To construct a cyclic quadrilateral given four sides *a*, *b*, *c* and *d*.

Created by Mr. Francis Hung on 20190108 Last updated: 2019-01-11

Let the cyclic quadrilateral be $PQRS$ with $PQ = a$, $QR = b$, $RS = c$, $SP = d$, $PR = x$, $\angle PQR = \theta$

Assume that the sum of any three sides is greater than the fourth side.

First, we find the length of a diagonal PR in terms of a, b, c and d.

$\angle PSR = 180° - \theta$ (opp. $\angle$s of cyclic quadrilateral)

$\cos\theta = \dfrac{a^2+b^2-x^2}{2ab}$ (cosine rule on ΔPQR)

$\cos(180° - \theta) = \dfrac{c^2+d^2-x^2}{2cd}$ (cosine rule on ΔPSR)

Using the fact that $\cos(180° - \theta) = -\cos\theta$,

$$\frac{a^2+b^2-x^2}{2ab} = -\frac{c^2+d^2-x^2}{2cd}$$

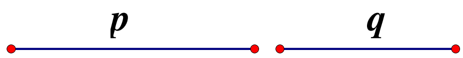
$$cd(a^2 + b^2 - x^2) + ab(c^2 + d^2 - x^2) = 0$$

$$(ab + cd)x^2 = a^2cd + b^2cd + c^2ab + d^2ab = ac(ad + bc) + bd(bc + ad) = (ac + bd)(ad + bc)$$

$$x = \sqrt{\frac{(ac+bd)(ad+bc)}{(ab+cd)}} \quad \cdots\cdots (1)$$

Lemma: Given the lengths of two line segments are p and q.

To construct $\sqrt{pq}$ by ruler and compasses.



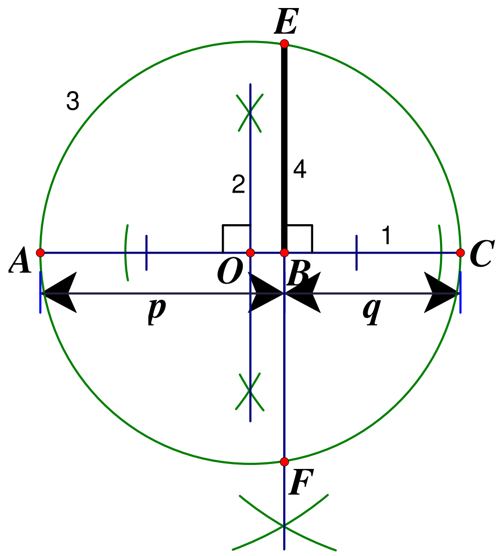
Construction steps:

(1) Draw a line segment ABC with $AB = p$, $BC = q$

(2) Draw the perpendicular bisector of AC.
O is the mid-point of AC.

(3) Use O as centre, OA as radius to draw a circle.

(4) Draw a chord EF through B and perpendicular to AC.



Then $BE = \sqrt{pq}$

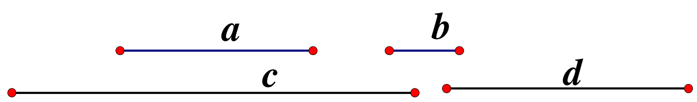
Proof: $BE = BF$ ($\perp$ from centre bisects chord)

$AB \times BC = BE \times BF$ (intersecting chords theorem)

$pq = BE^2$

$BE = \sqrt{pq}$

The proof is completed.



Construction steps (to construct a cyclic quadrilateral):

(1) Draw a line segment ABC with $AB = a$, $BC = b$

(2) Draw a semi-circle with AC as diameter.

