Individual Events

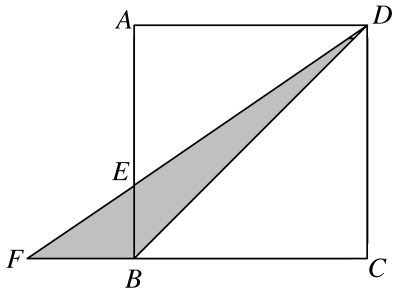
I1	***P***	40	**I2**	***P***	$\frac{99}{100}$	**I3**	***P***	12	**I4**	***P***	4
	Q	72		***Q***	1		***Q***	1		***Q***	8
	R	648		***R***	3		***R***	615		***R***	4
	S	40.5		***S***	$\frac{1}{12}$		***S***	60		***S***	10

Group Events

G1	***a***	21	**G2**	***a***	24	**G3**	***a***	2005	**G4**	***a***	4032
	b	2.5		***b***	52		***b***	2		***b***	2
	c	19		***c***	2005003		***c***	649		***c***	1
	d	300°		***d***	3		***d***	8		***d***	2

Individual Event 1

I1.1 In the following figure, $ABCD$ is a square of length 10 cm. AEB, FED and FBC are straight lines. The area of ΔAED is larger than that of ΔFEB by 10 cm^2. If the area of ΔDFB is P cm^2, find the value of P.



Let the area of ΔBDE be x.

Then area of $\Delta AED + x$ – (area of $\Delta BEF + x$) = 10

area of ΔABD – area of ΔBDF = 10

$\frac{1}{2}\cdot 10\times 10$ – area of ΔBDF = 10

area of ΔBDF = 40

Method 2

Area of ΔADE – area of ΔBFE = 10 (given)

$\Rightarrow$ Area of ΔADE + area of ΔAEF – area of ΔBFE – area of ΔAEF = 10

$\Rightarrow$ Area of ΔADF – area of ΔAFB = 10

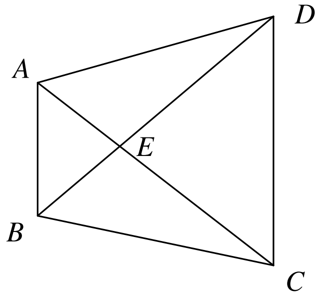
$\Rightarrow$ $\frac{1}{2}\cdot 10\times 10$ – area of ΔDFB = 10 $\Rightarrow$ Area of ΔDFB = 50 – 10 = 40

I1.2 Workman A needs 90 days to finish a task independently while workman B needs Q days for the same task. If they only need P days to finish the task when working together, find the value of Q.

$$\frac{1}{90}+\frac{1}{Q}=\frac{1}{40}$$

$Q = 72$

I1.3 In the following figure, $AB//CD$, the area of trapezium $ABCD$ is R cm^2. Given that the areas of ΔABE and ΔCDE are Q cm^2 and $4Q$ cm^2 respectively, find the value of R.



Reference: 1993 HI2, 1997 HG3, 2000 FI2.2, 2004 HG7, 2010HG4, 2013 HG2

It is easy to show that $\Delta ABE \sim \Delta CDE$ (equiangular)

$Q : 4Q = (AB)^2 : (CD)^2 \Rightarrow AB : CD = 1 : 2$

$AE : EC = AE : EC = BE : ED = 1 : 2$ (ratio of sides, ~ Δ's)

$S_{\Delta AEB} : S_{\Delta AED} = BE : ED = 1 : 2$ (the 2 Δs have the same heights)

$S_{\Delta AED} = 2Q$

$S_{\Delta AEB} : S_{\Delta BEC} = AE : EC = 1:2$ (the 2 Δs have the same heights)

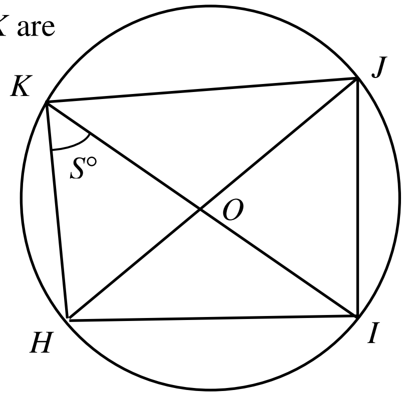
$S_{\Delta BEC} = 2Q$

$S_{ABCD} = Q + 4Q + 2Q + 2Q = 9Q = 648$

$R = 648$

I1.4 In the following figure, O is the centre of the circle, HJ and IK are diameters and $\angle HKI = S°$.

