Individual Events

I1	***a***	$\frac{2}{3}$	**I2**	***a***	12	**I3**	***P***	8	**I4**	***n***	9	**I5**	***a***	6
	b	0		***b***	36		***Q***	12		***b***	3		***b***	30
	c	3		***c***	12		***R***	4		***c***	8		***c***	4
	d	–6		***d***	5		***S***	70		***d***	62		***d***	4

Group Events

G1	***a***	180	**G2**	***a***	1	**G3**	***m***	–3	**G4**	***a***	99999919	**G5**	***a***	10	**Group Spare**	***a***	4
	b	7		***b***	2		***b***	1		***b***	1		***b***	9		***k***	2
	c	9		***c***	1		***c***	1.6		***c***	2		***c***	55		***d***	8.944
	d	4		***d***	120		***d***	2		***d***	1891		***d***	16		***r***	$\frac{25}{24}$

Individual Event 1 (1998 Sample Individual Event)

I1.1 Given that $\frac{3}{a}+\frac{1}{u}=\frac{7}{2}$ and $\frac{2}{a}-\frac{3}{u}=6$ are simultaneous equations in a and u. Solve for a.

3(1) + (2): $\frac{11}{a}=\frac{33}{2}$

$a=\frac{2}{3}$

I1.2 Three solutions of the equation $px + qy + bz = 1$ are $(0, 3a, 1)$, $(9a, –1, 2)$ and $(0, 3a, 0)$. Find the value of the coefficient b .

$$\begin{cases} 3aq+b=1\cdots\cdots(1) \\ 9ap-q+2b=1\cdots\cdots(2) \\ 3aq=1\cdots\cdots(3) \end{cases}$$

Sub. (3) into (1): $1 + b = 1$

$\Rightarrow b = 0$

I1.3 Find the value of c so that the graph of $y = mx + c$ passes through the two points $(b + 4, 5)$ and $(–2, 2)$.

The 2 points are: (4, 5) and (–2, 2). The slope is $\frac{5-2}{4-(-2)}=\frac{1}{2}$.

The line $y=\frac{1}{2}x+c$ passes through (–2, 2): $2 = –1 + c$

$\Rightarrow c = 3$

I1.4 The solution of the inequality $x^2 + 5x – 2c \le 0$ is $d \le x \le 1$. Find the value of d .

$x^2 + 5x – 6 \le 0$

$\Rightarrow (x + 6)(x – 1) \le 0$

$–6 \le x \le 1$

$d = –6$

Individual Event 2

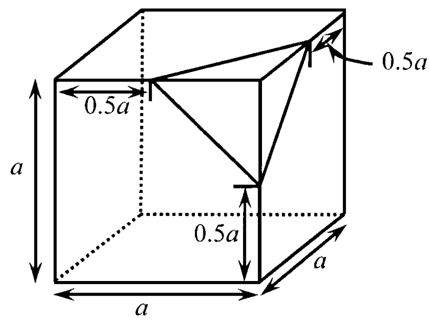
I2.1 By considering: $\frac{1^2}{1}=1$, $\frac{1^2+2^2}{1+2}=\frac{5}{3}$, $\frac{1^2+2^2+3^2}{1+2+3}=\frac{7}{3}$, $\frac{1^2+2^2+3^2+4^2}{1+2+3+4}=3$, find the value of a such that $\frac{1^2+2^2+\cdots+a^2}{1+2+\cdots+a}=\frac{25}{3}$.

The given is equivalent to: $\frac{1^2}{1}=\frac{3}{3}$, $\frac{1^2+2^2}{1+2}=\frac{5}{3}$, $\frac{1^2+2^2+3^2}{1+2+3}=\frac{7}{3}$, $\frac{1^2+2^2+3^2+4^2}{1+2+3+4}=\frac{9}{3}$

and $2\times1 + 1 = 3$, $2\times2 + 1 = 5$, $2\times3 + 1 = 7$, $2\times4 + 1 = 9$; so $2a + 1 = 25$

$\Rightarrow a = 12$

I2.2 A triangular pyramid is cut from a corner of a cube with side length a cm as the figure shown. If the volume of the pyramid is b cm^3, find the value of b.



$b=\frac{1}{3}\text{base area}\times\text{height}=\frac{1}{3}\left(\frac{\frac{1}{2}a\times\frac{1}{2}a}{2}\right)\times\frac{1}{2}a$

$=\frac{1}{48}a^3=\frac{1}{48}\cdot12^3=36$

I2.3 If the value of $x^2 + cx + b$ is not less than 0 for all real number x, find the maximum value of c

$x^2 + cx + 36 \geq 0$

$\Delta = c^2 - 4(36) \leq 0$

$\Rightarrow c \leq 12$

The maximum value of $c = 12$.

I2.4 If the units digit of 1997^{1997} is $c - d$, find the value of d.

