11-12 Individual	**1**	9	**2**	504510	**3**	8	**4**	23	**5**	8
	6	$\frac{293}{34}\left(=8\frac{21}{34}\right)$	**7**	6	**8**	4	**9**	$x = 13, y = 2$	**10**	$\frac{2041}{25}\left(=81\frac{16}{25}=81.64\right)$

11-12 Group	**1**	6	**2**	2037	**3**	$2 + 2^{1006}$	**4**	$2\sqrt{503}-1$	**5**	2012
	6	16	**7**	10	**8**	*124 see the remark	**9**	180	**10**	5

Individual Events

I1 Find the value of the unit digit of $2^2 + 3^2 + 4^2 + \cdots + 20122012^2$. (**Reference: 1996 HG10**)

$1^2 + 2^2 + 3^2 + \cdots + 10^2 \equiv 1 + 4 + 9 + 6 + 5 + 6 + 9 + 4 + 1 + 0 \pmod{10}$

$\equiv 5 \pmod{10}$

$2^2 + 3^2 + 4^2 + \cdots + 20122012^2$

$\equiv (1^2 +\cdots+ 10^2) + \cdots + (20122001^2 +\cdots+ 20122010^2) + 20122011^2 + 20122012^2 - 1^2 \pmod{10}$

$\equiv 5\times2012201 + 1 + 4 - 1 \pmod{10}$

$\equiv 9 \pmod{10}$

I2 Given that a, b and c are positive even integers which satisfy the equation $a + b + c = 2012$. How many solutions does the equation have?

Reference: 2001 HG2, 2006 HI6, 2010 HI1

Let $a = 2p$, $b = 2q$, $c = 2r$, where p, q, r are positive integers.

$a + b + c = 2012 \Rightarrow 2(p + q + r) = 2012 \Rightarrow p + q + r = 1006$

The question is equivalent to find the number of ways to put 1006 identical balls into 3 different boxes, and each box must contain at least one ball.

Align the 1006 balls in a row. There are 1005 gaps between these balls. Put 2 sticks into three of these gaps, so as to divide the balls into 3 groups.

The following diagrams show one possible division.

The three boxes contain 2 balls, 1003 balls and 1 ball. $p = 2$, $q = 1003$, $r = 1$.

The number of ways is equivalent to the number of choosing 2 gaps as sticks from 1005 gaps.

The number of ways is $C_2^{1005} = \frac{1005\times1004}{2} = 504510$.

I3 In Figure 1, $ABCD$ is a square. The coordinates of B and D are $(5, -1)$ and $(-3, 3)$ respectively. If $A(a, b)$ lies in the first quadrant, find the value of $a + b$.

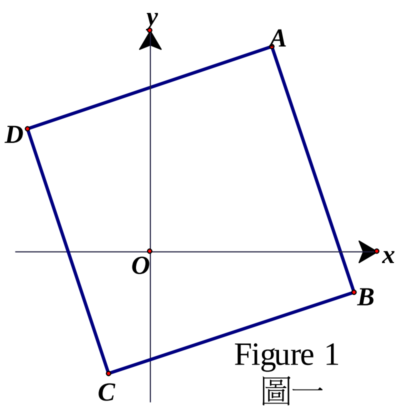


Figure 1
圖一

Mid-point of $BD = M(1, 1)$

$MB^2 = MA^2 \Rightarrow (a - 1)^2 + (b - 1)^2 = (5 - 1)^2 + (1+ 1)^2 = 20$

$a^2 + b^2 - 2a - 2b - 18 = 0$ …… (1)

$MA \perp MB \Rightarrow \frac{b-1}{a-1}\cdot\frac{-1-1}{5-1} = -1$

$b = 2a - 1$ …… (2)

Sub. (2) into (1): $a^2 + (2a - 1)^2 - 2a - 2(2a - 1) - 18 = 0$

$5a^2 - 10a - 15 = 0 \Rightarrow a^2 - 2a - 3 = 0 \Rightarrow (a - 3)(a + 1) = 0$

$a = 3$ or -1(rejected)

