21-22 Paper 1	**1**	291	**2**	4	**3**	281	**4**	$\frac{1}{10}$	**5**	16
	6	34	**7**	7	**8**	4	**9**	12.5	**10**	123
	11	$\frac{1}{6}$	**12**	6431205	**13**	23	**14**	120	**15**	$\frac{8}{15}$
21-22 Paper 2	**1**	2022	**2**	112	**3**	19	**4**	$\frac{1}{3}$	**5**	33
	6	120	**7**	$\frac{25}{4}$	**8**	$\frac{1}{1011}$				

Paper 1

1. α 及 β 是方程 $x^2 - 100x + k = 0$ 的實根。若 $\alpha - 7 = 30\beta$，求 k 的值。

α and β are the real roots of the equation $x^2 - 100x + k = 0$. If $\alpha - 7 = 30\beta$, find the value of k.

$\alpha + \beta = 100 \cdots\cdots (1)$, $\alpha\beta = k \cdots\cdots (2)$

$\alpha + \beta - 7 = 31\beta$

$100 - 7 = 31\beta$

$\beta = 3$

Put $\beta = 3$ into $x^2 - 100x + k = 0$

$3^2 - 100\times 3 + k = 0$

$k = 291$

2. 在圖一中，ACD 是一個三角形。B 是 CD 上的一點使 $AB = AC = 2$ 及 $AD = 4$。若 $BC : BD = 1 : 3$，求 CD 的長。

In Figure 1, ACD is a triangle. B is a point on CD such that $AB = AC = 2$ and $AD = 4$.

If $BC : BD = 1 : 3$, find the length of CD.

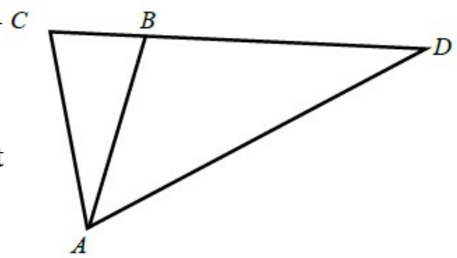


Figure 1 圖一

Let $BC = k$, $BD = 3k$, $\angle ADB = \alpha$

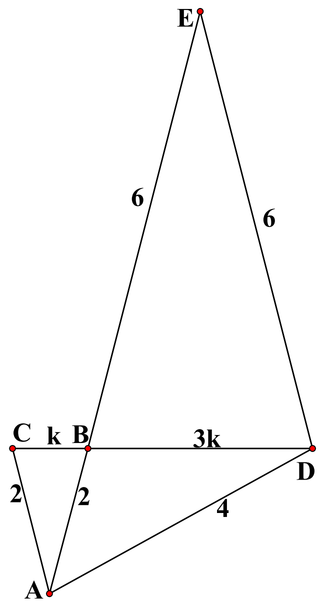
Apply cosine formula on ΔABD and ΔACD

$$\cos\alpha = \frac{4^2 + (3k)^2 - 2^2}{2(4)(3k)} = \frac{4^2 + (4k)^2 - 2^2}{2(4)(4k)}$$

$$\frac{12 + 9k^2}{24k} = \frac{12 + 16k^2}{32k}$$

$4 + 3k^2 = 3 + 4k^2$

$k = 1$, $CD = 4k = 4$



Method 2 Let $BC = k$, $BD = 3k$.

Produce AB to E so that $BE = 6$, join ED.

$\angle ABC = \angle DBE$ (vert. opp. $\angle$s)

$\frac{EB}{AB} = \frac{6}{2} = 3$, $\frac{BD}{BC} = \frac{3k}{k} = 3$

$\therefore \Delta BDE \sim \Delta BCA$ (ratio of 2 sides, included $\angle$s)

$\therefore \frac{ED}{AC} = 3 \Rightarrow ED = 6$

$\angle BAD = \angle DAE$ (common $\angle$s)

$\frac{AE}{AD} = \frac{2+6}{4} = 2$, $\frac{AD}{AB} = \frac{4}{2} = 2$

$\therefore \Delta ADE \sim \Delta ABD$ (ratio of 2 sides, included $\angle$s)

$\therefore \frac{DE}{BD} = \frac{6}{3k} = 2 \Rightarrow k = 1$

$CD = 4k = 4$

3. 在圖二中，$ABCD$ 是一個矩形。E是AC上的一點使$AE = 25$ 及 $CE = 144$。若 $p = AD + DE + CD$，求 p 的值。

In Figure 2, $ABCD$ is a rectangle. E is a point on AC such that $AE = 25$ and $CE = 144$. If $p = AD + DE + CD$, find the value of p . **Reference: 1998 FG1.3, 1999FG5.4**