Given that $\angle HKI + \angle HOI + \angle HJI = \frac{1}{4} R°$, find the value of S.



$S° + 2S° + S° = \frac{1}{4} \times 648°$

$\Rightarrow 4S° = 162°$ ($\angle$ at centre = $2\angle$ at $\odot^{ce}$)

$S = 40.5$

Individual Event 2

I2.1 Given that $P = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{99\times 100}$, find the value of P.

$P = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} - \frac{1}{100}$

$= 1 - \frac{1}{100} = \frac{99}{100}$

I2.2 Given that $99Q = P \times (1 + \frac{99}{100} + \frac{99^2}{100^2} + \frac{99^3}{100^3} + \ldots)$, find the value of Q.

$99Q = \frac{99}{100} \times (1 + \frac{99}{100} + \frac{99^2}{100^2} + \frac{99^3}{100^3} + \ldots)$

$= \frac{99}{100} \times \frac{1}{1 - \frac{99}{100}} = 99$

$Q = 1$

I2.3 Given that x and R are real numbers and $\frac{2x^2 + 2Rx + R}{4x^2 + 6x + 3} \le Q$ for all x, find the maximum value of R.

$4x^2 + 6x + 3 = (2x + 1.5)^2 + 0.75 > 0$

$\frac{2x^2 + 2Rx + R}{4x^2 + 6x + 3} \le 1$

$2x^2 + 2Rx + R \le 4x^2 + 6x + 3$

$2x^2 + 2(3 - R)x + 3 - R \ge 0$

$\Delta \le 0$

$(3 - R)^2 - 2(3 - R) \le 0$

$(3 - R)(1 - R) \le 0$

$1 \le R \le 3$

The maximum value of $R = 3$

I2.4 Given that $S = \log_{144} \sqrt[R]{2} + \log_{144} \sqrt[2R]{R}$, find the value of S.

$S = \frac{\frac{1}{3}\log 2}{\log 144} + \frac{\frac{1}{6}\log 3}{\log 144} = \frac{2\log 2 + \log 3}{6\log 144} = \frac{\log 12}{6\log 12^2} = \frac{\log 12}{12\log 12} = \frac{1}{12}$

Method 2

$S = \log_{144} \sqrt[R]{2} + \log_{144} \sqrt[2R]{R}$

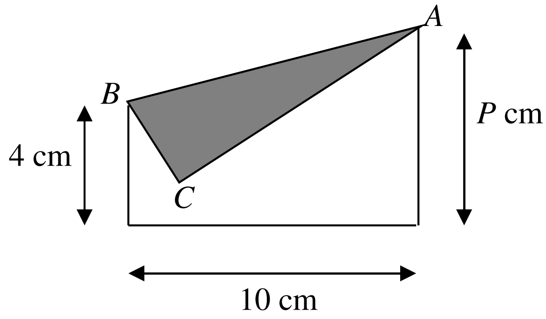
$= \log_{144}(\sqrt[3]{2} \cdot \sqrt[6]{3})$

$= \log_{144}(\sqrt[6]{12})$

$= \log_{144}(\sqrt[12]{144}) = \frac{1}{12}$

Individual Event 3

I3.1 A rectangular piece of paper is folded into the following figure. If the area of $\triangle ABC$ is $\frac{1}{3}$ of the area of the original rectangular piece of paper, find the value of P.



$BC = P - 4$, $AC = 10$, $\angle ACB = 90°$

$$\frac{(P-4)\cdot 10}{2} = \frac{1}{3}\times P\times 10$$

$\Rightarrow P = 12$

I3.2 If Q is the positive integral solution of the equation $\frac{P}{2}(4^x + 4^{-x}) - 35(2^x + 2^{-x}) + 62 = 0$, find the value of Q.

