Examples on Mathematical Induction: Inequality Series

Created by Mr. Francis Hung Last updated: September 1, 2021

1. Prove that $2\left(\sqrt{n+1}-\sqrt{n}\right)<\frac{1}{\sqrt{n}}<2\left(\sqrt{n}-\sqrt{n-1}\right)$ for all $n \geq 1$.

Hence deduce that $13<\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{11}}+\cdots+\frac{1}{\sqrt{100}}<14$.

$n = 1$, L.H.S. $= 2(\sqrt{2}-1) = 0.8 < \frac{1}{\sqrt{1}} = 1 < 2 = 2\left(\sqrt{1}-0\right)$

$\therefore$ It is true for $n = 1$

Suppose $2\left(\sqrt{k+1}-\sqrt{k}\right)<\frac{1}{\sqrt{k}}<2\left(\sqrt{k}-\sqrt{k-1}\right)$ for some positive integer k.

$$2\left(\sqrt{k+1}-\sqrt{k}\right)=2\left(\sqrt{k+1}-\sqrt{k}\right)\cdot\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}}=2\cdot\frac{k+1-k}{\sqrt{k+1}+\sqrt{k}}=\frac{2}{\sqrt{k+1}+\sqrt{k}}$$

$$>\frac{2}{\sqrt{k+1}+\sqrt{k+1}}=\frac{1}{\sqrt{k+1}}$$

On the other hand,

$$2\left(\sqrt{k+2}-\sqrt{k+1}\right)=2\cdot\frac{k+2-(k+1)}{\sqrt{k+2}+\sqrt{k+1}}=\frac{2}{\sqrt{k+2}+\sqrt{k+1}}<\frac{2}{\sqrt{k+1}+\sqrt{k+1}}=\frac{1}{\sqrt{k+1}}$$

Combine these two inequalities: $2\left(\sqrt{k+2}-\sqrt{k+1}\right)<\frac{1}{\sqrt{k+1}}<2\left(\sqrt{k+1}-\sqrt{k}\right)$

Put $n = 10$, $2\left(\sqrt{11}-\sqrt{10}\right)<\frac{1}{\sqrt{10}}<2\left(\sqrt{10}-\sqrt{9}\right)$

Put $n = 11$, $2\left(\sqrt{12}-\sqrt{11}\right)<\frac{1}{\sqrt{11}}<2\left(\sqrt{11}-\sqrt{10}\right)$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

Put $n = 100$, $2\left(\sqrt{101}-\sqrt{100}\right)<\frac{1}{\sqrt{100}}<2\left(\sqrt{100}-\sqrt{99}\right)$

Add up these equations: $2\left(\sqrt{101}-\sqrt{10}\right)<\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{11}}+\cdots+\frac{1}{\sqrt{100}}<2\left(\sqrt{100}-\sqrt{9}\right)= 14$

$2\left(\sqrt{101}-\sqrt{10}\right)>2\left(\sqrt{100}-\sqrt{12.25}\right)= 13$

$\therefore\ 13<\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{11}}+\cdots+\frac{1}{\sqrt{100}}<14$

2. Prove that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\geq\frac{13}{8}-\frac{1}{n}+\frac{1}{2n^2}$ for $n \geq 2$.

3. Prove by induction the following inequalities:

$$1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \cdots + n^3.$$

3. $n = 2$, L.H.S. $= 1^3 = 1$, middle $= \frac{2^4}{4} = 4$, R.H.S. $= 1^3 + 2^3 = 9 \Rightarrow$ L.H.S. < middle < R.H.S.

Suppose $1^3 + 2^3 + \ldots + (k-1)^3 < \frac{k^4}{4} < 1^3 + 2^3 + \ldots + k^3$ for some $k \geq 2$.