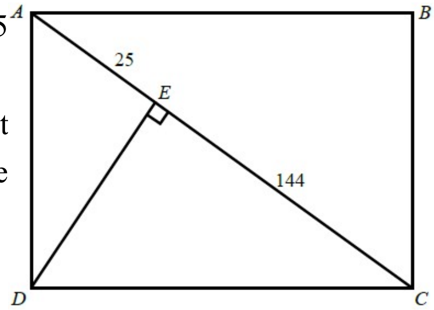


Figure 2 圖二

Let $\angle ADE = \alpha$, $\angle CDE = 90° - \alpha$

$\angle DAE = 90° - \alpha$, $\angle DCE = \alpha$ ($\angle$ sum of Δ)

$\tan \alpha = \frac{DE}{144} = \frac{25}{DE}$

$DE = 60$

$AD = \sqrt{25^2 + 60^2} = 65$ (Pythagoras' theorem on ΔADE)

$CD = \sqrt{60^2 + 144^2} = 156$ (Pythagoras' theorem on ΔCDE)

$p = 65 + 60 + 156 = 281$

4. 設 x、y 及 z 是非零數。若 $2^x = 3^y = 18^z$，求 $\frac{xz}{5y(x-z)}$ 的值。

Let x, y and z are non-zero numbers. If $2^x = 3^y = 18^z$, find the value of $\frac{xz}{5y(x-z)}$.

$2^x = 3^y = 18^z \Rightarrow x \log 2 = y \log 3 = z \log 18 = k$

$x = \frac{k}{\log 2}$, $y = \frac{k}{\log 3}$, $z = \frac{k}{\log 18}$

$$\frac{xz}{5y(x-z)} = \frac{\frac{k}{\log 2} \cdot \frac{k}{\log 18}}{\frac{5k}{\log 3}\left(\frac{k}{\log 2} - \frac{k}{\log 18}\right)}$$

$$= \frac{\log 3}{5(\log 18 - \log 2)}$$

$$= \frac{\log 3}{5 \log 9} = \frac{\log 3}{5 \times 2 \log 3}$$

$$= \frac{1}{10}$$

5. 設 $N = 24x + 216y$，其中x及y均為正整數。若N為完全立方數，求 $x + y$ 的最小值。

Let $N = 24x + 216y$, where both x and y are positive integers.
If N is a cube number, find the minimum value of $x + y$.
$N = 24(x + 9y) = 2^3 \times 3(x + 9y) = m^3$, where m is a positive integer.
$3(x + 9y) = k^3$, where k is a positive integer
For the least value of x and y, $x + 9y = 9n^3$, where n is a positive integer
$x = 9$, $1 + y = n^3$
$n = 1$, no positive integral solution for y
$n = 2$, $y = 7$
The minimum value of $x + y = 9 + 7 = 16$

6. 小馬參加數學比賽，解其中一條題目

$$\begin{cases} ax + by = -16 \\ cx + 20y = -224 \end{cases}$$，其中 a，b 及c 是實數。

題目的正確答案為 $x = 8$ 及 $y = -10$。
怎料小馬抄錯 c 的數值，得出答案 $x = 12$ 及 $y = -13$。求原題中 $a^2 + b^2 + c^2$ 的值。

John participated in a mathematics competition, in which one of the questions was to solve

$$\begin{cases} ax + by = -16 \\ cx + 20y = -224 \end{cases}$$, where a, b and c are real numbers.

The correct answer to the question was $x = 8$ and $y = -10$.
However, John copied a wrong value for c and then gave an answer of $x = 12$ and $y = -13$.
Find the value of $a^2 + b^2 + c^2$ in the original question.

Put $x = 8$ and $y = -10$ into $\begin{cases} ax + by = -16 \\ cx + 20y = -224 \end{cases}$.

$$\begin{cases} 8a - 10b = -16 \cdots\cdots (1) \\ 8c - 200 = -224 \cdots\cdots (2) \end{cases} \Rightarrow c = -3$$

Let the wrongly copied c be c'.

Put $x = 12$ and $y = -13$ into $\begin{cases} ax + by = -16 \\ c'x + 20y = -224 \end{cases}$.

$$\begin{cases} 12a - 13b = -16 \cdots\cdots (3) \\ 12c' - 260 = -224 \cdots\cdots (4) \end{cases} \Rightarrow c' = 3$$

(1)×13: $104a - 130b = -208$ ⋯⋯ (5)

(3) ×10: $120a - 130b = -160$ ⋯⋯ (6)

(6) – (5): $16a = 48$
$a = 3$
Sub. $a = 3$ into (1): $8\times3 - 10b = -16$
$b = 4$
$a^2 + b^2 + c^2 = 3^2 + 4^2 + 3^2 = 34$

7. 已知 $459 + x^3 = 3^y$，其中 x 及 y 均為正整數。求 y 的 最小值。

Given that $459 + x^3 = 3^y$, where both x and y are positive integers. Find the least value of y.
$3^5 = 243 < 459 < 3^6 = 729$
Try $y = 6$, $459 + x^3 = 729$
$x^3 = 270$, no solution
Try $y = 7$, $459 + x^3 = 2187$
$x^3 = 1728$
$x = 12$
$\therefore y = 7$

8. 在圖三中，D 為四邊形 $ABCE$ 內的一點使得 $AD \parallel BC$，$AB \perp AD$，$CD \perp DE$，$CD = ED$，$AD = 4$ cm 及 $BC = 6$ cm。若 ΔADE 的面積為 P cm^2，求 P 的值。

In Figure 3, D is a point inside the quadrilateral $ABCE$ such that $AD \parallel BC$, $AB \perp AD$, $CD \perp DE$, $CD = ED$, $AD = 4$ cm and $BC = 6$ cm . If P cm^2 is the area of ΔADE, find the value of P .

