Hong Kong Mathematics Olympiad (1999-2000)

Final Event (Individual) Example

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 對任意整數 m 及 n，$m \otimes n$ 之定義如下：$m \otimes n = m^n + n^m$。
若 $2 \otimes P = 100$，求 P 之值。
For all integers m and n, $m \otimes n$ is defined as $m \otimes n = m^n + n^m$.
If $2 \otimes P = 100$, find the value of P.

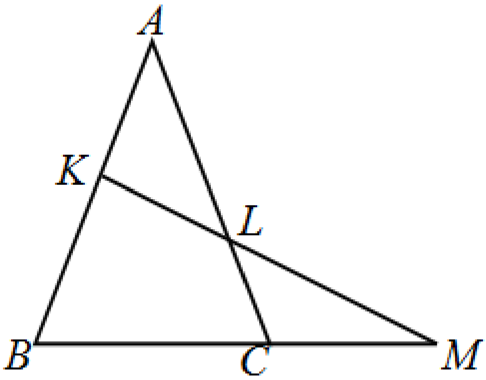
$P =$

(ii) 若 $\sqrt[3]{13Q+6P+1} - \sqrt[3]{13Q-6P-1} = \sqrt[3]{2}$，其中 $Q > 0$，求 Q 之值。
If $\sqrt[3]{13Q+6P+1} - \sqrt[3]{13Q-6P-1} = \sqrt[3]{2}$, where $Q > 0$, find the value of Q.

$Q =$

(iii) 在圖一，$AB = AC$ 和 $KL = LM$。若 $LC = Q - 6$ cm 及 $KB = R$ cm，求 R 之值。
In figure 1, $AB = AC$ and $KL = LM$. If $LC = Q - 6$ cm and $KB = R$ cm, find the value of R.

$R =$



圖一 Figure 1

(iv) 數列$\{a_n\}$的定義如下：$a_1 = R$，$a_{n+1} = a_n + 2n$ $(n \geq 1)$。若 $a_{100} = S$，求 S 之值。
The sequence $\{a_n\}$ is defined as $a_1 = R$, $a_{n+1} = a_n + 2n$ $(n \geq 1)$.
If $a_{100} = S$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐

+ Bonus score ☐

Total score ☐

Team No. ☐

Time ☐ Min. ☐ Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 $[x]$ 表示小數 x 的整數部份。

已知 $[3.126]+[3.126+\frac{1}{8}]+[3.126+\frac{2}{8}]+\ldots+[3.126+\frac{7}{8}]=P$，求 P 的值。

Let $[x]$ represents the integral part of the decimal number x. Given that

$[3.126]+[3.126+\frac{1}{8}]+[3.126+\frac{2}{8}]+\ldots+[3.126+\frac{7}{8}]=$ P, find the value of P.

$P =$

(ii) 設 $a+b+c=0$。已知 $\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}=P-3Q$，求 Q 的值。

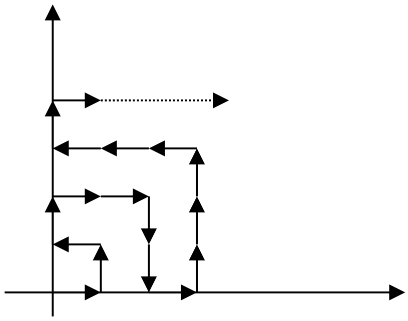
Let $a+b+c=0$. Given that $\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}=P-3Q$,

find the value of Q.

