Individual Events

I1	α	10	**I2**	α	7	**I3**	α	*686 see the remark	**I4**	α	3
	β	90		β	5		β	236328		β	2
	γ	10		γ	2		γ	*15 see the remark		γ	7
	δ	2047		δ	–56		δ	$\frac{15}{4}$		δ	1

Group Events

G1		$\frac{3}{5}$	**G2**		417	**G3**		$\sqrt{10}$	**G4**		1
		15			23			0			625
		34			–3			$\frac{1+\sqrt{5}}{2}$			1
		15			$12\sqrt{3}$			10			$\frac{3-\sqrt{5}}{2}$

Individual Event 1

I1.1 If $\left|x+\sqrt{5}\right|+\left|y-\sqrt{5}\right|+|z|=0$, determine $\alpha = x^2 + y^2 + z^2$.

Reference: 2005 FI4.1, 2006 FI4.2, 2009 FG1.4, 2013 FI1.4, 2015 HG4, 2015 FI1.1

Sum of non-negative terms = 0 ⇒ each term = 0 at the same time

$x+\sqrt{5}=0$ and $y-\sqrt{5}=0$ and $z = 0$

$x=-\sqrt{5}$ and $y=\sqrt{5}$ and $z = 0$

$\alpha = x^2 + y^2 + z^2 = 5 + 5 + 0 = 10$

I1.2 If β is the sum of all digits of the product $\underbrace{11111\cdots11}_{\alpha\text{ 1's}}\times\underbrace{99999\cdots99}_{\alpha\text{ 9's}}$, determine the value of β.

Reference: 2000 FI4.4

Observe the patterns 11 × 99 = 1089; 111×999 = 110889.

Deductively, $\underbrace{11111\cdots11}_{10\text{ 1's}}\times\underbrace{99999\cdots99}_{10\text{ 9's}} = \underbrace{11111\cdots11}_{9\text{ 1's}}0\underbrace{88888\cdots88}_{9\text{ 9's}}9$

β = the sum of all digits = 9 + 9×8 + 9 = 90

I1.3 Suppose that the real function f(x) satisfies f(xy) = f(x) f(y) for all real numbers x and y, and f(1) < 1. Determine the value of γ = f(β) + 100 – β .

Reference: 2013 FI4.1

f(1) = f(1) f(1)

⇒ f(1)[f(1) – 1] = 0

⇒ f(1) = 0 or 1 (rejected)

∵ f(1) = 0

f(x) = f (1×x) = f(1)f(x) = 0 for all real values of x.

γ = f(β) + 100 – β = 0 + 100 – 90 = 10

I1.4 If n is a positive integer and $f(n) = 2^n + 2^{n-1} + 2^{n-2} + \ldots + 2^2 + 2^1 + 1$, determine the value of δ = f(γ).

Reference: 2009 FI1.3, 2017 FI3.4

$f(n) = 2^{n+1} - 1$ (sum to n terms of a G.S. $a = 1$, $r = 2$, no. of terms = $n + 1$)

$\delta = f(10) = 2^{11} - 1 = 2047$

Individual Event 2

I2.1 If x_0, y_0, z_0 is a solution to the simultaneous equations below, $\begin{cases} x-y-z=-1 \\ y-x-z=-2 \\ z-x-y=-4 \end{cases}$

determine the value of $\alpha = x_0 + y_0 + z_0$.

(1) + (2) + (3): $-(x + y + z) = -7$

$\alpha = 7$

I2.2 If β is the reminder of $\underbrace{111\cdots111}_{100\ 1's} \div \alpha$, determine the value of β.

$111111 \div 7 = 15873$

$\underbrace{111\cdots111}_{100\ 1's} = \underbrace{111\cdots111}_{96\ 1's}0000 + 1111$

$= 7m + 7\times158 + 5$, where m is an integer

$\beta = 5$

I2.3 If γ is the remainder of $[(\beta - 2)^{100} + \beta^{50} + (\beta + 2)^{25}] \div 3$, determine the value of γ.

$3^{100} + 5^{50} + 7^{25} = 3^{100} + (6 - 1)^{50} + (6 + 1)^{25}$

$= 3^{100} + 6^n + 1 + 6^m + 1$, where m and n are integers

$\gamma = 2$

I2.4 If the equation $x^4 + ax^2 + bx + \delta = 0$ has four real roots with three of them being 1, γ and γ^2, determine the value of δ.

Reference: 2013 FI4.3

Let the fourth root be t.

$1 + 2 + 2^2 + t = \text{sum of roots} = -\dfrac{\text{coefficient of } x^3}{\text{coefficient of } x^4} = 0$

$t = -7$

$1\times2\times2^2\times(-7) = \text{product of roots} = \dfrac{\text{constant term}}{\text{coefficient of } x^4} = \delta$

$\delta = -56$

Individual Event 3

I3.1 Of the positive integers from 1 to 1000, including 1 and 1000, there are α of them that are not divisible by 5 or 7. Determine the value of α.

