game will be ended in five rounds)
= P(After 4 rounds, the result is 2 : 2)
= P(AABB, ABAB, ABBA, BAAB, BABA, BBAA)

$$= 6\times\frac{1}{2^4} = \frac{3}{8}$$

Example 5 Mark Six Lottery: There are altogether 49 numbered balls (from 1 to 49). Each ball is equal likely to be selected. 6 balls are drawn from the lottery, one by one without replacement. These are the 6 winning numbers of the lottery. Then one extra ball is drawn from the remaining 43 balls. This is the extra number. Each bet costs \$10, 6 numbers can be chosen at random.

(a) Find the probability of winning the first prize.

In order to win the first prize, all 6 winning numbers should be chosen in a single bet.

$$\text{Probability} = \frac{6}{49}\times\frac{5}{48}\times\frac{4}{47}\times\frac{3}{46}\times\frac{2}{45}\times\frac{1}{44} = \frac{1}{13983816} = 7.15\times10^{-8}$$

Thus in order to ensure to win the first prize, a player has to bet 13983816 times.

In other words, he has to use 13983816×\$10 = \$139838160.

(b) Find the probability of winning the second prize.

In order to win the second prize, any 5 of the 6 winning numbers + the special number should be chosen.

Let ✓ = your chosen no. matches a winning no.

e = your chosen no. matches the extra no.

× = your chosen no. does not match any winning no. or the extra no.

Your chosen no.	1st no	2nd no	3rd no	4th no	5th no	6th no
winning sequence	✓	✓	✓	✓	✓	e
	✓	✓	✓	✓	e	✓
	✓	✓	✓	e	✓	✓
	✓	✓	e	✓	✓	✓
	✓	e	✓	✓	✓	✓
	e	✓	✓	✓	✓	✓

$$\text{Probability} = 6\times\frac{6}{49}\times\frac{5}{48}\times\frac{4}{47}\times\frac{3}{46}\times\frac{2}{45}\times\frac{1}{44} = \frac{C_5^6\times C_1^1}{C_6^{49}} = \frac{1}{2330636} = 4.29\times10^{-7}$$

The probability is six times that of winning the first prize.

(c) Find the probability of winning the third prize.

Exactly 5 winning numbers should be chosen.

Your chosen no.	1st no	2nd no	3rd no	4th no	5th no	6th no
winning sequence	✓	✓	✓	✓	✓	×
	✓	✓	✓	✓	×	✓
	✓	✓	✓	×	✓	✓
	✓	✓	×	✓	✓	✓
	✓	×	✓	✓	✓	✓
	×	✓	✓	✓	✓	✓

$$\text{Probability} = 6\times\frac{6}{49}\times\frac{5}{48}\times\frac{4}{47}\times\frac{3}{46}\times\frac{2}{45}\times\frac{42}{44} = \frac{C_5^6\times C_1^{42}}{C_6^{49}} = \frac{3}{166474} = 1.80\times10^{-5}$$

It is 42 times the probability of winning the second prize.

(d) Find the probability of winning the fourth prize.

Exactly 4 winning numbers + the special number.

Your chosen no.	1st no	2nd no	3rd no	4th no	5th no	6th no		1st no	2nd no	3rd no	4th no	5th no	6th no
winning sequence	✓	✓	✓	✓	e	×	or	✓	✓	✓	✓	×	e
	✓	✓	✓	e	✓	×		✓	✓	✓	×	✓	e
	✓	✓	✓	e	×	✓		✓	✓	✓	×	e	✓
	✓	✓	e	✓	✓	×		✓	✓	×	✓	✓	e
	✓	✓	e	✓	×	✓		✓	✓	×	✓	e	✓
	✓	✓	e	×	✓	✓		✓	✓	×	e	✓	✓
	✓	e	✓	✓	✓	×		✓	×	✓	✓	✓	×
	✓	e	✓	✓	×	✓		✓	×	✓	✓	e	✓
	✓	e	✓	×	✓	✓		✓	×	✓	e	✓	✓
	✓	e	×	✓	✓	✓		✓	×	e	✓	✓	✓
	e	✓	✓	✓	✓	×		×	✓	✓	✓	✓	e
	e	✓	✓	✓	×	✓		×	✓	✓	✓	e	✓
	e	✓	✓	×	✓	✓		×	✓	✓	e	✓	✓
	e	✓	×	✓	✓	✓		×	✓	e	✓	✓	✓
	e	×	✓	✓	✓	✓		×	e	✓	✓	✓	✓

There are altogether 15×2 different cases.

$$\text{Probability} = 15\times 2\times\frac{6}{49}\times\frac{5}{48}\times\frac{4}{47}\times\frac{3}{46}\times\frac{1}{45}\times\frac{42}{44}=\frac{C_4^6\times C_1^{42}\times C_1^1}{C_6^{49}}=\frac{15}{332948}=4.51\times10^{-5}$$

It is just 2.5 times of the probability of winning the third prize!

