16-17 Individual	**1**	2	**2**	3	**3**	30	**4**	10	**5**	1
	6	–6049	**7**	79	**8**	$\frac{22}{35}$	**9**	90°	**10**	5
	11	44	**12**	$\frac{1}{42}$	**13**	20 or $3\sqrt{11}$ see the remark	**14**	16	**15**	$4\sqrt{5}$
16-17 Group	**1**	(8, 10)	**2**	60	**3**	45°	**4**	4	**5**	10
	6	16	**7**	4034	**8**	13 see the remark	**9**	3	**10**	11

Individual Events

I1 已知 $A2017B$ 是一個六位數，且可被 72 整除，求 A 的值。

Given that $A2017B$ is a 6-digit number which is divisible by 72, find the value of A .

Reference: 2001 FG1.3, 2003 FI4.1

$72 = 8\times9$, the number is divisible by 8 and 9 .

$\overline{17B}$ is divisible by 8, i.e. $B = 6$.

$A + 2 + 0 + 1 + 7 + 6 = 9m$, where m is an integer.

$16 + A = 9m$, $A = 2$

$A = 2$

I2 已知 $0 \le p \le 1$，求 $Q = 3p^2(1 - p) + 6p(1 - p)^2 + 3(1 - p)^3$ 的最大值。

Given that $0 \le p \le 1$, find the greatest value of $Q = 3p^2(1 - p) + 6p(1 - p)^2 + 3(1 - p)^3$.

$Q = 3p^2(1 - p) + 6p(1 - p)^2 + 3(1 - p)^3$

$\quad = 3(1 - p)(p + 1 - p)^2$

$Q = 3(1 - p) \le 3$

The maximum value of Q is 3.

I3 已知 ΔABC 的三條邊的長是 a 、b 和 c，其中 $3 \le a \le 5 \le b \le 12 \le c \le 15$，求當 ΔABC 的面積最大時，它的周界是多少？

Given that the three sides of ΔABC are of lengths a, b and c, where $3 \le a \le 5 \le b \le 12 \le c \le 15$, find the perimeter of ΔABC when its area attains the maximum value.

c is the longest side.

Area $=\frac{1}{2}\cdot ab\sin C \le \frac{1}{2}\cdot ab\cdot 1$ (Equality holds when $a^2 + b^2 = c^2$)

The largest area is attained when $a = 5$, $b = 12$, $c = 13$

Perimeter = 5 + 12 + 13 = 30

I4 設 B 及 C 為正整數，求 C 的最小值使得 $B^2 = C + 134$。

Let B and C be positive integers. Find the least value of C satisfying $B^2 = C + 134$.

C is the least when B is the least.

$B^2 = C + 134 \le 144 = 12^2$

When $B = 12$, $C = \underline{\mathbf{10}}$

I5 若把一組自然數之和 $1 + 2 + 3 + \cdots + 2015 + 2016 + 2017$ 除以 9，餘數是甚麼？

Determine the remainder when the sum of natural numbers $1 + 2 + 3 +\cdots+ 2015 +2016 +2017$ is divided by 9 .

$$\begin{aligned}1 + 2 + \cdots + 2017 &= \frac{1+2017}{2}\cdot 2017\\ &= 1009\times2017\\ &= (112\times9 + 1)(224\times9 + 1)\\ &= 9m + 1\end{aligned}$$

The remainder when divided by 9 is 1.

I6 已知 $a_0 = 2$，$a_1 = -1$ 及 $a_{n+1} = 2a_n - a_{n-1}$，其中 $n \geq 1$，求 a_{2017} 的值。

Given that $a_0 = 2$, $a_1 = -1$ and $a_{n+1} = 2a_n - a_{n-1}$, where $n \geq 1$, determine the value of a_{2017}.

The characteristics equation is $\lambda^2 = 2\lambda - 1 \Rightarrow \lambda = 1$

The general solution is $a_n = (An + B)\cdot 1^n = An + B$

$a_0 = 0 + B = 2$, $a_1 = A + 2 = -1 \Rightarrow A = -3$

$a_n = 2 - 3n \Rightarrow a_{2017} = 2 - 3\times 2017 = -6049$

I7 設 N 為完全立方數，已知 $N = 161x + 23y$，其中 x 和 y 均為正整數。求 $x + y$ 的最小值。

Let N be a perfect cube number. Given that $N = 161x + 23y$, where x and y are positive integers. Find the minimum value of $x + y$.

$161x + 23y = 23(7x + y) = m^3$

$7x + y = 23^2 = 529 = 7\times 75 + 4$

$x = 75$, $y = 4$

Minimum value of $x + y = 79$

I8 已知 ② $= 1\times 2\times 3\times 4$，③ $= 2\times 3\times 4\times 5$，④ $= 3\times 4\times 5\times 6$，… 及 $\frac{1}{⑮} - \frac{1}{⑰} = \frac{1}{⑰} \times A$，求 A 的值。

Given that ② $= 1\times 2\times 3\times 4$, ③ $= 2\times 3\times 4\times 5$, ④ $= 3\times 4\times 5\times 6$, … and $\frac{1}{⑮} - \frac{1}{⑰} = \frac{1}{⑰} \times A$, find the value of A.

$$\frac{1}{14\times 15\times 16\times 17} - \frac{1}{16\times 17\times 18\times 19} = \frac{1}{16\times 17\times 18\times 19} \times A$$

$$\frac{1}{14\times 15} - \frac{1}{18\times 19} = \frac{1}{18\times 19} \times A$$

$18\times 19 - 14\times 15 = 14\times 15A$

$132 = 210A$

$A = \frac{22}{35}$

I9 已知 $\sin x\cdot\cos x = 0$ 及 $\sin^3 x - \cos^3 x = 1$，其中 $90° \leq x < 180°$，求 x 的值。

Given that $\sin x\cdot\cos x = 0$ and $\sin^3 x - \cos^3 x = 1$, where $90° \leq x < 180°$, find the value of x.

