Platonic solids

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I Euler's Formula on Graph Theory

Let G be a connected plane graph with V vertices, E edges and F faces, then $V - E + F = 2$.

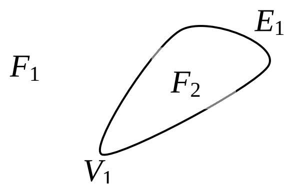
Proof: Case 1: Suppose G has no cycle. Hence G is a tree.

$E = V - 1$ (every vertices are connected.) $F = 1$ (G has no cycle.)

$V - E + F = V - (V - 1) + 1 = 2$

Case 2: Suppose G has a cycle, use induction on the number of edges.

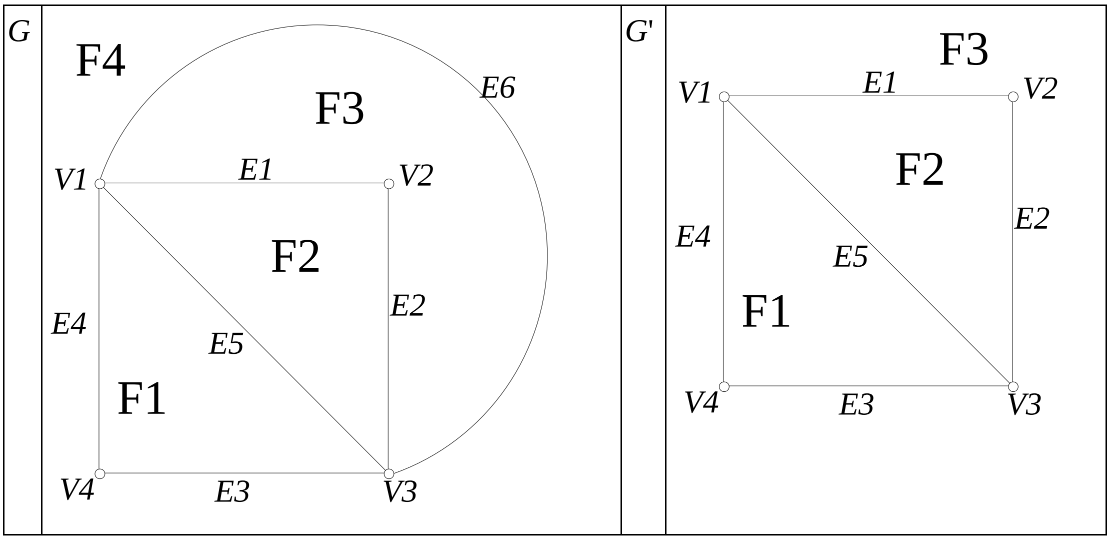
$E = 1 \Rightarrow V = 1, F = 2 \Rightarrow V - E + F = 1 - 1 + 2 = 2.$



It is true for $E = 1$

Suppose the formula to true for some positive integer k ($E = k$ edges).

For a connected plane graph with $k + 1$ edges (with at least one cycle). Remove an edge from this cycle to obtain a new graph called G'. (The following graph is an example.)



G' is a connected plane graph with one edge less. ($E' = k$)

$V' = V$, $F' = F - 1$

$V' - E' + F' = 2$ (by induction assumption)

$V - k + (F - 1) = 2$

$V - (k + 1) + F = 2$

$V - E + F = 2$

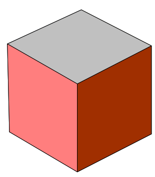
The formula is still true for $E = k + 1$.

By the principle of Mathematical Induction, Euler's formula is true for all number of edges.

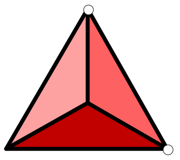
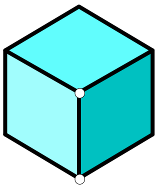
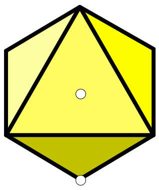
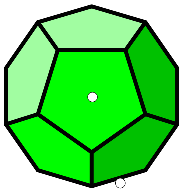
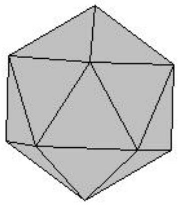
Example on Euler's formula

Given a cube, $V = 8$, $E = 12$, $F = 6$.

then $V - E + F = 2$.