When $n = k + 1$,

L.H.S. $= 1^3 + 2^3 + \ldots + (k-1)^3 + k^3 < \frac{k^4}{4} + k^3 < \frac{k^4+4k^3+6k^2+4k+1}{4} = \frac{(k+1)^4}{4}$

On the other hand,

R.H.S. = $1^3 + 2^3 + \ldots + k^3 + (k+1)^3 > \frac{k^4}{4} + (k+1)^3 = \frac{k^4 + 4(k^3 + 3k^2 + 3k + 1)}{4}$

$$> \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{4} = \frac{(k+1)^4}{4}$$

$\therefore$ If it is true for $n = k$, then it is also true for $n = k + 1$

By the principle of mathematical induction, $1^3 + 2^3 + \ldots + (n-1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \ldots + n^3$.

4. Let $a_1, a_2, \ldots, a_n$ be positive integers such that $1 < a_1 < a_2 < \ldots < a_n$. Prove that

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} < 1.$$

[Use $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.]

4. $1 < a_1 < a_2 < \cdots < a_n \Rightarrow 2 \le a_1, 3 \le a_2, \cdots, n+1 \le a_n$

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} < \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n+1)^2}$$

$$< \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} < 1$$

5. Prove that $1>\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}>\frac{1}{2}$ for $n>1$

Proof: We shall prove $1>\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}+\frac{1}{2n+1}$ and

$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}>\frac{1}{2}$ separately.

$n=2$, LHS = 1 , RHS = $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{47}{60}$

Suppose $1>\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}$

then $1+\frac{1}{2k+2}+\frac{1}{2k+3}>\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}+\frac{1}{2k+3}$

$$1+\frac{1}{2k+2}+\frac{1}{2k+3}-\frac{1}{k+1}>\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}+\frac{1}{2k+3}$$

$$1-\frac{1}{(2k+2)(2k+3)}>\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}+\frac{1}{2k+3}$$

$$1>\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}+\frac{1}{2k+3}$$

To prove $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}>\frac{1}{2}$

$n=2$, LHS $=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}>\frac{1}{2}=$ RHS

Suppose $\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}>\frac{1}{2}$

$$\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}>\frac{1}{2}-\frac{1}{k+1}+\frac{1}{2k+1}+\frac{1}{2k+2}$$

$$\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}>\frac{1}{2}+\frac{1}{(2k+1)(2k+2)}$$

$$\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}>\frac{1}{2}$$

By the principle of mathematical induction, if P(k) is true, then P(k+1) is also true.

Hence we have $1>\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}>\frac{1}{2}$ for $n>1$. Q.E.D.

6. Prove that $n>\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2^n}\geq\frac{n}{2}$ for all $n\geq 1$.

7. Prove that $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}>\frac{13}{24}$ for $n>1$.

8. Prove that $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\geq\sqrt{n}$ for all $n\geq 1$.

9. Prove that $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\ge 2\left(\sqrt{n+1}-1\right)$ for all $n \ge 1$. (same as Q1)

10. Prove that $1+\frac{1}{2}+\cdots+\frac{1}{n}\ge\frac{2n}{n+1}$ for all $n \ge 1$. (Deduce backwards)

10. $n = 1$, L.H.S. = 1, R.H.S. $=\frac{2}{1+1}=1$ $\therefore$ L.H.S. ≥ R.H.S. it is true for $n = 1$

Suppose $1+\frac{1}{2}+\cdots+\frac{1}{k}\ge\frac{2k}{k+1}$

$(2k + 1)(k + 2) - 2(k + 1)(k + 1)$
$= 2k^2 + 5k + 2 - (2k^2 + 4k + 2)$
$= k > 0$
$\therefore (2k + 1)(k + 2) > 2(k + 1)(k + 1)$ for $k > 0$
$\Rightarrow \frac{2k+1}{k+1}>\frac{2(k+1)}{k+2}$ …… (1)

When $n = k + 1$,

LHS $=1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}$

$\ge\frac{2k}{k+1}+\frac{1}{k+1}$

$=\frac{2k+1}{k+1}$

$>\frac{2(k+1)}{k+2}$ by (1)

$\therefore 1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}\ge\frac{2(k+1)}{k+2}$

It is also true for $n = k + 1$

If it is true for $n = k$ then it is true for $n = k +1$

By the principle of mathematical induction, it is true for all positive integers n.