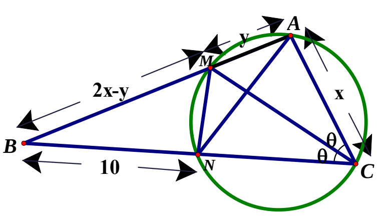
$\sin x = 0$ or $\cos x = 0$

$x = 0°$ (rejected), $180°$ (rejected) or $\underline{\mathbf{90°}}$

When $x = 90°$, $\sin^3 x - \cos^3 x = 1$

I10 如圖一，CM 是 $\angle ACB$ 的角平分綫，且 $AB = 2AC$。已知 ΔAMC 的外接圓與 BC 相交於 N。若 $BN = 10$，求 AM 的長度。

In Figure 1, CM is the angle bisector of $\angle ACB$ and $AB = 2AC$. Given that the circumscribed circle of ΔAMC intersects BC at N. If $BN = 10$, find the length of AM.



圖一 Figure 1

Let $AC = x$, $AM = y$, then $AB = 2x$, $BM = 2x - y$

Let $\angle ACM = \angle BCM = \theta$

$\angle MAC = \theta$ ($\angle$s in the same segment)

$\angle ANM = \theta$ ($\angle$s in the same segment)

$MN = y$ (sides opposite equal $\angle$s)

$\Delta BMN \sim \Delta BCA$ (equiangular)

$\frac{y}{10} = \frac{x}{2x} \Rightarrow AM = y = 5$

I11 已知 x 為一實數，求 $\sqrt{x(x+3)(x+6)(x+9)+2017}$ 的最小值。

Given that x is a real number, find the least value of $\sqrt{x(x+3)(x+6)(x+9)+2017}$.

Reference 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3, 2013 HI5

$$\begin{aligned}\sqrt{x(x+3)(x+6)(x+9)+2017} &= \sqrt{x(x+9)(x+3)(x+6)+2017}\\ &= \sqrt{(x^2+9x)(x^2+9x+18)+2017}\\ &= \sqrt{(x^2+9x)^2+18(x^2+9x)+9^2+1936}\\ &= \sqrt{(x^2+9x+9)^2+44^2} \geq 44\end{aligned}$$

The minimum value is 44 .

I12 已知 $\frac{x}{x^2-5x+1}=\frac{1}{2}$，求 $\frac{x^2}{x^4-5x^2+1}$ 的值。

Given $\frac{x}{x^2-5x+1}=\frac{1}{2}$, find the value of $\frac{x^2}{x^4-5x^2+1}$.

$$\frac{x^2-5x+1}{x}=2 \Rightarrow \frac{x^2+1}{x}-5=2 \Rightarrow x+\frac{1}{x}=7$$

$$\left(x+\frac{1}{x}\right)^2=49 \Rightarrow x^2+2+\frac{1}{x^2}=49 \Rightarrow x^2+\frac{1}{x^2}=47$$

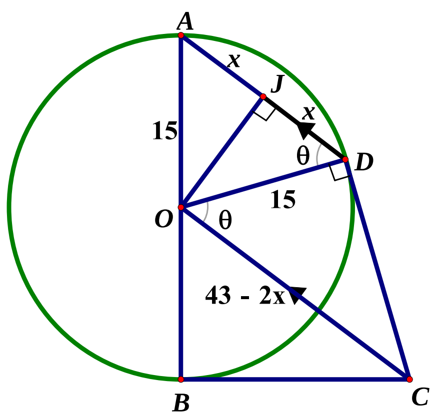
$$x^2-5+\frac{1}{x^2}=42 \Rightarrow \frac{x^4-5x^2+1}{x^2}=42$$

$$\Rightarrow \frac{x^2}{x^4-5x^2+1}=\frac{1}{42}$$

I13 如圖二，O 是圓 ADB 的圓心。BC 及 CD 分別是圓形在點 B 及 D 的切綫。$OC \mathbin{/\mkern-6mu/} AD$，$OA = 15$。若 $AD + OC = 43$，求 CD 的長。

As shown in Figure 2, O is the centre of the circle ADB. BC and CD are tangents to the circle at points B and D respectively. $OC \mathbin{/\mkern-6mu/} AD$, $OA = 15$.

If $AD + OC = 43$, find the length of CD .



圖二 Figure 2

Join OD. $OD \perp DC$ (tangent $\perp$ radius)

Draw $OJ \perp AD$. $\Delta OAJ \cong \Delta ODJ$ (R.H.S.)

Let $AJ = JD = x$ (corr. sides $\cong$ Δs), $OC = 43 - 2x$

Let $\angle ODA = \theta$, $\angle COD = \theta$ (alt. $\angle$s $AD \mathbin{/\mkern-6mu/} OC$)

$$\cos\theta = \frac{x}{15} = \frac{15}{43-2x}$$

$43x - 2x^2 = 225$

$2x^2 - 43x + 225 = 0$

$(x - 9)(2x - 25) = 0$

$x = 9$ or 12.5

When $x = 9$, $OC = 43 - 2x = 25$

$CD^2 = OC^2 - OD^2 = 25^2 - 15^2$ (Pythagoras' theorem)

$CD = 20$

When $x = 12.5$, $OC = 43 - 2x = 18$

$CD^2 = OC^2 - OD^2 = 18^2 - 15^2 = 99$ (Pythagoras' theorem)

$CD = 3\sqrt{11}$ (**Remark:** Candidates give answer with either 20 or $3\sqrt{11}$ will score the mark)

I14 若 $a + \log_2 b = a^2 + \log_2 b^3 - 10 = 3$，其中 $b > 1$，求 b 的值。

If $a + \log_2 b = a^2 + \log_2 b^3 - 10 = 3$, where $b > 1$, find the value of b.