(e) Find the probability of winning the fifth prize.

Exactly 4 winning numbers should be chosen.

Your chosen no.	1st no	2nd no	3rd no	4th no	5th no	6th no
	✓	✓	✓	✓	×	×
	✓	✓	✓	×	✓	×
	✓	✓	×	✓	✓	×
	✓	×	✓	✓	✓	×
	×	✓	✓	✓	✓	×
	✓	✓	✓	×	×	✓
	✓	✓	×	✓	×	✓
winning sequence	✓	×	✓	✓	×	✓
	×	✓	✓	✓	×	✓
	✓	✓	×	×	✓	✓
	✓	×	✓	×	✓	✓
	×	✓	✓	×	✓	✓
	✓	×	×	✓	✓	✓
	×	✓	×	✓	✓	✓
	×	×	✓	✓	✓	✓

There are altogether 15 different cases.

$$\text{Probability} = 15\times\frac{6}{49}\times\frac{5}{48}\times\frac{4}{47}\times\frac{3}{46}\times\frac{42}{45}\times\frac{41}{44}=\frac{C_4^6\times C_2^{42}}{C_6^{49}}=\frac{615}{665896}=9.24\times10^{-4}$$

It is 20.5 times that of winning the fourth prize.

(f) Find the probability of winning the sixth prize.

Exactly 3 winning numbers + the special number should be chosen.

Your chosen no.	1st	2nd	3rd	4th	5th	6th		1st	2nd	3rd	4th	5th	6th		1st	2nd	3rd	4th	5th	6th
	✓	✓	✓	×	×	e		✓	✓	✓	×	e	×		✓	✓	✓	e	×	×
	✓	✓	×	✓	×	e		✓	✓	×	✓	e	×		✓	✓	e	✓	×	×
	✓	✓	×	×	✓	e		✓	✓	×	e	✓	×		✓	✓	e	×	✓	×
	✓	✓	×	×	e	✓		✓	✓	×	e	×	✓		✓	✓	e	×	×	✓
	✓	×	✓	✓	×	e		✓	×	✓	✓	e	×		✓	e	✓	✓	×	×
	✓	×	✓	×	✓	e		✓	×	✓	e	✓	×		✓	e	✓	×	✓	×
	✓	×	✓	×	e	✓		✓	×	✓	e	×	✓		✓	e	✓	×	×	✓
	✓	×	×	✓	✓	e		✓	×	e	✓	✓	×		✓	e	×	✓	✓	×
	✓	×	×	✓	e	✓		✓	×	e	✓	×	✓		✓	e	×	✓	×	✓
winning sequence	✓	×	×	e	✓	✓	or	✓	×	e	×	✓	✓	or	✓	e	×	×	✓	✓
	×	✓	✓	✓	×	e		×	✓	✓	✓	e	×		e	✓	✓	✓	×	×
	×	✓	✓	×	✓	e		×	✓	✓	e	✓	×		e	✓	✓	×	✓	×
	×	✓	✓	×	e	✓		×	✓	✓	e	×	✓		e	✓	✓	×	×	✓
	×	✓	×	✓	✓	e		×	✓	e	✓	✓	×		e	✓	×	✓	✓	×
	×	✓	×	✓	e	✓		×	✓	e	✓	×	✓		e	✓	×	✓	×	✓
	×	✓	×	e	✓	✓		×	✓	e	×	✓	✓		e	✓	×	×	✓	✓
	×	×	✓	✓	✓	e		×	e	✓	✓	✓	×		e	×	✓	✓	✓	×
	×	×	✓	✓	e	✓		×	e	✓	✓	×	✓		e	×	✓	✓	×	✓
	×	×	✓	e	✓	✓		×	e	✓	×	✓	✓		e	×	✓	×	✓	✓
	×	×	e	✓	✓	✓		×	e	×	✓	✓	✓		e	×	×	✓	✓	✓

There are altogether 20×3 different cases.

$$\text{Probability} = 20\times 3\times\frac{6}{49}\times\frac{5}{48}\times\frac{4}{47}\times\frac{1}{46}\times\frac{42}{45}\times\frac{41}{44}=\frac{C_3^6\times C_2^{42}\times C_1^1}{C_6^{49}}=\frac{205}{166474}=1.23\times10^{-3}$$

It is $\frac{4}{3}$ times of that of winning the 5th prize.