Reference: 1993 FG8.3-4, 1994 FG8.1-2, 1998 HI6

Numbers divisible by 5: 5, 10, 15, $\cdots$, 1000, there are 200 numbers

Numbers divisible by 7: 7, 14, 21, $\cdots$, 994, there are 142 numbers

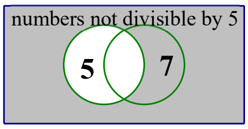
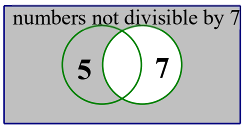
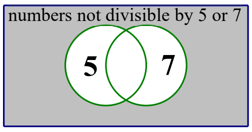
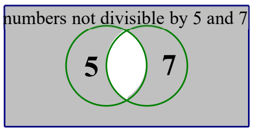
Numbers divisible by 35: 35, 70, $\cdots$, 980, there are 28 numbers

Numbers divisible by 5 or 7 = 200 + 142 – 28 = 314

Numbers that are not divisible by 5 or 7 = 1000 – 314 = 686

Remark: The original question is:

Of the positive integers from 1 to 1000, including 1 and 1000, there are α of them that are not divisible by 5 or not divisible by 7. Determine the value of α.

		 'numbers neither divisible by 5 nor 7' 'numbers not divisible by 5 and numbers not divisible by 7'	 'numbers not divisible by 5 or not divisible by 7'

I3.2 Determine the value of $\beta = 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{\alpha} (\alpha + 1)^2$.

Reference: 1997 HI5, 2002 FG2.3, 2004 HI1, 2015 FG4.1

$1^2 - 2^2 + 3^2 - 4^2 + \cdots + 685^2 - 686^2 + 687^2$

$= 1 + (3^2 - 2^2) + (5^2 - 4^2) + \cdots + (687^2 - 686^2)$

$= 1 + (3 + 2)(3 - 2) + (5 + 4)(5 - 4) + \cdots + (687 + 686)(687 - 686)$

$= 1 + 5 + 9 + \cdots + 1373$ (sum of 344 terms of an A.S., $a = 1$, $d = 4$)

$= \frac{1+1373}{2} \times 344$

$= 236328$

I3.3 If γ is the remainder of β divided by the 1993rd term of the following sequence:

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, $\cdots$. Determine the value of γ.

$1 + 2 + 3 + \cdots + 62 = \frac{1+62}{2} \times 62 = 1953$ and $1993 - 1953 = 40 < 63$

The 1993rd term of the sequence is 63.

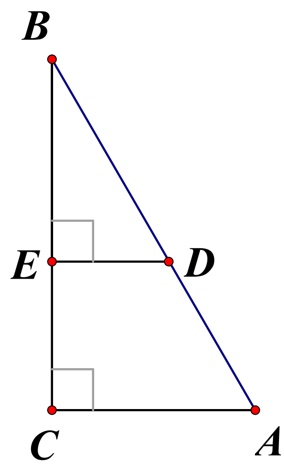
$236328 \div 63$, by division, the remainder is $\gamma = 15$.

Remark: The original question is:

Determine the remainder of β divided by the 1993rd term of the following sequence:

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, $\cdots$. γ is not mentioned.

I3.4 In the figure below, $BE = AC$, $BD = \frac{1}{2}$ and $DE + BC = 1$.



If δ is γ times the length of ED, determine the value of δ.

Let $DE = x$, $BE = y$

Then $AC = y$, $BC = 1 - x$

It is easy to show that $\Delta BED \sim \Delta BCA$ (equiangular)

$\frac{DE}{BE} = \frac{AC}{BC}$ (cor. sides, ~Δs) $\Rightarrow \frac{x}{y} = \frac{y}{1-x} \Rightarrow y^2 = x(1-x)$

$BE^2 + DE^2 = BD^2$ (Pythagoras' theorem)

$y^2 + x^2 = \frac{1}{4} \Rightarrow x(1-x) + x^2 = \frac{1}{4} \Rightarrow x = \frac{1}{4}$

$\because \gamma = 15$, $\delta = \gamma DE = \frac{15}{4}$

Individual Event 4

I4.1 Let α be the remainder of 2^{1000} divided by 13, determine the value of α.

Reference: 1972 American High School Mathematics Examination Q31, 2011 HI1

$13\times 5 = 64 + 1 \Rightarrow 2^6 = 13\times 5 - 1$

$2^{1000} = 2^4\cdot 2^{996} = 16\cdot(2^6)^{166} = (13 + 3)\cdot(13\times 5 - 1)^{166}$

$= (13 + 3)\cdot(13m + 1)$, by using binomial theorem

$= 13n + 3$, where n and m are integers

$\alpha = 3$

I4.2 Determine the value of $\beta = \dfrac{\left(7+4\sqrt{\alpha}\right)^{\frac{1}{2}} - \left(7-4\sqrt{\alpha}\right)^{\frac{1}{2}}}{\sqrt{\alpha}}$.

Reference: 2013 FI3.1

$\sqrt{7+4\sqrt{3}} = \sqrt{7+2\sqrt{12}} = \sqrt{4+3+2\sqrt{4\times 3}} = \sqrt{4}+\sqrt{3} = 2+\sqrt{3}$

$\sqrt{7-4\sqrt{3}} = 2-\sqrt{3}$

$\beta = \dfrac{\left(7+4\sqrt{3}\right)^{\frac{1}{2}} - \left(7-4\sqrt{3}\right)^{\frac{1}{2}}}{\sqrt{3}} = \dfrac{2+\sqrt{3}-2+\sqrt{3}}{\sqrt{3}} = 2$

I4.3 If $f(a) = a - \beta$ and $F(a, b) = b^2 + a$, determine the value of $\gamma = F(3, f(4))$.

