

Individual Events

I1	a	6	I2	P	2	I3	a	-2	I4	a	2	IS	P	84
	b	$2 + 4\sqrt{2}$		Q	6		b	9		b	11		Q	8
	c	7		R	56		c	$\frac{1}{24}$		c	462		R	$\frac{\sqrt{3}}{4}$
	d	11		S	2352		d	$-\frac{7}{18}$		d	*334 see the remark		S	$1 + \sqrt{2}$

Group Events

G1	a	47	G2	a	*2 see the remark	G3	a	-10	G4	P	500	GS	a	16
	b	101		b	4.5		b	0		Q	15		b	1
	c	43		c	15		c	2005		R	$\frac{1}{2}$		c	12
	d	0		d	1		d	2005		S	1		d	9

Individual Event 1

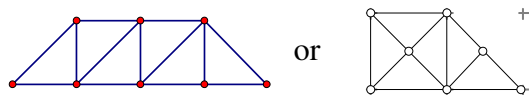
- I1.1** Given that there are a positive integers less than 200 and each of them has exactly three positive factors, find the value of a .

If $x = rs$, where r and s are positive integers, then the positive factors of x may be 1, r , s and x .

In order to have exactly three positive factors, $r = s = a$ prime number.

Possible $x = 4, 9, 25, 49, 121, 169$. $a = 6$.

- I1.2** If a copies of a right-angled isosceles triangle with hypotenuse $\sqrt{2}$ cm can be assembled to form a trapezium with perimeter equal to b cm, find the least possible value of b . (give the answer in surd form).



The perimeter $= 6 + 2\sqrt{2} \approx 8.8$ or $2 + 4\sqrt{2} \approx 7.7$

The least possible value of $b = 2 + 4\sqrt{2}$

- I1.3** If $\sin(c^2 - 3c + 17)^\circ = \frac{4}{b-2}$, where $0 < c^2 - 3c + 17 < 90$ and $c > 0$, find the value of c .

$$\sin(c^2 - 3c + 17)^\circ = \frac{4}{2 + 4\sqrt{2} - 2} = \frac{1}{\sqrt{2}}$$

$$c^2 - 3c + 17 = 45$$

$$c^2 - 3c - 28 = 0$$

$$(c - 7)(c + 4) = 0$$

$$c = 7 \text{ or } -4 \text{ (rejected)}$$

- I1.4** Given that the difference between two 3-digit numbers \overline{xyz} and \overline{zyx} is $700 - c$, where $x > z$. If d is the greatest value of $x + z$, find the value of d .

$$\overline{xyz} - \overline{zyx} = 700 - c$$

$$100x + 10y + z - (100z + 10y + x) = 700 - 7$$

$$99x - 99z = 693$$

$$x - z = 7$$

Possible answers: $x = 8, z = 1$ or $x = 9, z = 2$

d is the greatest value of $x + z = 9 + 2 = 11$

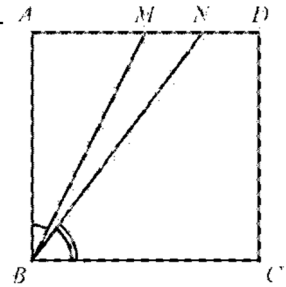
Individual Event 2

- 12.1** In Figure 1, $ABCD$ is a square, M is the mid-point of AD and N is the mid-point of MD . If $\angle CBN : \angle MBA = P : 1$, find the value of P .

Let $\angle ABM = \theta$, $\angle CBM = P\theta$. Let $AB = 4$, $AM = 2$, $MN = 1 = ND$.

$$\tan \theta = \frac{1}{2}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2 \times \frac{1}{2}}{1 - (\frac{1}{2})^2} = \frac{4}{3} = \tan P\theta, P = 2$$



- 12.2** Given that $ABCD$ is a rhombus on a Cartesian plane, and the co-ordinates of its vertices are $A(0, 0)$, $B(P, 1)$, $C(u, v)$ and $D(1, P)$ respectively. If $u + v = Q$, find the value of Q .

$\therefore ABCD$ is a rhombus, \therefore It is also a parallelogram

By the property of parallelogram, the diagonals bisect each other

Mid point of B, D = mid point of AC

$$\left(\frac{1+P}{2}, \frac{1+1}{2} \right) = \left(\frac{0+u}{2}, \frac{0+v}{2} \right)$$

$$u = 3, v = 3 \Rightarrow Q = u + v = 6$$

- 12.3** If $1 + (1 + 2) + (1 + 2 + 3) + \dots + (1 + 2 + 3 + \dots + Q) = R$, find the value of R .

$$R = 1 + (1 + 2) + (1 + 2 + 3) + \dots + (1 + 2 + 3 + \dots + 6)$$

$$R = 1 + 3 + 6 + 10 + 15 + 21 = 56$$

- 12.4** In the figure, EBC is an equilateral triangle, and A, D lie on EB and EC respectively. Given that $AD \parallel BC$, $AB = CD = R$ and $AC \perp BD$. If the area of the trapezium $ABCD$ is S , find the value of S .