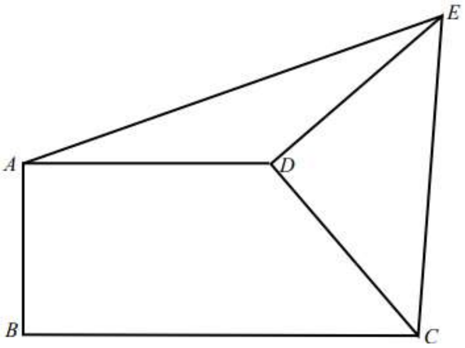


Figure 3 圖三

Let the foot of perpendicular from D to BC be E. Join DE.

$DE \perp BC$, $AD \perp DE$. Let $AB = DE = h$ cm, $\angle CDE = \theta$.

$AD = BE = 4$ cm (opp. sides of rectangle)

$CE = BC - BE = (6 - 4)$ cm $= 2$ cm

In ΔCDE, $\sin\theta = \dfrac{2}{CD}$

$\angle ADE + 90° + 90° + \theta = 360°$ ($\angle$s at a point)

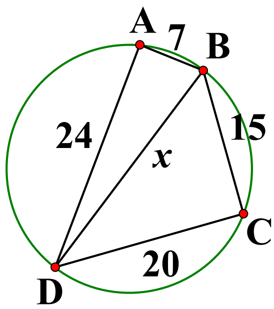
$\angle ADE = 180° - \theta$

$$P = \frac{1}{2}\cdot 4 \times DE \sin\left(180^\circ - \theta\right)$$

$$= 2 \times CD \sin\theta = 2 \times CD \times \frac{2}{CD} = 4$$

9. $ABCD$ 是一個圓內接四邊形，其中 $AB = 7$, $BC = 15$, $CD = 20$ and $DA = 24$。求圓 $ABCD$ 的半徑。

$ABCD$ is a cyclic quadrilateral with $AB = 7$, $BC = 15$, $CD = 20$ and $DA = 24$. Find the radius of the circle $ABCD$.



Let $BD = x$, $\angle BAD = \theta$, $\angle BCD = 180° - \theta$ (opp. $\angle$s cyclic quad.)

Apply cosine rules on ΔABD and ΔCBD:

$x^2 = 7^2 + 24^2 - 2\times7\times24 \cos\theta = 15^2 + 20^2 - 2\times15\times20 \cos(180° - \theta)$

$625 - 336 \cos\theta = 625 + 600 \cos\theta$

$\cos\theta = 0$, $\theta = 90°$

BD is the diameter (converse, $\angle$ in semi-circle)

$BD^2 = 7^2 + 24^2$ (Pythagoras' theorem)

$BD = 25$

Radius = 12.5

10. 已知 $a^2 + \dfrac{1}{a^2} = 7$，其中 $a > 0$。若 $b = a^5 + \dfrac{1}{a^5}$，求 b 的值。

Given that $a^2 + \dfrac{1}{a^2} = 7$, where $a > 0$. If $b = a^5 + \dfrac{1}{a^5}$, find the value of b .

$$a^2 + \frac{1}{a^2} + 2 = 7 + 2 = 9$$

$$\left(a + \frac{1}{a}\right)^2 = 3^2 \Rightarrow \because a > 0 \therefore\ a + \frac{1}{a} = 3$$

$$\left(a + \frac{1}{a}\right)\left(a^2 + \frac{1}{a^2}\right) = 3\times7 = 21 \Rightarrow a^3 + \frac{1}{a^3} + a + \frac{1}{a} = 21 \Rightarrow a^3 + \frac{1}{a^3} = 21 - 3 = 18$$

$$\left(a^2 + \frac{1}{a^2}\right)\left(a^3 + \frac{1}{a^3}\right) = 7\times18 = 126 \Rightarrow a^5 + \frac{1}{a^5} + a + \frac{1}{a} = 126 \Rightarrow a^5 + \frac{1}{a^5} = 126 - 3 = 123$$

11. x_1 及 x_2 是方程 $(\log 2x)(\log 3x) = a$ 的實根，其中 a 為實數。求 x_1x_2 的值。
x_1 and x_2 are the real roots of the equation $(\log 2x)(\log 3x) = a^2$, where a is a real number. Find the value of x_1x_2 .
$(\log 2+ \log x)(\log 3 + \log x) = a$
$(\log x)^2 + (\log 2 + \log 3) \log x + (\log 2)(\log 3) - a = 0$
$(\log x)^2 + (\log 6) \log x + (\log 2)(\log 3) - a = 0$
$\log x_1 + \log x_2 = -\log 6$
$\log(x_1 x_2) = \log\frac{1}{6}$
$x_1x_2 = \frac{1}{6}$

