17-18 Individual	**1**	1966	**2**	19	**3**	718	**4**	99999	**5**	$\frac{1}{2}$
	6	2.5	**7**	$2\sqrt{5}$	**8**	$2\left(\sqrt{3}-1\right)$	**9**	$\frac{61}{8}=7.625$	**10**	54
	11	13	**12**	16π	**13**	2000	**14**	$-\frac{2018}{2019}$	**15**	289578289
17-18 Group	**1**	$\frac{4}{3}$	**2**	3999766	**3**	10	**4**	275	**5**	–15
	6	5	**7**	$\frac{9}{16}$	**8**	2380	**9**	2475	**10**	$\frac{900}{7}=128\frac{4}{7}$

Individual Events

I1 若 a 及 b 均為實數，求 $a^2+b^2+12a-8b+2018$ 的最小值。

If a and b are real numbers, find the minimum value of $a^2+b^2+12a-8b+2018$.

Reference: 1999 HG7, 2001 HI3, 2012 HG5

$a^2+b^2+12a-8b+2018$ $=a^2+12a+36+b^2-8b+16+1966$ $=(a+6)^2+(b-4)^2+1966\ge 1966$ 最小值為 1966。	$a^2+b^2+12a-8b+2018$ $=a^2+12a+36+b^2-8b+16+1966$ $=(a+6)^2+(b-4)^2+1966\ge 1966$ The minimum value is 1966 .

I2 設 a 及 k 均為常數。若 $(6x^3+ax^2+7x-3)\div(2x^2+kx-1)$ 的商和餘式分別為 $3x+5$ 及 $-5x+2$，求 a 的值。

Let a and k be constants. If the quotient and the remainder of $(6x^3+ax^2+7x-3)\div(2x^2+kx-1)$ are $3x+5$ and $-5x+2$ respectively, find the value of a .

利用除法定理：被除數 = 除數×商 + 餘數 $6x^3+ax^2+7x-3=(2x^2+kx-1)\times(3x+5)-5x+2$ 比較兩邊的 x 的係數：$7=5k-3-5$ $\Rightarrow k=3$ 比較兩邊的 x^2 的係數：$a=2\times 5+3k$ 代 $k=3$ 入右式中： $a=10+9=19$	Using division algorithms: Dividend = divisor×quotient + remainder $6x^3+ax^2+7x-3=(2x^2+kx-1)\times(3x+5)-5x+2$ Compare coefficients of x on both sides: $7=5k-3-5\Rightarrow k=3$ Compare coefficients of x^2 on both sides: $a=2\times 5+3k$ Sub. $k=3$ into the right side: $a=10+9=19$

I3 在編制某雜誌中每頁的頁碼時，總共用去了 2,046 個數字，問該雜誌總共有多少頁？(假設該雜誌第一頁的頁碼是 1。)

In numbering the pages of a magazine, 2046 digits were used. How many pages are there in the magazine? (Assume the page number of the magazine starts from 1 .)

由第一頁至第九頁： 共 9 個數字 由第十頁至第九十九頁： 共$(99-9)\times 2=180$ 個數字 由第一百頁至第九百九十九頁： 共$(999-99)\times 3=2700$ 個數字 $9+180<2046<9+180+2700$ 假設該雜誌共有 x 頁，其中 $100<x<999$。 $9+180+(x-99)\times 3=2046$ $x=718$，該雜誌共有 718 頁。	Page 1 to 9: 9 digits Page 10 to 99: $(99-9)\times 2=180$ digits Page 100 to 999: $(999-99)\times 3=2700$ digits $9+180<2046<9+180+2700$ Suppose there are x pages in the magazine, where $100<x<999$. $9+180+(x-99)\times 3=2046$ $\Rightarrow x=718$, there 718 pages in the magazine.

I4 解 $\log\left(1+\frac{1}{1}\right)+\log\left(1+\frac{1}{2}\right)+\log\left(1+\frac{1}{3}\right)+\cdots+\log\left(1+\frac{1}{n}\right)=5$ 。

Solve $\log\left(1+\frac{1}{1}\right)+\log\left(1+\frac{1}{2}\right)+\log\left(1+\frac{1}{3}\right)+\cdots+\log\left(1+\frac{1}{n}\right)=5$.

$$\log 2+\log\frac{3}{2}+\log\frac{4}{3}+\cdots+\log\frac{n+1}{n}=5$$

$$\log\left(2\times\frac{3}{2}\times\frac{4}{3}\times\cdots\times\frac{n+1}{n}\right)=5$$