$\log_2 b = 3 - a$ …… (1)

$a^2 + \log_2 b^3 - 10 = 3$

$a^2 + 3 \log_2 b - 10 = 3$

Sub. (1) into the equation: $a^2 + 3(3 - a) - 10 = 3$

$a^2 - 3a - 4 = 0$

$(a - 4)(a + 1) = 0$

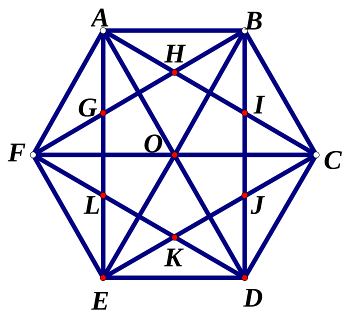
$a = 4$ or -1

Sub. $a = 4$ into (1): $\log_2 b = 3 - 4 = -1 \Rightarrow b = 2^{-1} < 1$ (rejected)

Sub. $a = -1$ into (1): $\log_2 b = 3 + 1 = 4 \Rightarrow b = 2^4 = 16$

I15 在圖三中，已知 $ABCDEF$ 為正六邊形，且它的面積是 $90\sqrt{3}$，求 GJ 的值。

In Figure 3, given that $ABCDEF$ is a regular hexagon and its area is $90\sqrt{3}$, find the length of GJ.



圖三 Figure 3

Let O be the centre. Let $AB = a$, OA, OB, OC, OD, OE, OF divides the hexagon $ABCDEF$ into 6 congruent equilateral triangles with sides a.

$$\frac{6}{2} \cdot a^2 \sin 60^\circ = 90\sqrt{3}$$

$\Rightarrow a^2 = 60$

$\Rightarrow a = \sqrt{60}$

In ΔOFG, $\angle GOF = 30° = \angle GFO$, $OF = AB = a$

$$\frac{a}{2OG} = \cos 30^\circ$$

$$\frac{\sqrt{60}}{2} = \frac{\sqrt{3}}{2} OG$$

$OG = \sqrt{20} = 2\sqrt{5}$

$GJ = 2OG = 4\sqrt{5}$

Group Events

G1 設 ΔABC 為一等腰直角三角形，頂點 A 及 B 的座標分別為 (–2, 0) 及 (18, 0)，且 C 的座標是正數。當 ΔABC 的面積為最小時，求 C 的座標。

Suppose that ΔABC is an isosceles right-angled triangle with the coordinates of the vertices A and B as (–2, 0) and (18, 0), respectively, and the coordinates of C having positive values. Determine the coordinates of C when the area of ΔABC attains its minimum.

When the area of ΔABC attains its minimum, AB is the hypotenuse, $AC = BC$, $AC \perp BC$.

Let M be the mid-point of $AB = (8, 0)$. Let the coordinates of C be $(8, y)$.

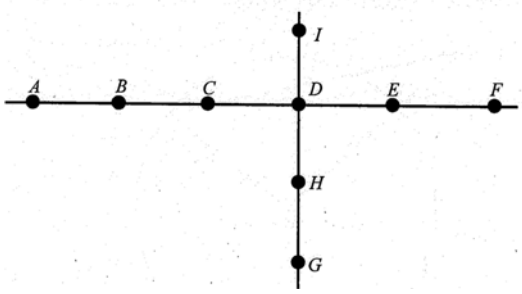
$$\frac{y}{8+2}\cdot\frac{y}{8-18}=-1$$

$y^2 = 100$

$y = 10$, the coordinates of C is $(8, 10)$.

G2 如圖一所示，點 A、B、C、D、E 及 F 均在一直綫上。點 G、H、D 及 I 在另一直綫上。揀選三點，可形成多少個三角形？

As shown in Figure 1, points A, B, C, D, E and F lie on the same straight line, and G, H, D and I lie on another straight line. How many triangles can be made by connecting any three points?



圖一 Figure 1

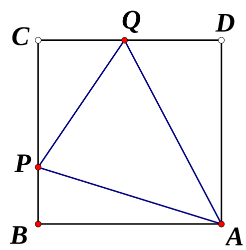
Number of triangles without $D = C_2^5\cdot C_1^3 + C_1^5\cdot C_2^3 = 45$

Number of triangles with $D = C_1^5\cdot C_1^3 = 15$

Total number of triangles = 45 + 15 = 60

G3 如圖二所示，P、Q 分別是正方形 $ABCD$ 的邊 BC 及 CD 上的點。已知 ΔPCQ 的周界的長等於正方形 $ABCD$ 的周界的長的 $\frac{1}{2}$，求 $\angle PAQ$ 的值。

As shown in Figure 2, P, Q are points on the sides BC and CD of a square $ABCD$. Given that the perimeter of ΔPCQ is $\frac{1}{2}$ of that of the square $ABCD$, find the value of $\angle PAQ$.