$b = 2(3) - 1 = 5$

$a + b = 3 + 5 = 8$

Method 2 $m_{BD}=\dfrac{3-(-1)}{-3-5}=-\dfrac{1}{2}$; $m_{AD}=\dfrac{b-3}{a+3}$

$\because \angle BDA = 45° \Rightarrow \tan 45° = 1$

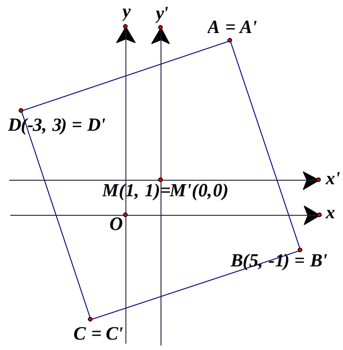
$$m_{AD}=\frac{b-3}{a+3}=\frac{m_{BD}+\tan 45^\circ}{1-m_{BD}\tan 45^\circ}=\frac{-\frac{1}{2}+1}{1-\left(-\frac{1}{2}\right)\cdot 1}=\frac{1}{3}$$

$3b - 9 = a + 3 \Rightarrow 3b - a = 12$ …… (1)

Mid-point of $BD = M(1, 1) \Rightarrow b = 2a - 1$ …… (2) (similar to **method 1**)

Solving (1) and (2) gives $a = 3$, $b = 5 \Rightarrow a + b = 8$

Method 3 by Mr. Jimmy Pang from Lee Shing Pik College



Mid-point of $BD = M(1, 1)$

Translate the coordinate system by $x' = x - 1$, $y' = y - 1$

The new coordinate of M is $M' = (1 - 1, 1 - 1) = (0, 0)$

The new coordinate of B is $B' = (5 - 1, -1 - 1) = (4, -2)$

Rotate B' about M' in anticlockwise direction through 90°

The new coordinate of $A' = (2, 4)$

Translate the coordinate system by $x = x' + 1$, $y = y' + 1$

The old coordinate of $A = (2 + 1, 4 + 1) = (3, 5) = (a, b)$

$\therefore a + b = 8$

I4 Find the number of places of the number $2^{20}\times 25^{12}$. (**Reference: 1982 FG10.1, 1992 HI17**)

$2^{20}\times 25^{12} = 2^{20}\times 5^{24} = 10^{20}\times 5^{4} = 625\times 10^{20}$

The number of places = 23

I5 Given that $\log_4 N = 1+\dfrac{1}{3}+\dfrac{1}{9}+\dfrac{1}{27}+\cdots$, find the value of N. **(Reference: 1994 HI1)**

$\log_4 N = 1+\dfrac{1}{3}+\dfrac{1}{9}+\dfrac{1}{27}+\cdots=\dfrac{1}{1-\frac{1}{3}}=\dfrac{3}{2}$ (sum to infinity of a geometric series, $a = 1$, $r=\dfrac{1}{3}$.)

$N = 4^{3/2} = 8$

I6 Given that a and b are distinct prime numbers, $a^2 - 19a + m = 0$ and $b^2 - 19b + m = 0$. Find the value of $\dfrac{a}{b}+\dfrac{b}{a}$. **(Reference: 1996 HG8, 1996FG7.1, 2001 FG4.4, 2005 FG1.2)**

a and b are prime distinct roots of $x^2 - 19x + m = 0$

$a + b$ = sum of roots = 19 (odd)

$\because$ a and b are prime number and all prime number except 2, are odd.

$\therefore a = 2, b = 17$ (or $a = 17, b = 2$)

$\dfrac{a}{b}+\dfrac{b}{a}=\dfrac{17}{2}+\dfrac{2}{17}=\dfrac{293}{34}\left(=8\dfrac{21}{34}\right)$

I7 Given that a, b and c are positive numbers, and $a + b + c = 9$. Suppose the maximum value among $a + b$, $a + c$ and $b + c$ is P, find the minimum value of P.

WLOG assume that $a + b = P$, $a + c \le P$, $c + a \le P$.

$18 = 2(a + b + c) = (a + b) + (b + c) + (c + a) \le 3P$

$6 \le P$

The minimum value of P is 6.

I8 If the quadratic equation $(k^2 - 4)x^2 - (14k + 4)x + 48 = 0$ has two distinct positive integral roots, find the value(s) of k.

Clearly $k^2 - 4 \neq 0$; otherwise, the equation cannot have two real roots.

Let the roots be α, β.

$\Delta = (14k + 4)^2 - 4(48)(k^2 - 4) = 2^2[(7k + 2)^2 - 48k^2 + 192] = 2^2(k^2 + 28k + 196) = [2(k + 14)]^2$

$$\alpha = \frac{14k+4+\sqrt{[2(k+14)]^2}}{2(k^2-4)} = \frac{7k+2+k+14}{k^2-4} = \frac{8k+16}{k^2-4} = \frac{8}{k-2},\ \beta = \frac{6k-12}{k^2-4} = \frac{6}{k+2}.$$

For positive integral roots, $k - 2$ is a positive factor of 8 and $k + 2$ is a positive factor of 6.