$1997^{1997} \equiv 7^{1997} \equiv 7^{4(499)+1} \equiv 7 \pmod{10}$

The units digit of 1997^{1997} is 7

$12 - d = 7$

$d = 5$

Individual Event 3

I3.1 The average of a, b c and d is 8. If the average of a, b, c, d and P is P, find the value of P .

$\frac{a+b+c+d}{4}=8$

$\Rightarrow a + b + c + d = 32$

$\frac{a+b+c+d+P}{5}=P$

$\Rightarrow 32 + P = 5P$

$P = 8$

I3.2 If the lines $2x + 3y + 2 = 0$ and $Px + Qy + 3 = 0$ are parallel, find the value of Q .

Their slopes are equal: $-\frac{2}{3}=-\frac{8}{Q}$

$Q = 12$

I3.3 The perimeter and the area of an equilateral triangle are Q cm and $\sqrt{3}R\,\text{cm}^2$ respectively. Find the value of R .

Perimeter = 12 cm, side = 4 cm

Area $=\frac{1}{2}\cdot 4^2 \sin 60^\circ = 4\sqrt{3}$

$R = 4$

I3.4 If $(1 + 2 + \ldots + R)^2 = 1^2 + 2^2 + \ldots + R^2 + S$, find the value of S .

$(1 + 2 + 3 + 4)^2 = 1^2 + 2^2 + 3^2 + 4^2 + S$

$100 = 30 + S$

$S = 70$

Individual Event 4

I4.1 If each interior angle of a n-sided regular polygon is 140°, find the value of n .

Reference: 1987 FG6.3

Each exterior angle is 40° (adj. ∠s on st. line)

$$\frac{360^\circ}{n} = 40^\circ$$

$n = 9$

I4.2 If the solution of the inequality $2x^2 - nx + 9 < 0$ is $k < x < b$, find the value of b .

$2x^2 - 9x + 9 < 0$

$(2x - 3)(x - 3) < 0$

$$\frac{3}{2} < x < 3$$

$\Rightarrow b = 3$

I4.3 If $cx^3 - bx + x - 1$ is divided by $x + 1$, the remainder is -7, find the value of c .

$f(x) = cx^3 - 3x + x - 1$

$f(-1) = -c + 3 - 1 - 1 = -7$

$c = 8$

I4.4 If $x + \frac{1}{x} = c$ and $x^2 + \frac{1}{x^2} = d$, find d .

$$x + \frac{1}{x} = 8$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 64$$

$$x^2 + \frac{1}{x^2} + 2 = 64$$

$d = 62$

Individual Event 5

I5.1 The volume of a hemisphere with diameter a cm is 18π cm^3, find the value of a .

$$\frac{1}{2}\cdot 4\pi\left(\frac{a}{2}\right)^2=18\pi$$

$a = 6$

I5.2 If $\sin 10a° = \cos(360° - b°)$ and $0 < b < 90$, find the value of b .

$\sin 60° = \cos(360° - b°)$

$360° - b° = 330°$

$b = 30$

I5.3 The triangle is formed by the x-axis and y-axis and the line $bx + 2by = 120$.

If the bounded area of the triangle is c, find the value of c .

$30x + 60y = 120$

$\Rightarrow x + 2y = 4$

x-intercept = 4, y-intercept = 2

$c=\frac{1}{2}\cdot 4\cdot 2=4$

I5.4 If the difference of the two roots of the equation $x^2 - (c + 2)x + (c + 1) = 0$ is d , find the value of d .

$x^2 - 6x + 5 = 0$

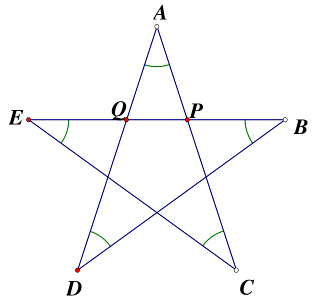
$\Rightarrow (x - 1)(x - 5) = 0$

$x = 1$ or 5

$\Rightarrow d = 5 - 1 = 4$

Group Event 1

G1.1 In the given diagram, $\angle A + \angle B + \angle C + \angle D + \angle E = a°$, find the value of a .



Reference: 1989 FI5.1, 2005 FI2.3

In ΔAPQ, $\angle B + \angle D = \angle AQP$ ……(1) (ext. $\angle$ of Δ)

$\angle C + \angle E = \angle APQ$ ……(2) (ext. $\angle$ of Δ)

$\angle A+\angle B + \angle C + \angle D + \angle E = \angle A + \angle AQP + \angle APQ$ (by (1) and (2))

$= 180°$ ($\angle$s sum of Δ)

$\therefore a = 180$

G1.2 There are x terms in the algebraic expression $x^6 + x^6 + x^6 + \ldots + x^6$ and its sum is x^b. Find the value of b .

