| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **10-11 Individual** | **1** | 11 | **2** | $\frac{26}{5}$ (= 5.2) | **3** | $\frac{1}{2}\sqrt{2010}$ | **4** | 20 | **5** | 237 |
| | **6** | 10 | **7** | $\sqrt{6}-\sqrt{2}$ | **8** | 7 | **9** | 36 | **10** | $24\sqrt{3}$ |
| **10-11 Group** | **1** | $a = 1006$, $b = 1002$, $c = 998$, $d = 996$, $e = 994$ | **2** | 15 | **3** | 5026 | **4** | 4 | **5** | 1005 |
| | **6** | $6\sqrt{2}$ | **7** | 125751 | **8** | 2012 | **9** | 99° | **10** | $2\sqrt{3}$ |

**Individual Events**

**I1** Find the remainder when $2^{2011}$ is divided by 13.

**Reference: 1972 American High School Mathematics Examination Q31, 2015 FI4.1**

$2^6 = 64 = 13\times5 - 1 \equiv -1 \bmod 13$; $2^{12} \equiv 1 \bmod 13$

$2011 = 12\times167 + 7$

$2^{2011} = 2^{12\times167+7} = (2^{12})^{167}\times2^7 \equiv 2^7 \equiv 2^6\times2 \equiv -1\times2 \equiv -2 \equiv 11 \bmod 13$

**I2** Given that $x^2 + y^2 = 1$, find the maximum value of $2x + 5y^2$.

**Reference: 2003 FG1.2, 2009 HI5**

$2x + 5y^2 = 2x + 5(1 - x^2) = -5x^2 + 2x + 5$

The maximum value $= \dfrac{4ac-b^2}{4a} = \dfrac{4(-5)\cdot5-2^2}{4(-5)} = \dfrac{26}{5}$ (= 5.2)

**I3** Given that $a + b = \sqrt{\sqrt{2011}+\sqrt{2010}}$ and $a - b = \sqrt{\sqrt{2011}-\sqrt{2010}}$, find the value of $ab$. (Give your answer in surd form)

$(a + b)^2 - (a - b)^2 = \sqrt{2011}+\sqrt{2010}-\sqrt{2011}+\sqrt{2010}$

$4ab = 2\sqrt{2010}$

$ab = \dfrac{1}{2}\sqrt{2010}$

**I4** In $\Delta ABC$, the ratio of the altitudes perpendicular to the three sides $AB$, $BC$ and $CA$ is 3 : 4 : 5. If the lengths of the three sides are integers, find the minimum value of $AB$.

Let $AB = c$, $BC = a$, $CA = b$. By calculating the areas in 3 different ways:

$3a = 4b = 5c$

$a : b : c = 20 : 15 : 12$

The minimum value of $AB = 20$

**I5** An integer $x$ minus 12 is the square of an integer. $x$ plus 19 is the square of another integer. Find the value of $x$.

$x - 12 = n^2$ …… (1); $x + 19 = m^2$ …… (2), where $m$, $n$ are integers.

(2) – (1): $(m + n)(m - n) = 31$

$\because$ 31 is a prime number

$\therefore m + n = 31$ and $m - n = 1$

$m = 16$, $n = 15$

$x = 15^2 + 12 = 237$

**I6** $A$, $B$ and $C$ pass a ball among themselves. $A$ is the first one to pass the ball to other one. In how many ways will the ball be passed back to $A$ after 5 passes?

Construct the following table:

| Number of passes | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $A$ | 0 | 1+1=2 | 1+1=2 | 3+3=6 | 5+5=10 |
| $B$ | 1 | 0+1=1 | 1+2=3 | 3+2=5 | 5+6=11 |
| $C$ | 1 | 0+1=1 | 1+2=3 | 3+2=5 | 5+6=11 |

There will be 10 ways for the ball to pass back to $A$.

**I7** Find the value of $\sqrt{7-\sqrt{12}-\sqrt{13-2\sqrt{12}}}$.

**Reference: 1993 FI1.4, 1999 HG3, 2001 FG2.1, 2015 FI4.2, 2015 FG3.1**

$$\begin{aligned}\sqrt{13-2\sqrt{12}} &= \sqrt{12-2\sqrt{1\times 12}+1}\\ &= \sqrt{\left(\sqrt{12}-1\right)^2}\\ &= \sqrt{12}-1\end{aligned}$$

$$\begin{aligned}\sqrt{7-\sqrt{12}-\sqrt{13-2\sqrt{12}}} &= \sqrt{7-\sqrt{12}-\sqrt{12}+1}\\ &= \sqrt{8-2\sqrt{12}}\\ &= \sqrt{6-2\sqrt{2\times 6}+2}\\ &= \sqrt{\left(\sqrt{6}-\sqrt{2}\right)^2}\\ &= \sqrt{6}-\sqrt{2}\end{aligned}$$

**I8** A school issues 4 types of raffle tickets with face values \$10, \$15, \$25 and \$40. Class A uses several one-hundred dollar notes to buy 30 raffle tickets, including 5 tickets each for two of the types and 10 tickets each for the other two types. How many one-hundred dollars notes Class A use to buy the raffle tickets?

Suppose $x$ tickets of \$10, $y$ tickets of \$15, $z$ tickets of \$25, $t$ tickets of \$40 are bought.