$Q =$

(iii) 在直角座標平面的第一象限中，把座標為整數的點按以下方法編號：

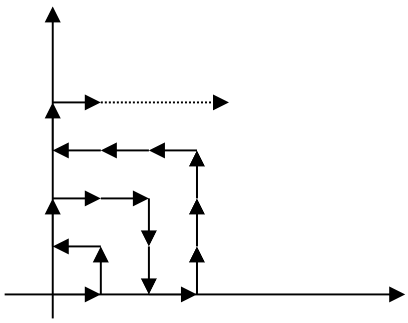
點 (0, 0) 為第 1 號，
點 (1, 0) 為第 2 號，
點 (1, 1) 為第 3 號，
點 (0, 1) 為第 4 號，
點 (0, 2) 為第 5 號，
點 (1, 2) 為第 6 號，
點 (2, 2) 為第 7 號，
點 (2, 1) 為第 8 號，
…………………………



已知 $(Q-1, Q)$ 點為第 R 號，求 R 的值。

$R =$

In the first quadrant of the rectangular co-ordinate plane, all integral points are numbered as follows,
point (0, 0) is numbered as 1,
point (1, 0) is numbered as 2,
point (1, 1) is numbered as 3,
point (0, 1) is numbered as 4,
point (0, 2) is numbered as 5,
point (1, 2) is numbered as 6,
point (2, 2) is numbered as 7,
point (2, 1) is numbered as 8,
………………………………



Given that point $(Q-1, Q)$ is numbered as R, find the value of R.

(iv) 當 $x+y=4$ 時，$3x^2+y^2$ 的最小值為 $\frac{R}{S}$，求 S 的值。

When $x+y=4$, the minimum value of $3x^2+y^2$ is $\frac{R}{S}$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event 2 (Individual)

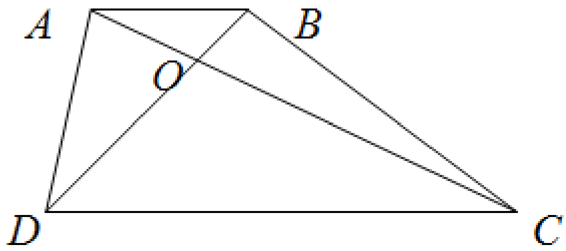
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如果 $\log_2(\log_4 P) = \log_4(\log_2 P)$ 及 $P \neq 1$，求 P 的值。
If $\log_2(\log_4 P) = \log_4(\log_2 P)$ and $P \neq 1$, find the value of P.

$P =$

(ii) 在梯形 $ABCD$ 中，$AB \parallel DC$。AC 和 BD 相交於 O。三角形 AOB 和 COD 的面積分別為 P 和 25。已知梯形的面積為 Q，求 Q 的值。

$Q =$



In the trapezium $ABCD$, $AB \parallel DC$. AC and BD intersect at O. The areas of triangles AOB and COD are P and 25 respectively. Given that the area of the trapezium is Q, find the value of Q.

(iii) 當 1999^Q 被 7 除時，餘數為 R。求 R 的值。
When 1999^Q is divided by 7, the remainder is R. Find the value of R.

$R =$

(iv) 如果 $111111111111 - 222222 = (R + S)^2$，求正數 S 的值。
If $111111111111 - 222222 = (R + S)^2$, find the positive value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $1 + 2 + 3 + \cdots + 1997 + 1998 + 1999 + 1998 + 1997 + \cdots + 3 + 2 + 1$ 的個位數是 P，求 P 的值。
Given that the units digit of $1+2+3+\ldots+1997+1998+1999+1998+1997+\ldots+3+2+1$ is P, find the value of P.

$P =$

(ii) 已知 $x + \frac{1}{x} = P$。如果 $x^6 + \frac{1}{x^6} = Q$，求 Q 的值。
Given that $x + \frac{1}{x} = P$. If $x^6 + \frac{1}{x^6} = Q$, find the value of Q.

$Q =$

(iii) 已知 $\frac{Q}{\sqrt{Q}+\sqrt{2Q}} + \frac{Q}{\sqrt{2Q}+\sqrt{3Q}} + \cdots + \frac{Q}{\sqrt{1998Q}+\sqrt{1999Q}} = \frac{R}{\sqrt{Q}+\sqrt{1999Q}}$，
求 R 的值。
Given that
$$\frac{Q}{\sqrt{Q}+\sqrt{2Q}} + \frac{Q}{\sqrt{2Q}+\sqrt{3Q}} + \cdots + \frac{Q}{\sqrt{1998Q}+\sqrt{1999Q}} = \frac{R}{\sqrt{Q}+\sqrt{1999Q}},$$
find the value of R.