$\angle ABC = \angle BCD = 60^\circ$, AC intersects BD at J , $AC \perp BD$.

$\triangle ACD \cong \triangle DBA$ (S.A.S.)

$AC = BD$ (corr. sides, $\cong \Delta$'s)

$\angle ABD = \angle DCA$ (corr. \angle s, $\cong \Delta$'s)

$\therefore \angle JBC = 60^\circ - \angle ABD = \angle JCB$

$\triangle JBC$ is a right-angled isosceles triangle.

$\angle JBC = \angle JCB = 45^\circ$

$BJ = CJ = y$, $AJ = AC - y = BD - y = DJ = x$

$\angle JAD = \angle JDA = 45^\circ$, $\angle ADC = 120^\circ$

Apply sine formula on $\triangle ACD$, $\frac{56}{\sin 45^\circ} = \frac{x+y}{\sin 120^\circ}$

$$x + y = 28\sqrt{6}, \text{ area} = \frac{1}{2}(x+y)^2 \sin 90^\circ = \frac{1}{2}(784)(6) = 2352$$

Method 2 $\angle EBC = 60^\circ = \angle ECB$

Draw $AF \perp BC$ and $DG \perp BC$, cutting BC at F and G .

$BF = CG = 56 \cos 60^\circ = 28$

$AF = DG = 56 \sin 60^\circ = 28\sqrt{3}$

AC intersects BD at J , $AC \perp BD$.

$\triangle ACD \cong \triangle DBA$ (S.A.S.)

$\angle ABD = \angle DCA$ (corr. \angle s, $\cong \Delta$'s)

$\therefore \angle JBC = 60^\circ - \angle ABD = \angle JCB$

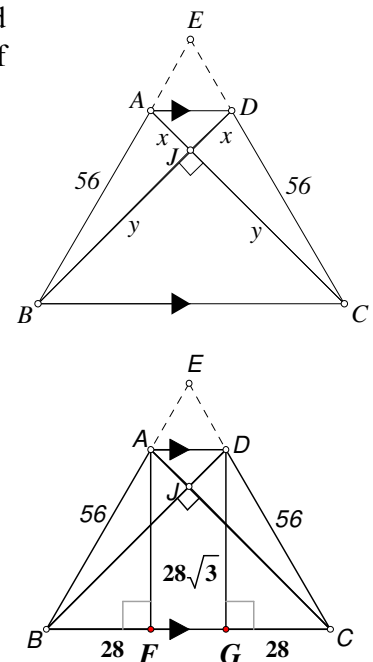
$\triangle JBC$ is a right-angled isosceles triangle.

$\angle JBC = \angle JCB = 45^\circ$

$CF = AF \cot 45^\circ = 28\sqrt{3}$

$FG = CF - CG = 28(\sqrt{3} - 1) = AD$

$$\text{Area of the trapezium } ABCD = S = \frac{28(\sqrt{3}-1) \times 2 + 28 \times 2}{2} \cdot (28\sqrt{3}) = 2352$$



Individual Event 3

I3.1 Let $x \neq \pm 1$ and $x \neq -3$. If a is the real root of the equation $\frac{1}{x-1} + \frac{1}{x+3} = \frac{2}{x^2-1}$, find the value of a .

$$\begin{aligned}\frac{1}{x+3} &= \frac{2}{x^2-1} - \frac{1}{x-1} \\ \frac{1}{x+3} &= \frac{2-(x+1)}{(x-1)(x+1)} \\ \frac{1}{x+3} &= \frac{1-x}{(x-1)(x+1)} \\ \frac{1}{x+3} &= -\frac{1}{x+1} \\ x+1 &= -x-3 \\ \Rightarrow x &= -2 = a\end{aligned}$$