$n + 1 = 10^5 = 100000$

$n = 99999$

I5 已知 $\dfrac{1-2^{-\frac{1}{x}}}{2^{-\frac{1}{x}}-2^{-\frac{2}{x}}}=4$ 。求 x 的值。

Given that $\dfrac{1-2^{-\frac{1}{x}}}{2^{-\frac{1}{x}}-2^{-\frac{2}{x}}}=4$. Find the value of x . (**Reference 2018 FG2.1**)

$$\frac{\left(1-\frac{1}{2^{\frac{1}{x}}}\right)}{\left(\frac{1}{2^{\frac{1}{x}}}-\frac{1}{2^{\frac{2}{x}}}\right)}\cdot\frac{2^{\frac{2}{x}}}{2^{\frac{2}{x}}}=4$$

$$\frac{2^{\frac{1}{x}}\left(2^{\frac{1}{x}}-1\right)}{2^{\frac{1}{x}}-1}=4$$

$$2^{\frac{1}{x}}=2^2$$

$$x=\frac{1}{2}$$

I6 若 x 為有理數，求 x 的值滿足聯立方程 $\begin{cases}y=2x^2-11x+15\\y=2x^3-17x^2+16x+35\end{cases}$ 。

If x is a rational number, find the value of x satisfying the simultaneous equations $\begin{cases}y=2x^2-11x+15\\y=2x^3-17x^2+16x+35\end{cases}$.

$2x^2 - 11x + 15 = (x - 3)(2x - 5)$

$2(3)^3 - 17(3)^2 + 16(3) + 35 = 54 - 153 + 48 + 35 = -86$

$2(2.5)^3 - 17(2.5)^2 + 16(2.5) + 35 = \frac{125}{4}-\frac{425}{4}+40+35=0$

$2x^3 - 17x^2 + 16x + 35 = 2x^2 - 11x + 15$

$(2x - 5)(x^2 - 6x - 7) - (2x - 5)(x + 3) = 0$

$(2x - 5)(x^2 - 6x - 7 - x - 3) = 0$

$(2x - 5)(x^2 - 7x - 10) = 0$

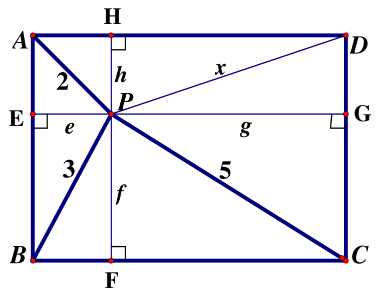
$x = 2.5$ or $x=\frac{7\pm\sqrt{89}}{2}$ (無理根，捨去 irrational roots, rejected)

$$\begin{array}{r}
x^2-6x-7\\
2x-5\,\big)\,\overline{2x^3-17x^2+16x+35}\\
\underline{2x^3-5x^2}\\
-12x^2+16x\\
\underline{-12x^2+30x}\\
-14x+35\\
\underline{-14x+35}
\end{array}$$

I7 如圖一所示，P 為長方形 $ABCD$ 內的一點，使得 $PA = 2$，$PB = 3$ 及 $PC = 5$。求 PD 的長度。

As shown in Figure 1, P is a point inside a rectangle $ABCD$ such that $PA = 2$, $PB = 3$ and $PC = 5$. Find the length of PD.

Reference: 1994 FG10.1-2, 2001 FG2.2, 2003 FI3.4

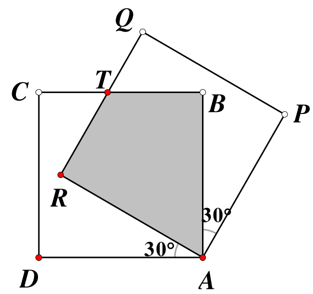


圖一 Figure 1

假設分別由 P 至 AB、BC、CD 及 DA 之垂足為 E、F、G 及 H。設 $PD = x$、$PE = e$、$PF = f$、$PG = g$、$PH = h$。 那麼，由畢氏定理可得知： $e^2 + h^2 = 2^2$ …… (1) $e^2 + f^2 = 3^2$ …… (2) $f^2 + g^2 = 5^2$ …… (3) $g^2 + h^2 = x^2$ …… (4) (1) + (3) – (2) – (4): $0 = 4 + 29 - 9 - x^2$ $PD = x = 2\sqrt{5}$	Let E, F, G, H be the feet of perpendiculars drawn from P onto AB, BC, CD and DA respectively. Let $PD = x$, $PE = e$, $PF = f$, $PG = g$, $PH = h$. Then by Pythagoras' theorem, $e^2 + h^2 = 2^2$ …… (1) $e^2 + f^2 = 3^2$ …… (2) $f^2 + g^2 = 5^2$ …… (3) $g^2 + h^2 = x^2$ …… (4) (1) + (3) – (2) – (4): $0 = 4 + 29 - 9 - x^2$ $PD = x = 2\sqrt{5}$