(g) Find the probability of winning the seventh prize.
Exactly 3 winning numbers should be chosen.

Your chosen no.	1^{st} no	2^{nd} no	3^{rd} no	4^{th} no	5^{th} no	6^{th} no
winning sequence	✓	✓	✓	×	×	×
	✓	✓	×	✓	×	×
	✓	✓	×	×	✓	×
	✓	✓	×	×	×	✓
	✓	×	✓	✓	×	×
	✓	×	✓	×	✓	×
	✓	×	✓	×	×	✓
	✓	×	×	✓	✓	×
	✓	×	×	✓	×	✓
	✓	×	×	×	✓	✓
	×	✓	✓	✓	×	×
	×	✓	✓	×	✓	×
	×	✓	✓	×	×	✓
	×	✓	×	✓	✓	×
	×	✓	×	✓	×	✓
	×	✓	×	×	✓	✓
	×	×	✓	✓	✓	×
	×	×	✓	✓	×	✓
	×	×	✓	×	✓	✓
	×	×	×	✓	✓	✓

There are altogether 20 different cases.

Probability $=20\times\frac{6}{49}\times\frac{5}{48}\times\frac{4}{47}\times\frac{42}{46}\times\frac{41}{45}\times\frac{40}{44}=\frac{C_3^6\times C_3^{42}}{C_6^{49}}=\frac{4100}{249711}=0.0164$

It is $\frac{40}{3}$ times of winning the 6th prize.

(h) Find the probability of winning some money.
Probability = P(1^{st} prize or 2^{nd} prize or ... or 7^{th} prize)

$=\frac{1}{13983816}+\frac{1}{2330636}+\frac{3}{166474}+\frac{15}{332948}+\frac{615}{665896}+\frac{205}{166474}+\frac{4100}{249711}=0.0186\approx\frac{1}{53.7}$

On the average, a person will win one time for every 54 bets.

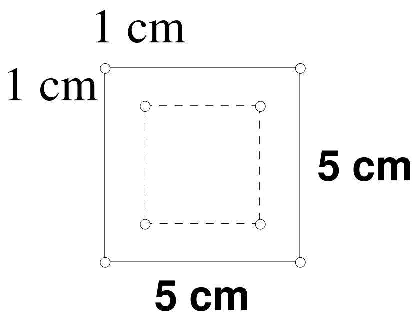
(i) Allocation of entry money
Every dollar spent on Mark Six is distributed as follows:
25% - Lottery Duty - paid to the Government
15% - Lottery fund - used for financing social welfare capital projects
6% - Commission -used to cover operating costs of Hong Kong Jockey Club
54% - Prize Fund -used to fund prize
As of 2011, the winning probability for each division is as follows:

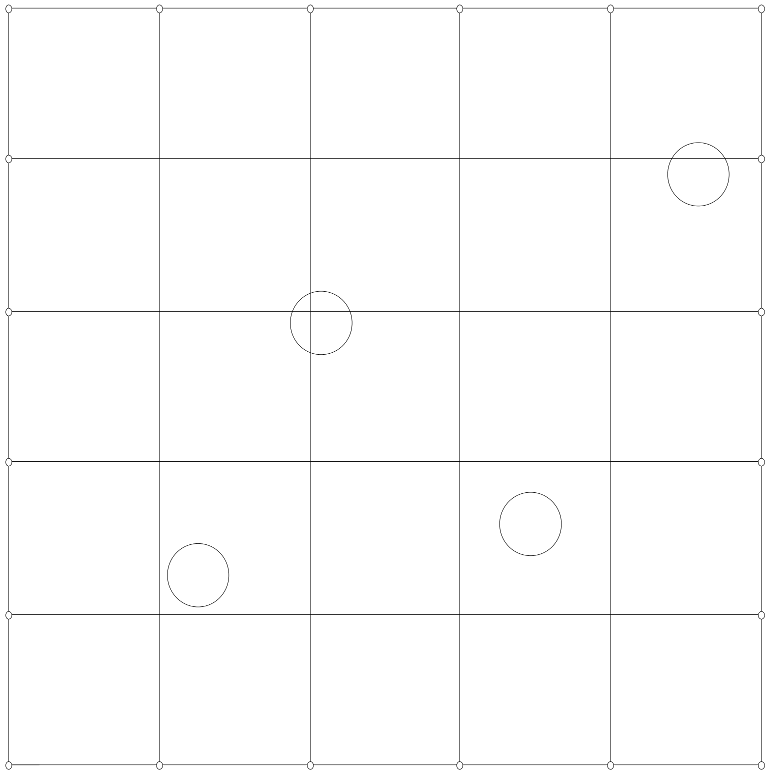
Prize	Criteria	Dividend	Probability
1st Division	All 6 drawn numbers	45% of prize fund after deduction	$\frac{1}{13983816}$
2nd Division	5 out of 6 drawn numbers, plus the extra number	15% of prize fund after deduction	$\frac{1}{2330636}$
3rd Division	5 out of 6 drawn numbers	40% of prize fund after deduction	$\frac{3}{166474}$
4th Division	4 out of 6 drawn numbers, plus the extra number	$9600	$\frac{15}{332948}$
5th Division	4 out of 6 drawn numbers	$640	$\frac{615}{665896}$
6th Division	3 out of 6 drawn numbers, plus the extra number	$320	$\frac{205}{166474}$
7th Division	3 out of 6 drawn numbers	$40	$\frac{4100}{249711}$

9.4 Continuous Probability

Example 6 A grid (網格) of rectangular squares each side is 5 cm. A circular coin of radius 1 cm is thrown onto the grid. Find the probability that the coin overlaps the grid.

Solution:





The centre of the coin must lie inside one of the square at random.

It <u>does not</u> overlap the boundary if the centre lies on the concentric square of side 3 cm.

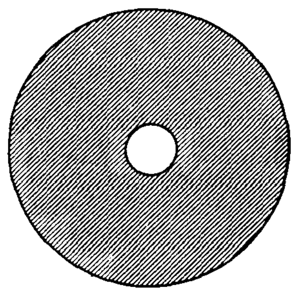
Probability

$$= \frac{\text{Area of small square}}{\text{area of big square}} = \frac{9}{25}$$

$$\therefore \text{P(overlaps)} = 1 - \frac{9}{25} = \frac{16}{25}$$

Exercise 3 HKCEE 1996 Q7

The figure shows a circular dartboard. Its surface consists of two concentric circles of radii 12 cm and 2 cm respectively.



(a) Find the area of the shaded region on the dartboard.

(b) Two darts are thrown and hit the dartboard. Find the probability that

(i) both darts hit the shaded region;

(ii) only one dart hits the shaded region.

Answers (a) 140π cm^2 (b) (i) $\frac{1225}{1296}$ (ii) $\frac{35}{648}$

Exercise 4 In a 3 cm × 3 cm square card box, there is a circular dark spot of diameter 1 cm located at the centre of the box. Katharine throws a coin of diameter 1.8 cm into the box. How likely will the coin fully cover the dark spot?

ABCD is the 3 cm × 3 cm square card box.

The circular dark spot (c_1) of radius 0.5 cm centre O.

J is the centre of coin (c_3) of radius 0.9 cm.

J will lie on the concentric square with side = 1.2 cm

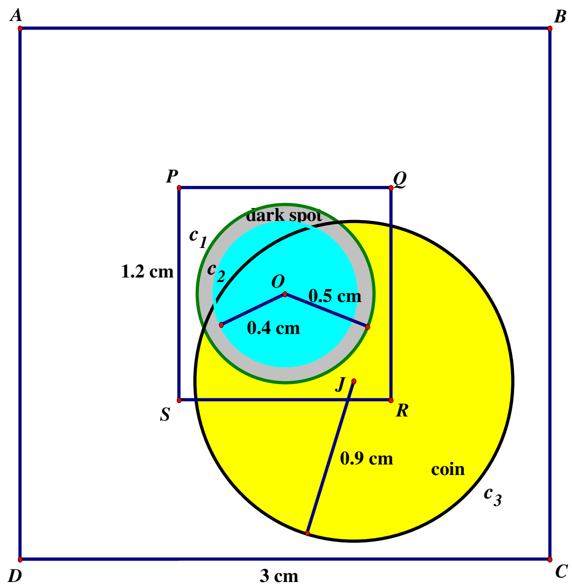
(c_2) is the concentric circle with radius 0.4 cm centre O.