I3.2 If $b > 1$, $f(b) = \frac{-a}{\log_2 b}$ and $g(b) = 1 + \frac{1}{\log_3 b}$. If b satisfies the equation

$$|f(b) - g(b)| + f(b) + g(b) = 3, \text{ find the value of } b.$$

Similar question: 2007 HI9

$$\left| \frac{2}{\log_2 b} - 1 - \frac{1}{\log_3 b} \right| + \frac{2}{\log_2 b} + 1 + \frac{1}{\log_3 b} = 3$$

$$|2\log_b 2 - \log_b b - \log_b 3| + 2\log_b 2 + \log_b 3 = 2$$

$$\left| \log_b \frac{4}{3b} \right| + \log_b 12 = \log_b b^2$$

$$\log \frac{4}{3b} = \pm \log \frac{b^2}{12}$$

$$\log \frac{4}{3b} = \log \frac{b^2}{12} \quad \text{or} \quad \log \frac{4}{3b} = \log \frac{12}{b^2}$$

$$b^3 = 16 \text{ or } b = 9$$

$$\text{When } b^3 = 16, (b^3)^2 = 256 < 1728 = 12^3$$

$$\Rightarrow b^2 < 12 \Rightarrow \left| \log \frac{4}{3b} \right| = \log \frac{b^2}{12} < 0 \text{ rejected.}$$

$$\text{When } b = 9, \left| \log \frac{4}{3b} \right| = \log \frac{b^2}{12} = \log \frac{81}{12} > 0 \text{ accepted.}$$

I3.3 Given that x_0 satisfies the equation $x^2 - 5x + (b-8) = 0$. If $c = \frac{x_0^2}{x_0^4 + x_0^2 + 1}$, find the value of c .

$$x^2 - 5x + 1 = 0, x^2 + 1 = 5x, x^4 + 2x^2 + 1 = 25x^2, x^4 + x^2 + 1 = 24x^2$$

$$c = \frac{x_0^2}{x_0^4 + x_0^2 + 1} = \frac{x_0^2}{24x_0^2} = \frac{1}{24}$$

I3.4 If -2 and $216c$ are the roots of the equation $px^2 + dx = 1$, find the value of d .

$$-2 \text{ and } 9 \text{ are roots of } px^2 + dx - 1 = 0$$

$$\text{Product of roots} = -\frac{1}{p} = -2 \times 9 \Rightarrow p = \frac{1}{18}$$

$$\text{Sum of roots} = -\frac{d}{p} = -2 + 9 \Rightarrow -18d = 7 \Rightarrow d = -\frac{7}{18}$$

Individual Event 4

I4.1 Let a be a real number.

If a satisfies the equation $\log_2(4^x + 4) = x + \log_2(2^{x+1} - 3)$, find the value of a .

$$\log_2(4^x + 4) = \log_2 2^x + \log_2(2^{x+1} - 3)$$

$$4^x + 4 = 2^x \cdot (2^{x+1} - 3)$$

$$(2^x)^2 + 4 = 2 \cdot (2^x)^2 - 3 \cdot 2^x$$

$$0 = (2^x)^2 - 3 \cdot 2^x - 4$$

$$(2^x - 4)(2^x + 1) = 0$$

$$2^x = 4, x = 2 = a$$

I4.2 Given that n is a natural number.

If $b = n^3 - 4n^2 - 12n + 144$ is a prime number, find the value of b . (**Reference: 2011 FI3.3**)

$$\text{Let } f(n) = n^3 - 8n^2 - 12n + 144$$

$$f(6) = 6^3 - 8 \cdot 6^2 - 12 \cdot 6 + 144 = 216 - 288 - 72 + 144 = 0$$

$\therefore f(6)$ is a factor

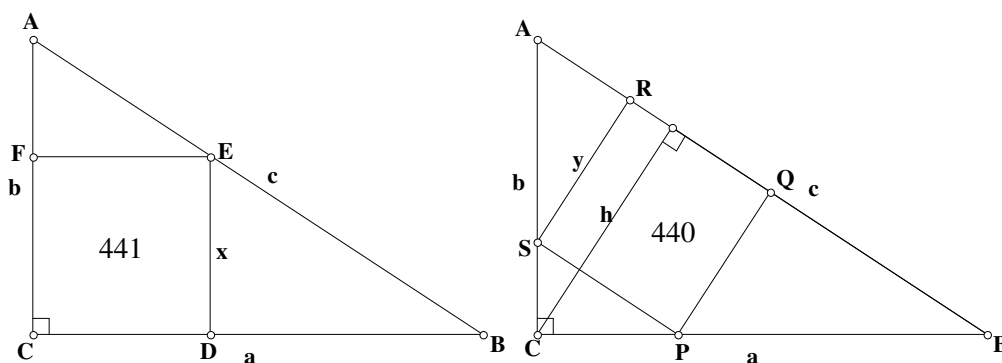
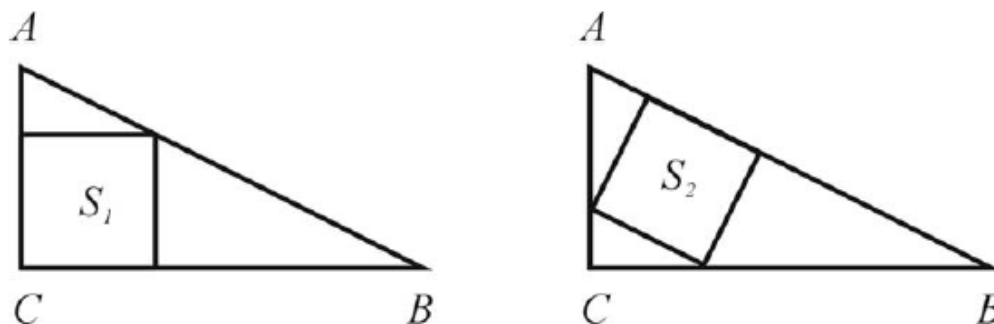
By division, $f(n) = (n - 6)(n - 6)(n + 4)$