I8 如圖二所示，兩個邊長為 x cm 的正方形於一角重疊。若兩個正方形的非重疊部分與重疊部分面積的比是 $a:1$，求 a 的值。

As shown in Figure 2, two squares with side x cm coincides at one corner. If the ratio of the non-overlapping area to the overlapping area of the two squares is $a : 1$, find the value of a.

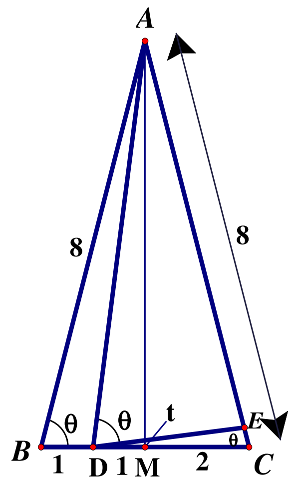


圖二 Figure 2

將兩個正方形的角重新命名為 $ABCD$ 及 $APQR$ 如圖所示。 假設 BC 與 QR 相交於 T。假設正方形的每邊邊長為 x。 $\angle B = \angle R = 90^\circ$ (正方形的性質) $\angle BAR = 60^\circ$ $\Delta ABT \cong \Delta ART$ (R.H.S.) $\therefore \angle BAT = \angle RAT = 30^\circ$ (全等三角形的對應角) $BT = RT = x \tan 30^\circ = \frac{x}{\sqrt{3}}$ $ABTR$ 的面積$= 2\times\frac{1}{2}x\cdot\frac{x}{\sqrt{3}} = \frac{x^2}{\sqrt{3}}$ 非重疊部分的面積$= 2\left(x^2 - \frac{x^2}{\sqrt{3}}\right) = 2x^2\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)$ 非重疊部分：重疊部分面積的比 $= 2x^2\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) : \frac{x^2}{\sqrt{3}} = 2\left(\sqrt{3}-1\right):1$ $a = 2\left(\sqrt{3}-1\right)$	Label the corners of the two squares as $ABCD$ and $APQR$ as shown. Suppose BC intersects QR at T. Let the length of side of each square be x. $\angle B = \angle R = 90^\circ$ (property of a square) $\angle BAR = 60^\circ$ $\Delta ABT \cong \Delta ART$ (R.H.S.) $\therefore \angle BAT = \angle RAT = 30^\circ$ (corr. $\angle$s $\cong \Delta$s) $BT = RT = x \tan 30^\circ = \frac{x}{\sqrt{3}}$ Area of $ABTR = 2\times\frac{1}{2}x\cdot\frac{x}{\sqrt{3}} = \frac{x^2}{\sqrt{3}}$ Area of the unshaded part $= 2\left(x^2 - \frac{x^2}{\sqrt{3}}\right) = 2x^2\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)$ The non-overlapping area : the overlapping area of the two squares $= 2x^2\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) : \frac{x^2}{\sqrt{3}} = 2\left(\sqrt{3}-1\right):1$ $a = 2\left(\sqrt{3}-1\right)$

I9 如圖三所示，ABC 是一個等腰三角形，其中 $AB = AC = 8$ 及 $BC = 4$。D 及 E 分別為 BC 及 AC 上的點使得 $BD = 1$ 及 $\angle ABC = \angle ADE$ 。求 AE 的值。

As shown in Figure 3, ABC is an isosceles triangle with $AB = AC = 8$ and $BC = 4$. D and E are points lying on BC and AC respectively such that $BD = 1$ and $\angle ABC = \angle ADE$. Find the length of AE.