圖二 Figure 2

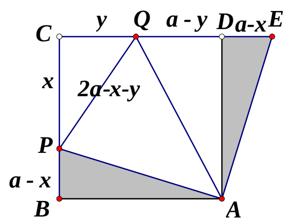
Reference: Dropbox/Data/My%20Web/Home_Page/Geometry/transform/Q5.pdf, 2006 HG7

Let $AB = BC = CD = DA = a$, perimeter of $\Delta PCQ = 2a$

Let $CP = x$, $CQ = y$, $BP = a - x$, $CQ = a - y$, $PQ = 2a - x - y$

Rotate ΔABP about A in clockwise direction by 90° to ΔADE

Then $\Delta ABP \cong \Delta ADE$; $DE = a - x$, $AP = AE$ (corr. sides $\cong$ Δ's)



$AQ = AQ$ (common side)

$PQ = 2a - x - y = (a - y) + (a - x) = QE$

$\therefore \Delta APQ \cong \Delta AEQ$ (S.S.S.)

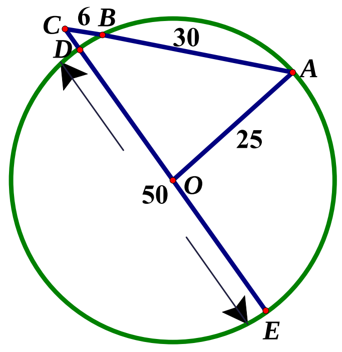
$\angle PAE = 90°$ (by rotation)

$\angle PAQ = \angle EAQ$ (corr. $\angle$s. $\cong$ Δ's)

$\angle PAQ = 45°$

G4 在圖三中，O 是圓心。弦 AB 及半徑 OD 的延綫相交於 C。已知 $OA = 25$、$AB = 30$ 及 $BC = 6$。求 CD 的長。

In Figure 3, O is the centre of the circle. Chord AB and radius OD are produced to meet at C. Given that $OA = 25$, $AB = 30$ and $BC = 6$, find the length of CD .



圖三 Figure 3

Produce CO to meet the circle again at E. DE = diameter = 50

By intersecting chords theorem, $CB \times CA = CD \times CE$

$6\times36 = CD\times(CD + 50)$

$CD^2 + 50CD - 216 = 0$

$(CD - 4)(CD + 54) = 0$

$CD = 4$

G5 設 Q 為所有能滿足不等式 $\dfrac{9p^2}{\left(\sqrt{3p+1}-1\right)^2}<3p+10$ 的整數 p 之和，求 Q 的值。

Let Q be the sum of all integers p satisfying the inequality $\dfrac{9p^2}{\left(\sqrt{3p+1}-1\right)^2}<3p+10$,

find the value of Q .

$3p + 1 \ge 0$ and $3p + 10 > 0$ and $\sqrt{3p+1}-1\neq 0$ and $9p^2<(3p+10)\left(3p+1-2\sqrt{3p+1}+1\right)$

$p\ge -\frac{1}{3}$ and $p \neq 0$ and $9p^2<(3p+10)(3p+2)-2(3p+10)\sqrt{3p+1}$

$p\ge -\frac{1}{3}$ and $p \neq 0$ and $9p^2<9p^2+36p+20-2(3p+10)\sqrt{3p+1}$

$p\ge -\frac{1}{3}$ and $p \neq 0$ and $(3p+10)\sqrt{3p+1}<18p+10$

$p\ge -\frac{1}{3}$ and $p \neq 0$ and $(3p + 10)^2(3p + 1) < (18p + 10)^2$

$p\ge -\frac{1}{3}$ and $p \neq 0$ and $27p^3 + 189p^2 + 360p + 100 < 324p^2 + 360p + 100$

$p\ge -\frac{1}{3}$ and $p \neq 0$ and $27p^3 - 135p^2 < 0$

$p\ge -\frac{1}{3}$ and $p < 5$

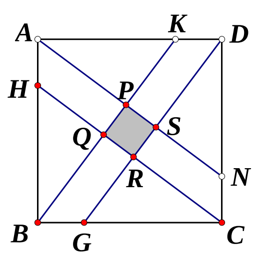
$p = 1, 2, 3$ or 4

Sum of all integers $p = 1 + 2 + 3 + 4 = 10$

G6 在圖四中，正方形 $ABCD$ 的邊長為 20。已知 $DK : KA = AH : HB = 1 : 3$ 及 $BK // GD$，$HC // AN$，求陰影部分 $PQRS$ 的面積。

In Figure 4, square $ABCD$ has sides of length 20 .

Given that $DK : KA = AH : HB = 1 : 3$ and $BK // GD$, $HC // AN$, find the area of shaded region $PQRS$. (**Reference 2009 HG6**)



圖四 Figure 4

$AK = 15 = HB$, $DK = 5 = AH$, $\angle KAB = 90° = \angle HBC$, $AB = BC$

$\Delta ABK \cong \Delta BCH$ (S.A.S.)

Let $\angle ABK = \theta = \angle BCH$ (corr. $\angle$s, $\cong \Delta$'s)

$\angle BHC = 90° - \theta$ ($\angle$ sum of Δ)

In ΔBQH, $\angle BQH = 180° - \theta - (90° - \theta) = 90°$ ($\angle$ sum of Δ)

$\because$ $BK // GD$, $HC // AN$ and $\angle BQH = 90°$

$\therefore$ $PQRS$ is a rectangle

$BK=\sqrt{15^2+20^2}=25$, $\cos\theta=\frac{20}{25}=\frac{4}{5}$

$PQ=AH\cos\theta=5\times\frac{4}{5}=4$

$PS=DK\cos\theta=5\times\frac{4}{5}=4$

Area of $PQRS=4\times4=16$

G7 已知對於實數 x_1、x_2、x_3、…、x_{2017}，

$$\sqrt{x_1-1}+\sqrt{x_2-1}+\sqrt{x_3-1}+\cdots+\sqrt{x_{2017}-1}=\frac{1}{2}\left(x_1+x_2+x_3+\cdots+x_{2017}\right)，$$

求 $x_1+x_2+x_3+\cdots+x_{2017}$ 的值。

It is given that for real numbers $x_1, x_2, x_3, \cdots, x_{2017}$,

$$\sqrt{x_1-1}+\sqrt{x_2-1}+\sqrt{x_3-1}+\cdots+\sqrt{x_{2017}-1}=\frac{1}{2}\left(x_1+x_2+x_3+\cdots+x_{2017}\right),$$

Find the value of $x_1+x_2+x_3+\cdots+x_{2017}$.