Theorem There are only five Platonic solids. Each face is formed by **identical regular polygons**. They are **tetrahedron**, **cube** (= **hexahedron**), **octahedron**, **icosahedron** and **dodecahedron**.

				
Tetrahedron	**Cube**	**Octahedron**	**Dodecahedron**	**Icosahedron**

We shall prove the **existence and uniqueness of Platonic solids** by Euler's formula.

On each vertex, there are m edges ($m \geq 3$). There are V vertices, so altogether Vm edges. Each edge is connected by two vertices $\Rightarrow Vm = 2E$ ....(1)

Each face is a regular polygon with n sides (edges) ($n \geq 3$). There are F faces, so altogether nF edges. Each edge is shared by two faces $\Rightarrow nF = 2E$ ....(2)

From (1) and (2), substitute V and F in terms of E, m, n into Euler's Formula: $V - E + F = 2$

$$\frac{2E}{m} - E + \frac{2E}{n} = 2$$

$$E\left(\frac{2}{m} + \frac{2}{n} - 1\right) = 2$$

$E(2m + 2n - mn) = 2mn$ ....(3)

$\because E$ is a positive integer and also RHS = $2mn$ is a positive integer

$\therefore 2m + 2n - mn$ is a positive integer.

$2m + 2n - mn = 4 - (m - 2)(n - 2)$ which is a positive integer.

$\because m - 2 > 0$ and $n - 2 > 0$,

Possible values of $(m - 2)(n - 2)$ may be 1, 2 or 3

When $(m - 2)(n - 2) = 1$,

$m - 2 = 1$, $n - 2 = 1$

$m = 3$, $n = 3$

Sub. into (3): $E(6 + 6 - 9) = 2(3)(3) \Rightarrow E = 6$

$$V = \frac{2E}{m} = \frac{2\times 6}{3} = 4$$

$F = \dfrac{2E}{n} = 4$, the solid is a **tetrahedron**, **each face is an equilateral triangle**. (**n = 3**)

When $(m - 2)(n - 2) = 2$,

$m - 2 = 2$, $n - 2 = 1$ or $m - 2 = 1$, $n - 2 = 2$

$m = 4$, $n = 3$ or $m = 3$, $n = 4$

Sub. into (3): $E(8 + 6 - 12) = 2(3)(4) \Rightarrow E = 12$

when $m = 3$, $n = 4$

$$V = \frac{2E}{m} = \frac{2\times 12}{3} = 8$$

$F = \dfrac{2E}{n} = 6$, **the solid is a cube, each face is a square.** (**$n = 4$**)

when $m = 4$, $n = 3$

$V = \dfrac{2E}{m} = \dfrac{2\times 12}{4} = 6$

$F = \dfrac{2E}{n} = 8$, the solid is an **octahedron**, **each face is an equilateral triangle.** (**$n = 3$**)

When $(m - 2)(n - 2) = 3$,

$m - 2 = 1$, $n - 2 = 3$ or $m - 2 = 3$, $n - 2 = 1$

$m = 3$, $n = 5$ or $m = 5$, $n = 3$

Sub. into (3): $E(6 + 10 - 15) = 2(3)(5) \Rightarrow E = 30$

when $m = 3$, $n = 5$

$V = \dfrac{2E}{m} = \dfrac{2\times 30}{3} = 20$

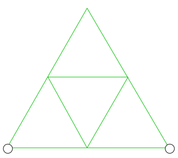
$F = \dfrac{2E}{n} = 12$, the solid is a **dodecahedron, each face is a regular pentagon.** (**$n = 5$**)

when $m = 5$, $n = 3$

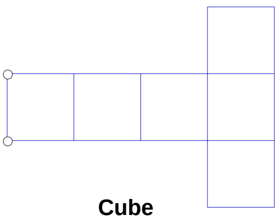
$V = \dfrac{2E}{m} = \dfrac{2\times 30}{5} = 12$

