Examples on Mathematical Induction: Sum of integers

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1. **AM 1968 Paper 1 Q9**

Prove that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$. Hence find the value of $1 + 3 + 5 + \cdots + (2n - 1)$.

$n = 1$, L.H.S. = 1, R.H.S. = $\dfrac{1(1+1)}{2} = 1$, it is true for $n = 1$

Suppose $1 + 2 + 3 + \cdots + k = \dfrac{k(k+1)}{2}$

$$\begin{aligned}1 + 2 + 3 + \cdots + k + (k + 1) &= \frac{k(k+1)}{2} + (k + 1)\\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}\\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

If it is true for $n = k$ then it is also true for $n = k + 1$

By the principle of mathematical induction, it is true for all positive integers n.

$$\begin{aligned}1 + 3 + 5 + \cdots + (2n - 1) &= 1 + 2 + 3 + 4 + \cdots + (2n - 1) + 2n - (2 + 4 + \cdots + 2n)\\ &= \frac{2n(2n+1)}{2} - 2(1 + 2 + 3 + \cdots + n)\\ &= \frac{2n(2n+1)}{2} - 2\,\frac{n(n+1)}{2}\\ &= n(2n + 1) - n(n + 1)\\ &= n(2n + 1 - n - 1)\\ &= n^2\end{aligned}$$

2. **Mathematics 1976 Paper 1 Q7**

John was asked to prove the statement $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. for all positive integers n.

His proof is as follows:

The statement is true when $n = 1$, because when line

$n = 1$. L.S. = R.S. = 1 $\cdots$ (1)

$n = k$ i.e. $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ $\cdots$ (2)

$1 + 2 + \cdots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1) = (k + 1)\left(\frac{1}{2}k + 1\right) = \frac{1}{2}(k+1)(k+2)$ $\cdots$ (3)

So, the statement is also true when $n = k + 1$ $\cdots$ (4)

$\therefore$ The statement is true when $n = 1$, $n = k$ and $n = k + 1$, hence by the principle of mathematical induction, the statement is true for all positive integers n. $\cdots$ (5)

Which two lines of John's proof are incorrect?

What would you write in the place of these two lines?

Line (2) and Line (5) are wrong. Correct statement should be :

Assume it it true for $n = k$, then $1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1)$ ………… (2)

$\therefore$ The statment is true when $n = 1$. If it is true for $n = k$, then it is also true for $n = k + 1$. Hence , by the Principle of Mathematical Induction the statement is true for all positive integers n . …………………………… (5)

3. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

$n = 1$, L.H.S. = 1, R.H.S. = $1^2 = 1$, it is true for $n = 1$

Suppose $1 + 3 + 5 + \cdots + (2k - 1) = k^2$.

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$

$$= (k + 1)^2$$

If it is true for $n = k$ then it is also true for $n = k + 1$

By the principle of mathematical induction, it is true for all positive integers n .