$b = n^3 - 8n^2 - 12n + 144$, it is a prime $\Rightarrow n - 6 = 1, n = 7, b = 11$

I4.3 In Figure 1, S_1 and S_2 are two different inscribed squares of the right-angled triangle ABC .

If the area of S_1 is $40b + 1$, the area of S_2 is $40b$ and $AC + CB = c$, find the value of c .

Reference: American Invitation Mathematics Examination 1987 Q15



Add the label D, E, F, P, Q, R, S as shown. $CDEF, PQRS$ are squares.

Let $DE = x, SR = y$, then $x = \sqrt{441} = 21, y = \sqrt{440}$. Let $BC = a, AC = b, AB = c = \sqrt{a^2 + b^2}$.

Let the height of the triangle drawn from C onto AB be h , then $ab = ch = 2 \text{ area of } \Delta \dots (*)$

$$\Delta AFE \sim \Delta ACB: \frac{b-x}{b} = \frac{x}{a} \Rightarrow x = \frac{ab}{a+b} = 21 \dots\dots\dots(1)$$

$$\triangle CSP \sim \triangle CAB: \frac{\text{height of } \triangle CSP \text{ from } C}{SP} = \frac{h}{c} \Rightarrow \frac{h-y}{y} = \frac{h}{c} \Rightarrow y = \frac{ch}{c+h} = \sqrt{440}$$

$$\text{By (*), } \frac{ab}{c + \frac{ab}{c}} = \sqrt{440} \Rightarrow \sqrt{440} = \frac{ab\sqrt{a^2+b^2}}{a^2+ab+b^2} \dots\dots(2)$$

From (1) $ab = 21(a+b) \dots\dots\dots (3)$, sub. (3) into (2):

$$\sqrt{440} = \frac{21(a+b)\sqrt{(a+b)^2-2ab}}{(a+b)^2-ab} = \frac{21(a+b)\sqrt{(a+b)^2-42(a+b)}}{(a+b)^2-21(a+b)} = \frac{21\sqrt{(a+b)^2-42(a+b)}}{(a+b)-21}$$

Cross multiplying and squaring both sides:

$$440[(a+b)^2-42(a+b)+441] = 441[(a+b)^2-42(a+b)]$$

$$(a+b)^2-42(a+b)-440 \times 441 = 0$$

$$(a+b-462)(a+b+420) = 0$$

$$AC + CB = a + b = 462$$

I4.4 Given that $241c + 214 = d^2$, find the positive value of d .

$$d^2 = 241 \times 462 + 214 = 111556$$

$$d = \sqrt{111556}$$

Reference: [昌爸工作坊圖解直式開平方](#)

Divide 111556 into 3 groups of numbers 11, 15, 56.

Find the maximum integer p such that $p^2 \leq 11 \Rightarrow p = 3$

$$11 - p^2 = 2$$

$$3 + 3 = 6$$

Find the maximum integer q such that $(60 + q)q \leq 215$

$$\Rightarrow q = 3$$

$$215 - 63 \times 3 = 26$$

$$60 + q + q = 66$$

Find the maximum integer r such that $(660 + r)r \leq 2656$

$$\Rightarrow r = 4$$

$$d = 334$$

Method 2: $d^2 = 241 \times 462 + 214 = 111556$

$$300^2 = 90000 < 111556 < 160000 = 400^2 \Rightarrow 300 < d < 400$$

$$330^2 = 108900 < 111556 < 115600 = 340^2 \Rightarrow 330 < d < 340$$

The unit digit of d^2 is 6 \Rightarrow the unit digit of d is 4 or 6

$$335^2 = 112225 \Rightarrow d = 334$$

or 111556 is not divisible by 3, but 336 is divisible by 3

$$\therefore d = 334$$

Remark: Original question: $\dots\dots$, find the value of d . $\Rightarrow d = \pm 334$