100 is an even number, the face values \$15 and \$25 are odd numbers.

Then either $(x, y, z, t) = (5, 10, 10, 5)$ or $(x, y, z, t) = (10, 5, 5, 10)$ will make an even sum.

Case 1 $(x, y, z, t) = (5, 10, 10, 5)$

Total cost = 10(5) + 15(10) + 25(10) + 40(5) = 650, which is not a multiple of 100, rejected.

Case 2 $(x, y, z, t) = (10, 5, 5, 10)$

Total coast = 10(10) + 15(5) + 25(5) + 40(10) = 700

$\therefore$ Class A uses 7 \$100 notes.

**I9** The length and the width of a rectangle are integers. If its area is larger than its perimeter by 9, find the perimeter.

Let the width be $x$ and the length be $y$.

$xy - 2(x + y) = 9$

$x(y - 2) - 2y + 4 = 13$

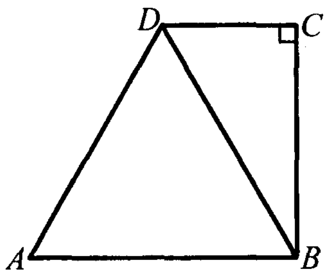
$(x - 2)(y - 2) = 13$

$x - 2 = 1, y - 2 = 13$

$x = 3, y = 15$

The perimeter = 2(3 + 15) = 36

**I10** In the figure, $ABCD$ is a trapezium with $\angle C = 90°$. If the area of the equilateral triangle $ABD$ is $16\sqrt{3}$, find the area of trapezium $ABCD$.




Area of $\Delta BCD = \frac{1}{2}$ area of $\Delta ABD$

Area of trapezium = $16\sqrt{3} + 8\sqrt{3} = 24\sqrt{3}$

**Group Events**

**G1** If $(1000 - a)(1000 - b)(1000 - c)(1000 - d)(1000 - e) = 24^2$, where $a$, $b$, $c$, $d$ and $e$ are even numbers and $a > b > c > d > e$, find the values of $a$, $b$, $c$, $d$ and $e$.

$24^2 = 2^6\times3^2 = -6\times(-2)\times2\times4\times6$

$= (1000 - 1006)(1000 - 1002)(1000 - 998)(1000 - 996)(1000 - 994)$

$a = 1006$, $b = 1002$, $c = 998$, $d = 996$, $e = 994$

**G2** $\overline{ab}$ denotes a two digit number with $a$ as tens digit and $b$ as the unit digit. $R_{\overline{ab}}$ is the remainder when $\overline{ab}$ is divided by $a + b$. Find the maximum value of $R_{\overline{ab}}$.

$\overline{ab} = 10a + b = (a + b)m + R_{\overline{ab}}$, where $0 \le R_{\overline{ab}} < a + b$

To find the maximum value of $R_{\overline{ab}}$.

We start searching $R_{\overline{ab}}$ from $\overline{ab} = 99$ downwards to 10.

$\overline{ab} = 99$, $99 \div (9 + 9) = 5 \cdots\cdots 9 = R_{\overline{ab}}$

$\overline{ab} = 98$, $98 \div (9 + 8) = 5 \cdots\cdots 13 = R_{\overline{ab}}$

$\overline{ab} = 89$, $89 \div (9 + 8) = 5 \cdots\cdots 4 = R_{\overline{ab}}$

$\overline{ab} = 97$, $97 \div (9 + 7) = 6 \cdots\cdots 1 = R_{\overline{ab}}$

$\overline{ab} = 88$, $88 \div (8 + 8) = 5 \cdots\cdots 8 = R_{\overline{ab}}$

$\overline{ab} = 79$, $79 \div (7 + 9) = 4 \cdots\cdots 15 = R_{\overline{ab}}$

The maximum value of the divisor is 16, which gives the maximum value of remainder = 15
For other divisors, which must be less than 16, the remainders must be less than 15.

$\therefore$ The maximum value of $R_{\overline{ab}}$ is 15.

**G3** Given that $a$, $b$ and $c$ are integers, and $a + b = 2011$, $c - a = 2010$, $a < b$. Find the greatest possible value of $a + b + c$.

$a + b + c = 2011 + c$

$c = a + 2010$

Maximum $a = 1005$, $b = 1006$, $c = 1005 + 2010 = 3015$

Maximum $a + b + c = 2011 + 3015 = 5026$

**G4** Given that $n$ is a positive integer and $n^4 - 18n^2 + 49$ is a prime number, find the value of $n$.

$n^4 - 18n^2 + 49 = (n^2 - 7)^2 - 4n^2 = (n^2 + 2n - 7)(n^2 - 2n - 7)$

$\because n^2 + 2n - 7 > n^2 - 2n - 7$

$n^2 - 2n - 7 = 1$ and $n^2 + 2n - 7$ is a prime

$n^2 - 2n - 8 = 0$

$(n - 4)(n + 2) = 0$

$n = 4$ only (verification: $n^2 + 2n - 7 = 17$ which is a prime)

**G5** Given that $\mathrm{f}(x) = \dfrac{4^x}{4^x + 2}$, where $x$ is a real number, find the value of

$$\mathrm{f}\left(\frac{1}{2011}\right)+\mathrm{f}\left(\frac{2}{2011}\right)+\mathrm{f}\left(\frac{3}{2011}\right)+\cdots+\mathrm{f}\left(\frac{2009}{2011}\right)+\mathrm{f}\left(\frac{2010}{2011}\right).$$