The coin will fully cover the dark spot if J lies on or inside (c_2).

The coin will not fully cover the dark spot if J lies outside (c_2).

$$\text{Required probability} = \frac{\text{area of } c_2}{\text{area of square PQRS}}$$

[Answer 0.349]



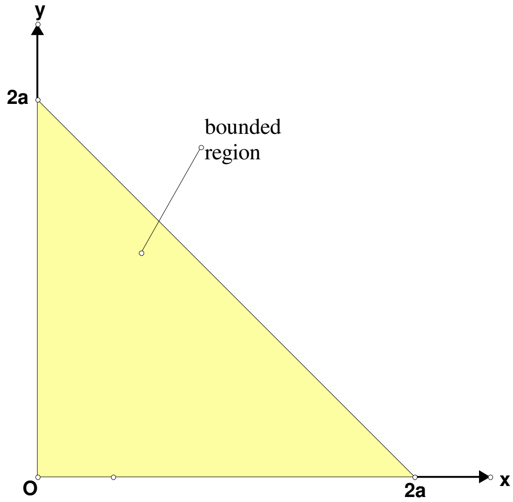
Example 7 Two points P and Q are chosen at random on a line segment AB of length $2a$. The line segment AB is then divided into 3 segments AP, PQ and QB. Find the probability that AP, PQ and QB from a triangle.

Solution: Let x, y, $2a - x - y$ be the length of the segments.

then $\begin{cases} 0 \le x \le 2a \\ 0 \le y \le 2a \\ 0 \le 2a - x - y \le 2a \end{cases}$

i.e. $\begin{cases} 0 \le x \le 2a \\ 0 \le y \le 2a \\ 0 \le x + y \le 2a \end{cases}$

the point (x, y) must lie inside this region regardless the 3 segments from a triangle or not.



Now, in order that they form a triangle, the 3 segments have to satisfy the triangle inequality. i.e. the sum of any 2 sides must greater than the third.

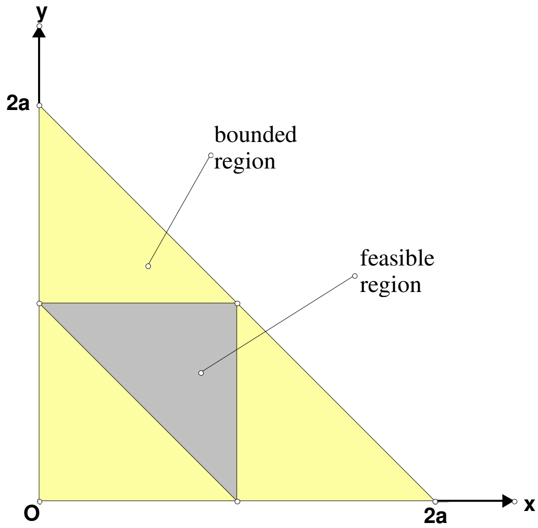
$\begin{cases} x + y - (2a - x - y) \ge 0 \\ x + (2a - x - y) - y \ge 0 \\ y + (2a - x - y) - x \ge 0 \end{cases}$

i.e. $\begin{cases} x + y \ge a \\ y \le a \\ x \le a \end{cases}$

P(form a triangle)

$= \dfrac{\text{area of feasible region}}{\text{area of bounded region}}$

= ______________



Example 8 Peter and Mary agree to meet at Tsuen Wan MTR between 8:00 am and 9:00 am. When one of them arrives first, they agree to wait for the other for at most 10 minutes. What is the probability they will meet each other?

Solution Suppose Peter arrives at x minutes past 8 o'clock and Mary arrives at y minutes past 8 o'clock.

Then (x, y) is any point inside the bounded region.

