Angle bisector theorem

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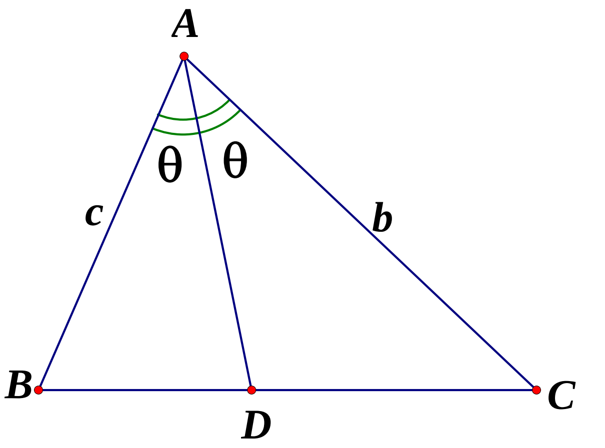


In the figure, $AC = b$, $AB = c$, AD is the angle bisector of $\angle A$, cutting BC at D. $\angle BAD = \angle CAD = \theta$.

Then $\frac{BD}{DC} = \frac{c}{b}$.

Proof: Let $\angle ADB = \alpha$, $\angle ADC = 180° - \alpha$ (adj. $\angle$s on st. line)

Apply sine rule on ΔABD and ΔACD.

$\frac{BD}{\sin\theta} = \frac{c}{\sin\alpha}$ …… (1) and $\frac{DC}{\sin\theta} = \frac{b}{\sin(180^\circ - \alpha)}$ …… (2)

Using the fact that $\sin(180° - \alpha) = \sin\alpha$, (1) ÷ (2): $\frac{BD}{DC} = \frac{c}{b}$

Converse of angle bisector theorem, if $\frac{BD}{DC} = \frac{c}{b}$, then AD is the angle bisector of $\angle A$.

Proof: Apply sine rule on ΔABD and ΔACD.

$\frac{BD}{\sin\angle BAD} = \frac{c}{\sin\alpha}$ …… (3) and $\frac{DC}{\sin\angle CAD} = \frac{b}{\sin(180^\circ - \alpha)}$ …… (4)

Using the fact that $\sin(180° - \alpha) = \sin\alpha$, (3) ÷ (4): $\frac{BD}{DC} \cdot \frac{\sin\angle CAD}{\sin\angle BAD} = \frac{c}{b}$

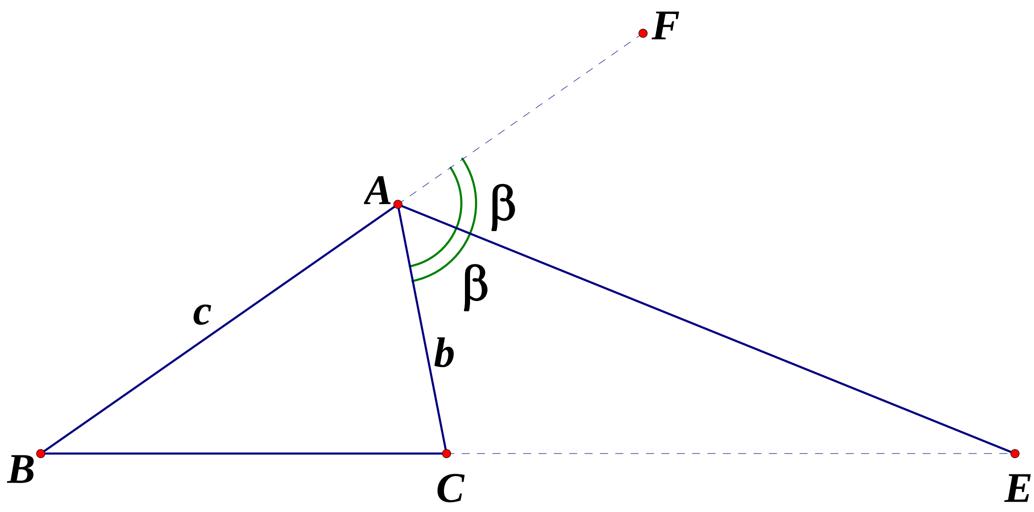
Given that $\frac{BD}{DC} = \frac{c}{b}$. $\therefore$ $\frac{\sin\angle CAD}{\sin\angle BAD} = 1$

$\sin\angle BAD = \sin\angle CAD$

$\angle BAD = \angle CAD$

AD is the angle bisector of $\angle A$.

External angle bisector theorem



In the figure, $AC = b < AB = c$, AE is the external angle bisector of $\angle BAC$, cutting BC produced at E. BA is produced to F. $\angle CAE = \angle FAE = \beta$, $\angle BAE = 180° - \beta$ (adj. $\angle$s on st. line)

Then $\dfrac{BE}{EC} = \dfrac{c}{b}$.

Proof: Apply sine rule on ΔABE and ΔACE.

$\dfrac{BE}{\sin(180^\circ - \beta)} = \dfrac{c}{\sin \angle AEC}$ …… (5) and $\dfrac{EC}{\sin \beta} = \dfrac{b}{\sin \angle AEC}$ …… (6)

Using the fact that $\sin(180° - \beta) = \sin \beta$, (5) ÷ (6): $\dfrac{BE}{EC} = \dfrac{c}{b}$

Converse of external angle bisector theorem, if $\dfrac{BE}{EC} = \dfrac{c}{b}$, then AE is the external angle bisector of $\angle BAC$.

Proof: Apply sine rule on ΔABE and ΔACE.

$\dfrac{BE}{\sin(180^\circ - \angle EAF)} = \dfrac{c}{\sin \angle AEC}$ …… (7) and $\dfrac{EC}{\sin \angle CAE} = \dfrac{b}{\sin \angle AEC}$ …… (8)

Using the fact that $\sin(180° - \angle EAF) = \sin \angle EAF$, (7) ÷ (8): $\dfrac{BE}{EC} \cdot \dfrac{\sin \angle CAE}{\sin \angle EAF} = \dfrac{c}{b}$

Given that $\dfrac{BE}{EC} = \dfrac{c}{b}$. $\therefore$ $\dfrac{\sin \angle CAE}{\sin \angle EAF} = 1$

$\sin \angle CAE = \sin \angle EAF$

$\angle CAE = \angle EAF$

AE is the external angle bisector of $\angle BAC$.