The arithmetic mean is not less than the geometric mean (AM ≥ GM)

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Theorem 1

Let a, b be two non-negative numbers.

The arithmetic mean is $\frac{a+b}{2}$; the geometric mean is $\sqrt{ab}$

Then $\frac{a+b}{2} \geq \sqrt{ab}$

Proof: $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$

$$a+b-2\sqrt{ab} \geq 0$$

$$a+b \geq 2\sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Theorem 2

Let a, b, c, d be four non-negative numbers.

The arithmetic mean is $\frac{a+b+c+d}{4}$; the geometric mean is $\sqrt[4]{abcd}$

Then $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$

Proof: Let $A = \frac{a+b}{2}$, $B = \frac{c+d}{2}$, then both A and B are non-negative numbers.

By the result of theorem 1, $\frac{A+B}{2} \geq \sqrt{AB}$

$$\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2} \geq \sqrt{\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)}$$

But $\frac{a+b}{2} \geq \sqrt{ab}$; $\frac{c+d}{2} \geq \sqrt{cd}$

$$\therefore \quad \frac{\frac{a+b}{2}+\frac{c+d}{2}}{2} \geq \sqrt{\sqrt{ab}\sqrt{cd}}$$

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

Theorem 3

Let a, b, c, d, e, f, g, h be eight non-negative numbers.

The arithmetic mean is $\dfrac{a+b+c+d+e+f+g+h}{8}$;

the geometric mean is $\sqrt[8]{abcdefgh}$

Then $\dfrac{a+b+c+d+e+f+g+h}{8} \geq \sqrt[8]{abcdefgh}$

Proof: Let $A = \dfrac{a+b+c+d}{4}$, $B = \dfrac{e+f+g+h}{4}$,

then both A and B are non-negative numbers.

By the result of theorem 1, $\dfrac{A+B}{2} \geq \sqrt{AB}$

$$\frac{\frac{a+b+c+d}{4}+\frac{e+f+g+h}{4}}{2} \geq \sqrt{\left(\frac{a+b+c+d}{4}\right)\left(\frac{e+f+g+h}{4}\right)}$$

By theorem 2, $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$; $\dfrac{e+f+g+h}{4} \geq \sqrt[4]{efgh}$

$$\therefore \frac{\frac{a+b+c+d}{4}+\frac{e+f+g+h}{4}}{2} \geq \sqrt{\sqrt[4]{abcd}\sqrt[4]{efgh}}$$

$$\frac{a+b+c+d+e+f+g+h}{8} \geq \sqrt[8]{abcdefgh}$$

Theorem 4 Let a, b, c be three non-negative numbers.

The arithmetic mean is $\dfrac{a+b+c}{3}$; the geometric mean is $\sqrt[3]{abc}$

Then $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$

Proof: **Method 1** let $d = \dfrac{a+b+c}{3}$, by the result of theorem 2

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

$$\frac{a+b+c+\frac{a+b+c}{3}}{4} \geq \sqrt[4]{abc\frac{a+b+c}{3}}$$

$$\frac{a+b+c}{3} \geq \sqrt[4]{abc}\left(\frac{a+b+c}{3}\right)^{\frac{1}{4}}$$

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}} \geq \sqrt[4]{abc}$$

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

Method 2

$$x^2 + y^2 + z^2 - xy - yz - zx$$
$$= \frac{1}{2}(2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx)$$
$$= \frac{1}{2}[(x^2 - 2xy + y^2) + (y^2 - 2yz + z^2) + (z^2 - 2zx + x^2)]$$
$$= \frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] \ge 0$$