Let $t = 2^x + 2^{-x}$, then $t^2 = 4^x + 4^{-x} + 2$

$\Rightarrow 4^x + 4^{-x} = t^2 - 2$

The equation becomes $6(t^2 - 2) - 35t + 62 = 0$

$6t^2 - 35t + 50 = 0$

$(2t - 5)(3t - 10) = 0$

$t = \frac{5}{2}$ or $\frac{10}{3}$

$2^x + 2^{-x} = \frac{5}{2}$ or $2^x + 2^{-x} = \frac{10}{3}$

$2^x + \frac{1}{2^x} = \frac{5}{2}$ or $2^x + \frac{1}{2^x} = \frac{10}{3}$

$2(2^x)^2 + 2 = 5(2^x)$ or $3(2^x)^2 + 3 = 10(2^x)$

$2(2^x)^2 - 5(2^x) + 2 = 0$ or $3(2^x)^2 - 10(2^x) + 3 = 0$

$(2\cdot 2^x - 1)(2^x - 2) = 0$ or $(3\cdot 2^x - 1)(2^x - 3) = 0$

$2^x = \frac{1}{2}, 2, \frac{1}{3}$ or 3

For positive integral solution $x = 1$; $Q = 1$

I3.3 Let $[a]$ be the largest integer not greater than a. For example, $[2.5] = 2$.
If $R = [\sqrt{1}] + [\sqrt{2}] + \dots + [\sqrt{99Q}]$, find the value of R.

$$R = [\sqrt{1}] + [\sqrt{2}] + \dots + [\sqrt{99}] = 1+1+1+\underbrace{2+\cdots+2}_{\text{5 times}}+\underbrace{3+\cdots+3}_{\text{7 times}}+\cdots+\underbrace{9+\cdots+9}_{\text{19 times}}$$

$R = 3\times 1 + 5\times 2 + 7\times 3 + \dots + 19\times 9$

$R = (2\times 1+1)\times 1 + (2\times 2+1)\times 2+(2\times 3+1)\times 3 + \dots + (2\times 9+1)\times 9$

$R = 2\times 1^2 + 1 + 2\times 2^2 + 2 + 2\times 3^2 + 3 + \dots + 2\times 9^2 + 9$

$R = 2\times(1^2 + 2^2 + 3^2 + \dots + 9^2) + (1 + 2 + 3 + \dots + 9)$

$$R = 2\times\frac{1}{6}\cdot 9(9+1)(2\times 9+1)+\frac{(1+9)9}{2} = 3\times 10\times 19 + 45 = 570 + 45 = 615$$

I3.4 In a convex polygon, other than the interior angle A, the sum of all the remaining interior angles is equal to $4R°$. If $\angle A = S°$, find the value of S.

Reference: 1989 HG2, 1990 FG10.3-4, 1992 HG3, 2013 HI6

$4\times 615 + S = 180\times(n - 2)$

$S = 180(n - 2) - 2460$

$\because$ The polygon is convex

$\therefore S < 180$. $S = 180(14) - 2460 = 60$

Individual Event 4

I4.1 Given that $f(x) = (x^2 + x - 2)^{2002} + 3$ and $f\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right) = P$, find the value of P.

$x = \frac{\sqrt{5}}{2}-\frac{1}{2}$

$\Rightarrow 2x = \sqrt{5}-1$

$\Rightarrow (2x + 1) = \sqrt{5}$

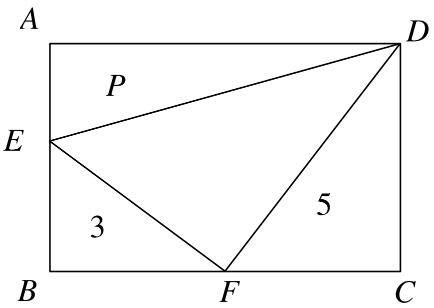
$\Rightarrow (2x + 1)^2 = 5$

$\Rightarrow 4x^2 + 4x - 4 = 0$

$\Rightarrow x^2 + x = 1$

$f(x) = (x^2 + x - 2)^{2002} + 3 = (1 - 2)^{2002} + 3 = 1 + 3 = 4$

I4.2 In the following figure, $ABCD$ is a rectangle. E and F are points on AB and BC respectively. The areas of triangles AED, EBF and FCD are P, 3 and 5 respectively. If the area of ΔEFD is Q, find the value of Q.



Let $AE = x$, $CF = y$, $AD = b$, $CD = a$.

