02-03 Individual	**1**	*8 see the remark	**2**	$-\frac{1}{2}$	**3**	9	**4**	6	**5**	8
	6	$\frac{3}{4}$	**7**	8	**8**	55	**9**	9	**10**	4

02-03 Group	**1**	$\frac{63}{2525}$	**2**	2	**3**	$\sqrt{3}$	**4**	$\frac{3\sqrt{5}}{4}$	**5**	-1
	6	$\frac{9}{8}$	**7**	10	**8**	40	**9**	20.56	**10**	10

I1 Let f be a function such that for all integers m and n, f(m) is an integer and f(mn) = f(m) f(n). It is given that f(m) > f(n) when 9 > m > n, f(2) = 3 and f(6) > 22, find the value of f(3).

Reference: 2012 FI4.3

$f(4) = f(2)\times f(2) = 9$

$f(6) = f(2)\times f(3) = 3f(3) > 22 \Rightarrow f(3) > \frac{22}{3}$

$\because 3 < 4 \therefore \frac{22}{3} < f(3) < f(4) \Rightarrow \frac{22}{3} < f(3) < 9$

f(3) is an integer $\Rightarrow f(3) = 8$

Remark: The old version of the question was:

Let f be a function such that for all integers m and n, f(m) is an integer and f(mn) = f(m) f(n).

It is given that f(m) > f(n) when m > n, f(2) = 3 and f(6) > 22, find the value of f(3).

The old version of the question was wrong because it can be proved that f(15) > f(16).

Proof: $f(4) = f(2)\times f(2) = 9$

$f(6) = f(2)\times f(3) = 3f(3) > 22 \Rightarrow f(3) > \frac{22}{3}$

$\because 3 < 4 \therefore \frac{22}{3} < f(3) < f(4) \Rightarrow \frac{22}{3} < f(3) < 9$

f(3) is an integer $\Rightarrow f(3) = 8$

$f(8) = f(4)\times f(2) = 27$; $f(16) = f(8)\times f(2) = 81$

$f(9) = f(3)\times f(3) = 64$; $f(12) = f(4)\times f(3) = 72$

$f(9) < f(10) < f(12)$

$\Rightarrow 64 < f(10) < 72$

$\Rightarrow 64 < f(2)\times f(5) < 72$

$\Rightarrow 64 < 3f(5) < 72$

$\Rightarrow 21\frac{1}{3} < f(5) < 24$

f(5) = 22 or 23

$\Rightarrow f(15) = f(3) \times f(5) = 8f(5)$

$= 176 \text{ or } 184 > 81$

$= f(16)$, which is a contradiction

I2 If $P = \frac{1}{4}$, find the value of $P \log_2 P$.

$$P \log_2 P = \frac{1}{4}\log_2 \frac{1}{4} = \frac{1}{4}\log_2 2^{-2} = \frac{1}{4}\times(-2) = -\frac{1}{2}$$

I3 If $0 \le x \le 1$, find the maximum value of $\left[\log_{10}\left(\frac{99999x+1}{1000}\right)\right]^2$.

$0 \le 99999x \le 99999$

$\Rightarrow 1 \le 99999x + 1 \le 100000$

$\Rightarrow \frac{1}{1000} \le \frac{99999x+1}{1000} \le 100$

$-3 \le \log_{10}\left(\frac{99999x+1}{1000}\right) \le 2$

$\Rightarrow \left[\log_{10}\left(\frac{99999x+1}{1000}\right)\right]^2 \le 9$

I4 Given that a quadratic equation $a(x + 1)(x + 2) + b(x + 2)(x + 3) + c(x + 3)(x + 1) = 0$ has roots 0 and 1, and $k = \frac{a}{b}$, find the value of k.

Put $x = 0$, $2a + 6b + 3c = 0$ ....(1)

Put $x = 1$, $6a + 12b + 8c = 0$

$\Rightarrow 3a + 6b + 4c = 0$ ....(2)

$3(2) - 4(1)$: $a - 6b = 0$

$k = \frac{a}{b} = 6$

I5 There are n persons in the classroom. If each person in the classroom shakes hands exactly once with each other person in the classroom and there are altogether 28 handshakes. Find the value of n.