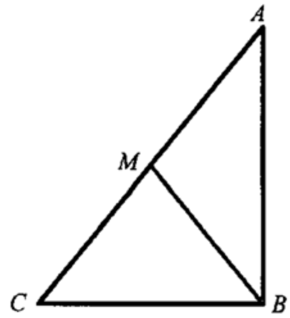
**Reference: 2004 FG4.1, 2012 FI2.2**

$$\mathrm{f}(x)+\mathrm{f}(1-x)=\frac{4^x}{4^x+2}+\frac{4^{1-x}}{4^{1-x}+2}=\frac{4+4+2\cdot4^x+2\cdot4^{1-x}}{4+4+2\cdot4^x+2\cdot4^{1-x}}=1$$

$$\mathrm{f}\left(\frac{1}{2011}\right)+\mathrm{f}\left(\frac{2}{2011}\right)+\mathrm{f}\left(\frac{3}{2011}\right)+\cdots+\mathrm{f}\left(\frac{2009}{2011}\right)+\mathrm{f}\left(\frac{2010}{2011}\right)$$

$$=\mathrm{f}\left(\frac{1}{2011}\right)+\mathrm{f}\left(\frac{2010}{2011}\right)+\mathrm{f}\left(\frac{2}{2011}\right)+\mathrm{f}\left(\frac{2009}{2011}\right)+\cdots+\mathrm{f}\left(\frac{1005}{2011}\right)+\mathrm{f}\left(\frac{1006}{2011}\right)=1005$$

**G6** In the figure below, $M$ is a point on $AC$, $AM = MC = BM = 3$. Find the maximum value of $AB + BC$. **Reference: 2007 HG8**




$M$ is the centre of the circle $ABC$ with $AC$ as diameter.

$\angle ABC = 90°$ ($\angle$ in semi-circle)

Let $\angle ACB = x$, $AB = 6 \sin \theta$, $BC = 6 \cos \theta$

$AB + BC = 6 \sin \theta + 6 \cos \theta$

$= 6\sqrt{2}\sin(\theta + 45^{\circ}) \leq 6\sqrt{2}$

$\therefore$ The maximum value is $6\sqrt{2}$.

**G7** Given that $n! = n\times(n-1)\times(n-2)\times \ldots \times3\times2\times1$ and $\frac{2011!}{10^k}$ is an integer, where $k$ is a positive integer. If $S$ is the sum of all possible values of $k$, find the value of $S$.

**Reference: 1990 HG6, 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2012 FI1.4, 2012 FG1.3**

When each factor of 5 is multiplied by 2, a trailing zero will appear in $n!$.

The number of factors of 2 is clearly more than the number of factors of 5 in 2011!

It is sufficient to find the number of factors of 5.

5, 10, 15, … , 2010; altogether 402 numbers, each have at least one factor of 5.

25, 50, 75, … , 2000; altogether 80 numbers, each have at least two factors of 5.

125, 250, 375, … , 2000; altogether 16 numbers, each have at least three factors of 5.

625, 1250, 1875; altogether 3 numbers, each have at least four factors of 5.

$\therefore$ Total number of factors of 5 is $402 + 80 + 16 + 3 = 501$

There are 501 trailing zeros of 2011! $\Rightarrow$ If $\frac{2011!}{10^k}$ is an integer, then $k = 1, 2, \ldots , 501$.

$S = 1 + 2 + \ldots + 501 = \frac{1+501}{2}\times 501 = 125751$

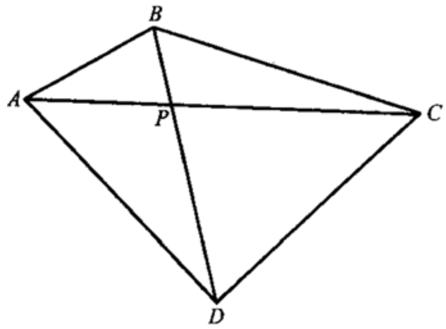
**G8** Given that $a$, $b$, $c$ and $d$ are non-negative integers and $ac + bd + ad + bc = 2011$. Find the value of $a + b + c + d$.

$(a + b)(c + d) = 2011$, which is a prime number

$a + b = 2011$ and $c + d = 1$ or vice versa.

$a + b + c + d = 2012$

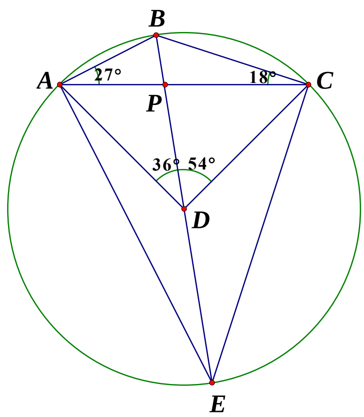
**G9** As shown in the figure, $ABCD$ is a convex quadrilateral, $\angle BAC = 27°$, $\angle BCA = 18°$, $\angle BDC = 54°$, $\angle BDA = 36°$. The diagonals $AC$ and $BD$ intersect at $P$. Find $\angle CPB$.




**Reference: 2003 HG8, 2014 FG2.4**

Draw a circumscribed circle $ABC$.