$F = \dfrac{2E}{n} = 20$, the solid is an **icosahedron**, **each face is an equilateral triangle.** (**$n = 3$**)

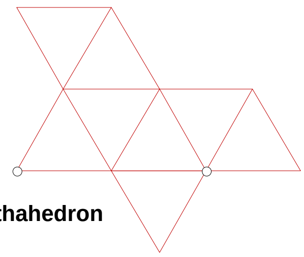
The following diagrams are **the nets (plane developments) of the five Platonic solids**.



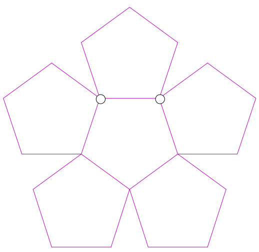
Tetrahedron



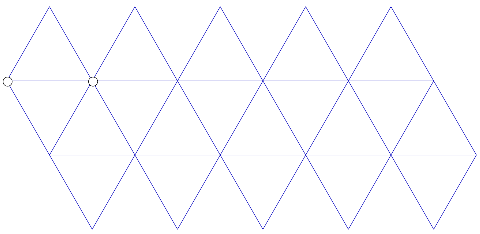
Cube



Octhahedron



Dodecahedron (two needed)



Icosahedron

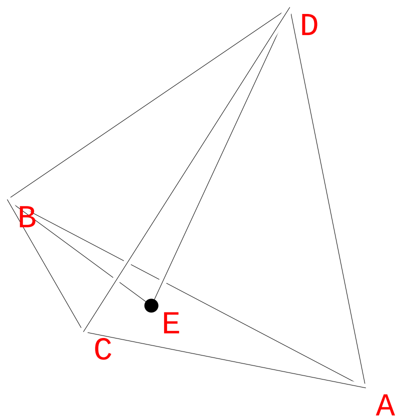
Let one side of the Platonic solids be a. **Express the volumes of the figures in terms of a.**

Let $ABCD$ be a regular tetrahedron with side a.

DE is the altitude from D onto the plane ABC.

E is the centroid of ΔABC.

It is easy to find the length of median of ΔABC.



median $= a \sin 60^\circ = \dfrac{\sqrt{3}}{2} a$

$BE = \dfrac{2}{3}$ median $= \dfrac{\sqrt{3}}{3} a$

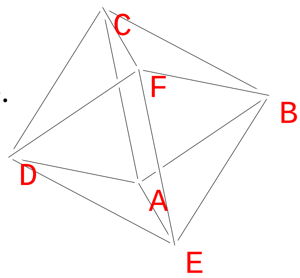
$$DE = \sqrt{BD^2 - BE^2} = \sqrt{a^2 - \frac{a^2}{3}} = \sqrt{\frac{2}{3}}\, a$$

$$\text{Volume} = \frac{1}{3}\text{base area} \times \text{height} = \frac{1}{3} \times \frac{1}{2} a^2 \sin 60^\circ \times \sqrt{\frac{2}{3}} a = \frac{\sqrt{2}}{12} a^3$$

For a square with side a, the volume is a^3

For a regular octahedron with side a,

it can be cut into two identical right pyramids with square base.