11. If $x > 0$, prove that $x^n + x^{n-2} + x^{n-4} + \cdots + \frac{1}{x^{n-4}} + \frac{1}{x^{n-2}} + \frac{1}{x^n} \ge n+1$ for $n > 0$.

(Hint: you have to prove for the cases where n is odd or even separately.)

11. If $x > 0$, prove that $x^n + x^{n-2} + x^{n-4} + \cdots + \frac{1}{x^{n-4}} + \frac{1}{x^{n-2}} + \frac{1}{x^n} \ge n+1$ for $n > 0$.

$n = 1$, $x + \frac{1}{x} - 2 = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \ge 0$

$\therefore x + \frac{1}{x} \ge 2$, it is true for $n = 1$

$n = 2$, $x^2 + 1 + \frac{1}{x^2} - 3 = x^2 - 2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 \ge 0$

$\therefore x^2 + 1 + \frac{1}{x^2} \ge 2 + 1$, it is true for $n = 2$

Suppose it is true for $n = 2k$,

i.e. $x^{2k} + x^{2k-2} + x^{2k-4} + \cdots + 1 + \cdots + \frac{1}{x^{2k-4}} + \frac{1}{x^{2k-2}} + \frac{1}{x^{2k}} \ge 2k+1$

Add $x^{2k+2} + \frac{1}{x^{2k+2}}$ to both sides.

$$x^{2k+2} + x^{2k} + x^{2k-2} + x^{2k-4} + \cdots + 1 + \cdots + \frac{1}{x^{2k-4}} + \frac{1}{x^{2k-2}} + \frac{1}{x^{2k}} + \frac{1}{x^{2k+2}}$$

$\ge 2k + 1 + x^{2k+2} + \frac{1}{x^{2k+2}} = 2k + 3 + \left(x^{k+1} - \frac{1}{x^{k+1}}\right)^2 \ge 2k + 3 = \text{R.H.S.}$

It is also true for $n = 2k + 2$.

Suppose it is true for $n = 2k - 1$,

i.e. $x^{2k-1} + x^{2k-3} + x^{2k-5} + \cdots + x + \frac{1}{x} + \cdots + \frac{1}{x^{2k-5}} + \frac{1}{x^{2k-3}} + \frac{1}{x^{2k-1}} \ge 2k$

Add $x^{2k+1} + \frac{1}{x^{2k+1}}$ to both sides.

$$x^{2k+1} + x^{2k-1} + x^{2k-3} + x^{2k-5} + \cdots + x + \frac{1}{x} + \cdots + \frac{1}{x^{2k-5}} + \frac{1}{x^{2k-3}} + \frac{1}{x^{2k-1}} + \frac{1}{x^{2k+1}}$$

$\ge 2k + x^{2k+1} + \frac{1}{x^{2k+1}} = 2k + 2 + \left(\sqrt{x}^{2k+1} - \frac{1}{\sqrt{x}^{2k+1}}\right)^2 \ge 2k + 2 = \text{R.H.S.}$

It is also true for $n = 2k + 3$.

By the principle of mathematical induction, $P(n)$ is true for all positive integer n.

12. Let $a_1, a_2, \cdots, a_n \geq 0$, prove that $\dfrac{a_1+\cdots+a_n}{n} \geq (a_1 \cdots a_n)^{1/n}$ for $n \geq 1$.

(Prove for $n = 2^m$ first.)

12. First we shall prove by induction that If $a_1, a_2, \cdots, a_{2^n} \geq 0$, $\dfrac{a_1+a_2+\cdots+a_{2^n}}{2^n} \geq \sqrt[2^n]{a_1 a_2 \cdots a_{2^n}}$ for all non-negative integer n and equality holds if and only if $a_1 = a_2 = \ldots = a_{2^n}$.

$n = 0$, L.H.S. $= a_1 =$ R.H.S., equality holds obviously.
Suppose it is true for $n = k$ for some non-negative integer k.
If $a_1, a_2, \cdots, a_{2^k} \geq 0$, then $\dfrac{a_1+a_2+\cdots+a_{2^k}}{2^k} \geq \sqrt[2^k]{a_1 a_2 \cdots a_{2^k}}$ … (1)

and equality holds if and only if $a_1 = a_2 = \cdots = a_{2^k}$.