12. 由數字 0，1，2，3，4，5，6 組成一個沒有重複數字的 7 位數。若這個數可以被 55 整除，求這個數的最大值。
A 7-digit number is formed by putting the numerals 0, 1, 2, 3, 4, 5, 6 together without repetition. If this number is divisible by 55, find its largest possible value.
Let the 7-digit number be $\overline{\text{abcdefg}}$
55 = 5×11, the units digit = 0 or 5. g = 0 or 5
This number is divisible by 11 ⇒ a + c + e + g – (b + d + f) = 11m, where m is an integer
To maximize the number, let a = 6 .
If g = 0, b = 5, then 6 + c + e + 0 – (5 + d + f) = 11m
1 + c + e – (d + f) = 11m, where c, d, e, f = 1, 2, 3, 4
∵ c + d + e + f = 10
∴ 1 + c + e – [10 – (c + d)] = 11m
2(c + d) = 11m + 9
L.H.S. is even, R.H.S. is odd, which is a contradiction
If g = 5, b = 4, then 6 + c + e + 5 – (4 + d + f) = 11m
c + e – (d + f) = 11(m – 1) + 4, where c, e, d, f = 0, 1, 2, 3
By trial and error, c = 3, e = 2, d = 1, f = 0
3 + 2 – (1 + 0) = 4, which satisfies the equation
The largest possible 7-digit number is 6431205.

13. 已知 $a^{2x} - b^{2y} = 1672$，其中 a、b、x 及 y 為正整數。求 $ax + by$ 的最小值。
Given that $a^{2x} - b^{2y} = 1672$, where a , b , x and y are positive integers.
Find the minimum value of $ax + by$.
$(a^x - b^y)(a^x + b^y) = 2^3\times11\times19 = m\times n$, where $m < n$ are integers
To minimize $ax - by$, m, n must be as close as possible
Let $m = 2\times19 = 38$, $n = 2^2\times11 = 44$
$a^x - b^y = 38 \cdots\cdots (1)$
$a^x + b^y = 44 \cdots\cdots (2)$
$[(1) + (2)]\div2 \qquad a^x = 41$, no integral solution
Let $m = 2\times11 = 22$, $n = 2^2\times19 = 76$
$a^x - b^y = 22 \cdots\cdots (3)$
$a^x + b^y = 76 \cdots\cdots (4)$
$[(3) + (4)]\div2 \qquad a^x = 49$
$a = 7, x = 2$
$[(4) - (3)]\div2 \qquad b^y = 27$
$b = 3, y = 3$
The minimum value of $ax + by = 7\times2 + 3\times3 = 23$.

14. 設 a、b 及 c 為非零數字。有多少個三位數 $\overline{abc}$ 使得 $\overline{ab}<\overline{bc}<\overline{ca}$？

Let a, b and c are non-zero digits. How many three digit numbers $\overline{abc}$ are there such that $\overline{ab}<\overline{bc}<\overline{ca}$?

$\overline{abc}$ = 112, 113, ⋯ , 119 (8)

123, 124, ⋯ , 129 (7)

134, 135, ⋯ , 129 (6)

145, 146, ⋯ , 149 (5)

………………………

189 (1)

223, 224, ⋯ , 229 (7)

234, 235, ⋯ , 239 (6)

………………………

289 (1)

334, 335, ⋯ , 339 (6)

………………………

389 (1)

445, ⋯ , 449 (5)

…………………

489 (1)

…………………

889 (1)

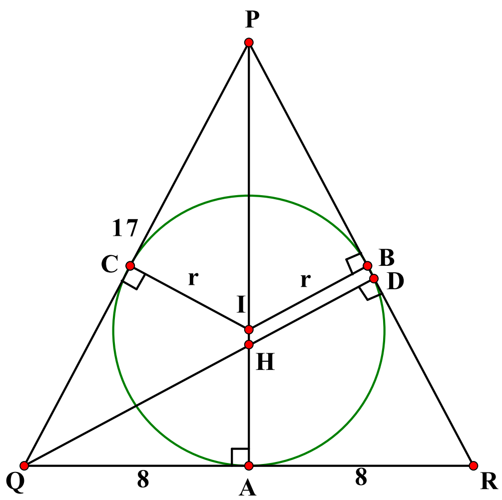
Total = (8 + 7 + 6 + ⋯ + 1) + (7 + 6 + ⋯ + 1) + (6 + 5 + ⋯ + 1) + ⋯ + 1

= 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1

= 120

15. PQR 是一個等腰三角形，其中 $PQ = PR = 17$ and $QR = 16$。將 I 及 H 分別記為 PQR 的內心及垂心。求 IH 長度的值。

PQR is an isosceles triangle with $PQ = PR = 17$ and $QR = 16$. Denote the in-centre and the orthocentre of PQR by I and H respectively. Find the length of HI.