$R =$

(iv) 設 $f(0) = 0$；$f(n) = f(n-1) + 3$ 當 $n = 1$、2、3、4、$\cdots$。
如果 $2f(S) = R$，求 S 的值。
Let $f(0) = 0$; $f(n) = f(n-1) + 3$ when $n = 1, 2, 3, 4, \cdots$.
If $2f(S) = R$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

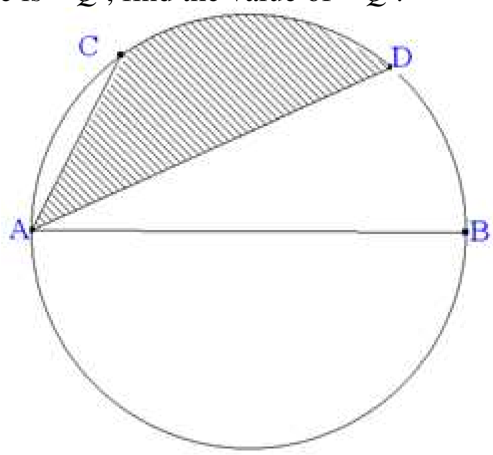
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 假設 $a+\frac{1}{a+1}=b+\frac{1}{b-1}-2$，其中 $a\neq -1$，$b\neq 1$ 和 $a-b+2\neq 0$。
已知 $ab-a+b=P$，求 P 的值。
Suppose $a+\frac{1}{a+1}=b+\frac{1}{b-1}-2$, where $a\neq -1$, $b\neq 1$, and $a-b+2\neq 0$.
Given that $ab-a+b=P$, find the value of P.

$P =$

(ii) 在下圖中，AB 為圓的直徑。C 和 D 把弧 AB 分為三等份。斜綫面積為 P。
若圓的面積為 Q，求 Q 的值。
In the following figure, AB is a diameter of the circle. C and D divide the arc AB into three equal parts. The shaded area is P.
If the area of the circle is Q, find the value of Q.

$Q =$



(iii) 已知兩個 Q 位數 $1111\cdots11$ 和 $9999\cdots99$ 的乘積中有 R 個數字是奇數，
求 R 的值。
Given that there are R odd numbers in the digits of the product of the two Q-digit numbers $1111\cdots11$ and $9999\cdots99$, find the value of R.

$R =$

(iv) 設 a_1、a_2、$\cdots$、a_R 為正整數，其中 $a_1 < a_2 < a_3 <\cdots< a_{R-1} < a_R$。
已知這 R 個正整數的和為 90 及 a_1 的最大值為 S，求 S 的值。
Let $a_1, a_2, \cdots, a_R$ be positive integers such that $a_1 < a_2 < a_3 <\cdots< a_{R-1} < a_R$.
Given that the sum of these R integers is 90 and the maximum value of a_1 is S, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如果 $\left(\dfrac{1\times2\times4+2\times4\times8+3\times6\times12+\cdots+1999\times3998\times7996}{1^3+2^3+3^3+\cdots+1999^3}\right)^{\frac{1}{3}}=P$，求 P 的值。

If $\left(\dfrac{1\times2\times4+2\times4\times8+3\times6\times12+\cdots+1999\times3998\times7996}{1^3+2^3+3^3+\cdots+1999^3}\right)^{\frac{1}{3}}=P$,
find the value of P.