Also, if $a_{2^k+1}, \cdots, a_{2^{k+1}} \geq 0$, then $\dfrac{a_{2^k+1}++\cdots+a_{2^{k+1}}}{2^k} \geq \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}}$ ⋯ (2)

and equality holds if and only if $a_{2^k+1} = \cdots = a_{2^{k+1}}$.

When $n = k + 1$, $a_1, a_2, \ldots, a_{2^k}, a_{2^k+1}, \cdots, a_{2^{k+1}} \geq 0$,

$$\frac{a_1+a_2+\cdots+a_{2^k}+a_{2^k+1}+\cdots+a_{2^{k+1}}}{2^{k+1}} = \frac{\dfrac{a_1+a_2+\cdots+a_{2^k}}{2^k}+\dfrac{a_{2^k+1}++\cdots+a_{2^{k+1}}}{2^k}}{2}$$

$$\geq \frac{\sqrt[2^k]{a_1 a_2 \cdots a_{2^k}}+\sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}}}{2} \quad \text{by (1) and (2)}$$

$$\geq \sqrt{\sqrt[2^k]{a_1 a_2 \cdots a_{2^k}} \cdot \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}}} \quad \text{by theorem 1, } \frac{A+B}{2} \geq \sqrt{AB}$$

$$= \sqrt[2^{k+1}]{a_1 \cdots a_{2^k} \cdot a_{2^k+1} \cdots a_{2^{k+1}}}$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_{2^k} = a_{2^k+1} = \cdots = a_{2^{k+1}}$.

$\therefore$ It is also true for $n = k + 1$.
By mathematical induction, the statement is true for all non-negative integer n.
Now, if n is any non-negative integer $\neq 2^{\ell}$, we can find the smallest non-negative integer m so that $0 \leq n < 2^m$. In fact, $m = \left[\dfrac{\log n}{\log 2}\right] + 1$, where $[x]$ is the greatest integer less than or equal to x.

If $a_1, \cdots, a_n \ge 0$. Let $a_{n+1} = \cdots = a_{2^m} = \frac{a_1 + \cdots + a_n}{n} = \overline{a} \ge 0$.

By the induction result above, $\frac{a_1 + a_2 + \cdots + a_{2^m}}{2^m} \ge \sqrt[2^m]{a_1 a_2 \cdots a_{2^m}}$ for all non-negative integer n and equality holds if and only if $a_1 = a_2 = \cdots = a_{2^m}$.

$$\therefore \quad \frac{a_1 + a_2 + \cdots + a_n + \overbrace{\overline{a} + \cdots + \overline{a}}^{2^m - n \text{ terms}}}{2^m} \ge \sqrt[2^m]{a_1 a_2 \cdots a_n \underbrace{\overline{a} \cdots \overline{a}}_{2^m - n \text{ factors}}}$$

$$\frac{n\overline{a} + (2^m - n)\overline{a}}{2^m} \ge \sqrt[2^m]{a_1 a_2 \cdots a_n} \cdot (\overline{a})^{\frac{2^m - n}{2^m}}$$

$$\frac{2^m \overline{a}}{2^m} \ge \sqrt[2^m]{a_1 a_2 \cdots a_n} \cdot (\overline{a})^{1 - \frac{n}{2^m}}$$

$$\overline{a} \ge \sqrt[2^m]{a_1 a_2 \cdots a_n} \cdot \frac{\overline{a}}{(\overline{a})^{\frac{n}{2^m}}}$$

$$(\overline{a})^{\frac{n}{2^m}} \ge (a_1 a_2 \cdots a_n)^{\frac{1}{2^m}}$$

$$\overline{a} \ge (a_1 a_2 \cdots a_n)^{\frac{1}{m}}$$

$\frac{a_1 + a_2 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \cdots a_n}$; equality holds if and only if $a_1 = a_2 = \cdots = a_n = \overline{a}$