Let A be the mid-point of QR. $QA = AR = 8$. $PA \perp QR$

$PA=\sqrt{17^2-8^2}=15$ (Pythagoras' theorem)

I and H lies on PA (altitude of the isosceles triangle)

Draw the inscribed circle with radius r, touching ΔPQR at A, B and C respectively as shown in the figure.

$IA \perp QR$, $IB \perp PR$, $IC \perp PQ$ (tangent ⊥ radius)

Area of ΔIQR + area of ΔIPR + area of ΔIPQ = Area of ΔPQR

$$\frac{1}{2}\cdot 16r+\frac{1}{2}\cdot 17r+\frac{1}{2}\cdot 17r=\frac{1}{2}\cdot 16\times 15$$

$$r=IA=\frac{24}{5}$$

Join QH and produce it to cut PR at D. $QD \perp PR$.

$ADPQ$ is a cyclic quadrilateral (converse ∠s in the same segment)

Let $\angle AQH = \theta$, then $\angle APR = \theta$ (∠s in the same segment)

In ΔAHQ, $HA = 8\tan\theta$

In ΔAPR, $\tan\theta=\frac{8}{15}$

$$\therefore\ HA=8\times\frac{8}{15}=\frac{64}{15}$$

$$IH=IA-HA=\frac{24}{5}-\frac{64}{15}=\frac{8}{15}$$

Paper 2

1. 設 $\frac{A}{2022}=\frac{1}{1+1\times 2\times 3\times\cdots\times 2022}+\frac{1}{1+\frac{1}{1\times 2\times 3\times\cdots\times 2022}}$。求 A 的值。

Let $\frac{A}{2022}=\frac{1}{1+1\times 2\times 3\times\cdots\times 2022}+\frac{1}{1+\frac{1}{1\times 2\times 3\times\cdots\times 2022}}$. Find the value of A.

$\frac{A}{2022}=\frac{1}{1+1\times 2\times 3\times\cdots\times 2022}+\frac{1\times 2\times 3\times\cdots\times 2022}{1+1\times 2\times 3\times\cdots\times 2022}=1$

$A = 2022$

2. $\overline{AB}$ 和 $\overline{CB}$ 均為兩位正整數，其中 A、B 和 C 是不同的數字。設 $d=\overline{AB}+\overline{CB}$。若 $\overline{AB}\times\overline{CB}=\overline{BCBB}$ 是四位數，求 d 的值。

Both $\overline{AB}$ and $\overline{CB}$ are two-digit positive integers, where A, B and C are different digits. Let $d=\overline{AB}+\overline{CB}$. If $\overline{AB}\times\overline{CB}=\overline{BCBB}$ is a four-digit number, find the value of d.

$B \neq 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$ $\therefore$ $B = 1, 5$ or 6

$(10A + B)(10C + B) = 1000B + 100C + 10B + B$

$100AC + 10(A + C)B + B^2 = 1000B + 100C + 10B + B$

$100(A - 1)C = [1011 - 10(A + C) - B]B$

L.H.S. is multiple of 100 $\Rightarrow$ $1011 - 10(A + C) - B$ is a multiple of 100

$\Rightarrow B = 1$

$100(A - 1)C = 1011 - 10(A + C) - 1 = 10(101 - A - C)$

$10(A - 1)C = 101 - A - C$

$A - 9C + 10AC = 101$

$A(1 + 10C) - 9C = 101$

$10A(1 + 10C) - 90C = 1010$

$10A(1 + 10C) - 9(1 + 10C) = 1001 = 7\times 11\times 13$

$(10A - 9)(10C + 1) = 7\times 11\times 13$

$(10A - 9, 10C + 1) = (11, 91) \Rightarrow A = 2, C = 9$

$(10A - 9, 10C + 1) = (91, 11) \Rightarrow A = 10, C = 1$ (rejected)

$d = 21 + 91 = 112$

3. 假設方程 $x^2y-2x^2-3y-13=0$ 只有一對正整數解 (x_0, y_0)。若 $a=y_0-x_0$，求 a 的值。

Suppose the equation $x^2y - 2x^2 - 3y - 13 = 0$ has only one pair of positive integral solution (x_0, y_0). If $a = y_0 - x_0$, find the value of a.

$x^2y - 2x^2 - 3y - 13 = 0$

$x^2(y - 2) - 3(y - 2) = 19$

$(x^2 - 3)(y - 2) = 19$

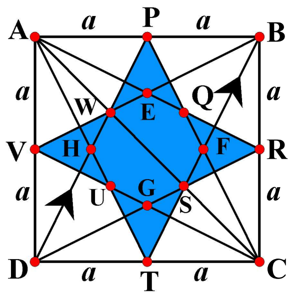
$\begin{cases} x^2-3=1 \\ y-2=19 \end{cases}$ or $\begin{cases} x^2-3=19 \\ y-2=1 \end{cases}$

$x = 2$, $y = 21$ or no positive integral solution

$a = 21 - 2 = 19$

4. 圖一所示為一正方形。每一條邊的中點都連接對邊的兩端點，由此形成一個四角星(著色部分)。求 $\frac{\text{四角星的面積}}{\text{正方形的面積}}$ 的值。

Figure 1 shows a square. The mid-point of each side is joined to the two end points of the opposite side and a four-pointed star is thus formed (the shaded part). Find the value of $\frac{\text{Area of the four point star}}{\text{Area of the square}}$.