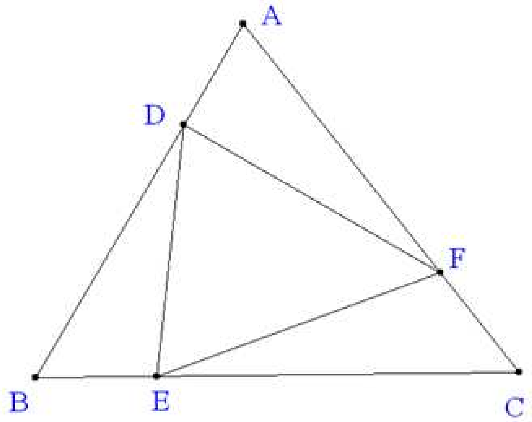
$P =$

(ii) 如果 $(x-P)(x-2Q)-1=0$ 有兩個整數根，求 Q 的值。
If $(x-P)(x-2Q)-1=0$ has two integral roots, find the value of Q.

$Q =$

(iii) 已知 ΔABC 的面積為 $3Q$；D、E 和 F 分別為 AB、BC 和 CA 上的點使得 $AD=\frac{1}{3}AB$，$BE=\frac{1}{3}BC$，$CF=\frac{1}{3}CA$。如果 ΔDEF 的面積為 R，求 R 的值。
Given that the area of the ΔABC is $3Q$; D, E and F are the points on AB, BC and CA respectively such that $AD=\frac{1}{3}AB$, $BE=\frac{1}{3}BC$, $CF=\frac{1}{3}CA$.
If the area of ΔDEF is R, find the value of R.

$R =$



(iv) 已知 $(Rx^2-x+1)^{1999}\equiv a_0+a_1x+a_2x^2+\cdots+a_{3998}x^{3998}$。
設 $S=a_0+a_1+a_2+\cdots+a_{3997}$，求 S 的值。
Given that $(Rx^2-x+1)^{1999}\equiv a_0+a_1x+a_2x^2+\cdots+a_{3998}x^{3998}$.
If $S=a_0+a_1+a_2+\cdots+a_{3997}$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event (Group) Example

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 $x * y = x + y - xy$，其中 x, y 為實數，若 $a = 1 * (0 * 1)$，求 a 之值。
Let $x * y = x + y - xy$, where x, y are real numbers.
If $a = 1 * (0 * 1)$, find the value of a.

$a =$

(ii) 在圖一，AB 平行於 DC，$\angle ACB$ 為一直角，$AC = CB$ 及 $AB = BD$.，
若 $\angle CBD = b°$，求 b 之值。
In figure 1, AB is parallel to DC, $\angle ACB$ is a right angle, $AC = CB$ and $AB = BD$.
If $\angle CBD = b°$, find the value of b.

$b =$

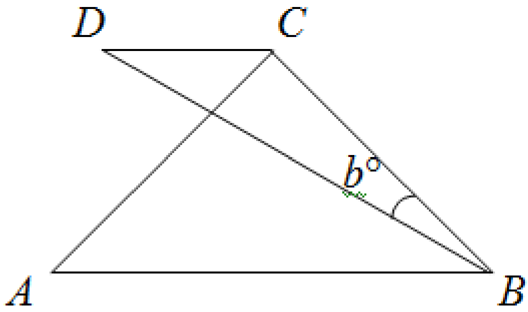


Figure 1 圖一

(iii) 設 x、y 為非零實數，若 x 是 y 的 250%，而 $2y$ 是 x 的 c %，求 c 之值。
Let x, y be non-zero real numbers.
If x is 250% of y and $2y$ is $c\%$ of x, find the value of c.

$c =$

(iv) 若 $\log_p x = 2$，$\log_q x = 3$，$\log_r x = 6$ 及 $\log_{pqr} x = d$，求 d 之值。
If $\log_p x = 2$, $\log_q x = 3$, $\log_r x = 6$ and $\log_{pqr} x = d$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)

Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知整數 n 除 81849、106392 及 124374 得出的餘數相等，求 n 的最大值 a。
Given that when 81849, 106392 and 124374 are divided by an integer n, the remainders are equal. If a is the maximum value of n, find a.

$a =$

(ii) 設 $x=\frac{1-\sqrt{3}}{1+\sqrt{3}}$ 及 $y=\frac{1+\sqrt{3}}{1-\sqrt{3}}$。如果 $b=2x^2-3xy+2y^2$，求 b 的值。
Let $x=\frac{1-\sqrt{3}}{1+\sqrt{3}}$ and $y=\frac{1+\sqrt{3}}{1-\sqrt{3}}$. If $b=2x^2-3xy+2y^2$, find the value of b.