In the figure, $BD = \sqrt{a^2 + a^2} = \sqrt{2}\, a$

$$\text{Height} = \sqrt{a^2 - \left(\frac{\sqrt{2}a}{2}\right)^2} = \frac{\sqrt{2}}{2} a$$

$$\text{Total volume} = 2 \times \frac{1}{3} \times a^2 \times \frac{\sqrt{2}}{2} a = \frac{\sqrt{2}}{3} a^3$$

The following figure shows a regular dodecahedron and a regular face:

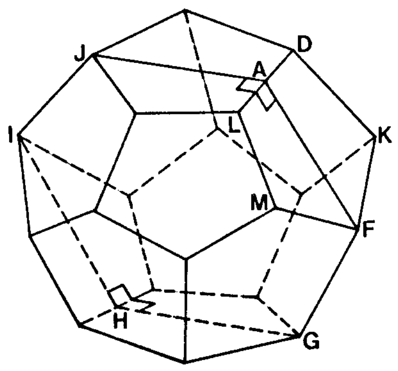


Figure 1 (a)

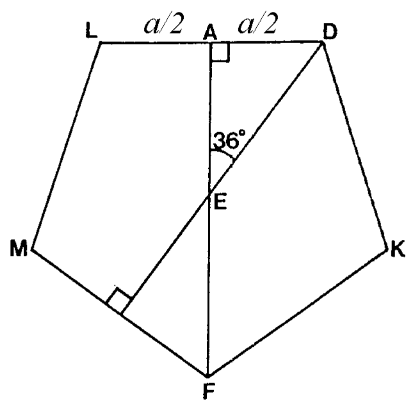


Figure 1 (b)

Let the angle between two adjacent sides be 2θ. We can use figure 1(a) and figure 1(b) to find θ.

In figure 1(b), ΔAED, $AE = \dfrac{a}{2\tan 36^\circ}$

$$EF = ED = \frac{a}{2\sin 36^\circ}$$

$$AF = AE + EF = \frac{a}{2\tan 36^\circ} + \frac{a}{2\sin 36^\circ}$$

$$= \frac{a}{2}\left(\frac{1+\cos 36^\circ}{\sin 36^\circ}\right) = \frac{a}{2\tan 18^\circ}$$, by using the formula $t = \tan\dfrac{\theta}{2}$.

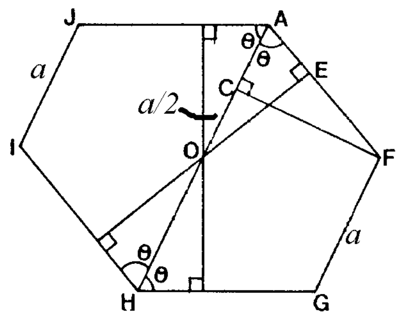


Figure 1(c)

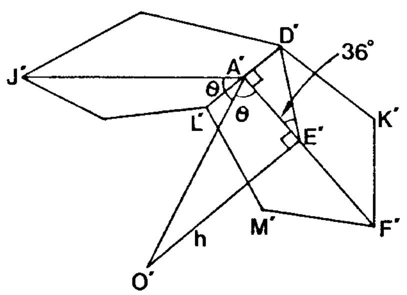


Figure 1(d)

In figure 1(c), which is the cross section of figure 1(a),

$$\cos\theta = \frac{AE}{AO} = \frac{AE}{AC+CO} = \frac{AE}{AF\cos\theta + CO} = \frac{\frac{a}{2\tan 36^\circ}}{\frac{a\cos\theta}{2\tan 18^\circ} + \frac{a}{2}} = \frac{\tan 18^\circ}{\tan 36^\circ \cos\theta + \tan 36^\circ \tan 18^\circ}$$

$(\tan 36^\circ)\cos^2\theta + (\tan 36^\circ \tan 18^\circ)\cos\theta - \tan 18^\circ = 0$

$$\cos\theta = \frac{-\tan 18^\circ \tan 36^\circ \pm \sqrt{\left(\tan 18^\circ \tan 36^\circ\right)^2 + 4\tan 18^\circ \tan 36^\circ}}{2\tan 36^\circ} = -\frac{1}{2}\left(\tan 18^\circ \pm \tan 54^\circ\right)$$