There are altogether ${}_nC_2$ hand-shaking: $\frac{n(n-1)}{2} = 28$

$\Rightarrow n = 8$

I6 If for any $0 < x < \frac{\pi}{2}$, $\cot\frac{1}{4}x - \cot x \equiv \frac{\sin kx}{\left(\sin\frac{1}{4}x\right)(\sin x)}$, where k is a constant, find the value of k

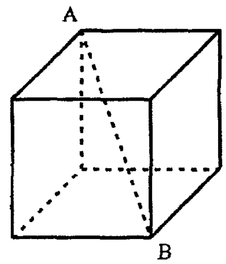
$$\frac{\cos\frac{1}{4}x}{\sin\frac{1}{4}x} - \frac{\cos x}{\sin x} \equiv \frac{\sin kx}{\left(\sin\frac{1}{4}x\right)(\sin x)}$$

$$\frac{\cos\frac{1}{4}x\sin x - \cos x\sin\frac{1}{4}x}{\left(\sin\frac{1}{4}x\right)(\sin x)} \equiv \frac{\sin kx}{\left(\sin\frac{1}{4}x\right)(\sin x)}$$

$$\frac{\sin\left(x - \frac{1}{4}x\right)}{\left(\sin\frac{1}{4}x\right)(\sin x)} \equiv \frac{\sin kx}{\left(\sin\frac{1}{4}x\right)(\sin x)}$$

$k = \frac{3}{4}$

I7 In Figure 1, AB is a diagonal of the cube and $AB = \sqrt{12}$ cm. If the volume of the cube is M cm^3, find the value of M.



Reference: 1992 HI14, 1995 FI5.2

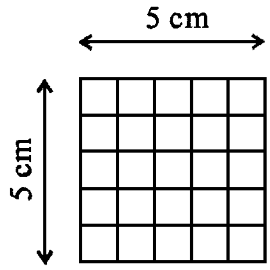
Let the side length of the square be x cm

$x^2 + x^2 + x^2 = AB^2 = 12$

$x = 2$

$M = 2^3 = 8$

I8 In Figure 2, a square with area equal to 25 cm^2 is divided into 25 small squares with side length equal to 1 cm. If the total number of different squares in the figure is K, find the value of K.



The number of squares with side = 1 is 25

The number of squares with side = 2 is 16

The number of squares with side = 3 is 9

The number of squares with side = 4 is 4

The number of squares with side = 5 is 1

K = total number of squares = 1 + 4 + 9 + 16 + 25 = 55

I9 It is given that the 6-digit number $N = \overline{x1527y}$ is a multiple of 4, and the remainder is 5 when N is divided by 11. Find the value of $x + y$.

N is a multiple of 4

$\Rightarrow$ the last two digit of N must be divisible by 4

$\Rightarrow y = 2$ or 6

When N is divided by 11, the remainder is 5

$\Rightarrow (N - 5)$ is divisible by 11

When $y = 2$, $(N-5) = \overline{x15267}$, it is divisible by 11

$\Rightarrow x+5+6 - (1+2+7) = 11m$

$\Rightarrow x = 10$ rejected

When $y = 6$, $(N-5) = \overline{x15271}$.

It is divisible by 11 $\Rightarrow x+5+7 - (1+2+1) = 11m$

$\Rightarrow x = 3$

$x + y = 3 + 6 = 9$

I10 The sides of a triangle have lengths 7.5 cm, 11 cm and x cm respectively. If x is an integer, find the minimum value of x.

Triangle inequality gives:

$7.5 + 11 > x \cdots (1)$

$7.5 + x > 11 \cdots (2)$

$11 + x > 7.5 \cdots (3)$

Solve (1), (2), (3) gives

$18.5 > x$, $x > 3.5$ and $x > -3.5$

$\therefore 3.5 < x < 18.5$

For integral value of x, the minimum value is 4.

Group Events

G1 If $k=\frac{1}{4\times5\times6}+\frac{1}{5\times6\times7}+\frac{1}{6\times7\times8}+\cdots+\frac{1}{99\times100\times101}$, find the value of k.

Reference: 2014 FI3.2

$$\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}=\frac{(r+2)-r}{r(r+1)(r+2)}=2\cdot\frac{1}{r(r+1)(r+2)}$$

Put $r = 4$, $\frac{1}{4\times5}-\frac{1}{5\times6}=2\cdot\frac{1}{4\times5\times6}$

Put $r = 5$, $\frac{1}{5\times6}-\frac{1}{6\times7}=2\cdot\frac{1}{5\times6\times7}$

....