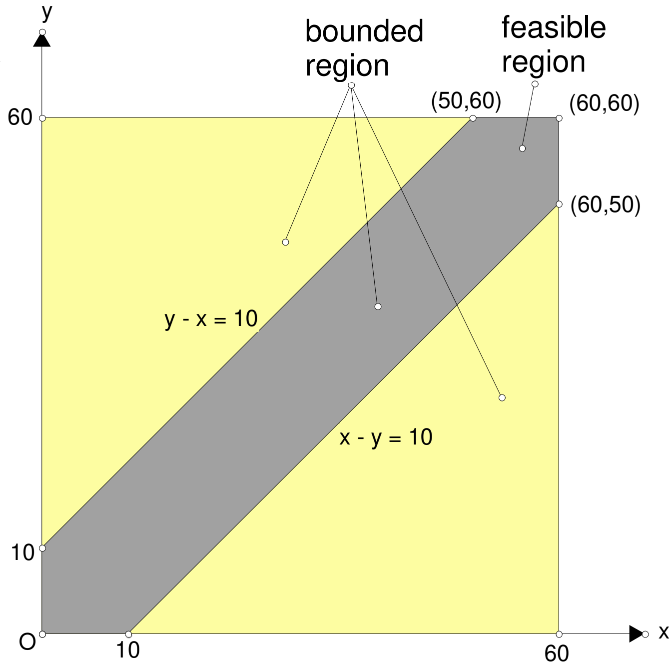
They will meet each other if and only if $-10 \le x - y \le 10$

P(they will meet)

$= \dfrac{\text{area of feasible region}}{\text{area of bounded region}}$

$= \dfrac{60 \times 60 - 2 \times \frac{1}{2} \times 50 \times 50}{60 \times 60}$

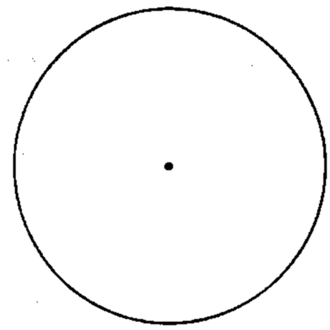
$= \dfrac{11}{36}$



Class Activity 2 NTCM Student Math Note, May, 1985.

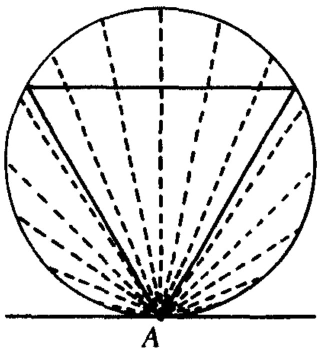
Sometimes more than one theoretical probability can be computed for the same problem. In the nineteenth century, the French mathematician Joseph Bertrand worked on such a problem:

In the circle, draw a chord of any length you like. (Draw it before you read any further.) Now, use your compass and straightedge to construct an equilateral triangle that is inscribed in a circle.

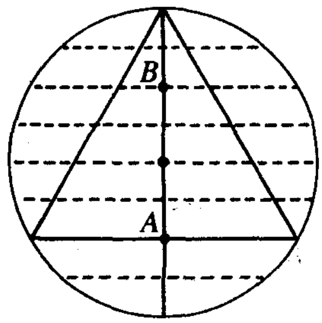


Compare the length of the random chord you drew to the length of the side of the triangle. Which is longer? Thirty mathematics students did this, and 23 of them (about 75%) drew chords that had a length shorter than a side of the triangle. For them, chance appears to favor random chords shorter than the side of the triangle.

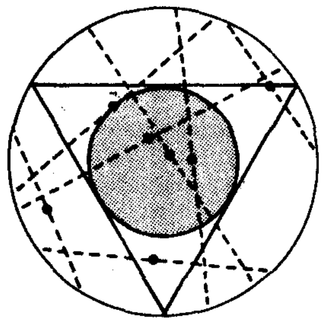
Computing the theoretical probability in this problem is interesting in that there are three ways to do the computation, depending on how the problem is interpreted. All three techniques yield different answers: $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{1}{4}$. The geometrical diagrams below are the basis for computing each of the different probabilities. This paradox of three answers for this problem stems from there being no precise definition of how to draw a random chord. See if you can figure out how each of these probabilities was obtained.



Probability of $\frac{1}{3}$



Probability of $\frac{1}{2}$


Probability of $\frac{1}{4}$

9.5 Probability involving the use of infinite geometric sum

e.g. 9 Three players throw a die one by one and the first to throw a 'six' wins. Find the probability that

(a) the first player wins,

(b) the second player wins.

Solution (a) The required probability

$$= \frac{1}{6} + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^6\left(\frac{1}{6}\right) + \cdots$$

$$= \frac{a}{1-r}$$

= ________________________

(b) The required probability

= ________________________

= ________________________

(Ans. = $\frac{30}{91}$)

Exercise 5 In throwing 3 dice, the one who first gets an eleven will win a prize. A throws the dice first then B and then C. If they do not get the eleven, they will go on the same order until one of them gets the prize. Find the probability of each of them to get the prize.

Ans. $P(A) = \frac{64}{169}$, $P(B) = \frac{56}{169}$

END