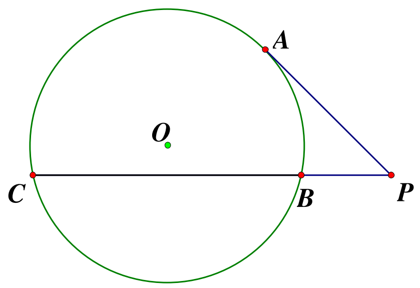
$b =$

(iii) 已知 c 為正數，如果只有一條直綫穿過點 A(1, c)且與曲綫 C: $x^2+y^2-2x-2y-7=0$ 相交於一點，求 c 的值。
Given that c is a positive number. If there is only one straight line which passes through point A(1, c) and meets the curve C: $x^2+y^2-2x-2y-7=0$ at only one point, find the value of c.

$c =$

(iv) 在圖一，PA 切圓於 A，O 為圓心。如果 $PA=6$，$BC=9$，$PB=d$，求 d 的值。
In Figure 1, PA touches the circle with centre O at A. If $PA=6$, $BC=9$, $PB=d$, find the value of d.

$d =$



圖一
Figure 1

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

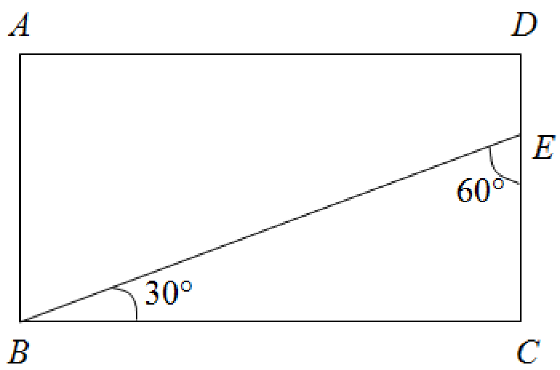
(i) 如果 191 為兩個連續平方數之差，而 a 為其中最小的平方數，求 a 的值。
If 191 is the difference of two consecutive perfect squares, find the value of the smallest square number, a.

$a =$

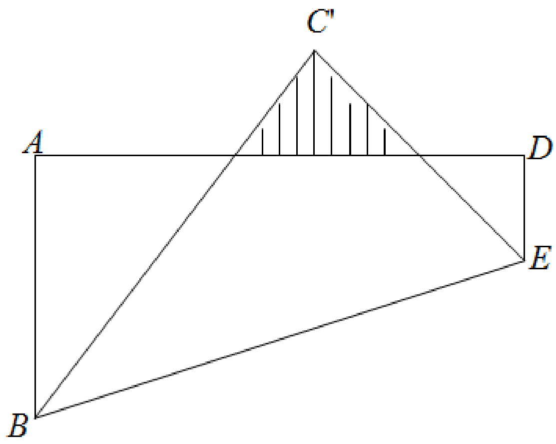
(ii) 在圖二(a)，$ABCD$ 是一長方形。$DE : EC = 1 : 5$，且 $DE = 12^{\frac{1}{4}}$。ΔBCE 沿 BE 摺去另一方。設 b 為圖二(b)中陰影部份的面積，求 b 的值。

In Figure 2(a), $ABCD$ is a rectangle. $DE:EC = 1: 5$, and $DE = 12^{\frac{1}{4}}$. ΔBCE is folded along the side BE. If b is the area of the shaded part as shown in Figure 2(b), find the value of b.

$b =$



圖二(a) Figure 2(a)



圖二(b) Figure 2(b)

(iii) 設曲綫 $y = x^2 - 7x + 12$ 與 x 軸的交點為 A 及 B，而與 y 軸的交點為 C。如果 c 是 ΔABC 的面積，求 c 的值。
Let the curve $y = x^2 - 7x + 12$ intersect the x-axis at points A and B, and intersect the y-axis at C. If c is the area of ΔABC, find the value of c.