Produce $BD$ to cut the circle at $E$.




$\angle BEA = 18° = \angle BCA$ ($\angle$s in the same segment)

$\angle BEC = 27° = \angle BAC$ ($\angle$s in the same segment)

$\angle DAE = 36° - 18° = 18°$ (ext. $\angle$ of $\Delta ADE$)

$\angle DCE = 54° - 27° = 27°$ (ext. $\angle$ of $\Delta ADE$)

$\therefore$ $DA = DE$ and $DC = DE$ (sides opp. eq. $\angle$s)

$\therefore$ $D$ is the centre of the circle

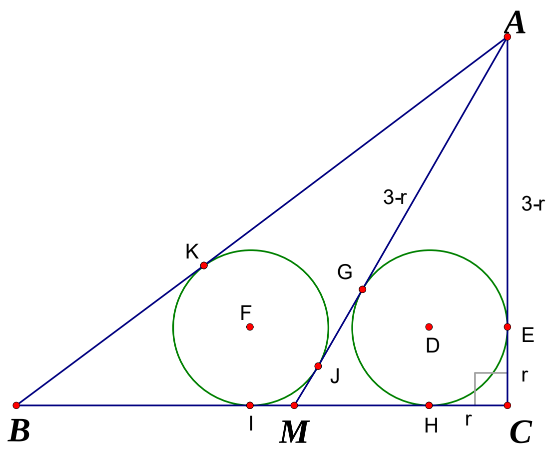
$\therefore$ $DA = DB = DC$ (radii)

$\angle DCB = \angle DBC = (180° - 54°)\div 2 = 63°$ ($\angle$ sum of isos. $\Delta BCD$)

$\angle DCP = 63° - 18° = 45°$

$\angle CPB = 45° + 54° = 99°$ (ext. $\angle$ of $\Delta CDP$)

**G10** As shown in the figure, $AC = 3$, $BC = 4$ and $\angle C = 90°$. $M$ is a point on $BC$ such that the radii of the incircles in $\Delta ABM$ and $\Delta ACM$ are equal. Find the length of $AM$.




Let $D$, $F$ be the centres of the circles, with radii $r$, the points of contact be $E$, $G$, $H$, $I$, $J$, $K$ as shown. Let $AM = x$, $AB = 5$, $CH = CE = r$.

$AE = 3 - r = AG$ (tangent from ext. point)

$S_{\Delta ABF}+S_{\Delta AMF}+S_{\Delta BMF}+S_{\Delta AMD}+S_{\Delta ACD}+S_{\Delta CMD}=S_{\Delta ABC}$

$$\frac{3+4+5}{2}\cdot r+\frac{2x\cdot r}{2}=\frac{3\times 4}{2}$$

$r(6 + x) = 6$

$$r=\frac{6}{6+x} \quad \cdots\cdots (1)$$

In $\Delta ACM$, $MG = x + r - 3 = MH$ (tangent from ext. point)

$$MC = x + 2r - 3 = x+2\times\frac{6}{6+x}-3 \text{ by (1)}$$

$$=\frac{(x-3)(6+x)+12}{6+x}=\frac{x^2+3x-6}{6+x}$$

$AM^2 - AC^2 = MC^2$ (Pythagoras' Theorem)

$$x^2-9=\left(\frac{x^2+3x-6}{6+x}\right)^2$$

$(x^2 - 9)(x + 6)^2 = (x^2 + 3x - 6)^2$

$(x^2 - 9)(x^2 + 12x + 36) = x^4 + 6x^3 - 3x^2 - 36x + 36$

$6x^3 + 30x^2 - 72x - 360 = 0$

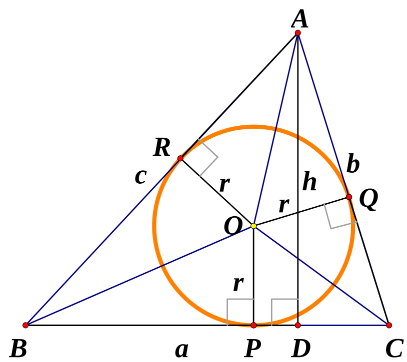
$x^3 + 5x^2 - 12x - 60 = 0$

$x(x^2 - 12) + 5(x^2 - 12) = 0$

$(x + 5)(x^2 - 12) = 0$

$AM = x = 2\sqrt{3}$

**Method 2 Lemma** In the figure, given a triangle $ABC$. $AD\perp BC$, $AD = h$, $O$ is the centre of the inscribed circle with radius $r$. $s=\frac{1}{2}(a + b + c)$




Then $r=\frac{\text{Area of }\Delta ABC}{s}=\frac{ah}{a+b+c}$

Proof: Area $\Delta OBC$ + area $\Delta OCA$ + area $\Delta OAB$ = Area of $\Delta ABC$

$$\frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr=\frac{1}{2}ah$$

$$\therefore\ r=\frac{\text{Area of }\Delta ABC}{s}=\frac{ah}{a+b+c}$$

Let $CM = t$, then $BM = 4 - t$. Now apply the above lemma to $\Delta ACM$ and $\Delta ABM$ respectively.