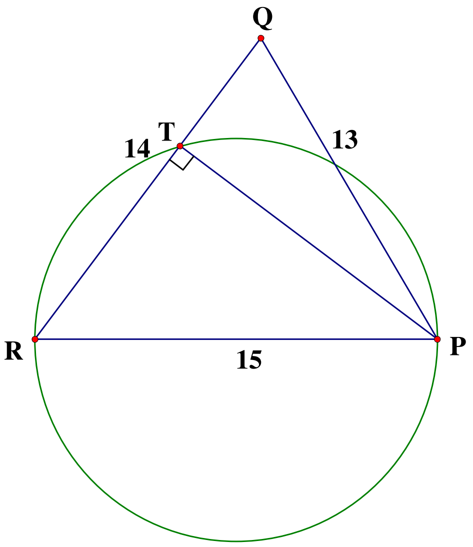


圖三 Figure 3

假設 M 為 BC 的中點。 $BM = MC = 2$ $\Delta ABM \cong \Delta ACM$ (S.S.S.) $\angle AMB = \angle AMC = 90°$ (全等三角形的對應角) Let $\angle ABM = \theta = \angle ACM$ (等腰三角形底角) $\angle ADE = \theta = \angle ACB$ 在 ΔABM 中， $\cos\theta = \frac{2}{8} = \frac{1}{4}$ 於ΔABD 應用餘弦定理： $AD^2 = 1^2 + 8^2 - 2\times1\times8\cos\theta$ $AD = \sqrt{61}$ 設 $\angle CDE = t$ $\angle AED = t + \theta = \angle ADC$ (ΔCDE 的外角) $\Delta ADE \sim \Delta ACD$ (等角) $\frac{AE}{AD} = \frac{AD}{AC}$ (相似三角形的對應邊) $\frac{AE}{\sqrt{61}} = \frac{\sqrt{61}}{8}$ $AE = \frac{61}{8} = 7.625$	Let M be the mid-point of BC. $BM = MC = 2$ $\Delta ABM \cong \Delta ACM$ (S.S.S.) $\angle AMB = \angle AMC = 90°$ (corr. $\angle$s $\cong \Delta$'s) Let $\angle ABM = \theta = \angle ACM$ (base $\angle$s isos. Δ) $\angle ADE = \theta = \angle ACB$ In ΔABM, $\cos\theta = \frac{2}{8} = \frac{1}{4}$ Apply cosine formula on ΔABD: $AD^2 = 1^2 + 8^2 - 2\times1\times8\cos\theta$ $AD = \sqrt{61}$ Let $\angle CDE = t$ $\angle AED = t + \theta = \angle ADC$ (ext. $\angle$ of ΔCDE) $\Delta ADE \sim \Delta ACD$ (equiangular) $\frac{AE}{AD} = \frac{AD}{AC}$ (corr. sides, ~Δ's) $\frac{AE}{\sqrt{61}} = \frac{\sqrt{61}}{8}$ $AE = \frac{61}{8} = 7.625$

I10 PQR 是一個三角形，其中 $PQ = 13$、$QR = 14$ 及 $PR = 15$。以 PR 為直徑繪畫出圓 C，C 相交 QR 於點 T。求 ΔPTR 的面積。

PQR is a triangle with $PQ = 13$, $QR = 14$ and $PR = 15$. The circle C is drawn with diameter PR. C intersects QR at a point T. Find the area of ΔPTR.



$\angle PTR = 90°$ (半圓上的圓周角) $\cos \angle PTR = \dfrac{14^2+15^2-13^2}{2\times 14\times 15} = \dfrac{3}{5}$ $\sin \angle PTR = \dfrac{4}{5}$ $RT = PR \cos \angle PTR = 15\times\dfrac{3}{5} = 9$ ΔPRT 的面積$=\dfrac{1}{2}RP\cdot RT\sin\angle PRT$ $=\dfrac{1}{2}\times 15\times 9\times\dfrac{4}{5}$ $= 54$	$\angle PTR = 90°$ ($\angle$ in semi-circle) $\cos \angle PTR = \dfrac{14^2+15^2-13^2}{2\times 14\times 15} = \dfrac{3}{5}$ $\sin \angle PTR = \dfrac{4}{5}$ $RT = PR \cos \angle PTR = 15\times\dfrac{3}{5} = 9$ Area of $\Delta PRT = \dfrac{1}{2}RP\cdot RT\sin\angle PRT$ $=\dfrac{1}{2}\times 15\times 9\times\dfrac{4}{5}$ $= 54$

I11 求 $3^x + 5 + \dfrac{36}{3^x+4}$ 的最小值。

Find the minimum value of $3^x + 5 + \dfrac{36}{3^x+4}$.