$c =$

(iv) 設 $f(x) = 41x^2 - 4x + 4$，$g(x) = -2x^2 + x$。如果 $f(x) + kg(x) = 0$ 只有一個根，求 k 的最小值 d。
Let $f(x) = 41x^2 - 4x + 4$ and $g(x) = -2x^2 + x$. If d is the smallest value of k such that $f(x) + kg(x) = 0$ has a single root, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)

Final Event 3 (Group)

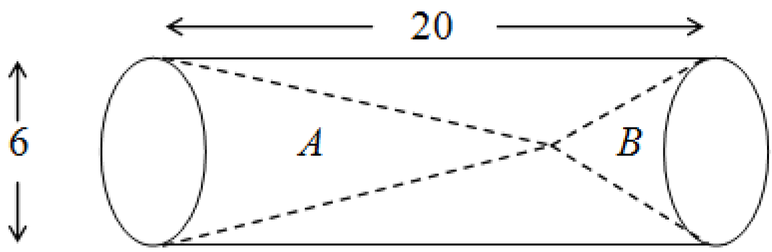
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 $a=\sqrt{1997\times1998\times1999\times2000+1}$，求 a 的值。
Let $a=\sqrt{1997\times1998\times1999\times2000+1}$, find the value of a.

$a =$

(ii) 在圖三，圓管的長為 20 及直徑為 6，內有兩個圓錐體 A 和 B。A 及 B 的體積比例為 3 : 1。如果 b 是 B 的高度，求 b 的值。
In Figure 3, A and B are two cones inside a cylindrical tube with length of 20 and diameter of 6. If the volumes of A and B are in the ratio 3:1 and b is the height of the cone B, find the value of b.

$b =$



圖三 Figure 3

(iii) 現有點 $A\left(\frac{\sqrt{10}}{2},\frac{\sqrt{10}}{2}\right)$ 和圓 $C: x^2+y^2=1$。
如果 c 是通過點 A 與圓相切直綫的最大斜率，求 c 的值。
If c is the largest slope of the tangents from the point $A\left(\frac{\sqrt{10}}{2},\frac{\sqrt{10}}{2}\right)$ to the circle C: $x^2+y^2=1$, find the value of c.

$c =$

(iv) 在座標平面的原點上有一點 P。假如擲出骰子的點數 n 是偶數，P 在 x 方向右前進 n；如果 n 是奇數，P 在 y 方向上前進 n。如果有 d 種不同擲法使得 P 到達點 (4, 4)，求 d 的值。
P is a point located at the origin of the coordinate plane. When a dice is thrown and the number n shown is even, P moves to the right by n. If n is odd, P moves upward by n. Find the value of d, the total number of tossing sequences for P to move to the point (4, 4).

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)

Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如果 a 是一個三位數，駁在 504 之後，新組成的六位數可被 7、9、11 整除，求 a 的值。
Let a be a 3-digit number. If the 6-digit number formed by putting a at the end of the number 504 is divisible by 7, 9, and 11, find the value of a.

a =

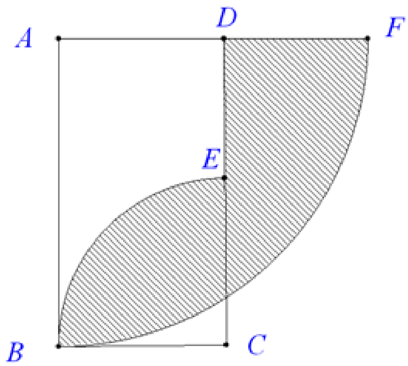
(ii) 在圖四，$ABCD$ 為長方形，

$$AB=\sqrt{\frac{8+\sqrt{64-\pi^2}}{\pi}}，BC=\sqrt{\frac{8-\sqrt{64-\pi^2}}{\pi}}。$$

BE、BF 分別是以 C、A 為圓心的弧。
若 b 是陰影部份之面積，求 b 的值。
In Figure 4, $ABCD$ is a rectangle with

$$AB=\sqrt{\frac{8+\sqrt{64-\pi^2}}{\pi}} \text{ and } BC=\sqrt{\frac{8-\sqrt{64-\pi^2}}{\pi}}.$$

BE and BF are the arcs of circles with centres at C and A respectively. If b is the total area of the shaded parts, find the value of b.