If $x, y, z \ge 0$,

$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \ge 0$

$x^3 + y^3 + z^3 - 3xyz \ge 0$

$x^3 + y^3 + z^3 \ge 3xyz \cdots\cdots (*)$

Let $x = \sqrt[3]{a}$, $y = \sqrt[3]{b}$, $z = \sqrt[3]{c}$

By (*), $\left(\sqrt[3]{a}\right)^3 + \left(\sqrt[3]{b}\right)^3 + \left(\sqrt[3]{c}\right)^3 \ge 3\left(\sqrt[3]{a}\right)\left(\sqrt[3]{b}\right)\left(\sqrt[3]{c}\right)$

$$\frac{a + b + c}{3} \ge \sqrt[3]{abc}$$

Theorem 5 Let n be a positive integer. If $a_1, a_2, \cdots, a_n \ge 0$, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \cdots a_n}$$; equality holds if and only if $a_1 = a_2 = \cdots = a_n$

Proof: First we shall prove by induction that if $a_1, a_2, \cdots, a_{2^n} \ge 0$, $\frac{a_1 + a_2 + \cdots + a_{2^n}}{2^n} \ge \sqrt[2^n]{a_1 a_2 \cdots a_{2^n}}$

for all non-negative integer n and equality holds if and only if $a_1 = a_2 = \cdots = a_{2^n}$.

$n = 0$, L.H.S. $= a_1 =$ R.H.S., equality holds obviously.

Suppose it is true for $n = k$ for some non-negative integer k.

If $a_1, a_2, \cdots, a_{2^k} \ge 0$, then $\frac{a_1 + a_2 + \cdots + a_{2^k}}{2^k} \ge \sqrt[2^k]{a_1 a_2 \cdots a_{2^k}} \cdots (1)$

and equality holds if and only if $a_1 = a_2 = \cdots = a_{2^k}$.

Also, if $a_{2^k+1}, \cdots, a_{2^{k+1}} \ge 0$, then $\frac{a_{2^k+1} + \cdots + a_{2^{k+1}}}{2^k} \ge \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}} \cdots (2)$

and equality holds if and only if $a_{2^k+1} = \cdots = a_{2^{k+1}}$.

When $n = k + 1$, $a_1, a_2, \ldots, a_{2^k}, a_{2^k+1}, \ldots, a_{2^{k+1}} \ge 0$,

$$\frac{a_1 + a_2 + \cdots + a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}}}{2^{k+1}} = \frac{\frac{a_1 + a_2 + \cdots + a_{2^k}}{2^k} + \frac{a_{2^k+1} + \cdots + a_{2^{k+1}}}{2^k}}{2}$$

$$\ge \frac{\sqrt[2^k]{a_1 a_2 \cdots a_{2^k}} + \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}}}{2} \quad \text{by (1) and (2)}$$

$$\ge \sqrt{\sqrt[2^k]{a_1 a_2 \cdots a_{2^k}} \cdot \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}}} \quad \text{by theorem 1, } \frac{A + B}{2} \ge \sqrt{AB}$$

$$= \sqrt[2^{k+1}]{a_1 \cdots a_{2^k} \cdot a_{2^k+1} \cdots a_{2^{k+1}}}$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_{2^k} = a_{2^k+1} = \cdots = a_{2^{k+1}}$.

$\therefore$ It is also true for $n = k + 1$.

By mathematical induction, the statement is true for all non-negative integer n.

Now, if n is any non-negative integer $\neq 2^{\ell}$, we can find the smallest non-negative integer m so that $0 \le n < 2^m$. In fact, $m = \left[\frac{\log n}{\log 2}\right] + 1$, where $[x]$ is the greatest integer less than or equal to x.

If $a_1, \cdots, a_n \ge 0$. Let $a_{n+1} = \cdots = a_{2^m} = \frac{a_1 + \cdots + a_n}{n} = \bar{a} \ge 0$.