$x_1\ge1, x_2\ge1,\cdots, x_{2017}\ge1$ (otherwise, $\sqrt{x_1-1}+\sqrt{x_2-1}+\sqrt{x_3-1}+\cdots+\sqrt{x_{2017}-1}$ is undefined)

For $1\le i\le2017$, $\sqrt{x_i-1}\le\frac{1}{2}\left(x_i-1+1\right)=\frac{1}{2}x_i$ (A.M. ≥ G.M., equality holds when $x_i=2$)

$$\sqrt{x_1-1}+\sqrt{x_2-1}+\sqrt{x_3-1}+\cdots+\sqrt{x_{2017}-1}=\frac{1}{2}\left(x_1+x_2+x_3+\cdots+x_{2017}\right)$$

$\Rightarrow x_1=x_2=\cdots=x_{2017}=2$

$x_1+x_2+x_3+\cdots+x_{2017}=2\times2017=4034$

G8 設正整數 T 能滿足條件：T 的數字的積是 $T^2-11T-23$。求該等正整數之和，S 的值。

Let positive integers, T, satisfy the condition: the product of the digits of T is $T^2-11T-23$.

Find the sum S, of all such positive integers.

Let $y=T^2-11T-23=(T-5.5)^2-53.25$, y is decreasing for $T<5.5$, increasing for $T>5.5$

If $1\le T\le5$, then $T=T^2-11T-23<1^2-11-23<0$, which is impossible

$y>0\Leftrightarrow(T-5.5)^2-53.25>0\Leftrightarrow T-5.5>\sqrt{53.25}>\sqrt{42.25}=6.5\Leftrightarrow T>12$

When $T=13$, $y=13^2-11\times13-23=3=1\times3=$ product of digits

$\therefore$ $T=13$ is one possible solution

Δ of y $(=T^2-11T-23)$ is $(-11)^2-4(-23)=213$, which is not a perfect square

$\therefore$ y cannot be a composite number

However, $y=T^2-11T-23=$ product of its digits of T

$\Rightarrow y=1\times$ prime number

$\therefore$ $1<y=$ prime number <9

$\because$ y is strictly increasing for $T>5.5$

When $T=14$, $T^2-11T-23=14^2-11\times14-23=19$, which is a two-digit number

$\therefore$ There is no solution for $T\ge14$

$\therefore$ There is only one possible solution $T=13$ which satisfies $1<y<9$

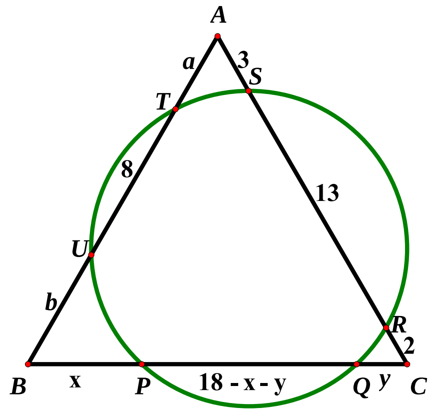
$S=$ sum of all such positive integers $=13$

Remark Original version ⋯ product of the digits of $T=T^2-11T-23$ ⋯

Somebody will confuse that $T=T^2-11T-23$.

G9 在圖五中，ABC 是一個等邊三角形且與一圓相交於六點：P、Q、R、S、T 及 U。若 $AS = 3$，$SR = 13$，$RC = 2$ 及 $UT = 8$，求 $BP - QC$ 的值。

In Figure 5, ABC is an equilateral triangle intersecting the circle at six points P, Q, R, S, T and U. If $AS = 3$, $SR = 13$, $RC = 2$ and $UT = 8$, find the value of $BP - QC$.



圖五 Figure 5

Reference: 2015 HG9

Let $AT = a$, $BU = b$, $BP = x$, $QC = y$, $PQ = 18 - x - y$

By intersecting chord theorem,

$a(a + 8) = 3\times(3 + 13)$

$a^2 + 8a - 48 = 0$

$(a - 4)(a + 12) = 0$

$a = 4$ or -12 (rejected)

$AC = 3 + 13 + 2 = 18 = AB = BC$

$b = 18 - 4 - 8 = 6$

$x(x + 18 - x - y) = 6\times(6 + 8)$

$x(18 - y) = 84 \cdots\cdots (1)$

$y(y + 18 - x - y) = 2\times(2 + 13)$

$y(18 - x) = 30 \cdots\cdots (2)$

(1) – (2): $18(x - y) = 54$

$BP - QC = x - y = 3$

G10 已知方程 $a^2x^2 - (4a - 3a^2)x + 2a^2 - a - 21 = 0$ (其中 $a > 0$) 最少有一個整數根，求所有 a 的可能整數值之和。

It is given that the equation $a^2x^2 - (4a - 3a^2)x + 2a^2 - a - 21 = 0$ (where $a > 0$) has at least one integral root. Find the sum of all possible integral values of a.