設 $y = 3^x + 4$，則此該表達式變成： $y+\dfrac{36}{y}+1 \;\geq\; 2\sqrt{y\times\dfrac{36}{y}}+1$ (A.M. ≥ G.M.) $= 13$ 等式成立當 $y = \dfrac{36}{y}$; 即 $y = 6$ $3^x + 4 = 6$ $\Rightarrow x = \log 2 \div \log 3$ $\therefore$ 最小值為 13。	Let $y = 3^x + 4$, then the expression becomes: $y+\dfrac{36}{y}+1 \;\geq\; 2\sqrt{y\times\dfrac{36}{y}}+1$ (A.M. ≥ G.M.) $= 13$ Equality holds when $y = \dfrac{36}{y}$; 即 i.e. $y = 6$ $3^x + 4 = 6$ $\Rightarrow x = \log 2 \div \log 3$ $\therefore$ The minimum value is 13 .

The following method is suggested by Mr. Ma Shing, a secondary school teacher:

Let $y = 3^x + 4$ and $T = y + \dfrac{36}{y} + 1 \geq 0$, then the equation becomes: $yT = y^2 + y + 36$

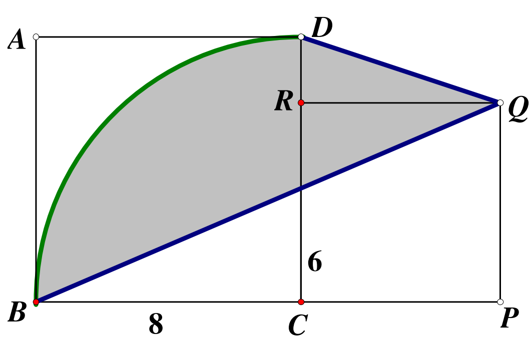
$\Rightarrow y^2 + (1 - T)y + 36 = 0$, a quadratic equation in y. For real values of y, $\Delta = (1 - T)^2 - 4(36) \geq 0$

$\Rightarrow T - 1 \geq 12$ or $T - 1 \leq -12$ (rejected) $\Rightarrow T \geq 13$.

The minimum value is 13 .

I12 如圖四所示，$ABCD$ 及 $PQRC$ 為兩個連接的正方形。以 C 為圓心及 CB 為半徑繪畫出弧 BD。已知 $BC = 8$ 及 $RC = 6$。求弧 BD 及綫段 DQ 與 BQ 所圍成的區域的面積。

As shown in Figure 4, two squares $ABCD$ and $PQRC$ are joined together. An arc BD is drawn with centre C and radius CB. Given that $BC = 8$ and $RC = 6$. Find the area of the region bounded by the arc BD, line segments DQ and BQ.



圖四 Figure 4

Reference: 2000 FI4.2, 2004 HI9, 2005 HG7

陰影面積 $=$ 弓形 $BD + S_{\Delta BRD} + S_{\Delta QRD} + S_{\Delta BRQ}$ $=\frac{1}{4}\pi\cdot 8^2-\frac{1}{2}\cdot 8^2+\frac{1}{2}(8-6)\cdot 8+\frac{1}{2}(8-6)\cdot 6+\frac{1}{2}\cdot 6\times 6$ $= 16\pi - 32 + 8 + 6 + 18 = 16\pi$ **方法二** 連接 BD 及 CQ。 $\angle CBD = \angle PCQ = 45°$ (正方形的性質) $\therefore BD \parallel CQ$ (對應角相等) $S_{\Delta BDQ} = S_{\Delta BDC}$ (兩三角形同底同高) 陰影面積 $=$ 弓形 $BD + S_{\Delta BDQ}$ $=$ 弓形 $BD + S_{\Delta BDC}$ $=$ 扇形 $BDC = 16\pi$	Shaded area = segment $BD + S_{\Delta BRD} + S_{\Delta QRD} + S_{\Delta BRQ}$ $=\frac{1}{4}\pi\cdot 8^2-\frac{1}{2}\cdot 8^2+\frac{1}{2}(8-6)\cdot 8+\frac{1}{2}(8-6)\cdot 6+\frac{1}{2}\cdot 6\times 6$ $= 16\pi - 32 + 8 + 6 + 18 = 16\pi$ **Method 2** Join BD and CQ. $\angle CBD = \angle PCQ = 45°$ (property of a square) $\therefore BD \parallel CQ$ (corr. $\angle$s eq.) $S_{\Delta BDQ} = S_{\Delta BDC}$ (same bases and same heights) Shaded area = segment $BD + S_{\Delta BDQ}$ = segment $BD + S_{\Delta BDC}$ = sector $BDC = 16\pi$

I13 一個四位數可以透過把它的所有數字加起來，變成另一個數。例如：1234 可以變成 10，因為 $1 + 2 + 3 + 4 = 10$。究竟從 1998 至 4998 (包括此兩個數)有多少個四位數，經上述變換後不可以被 3 整除？

A 4-digit number can be transformed into another number by adding its digits. For example, 1234 is transformed into 10 as $1 + 2 + 3 + 4 = 10$. How many transformed numbers from 1998 to 4998 inclusive are **NOT** divisible by 3?