$k - 2 = 1, 2, 4, 8$ and $k + 2 = 1, 2, 3, 6$

$k = 3, 4, 6, 10$ and $k = -1, 0, 1, 4$

$\therefore k = 4$ only

Method 2 provided by Mr. Jimmy Pang from Po Leung Kuk Lee Shing Pik College

The quadratic equation can be factorised as: $[(k - 2)x - 8][(k + 2)x - 6] = 0$

$\because k \neq 2$ and $k \neq -2$ $\therefore x = \dfrac{8}{k-2}$ or $\dfrac{6}{k+2}$

By similar argument as before, for positive integral root, $k = 4$ only.

I9 Given that x, y are positive integers and $x > y$, solve $x^3 = 2189 + y^3$.

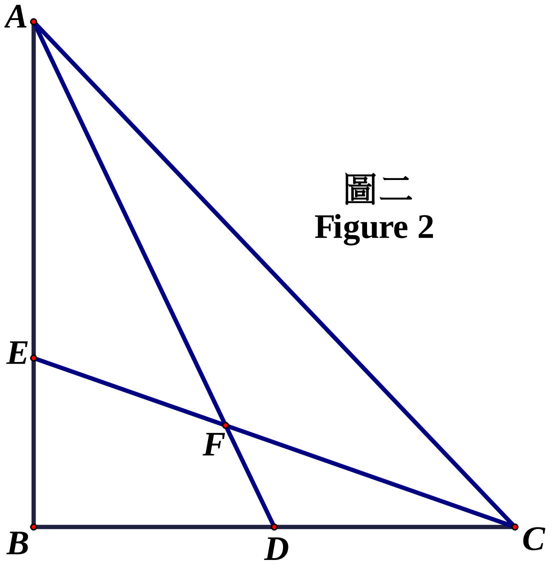
$x^3 - y^3 = (x - y)(x^2 + xy + y^2) = 2189 = 11\times199$ and both 11 and 199 are primes.

$x^2 + xy + y^2 = (x - y)^2 + 3xy$

$x - y$	$(x - y)^2 + 3xy$	xy	x	y
1	$2189 = 1 + 3xy$	729.33 (rejected)		
11	$199 = 121 + 3xy$	26	13	2
199	$11 = 199^2 + 3xy$	– (rejected)		
2189	$1 = 2189^2 + 3xy$	– (rejected)		

$\therefore x = 13, y = 2$

I10 In figure 2, $AE = 14$, $EB = 7$, $AC = 29$ and $BD = DC = 10$.
Find the value of BF^2.
Reference: 2005 HI5, 2009 HG8



圖二
Figure 2

$AB = 14 + 7 = 21$, $BC = 10 + 10 = 20$

$AB^2 + BC^2 = 21^2 + 20^2 = 841 = 29^2 = AC^2$

$\therefore \angle ABC = 90°$ (converse, Pythagoras' theorem)

Let $BF = a$, $\angle CBF = \theta$, $\angle ABF = 90° - \theta$

Area of ΔBEF + area of ΔBCF = area of ΔBCE

$$\frac{1}{2}\cdot 20\times a\sin\theta+\frac{1}{2}\cdot a\times 7\cos\theta=\frac{20\times 7}{2}$$

$20a \sin \theta + 7a \cos \theta = 140$ …… (1)

Area of ΔBDF + area of ΔABF = area of ΔABD

$$\frac{1}{2}\cdot 21\times a\cos\theta+\frac{1}{2}\cdot a\times 10\sin\theta=\frac{10\times 21}{2}$$

$21a \cos \theta + 10a \sin \theta = 210$ …… (2)

$2(2) - (1)$: $35\ a \cos \theta = 280$

$a \cos \theta = 8$ …… (3)

$3(1) - (2)$: $50\ a \sin \theta = 210$

$a \sin \theta = \dfrac{21}{5}$ …… (4)

$(3)^2 + (4)^2$: $BF^2 = a^2 = 8^2 + \left(\dfrac{21}{5}\right)^2 = \dfrac{2041}{25} \left(= 81\dfrac{16}{25} = 81.64\right)$

Method 2 $\angle ABC = 90°$ (similar to **method 1**)

Regard B as the origin, BC as the x-axis, BA as the y-axis, then $\vec{d} = 10\mathbf{i}$, $\vec{c} = 20\mathbf{i}$, $\vec{e} = 7\mathbf{j}$, $\vec{a} = 21\mathbf{j}$

Suppose F divides AD in the ratio p and $1 - p$.