Reference: 1985 FI3.3, 1990 HI3, 2013 FI3.2

$f(4) = 4 - 2 = 2$

$\gamma = F(3, f(4)) = F(3, 2) = 2^2 + 3 = 7$

I4.4 If δ is the product of all real roots of $x^{\log_\gamma x} = 10$, determine the value of δ.

$x^{\log_7 x} = 10$

$\log_7 x \log x = \log 10$

$\dfrac{(\log x)^2}{\log 7} = 1$

$\log x = \pm\sqrt{\log 7}$

$x = 10^{\sqrt{\log 7}}$ or $10^{-\sqrt{\log 7}}$

Product of roots $= 10^{\sqrt{\log 7}} \times 10^{-\sqrt{\log 7}} = 1$

Group Event 1

G1.1 Simplify $\left(\dfrac{1\times3\times9+2\times6\times18+\cdots+n\times3n\times9n}{1\times5\times25+2\times10\times50+\cdots+n\times5n\times25n}\right)^{\frac{1}{3}}$.

Reference: 2000 FI5.1

$$\left(\dfrac{1\times3\times9+2\times6\times18+\cdots+n\times3n\times9n}{1\times5\times25+2\times10\times50+\cdots+n\times5n\times25n}\right)^{\frac{1}{3}}$$

$$=\left[\dfrac{3^3\left(1^3+2^3+\cdots+n^3\right)}{5^3\left(1^3+2^3+\cdots+n^3\right)}\right]^{\frac{1}{3}}$$

$$=\dfrac{3}{5}$$

G1.2 Among 50 school teams joining the HKMO, no team answered all four questions correctly in the paper of a group event. If the first question was solved by 45 teams, the second by 40 teams, the third by 35 teams and the fourth by 30 teams. How many teams solved both the third and the fourth questions?

$\because$ No team answered all four questions correctly

$\therefore$ Each team can solve at most three questions.

The maximum number of solved questions = 50×3 = 150

The actual number of solved questions = 45 + 40 + 35 + 30 = 150

$\therefore$ Each team can solve exactly three questions.

Number of teams that cannot solve the first question = (50 – 45) teams = 5 teams

$\Rightarrow$ These 5 teams can solve Q2, Q3 and Q4 but not Q1.

Number of teams that cannot solve the second question = (50 – 40) teams =10 teams

$\Rightarrow$ These 10 teams can solve Q1, Q3 and Q4 but not Q2.

Number of teams that cannot solve the third question = (50 – 35) teams = 15 teams

$\Rightarrow$ These 15 teams can solve Q1, Q2 and Q4 but not Q3.

Number of teams that cannot solve the fourth question = (50 – 30) teams = 20 teams

$\Rightarrow$ These 20 teams can solve Q1, Q2 and Q3 but not Q4.

Number of school teams solved both the third and the fourth questions = 5 + 10 = 15

Remark We cannot use the Venn diagram on the right with explanation below:

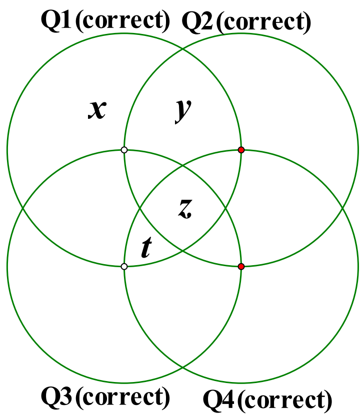
x = school teams that can solve Q1 but not Q2, Q3 nor Q4.

y = school teams that can solve Q1, Q2 but not Q3 nor Q4.

z = school teams that can solve all four questions.

t = school teams that can solve Q1, Q3 and Q4 but not Q2

However, we could not find any part in Venn diagram representing school teams that can solve Q1, Q4 but not Q2 nor Q3 !!!



G1.3 Let n be the product 3659893456789325678 and 342973489379256. Determine the number of digits of n. **(Reference: 2013 FG4.1)**

