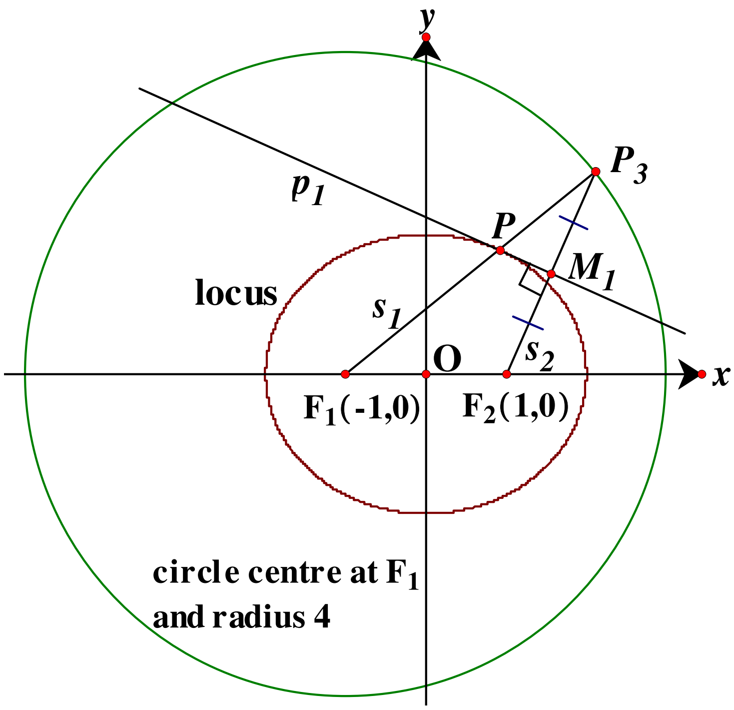
Locus Ellipse

Created by Mr. Francis Hung on 20060423 Last updated: 2021-08-29

In the figure, $F_1(-1, 0)$ and $F_2(1, 0)$ are two fixed points. A circle with F_1 as centre and radius = 4 is drawn. P_3 is a variable point on the circle. A line segment s_2 joining the points F_2 and P_3. p_1 is the perpendicular bisector of F_2P_3 through the mid point M_1. If p_1 intersects the radius F_1P_3 at P, find the locus of P.



Method 1

The equation of circle c_1: $(x + 1)^2 + y^2 = 4^2$

Using parametric form, $P_3 = (-1 + 4\cos\theta, 4\sin\theta)$

$M_1 = (2\cos\theta, 2\sin\theta)$

The perpendicular bisector p_1 is: $\dfrac{y-2\sin\theta}{x-2\cos\theta}\cdot\dfrac{4\sin\theta}{-2+4\cos\theta}=-1$

$(y - 2\sin\theta)(2\sin\theta) + (x - 2\cos\theta)(-1 + 2\cos\theta) = 0$

$(-1 + 2\cos\theta)x + 2\sin\theta\, y - 4\sin^2\theta + 2\cos\theta - 4\cos^2\theta = 0$

$-x + 2\cos\theta\, x + 2\sin\theta\, y - 4 + 2\cos\theta = 0$

$2\cos\theta\,(x + 1) + 2\sin\theta\, y = x + 4 \cdots\cdots\cdots (1)$

The radius s_1 is $\dfrac{y-0}{x+1}=\dfrac{4\sin\theta-0}{-1+4\cos\theta+1}$

$\dfrac{y}{x+1}=\dfrac{\sin\theta}{\cos\theta}$

$2\sin\theta\,(x + 1) - 2\cos\theta\, y = 0 \cdots\cdots\cdots(2)$

We try to eliminate θ from equations (1) and (2)

$(1)^2 + (2)^2$: $4(x + 1)^2 + 4y^2 = (x + 4)^2$

$4(x^2 + 2x + 1) + 4y^2 = x^2 + 8x + 16$

$3x^2 + 4y^2 = 12$

$\dfrac{x^2}{4}+\dfrac{y^2}{3}=1$

Method 2

By perpendicular bisector theorem, $\Delta PM_1P_3 \cong \Delta PM_1F_2$ (S.A.S.)

$\therefore PP_3 = PF_2$ (corr. sides $\cong \Delta$s)

F_1P_3 = radius = 4 $\Rightarrow F_1P + PP_3 = 4 \Rightarrow F_1P + PF_2 = 4$

Let $P = (x, y) \Rightarrow \sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2} = 4$

$$\sqrt{(x+1)^2 + y^2} = 4 - \sqrt{(x-1)^2 + y^2}$$

$$\left[\sqrt{(x+1)^2 + y^2}\right]^2 = \left[4 - \sqrt{(x-1)^2 + y^2}\right]^2$$

$$(x + 1)^2 + y^2 = 16 - 8\sqrt{(x-1)^2 + y^2} + (x - 1)^2 + y^2$$

$$x^2 + 2x + 1 + y^2 = 16 - 8\sqrt{(x-1)^2 + y^2} + x^2 - 2x + 1 + y^2$$

$$8\sqrt{(x-1)^2 + y^2} = 16 - 4x$$

$$2\sqrt{(x-1)^2 + y^2} = 4 - x$$

$$4(x^2 - 2x + 1 + y^2) = 16 - 8x + x^2$$

$$3x^2 + 4y^2 = 12$$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$