Also, F divides EC in the ratio $t : 1 - t$.

$\vec{f} = p\,\vec{d} + (1 - p)\,\vec{a} = 10p\mathbf{i} + 21(1 - p)\mathbf{j}$ …… (1)

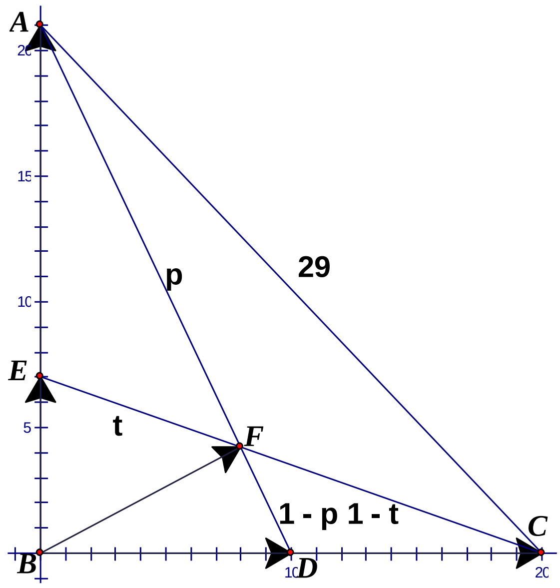
$\vec{f} = t\vec{c} + (1 - t)\,\vec{e} = 20t\mathbf{i} + 7(1 - t)\mathbf{j}$ …… (2)

Compare coefficients:

$10p = 20t$ and $21(1 - p) = 7(1 - t)$

$\Rightarrow p = 2t$ …… (3) and $3(1 - p) = 1 - t$ …… (4)

$3(3) + (4)$: $3 = 5t + 1 \Rightarrow t = \dfrac{2}{5}$



$BF^2 = |\vec{f}|^2 = |20(\frac{2}{5})\mathbf{i} + 7(1 - \frac{2}{5})\mathbf{j}|^2$

$= |8\mathbf{i} + \frac{21}{5})\mathbf{j}|^2 = 8^2 + \left(\dfrac{21}{5}\right)^2 = \dfrac{2041}{25}$

Method 3 $\angle ABC = 90°$ (converse, Pythagoras' theorem, similar to **method 1**)
Regard B as the origin, BC as the x-axis, BA as the y-axis, then

Equation of AD: $\frac{x}{10}+\frac{y}{21}=1$…… (1); equation of EC: $\frac{x}{20}+\frac{y}{7}=1$…… (2)

2(2) – (1): $\frac{5y}{21}=1\Rightarrow y=\frac{21}{5}$; 3(1) – (2): $\frac{x}{4}=2\Rightarrow x=8$

$BF^2 = x^2 + y^2 = 8^2+\left(\frac{21}{5}\right)^2=\frac{2041}{25}\left(=81\frac{16}{25}=81.64\right)$

Method 4 $\angle ABC = 90°$ (similar to **method 1**)
Regard B as the origin, BC as the x-axis, BA as the y-axis, then
$A(0, 21)$, $E(0, 7)$, $C(20, 0)$, $D(10, 0)$.
Let $AF : FD = r : s$
Apply Menelaus theorem on ΔABD with EFC.

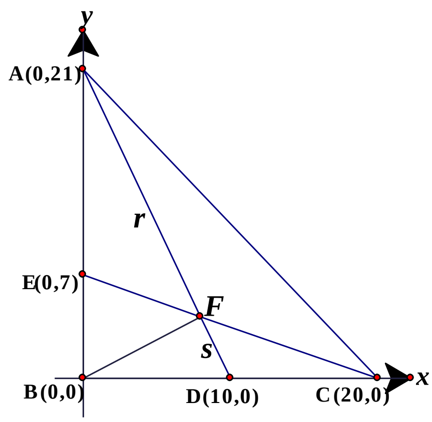
$\frac{AE}{EB}\cdot\frac{BC}{CD}\cdot\frac{DF}{FA}=-1$

$\frac{14}{7}\cdot\frac{20}{-10}\cdot\frac{s}{r}=-1\Rightarrow\frac{s}{r}=\frac{1}{4}$

By the point of division formula,

$F=\left(\frac{1\times0+4\times10}{5},\frac{1\times21+4\times0}{5}\right)=\left(8,\frac{21}{5}\right)$