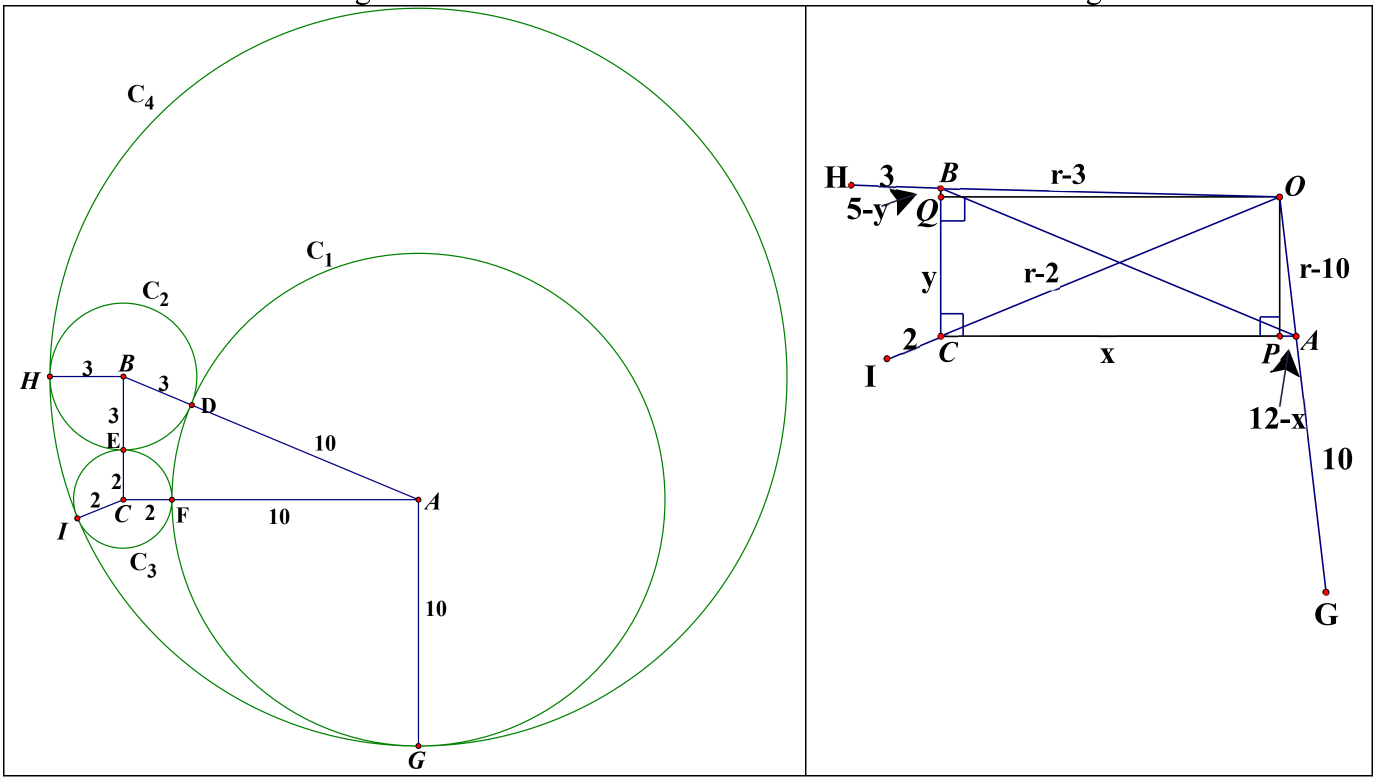
Let x = 3 659 893 456 789 325 678, y = 342 973 489 379 256

$x = 3.7\times10^{18}$, $y = 3.4\times10^{14}$ (correct to 2 sig. fig.)

$n = xy = 3.7\times10^{18}\times3.4\times10^{14} = 12.58\times10^{32} = 1.258\times10^{33}$

The number of digits of n is 34.

G1.4 Three circles of radii 2, 3 and 10 units are placed inside another big circle in such a way that all circles are touching one another. Determine the value of the radius of the big circle.



Let A be the centre of circle C_1 with radius 10, B be the centre of circle C_2 with radius 3, C be the centre of circle C_3 with radius 2. Join AB, BC, AC.

Suppose C_1 and C_2 touch each other at D, C_2 and C_3 touch each other at E, C_3 and C_1 touch each other at F. Then A, D, B are collinear, B, E, C are collinear, C, F, A are collinear.

$AB = 10 + 3 = 13$, $BC = 3 + 2 = 5$, $AC = 10 + 2 = 12$

$BC^2 + AC^2 = 5^2 + 12^2 = 25 + 144 = 169 = 13^2 = AB^2$

$\therefore \angle ACB = 90°$ (converse, Pythagoras' theorem)

Let O be the centre of circle C_4 with radius r circumscribing all three circles C_1, C_2, C_3 at G, H and I respectively. Then O, A, G are collinear, O, B, H are collinear, O, C, I are collinear.

$AG = 10$, $BH = 3$, $CI = 2$, $OA = r - 10$, $OB = r - 3$, $OC = r - 2$.

Let P and Q be the feet of perpendiculars drawn from O onto AC and AB respectively.

Then $OPCQ$ is a rectangle.

Let $CP = x = QO$ (opp. sides of rectangle), $CQ = y = PO$ (opp. sides of rectangle)

$AP = 12 - x$, $BQ = 5 - y$.

In ΔOCP, $x^2 + y^2 = (r - 2)^2$ …… (1) (Pythagoras' theorem)

In ΔOAP, $(12 - x)^2 + y^2 = (r - 10)^2$ …… (2) (Pythagoras' theorem)

In ΔOBQ, $x^2 + (5 - y)^2 = (r - 3)^2$ …… (3) (Pythagoras' theorem)

(1) – (2): $24x - 144 = 16r - 96 \Rightarrow x = \dfrac{2r+6}{3}$ …… (4)

(1) – (3): $10y - 25 = 2r - 5 \Rightarrow y = \dfrac{r+10}{5}$ …… (5)

Sub. (4) and (5) into (1): $\left(\dfrac{2r+6}{3}\right)^2 + \left(\dfrac{r+10}{5}\right)^2 = (r-2)^2$

$25(4r^2 + 24r + 36) + 9(r^2 + 20r + 100) = 225(r^2 - 4r + 4)$

$116r^2 - 1680r - 900 = 0 \Rightarrow 29r^2 - 420r - 225 = 0$

$(r - 15)(29r + 15) = 0 \Rightarrow r = 15$, the radius of the big circle is 15.