$$r=\frac{3t}{t+3+\sqrt{9+t^2}} \quad \cdots\cdots (1)$$

$$r=\frac{3(4-t)}{4-t+5+\sqrt{9+t^2}} \quad \cdots\cdots (2)$$

$$(1) = (2) \Rightarrow \frac{3t}{t+3+\sqrt{9+t^2}}=\frac{3(4-t)}{9-t+\sqrt{9+t^2}}$$

$$\frac{t}{4-t}=\frac{t+3+\sqrt{9+t^2}}{9-t+\sqrt{9+t^2}}$$

$$\frac{2t-4}{4}=\frac{2t-6}{12+2\sqrt{9+t^2}} \quad (\because \ \frac{A}{B}=\frac{C}{D}\Rightarrow\frac{A-B}{A+B}=\frac{C-D}{C+D})$$

$$\frac{t-2}{2}=\frac{t-3}{6+\sqrt{9+t^2}}$$

$$(t-2)\left(6+\sqrt{9+t^2}\right)=2t-6$$

$$6t-12+(t-2)\sqrt{9+t^2}=2t-6$$

$$(t-2)\sqrt{9+t^2}=6-4t$$

$(t^2-4t+4)(9+t^2)=36-48t+16t^2$

$t^4-4t^3+4t^2+9t^2-36t+36=36-48t+16t^2$

$t^4-4t^3-3t^2+12t=0$

$t[t^2(t-4)-3(t-4)]=0$

$t(t-4)(t^2-3)=0$

$\therefore$ $t=0$ (rejected), 4 (rejected because $CM<BC=4$) or $\sqrt{3}$.

$$AM=\sqrt{AC^2+CM^2}=\sqrt{3^2+\sqrt{3}^2}=2\sqrt{3}$$

**Geometrical Construction**

**1.** Given a straight line $L$, and two points $P$ and $Q$ lying on the same side of $L$. Mark a point $T$ on $L$ so that the sum of the lengths of $PT$ and $QT$ is minimal. (Hint: Consider the reflection image of $P$ about the line $L$.)

**Reference: 2008 Heat Sample construction Q2**


Figure 1

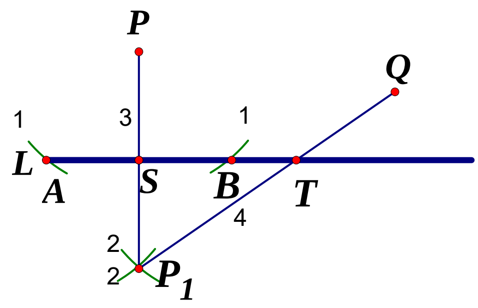



Figure 2

Let $A$ be one of the end point of $L$ nearer to $P$.

(1) Use $P$ as centre, $PA$ as radius to draw an arc, which intersects $L$ at $A$ and $B$.

(2) Use $A$ as centre, $AP$ as radius to draw an arc. Use $B$ as centre, $BP$ as radius to draw an arc. The two arcs intersect at $P_1$.

(3) Join $PP_1$ which intersects $L$ at $S$.

(4) Join $P_1Q$ which intersects $L$ at $T$ 。

| | |
|---|---|
| $AP = AP_1$ | (same radii) |
| $AB = AB$ | (common side) |
| $BP = BP_1$ | (same radii) |
| $\Delta APB \cong \Delta AP_1B$ | (S.S.S.) |
| $\angle PBA = \angle P_1BA$ | (corr. sides $\cong\Delta$'s) |
| $BS = BS$ | (common side) |
| $\Delta PBS \cong \Delta P_1BS$ | (S.A.S.) |
| $\angle BSP = \angle BSP_1$ | (corr. sides $\cong\Delta$'s) |
| $= 90°$ | (adj. $\angle$s on st. line) |
| $SP = SP_1$ | (corr. sides $\cong\Delta$'s) |
| $ST = ST$ | (common side) |
| $\Delta PST \cong \Delta P_1ST$ | (S.A.S.) |
| $PT = P_1T$ | (corr. sides $\cong$ $\Delta$'s) |

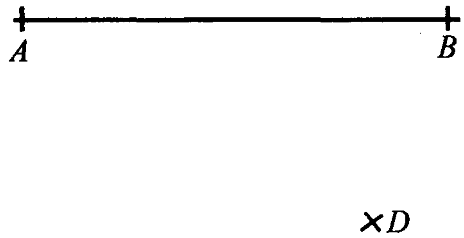
$PT + QT = P_1T + QT$

It is known that $P_1T + QT$ is a minimum when $P_1$, $T$, $Q$ are collinear.

$\therefore$ $T$ is the required point.

**2.** The figure shows a line segment $AB$. $D$ is a fixed point such that $A$, $B$, $D$ are not collinear. Construct a triangle $ABC$ so that $C$, $B$ and $D$ are collinear and $AC - BC = BD$.

如圖所示為一綫段 $AB$。$D$ 為一固定點，且 $A$、$B$、$D$ 不共綫。試作$\Delta ABC$，使得 $C$、$B$ 及 $D$ 共綫，及 $AC - BC = BD$。



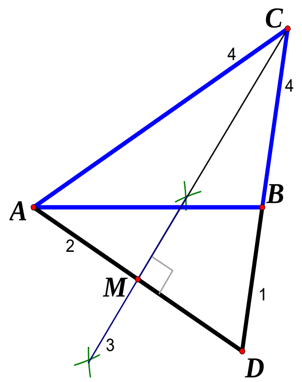

**Remark:** The wording in both versions are ambiguous, so I have changed it.
The original question is:

Figure 2 shows a line segment $AB$ which is a $\Delta ABC$ and $D$ is any point not lying on $AB$. If the difference between the other two sides of $\Delta ABC$ (i.e. $AC - BC$) is equal to $BD$ and $C$, $B$ and $D$ are collinear, construct $\Delta ABC$.