$BF^2=64+\frac{441}{25}=\frac{2041}{25}$



Method 5 (Provided by Mr. Lee Chun Yu, James from St. Paul's Co-educational College)

Area of $\Delta ABC=\frac{1}{2}\cdot21\cdot20=210$

Area of ΔABD : area of ΔADC = 1 : 1 = Area of ΔBDF : area of ΔDCF
$\therefore$ Area of ΔABF : area of ΔACF = 1 : 1
Area of ΔCEB : area of ΔCEA = 1 : 2 = Area of ΔFEB : area of ΔFEA
$\therefore$ Area of ΔCFB : area of ΔCFA = 1 : 2
$\therefore$ Area of ΔABF : area of ΔACF : area of ΔBCF = 2 : 2 : 1

Area of $\Delta BCF=210\times\frac{1}{2+2+1}=42$

Distance from F to $BC=\frac{42\times2}{20}=4.2$

Area of $\Delta ABF=210\times\frac{2}{2+2+1}=84$

Distance from F to $AB=\frac{84\times2}{14+7}=8$

$BF^2 = 8^2 + 4.2^2 = 81.64$ (Pythagoras' theorem)

Group Events

G1 Given that x, y and z are three consecutive positive integers, and $\frac{y}{x}+\frac{z}{x}+\frac{x}{y}+\frac{z}{y}+\frac{x}{z}+\frac{y}{z}$ is an integer. Find the value of $x + y + z$.

$x = y - 1$, $z = y + 1$

$$\frac{y}{x}+\frac{z}{x}+\frac{x}{y}+\frac{z}{y}+\frac{x}{z}+\frac{y}{z}=\frac{y+y+1}{y-1}+\frac{y-1+y+1}{y}+\frac{y-1+y}{y+1}$$

$$=\frac{(2y+1)y(y+1)+2y(y^2-1)+(2y-1)y(y-1)}{(y-1)y(y+1)}$$

$$=\frac{2y^3+3y^2+y+2y^3-2y+2y^3-3y^2+y}{(y-1)y(y+1)}=\frac{6y^3}{(y-1)y(y+1)}=\frac{6y^2}{y^2-1}$$

Clearly $y^2 - 1$ does not divide y^2, so $y + 1$ and $y - 1$ are factors of 6.

$y - 1 = 1 \Rightarrow y = 2$, $y + 1 = 3 \Rightarrow x + y + z = 6$

$y - 1 = 2 \Rightarrow y = 3$, but $y + 1 = 4$ which is not a factor of 6, rejected.

$y - 1 = 3$ or 6 are similarly rejected.

Method 2 $x = y - 1$, $z = y + 1$

$$\frac{y}{x}+\frac{z}{x}+\frac{x}{y}+\frac{z}{y}+\frac{x}{z}+\frac{y}{z}=\frac{y+y+1}{y-1}+\frac{y-1+y+1}{y}+\frac{y-1+y}{y+1}=6+\frac{3}{y-1}-\frac{3}{y+1}$$

$\frac{3}{y-1}$ is an integer $\Rightarrow y > 1$ …… (1); $\frac{3}{y+1}$ is an integer $\Rightarrow y \le 2$ …… (2)

Solving (1) and (2) gives $y = 2$, $x = 1$, $z = 3 \Rightarrow x + y + z = 6$

G2 Given that x is a real number and $\sqrt{x-2012}+\sqrt{(5-x)^2}=x$. Find the value of x.

If $5 \ge x$, the equation is equivalent to $\sqrt{x-2012}+5-x=x$

$\sqrt{x-2012}=2x-5$

$x - 2012 = 4x^2 - 20x + 25$

$4x^2 - 21x + 2037 = 0$

$\Delta = 21^2 - 4(4)(2037) < 0 \Rightarrow$ no real solution, rejected

If $5 < x$, then the equation becomes $\sqrt{x-2012}+x-5=x$

$x - 2012 = 25$

$x = 2037$

G3 Evaluate $\sqrt{2^2+2^{1008}+2^{2012}}$. (Answer can be expressed in index form.)

$$\begin{aligned}\sqrt{2^2+2^{1008}+2^{2012}}&=2\cdot\sqrt{1+2^{1006}+2^{2010}}\\&=2\cdot\sqrt{1+2\times 2^{1005}+\left(2^{1005}\right)^2}\\&=2\cdot\sqrt{\left(1+2^{1005}\right)^2}\\&=2+2^{1006}.\end{aligned}$$

G4 Evaluate $\frac{1}{\sqrt{2012}+\sqrt{2011}}+\frac{1}{\sqrt{2011}+\sqrt{2010}}+\cdots+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{1}}$.