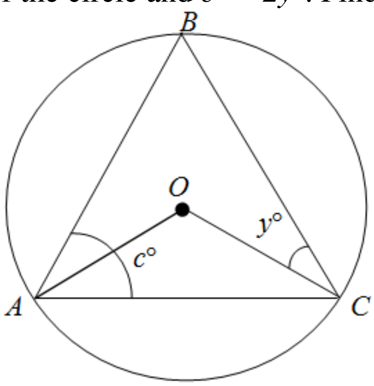


圖四 Figure 4

b =

(iii) 在圖五，O 為圓心，$c° = 2y°$，求 c 的值。
In Figure 5, O is the centre of the circle and $c° = 2y°$. Find the value of c.

c =



圖五 Figure 5

(iv) A、B、C、D、E、F、G 七個人圍圓桌而坐。
如果 B 及 G 都與 C 相鄰而坐的坐法總數為 d，求 d 的值。
A, B, C, D, E, F, G are seven people sitting around a circular table.
If d is the total number of ways that B and G must sit next to C, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

Team No.

\+ Bonus score

Time

Total score

Min. Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)

Final Event 5 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如果 a 是可被 810 整除的最小立方數，求 a 的值。
If a is the smallest cubic number divisible by 810, find the value of a.

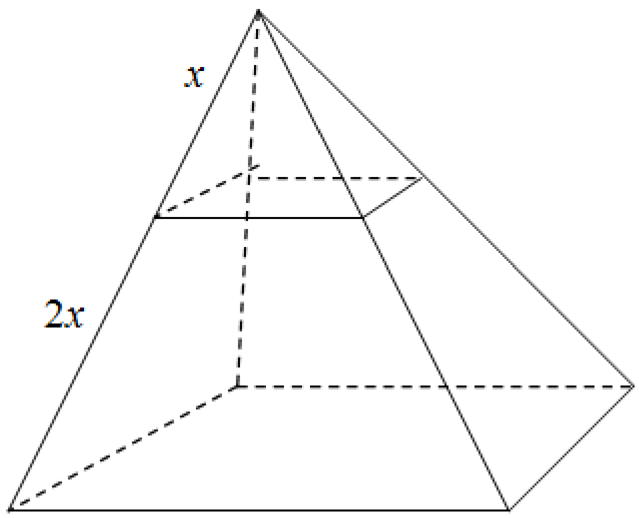
$a =$

(ii) 設 b 是函數 $y = |x^2 - 4| - 6x$（其中 $-2 \le x \le 5$）的最大值，求 b 的值。
Let b be the maximum of the function $y = |x^2 - 4| - 6x$ (where $-2 \le x \le 5$), find the value of b.

$b =$

(iii) 圖六為一個正方形底的錐體。若從底部向上並在 $\frac{2}{3}$ 之高度平行橫切，並設 $1 : c$ 為上面細錐與餘下底部體積的比，求 c 的值。
In Figure 6, a square-based pyramid is cut into two shapes by a cut running parallel to the base and made $\frac{2}{3}$ of the way up. Let $1 : c$ be the ratio of the volume of the small pyramid to that of the truncated base, find the value of c.

$c =$



圖六 Figure 6

(iv) 如果 $\cos^6\theta + \sin^6\theta = 0.4$，及 $d = 2 + 5\cos^2\theta\sin^2\theta$，求 d 的值。
If $\cos^6\theta + \sin^6\theta = 0.4$ and $d = 2 + 5\cos^2\theta\sin^2\theta$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.