Method 3

Observe the number patterns:

$$34^2 = 1156$$

$$334^2 = 111556$$

$$3334^2 = 11115556$$

$\dots\dots\dots$

$$\therefore d = 334$$

$$\text{Also, } 33^2 = 1089$$

$$333^2 = 110889$$

$$3333^2 = 11108889$$

$\dots\dots\dots$

$$3 \times 4 = 12$$

$$33 \times 34 = 1122$$

$$333 \times 334 = 111222$$

$$3333 \times 3334 = 11112222$$

$\dots\dots\dots$

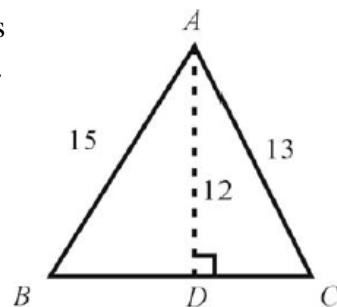
Individual Event (Spare)

IS.1 In figure 1, $\triangle ABC$ is an acute triangle, $AB = 15$, $AC = 13$, and its altitude $AD = 12$. If the area of the $\triangle ABC$ is P , find the value of P .

$$BD = \sqrt{15^2 - 12^2} = 9$$

$$CD = \sqrt{13^2 - 12^2} = 5$$

$$P = \text{area of } \triangle = \frac{1}{2}(9+5) \times 12 = 84$$



IS.2 Given that x and y are positive integers. If $x^4 + y^4$ is divided by $x + y$, the quotient is $P + 13$ and the remainder is Q , find the value of Q .

$$P + 13 = 84 + 13 = 97$$

$$x^4 + y^4 = 97(x + y) + Q, 0 \leq Q < x + y$$

Without loss of generality, assume $x \geq y$,

$$x^4 < x^4 + y^4 = 97(x + y) + Q < 98(x + y) \leq 98(2x) = 196x$$

$$x^3 < 196$$

$$x \leq 5$$

$$\text{On the other hand, } x^4 + y^4 = 97(x + y) + Q = x^3(x + y) - y(x^3 - y^3)$$

$$\Rightarrow (x^3 - 97)(x + y) = y(x^3 - y^3) + Q$$

$$\text{RHS} \geq 0 \Rightarrow \text{LHS} \geq 0 \Rightarrow x^3 \geq 97$$

$$x \geq 5$$

$$\therefore x = 5$$

$$5^4 + y^4 = 625 + y^4 \text{ is divided by } 5 + y, \text{ the quotient is } 97 \text{ and the remainder is } Q, 1 \leq y \leq 5$$

$$625 + y^4 = 97(5 + y) + Q, 0 \leq Q < 5 + y \leq 10$$

$$y^4 + 140 = 97y + Q$$

$$97y \leq y^4 + 140 \leq 97y + 9$$

$$131 \leq 97y - y^4 \leq 140$$

By putting $y = 1, 2, 3, 4$ and 5 into the above inequalities, only $y = 4$ satisfies it.

$$5^4 + 4^4 = 881 = (5 + 4) \times 97 + 8, Q = 8$$

IS.3 Given that the perimeter of an equilateral triangle equals to that of a circle with radius $\frac{12}{Q}$ cm.

If the area of the triangle is $R\pi^2$ cm², find the value of R .

$$\text{Radius of circle} = \frac{12}{8} \text{ cm} = 1.5 \text{ cm} \Rightarrow \text{circumference} = 2 \times \pi \times 1.5 \text{ cm} = 3\pi \text{ cm}$$

$$\text{Side of the equilateral triangle} = \pi \text{ cm}$$

$$\text{Area of the triangle} = \frac{1}{2} \pi^2 \sin 60^\circ \text{ cm}^2 = \frac{\sqrt{3}}{4} \pi^2 \text{ cm}^2 \Rightarrow R = \frac{\sqrt{3}}{4}$$

IS.4 Let $W = \frac{\sqrt{3}}{2R}$, $S = W + \frac{1}{W + \frac{1}{W + \frac{1}{W + \dots}}}$, find the value of S .

$$W = 2, S = 2 + \frac{1}{S} \Rightarrow S^2 - 2S - 1 = 0, S = \frac{1 \pm \sqrt{2}}{2}, S > 0, \therefore S = 1 + \sqrt{2} \text{ only}$$

Group Event 1**G1.1** Given that a is an integer. If $50!$ is divisible by 2^a , find the largest possible value of a .

2, 4, 6, 8, ..., 50 are divisible by 2, there are 25 even integers.

Reference: 1990 HG6, 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FI1.4, 2012 FG1.3

4, 8, ..., 48 are divisible by 4, there are 12 multiples of 4.

8, ..., 48 are divisible by 8, there are 6 multiples of 8.