(Answer can be expressed in surd form.)

$$\frac{1}{\sqrt{2012}+\sqrt{2011}}+\frac{1}{\sqrt{2011}+\sqrt{2010}}+\cdots+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{1}}$$

$$=\frac{\sqrt{2012}-\sqrt{2011}}{2012-2011}+\frac{\sqrt{2011}-\sqrt{2010}}{2011-2010}+\cdots+\frac{\sqrt{3}-\sqrt{2}}{3-2}+\frac{\sqrt{2}-\sqrt{1}}{2-1}$$

$$=\sqrt{2012}-1$$

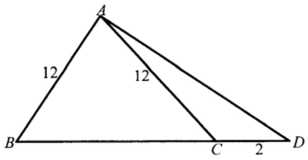
$$=2\sqrt{503}-1$$

G5 Find the minimum value of $x^2 + y^2 - 10x - 6y + 2046$. **Reference 1999 HG7, 2001 HI3, 2018 HI1**

$x^2 + y^2 - 10x - 6y + 2046$

$= (x - 5)^2 + (y - 3)^2 + 2012 \geq 2012$

G6 In Figure 3, ΔABC is an isosceles triangle. Suppose $AB = AC = 12$. If D is a point on BC produced such that $\angle DAB = 90°$ and $CD = 2$, find the length of BC.



Let $\angle ABC = \theta = \angle ACB$ (base $\angle$, isos. Δ)

$\angle ACD = 180° - \theta$ (adj. $\angle$s on st. line)

$BD = 12 \sec \theta$

$BC = 2\times12 \cos \theta = BD - 2 = 12 \sec \theta - 2$

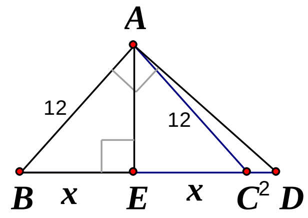
$12 \cos^2 \theta + \cos \theta - 6 = 0$

$(3 \cos \theta - 2)(4 \cos \theta + 3) = 0$

$\cos \theta = \frac{2}{3}$ or $-\frac{3}{4}$ (rejected)

$BC = 2\times12 \cos \theta = 16$

Method 2 Draw $AE \perp BD$.



$\Delta ABE \cong \Delta ACE$ (R.H.S.) Let $BE = x = EC$.

$\cos B = \frac{x}{12} = \frac{12}{2x+2} \Rightarrow \frac{x}{12} = \frac{6}{x+1}$

$x^2 + x - 72 = 0$

$\Rightarrow (x - 8)(x + 9) = 0$

$\Rightarrow x = 8$

$\Rightarrow BC = 2x = 16$

G7 Given that $a^x = b^y = c^z = 30^w$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}$, where a, b, c are positive integers ($a \leq b \leq c$) and x, y, z, w are real numbers, find the value of $a + b + c$.

$\log a^x = \log b^y = \log c^z = \log 30^w$

$x \log a = y \log b = z \log c = w \log 30$

$\frac{1}{w} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{\log a}{w \log 30} + \frac{\log b}{w \log 30} + \frac{\log c}{w \log 30} = \frac{\log abc}{w \log 30}$

$abc = 30$

$\because$ $a \neq 1$ and $b \neq 1$ (otherwise $x \log a = y \log b = z \log c = w \log 30 \Rightarrow 0 = w \log 30 \Rightarrow w = 0$)

$\therefore$ $a = 2$, $b = 3$, $c = 5$

$a + b + c = 10$

G8 Given that the roots of the equation $x^2 + px + q = 0$ are integers and $q > 0$. If $p + q = 60$, find the value of q.

Let the roots be α and β.

$\alpha + \beta = -p$ …… (1)

$\alpha\beta = q > 0$ …… (2)

$p + q = 60$

$\Rightarrow -(\alpha + \beta) + \alpha\beta = 60$

$1 - \alpha - \beta(1 - \alpha) = 61$

$(\alpha - 1)(\beta - 1) = 61$, which is a prime

$\alpha - 1 = -1$, $\beta - 1 = -61$ or $\alpha - 1 = 1$, $\beta - 1 = 61$

When $\alpha - 1 = -1$, $\beta - 1 = -61$

$\Rightarrow \alpha = 0$, $\beta = -60$

$\Rightarrow \alpha\beta = q = 0$ (contradicting to the given condition $q > 0$, $\therefore$ rejected)