已給一正整數，易證該數能被 3 整除的充分及必要條件是該數的數位之和是 3 的倍數。 我們只須數一數由 1998 至 4998 之間的 3 的倍數。 $1998 = 3\times 666$, $4998 = 3\times 1666$ $\therefore$ 3 的倍數數量是 $1666 - 666 + 1 = 1001$ 不能被 3 整除的數目有 $4998 - 1998 + 1 - 1001 = 2000$	It is easy to show that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3. It suffices to count the number of multiples of 3 from 1998 to 4998. $1998 = 3\times 666$, $4998 = 3\times 3666$ $\therefore$ The number of multiples of 3 is $1666 - 666 + 1 = 1001$ Number of integers which are **NOT** divisible by 3 is $4998 - 1998 + 1 - 1001 = 2000$

I14 對任意實數 x ($x \neq 1$)，定義函數 $\mathrm{f}(x)=\frac{x}{1-x}$ 及 $\mathrm{f}\circ\mathrm{f}(x)=\mathrm{f}(\mathrm{f}(x))$。

求 $2017\underbrace{\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}\circ\cdots\circ\mathrm{f}}_{2018個f}(2018)$ 的值。

For any real number x ($x \neq 1$), define a function $\mathrm{f}(x)=\frac{x}{1-x}$ and $\mathrm{f}\circ\mathrm{f}(x) = \mathrm{f}(\mathrm{f}(x))$.

Find the value of $2017\underbrace{\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}\circ\cdots\circ\mathrm{f}}_{2018\text{ copies of f}}(2018)$.

Reference: 1997 HG2

$\mathrm{f}\circ\mathrm{f}(x)=\mathrm{f}(\mathrm{f}(x))=\mathrm{f}\left(\frac{x}{1-x}\right)=\frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}=\frac{x}{1-x-x}=\frac{x}{1-2x}$ $\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}(x)=\mathrm{f}\left(\frac{x}{1-2x}\right)=\frac{\frac{x}{1-x}}{1-\frac{2x}{1-x}}=\frac{x}{1-x-2x}=\frac{x}{1-3x}$ 聲明： $\underbrace{\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}\circ\cdots\circ\mathrm{f}}_{n\text{ copies of f}}(x)=\frac{x}{1-nx}$ 證明：利用數學歸納法，$n=1$、2、3，上文已證。 假設 $\underbrace{\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}\circ\cdots\circ\mathrm{f}}_{k\text{ 個 f}}(x)=\frac{x}{1-kx}$ 成立。 $\underbrace{\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}\circ\cdots\circ\mathrm{f}}_{k+1\text{ 個 f}}(x)=\mathrm{f}\left(\frac{x}{1-kx}\right)=\frac{\frac{x}{1-x}}{1-\frac{kx}{1-x}}=\frac{x}{1-(k+1)x}$ 如果 $n = k$ 等式成立，$n = k + 1$ 時等式依然成立。 由數學歸納法的原則，對於所有正整數，該等式成立。 $2017\underbrace{\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}\circ\cdots\circ\mathrm{f}}_{2018個f}(2018)$ $=2017\times\frac{2018}{1-2018\times2018}$ $=2017\times\frac{2018}{(1-2018)\times(1+2018)}$ $=-\frac{2018}{2019}$	$\mathrm{f}\circ\mathrm{f}(x)=\mathrm{f}(\mathrm{f}(x))=\mathrm{f}\left(\frac{x}{1-x}\right)=\frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}=\frac{x}{1-x-x}=\frac{x}{1-2x}$ $\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}(x)=\mathrm{f}\left(\frac{x}{1-2x}\right)=\frac{\frac{x}{1-x}}{1-\frac{2x}{1-x}}=\frac{x}{1-x-2x}=\frac{x}{1-3x}$ Claim: $\underbrace{\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}\circ\cdots\circ\mathrm{f}}_{n\text{ copies of f}}(x)=\frac{x}{1-nx}$ Proof: By M.I.. n = 1, 2, 3, proved above Suppose $\underbrace{\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}\circ\cdots\circ\mathrm{f}}_{k\text{ copies of f}}(x)=\frac{x}{1-kx}$ $\underbrace{\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}\circ\cdots\circ\mathrm{f}}_{k+1\text{ copies of f}}(x)=\mathrm{f}\left(\frac{x}{1-kx}\right)=\frac{\frac{x}{1-x}}{1-\frac{kx}{1-x}}=\frac{x}{1-(k+1)x}$ If it is true for $n=k$, then it is also true for $n=k+1$ By the principle of mathematical induction, the formula is true for all positive integer n . $2017\underbrace{\mathrm{f}\circ\mathrm{f}\circ\mathrm{f}\circ\cdots\circ\mathrm{f}}_{2018個f}(2018)$ $=2017\times\frac{2018}{1-2018\times2018}$ $=2017\times\frac{2018}{(1-2018)\times(1+2018)}$ $=-\frac{2018}{2019}$