$\Delta = (4a - 3a^2)^2 - 4a^2(2a^2 - a - 21)$

$\Delta = 16a^2 - 24a^3 + 9a^4 - 8a^4 + 4a^3 + 84a^2$

$\Delta = a^4 - 20a^3 + 100a^2 = a^2(a - 10)^2$

$$x = \frac{(4a-3a^2) \pm \sqrt{a^2(a-10)^2}}{2a^2}$$

$$= \frac{(4a-3a^2) \pm a(a-10)}{2a^2}$$

$$= \frac{(4-3a) \pm (a-10)}{2a}$$

$$x = \frac{-6-2a}{2a} \text{ or } \frac{14-4a}{2a}$$

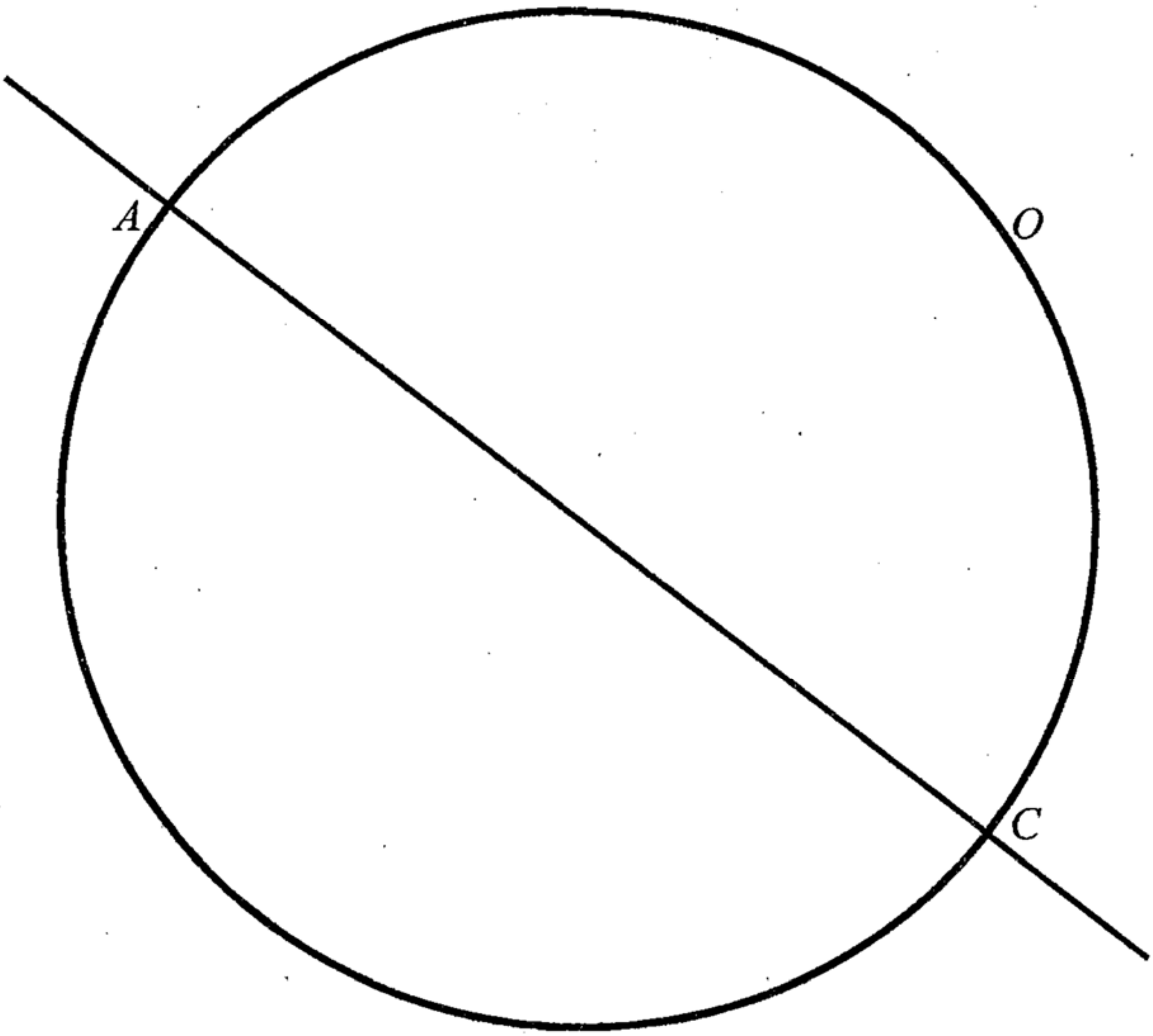
$$x = -\frac{3}{a} - 1 \text{ or } \frac{7}{a} - 2$$

$a = 1, 3, 7$

Sum of all possible integral values of $a = 1 + 3 + 7 = 11$

Geometrical Construction

1. 如下圖，已知一圓 O 的其中一條直徑為 AC。
求作圓上兩點 B、D 使得 $ABCD$ 成為一個正方形。
As shown in the figure below, given that O is a circle with a diameter AC. Construct two points B, D on the circle such that $ABCD$ form a square.