$\alpha - 1 = 1$, $\beta - 1 = 61$

$\Rightarrow \alpha = 2$, $\beta = 62$, $q = \alpha\beta = 124$

> **Remark** the original question is: Given that the roots of the equation $x^2 + px + q = 0$ are integers and $p, q > 0$. If $p + q = 60$, find the value of q.
>
> $\alpha + \beta = -p < 0$ …… (1), $\alpha\beta = q > 0$ …… (2)
>
> $\Rightarrow \alpha < 0$ and $\beta < 0$ and $(\alpha - 1)(\beta - 1) = 61$
>
> $\Rightarrow \alpha - 1 = -1$, $\beta - 1 = -61 \Rightarrow \alpha = 0$, $\beta = -60$
>
> $\Rightarrow \alpha\beta = q = 0$ (rejected), no solution

G9 Evaluate $\sin^2 1° + \sin^2 2° + \sin^2 3° + \ldots + \sin^2 359° + \sin^2 360°$. (**Reference 2010 FG1.1**)

$\sin^2 1° + \sin^2 89° = \sin^2 1° + \cos^2 1° = 1$

$\sin^2 2° + \sin^2 88° = \sin^2 2° + \cos^2 2° = 1$

…………………………………………

$\sin^2 44° + \sin^2 46° = \sin^2 44° + \cos^2 44° = 1$

$\sin^2 1° + \sin^2 2° + \cdots + \sin^2 89° = (\sin^2 1° + \sin^2 89°) + \cdots + (\sin^2 44° + \sin^2 46°) + \sin^2 45°$

$= 44.5$

$\sin^2 1° + \sin^2 2° + \sin^2 3° + \cdots + \sin^2 359° + \sin^2 360°$

$= 44.5 + \sin^2 90° + 44.5 + 0 + 44.5 + \sin^2 270° + 44.5 + 0 = 180$

G10 In a gathering, originally each guest will shake hands with every other guest, but Steven only shakes hands with people whom he knows. If the total number of handshakes in the gathering is 60, how many people in the gathering does Steven know? (Note: when two persons shake hands with each other, the total number of handshakes will be one (not two).)

Suppose there are n persons and Steven knows m persons (where $n > m$).

If everyone shakes hands with each other, then the total number of hand-shaking $= C_2^n$

In this case, Steven shakes hands with $n - 1$ persons. However, he had made only m hand-shaking.

Method 1

$C_2^{n-1} < 60 \le C_2^n$

By trial and error, $C_2^{11} = \dfrac{11\times 10}{2} = 55 < 60 \le C_2^{12} = \dfrac{12\times 11}{2} = 66$

$n = 12$

$m = 60 - 55 = 5$

Method 2

$\therefore\ C_2^n - (n-1) + m = 60$

$m = 59 + n - \dfrac{n(n-1)}{2}$

$\because\ 0 \le m < n\ \therefore\ 0 \le \dfrac{1}{2}(118 + 3n - n^2) < n$

$n^2 - 3n - 118 \le 0$ and $n^2 - n - 118 > 0$

$(n - 1.5)^2 - 120.25 \le 0$ and $(n - 0.5)^2 - 118.25 > 0$

$\left(n - 1.5 - \sqrt{120.25}\right)\left(n - 1.5 + \sqrt{120.25}\right) \le 0$ and $\left(n - 0.5 - \sqrt{118.25}\right)\left(n - 0.5 + \sqrt{118.25}\right) > 0$

$\left(1.5 - \sqrt{120.25} \le n \le 1.5 + \sqrt{120.25}\right)$ and $\left(n < 0.5 - \sqrt{118.25}\ \text{or}\ n > 0.5 + \sqrt{118.25}\right)$