By the induction result above, $\frac{a_1 + a_2 + \cdots + a_{2^m}}{2^m} \ge \sqrt[2^m]{a_1 a_2 \cdots a_{2^m}}$ for all non-negative integer n and equality holds if and only if $a_1 = a_2 = \cdots = a_{2^m}$.

$$\therefore \quad \frac{a_1 + a_2 + \cdots + a_n + \overbrace{\bar{a} + \cdots + \bar{a}}^{2^m - n \text{ terms}}}{2^m} \ge \sqrt[2^m]{a_1 a_2 \cdots a_n \underbrace{\bar{a} \cdots \bar{a}}_{2^m - n \text{ factors}}}$$

$$\frac{n\bar{a} + (2^m - n)\bar{a}}{2^m} \ge \sqrt[2^m]{a_1 a_2 \cdots a_n} \cdot (\bar{a})^{\frac{2^m - n}{2^m}}$$

$$\frac{2^m \bar{a}}{2^m} \ge \sqrt[2^m]{a_1 a_2 \cdots a_n} \cdot (\bar{a})^{1 - \frac{n}{2^m}}$$

$$\bar{a} \ge \sqrt[2^m]{a_1 a_2 \cdots a_n} \cdot \frac{\bar{a}}{(\bar{a})^{\frac{n}{2^m}}}$$

$$(\bar{a})^{\frac{n}{2^m}} \ge (a_1 a_2 \cdots a_n)^{\frac{1}{2^m}}$$

$$\bar{a} \ge (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$$

$\frac{a_1 + a_2 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \cdots a_n}$; equality holds if and only if $a_1 = a_2 = \cdots = a_n = \bar{a}$

Theorem Prove that $\left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n$ for $n \ge 1$.

Method 1 HKAL 2003 Paper 1 Q10

Let $a_1 = 1$, $a_2 = a_3 = \cdots = a_{n+1} = 1 + \frac{1}{n}$; then by A.M. ≥ G.M.

$$\frac{1 + n\left(1 + \frac{1}{n}\right)}{n+1} > \sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n}$$

$$\frac{n+2}{n+1} > \left(1 + \frac{1}{n}\right)^{\frac{n}{n+1}}$$

$$\left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n$$

Method 2 (Binomial theorem)

$$\left(1 + \frac{1}{n}\right)^n = \sum_{r=0}^{n} C_r^n \left(\frac{1}{n}\right)^r = 1 + \sum_{r=1}^{n} C_r^n \left(\frac{1}{n}\right)^r$$

$$= 1 + \sum_{r=1}^{n} \frac{n(n-1)\cdots(n-r+1)}{r!} \cdot \left(\frac{1}{n}\right)^r = 1 + \sum_{r=1}^{n} \frac{1}{r!} \cdot \left(1 - \frac{0}{n}\right)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{r-1}{n}\right)$$

$$= 1 + \sum_{r=1}^{n} \left[\frac{1}{r!} \cdot \prod_{k=0}^{r-1} \left(1 - \frac{k}{n}\right)\right]$$

$$<1+\sum_{r=1}^{n}\left[\frac{1}{r!}\cdot\prod_{k=0}^{r-1}\left(1-\frac{k}{n+1}\right)\right]$$

$$<1+\sum_{r=1}^{n+1}\left[\frac{1}{r!}\cdot\prod_{k=0}^{r-1}\left(1-\frac{k}{n+1}\right)\right]=\left(1+\frac{1}{n+1}\right)^{n+1}$$

Method 3 (**Bernoulli inequality**)

Claim: If $x \ge -1$, then $(1+x)^n \ge 1 + nx$, $\forall n \in \mathbb{N}$

Proof: Induction on n. $n = 1$, $(1+x)^1 = 1 + x$, the result is obvious.

Suppose $(1+x)^k \ge 1 + kx$

Multiply both sides by $(1 + x)$, which is non-negative.

$(1+x)^{n+1} \ge (1+nx)(1+x)$

$(1+x)^{n+1} \ge 1 + (n+1)x + nx^2 \ge 1 + (n+1)x$

By MI, if $x \ge -1$, then $(1+x)^n \ge 1 + nx$, $\forall n \in \mathbb{N}$

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}}=\left[\frac{n(n+2)}{(n+1)^2}\right]^n\left(1+\frac{1}{n+1}\right)=\left[1-\frac{1}{(n+1)^2}\right]^n\left(1+\frac{1}{n+1}\right)$$

$$\ge\left[1-\frac{n}{(n+1)^2}\right]\left(1+\frac{1}{n+1}\right)\quad \textbf{(Bernoulli inequality)}$$

$$=\left[\frac{n^2+n+1}{(n+1)^2}\right]\left(\frac{n+2}{n+1}\right)=\frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1}>1$$

$$\therefore\ \left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^{n}\quad \text{for } n\ge 1$$

Method 4 1978 Paper 1 Q1

(a) Let a and b be two distinct positive real numbers. Show that for any positive integer n,

$$a^{n+1} - a^n b > ab^n - b^{n+1}.$$

(b) Hence show by induction that $b^n[(n+1)a - nb] < a^{n+1}$ for any positive integer n.