Group Event 2

G2.1 On a 3×3 grid of 9 squares, each squares is to be painted with either Red or Blue. If α is the total number of possible colouring in which no 2×2 grid consists of only Red squares, determine the value of α.

If there is no restriction, number of possible colouring = 2^9 = 512

(1)If all 9 squares are painted as red, number of colouring = 1

(2)If there are exactly three 2×2 grid consists of only Red squares, possible pattern may be:

$\begin{array}{|c|c|c|}\hline R&R&B\\\hline R&R&R\\\hline R&R&R\\\hline\end{array}$

90° rotation gives another possible pattern
Number of colouring = 4

(3)If there are exactly two 2×2 grid consists of only Red squares, possible pattern may be:

$\begin{array}{|c|c|c|}\hline B&R&R\\\hline R&R&R\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline R&R&B\\\hline R&R&R\\\hline B&R&R\\\hline\end{array}$

Number of colouring = 2

(4)

$\begin{array}{|c|c|c|}\hline R&R&R\\\hline R&R&B\\\hline R&R&R\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline R&R&R\\\hline R&R&B\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline R&R&B\\\hline R&R&R\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline R&R&B\\\hline R&R&B\\\hline R&R&R\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline R&R&B\\\hline R&R&B\\\hline R&R&B\\\hline\end{array}$

90° rotation gives another possible pattern
Number of colouring = 4×5 = 20

(5)If there is exactly one 2×2 grid consists of only Red squares, possible pattern may be:

$\begin{array}{|c|c|c|}\hline B&B&B\\\hline R&R&B\\\hline R&R&B\\\hline\end{array}$

90° rotation gives another possible pattern
Number of colouring = 4

(6)

$\begin{array}{|c|c|c|}\hline R&B&B\\\hline R&R&B\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline B&R&B\\\hline R&R&B\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline B&B&R\\\hline R&R&B\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline B&B&B\\\hline R&R&R\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline B&B&B\\\hline R&R&B\\\hline R&R&R\\\hline\end{array}$

90° rotation gives another possible pattern
Number of colouring = 4×5 = 20

(7)

$\begin{array}{|c|c|c|}\hline R&B&R\\\hline R&R&B\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline R&B&B\\\hline R&R&R\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline R&B&B\\\hline R&R&B\\\hline R&R&R\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline B&R&R\\\hline R&R&B\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline B&R&B\\\hline R&R&R\\\hline R&R&B\\\hline\end{array}$

90° rotation gives another possible pattern
Number of colouring = 4×5 = 20

(8)

$\begin{array}{|c|c|c|}\hline B&R&B\\\hline R&R&B\\\hline R&R&R\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline B&B&R\\\hline R&R&R\\\hline R&R&B\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline B&B&R\\\hline R&R&B\\\hline R&R&R\\\hline\end{array}$

90° rotation gives another possible pattern
Number of colouring = 4×3 = 12

(9)

$\begin{array}{|c|c|c|}\hline B&R&R\\\hline R&R&B\\\hline R&R&R\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline R&B&R\\\hline R&R&B\\\hline R&R&R\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline R&B&R\\\hline R&R&R\\\hline R&R&B\\\hline\end{array}$

90° rotation gives another possible pattern
Number of colouring = 4×3 = 12

$\therefore$ Total number of possible colouring in which no 2×2 grid consists of only Red squares
= No restriction – all 9 red squares – exactly 3 2×2 red grid – exactly 2 2×2 red grid – exactly 1 2×2 red grid
= 512 – 1 – 4 – 2 – 20 – 4 – 20 – 20 – 12 – 12 = 417

Method 2 (a)All 9 blue squares = 1 pattern. (b)8 blue squares + 1 red squares = 9 patterns.
(c)7B+2R $= C_2^9$ = 36 patterns, (d)6B+3R $= C_3^9$ = 84 patterns, (e)5B+4R $= C_4^9 - 4$ = 122 patterns
(f)4B+5R $= C_5^9 - 4\times 5$ = 106 patterns, (g)3B+6R $= C_6^9 - 4\times C_2^5 + 4$ = 48 patterns
(h)2B+7R = 8 + 2 = 10 patterns, (i)1B+8R = 1 pattern
Total number of different patterns = 1 + 9 + 36 + 84 + 122 + 106 + 48 + 10 + 1 = 417

G2.2 If the sum of 25 consecutive positive integers is the product of 3 prime numbers, what is the minimum sum of these 3 prime numbers?

Let the smallest positive integer be x. We use the formula: $S(n)=\frac{n}{2}\left[2a+(n-1)d\right]$.

$\frac{25}{2}(2x+24\times 1)=25(x+12)=5\times 5\times(x+12)=$ product of 3 prime numbers

The minimum prime for $x+12$ is 13. The minimum sum of these 3 prime numbers is 23.