$x \cdot x^6 = x^b$

$x^7 = x^b$

$b = 7$

G1.3 If $1 + 3 + 3^2 + 3^3 + \ldots + 3^8 = \frac{3^c - 1}{2}$, find the value of c .

$\frac{3^9 - 1}{2} = \frac{3^c - 1}{2}$

$c = 9$

G1.4 16 cards are marked from 1 to 16 and one is drawn at random.

If the chance of it being a perfect square number is $\frac{1}{d}$, find the value of d .

Reference: 1995 HI4

Perfect square numbers are 1, 4, 9, 16.

Probability $= \frac{4}{16} = \frac{1}{d}$

$d = 4$.

Group Event 2

G2.1 If the sequence 1, 6 + 2a, 10 + 5a, … forms an A.P., find the value of a .

$$6+2a=\frac{1+10+5a}{2}$$

$12 + 4a = 11 + 5a$

$\Rightarrow a = 1$

G2.2 If $(0.0025\times40)^b=\frac{1}{100}$, find the value of b .

$$\left(\frac{1}{400}\times40\right)^b=\frac{1}{100}$$

$$\Rightarrow\frac{1}{10^b}=\frac{1}{10^2}$$

$b = 2$

G2.3 If c is an integer and $c^3+3c+\frac{3}{c}+\frac{1}{c^3}=8$, find the value of c .

$$\left(c+\frac{1}{c}\right)^3=2^3$$

$$\Rightarrow\left(c+\frac{1}{c}-2\right)\left[\left(c+\frac{1}{c}\right)^2+2\left(c+\frac{1}{c}\right)+4\right]=0$$

$c^2 - 2c + 1 = 0$ or $\left(c+\frac{1}{c}\right)^2+2\left(c+\frac{1}{c}\right)+4=0$

$\Rightarrow c = 1$ or no real solution ($\because \Delta = 2^2 - 4(2)(4) < 0$)

$\therefore c = 1$

G2.4 There are d different ways for arranging 5 girls in a row. Find the value of d .

First position has 5 choices; 2nd position has 4 choices, …, the last position has 1 choice.

$d = 5\times4\times3\times2\times1 = 120$

Group Event 3

G3.1 Let m be an integer satisfying the inequality $14x - 7(3x - 8) < 4(25 + x)$.
Find the least value of m.
$14x - 21x + 56 < 100 + 4x$
$-44 < 11x$
$\Rightarrow -4 < x$
$m = -3$

G3.2 It is given that $f(x) = \frac{1}{3}x^3 - 2x^2 + \frac{2}{3}x^3 + 3x^2 + 5x + 7 - 4x$. If $f(-2) = b$, find the value of b.
$f(x) = x^3 + x^2 + x + 7$
$b = f(-2) = -8 + 4 - 2 + 7 = 1$

G3.3 It is given that $\log\frac{x}{2} = 0.5$ and $\log\frac{y}{5} = 0.1$. If $\log xy = c$, find the value of c.
$\log\frac{x}{2} + \log\frac{y}{5} = 0.5 + 0.1$
$\log xy - 1 = 0.6$
$\Rightarrow c = \log xy = 1.6$

G3.4 Three prime numbers d, e and f which are all less than 10, satisfy the two conditions $d + e = f$ and $d < e$. Find the value of d.
Possible prime numbers are 2, 3, 5, 7.
$2 + 3 = 5$ or $2 + 5 = 7$
$\therefore d = 2$

Group Event 4

G4.1 It is given that $a = 103\times97\times10009$, find the value of a.

$a = (100 + 3)(100 - 3)\times10009$

$= (10000 - 9)\times(10000 + 9)$

$= 100000000 - 81$

$a = 99999919$

G4.2 It is given that $1 + x + x^2 + x^3 + x^4 = 0$. If $b = 2 + x + x^2 + x^3 + x^4 + \cdots + x^{1989}$, find the value of b.

Reference: 2014 HI7

$b = 1 + (1 + x + x^2 + x^3 + x^4) + x^5(1 + x + x^2 + x^3 + x^4) + \cdots + x^{1985}(1 + x + x^2 + x^3 + x^4) = 1$

G4.3 It is given that m and n are two natural numbers and both are not greater than 10.

If c is the number of pairs of m and n satisfying the equation $mx = n$, where $\frac{1}{4} < x < \frac{1}{3}$,

find the value of c.