圖 2 所示為$\Delta ABC$ 的其中一條邊 $AB$ 及 $D$ 為一非綫段 $AB$ 上的任意點。若$\Delta ABC$ 的其餘兩條邊(即 $AC - BC$)的長度差距等如 $BD$，且 $C$、$B$ 及 $D$ 共綫，試構作$\Delta ABC$。

(1) Join $DB$ and produce it further.
(2) Join $AD$.
(3) Construct the perpendicular bisector of $AD$, cutting $DB$ produced at $C$. Let $M$ be the mid-point of $AD$.
(4) Join $AC$, $BC$.

$\Delta ABC$ is the required triangle.




Proof: $\Delta ACM \cong \Delta DCM$ (S.A.S.)

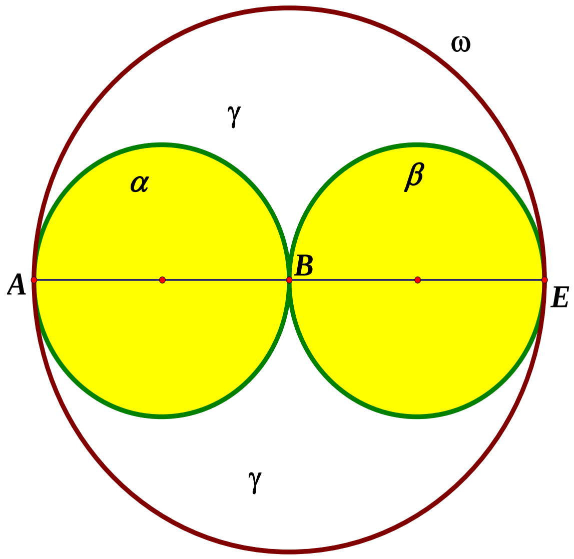
$AC = DC$ (corr. sides, $\cong \Delta$s)

$BC = DC - BD = AC - BD$

$\therefore BD = AC - BC$.

**Discussion: There are some positions on the plane for which $\Delta ABC$ is not constructible.**

Use $B$ as centre $BA$ as radius to draw a circle.
Let $AE$ be the diameter of this circle. $AB = BE$.
Draw 2 circles with $AB$, $BE$ as diameters.
Let the region (including the boundary) bounded by the circles with $AB$, $BE$ as diameters be $\alpha$ and $\beta$ respectively.
Let the region inside the great circle with $AE$ as diameter but not in $\alpha$ and $\beta$ be $\gamma$.
Let the region on or outside the great circle be $\omega$.




**When $D$ lies on $\omega$**, then $BD \geq AB$
So $AC - BC \geq AB$
$AC \geq AB + BC$
This inequality violates the triangle law:
The sum of 2 sides of a triangle is larger than the third side.
i.e. $AC \geq AB + BC > AC$, which is false.

$\therefore$ **When $D$ lies on or outside the circle, $\Delta ABC$ is not constructible.**

**When $D$ lies on α,**

$\Delta AC_1M \cong \Delta DC_1M$ (S.A.S.)

$\angle C_1AM = \angle C_1DM$ (corr. ∠s, ≅ Δs)

$\angle ADC_1 = \angle BAD + \angle ABD$ (ext. ∠ of $\Delta ABD$)

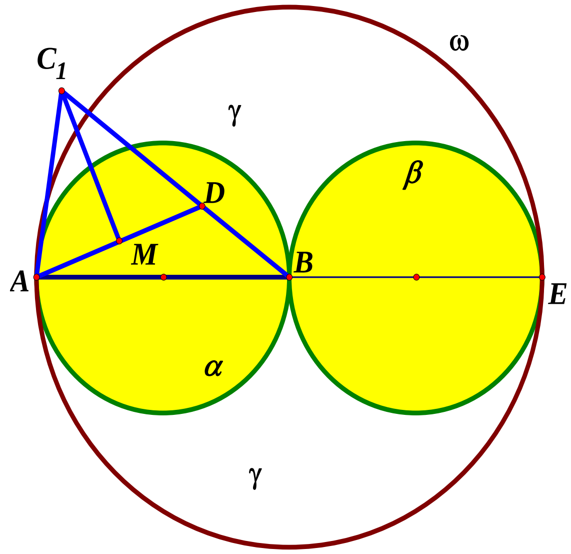
$> \angle ABD$

$\therefore\ \angle C_1AM = \angle ADC_1 > \angle ABD$

$\angle C_1AB > \angle ABD$

$AC_1 < BC_1$ (greater sides opp. greater ∠s)

The contradicts to the fact that $BD = AC_1 - BC_1$ because $AC_1 = BD + BC_1 > BC_1$




In this case the steps are as follows:

(1) Join $DB$ and produce it further.