Put $r = 99$, $\frac{1}{99\times100}-\frac{1}{100\times101}=2\cdot\frac{1}{99\times100\times101}$

Add up these equations, $k=\frac{1}{4\times5\times6}+\frac{1}{5\times6\times7}+\frac{1}{6\times7\times8}+\cdots+\frac{1}{99\times100\times101}$

$$=\frac{1}{2}\left(\frac{1}{4\times5}-\frac{1}{100\times101}\right)=\frac{1}{2}\cdot\frac{504}{10100}=\frac{63}{2525}$$

G2 Suppose $x^y+x^{-y}=2\sqrt{2}$ and $x^y-x^{-y}=k$, where $x > 1$ and $y > 0$, find the value of k.

$x^y+\frac{1}{x^y}=2\sqrt{2}$

$\Rightarrow\left(x^y+\frac{1}{x^y}\right)^2=8$

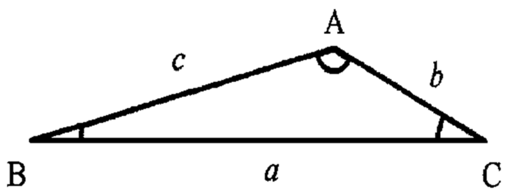
$\Rightarrow x^{2y}+\frac{1}{x^{2y}}+2=8$

$\Rightarrow x^{2y}+\frac{1}{x^{2y}}-2=4$

$\Rightarrow\left(x^y-\frac{1}{x^y}\right)^2=4$

$k=x^y-x^{-y}=x^y-\frac{1}{x^y}=2$

G3 In Figure 1, $\angle A : \angle B : \angle C = 3 : 2 : 1$, $a : b : c = 2 : k : 1$, find the value of k.



Let $\angle A = 3t$, $\angle B = 2t$, $\angle C = t$

$3t + 2t + t = 180°$ ($\angle$s sum of Δ)

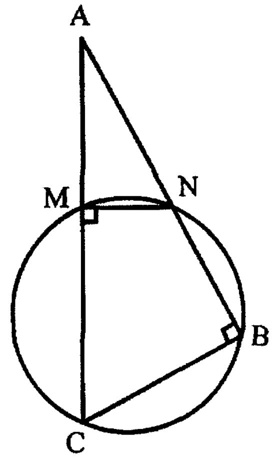
$t = 30°$, $\angle A = 90°$, $\angle B = 60°$, $\angle C = 30°$

By sine formula, $a : b : c = \sin 90° : \sin 60° : \sin 30°$

$$=1:\frac{\sqrt{3}}{2}:\frac{1}{2}=2:\sqrt{3}:1$$

$k=\sqrt{3}$

G4 In Figure 1, AMC and ANB are straight lines, $\angle NMC = \angle NBC = 90°$, $AB = 4$, $BC = 3$, areas of ΔAMN and ΔABC are in the ratio $1 : 4$. Find the radius of the circle $BNMC$.



It is easy to show that $\Delta AMN \sim \Delta ABC$ (equiangular)

$$\frac{\text{Area of } \Delta\text{AMN}}{\text{Area of } \Delta\text{ABC}} = \left(\frac{MN}{BC}\right)^2 = \frac{1}{4}$$

$$\frac{MN}{BC} = \frac{1}{2} = \frac{AN}{AC} \Rightarrow \frac{AN}{5} = \frac{1}{2}, AN = 2.5$$

$NB = AB - AN = 4 - 2.5 = 1.5$

$$NC = \text{diameter} = \sqrt{1.5^2 + 3^2} = \frac{3}{2}\sqrt{5}, \text{ radius} = \frac{3\sqrt{5}}{4}$$

G5 If the equation $x^2 + ax + 3b - a + 2 = 0$ has real root(s) for any real number a, find the maximum value of b.

$\Delta = a^2 - 4(3b - a + 2)$

$\quad = a^2 + 4a - 8 - 12b \geq 0$

$(a + 2)^2 - 12(1 + b) \geq 0$

$-12(1 + b) \geq 0$

$b \leq -1$

G6 Suppose the parabola $y = 4x^2 - 5x + c$ intersects the x-axis at $(\cos\theta, 0)$ and $(\cos\phi, 0)$ respectively. If θ and ϕ are two acute angles of a right-angled triangle, find the value of c.