I15 設 $N^2=\overline{abcdefabc}$ 為一個 9 位整數，其中 N 是 4 個相異質數的積及 a、b、c、d、e、f 均為非零數字且滿足 $\overline{def}=2\times\overline{abc}$。求 N^2 的最小值。

Let $N^2=\overline{abcdefabc}$ be a nine-digit positive integer, where N is the product of four distinct primes and a, b, c, d, e, f are non-zero digits that satisfy $\overline{def}=2\times\overline{abc}$. Find the least value of N^2.

$N^2=\overline{abcdefabc}=1000000\,\overline{abc}+1000\,\overline{def}+\overline{abc}$ $=1000001\,\overline{abc}+2000\,\overline{abc}$ $=1002001\,\overline{abc}$ $=1001^2\times\overline{abc}=(7\times11\times13)^2\times\overline{abc}$ $\therefore\ \overline{abc}=$ 一個三位數不等於 7、11 或 13 的質數的平方 $\overline{abc}$ 的最小值$=17^2=289$	$N^2=\overline{abcdefabc}=1000000\,\overline{abc}+1000\,\overline{def}+\overline{abc}$ $=1000001\,\overline{abc}+2000\,\overline{abc}$ $=1002001\,\overline{abc}$ $=1001^2\times\overline{abc}=(7\times11\times13)^2\times\overline{abc}$ $\therefore\ \overline{abc}=$ a 3-digit number which is the square of a prime number different from 7, 11 and 13 The least possible $\overline{abc}=17^2=289$ The least possible value of N^2

N^2 的最小值 $= 1002001\times289 = 289578289$	$= 1002001\times289 = 289578289$

Group Events

G1 設 $\mathrm{f}(x)$ 為二次多項式，其中 $\mathrm{f}(1)=\frac{1}{2}$，$\mathrm{f}(2)=\frac{1}{6}$，$\mathrm{f}(3)=\frac{1}{12}$。求 $\mathrm{f}(6)$ 的值。

Let $\mathrm{f}(x)$ be a polynomial of degree 2, where $\mathrm{f}(1)=\frac{1}{2}$, $\mathrm{f}(2)=\frac{1}{6}$, $\mathrm{f}(3)=\frac{1}{12}$. Find the value of $\mathrm{f}(6)$.

