# Trigonometric inequality

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1. In $\Delta ABC$, prove the following inequalities:

   (a) $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$

   (b) $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$

   (c) $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2} \le \frac{3}{2}$

   (d) $1 < \cos A + \cos B + \cos C \le \frac{3}{2}$

   (e) $\cos A \cos B \cos C \le \frac{1}{8}$

   (f) $\tan^2 A + \tan^2 B + \tan^2 C \ge 9$ if $\Delta ABC$ is acute;

   (g) $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2} \ge 1$

   (h) $\cot^2 A + \cot^2 B + \cot^2 C \ge 1$

   (i) $\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+\cot^2\frac{C}{2} \ge 9$

   (j) $\sec^2 A + \sec^2 B + \sec^2 C \ge 12$

2. (a) For any real numbers $x$, $y$, and $z$, prove that $x^2 + y^2 + z^2 \ge xy + yz + zx$.

   (b) Let $a$, $b$ and $c$ be the angles of a triangle. Prove that

   $$\tan\frac{a}{2}\tan\frac{b}{2}+\tan\frac{b}{2}\tan\frac{c}{2}+\tan\frac{c}{2}\tan\frac{a}{2}=1.$$

   (c) By using the results of (a) and (b), prove that $\tan^2\frac{a}{2}+\tan^2\frac{b}{2}+\tan^2\frac{c}{2} \ge 1$

1. In $\Delta ABC$, $A + B + C = \pi$, $0 < A + B + C < \pi$

(a) To prove that $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$

LHS = $\sin A + \sin B + \sin C$

$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin C$

$\le 2\cos\frac{C}{2} + \sin C$, equality holds when $A = B$

Let $f(x) = 2\cos\frac{x}{2} + \sin x$

$f'(x) = -\sin\frac{x}{2} + \cos x$

Let $f'(x) = 0 \Rightarrow \sin\frac{x}{2} = \cos x$

$\Rightarrow \sin\frac{x}{2} = 1 - 2\sin^2\frac{x}{2}$

$\Rightarrow 2\sin^2\frac{x}{2} + \sin\frac{x}{2} - 1 = 0$

$\Rightarrow \left(2\sin\frac{x}{2} - 1\right)\left(\sin\frac{x}{2} - 1\right) = 0$

$\Rightarrow \frac{x}{2} = \frac{\pi}{6}$ or $\frac{x}{2} = \frac{\pi}{2}$ (rejected)

$\Rightarrow \frac{x}{2} = \frac{\pi}{3}$

$f''(x) = -\frac{1}{2}\cos\frac{x}{2} - \sin x$

$f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}\cos\frac{\pi}{6} - \sin\frac{\pi}{3} = -\frac{3\sqrt{3}}{4} < 0$

$\therefore$ $f(x) \le f\left(\frac{\pi}{3}\right)$, which is an absolute maximum for $0 < x < \pi$.

$2\cos\frac{x}{2} + \sin x \le 2\cos\frac{\pi}{6} + \sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$

$\therefore$ $\sin A + \sin B + \sin C \le 2\cos\frac{C}{2} + \sin C \le 2\cos\frac{\pi}{6} + \sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$

Equality holds when $C = \frac{\pi}{3}$ and $A = B$.

**Method 2**

$\sin A + \sin B + \sin C + \sin\frac{\pi}{3}$

$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 2\sin\frac{C+\frac{\pi}{3}}{2}\cos\frac{C-\frac{\pi}{3}}{2}$

$\le 2\sin\frac{A+B}{2} + 2\sin\frac{C+\frac{\pi}{3}}{2}$, equality holds when $A = B = C = \frac{\pi}{3}$

$= 4\sin\frac{A+B+C+\frac{\pi}{3}}{4}\cos\frac{A+B-C-\frac{\pi}{3}}{4}$

$\le 4\sin\frac{\pi+\frac{\pi}{3}}{4}=2\sqrt{3}$, equality holds when $A+B=C+\frac{\pi}{3}$

$\therefore\ \sin A+\sin B+\sin C\le 2\sqrt{3}-\sin\frac{\pi}{3}=2\sqrt{3}-\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{2}$

(b) To prove that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$

$\text{LHS}=\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$=-\frac{1}{2}\left(\cos\frac{A+B}{2}-\cos\frac{A-B}{2}\right)\sin\frac{C}{2}$

$=\frac{1}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)\sin\frac{C}{2}$

$\le\frac{1}{2}\left(1-\sin\frac{C}{2}\right)\sin\frac{C}{2}$ (equality holds when $A=B$)