G2.3 Determine the sum of all real roots of the following equation $|x+3|-|x-1|=x+1$.

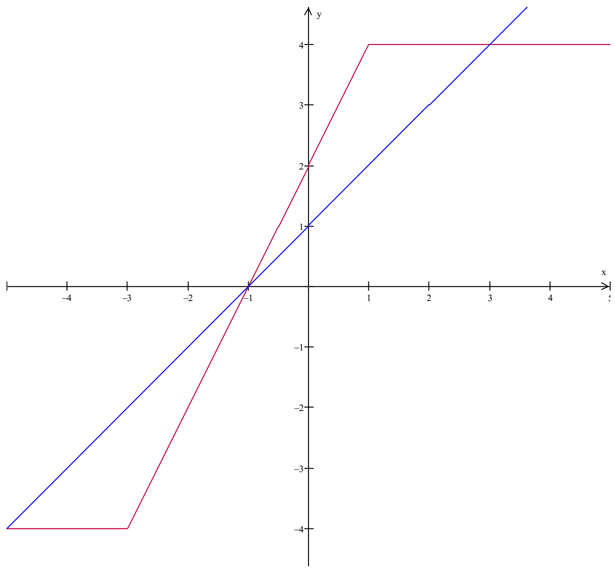
When $x\le -3$, the equation becomes $-x-3-(1-x)=x+1 \Rightarrow x=-5$

When $-3<x\le 1$, the equation becomes $x+3-(1-x)=x+1 \Rightarrow x=-1$

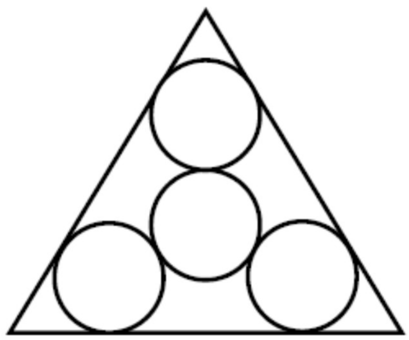
When $1<x$, the equation becomes $x+3-(x-1)=x+1 \Rightarrow x=3$

$\therefore$ Sum of all real roots $=-5+(-1)+3=-3$

A graph is given below:



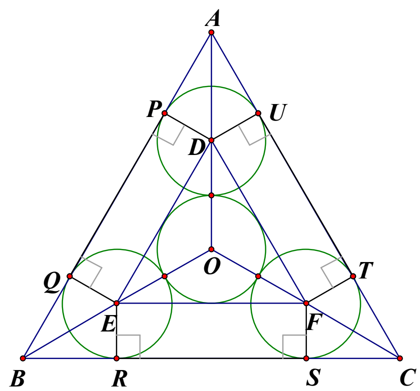
G2.4 In the figure below, there are 4 identical circles placed inside an equilateral triangle. If the radii of the circles are 1 unit, what is the value of the area of the triangle?



Let the triangle be ABC, O is the centre of the middle circle, D, E, F are the centres of the other 3 circles respectively.

Let P, Q, R, S, T, U be the points of contact as shown.

$DP\perp AB$, $EQ\perp AB$, $ER\perp BC$, $FS\perp BC$, $FT\perp AC$, $DU\perp AC$ (tangent $\perp$ radius)



$DP=EQ=ER=FS=FT=DU=1$ (radii)

$OD=OE=OF=2$ (radii 1 + radii 1)

$\Delta ODE\cong\Delta OEF\cong\Delta OFD$ (S.S.S.)

$\angle DOE=\angle EOF=\angle FOD$ (corr. $\angle$s $\cong\Delta$s)

$\angle DOE+\angle EOF+\angle FOD=360^\circ$ ($\angle$s at a point)

$\therefore\ \angle DOE=\angle EOF=\angle FOD=120^\circ$

$DPQE$, $ERSF$, $FTUD$ are rectangles (opp. sides are eq. and //)

$DE=EF=FD=2\times 2\sin 60^\circ=2\sqrt{3}=PQ=RS=TU$

In ΔADU, $\angle DAU=30^\circ$, $DU=1$, $DU\perp AU$,

$AU=1\tan 60^\circ=\sqrt{3}$

$\therefore\ AB=BC=CA=2\sqrt{3}+2\sqrt{3}=4\sqrt{3}$

Area of $\Delta ABC=\frac{1}{2}\cdot\left(4\sqrt{3}\right)^2\sin 60^\circ=12\sqrt{3}$

Group Event 3

G3.1 Simplify $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}$.

Reference: 1993 FI1.4, 1999 HG3, 2001 FG2.1, 2011 HI7, 2015 FI4.2

$$\sqrt{3+\sqrt{5}}=\sqrt{\frac{6+2\sqrt{5}}{2}}=\frac{1}{\sqrt{2}}\cdot\sqrt{5+2\sqrt{5}+1}=\frac{1}{\sqrt{2}}\cdot\left(\sqrt{5}+1\right)$$

$$\sqrt{3-\sqrt{5}}=\sqrt{\frac{6-2\sqrt{5}}{2}}=\frac{1}{\sqrt{2}}\cdot\sqrt{5-2\sqrt{5}+1}=\frac{1}{\sqrt{2}}\cdot\left(\sqrt{5}-1\right)$$

$$\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}=\frac{1}{\sqrt{2}}\cdot\left(\sqrt{5}+1\right)+\frac{1}{\sqrt{2}}\cdot\left(\sqrt{5}-1\right)=\frac{1}{\sqrt{2}}\cdot\left(2\sqrt{5}\right)=\sqrt{10}$$

G3.2 Let p be a prime and m be an integer. If $p(p + m) + 2p = (m + 2)^3$, find the greatest possible value of m.