Using the notations in the figure, the square $ABCD$ has length = $2a$.

Area = $(2a)^2 = 4a^2$

$\angle VAW = 45°$ (Property of a square)

$\angle ADC = 90°$ (Property of a square)

$AC = 2\sqrt{2}a$ (Pythagoras' theorem)

$\Delta TDP \cong \Delta CTB$ (S.A.S.)

$\angle TDP = \angle CTB$ (corr. $\angle$s, $\cong \Delta$s)

$PD \parallel BT$ (corr. $\angle$s eq.)

$AW = WS = SC = \frac{2\sqrt{2}a}{3}$ (Intercept theorem)

Area of the star = $4a^2 - 8\times$ area of ΔAVW

$$= 4a^2 - 8\times\frac{1}{2}a\cdot\frac{2\sqrt{2}}{3}a\sin 45^\circ$$

$$= \frac{4a^2}{3}$$

$$\frac{\text{Area of the four point star}}{\text{Area of the square}} = \frac{\frac{4a^2}{3}}{4a^2} = \frac{1}{3}$$

5. $VABC$ 為一個錐體，其中 $VA = VB = VC$ 及 $AB = BC = CA = a$ m。設它的高為 h m 及它的總表面積及體積相等。若 a 和 h 均為正整數，求 h 的可能值之和。

$VABC$ is a right pyramid with $VA = VB = VC$ and $AB = BC = CA = a$ m . Let its height be h m and its total surface area and volume are the same. If a and h are both positive integers, find the sum of all possible values of h .

The base ΔABC is an equilateral triangle with area $= \frac{1}{2}a^2\sin 60^\circ\text{ m}^2 = \frac{\sqrt{3}}{4}a^2\text{ m}^2$

Let the projection of V on ABC be O, thewn O is the centriod of ΔABC.

$\angle AOB = \angle BOC = \angle COA = 120°$ ($\angle$s at a point)

$$OA = OB = OC = \frac{\frac{a}{2}}{\cos 30^\circ} = \frac{a}{\sqrt{3}}$$

$VO \perp$ the base ΔABC.

In ΔVOA, $\angle AOV = 90°$, $OA^2 + VO^2 = VA^2$ (Pythagoras' theorem)

$$VA^2 = \left(\frac{a}{\sqrt{3}}\right)^2 + h^2 = \frac{a^2}{3} + h^2$$

Let M be the mid-point of AB. $AM = MB = \frac{a}{2}$, $VM \perp AB$.

In ΔVAM, $VM^2 + AM^2 = VA^2$ (Pythagoras' theorem)

$VM^2 + \frac{a^2}{4} = \frac{a^2}{3} + h^2$

$VM = \sqrt{\frac{a^2}{12} + h^2}$

Area of a lateral face $= \frac{1}{2} AB \times VM = \frac{1}{2} a \times \sqrt{\frac{a^2}{12} + h^2}$

Total surface area = volume

$\frac{3}{2} a \times \sqrt{\frac{a^2}{12} + h^2} + \frac{\sqrt{3}}{4} a^2 = \frac{1}{3} \cdot \frac{\sqrt{3}}{4} a^2 h$

$18 \cdot \sqrt{\frac{a^2}{12} + h^2} + 3\sqrt{3}a = \sqrt{3}ah$

$18 \cdot \sqrt{\frac{a^2}{12} + h^2} = \sqrt{3}(h-3)a$

$108\left(\frac{a^2}{12} + h^2\right) = (h-3)^2 a^2$

$9a^2 + 108h^2 = (h^2 - 6h + 9)a^2$

$a^2h^2 - 6a^2h - 108h^2 = 0$

$a^2h - 6a^2 - 108h = 0 \cdots\cdots (*)$

$a^2(h - 6) = 108h$

$a^2 = \frac{108h}{h-6} \cdots\cdots (1) \Rightarrow h > 6$

When $h = 7$, no integral solution for a

When $h = 8$, no integral solution for a

When $h = 9$, $a^2 = \frac{108 \times 9}{9-6} \Rightarrow a = 18$

(*) can be rearranged as $h = \frac{6a^2}{a^2 - 108} \cdots\cdots (2) \Rightarrow a > 10$

$h \ge 9 \Rightarrow 6a^2 \ge 9(a^2 - 108)$

$9 \times 36 \ge a^2 \Rightarrow 18 \ge a > 10$

From (2), a must be a multiple of 6

Put $a = 12$ into (2), $h = \frac{6 \times 12^2}{12^2 - 108} = 24$

Sum of all possible values of $h = 9 + 24 = 33$

6. 圖二中，$ABCD$ 是平行四邊形。E 為 BC 的中點，AE 和 BD 相交於 H，AC 和 DE 相交於 F，AC 和 BD 相交於 G。若四邊形 $EFGH$ 的面積及 $ABCD$ 的面積分別為 10 cm^2 及 $k\text{ cm}^2$，求k的值。
In Figure 2, $ABCD$ is a parallelogram. E is the mid-point of BC, AE and BD intersect at H, AC and DE intersect at F, AC and BD intersect at G. If the area of the quadrilateral $EFGH$ and $ABCD$ are 10 cm^2 and $k\text{ cm}^2$ respectively, find the value of k.