(3) Draw a line segment BE perpendicular to AC, cutting the semi-circle at E.
By the result of the lemma, $BE = \sqrt{ac}$

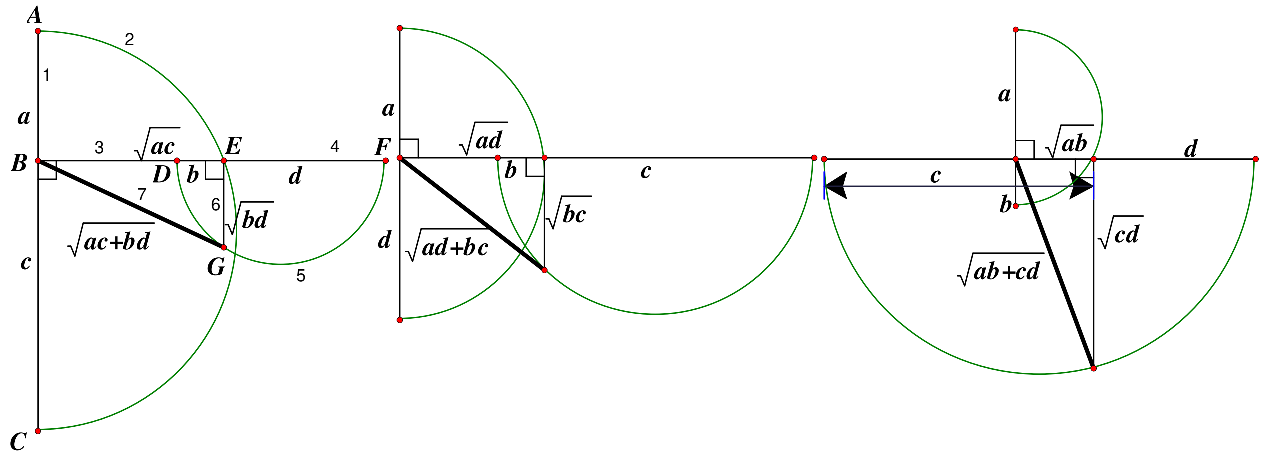
(4) Locate D and F (on the opposite sides of E) on BE produced so that $DE = b$, $EF = d$.

(5) Draw a semi-circle with DF as diameter.

(6) Draw a line segment FG perpendicular to DF, cutting the semi-circle at G.
By the result of the lemma, $EG = \sqrt{bd}$

(7) Join BG.
$BE^2 + EG^2 = BG^2$ (Pythagoras' theorem)
$BG = \sqrt{ac+bd}$

To construct a cyclic quadrilateral given 4 sides.



Similarly, line segments with lengths $\sqrt{ad+bc}$ and $\sqrt{ab+cd}$ are constructed.

(8) Draw a line segment $HJ = \sqrt{ab+cd}$.

(9) Locate a point I on HJ (or HJ produced) so that $HI = \sqrt{ad+bc}$.

(10) Draw a line segment $HL = \sqrt{ac+bd}$ in any direction which is not parallel to HJ.

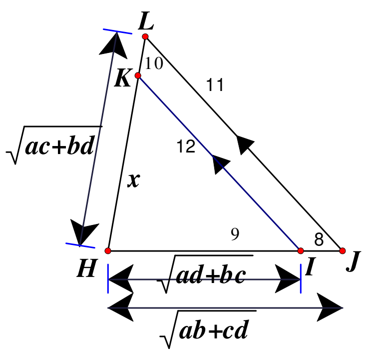
(11) Join JL.

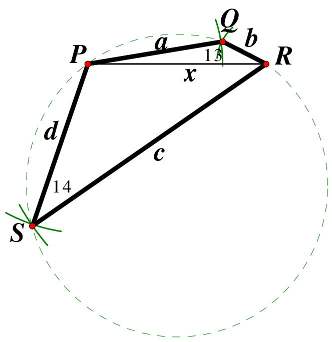
(12) Draw a line segment IK parallel to JL, cutting HL at K. Let $HK = x$.

$\Delta HIK \sim \Delta HJL$ (equiangular)

$$\frac{\sqrt{ad+bc}}{\sqrt{ab+cd}} = \frac{x}{\sqrt{ac+bd}} \qquad \text{(corr. sides, } \sim\Delta\text{s)}$$

$x = \sqrt{\dfrac{(ac+bd)(ad+bc)}{(ab+cd)}}$, which satisfies equation (1) on page 1.





(13) Construct a triangle PQR with $PR = x$, $PQ = a$, $QR = b$.

(14) Construct another triangle PSR on the opposite side of PR with $RS = c$, $SP = d$.

Then $PQRS$ is the required cyclic quadrilateral.

To construct a cyclic quadrilateral given 4 sides.