16, ..., 48 are divisible by 16, there are 3 multiples of 16.

32 is the only multiple of 32.

$$a = 25 + 12 + 6 + 3 + 1 = 47$$

G1.2 Let $[x]$ be the largest integer not greater than x . For example, $[2.5] = 2$.If $b = \left\lceil 100 \times \frac{11 \times 77 + 12 \times 78 + 13 \times 79 + 14 \times 80}{11 \times 76 + 12 \times 77 + 13 \times 78 + 14 \times 79} \right\rceil$, find the value of b .

$$\begin{aligned} & 100 \times \frac{11 \times 77 + 12 \times 78 + 13 \times 79 + 14 \times 80}{11 \times 76 + 12 \times 77 + 13 \times 78 + 14 \times 79} \\ &= 100 \times \frac{11 \times 76 + 12 \times 77 + 13 \times 78 + 14 \times 79 + 11 + 12 + 13 + 14}{11 \times 76 + 12 \times 77 + 13 \times 78 + 14 \times 79} \\ &= 100 \times \left(1 + \frac{11 + 12 + 13 + 14}{11 \times 76 + 12 \times 77 + 13 \times 78 + 14 \times 79} \right) \\ &= 100 + \frac{11 \times 100 + 12 \times 100 + 13 \times 100 + 14 \times 100}{11 \times 76 + 12 \times 77 + 13 \times 78 + 14 \times 79} \\ & 11 \times 76 + 12 \times 77 + 13 \times 78 + 14 \times 79 < 11 \times 100 + 12 \times 100 + 13 \times 100 + 14 \times 100 < 2(11 \times 76 + 12 \times 77 + 13 \times 78 + 14 \times 79) \\ & 1 < \frac{11 \times 77 + 12 \times 78 + 13 \times 79 + 14 \times 80}{11 \times 76 + 12 \times 77 + 13 \times 78 + 14 \times 79} < 2 \\ & 101 < 100 \times \frac{11 \times 77 + 12 \times 78 + 13 \times 79 + 14 \times 80}{11 \times 76 + 12 \times 77 + 13 \times 78 + 14 \times 79} < 102, b = 101 \end{aligned}$$

G1.3 If there are c multiples of 7 between 200 and 500, find the value of c .

$$\frac{200}{7} = 28.6, \text{ the least multiple of 7 is } 7 \times 29 = 203$$

$$\frac{500}{7} = 71.4, \text{ the greatest multiple of 7 is } 7 \times 71 = 497$$

$$497 = a + (c - 1)d = 203 + (c - 1) \cdot 7; c = 43$$

G1.4 Given that $0 \leq x_0 \leq \frac{\pi}{2}$ and x_0 satisfies the equation $\sqrt{\sin x + 1} - \sqrt{1 - \sin x} = \sin \frac{x}{2}$. If $d = \tan x_0$, find the value of d .

$$\left(\sqrt{\sin x + 1} - \sqrt{1 - \sin x} \right)^2 = \left(\sin \frac{x}{2} \right)^2$$

$$1 + \sin x + 1 - \sin x - 2\sqrt{1 - \sin^2 x} = \frac{1 - \cos x}{2}$$

$$2(2 - 2 \cos x) = 1 - \cos x$$

$$\cos x = 1$$

$$x_0 = 0$$

$$d = \tan x_0 = 0$$

Group Event 2**G2.1** If the tens digit of 5^{5^5} is a , find the value of a .

$$5^{5^5} = \dots\dots\dots 125, \text{ the tens digit} = a = 2$$

Remark Original question: If the tenth-place digit $\dots\dots\dots$, this is the position of the first digit to the right of the decimal point.

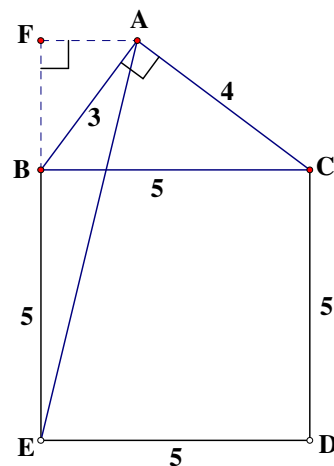
G2.2 In Figure 1, $\triangle ABC$ is a right-angled triangle, $AB = 3$ cm, $AC = 4$ cm and $BC = 5$ cm. If $BCDE$ is a square and the area of $\triangle ABE$ is b cm², find the value of b .

$$\cos B = \frac{AB}{BC} = \frac{3}{5}$$

Height of $\triangle ABE$ from A

$$= AF = AB \sin(90^\circ - B) = 3 \times \frac{3}{5} = \frac{9}{5}$$

$$\begin{aligned} b &= \frac{1}{2} \cdot AF \times BE \\ &= \frac{1}{2} \cdot 5 \times \frac{9}{5} = \frac{9}{2} \end{aligned}$$



G2.3 Given that there are c prime numbers less than 100 such that their unit digits are not square numbers, find the values of c .