Then $BE = a - x$, $BF = b - y$

Given the area of $\Delta ADE = 4 \Rightarrow bx = 8$ ………..(1)

the area of $\Delta CDF = 5 \Rightarrow ay = 10$……….(2)

The area of $\Delta BEF = 3 \Rightarrow (a - x)(b - y) = 6$

$\Rightarrow ab - bx - ay + xy = 6$ ……..(3)

Sub. (1), (2) into (3) $ab - 8 - 10 + xy = 6$

Sub. (1), (2) into the equation again: $ab - 18 + \frac{80}{ab} = 6$

Solving for ab, $ab = 20$ or 4 (rejected)

The area of $\Delta DEF = 20 - 3 - 4 - 5 = 8$

$Q = 8$

I4.3 It is given that x and y are positive integers. If the number of solutions (x, y) of the inequality $x^2 + y^2 \le Q$ is R, find the value of R. **(Reference: 2007 FG1.2)**

$x^2 + y^2 \le 8$

$\Rightarrow (x, y) = (1, 1), (1, 2), (2, 1), (2, 2)$

$R = 4$

I4.4 It is given that α and β are roots of the equation $x^2 - ax + a - R = 0$, where a is real. If the minimum value of $(\alpha + 1)^2 + (\beta + 1)^2$ is S, find the value of S.

$x^2 - ax + a - 4 = 0$; $\alpha + \beta = a$, $\alpha\beta = a - 4$

$(\alpha+1)^2 + (\beta+1)^2 = \alpha^2 + 2\alpha + 1 + \beta^2 + 2\beta + 1$

$= (\alpha + \beta)^2 - 2\alpha\beta + 2(\alpha + \beta) + 2$

$= a^2 - 2(a - 4) + 2a + 2 = a^2 + 10 \ge 10$

The minimum value is $S = 10$.

Group Event 1

G1.1 Assume that the curve $x^2 + 3y^2 = 12$ and the straight line $mx + y = 16$ intersect at only one point. If $a = m^2$, find the value of a.

Sub. $y = 16 - mx$ into $x^2 + 3y^2 = 12$

$\Rightarrow x^2 + 3(16 - mx)^2 = 12$

$x^2 + 3(256 - 32mx + m^2x^2) = 12$

$\Rightarrow (1 + 3m^2)x^2 - 96mx + 756 = 0$

The straight line is a tangent $\Rightarrow \Delta = (-96m)^2 - 4(1 + 3m^2)756 = 0$

$576m^2 - 189(1 + 3m^2) = 0$

$\Rightarrow 64m^2 - 21(1 + 3m^2) = 0$

$\Rightarrow a = m^2 = 21$

G1.2 It is given that $x + y = 1$ and $x^2 + y^2 = 2$. If $x^3 + y^3 = b$, find the value of b.

Reference: 2011 FI2.2

$(x + y)^2 = 1 \Rightarrow x^2 + y^2 + 2xy = 1$

$\Rightarrow 2 + 2xy = 1$

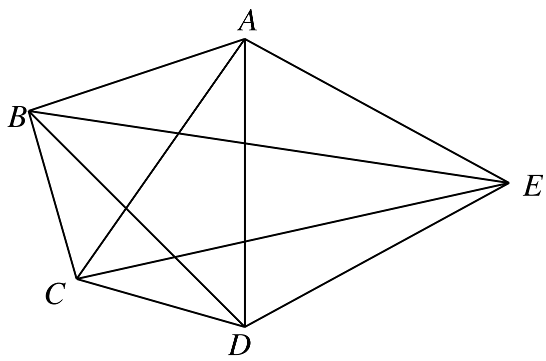
$\Rightarrow xy = -\frac{1}{2}$

$b = x^3 + y^3$

$= (x + y)(x^2 + y^2 - xy)$

$= 1(2 + \frac{1}{2}) = \frac{5}{2}$

G1.3 In the following figure, $AC = AD = AE = ED = DB$ and $\angle BEC = c°$. Given that $\angle BDC = 26°$ and $\angle ADB = 46°$, find the value of c.



ΔADE is an equilateral triangle.

$\angle DAE = \angle ADE = \angle AED = 60°$

$\because BD = DE$ and $\angle BDE = 46° + 60° = 106°$

$\therefore \angle BED = (180° - 106°) \div 2 = 37°$ ($\angle$s sum of Δ)

$\angle AEB = 60° - 37° = 23°$

$\angle ADC = 26° + 46° = 72°$

$\because AC = AD$ and $\angle ADC = 72° = \angle ACD$ (base $\angle$, isos. Δ)

$\therefore \angle CAD = 180° - 72° \times 2 = 36°$ ($\angle$s sum of Δ)

$\because AC = AE$ and $\angle CAE = 36° + 60° = 96°$

$\therefore \angle AEC = (180° - 96°) \div 2 = 42°$ ($\angle$s sum of Δ)

$\angle CED = 60° - 42° = 18°$

$\angle BCE = 60° - 18° - 23° = 19°$

$c = 19$

G1.4 It is given that $4\cos^4\theta + 5\sin^2\theta - 4 = 0$, where $0° < \theta < 360°$. If the maximum value of θ is d, find the value of d.