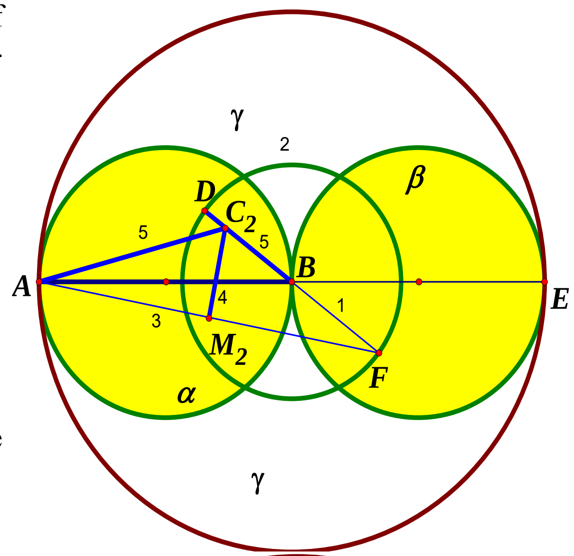
(2) Use $B$ as centre, $BD$ as radius to draw a circle, cutting $DB$ produced at $F$.

(3) Join $AF$.

(4) Construct the perpendicular bisector of $AF$, cutting $DB$ at $C_2$. Let $M_2$ be the mid-point of $AF$.

(5) Join $AC_2$, $BC_2$.

$\Delta ABC_2$ is the required triangle.




$\angle ABC_2 = \angle BAF + \angle AFB$ (ext. ∠ of $\Delta ABF$)

$> \angle AFB$

$= \angle FAC_2$ (corr. ∠s, ≅ Δs)

$> \angle BAC_2$

$\therefore\ AC_2 > BC_2$ (greater sides opp. greater ∠s)

The triangle $ABC$ is constructible.

**When $D$ lies on β**, relabel $D$ as $F$ and $F$ as $D$, the construction steps are shown above.

**There is only one possible triangle**.

**When $D$ lies on the boundary of α,**

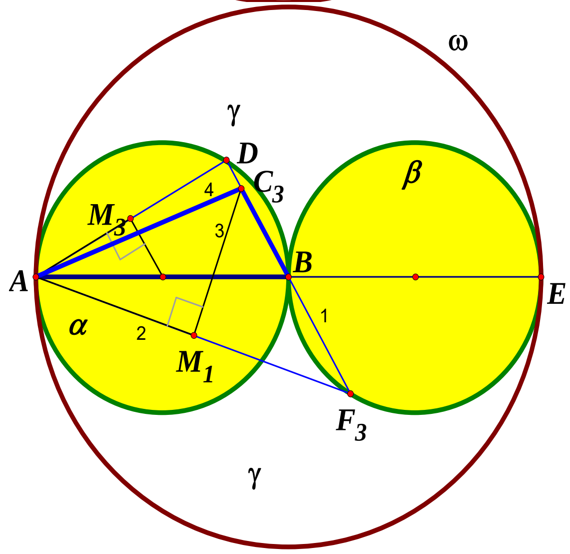
$\angle ADB = 90°$ (∠ in semi-circle)

The perpendicular bisector of $AD$ is parallel to $DB$. ∴ It will not intersect $DB$ produced.

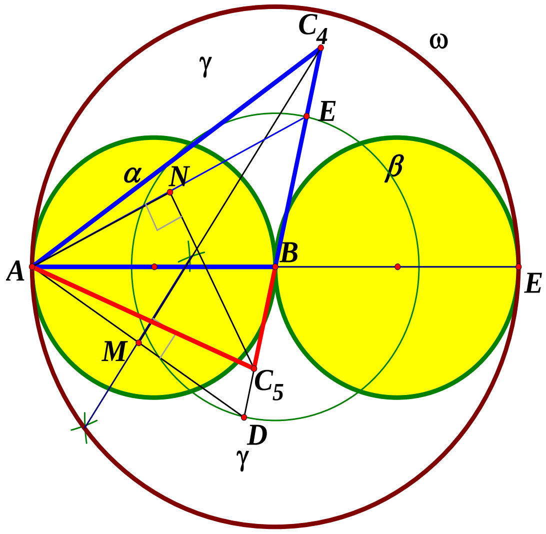
Suppose $DB$ produced intersects the boundary of β at $F_3$. Then the perpendicular bisector of $AF_3$ intersects $DB$ at $C_3$.

$\Delta ABC_3$ is the required triangle.

Proof: omitted.

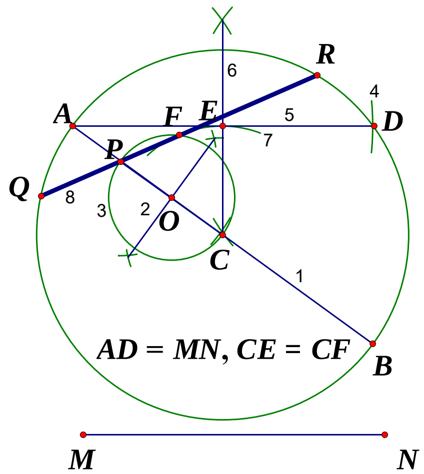


**When $D$ lies on $\gamma$,**
Use $B$ as centre, $BD$ as radius to draw a circle, cutting $DB$ produced at $E$ on $\gamma$.
The perpendicular bisector of $AD$ cuts $DB$ produced at $C_4$. The perpendicular bisector of $AE$ cut $BD$ at $C_5$.
$\therefore$ **We can construct 2 possible triangles** $\Delta ABC_4$ and $\Delta ABC_5$ satisfying the given conditions.




