Problem on a 5-12-13 triangle

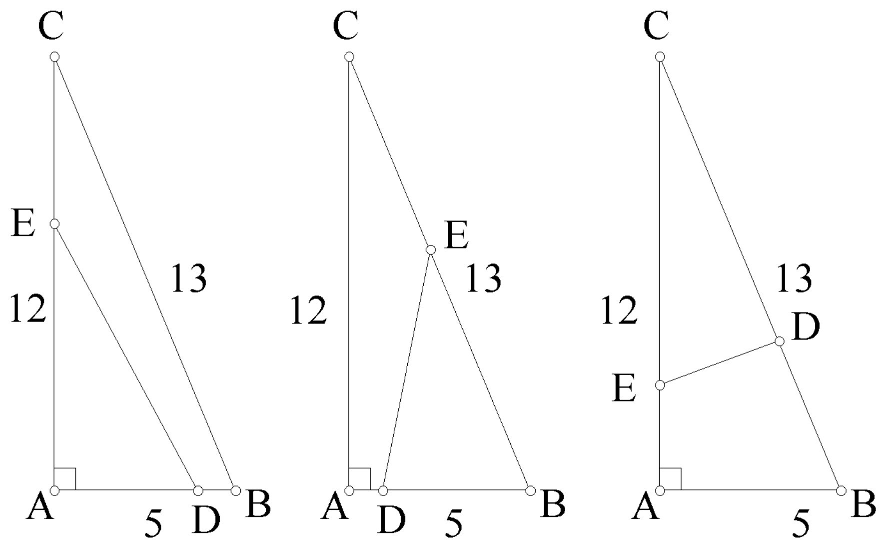
Created by Mr. Francis Hung Last updated: April 23, 2011

Given a triangle ABC. $AB = 5$, $AC = 12$, $BC = 13$. D and E are points on the sides of the triangle such that DE separates ΔABC into 2 parts of equal areas. Determine the minimum length of DE.

Solution: Clearly $\angle BAC = 90°$ (Converse Pythagoras' Theorem) and Area of $\Delta ABC = \frac{1}{2}\cdot 5\times 12 = 30$

$\sin B = \frac{12}{13}$, $\cos B = \frac{5}{13}$; $\sin C = \frac{5}{13}$, $\cos C = \frac{12}{13}$

There are three different cases:



Case 1 **Case 2** **Case 3**

Case 1: D lies on AB and E lies on AC.

Let $AD = x$, $AE = y$. Area of $\Delta ADE = \frac{1}{2}\cdot xy = \frac{1}{2}$ Area of $\Delta ABC = 15 \Rightarrow xy = 30$

In ΔADE, $DE^2 = x^2 + y^2$ (Pythagoras' Theorem)

$DE^2 \ge 2xy = 60$ (AM $\ge$ GM)

Case 2: D lies on AB and E lies on BC.

Let $BD = x$, $BE = y$. Area of $\Delta BDE = \frac{1}{2}\cdot xy \sin B = \frac{1}{2}$ Area of $\Delta ABC = 15 \Rightarrow xy = \frac{65}{2}$

Apply cosine formula on ΔBDE:

$$DE^2 = x^2 + y^2 - 2xy\cos B = x^2 + y^2 - 2\times\frac{65}{2}\times\frac{5}{13} = x^2 + y^2 - 25$$

$DE^2 \ge 2xy - 25 = 65 - 25 = 40$

Case 3: D lies on BC and E lies on AC.

Let $CD = x$, $CE = y$. Area of $\Delta CDE = \frac{1}{2}\cdot xy \sin C = \frac{1}{2}$ Area of $\Delta ABC = 15 \Rightarrow xy = 78$

Apply cosine formula on ΔCDE:

$$DE^2 = x^2 + y^2 - 2xy\cos C = x^2 + y^2 - 2\times 78\times\frac{12}{13} = x^2 + y^2 - 144$$

$DE^2 \ge 2xy - 144 = 156 - 144 = 12$

Combine the 3 cases, minimum of $DE = \sqrt{12} = 2\sqrt{3}$