$\le\frac{1}{2}\left(\frac{1-\sin\frac{C}{2}+\sin\frac{C}{2}}{2}\right)^2$ (GM ≤ AM: $ab\le\left(\frac{a+b}{2}\right)^2$ for $a\ge 0$, $b\ge 0$)

$=\frac{1}{8}$ (equality holds when $1-\sin\frac{C}{2}=\sin\frac{C}{2}$, i.e. $C=\frac{\pi}{3}$)

(c) To prove that $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2}$

$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}+\sin\frac{\pi}{6}$

$=\ 2\sin\frac{\frac{A}{2}+\frac{B}{2}}{2}\cos\frac{\frac{A}{2}-\frac{B}{2}}{2}+2\sin\frac{\frac{C}{2}+\frac{\pi}{6}}{2}\cos\frac{\frac{C}{2}-\frac{\pi}{6}}{2}$

$\le\ 2\sin\frac{\frac{A}{2}+\frac{B}{2}}{2}+2\sin\frac{\frac{C}{2}+\frac{\pi}{6}}{2}$ equality holds when $A=B$ and $C=\frac{\pi}{3}$.

$=\ 4\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}+\frac{\pi}{6}}{4}\cos\frac{\frac{A}{2}+\frac{B}{2}-\frac{C}{2}-\frac{\pi}{6}}{4}$

$\le\ 4\sin\frac{\pi}{6}=2$, equality holds when $\frac{A}{2}+\frac{B}{2}=\frac{C}{2}+\frac{\pi}{6}$

$\therefore\ \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le 2-\sin\frac{\pi}{6}=\frac{3}{2}$

(d) To prove that $1<\cos A+\cos B+\cos C\le\frac{3}{2}$

$\cos A+\cos B+\cos C=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\cos C$

$=2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}$

$=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)+1$

$=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)+1$

$$=-4\sin\frac{C}{2}\sin\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\sin\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}+1$$

$$=4\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2}+1$$

$$\le 4\times\frac{1}{8}+1=\frac{3}{2}\quad\text{by the result of 1(b)}$$

On the other hand, $1<4\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2}+1=\cos A+\cos B+\cos C$

$\therefore\ 1<\cos A+\cos B+\cos C\le\frac{3}{2}$

(e) To prove that $\cos A\cos B\cos C\le\frac{1}{8}$

LHS = $\cos A\cos B\cos C$

$$=\frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]\cos C$$

$$\le\frac{1}{2}(-\cos C+1)\cos C\text{, equality holds when } A=B$$

$$\le\frac{1}{2}\left(\frac{-\cos C+1+\cos C}{2}\right)^2\text{, (AM} \ge \text{GM), equality holds when } C=\frac{\pi}{3}$$

$$\le\frac{1}{8}$$

(f) To prove that $\tan^2 A+\tan^2 B+\tan^2 C\ge 9$ if $\Delta ABC$ is acute

Method 1: $\tan^2 A+\tan^2 B+\tan^2 C=\sec^2 A+\sec^2 B+\sec^2 C-3\ge 9$, by Q1 (*j*)

Method 2: $\tan(A+B)=\tan(\pi-C)$

$$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$

$\tan A+\tan B=-\tan C(1-\tan A\tan B)$

$\tan A+\tan B+\tan C=\tan A\tan B\tan C\ \cdots\cdots\ (*)$

$\because\ A, B, C$ are acute $\therefore\ \tan A, \tan B, \tan C>0$

$$\frac{\tan A+\tan B+\tan C}{3}\ge(\tan A\tan B\tan C)^{1/3}\quad(\text{AM}\ge\text{GM})$$

$$\frac{\tan A\tan B\tan C}{3}\ge(\tan A\tan B\tan C)^{1/3}$$

$$(\tan A\tan B\tan C)^{2/3}\ge 3$$

$\tan A\tan B\tan C\ge 3^{3/2}$, equality holds when $A=B=C=\frac{\pi}{3}$

$\tan^2 A+\tan^2 B+\tan^2 C\ge 3(\tan A\tan B\tan C)^{2/3}$ (AM ≥ GM)

$\ge 3\times\left(3^{3/2}\right)^{2/3}=9$, by the above result

(g) To prove that $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge 1$

$A+B+C=\pi,\quad \tan\frac{C}{2}=\tan\left(\frac{\pi}{2}-\frac{A+B}{2}\right)$

$$=\frac{1-\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}$$

$\Rightarrow \tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$

$\therefore \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{A}{2}\tan\frac{C}{2}=1$ …… (*)

By Cauchy's Schwarz's Inequality (Square Product ≥ Product Square)

$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \ge (ax + by + cz)^2$ (equality holds when $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$)