The prime are: $\{2, 3, 5, 7, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, 97\}$

$$c = 15$$

G2.4 If the lines $y = x + d$ and $x = -y + d$ intersect at the point $(d - 1, d)$, find the value of d .

$$x = -(x + d) + d$$

$$x = 0 = d - 1$$

$$d = 1$$

Group Event 3

G3.1 If a is the smallest real root of the equation $\sqrt{x(x+1)(x+2)(x+3)+1} = 71$, find the value of a .

Reference: 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2012 FI2.3

Let $t = x + 1.5$, then the equation becomes $\sqrt{(t-1.5)(t-0.5)(t+0.5)(t+1.5)+1} = 71$

$$\sqrt{\left(t^2 - \frac{9}{4}\right)\left(t^2 - \frac{1}{4}\right) + 1} = 71$$

$$\sqrt{t^4 - \frac{5}{2}t^2 + \frac{9}{16}} + 1 = 71$$

$$\Rightarrow \sqrt{t^4 - \frac{5}{2}t^2 + \frac{25}{16}} = 71$$

$$\Rightarrow \sqrt{\left(t^2 - \frac{5}{4}\right)^2} = 71$$

$$t^2 - \frac{5}{4} = 71 \Rightarrow t^2 = \frac{289}{4} \Rightarrow t = \frac{17}{2} \quad \text{or} \quad t = -\frac{17}{2}$$

$$x = t - 1.5 = \pm \frac{17}{2} - \frac{3}{2} = 7 \text{ or } -10$$

$a = \text{the smallest root} = -10$

G3.2 Given that p and q are prime numbers satisfying the equation $18p + 30q = 186$.

If $\log_8 \frac{p}{3q+1} = b \geq 0$, find the value of b .

$$18p + 30q = 186 \Rightarrow 3p + 5q = 31$$

Note that the number “2” is the only prime number which is even.

$$3p + 5q = 31 = \text{odd number} \Rightarrow \text{either } p = 2 \text{ or } q = 2$$

$$\text{If } p = 2, \text{ then } q = 5; b = \log_8 \frac{p}{3q+1} = \log_8 \frac{2}{3 \times 5 + 1} = \log_8 \frac{1}{16} < 0 \text{ (rejected)}$$

$$\text{If } q = 2, \text{ then } p = 7; b = \log_8 \frac{p}{3q+1} = \log_8 \frac{7}{3 \times 2 + 1} = 0$$

G3.3 Given that for any real numbers x, y and z , \oplus is an operation satisfying

$$(i) \quad x \oplus 0 = 1, \text{ and}$$

$$(ii) \quad (x \oplus y) \oplus z = (z \oplus xy) + z.$$

If $1 \oplus 2004 = c$, find the value of c .

$$c = 1 \oplus 2004$$

$$= (1 \oplus 0) \oplus 2004$$

$$= (2004 \oplus 0) + 2004$$

$$= 1 + 2004 = 2005$$

G3.4 Given that $f(x) = (x^4 + 2x^3 + 4x - 5)^{2004} + 2004$. If $f(\sqrt{3}-1) = d$, find the value of d .

$$x = \sqrt{3} - 1$$

$$(x+1)^2 = (\sqrt{3})^2$$

$$x^2 + 2x - 2 = 0$$

$$\text{By division, } x^4 + 2x^3 + 4x - 5 = (x^2 + 2x - 2)(x^2 + 2) - 1 = -1$$

$$d = f(\sqrt{3}-1)$$

$$= f(x)$$

$$= (x^4 + 2x^3 + 4x - 5)^{2004} + 2004$$

$$= (-1)^{2004} + 2004 = 2005$$

Group Event 4

G4.1 If $f(x) = \frac{4^x}{4^x + 2}$ and $P = f\left(\frac{1}{1001}\right) + f\left(\frac{2}{1001}\right) + \cdots + f\left(\frac{1000}{1001}\right)$, find the value of P .

Reference: 2011 HG5, 2012 FI2.2

$$f(x) + f(1-x) = \frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2} = \frac{4 + 4 + 2 \cdot 4^x + 2 \cdot 4^{1-x}}{4 + 4 + 2 \cdot 4^x + 2 \cdot 4^{1-x}} = 1$$

$$\begin{aligned} P &= f\left(\frac{1}{1001}\right) + f\left(\frac{2}{1001}\right) + \cdots + f\left(\frac{1000}{1001}\right) \\ &= f\left(\frac{1}{1001}\right) + f\left(\frac{1000}{1001}\right) + f\left(\frac{2}{1001}\right) + f\left(\frac{999}{1001}\right) + \cdots + f\left(\frac{500}{1001}\right) + f\left(\frac{501}{1001}\right) = 500 \end{aligned}$$

G4.2 Let $f(x) = |x - a| + |x - 15| + |x - a - 15|$, where $a \leq x \leq 15$ and $0 < a < 15$.