$4\cos^4\theta + 5\sin^2\theta - 4 = 0 \Rightarrow 4\cos^4\theta + 5(1 - \cos^2\theta) - 4 = 0 \Rightarrow 4\cos^4\theta - 5\cos^2\theta + 1 = 0$

$(4\cos^2\theta - 1)(\cos^2\theta - 1) = 0$

$\cos^2\theta = \frac{1}{4}$ or 1

$\Rightarrow \cos\theta = \frac{1}{2}, -\frac{1}{2}$, 1 or -1.

$\theta = 60°, 300°, 120°, 180°, 240°$.

The maximum value of $\theta = 300°$

$d = 300°$

Group Event 2

G2.1 It is given that the lengths of the sides of a triangle are 6, 8, and 10.

If the area of the triangle is a, find the value of a.

$6^2 + 8^2 = 36 + 64 = 100 = 10^2$

It is a right angled triangle.

The area of the triangle $= 6 \times 8 \div 2 = 24$

$a = 24$

G2.2 Given that $f\left(x+\frac{1}{x}\right)=x^3+\frac{1}{x^3}$ and $f(4) = b$, find the value of b.

Reference: 1987 FG8.2, 2002 HI10

Let $y = x + \frac{1}{x}$

$y^2 - 2 = x^2 + \frac{1}{x^2}$

$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$

$= y(y^2 - 3) = y^3 - 3y$

$f(y) = y^3 - 3y$

$b = f(4) = 4^3 - 3(4) = 52$

G2.3 Given that $2002^2 - 2001^2 + 2000^2 - 1999^2 + \dots + 4^2 - 3^2 + 2^2 - 1^2 = c$, find the value of c.

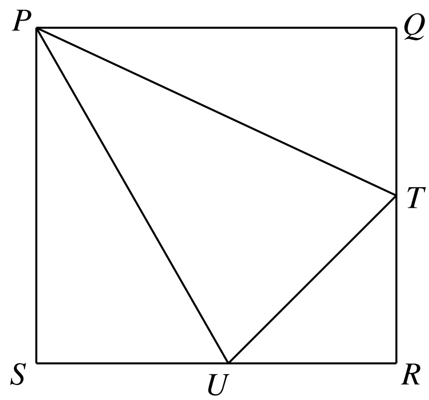
Reference: 1997 HI5, 2004 HI1, 2015 FI3.2, 2015 FG4.1

$c = (2002+ 2001)(2002 - 2001) + (2000 + 1999)(2000 - 1999) +\dots+ (4+3)(4-3) + (2+1)(2-1)$

$c = 4003 + 3999 + \dots + 7 + 3$

$= \frac{4003+3}{2}\times 1001 = 2005003$

G2.4 $PQRS$ is a square, PTU is an isosceles triangle, and $\angle TPU = 30°$. Points T and U lie on QR and RS respectively. The area of ΔPTU is 1. If the area of $PQRS$ is d, find the value of d.



Let $PT = a = PU$

$\frac{1}{2}a^2 \sin 30^\circ = 1$

$\Rightarrow a = 2$

$\Delta PSU \cong \Delta PQT$ (RHS)

Let $PS = x = PQ$; $SU = y = QT$

$\angle SPU = \angle QPT = 30°$ (corr. $\angle$s $\cong \Delta$)

$x = PU \cos 30° = 2\times\frac{\sqrt{3}}{2}=\sqrt{3}$

d = area of $PQRS = \sqrt{3}^2 = 3$

Group Event 3

G3.1 If $\dfrac{2002^3+4\times2002^2+6006}{2002^2+2002}=a$, find the value of a.

$a=\dfrac{2002\left(2002^2+4\times2002+3\right)}{2002(2002+1)}$

$=\dfrac{(2002+1)(2002+3)}{2002+1}$

$= 2005$

G3.2 It is given that the real numbers x and y satisfy the relation $y=\dfrac{x}{2x-1}$.

If the minimum value of $\dfrac{1}{x^2}+\dfrac{1}{y^2}$ is b, find the value of b.