$0.5 + \sqrt{118.25} < n \le 1.5 + \sqrt{120.25}$

$10.5 = \sqrt{110.25} < \sqrt{118.25}$, $\sqrt{120.25} < \sqrt{121} = 11$

$\Rightarrow 11 < n \le 12.5$

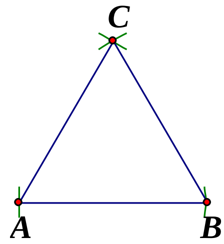
For integral value, $n = 12$

$\dfrac{12\cdot 11}{2} - (12 - 1) + m = 60$

$m = 5$

Geometrical Construction

1. In the space provided, construct an equilateral triangle *ABC* with sides equal to the length of *MN* below.



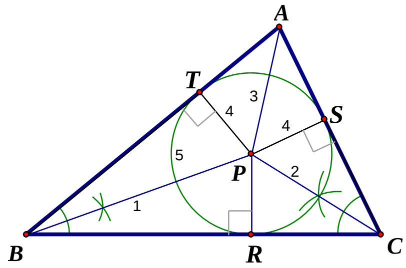
(1) 作綫段 AB，使得 $AB = MN$。

(2) 以 A 為圓心，AB 為半徑作一弧；以 B 為圓心，BA 為半徑作一弧；兩弧相交於 C。

(3) 連接 AC 及 BC。

ΔABC 為等邊三角形。

2. As shown in Figure 1, construct a circle inside the triangle *ABC*, so that *AB*, *BC* and *CA* are tangents to the circle. **Reference: 2009 HSC 1, 2014 HC1, 2019 HC3**



(1) 作$\angle ABC$ 的角平分綫。

(2) 作$\angle ACB$ 的角平分綫。P 為兩條角平分綫的交點。

(3) 連接 AP。

(4) 分別過 P 作垂直綫至 BC、AC 及 AB，R、S、T 為對應的垂足。

$\Delta BPR \cong \Delta BPT$ (A.A.S.)

$\Delta CPR \cong \Delta CPS$ (A.A.S.)

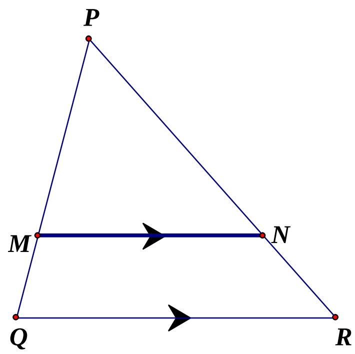
$PT = PR = PS$ (全等三角形的對應邊)

(5) 若 P 至 BC 的垂足為 R，以 P 為圓心，PR 為半徑作一圓，此圓內切於三角形的三邊，稱為**內切圓 (inscribed circle)**。(切綫⊥半徑的逆定理)

3. Figure 2 shows a triangle *PQR*. Construct a line *MN* parallel to *QR* so that

(*i*) *M* and *N* lie on *PQ* and *PR* respectively; and

(*ii*) the area of $\Delta PMN = \frac{1}{2} \times$ the area of ΔPQR.



首先，我們計算 MN 和 QR 的關係：

$\angle QPR = \angle MPN$ (公共角)

$\angle PQR = \angle PMN$ (QR // MN, 對應角)

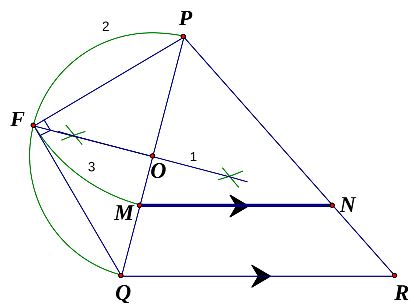
$\angle PRQ = \angle PNM$ (QR // MN, 對應角)

$\therefore \Delta PQR \sim \Delta PMN$ (等角)

$$\frac{\Delta PMN\text{的面積}}{\Delta PQR\text{的面積}} = \frac{1}{2} = \left(\frac{PM}{PQ}\right)^2$$

$$\frac{PM}{PQ} = \frac{1}{\sqrt{2}} \Rightarrow PM = \frac{PQ}{\sqrt{2}}$$

作圖步驟:



(1) 利用垂直平分綫，找 PQ 之中點 O。

(2) 以 O 為圓心 $OP = OQ$ 為半徑，向外作一半圓，與剛才的垂直平分綫相交於 F。

$\angle PFQ = 90°$ (半圓上的圓周角)

ΔPFQ 為一個直角等腰三角形

$\angle QPF = 45°$

$PF = PQ \sin 45° = \frac{PQ}{\sqrt{2}}$

(3) 以 P 為圓心，PF 為半徑，作一圓弧，交 PQ 於 M。$PM = \frac{PQ}{\sqrt{2}}$。

(4) 自 M 作一綫段平行於 QR，交 PR 於 N，則ΔPMN 平分ΔPQR 的面積。

Percentage of correct questions

1	34.48%	**2**	12.58%	**3**	32.66%	**4**	35.40%	**5**	30.73%
6	30.83%	**7**	42.19%	**8**	13.08%	**9**	25.96%	**10**	4.26%
1	57.49%	**2**	35.22%	**3**	38.06%	**4**	48.99%	**5**	53.44%
6	18.62%	**7**	16.60%	**8**	6.07%	**9**	30.77%	**10**	42.51%