(all quantities are positive)

$$\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)\left(\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+\tan^2\frac{A}{2}\right)\ge\left(\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{A}{2}\tan\frac{C}{2}\right)^2=1$$

$\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)^2\ge 1$, by (*) equality holds when $\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}=\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}=\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}$

$\therefore \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge 1$, equality holds when $A = B = C = \frac{\pi}{3}$

(h) To prove that $\cot^2 A + \cot^2 B + \cot^2 C \ge 1$

Note that $\cot(A + B) = \frac{\cot A\cot B-1}{\cot A+\cot B}$

Simplifying and cross multiplying: $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$ … (*)

Use the same method as Q1($g$), Cauchy's Schwarz's Inequality SP ≥ PS

$(\cot^2 A+\cot^2 B+\cot^2 C)(\cot^2 B+\cot^2 C+\cot^2 A) \ge(\cot A \cot B+\cot B \cot C+\cot C \cot A)^2$

$(\cot^2 A + \cot^2 B + \cot^2 C)^2 \ge 1$ by (*) equality holds when $\frac{\cot A}{\cot B}=\frac{\cot B}{\cot C}=\frac{\cot C}{\cot A}$

$\cot^2 A + \cot^2 B + \cot^2 C \ge 1$, equality holds when $A = B = C = \frac{\pi}{3}$

(i) To prove that $\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+\cot^2\frac{C}{2}\ge 9$

$\cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(\frac{\pi}{2}-\frac{C}{2}\right)$

$\frac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{A}{2}+\cot\frac{B}{2}}=\frac{1}{\cot\frac{C}{2}}$

Cross multiplying: $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$ …… (*)

$\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{3}\ge\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^{1/3}$ (AM ≥ GM)

$\frac{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}{3}\ge\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^{1/3}$, equality holds when $\cot\frac{A}{2}=\cot\frac{B}{2}=\cot\frac{C}{2}$

$\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^{2/3}\ge 3$ …… (1), equality holds when $A = B = C = \frac{\pi}{3}$

Now $\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+\cot^2\frac{C}{2}\ge 3\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^{2/3}\ge 9$ (AM ≥ GM), by (1)

(j) To prove that $\sec^2 A + \sec^2 B + \sec^2 C \ge 12$

$\frac{\sec^2 A+\sec^2 B+\sec^2 C}{3}\ge\left(\frac{1}{\cos^2 A\cos^2 B\cos^2 C}\right)^{1/3}$ (AM ≥ GM)

$\ge 8^{2/3} = 4$ by the result of Q1(e)

$\therefore \sec^2 A + \sec^2 B + \sec^2 C \ge 12$

2. (a) $x^2+y^2+z^2-(xy+yz+zx)$

$=\frac{1}{2}\left[2x^2+2y^2+2z^2-2(xy+yz+zx)\right]$

$=\frac{1}{2}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]$

$=\frac{1}{2}\left[\text{sum of three squares}\right] \ge 0$

$x^2+y^2+z^2 \ge xy+yz+zx$

(b) $a+b+c=180°$

$\frac{a+b+c}{2}=90^\circ$

$\frac{c}{2}=90^\circ-\left(\frac{a}{2}+\frac{b}{2}\right)$

$\cot\frac{c}{2}=\cot\left[90^\circ-\left(\frac{a}{2}+\frac{b}{2}\right)\right]$

$\frac{1}{\tan\frac{c}{2}}=\tan\left(\frac{a}{2}+\frac{b}{2}\right)$

$\frac{1}{\tan\frac{c}{2}}=\frac{\tan\frac{a}{2}+\tan\frac{b}{2}}{1-\tan\frac{a}{2}\tan\frac{b}{2}}$

$1-\tan\frac{a}{2}\tan\frac{b}{2}=\tan\frac{a}{2}\tan\frac{c}{2}+\tan\frac{b}{2}\tan\frac{c}{2}$

$\tan\frac{a}{2}\tan\frac{b}{2}+\tan\frac{b}{2}\tan\frac{c}{2}+\tan\frac{c}{2}\tan\frac{a}{2}=1$

(b) Let $x=\tan\frac{a}{2}$, $y=\tan\frac{b}{2}$, $z=\tan\frac{c}{2}$

by (a) $\tan^2\frac{a}{2}+\tan^2\frac{b}{2}+\tan^2\frac{c}{2}\ge\tan\frac{a}{2}\tan\frac{b}{2}+\tan\frac{b}{2}\tan\frac{c}{2}+\tan\frac{a}{2}\tan\frac{c}{2}=1$

$\therefore\ \tan^2\frac{a}{2}+\tan^2\frac{b}{2}+\tan^2\frac{c}{2}\ge 1$