$\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{1}{x^2}+\dfrac{(2x-1)^2}{x^2}$

$=\dfrac{4x^2-4x+2}{x^2}$

Let $T=\dfrac{4x^2-4x+2}{x^2}$

$Tx^2 = 4x^2 - 4x + 2$

$(T-4)x^2 + 4x - 2 = 0$

$\Delta = 4^2 + 4\times2(T-4) \ge 0$

$2 + T - 4 \ge 0$

$\Rightarrow T \ge 2$

The minimum value is 2

$b = 2$

G3.3 Suppose two different numbers are chosen randomly from the 50 positive integers 1, 2, 3, … , 50, and the sum of these two numbers is not less than 50. If the number of ways of choosing these two numbers is c, find the value of c.

Reference: 2011 FG2.2

Possible combinations may be: (1, 49), (1, 50),

(2, 48), (2, 49), (2, 50),

(3, 47), (3, 48), (3, 49), (3, 50),

....

(24, 26), (24, 27), … , (24, 50),

(25, 26), (25, 27), … , (25, 50),

(26, 27), … , (26, 50)

....

(49, 50)

Total number of combinations = (2 + 3 + … + 25) + 25 + (24 + 23 + … + 1)

= (1 + 2 + … + 24)×2 + 24 + 25

= 25×24 + 49 = 649

G3.4 Given that $x-y=1+\sqrt{5}$, $y-z=1-\sqrt{5}$. If $x^2+y^2+z^2-xy-yz-zx=d$, find the value of d.

$2d=(x-y)^2+(y-z)^2+(z-x)^2=(1+\sqrt{5})^2+(1-\sqrt{5})^2+[(z-y)-(x-y)]^2$

$2d=1+2\sqrt{5}+5+1-2\sqrt{5}+5+[-1+\sqrt{5}-(1+\sqrt{5})]^2=12+4=16$

$d = 8$

Group Event 4

G4.1 If a is the sum of all the positive factors of 2002, find the value of a.

Reference 1993 HI8, 1994 FI3.2, 1997 HI3, 1998 HI10, 1998 FI1.4, 2005 FI4.4

$2002 = 2\times7\times11\times13$

The positive factors may be $2^a 7^b 11^c 13^d$, where $0 \le a, b, c, d \le 1$ are integers.

The sum of all positive factors are $(1 + 2)(1 + 7)(1 + 11)(1 + 13) = 3\times8\times12\times14 = 4032 = a$

G4.2 It is given that $x > 0$, $y > 0$ and $\sqrt{x}(\sqrt{x}+\sqrt{y})=3\sqrt{y}(\sqrt{x}+5\sqrt{y})$. If $b=\dfrac{2x+\sqrt{xy}+3y}{x+\sqrt{xy}-y}$,

find the value of b.

$\sqrt{x}(\sqrt{x}+\sqrt{y})=3\sqrt{y}(\sqrt{x}+5\sqrt{y})$

$\Rightarrow x+\sqrt{xy} = 3\sqrt{xy} + 15y$

$\Rightarrow x - 2\sqrt{xy} - 15y = 0$

$\left(\sqrt{x}+3\sqrt{y}\right)\left(\sqrt{x}-5\sqrt{y}\right)=0$

$\Rightarrow \sqrt{x}=5\sqrt{y} \quad \Rightarrow x = 25y$

$$b=\frac{2x+\sqrt{xy}+3y}{x+\sqrt{xy}-y}=\frac{50y+\sqrt{25y^2}+3y}{25y+\sqrt{25y^2}-y}=\frac{58y}{29y}=2$$

$b = 2$

G4.3 Given that the equation $\left|\,|x-2|-1\right| = c$ has only 3 integral solutions, find the value of c.

Reference: 2005 FG4.2, 2009 HG9, 2012 FG4.2, 2017 FG1.2

$|x-2| - 1 = \pm c$

$\Rightarrow |x-2| = 1\pm c$

In order that it has only 3 integral solutions

$c = 1$

G4.4 If d is the positive real root of the equation $\dfrac{1}{2}\left\{\dfrac{1}{2}\left[\dfrac{1}{2}\left(\dfrac{1}{2}x^2+2\right)+2\right]+2\right\}=2$,

find the value of d.

$$\frac{1}{2}\left\{\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]+2\right\}=2$$

$$\Rightarrow\left\{\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]+2\right\}=4$$

$$\Rightarrow\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]=2$$

$$\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]=4$$

$$\Rightarrow\frac{1}{2}\left(\frac{1}{2}x^2+2\right)=2$$

$$\Rightarrow\frac{1}{2}x^2+2=4$$

$$\Rightarrow\frac{1}{2}x^2=2$$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm2$

d = the positive real root = 2