$\cos\theta > 0 \Rightarrow \cos\theta = \dfrac{1}{2}\left(\tan 54^\circ - \tan 18^\circ\right)$

$$= \frac{1}{2}\left(\frac{\sin 54^\circ}{\cos 54^\circ} - \frac{\sin 18^\circ}{\cos 18^\circ}\right) = \frac{1}{2}\left(\frac{\sin 54^\circ \cos 18^\circ - \cos 54^\circ \sin 18^\circ}{\cos 54^\circ \cos 18^\circ}\right)$$

$$= \frac{1}{2}\cdot\frac{\sin\left(54^\circ - 18^\circ\right)}{\cos 54^\circ \cos 18^\circ} = \frac{1}{2\cos 18^\circ}$$

$\theta = 58^\circ 17'$; $2\theta = 116.6^\circ$, the angle between two adjacent sides is 116.6°.

$$\tan\theta=\sqrt{\sec^2\theta-1}=\sqrt{\left(2\cos 18^\circ\right)^2-1}$$

$$=\sqrt{2\left(2\cos^2 18^\circ-1\right)+1}$$

$$=\sqrt{2\cos 36^\circ+1}$$

$$=\sqrt{2\times\frac{1+\sqrt{5}}{4}+1}$$

$$=\sqrt{\frac{6+2\sqrt{5}}{4}}$$

$$=\sqrt{\frac{\left(\sqrt{5}+1\right)^2}{2^2}}$$

$$=\frac{\sqrt{5}+1}{2}$$

A regular dodecahedron can be regarded as 12 identical right pyramids with regular pentagonal base, side = a. If the area of the pentagonal base is B and the height of the cone is h, then the volume of the regular dodecahedron is

$$12\left(\frac{1}{3}\right)Bh = 4\times 5\times\frac{1}{2}\times a\times AE\times OE$$

$$= 10a\times\frac{a}{2\tan 36^\circ}\times AE\tan\theta$$

$$= 10a\times\left(\frac{a}{2\tan 36^\circ}\right)^2\times\tan\theta$$

$$=\frac{5a^3}{2\left(\sec^2 36^\circ-1\right)}\times\frac{\sqrt{5}+1}{2}$$

$$=\frac{5\left(\sqrt{5}+1\right)a^3}{4\left[\left(\frac{4}{\sqrt{5}+1}\right)^2-1\right]}$$

$$=\frac{5\left(\sqrt{5}+1\right)^3a^3}{4\left[16-\left(\sqrt{5}+1\right)^2\right]}$$

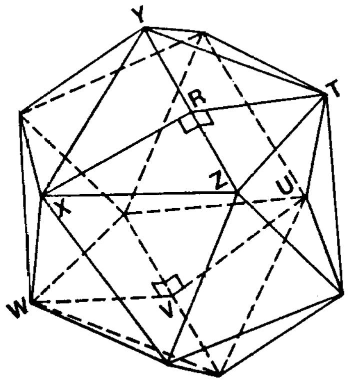
$$=\frac{5\left(5\sqrt{5}+3\times 5+3\times\sqrt{5}+1\right)a^3}{4\left[16-\left(6+2\sqrt{5}\right)\right]}$$

$$=\frac{5\left(8\sqrt{5}+16\right)a^3}{4\left(10-2\sqrt{5}\right)}$$

$$=\frac{5\left(\sqrt{5}+2\right)\left(5+\sqrt{5}\right)a^3}{\left(5-\sqrt{5}\right)\left(5+\sqrt{5}\right)}$$

$$=\frac{\left(15+7\sqrt{5}\right)a^3}{4}$$

The following figure shows a regular icosahedron and a regular face:



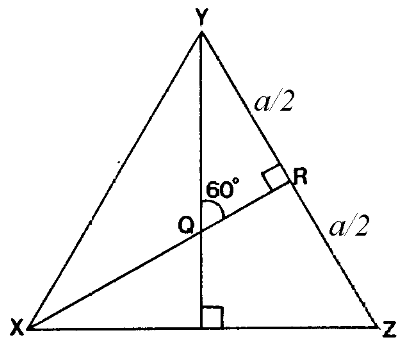


Figure 2 (*a*) Figure 2 (*b*)