(c) Using (b) or otherwise, show that for any positive integer n, $\left(1+\frac{1}{n}\right)^{n}<\left(1+\frac{1}{n+1}\right)^{n+1}$.

(a) $a^{n+1} - a^n b - (ab^n - b^{n+1}) = a(a^n - b^n) - b(a^n - b^n)$

$= (a - b)(a^n - b^n)$

$\because$ a and b are two distinct positive real numbers

$\therefore$ If $a > b$, then $(a - b)(a^n - b^n) > 0$; if $a < b$, then $(a - b)(a^n - b^n) > 0$.

$\therefore$ $a^{n+1} - a^n b > ab^n - b^{n+1}$

(b) Let $P(n) \equiv$ "$b^n[(n+1)a - nb] < a^{n+1}$ for all positive integer n."

$n = 1$, L.H.S. $= b(2a - b) = 2ab - b^2 = -(a - b)^2 + a^2 < a^2 =$ R.H.S.

P(1) is true

Suppose $P(k)$ is true for some positive integer k; i.e. $b^k[(k+1)a - kb] < a^{k+1}$

When $n = k + 1$,

L.H.S. $= b^{k+1}[(k+2)a - (k+1)b]$

$= b \cdot b^k[(k + 1)a + a - kb - b]$

$= b \cdot b^k[(k + 1)a - kb] + b^{k+1}(a - b)$

$< ba^{k+1} + ab^{k+1} - b^{k+2}$ (By induction assumption)

$< a^{k+1} \cdot b + a^{k+2} - a^{k+1} \cdot b$ (By the result of (a))

$= a^{k+2}$ = R.H.S.

$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

(c) Let $a = 1 + \frac{1}{n+1}$; $b = 1 + \frac{1}{n}$.

By (b), $\left(1+\frac{1}{n}\right)^n \left[(n+1)\cdot\left(1+\frac{1}{n+1}\right) - n\left(1+\frac{1}{n}\right)\right] < \left(1+\frac{1}{n+1}\right)^{n+1}$.

$\left(1+\frac{1}{n}\right)^n [n+1+1-(n+1)] < \left(1+\frac{1}{n+1}\right)^{n+1}$

$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$.

Method 5 (Differentiation) Advanced Level Pure Mathematics by K.S. Ng and Y.K. Kwok Calculus and Analytical Geometry P. 255 Example 5-44

Let n be a positive integer. Define $f(x) = \frac{(x+n+1)^{n+1}}{(x+n)^n}$ for $x \ge 0$.

Show that $f(x)$ is a strictly increasing function. Hence show that $\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$.

For $x > 0$,

$$f'(x) = \frac{(n+1)(x+n)^n (x+n+1)^n - n(x+n+1)^{n+1}(x+n)^{n-1}}{(x+n)^{2n}}$$

$$= \frac{(x+n+1)^n[(n+1)(x+n) - n(x+n+1)]}{(x+n)^{n+1}} = \frac{x(x+n+1)^n}{(x+n)^{n+1}} > 0$$

$\because$ $f(x)$ is continuous for $x \ge 0$, we can conclude that $f(x)$ is strictly increasing for $x \ge 0$

and so $f(0) < f(1)$

That is, $\frac{(n+1)^{n+1}}{n^n} < \frac{(1+n+1)^{n+1}}{(1+n)^n}$

$\frac{(n+1)^n}{n^n} < \frac{(1+n+1)^{n+1}}{(1+n)^{n+1}}$

$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$