$p(p + m + 2) = (m + 2)^3$

If m is even and p is odd, then odd×(odd + even + 2) = (even + 2)^3 $\Rightarrow$ LHS $\neq$ RHS !!!

If m is odd and p is odd, then odd×(odd + odd + 2) = (odd + 2)^3 $\Rightarrow$ LHS $\neq$ RHS !!!

In all cases, p must be even.

$\because$ the only even prime is 2 $\therefore$ $p = 2$

$2(m + 4) = (m + 2)^3$

LHS is even $\Rightarrow$ $(m+2)^3$ is even$\Rightarrow$$m + 2$ is even$\Rightarrow$RHS is divisible by 8$\Rightarrow$LHS is divisible by 8

$\Rightarrow$ $m + 4 = 4n$, where n is an integer $\Rightarrow$ $m + 2 = 4n - 2$

Put $m + 2 = 4n - 2$ into the equation: $2(4n) = (4n - 2)^3$

$n = (2n - 1)^3$

$\Rightarrow$ $n = 1$, $m = 0$ (This is the only solution, $n < (2n - 1)^3$ for $n > 1$ and $n > (2n - 1)^3$ for $n < 1$)

G3.3 Determine a root to $x=\left(x-\frac{1}{x}\right)^{\frac{1}{2}}+\left(1-\frac{1}{x}\right)^{\frac{1}{2}}$.

$$x-\sqrt{1-\frac{1}{x}}=\sqrt{x-\frac{1}{x}}\Rightarrow\left(x-\sqrt{1-\frac{1}{x}}\right)^2=\left(\sqrt{x-\frac{1}{x}}\right)^2$$

$$x^2-2x\sqrt{1-\frac{1}{x}}+1-\frac{1}{x}=x-\frac{1}{x}$$

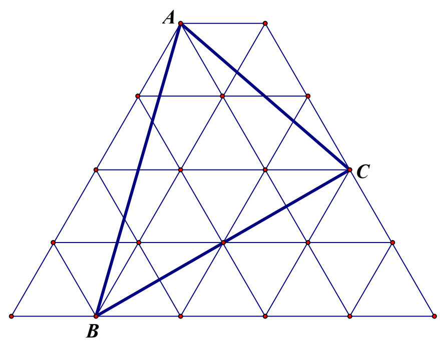
$$x^2-x+1=2\sqrt{x^2-x}\Rightarrow\left(x^2-x\right)-2\sqrt{x^2-x}+1=0$$

$$\left(\sqrt{x^2-x}-1\right)^2=0$$

$$\sqrt{x^2-x}=1\Rightarrow x^2-x-1=0$$

$$x=\frac{1+\sqrt{5}}{2}\quad\text{or}\quad\frac{1-\sqrt{5}}{2}\quad\text{(rejected as } x>0\text{)}$$

G3.4 In the figure below, the area of each small triangle is 1. Determine the value of the area of the triangle ABC.



Total number of equilateral triangles = 24

Area of ABC

$=24-\frac{1}{2}\cdot4-\frac{1}{2}\cdot6-1-\frac{1}{2}\cdot4-6$

$= 10$

Group Event 4

G4.1 Let $b = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \cdots - 2012^2 + 2013^2$.

Determine the remainder of b divided by 2015 .

Reference: 1997 HI5, 2002 FG2.3, 2004 HI1, 2015 FI3.2

$b = 1 + (3 - 2)(3 + 2) + (5 - 4)(5 + 4) + \cdots + (2013 - 2012)(2012 + 2013)$

$b = 1 + 5 + 9 + \cdots + 4025$

This is an arithmetic series with $a = 1$, $d = 4$.

$1 + (n - 1)\times 4 = 4025$

$\Rightarrow n = 1007$

$b = \frac{1007}{2}(1 + 4025)$

$= 1007\times 2013$

$= 1007\times(2015 - 2)$

$= 1007\times 2015 - 2014$

$= 1006\times 2015 + 1$

Remainder = 1

G4.2 There are positive integers with leading digits being 6 and upon removing this leading digit, the resulting integer is $\frac{1}{25}$ of the original value. Determine the least of such positive integers.

Let the original number be x.