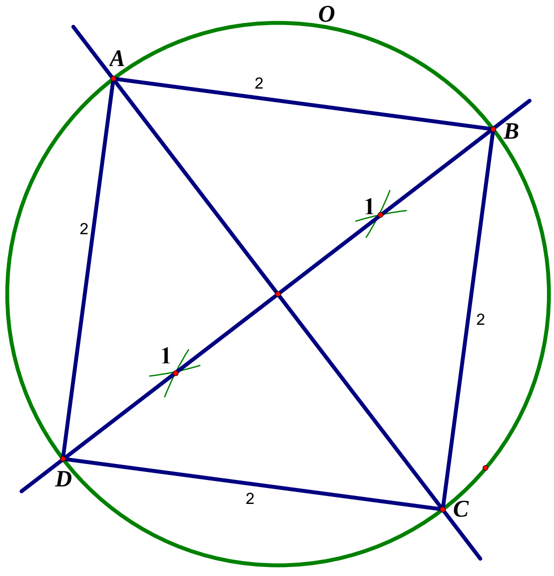
作圖方法如下(圖一)：

(1) 作 AC 的垂直平分綫，交圓 O 於 B 及 D。

(2) 連接 AB、BC、CD 及 DA。

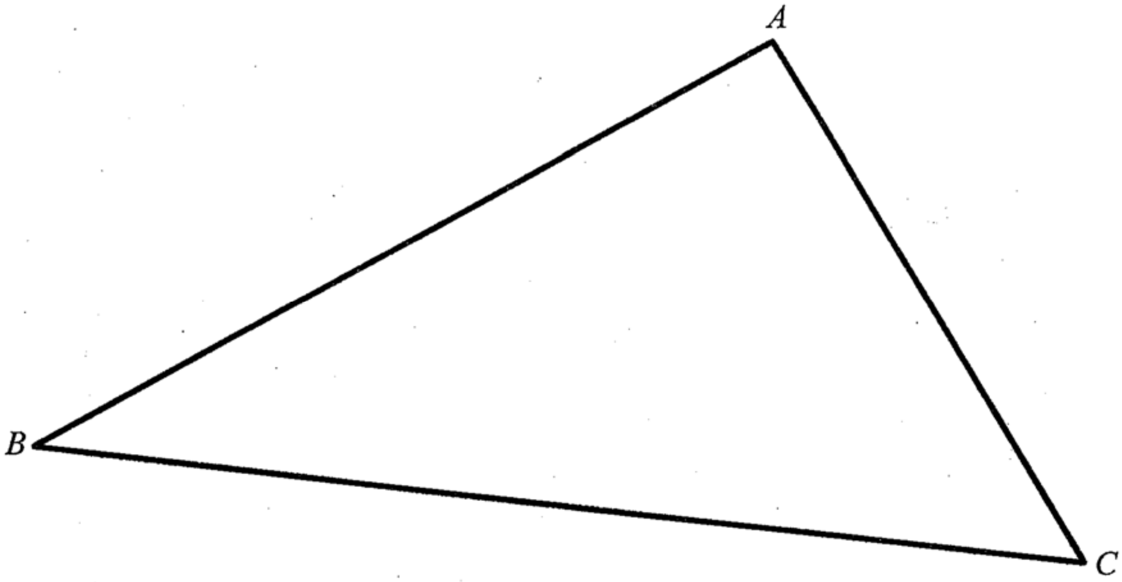
$ABCD$ 便是所需的正方形，作圖完畢。

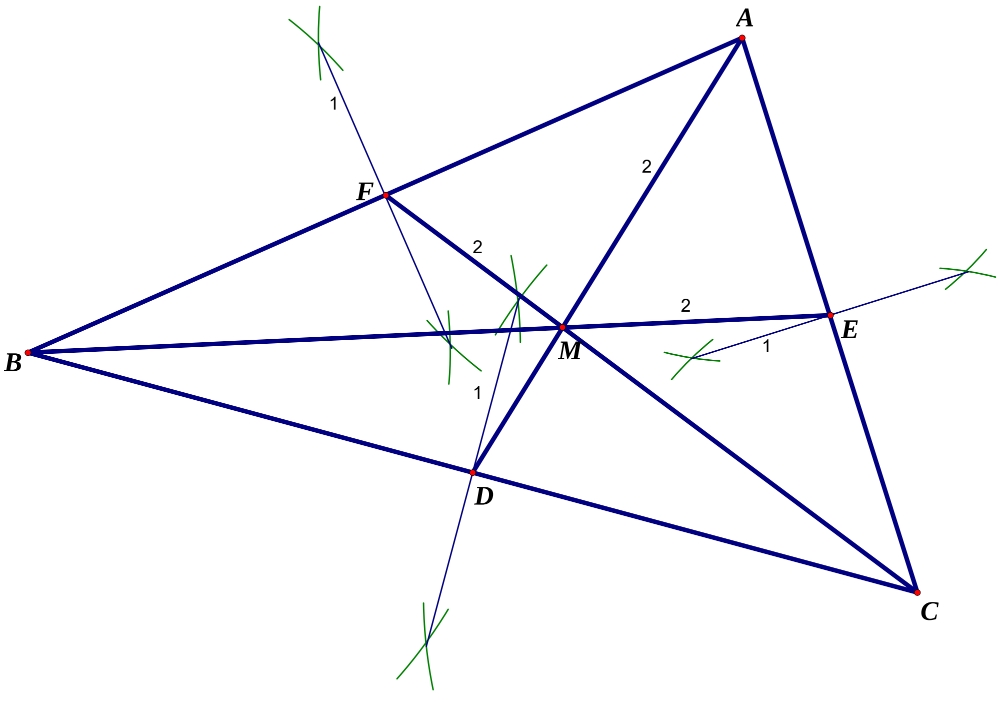
證明從略。



圖一

2. 已知ΔABC，如下圖所示。
求作一點 M，使得 MA、MB 及 MC 三條綫段將ΔABC 的面積三等分。
Given ΔABC as shown in the figure below. Construct a point M such that the line segments MA, MB, MC will divide the area of ΔABC into 3 equal parts.