We shall prove that the quadrilateral formed is cyclic.

$$\cos\angle PQR=\frac{a^2+b^2-x^2}{2ab}=\frac{a^2+b^2-\frac{(ac+bd)(ad+bc)}{(ab+cd)}}{2ab}$$

$$=\frac{(a^2+b^2)(ab+cd)-(ac+bd)(ad+bc)}{2ab(ab+cd)}$$

$$=\frac{a^3b+a^2cd+ab^3+b^2cd-(a^2cd+abc^2+abd^2+b^2cd)}{2ab(ab+cd)}$$

$$=\frac{a^3b+ab^3-(abc^2+abd^2)}{2ab(ab+cd)}$$

$$=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$$

$$\cos\angle PSR=\frac{c^2+d^2-x^2}{2cd}=\frac{c^2+d^2-\frac{(ac+bd)(ad+bc)}{(ab+cd)}}{2cd}$$

$$=\frac{(c^2+d^2)(ab+cd)-(ac+bd)(ad+bc)}{2cd(ab+cd)}$$

$$=\frac{abc^2+c^3d+abd^2+cd^3-(a^2cd+abc^2+abd^2+b^2cd)}{2cd(ab+cd)}$$

$$=\frac{c^3d+cd^3-(a^2cd+b^2cd)}{2cd(ab+cd)}$$

$$=\frac{c^2+d^2-a^2-b^2}{2(ab+cd)}$$

$\therefore$ $\cos\angle PQR=-\cos\angle PSR$

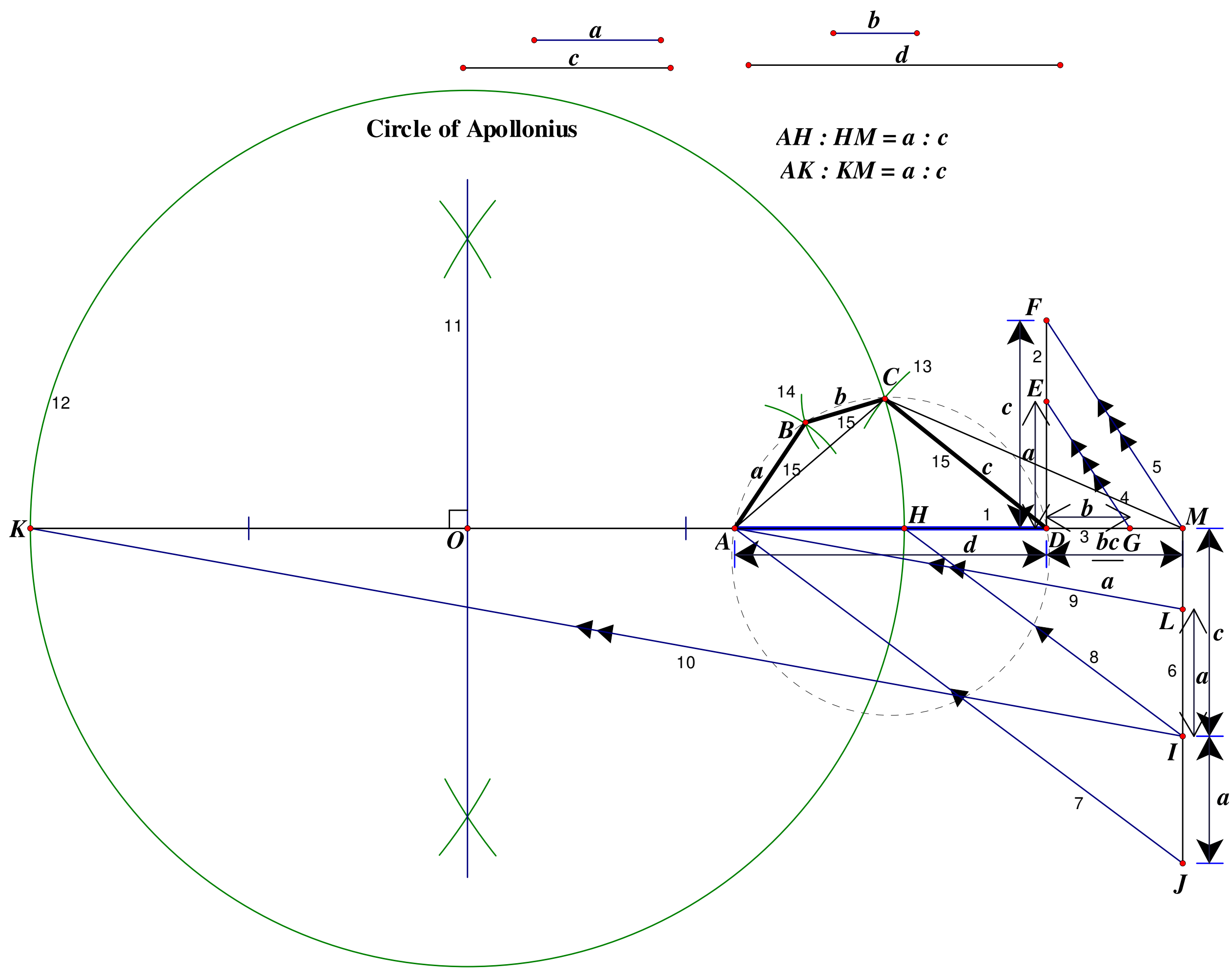
$\angle PQR = 180° - \angle PSR$

$\angle PQR + \angle PSR = 180°$

$\therefore$ $PQRS$ is a cyclic quadrilateral. (opp. $\angle$s supp.)

To construct a cyclic quadrilateral given 4 sides.

Method 2



Let AD be the longest and $a + b + c > d$. The construction steps are as follows:

(1) Draw $AD = d$, and extend it to both ways.

(2) Draw a line through D, which is not parallel to AD. Locate points E and F on the same side of AD so that $DE = a$ and $DF = c$.

(3) Mark a point G on AD produced so that $DG = b$.

(4) Join EG.

(5) Draw a line through F and parallel to EG, cutting AD produced at M.

(6) Draw a line through M, which is not parallel to AD. Locate points I, J and L on the same side of AD so that $MI = c$ and $IJ = a = IL$.

(7) Join AJ.

(8) Draw a line through I and parallel to AJ, cutting AD at H.

(9) Join AL.