$x = 6\times 10^n + y$, where $y < 10^n$ and $y = \frac{1}{25}x$

$x = 6\times 10^n + \frac{1}{25}x$

$24x = 150\times 10^n$

$4x = 25\times 10^n$

4 is not a factor of 25, so 4 must be a factor of 10^n

Least possible $n = 2$

The least positive x is $25\times 10^2 \div 4 = 625$

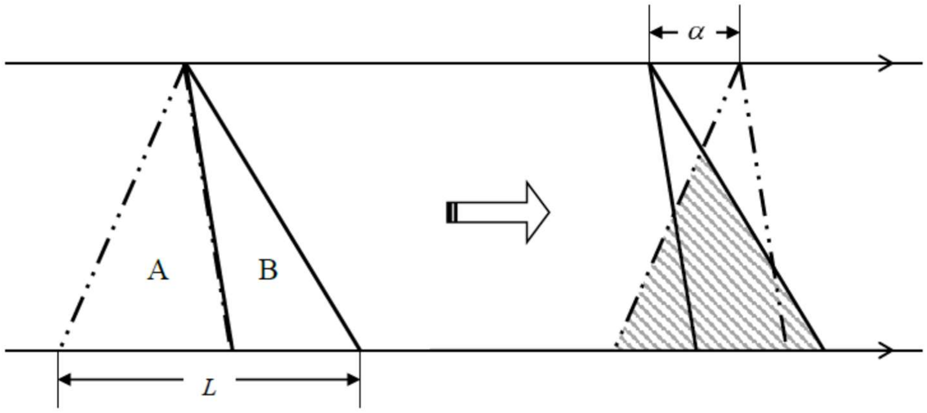
G4.3 If $x + \frac{1}{x} = 1$, determine the value of $x^5 + \frac{1}{x^5}$.

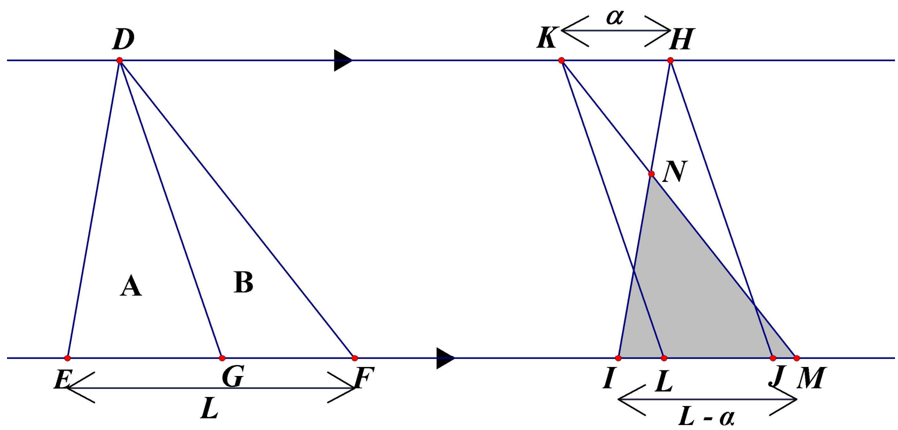
$\left(x + \frac{1}{x}\right)^2 = 1 \Rightarrow x^2 + 2 + \frac{1}{x^2} = 1 \Rightarrow x^2 + \frac{1}{x^2} = -1$

$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) = 1\times(-1) = -1 \Rightarrow x^3 + \frac{1}{x^3} + x + \frac{1}{x} = -1 \Rightarrow x^3 + \frac{1}{x^3} = -2$

$\left(x^2 + \frac{1}{x^2}\right)\left(x^3 + \frac{1}{x^3}\right) = (-1)\times(-2) = 2 \Rightarrow x^5 + \frac{1}{x^5} + x + \frac{1}{x} = 2 \Rightarrow x^5 + \frac{1}{x^5} = 1$

G4.4 In the figure below, when triangle A shifts α units to the right, the area of shaded region is $\dfrac{\alpha}{L}$ times of the total area of the triangles A and B. Determine the value of $\dfrac{\alpha}{L}$.





Let the original triangle be DEF. G is a point on EF with $EF = L$.
ΔDEG is translated to ΔHIJ by α units, $\Delta DEG \cong \Delta HIJ$, $\Delta DGF \cong \Delta KLM$, $EF = L$, $HK = \alpha$.
Let HI intersects KM at N, $IM = L - \alpha$
Consider ΔDEF in the left figure and ΔNIM in the right figure.

$\angle DEF = \angle NIM$ (corr. $\angle$s, DE // HI)

$\angle DFE = \angle NMI$ (corr. $\angle$s, DF // KM)

$\therefore$ $\Delta DEF \sim \Delta NIM$ (equiangular)

$\dfrac{S_{\Delta NIM}}{S_{\Delta DEF}} = \left(\dfrac{L-\alpha}{L}\right)^2$ (ratio of areas of ~Δs)

$\dfrac{\alpha}{L} = \left(1 - \dfrac{\alpha}{L}\right)^2$ (Given $\dfrac{S_{\Delta NIM}}{S_{\Delta DEF}} = \dfrac{\alpha}{L}$)

$\left(\dfrac{\alpha}{L}\right)^2 - 3\left(\dfrac{\alpha}{L}\right)^2 + 1 = 0$, this is a quadratic equation in $\dfrac{\alpha}{L}$.

$\dfrac{\alpha}{L} = \dfrac{3+\sqrt{5}}{2}$ or $\dfrac{3-\sqrt{5}}{2}$

From the figure, $\dfrac{\alpha}{L} < 1$ and $\dfrac{3+\sqrt{5}}{2} > 1$

$\therefore$ $\dfrac{\alpha}{L} = \dfrac{3-\sqrt{5}}{2}$ only