If Q is the smallest value of $f(x)$, find the value of Q .

Reference: 1994 HG1, 2001 HG9, 2008 HI8, 2008 FI1.3, 2010 HG6, 2011 FGS.1, 2012 Final G2.3

$$f(x) = x - a + 15 - x + 15 - x + a = 30 - x \geq 30 - 15 = 15 = Q$$

G4.3 If $2^m = 3^n = 36$ and $R = \frac{1}{m} + \frac{1}{n}$, find the value of R .

Reference: 2001 HI1, 2003 FG2.2, 2004 FG4.3, 2005 HI9, 2006 FG4.3

$$\log 2^m = \log 3^n = \log 36$$

$$m \log 2 = n \log 3 = \log 36$$

$$m = \frac{\log 36}{\log 2}; n = \frac{\log 36}{\log 3}$$

$$\frac{1}{m} + \frac{1}{n} = \frac{\log 2}{\log 36} + \frac{\log 3}{\log 36} = \frac{\log 6}{\log 36} = \frac{\log 6}{2 \log 6} = \frac{1}{2}$$

G4.4 Let $[x]$ be the largest integer not greater than x , for example, $[2.5] = 2$.

If $a = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{2004^2}$ and $S = [a]$, find the value of a .

$$\begin{aligned} 1 < a &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{2004^2} < 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{2003 \times 2004} \\ &= 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2003} - \frac{1}{2004} \\ &= 2 - \frac{1}{2004} < 2 \end{aligned}$$

$$S = [a] = 1$$

Group Event (Spare)

GS.1 For all integers n , F_n is defined by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$ and $F_1 = 1$.

If $a = F_{-5} + F_{-4} + \dots + F_4 + F_5$, find the value of a .

$$F_2 = 0 + 1 = 1, F_3 = 1 + 1 = 2, F_4 = 1 + 2 = 3, F_5 = 2 + 3 = 5$$

$$F_{-1} + 0 = 1 \Rightarrow F_{-1} = 1, F_{-2} + 1 = 0 \Rightarrow F_{-2} = -1, F_{-3} + (-1) = 1 \Rightarrow F_{-3} = 2,$$

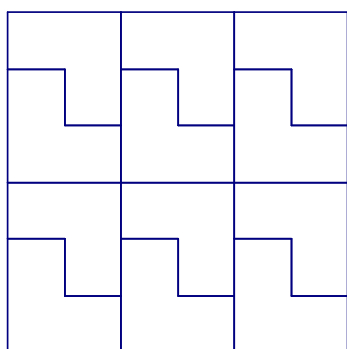
$$F_{-4} + 2 = -1 \Rightarrow F_{-4} = -3, F_{-5} + (-3) = 2 \Rightarrow F_{-5} = 5$$

$$a = F_{-5} + F_{-4} + \dots + F_4 + F_5 = 5 - 3 + 2 - 1 + 1 + 0 + 1 + 1 + 2 + 3 + 5 = 16$$

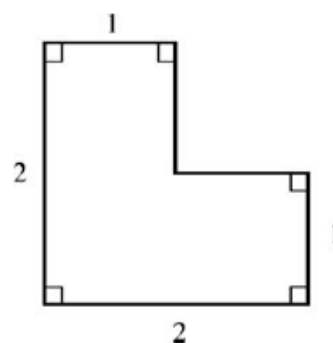
GS.2 Given that x_0 satisfies the equation $x^2 + x + 2 = 0$. If $b = x_0^4 + 2x_0^3 + 3x_0^2 + 2x_0 + 1$, find the value of b .

$$\begin{aligned} \text{By division, } b &= x_0^4 + 2x_0^3 + 3x_0^2 + 2x_0 + 1 \\ &= (x_0^2 + x_0 + 2)(x_0^2 + x_0) + 1 = 1 \end{aligned}$$

GS.3 Figure 1 shows a tile. If C is the minimum number of tiles required to tile a square, find the value of C .



$$C = 12$$



GS.4 If the line $5x + 2y - 100 = 0$ has d points whose x and y coordinates are both positive integers, find the value of d .

$$(x, y) = (18, 5), (16, 10), (14, 15), (12, 20), (10, 25), (8, 30), (6, 35), (4, 40), (2, 45)$$

$$d = 9$$