Locus Example 1

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Example 1 on locus

Given a circle C: $x^2 + y^2 + 2y - 4 = 0$, two perpendicular tangents intersect at P. Find the locus of P.

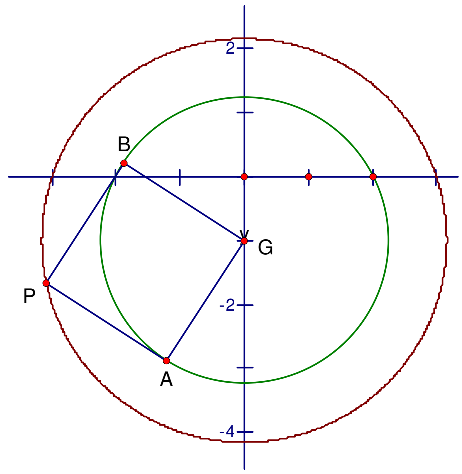
Method 1

centre = (0, –1), radius = $\sqrt{5}$

We first find the equation of tangent given slope m.

Let the equation of tangent be $y = mx + c$

Distance from centre to the line $mx - y + c = 0$ equal to the radius.



$$\left|\frac{m\times 0+1+c}{\sqrt{1+m^2}}\right|=\sqrt{5}$$

$$c=\pm\sqrt{5\left(1+m^2\right)}-1$$

$$y=mx\pm\sqrt{5\left(1+m^2\right)}-1$$

$$y-mx+1=\pm\sqrt{5\left(1+m^2\right)}$$

$(y - mx + 1)^2 = 5(1 + m^2)$

$y^2 - 2mxy + m^2x^2 + 2y - 2mx + 1 = 5 + 5m^2$

$(x^2 - 5)m^2 - (2xy + 2x)m + y^2 + 2y - 4 = 0$

This is a quadratic equation in m. It has two possible slopes m_1, m_2.

They are the slopes of the two tangents. Given that the two tangents are perpendicular.

$m_1m_2 = -1$

$$\frac{y^2+2y-4}{x^2-5}=-1$$

$x^2 + y^2 + 2y - 9 = 0$

Method 2

Let G be the centre, A, B be the points of contact of two perpendicular tangents. Suppose the two tangents meet at P. Then it can be easily proved that $GAPB$ is a square.

$GA = GB =$ radius $=\sqrt{5}$

$GP=\sqrt{GA^2+AP^2}=\sqrt{GA^2+GA^2}=\sqrt{2}GA=\sqrt{2}\times\sqrt{5}=\sqrt{10}$ which is a constant.

Therefore, the locus is a circle with $G(0, -1)$ as centre and radius $\sqrt{10}$.

It equation is $x^2 + (y + 1)^2 = 10$

$x^2 + y^2 + 2y - 9 = 0$