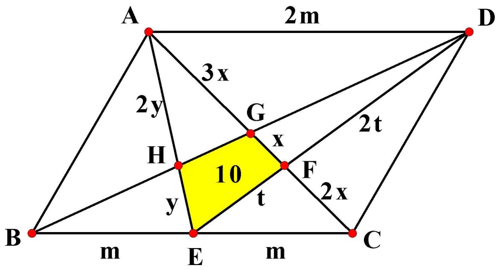


Figure 2 圖二

E is the mid-point of BC. Let $BE = m = EC$, $AD = 2m$
$\Delta ADH \sim \Delta EBH$ (equiangular)
$AH : HE = AD : BE = 2 : 1$ (corr. sides, ~Δs)
Let $AH = 2y$, $HE = y$
$\Delta ADF \sim \Delta CEF$ (equiangular)
$AF : CF = DF : EF = 2 : 1$ (corr. sides, ~Δs)
Let $AF = 4x$, $CF = 2x$, $DF = 2t$, $EF = t$, $AC = AF + FC = 6x$
$AG = GC = 3x$, $GF = GC - FC = x$
Let S represents the area. $\Delta ACD \cong \Delta CAB$ (S.S.S.)

$$S_{\Delta ACD} = S_{\Delta CAB} = \frac{k}{2}$$

ΔADG, ΔGDF and ΔCDF have the same height but different bases AG, GF and FC.
$S_{\Delta ADG} : S_{\Delta GDF} : S_{\Delta CDF} = AG : GF : FC = 3 : 1 : 2$

$$S_{\Delta ADG} = \frac{k}{2}\times\frac{3}{6} = \frac{k}{4},\ S_{\Delta GDF} = \frac{k}{2}\times\frac{1}{6} = \frac{k}{12},\ S_{\Delta CDF} = \frac{k}{2}\times\frac{2}{6} = \frac{k}{6}$$

Height of ΔADE = height of //-gram $ABCD$ and they have common base AD.

$$\therefore \text{ Area of } \Delta ADE = \frac{1}{2}\times \text{area of //-gram} = \frac{k}{2}$$

ΔADH and ΔDEH have the same height but different bases AH and HE.

$$S_{\Delta DEH} = \frac{k}{2}\times\frac{1}{3} = \frac{k}{6}$$

$$S_{EFGH} = S_{\Delta DEH} - S_{\Delta GDF}$$

$$10 = \frac{k}{6} - \frac{k}{12}$$

$k = 120$

7. 已知 $x+y+z=1\cdots(1)$，$x^2+y^2+z^2=2\cdots(2)$及$x^3+y^3+z^2=3\cdots(3)$。求$x^4+y^4+z^4$ 的值。

Given that $x+y+z=1$, $x^2+y^2+z^2=2$ and $x^3+y^3+z^2=3$. Find the value of $x^4+y^4+z^4$.

$(1)^2$: $(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)=1$

$2+2(xy+yz+zx)=1\Rightarrow xy+yz+zx=-\frac{1}{2}\quad\cdots(4)$

$x^3+y^3+z^3-3xyz=(x+y+z)[x^2+y^2+z^2-(xy+yz+zx)]$

$3-3xyz=1\times[2-(-\frac{1}{2})]$

$xyz=\frac{1}{6}\cdots(5)$

$(4)^2$: $(xy+yz+zx)^2=\left(-\frac{1}{2}\right)^2$

$x^2y^2+y^2z^2+z^2x^2+2xyz(x+y+z)=\frac{1}{4}$

$x^2y^2+y^2z^2+z^2x^2+2\times\frac{1}{6}\times1=\frac{1}{4}$

$x^2y^2+y^2z^2+z^2x^2=-\frac{1}{12}\quad\cdots(5)$

$(2)^2$: $(x^2+y^2+z^2)^2=2^2$

$x^4+y^4+z^4+2(x^2y^2+y^2z^2+z^2x^2)=4$

$x^4+y^4+z^4+2\times\left(-\frac{1}{12}\right)=4$

$x^4+y^4+z^4=\frac{25}{4}$

Method 2 (Newton's formulae for the sums of powers of the roots)