(10) Draw a line through I and parallel to AL, cutting DA produced at K.

(11) Draw the perpendicular bisector of HK, O is the mid-point of HK.

(12) Use O as centre, OH as radius to draw the Circle of Apollonius.

(13) Use D as centre, c as radius to draw an arc, cutting the circle of Apollonius at C.

(14) C as centre, b as radius to draw an arc; use A as centre, a as radius to draw another arc. The two arcs intersect at two points, of which one of the intersection B, a convex quadrilateral $ABCD$ is formed.

(15) Join AB, BC and CD.

Then $ABCD$ is the required cyclic quadrilateral.

To construct a cyclic quadrilateral given 4 sides.

Proof: By steps (2) to (5), $\Delta DEG \sim \Delta DFM$ (equiangular)

$\dfrac{DM}{DG} = \dfrac{DF}{DE}$ (corr. sides, ~Δs)

$DM = \dfrac{bc}{a}$ ⋯⋯ (1)

By steps (7) to (8), HI // AJ

$AH : HM = JI : IM$ (theorem of equal ratios)

$AH : HM = a : c$ ⋯⋯ (2)

By steps (9) to (10), $\Delta KMI \sim \Delta AML$ (equiangular)

$\dfrac{AM}{KM} = \dfrac{LM}{IM}$ (corr. sides, ~Δs)

$1 - \dfrac{AM}{KM} = 1 - \dfrac{c-a}{c}$

$\dfrac{AK}{KM} = \dfrac{a}{c}$ ⋯⋯ (3)

Join AC and CM.

By the property of the Circle of Apollonius, $AC : CM = AH : HM$

http://www2.hkedcity.net/citizen_files/aa/gi/fh7878/public_html/Geometry/construction/triangle/Equilateral_tri_on_triABC.pdf

page 4 theorem 3

Consider ΔABC and ΔCDM

$\therefore AC : CM = a : c$ (by (2))

$AB : CD = a : c$

$BC : DM = b : \dfrac{bc}{a} = a : c$

$\Delta ABC \sim \Delta CDM$ (3 sides proportional)

$\angle CDM = \angle ABC$ (corr. $\angle$s ~Δs)

$\therefore ABCD$ is a cyclic quadrilateral (ext. $\angle$ = int. opp. $\angle$)