$\theta + \phi = 90°$

$\Rightarrow \phi = 90° - \theta$

$\cos\theta + \cos\phi = \frac{5}{4}$

$\Rightarrow \cos\theta + \cos(90° - \theta) = \frac{5}{4}$

$\Rightarrow \cos\theta + \sin\theta = \frac{5}{4}$

$(\cos\theta + \sin\theta)^2 = \frac{25}{16}$

$\Rightarrow \cos^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = \frac{25}{16}$

$\Rightarrow \sin\theta\cos\theta = \frac{9}{32} = \frac{c}{4}$

$c = \frac{9}{8}$

G7 Suppose the straight line $y + 3x - 4 = 0$ intersects the parabola $y = x^2$ at points A and B respectively. If O is the origin, find the area of ΔOAB.

Sub. $y = 4 - 3x$ into $y = x^2$

$x^2 = 4 - 3x$

$\Rightarrow x^2 + 3x - 4 = 0$

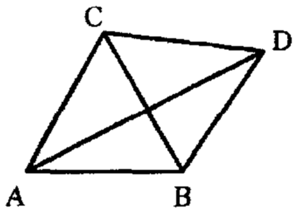
$\Rightarrow (x - 1)(x + 4) = 0$

$\Rightarrow x = 1$ or -4

When $x = 1$, $y = 1$; when $x = -4$, $y = 16$

Area of $\Delta OAB = \frac{1}{2}\begin{vmatrix} 1 & 1 \\ -4 & 16 \end{vmatrix} = 10$

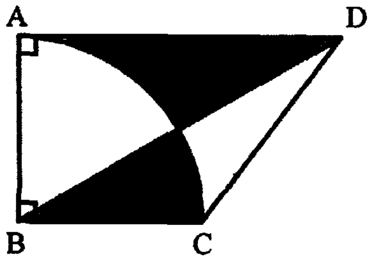
G8 In Figure 3, $AC = BC = CD$, $\angle ACB = 80°$.
If $\angle ADB = x°$, find the value of x. **Reference 2011 HG9, 2014 FG2.4**



We can use C as the centre, $AC = BC = CD$ as the radius to draw a circle to pass through A, B, D.

$x = \frac{1}{2}\angle ACB = 40$ ($\angle$ at centre twice $\angle$ at $\odot^{\text{ce}}$)

G9 In Figure 4, the sector ABC is one quarter of a circle with radius 4 cm. Suppose the areas of the two shaded parts are equal. Let the area of the trapezium $ABCD$ be A cm^2, find the area of A. (Take $\pi = 3.14$)



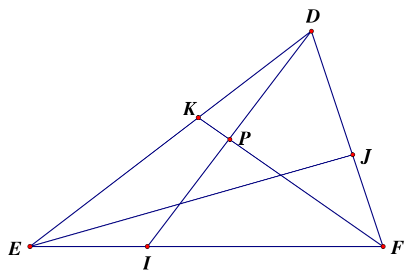
Let $AD = x$ cm

Area of ΔABD = area of sector ABC

$\frac{1}{2}\cdot 4\cdot x = \frac{1}{4}\cdot \pi(4)^2 \Rightarrow x = 2\pi$

$A = \frac{1}{2}\cdot(4+x)\cdot 4 = 2(4 + 2\times 3.14) = 20.56$

G10 In Figure 5, the area of ΔDEF is 30 cm^2. EIF, DJF and DKE are straight lines. P is the intersection point of DI and FK. Let $EI : IF = 1: 2$, $FJ : JD = 3 : 4$, $DK : KE = 2 : 3$. Let the area of ΔDFP be B cm^2, find the value of B.



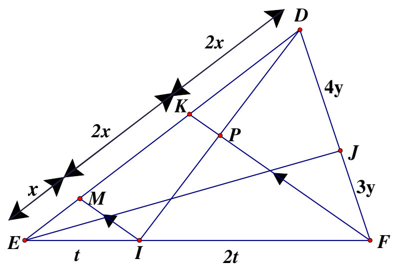
Let $EI = t$, $IF = 2t$, $DK = 2x$, $KE = 3x$

Draw a line IM parallel to KF cutting DE at M.

By the theorem of equal ratio, $\frac{EM}{MK} = \frac{EI}{IF} = \frac{1}{2}$

$\therefore EM = x$, $MK = 2x$

$DP = PI$ (Intercept theorem)



Area of $\Delta DIF = \frac{2}{3}$ Area of ΔDFE = 20 cm^2

Area of $\Delta DFP = \frac{1}{2}$ Area of ΔDIF = 10 cm^2

$\Rightarrow B = 10$