$\frac{1}{4} < \frac{m}{n} < \frac{1}{3} \Rightarrow \frac{n}{4} < m < \frac{n}{3}$

$\begin{cases} 4m - n > 0 \\ 3m - n < 0 \end{cases}$

$3m < n < 4m$

$1 \le m \Rightarrow 3 \le 3m < n < 4m \le 4\times10 = 40$

Possible $n = 4, 5, 6, \dots, 10$

when $n = 4$, $\frac{4}{4} < m < \frac{4}{3}$ no solution

when $n = 5$, $\frac{5}{4} < m < \frac{5}{3}$ no solution

when $n = 6$, $\frac{6}{4} < m < \frac{6}{3}$ no solution

when $n = 7$, $\frac{7}{4} < m < \frac{7}{3} \Rightarrow m = 2, x = \frac{2}{7}$

when $n = 8$, $\frac{8}{4} < m < \frac{8}{3}$ no solution

when $n = 9$, $\frac{9}{4} < m < \frac{9}{3}$ no solution

when $n = 10$, $\frac{10}{4} < m < \frac{10}{3} \Rightarrow m = 3, x = \frac{3}{10}$

$c = 2$ (There are 2 solutions.)

G4.4 Let x and y be real numbers and define the operation * as $x*y = px^y + q + 1$.

It is given that $1*2 = 869$ and $2*3 = 883$. If $2*9 = d$, find the value of d.

$\begin{cases} p + q + 1 = 869 \\ 8p + q + 1 = 883 \end{cases}$

$(2) - (1)$: $7p = 14$

$p = 2, q = 866$

$\Rightarrow d = 2\times2^9 + 866 + 1 = 1891$

Group Event 5

G5.1 If a is a positive multiple of 5, which gives remainder 1 when divided by 3, find the smallest possible value of a . **(Reference: 1998 FSG.1)**

$a = 5k = 3m + 1$

$5\times2 = 3\times3 + 1$

The smallest possible $a = 10$.

G5.2 If $x^3 + 6x^2 + 12x + 17 = (x + 2)^3 + b$, find the value of b .

Reference: 1998 FG1.4

$(x + 2)^3 + b = x^3 + 6x^2 + 12x + 8 + b$

$b = 9$

G5.3 If c is a 2 digit positive integer such that sum of its digits is 10 and product of its digit is 25, find the value of c . **(Reference: 1998 FSG.3)**

$c = 10x + y$, where $0 < x < 10$, $0 \le y < 10$.

$x + y = 10$

$xy = 25$

Solving these two equations gives $x = y = 5$; $c = 55$

G5.4 Let $S_1, S_2, \ldots, S_{10}$ be the first ten terms of an A.P., which consists of positive integers.

If $S_1 + S_2 + \ldots + S_{10} = 55$ and $(S_{10} - S_8) + (S_9 - S_7) + \cdots\cdots + (S_3 - S_1) = d$, find the value of d .

Reference: 1998 FSG.4

Let the general term be $S_n = a + (n - 1)t$

$$\frac{10}{2}[2a + (10-1)t] = 55$$

$\Rightarrow 2a + 9t = 11$

$\because$ a, t are positive integers, $a = 1$, $t = 1$

$d = (S_{10} - S_8) + (S_9 - S_7) + \cdots\cdots + (S_3 - S_1)$

$= [a + 9t - (a + 7t)] + [a + 8t - (a + 6t)] + \cdots\cdots + (a + 2t - a)$

$d = 2t + 2t + 2t + 2t + 2t + 2t + 2t + 2t = 16t = 16$

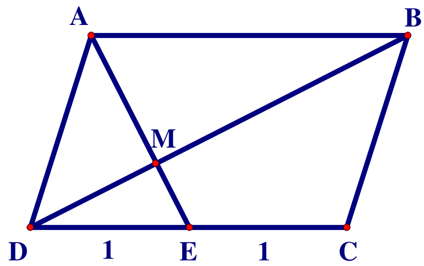
Group Spare

GS.1 *ABCD* is a parallelogram and *E* is the midpoint of *CD*. If the ratio of the area of the triangle *ADE* to the area of the parallelogram *ABCD* is $1 : a$, find the value of a .

$1 : a = 1 : 4$

$a = 4$

GS.2 *ABCD* is a parallelogram and *E* is the midpoint of *CD*. *AE* and *BD* meet at *M*. If $DM : MB = 1 : k$, find the value of k .



It is easy to show that $\Delta ABM \sim \Delta EDM$ (equiangular)

$DM : MB = DE : AB = 1 : 2$

$k = 2$

GS.3 If the square root of 5 is approximately 2.236, the square root of 80 with the same precision is d. Find the value of d .

$\sqrt{80} = \sqrt{16\times 5} = 4\sqrt{5} = 4\times 2.236 = 8.944$

GS.4 A square is changed into a rectangle by increasing its length by 20% and decreasing its width by 20% .

If the ratio of the area of the rectangle to the area of the square is $1 : r$, find the value of r .

Let the side of the square be x .

Ratio of areas $= 1.2x\cdot 0.8x : x^2$

$$= 0.96 : 1 = 1 : \frac{25}{24}$$

$$r = \frac{25}{24}$$