Let $S_n=x^n+y^n+z^n$, then $S_0=3$, $S_1=1$, $S_2=2$, $S_3=3$

Let x, y and z be the roots of $t^3+a_1t^2+a_2t+a_3=0$

Let $f(t)=(t-x)(t-y)(t-z)\equiv t^3+a_1t^2+a_2t+a_3$

$\ln f(t)=\ln(t-x)+\ln(t-y)+\ln(t-z)=\ln(t^3+a_1t^2+a_2t+a_3)$

Differentiate both sides w.r.t. t:

$$\frac{f'(t)}{f(t)}=\frac{1}{t-x}+\frac{1}{t-y}+\frac{1}{t-z}=\frac{3t^2+2a_1t+a_2}{t^3+a_1t^2+a_2t+a_3}$$

Replace t by $\frac{1}{u}$: $\frac{u}{1-ux}+\frac{u}{1-uy}+\frac{u}{1-uz}=\frac{\left(3+2a_1u+a_2u^2\right)u}{1+a_1u+a_2u^2+a_3u^3}$

$$\frac{1}{1-ux}+\frac{1}{1-uy}+\frac{1}{1-uz}=\frac{3+2a_1u+a_2u^2}{1+a_1u+a_2u^2+a_3u^3}$$

$$\sum_{n=0}^{\infty}(xu)^n+\sum_{n=0}^{\infty}(yu)^n+\sum_{n=0}^{\infty}(zu)^n=\frac{3+2a_1u+a_2u^2}{1+a_1u+a_2u^2+a_3u^3}$$

$$\sum_{n=0}^{\infty}S_nu^n=\frac{3+2a_1u+a_2u^2}{1+a_1u+a_2u^2+a_3u^3}$$

$$\left(1+a_1u+a_2u^2+a_3u^3\right)\left(\sum_{n=0}^{\infty}S_nu^n\right)=3+2a_1u+a_2u^2$$

Compare coefficients of u: $a_1S_0+S_1=2a_1\Rightarrow a_1=-1$

Compare coefficients of u^2: $a_2S_0 + a_1S_1 + S_2 = a_2 \Rightarrow 2a_2 - 1 + 2 = 0 \Rightarrow a_2 = -\frac{1}{2}$

Compare coefficents of u^3: $a_3S_0 + a_2S_1 + a_1S_2 + S_3 = 0 \Rightarrow 3a_3 - \frac{1}{2} - 1\times 2 + 3 = 0 \Rightarrow a_3 = -\frac{1}{6}$

Compare coefficents of u^4: $a_3S_1 + a_2S_2 + a_1S_3 + S_4 = 0$

$\Rightarrow -\frac{1}{6} - \frac{1}{2}\times 2 - 1\times 3 + S_4 = 0$

$\Rightarrow a_3 = \frac{25}{4}$

8. 對所有正整數 $n > 1$，函數 f 定義如下：
f $(1) = 2021$ 及 $f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n)$。
求 f(2021) 的值。
For all positive integers $n > 1$, a function f is defined as
$f(1) = 2021$ and $f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n)$.
Find the value of f(2021) .

Reference: 2013 HG10, 2014 FG1.4

$f(1) + f(2) + \cdots + f(n) = n^2 f(n) \Rightarrow f(n) = \frac{f(1)+f(2)+\cdots+f(n-1)}{n^2-1}$

$f(2) = \frac{f(1)}{3} = \frac{2021}{3}$

$f(3) = \frac{f(1)+f(2)}{8} = \frac{2021+\frac{2021}{3}}{8} = \frac{1+\frac{1}{3}}{8}\cdot 2021 = \frac{1}{6}\cdot 2021$

$f(4) = \frac{f(1)+f(2)+f(3)}{15} = \frac{2021+\frac{2021}{3}+\frac{2021}{6}}{15} = \frac{\frac{3}{2}}{15}\cdot 2021 = \frac{1}{10}\cdot 2021$

It is observed that the answer is 2021 divided by the n^{th} triangle number.

Claim: $f(n) = \frac{2}{n(n+1)}\cdot 2021$ for $n \geq 1$

$n = 1, 2, 3, 4$, proved above.

Suppose $f(k) = \frac{2}{k(k+1)}\cdot 2021$ for $k = 1, 2, \cdots, m$ for some positive integer m .

$$f(m+1) = \frac{f(1)+f(2)+\cdots+f(m)}{(m+1)^2-1} = \frac{\frac{2}{1\times 2}+\frac{2}{2\times 3}+\frac{2}{3\times 4}+\frac{2}{4\times 5}+\frac{2}{5\times 6}+\cdots+\frac{2}{m(m+1)}}{m(m+2)}\cdot 2021$$

$$= 2\cdot\frac{\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{m}-\frac{1}{m+1}\right)}{m(m+2)}\cdot 2021$$

$$= 2\cdot\frac{1-\frac{1}{m+1}}{m(m+2)}\cdot 2021 = \frac{2}{(m+1)(m+2)}\cdot 2021$$

$\therefore$ It is also true for m. By the principle of mathematical induction, the formula is true.

$f(2021) = \frac{2}{2021\times 2022}\cdot 2021 = \frac{1}{1011}$