**3.** Figure 3 shows a circle of centre $C$ and a line segment $MN$. $P$ is a point lies inside the circle. Construct a chord $QR$ with points $Q$ and $R$ on the circumference of the circle so that it passes through $P$ and its length is equal to that of $MN$.

(1) Join $CP$ and produce it to 2 ends $A$, $B$ of the diameter of the circle as shown.
(2) Draw the perpendicular bisector of $CP$, $O$ is the mid-point of $CP$.
(3) Use $O$ as centre, $OC$ as radius to draw a circle.
(4) Use $A$ as centre, $MN$ as radius to draw an arc, cutting the given circle at $D$.
(5) Join $AD$.
(6) Draw the perpendicular bisector of $AD$, $E$ is the mid-point of $AD$.
(7) Use $C$ as centre, $CE$ as radius to draw an arc, cutting the circle in step (3) at $F$.
(8) Join $PF$ and produce it to cut the circle at $Q$ and $R$. Then $QR$ is the required chord.




Proof: $\angle PFC = 90°$ ($\angle$ in semi-circle)
$CE = CF$ (by construction)
$QR = AD = MN$ (chords eq. distance from centre are equal)

**Method 2**: (Provided by Tsuen Wan Government Secondary School Tam Lok Him)

Let the radius of the given circle be $R$, the distance between $CP$ be $r$.

(1) Use $M$ as centre and $R$ as radius to draw an arc, use $N$ as centre and $R$ as radius to draw an arc. The two arcs intersect at $A$.

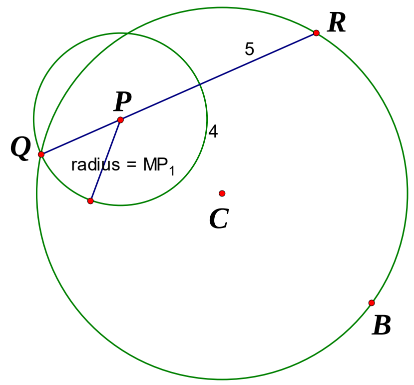
(2) Use $A$ as centre, $R$ as radius to draw a circle $C_1$. The circle $C_1$ must pass through $M$, $N$.

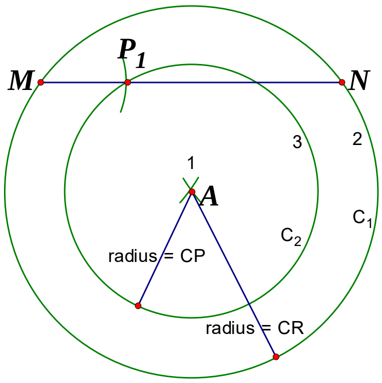
(3) Use $A$ as centre, $r$ as radius to draw a circle $C_2$, which cuts $MN$ at $P_1$.

(4) On the given circle, use $P$ as centre, $MP_1$ as radius to draw a circle, which cuts the given circle at $Q$.

(5) Join $QP$ and produce it to cut the given circle at $R$.

Then $QPR$ is the required chord.




Proof: $QP = MP_1$ (by construction)

$CP = AP_1$ (by construction)

$CQ = AM$ (by construction)

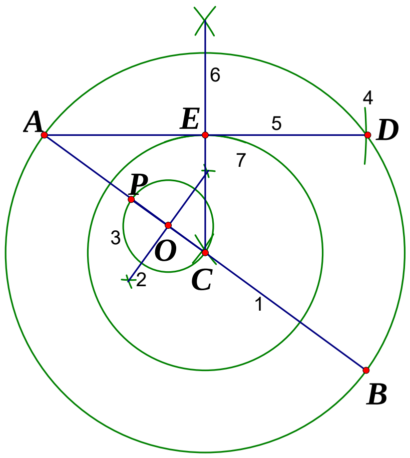
$\Delta CPQ \cong \Delta AP_1M$ (S.S.S.)

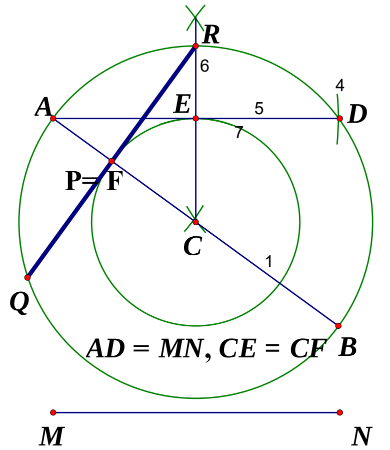
height of $\Delta CPQ$ = height of $\Delta AP_1M$

$QR = MN$ (chords eq. distance from centre are equal)

**Remark** There are some positions of $P$ inside the circle for which the chord is not constructible. The circle in step 7 may not cut the circle in step 3. At the limiting position, $CQ^2 = PQ^2 + CP^2$

$$4r^2 = QR^2 + 4CP^2 \Rightarrow QR = 2\sqrt{r^2 - CP^2}$$




$\therefore$ For fixed position $P$, the length of chord must satisfy $2\sqrt{r^2 - CP^2} \le MN \le 2r$.

Percentage of correct questions

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