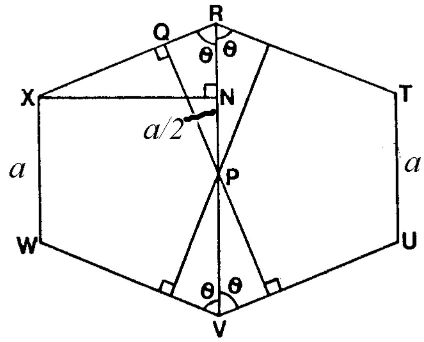
Let the angle between two adjacent sides be 2θ. We can use figure 2(*a*) and figure 2(*b*) to find θ.

In figure 2(*b*), Δ*YQR*, $RQ = \dfrac{a}{2\tan 60^\circ} = \dfrac{a}{2\sqrt{3}}$

$QX = QY$

$= \dfrac{a}{2\sin 60^\circ} = \dfrac{a}{\sqrt{3}}$

$RX = \dfrac{a}{2}\tan 60^\circ = \dfrac{\sqrt{3}}{2}a$



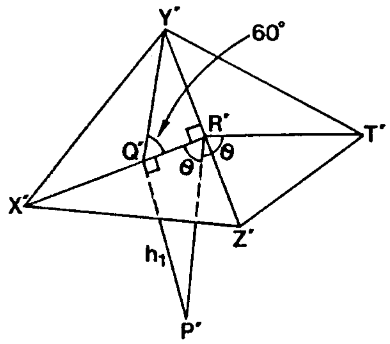


Figure 2(*c*) Figure 2(*d*)

In the figure 2(*c*), $\cos\theta = \dfrac{RQ}{RP}$

$= \dfrac{RQ}{RN + NP}$

$= \dfrac{RQ}{RX\cos\theta + NP}$

$= \dfrac{\frac{a}{2\sqrt{3}}}{\frac{\sqrt{3}}{2}a\cos\theta + \frac{a}{2}} = \dfrac{1}{3\cos\theta + \sqrt{3}}$

$3\cos^2\theta + \sqrt{3}\cos\theta - 1 = 0$

$\cos\theta = \dfrac{-\sqrt{3} \pm \sqrt{3+12}}{6} = \dfrac{\sqrt{15} - \sqrt{3}}{6}$

= 0.3568 or –0.9342 (rejected)

θ = 69°6'

The angle between two adjacent sides is 2θ = 138.189°

$$\tan^2\theta = \sec^2\theta - 1$$

$$= \frac{36}{\left(\sqrt{15}-\sqrt{3}\right)^2} - 1$$

$$= \frac{36}{3\left(\sqrt{5}-1\right)^2} - 1$$

$$= \frac{12}{6-2\sqrt{5}} - 1$$

$$= \frac{6}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}} - 1$$

$$= \frac{3}{2}\left(3+\sqrt{5}\right) - 1$$

$$= \frac{1}{2}\left(7+3\sqrt{5}\right)$$

$$= \frac{1}{4}\left(14+6\sqrt{5}\right)$$

$$= \left[\frac{1}{2}\left(3+\sqrt{5}\right)\right]^2$$

$$\tan\theta = \frac{1}{2}\left(3+\sqrt{5}\right)$$

The regular icosahedron can be regarded as 20 identical right pyramids with equilateral triangular base, side = a. The volume of the regular icosahedron is

$$20\times\frac{1}{3}\times\frac{1}{2}\times a^2\sin 60°\times PQ = \frac{5a^2}{\sqrt{3}}\times RQ\tan\theta$$

$$= \frac{5a^2}{\sqrt{3}}\times\frac{a}{2\sqrt{3}}\times\frac{1}{2}\left(3+\sqrt{5}\right)$$

$$= \frac{5}{12}\left(3+\sqrt{5}\right)a^3$$

參考資料：等周問題 香港時代圖書有限公司，1975 年