作圖方法如下(圖二)：

(1) 作 BC 的垂直平分綫，D 為中點，作 AC 的垂直平分綫，E 為中點，作 AB 的垂直平分綫，F 為中點。

(2) 連接中綫 AB、BE 及 CF，交於形心 M。

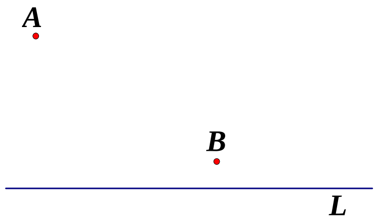
MA、MB 及 MC 三條綫段將ΔABC 的面積三等分，作圖完畢。

證明從略。

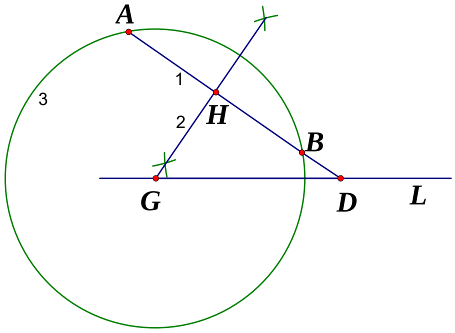
3. 已知 A、B 兩點和直綫 ℓ，如下圖所示。求作一圓過 A、B 兩點且與 ℓ 相切。

Given two points A, B and a straight line ℓ as shown in the figure below. Construct a circle which passes through A and B, and is tangent to the straight line ℓ.

Reference: C:\Users\twhung.CLSMSS.002\Dropbox\Data\My Web\Home_Page\Geometry\7 Construction by ruler and compasses\circle/circle_through_A_B_touch_L.pdf



圖三



圖四

作圖方法如下(圖三、圖四及圖五)：

(1) 連接 AB，其延長綫交 L 於 D。

(2) 作 AB 的垂直平分綫，交 L 於 G，H 為 AB 的中點。

(3) 以 G 為圓心，GA 為半徑作一圓。(圖四)

(4) 作 GD 的垂直平分綫，M 為 GD 的中點。

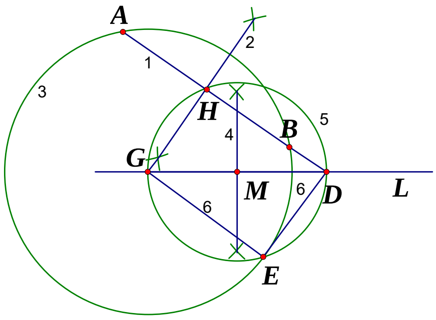
(5) 以 M 為圓心，MG 為半徑作一圓，交步驟(3)的圓於 E。

(6) 連接 EG、DE。(圖)

(7) 以 D 為圓心，DE 為半徑作一圓，交 L 於 F(在 D 與 G 之間)及 C(在 GD 之延長部分)。

(8) 過 F 作一綫段垂直於 L，且交 GH 的延綫於 O，過 C 作一綫段垂直於 L，且交 GH 的延長綫於 Q。

(9) 以 O 為圓心，OA 為半徑作一圓；以 Q 為圓心，QA 為半徑作一圓。(圖五)



圖五

作圖完畢，證明如下：

$\angle AHG = \angle BHG = 90^\circ$ (由作圖所得)

$GH = GH$ (公共邊)

$AH = HB$ (由作圖所得)

$\therefore \Delta AGH \cong \Delta BGH$ (S.A.S.)

$GA = GB$ (全等三角形的對應邊)

$\therefore$ 步驟(3)的圓經過 A、B。

利用相同方法，可證明步驟(9)的二圓皆經過 A、B。

$\angle GED = 90^\circ$ (半圓上的圓周角)

$\therefore DE$ 切步驟(3)的圓於 E。 (切綫垂直於半徑的逆定理)

$DA \times DB = DE^2$ (相交弦定理)

$\because DE = DF = DC$ (半徑)

$\therefore DA \times DB = DF^2$ 及 $DA \times DB = DC^2$

$\therefore DF$ 切圓 ABF 於 F 及 DC 切圓 ABC 於 C (相交弦定理的逆定理)

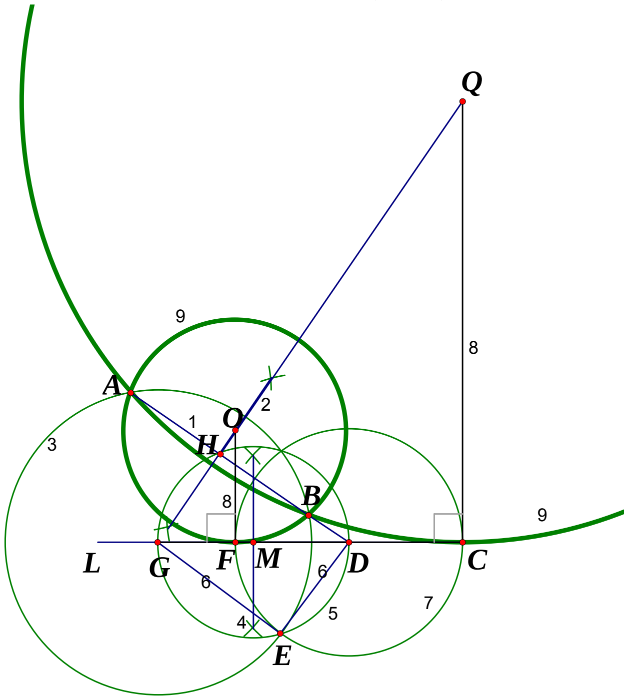


圖 2