Reference 2003 FG4.2

設 $\mathrm{f}(x) = ax^2 + bx + c$, 則 $\mathrm{f}(1) = a + b + c = \frac{1}{2} \quad \cdots(1)$ $\mathrm{f}(2) = 4a + 2b + c = \frac{1}{6} \quad \cdots(2)$ $\mathrm{f}(3) = 9a + 3b + c = \frac{1}{12} \quad \cdots(3)$ $(2) - (1)$: $3a + b = -\frac{1}{3} \quad \cdots(4)$ $(3) - (2)$: $5a + b = -\frac{1}{12} \quad \cdots(5)$ $(5) - (4)$: $2a = \frac{1}{4} \Rightarrow a = \frac{1}{8}$ 代 $a = \frac{1}{8}$ 入 (4): $\frac{3}{8} + b = -\frac{1}{3} \Rightarrow b = -\frac{17}{24}$ 代 $a = \frac{1}{8}$、$b = -\frac{17}{24}$ 入(1): $\frac{1}{8} - \frac{17}{24} + c = \frac{1}{2} \Rightarrow c = \frac{13}{12}$ $\mathrm{f}(x) = \frac{1}{8}x^2 - \frac{17}{24}x + \frac{13}{12}$ $\mathrm{f}(6) = \frac{1}{8}\times 6^2 - \frac{17}{24}\times 6 + \frac{13}{12}$ $= \frac{9}{2} - \frac{17}{4} + \frac{13}{12} = \frac{54}{12} - \frac{51}{12} + \frac{13}{12} = \frac{4}{3}$	Let $\mathrm{f}(x) = ax^2 + bx + c$, then $\mathrm{f}(1) = a + b + c = \frac{1}{2} \quad \cdots(1)$ $\mathrm{f}(2) = 4a + 2b + c = \frac{1}{6} \quad \cdots(2)$ $\mathrm{f}(3) = 9a + 3b + c = \frac{1}{12} \quad \cdots(3)$ $(2) - (1)$: $3a + b = -\frac{1}{3} \quad \cdots(4)$ $(3) - (2)$: $5a + b = -\frac{1}{12} \quad \cdots(5)$ $(5) - (4)$: $2a = \frac{1}{4} \Rightarrow a = \frac{1}{8}$ Sub. $a = \frac{1}{8}$ into (4): $\frac{3}{8} + b = -\frac{1}{3} \Rightarrow b = -\frac{17}{24}$ Sub. $a = \frac{1}{8}, b = -\frac{17}{24}$ into (1): $\frac{1}{8} - \frac{17}{24} + c = \frac{1}{2} \Rightarrow c = \frac{13}{12}$ $\mathrm{f}(x) = \frac{1}{8}x^2 - \frac{17}{24}x + \frac{13}{12}$ $\mathrm{f}(6) = \frac{1}{8}\times 6^2 - \frac{17}{24}\times 6 + \frac{13}{12}$ $= \frac{9}{2} - \frac{17}{4} + \frac{13}{12} = \frac{54}{12} - \frac{51}{12} + \frac{13}{12} = \frac{4}{3}$
方法二 設 $\mathrm{F}(x) = x(x + 1)\,\mathrm{f}(x) - 1$， 這是一多項式，其冪次為 4。 $\mathrm{F}(1) = 2\,\mathrm{f}(1) - 1 = 0 \Rightarrow (x - 1)$ 為 $\mathrm{F}(x)$ 的因式 $\mathrm{F}(2) = 6\,\mathrm{f}(2) - 1 = 0 \Rightarrow (x - 2)$ 為 $\mathrm{F}(x)$ 的因式 $\mathrm{F}(3) = 12\,\mathrm{f}(3) - 1 = 0 \Rightarrow (x - 3)$為 $\mathrm{F}(x)$ 的因式 $\mathrm{F}(x) = (x-1)(x-2)(x-3)(ax + b) = x(x + 1)\,\mathrm{f}(x) - 1$ $\mathrm{F}(0) = (-1)(-2)(-3)(0 + b) = 0(0 + 1)\,\mathrm{f}(0) - 1$ $\Rightarrow b = \frac{1}{6}$ $\mathrm{F}(-1) = (-2)(-3)(-4)(-a + b) = -1\times 0\times \mathrm{f}(-1) - 1$ $-24\left(-a + \frac{1}{6}\right) = -1 \Rightarrow a = \frac{1}{8}$ $\mathrm{F}(6) = (6-1)(6-2)(6-3)(\frac{1}{8}\cdot 6 + \frac{1}{6}) = 6\times 7\mathrm{f}(6) - 1$ $5\times 4\times 3\times\left(\frac{3}{4} + \frac{1}{6}\right) = 42\mathrm{f}(6) - 1$ $\mathrm{f}(6) = \frac{4}{3}$	**Method 2** Let $\mathrm{F}(x) = x(x + 1)\,\mathrm{f}(x) - 1$ This is a polynomial of degree 4 $\mathrm{F}(1) = 2\,\mathrm{f}(1) - 1 = 0 \Rightarrow (x - 1)$ is a factor of $\mathrm{F}(x)$ $\mathrm{F}(2) = 6\,\mathrm{f}(2) - 1 = 0 \Rightarrow (x - 2)$ is a factor of $\mathrm{F}(x)$ $\mathrm{F}(3) = 12\mathrm{f}(3) - 1 = 0 \Rightarrow (x - 3)$ is a factor of $\mathrm{F}(x)$ $\mathrm{F}(x) = (x-1)(x-2)(x-3)(ax + b) = x(x + 1)\,\mathrm{f}(x) - 1$ $\mathrm{F}(0) = (-1)(-2)(-3)(0 + b) = 0(0 + 1)\,\mathrm{f}(0) - 1$ $\Rightarrow b = \frac{1}{6}$ $\mathrm{F}(-1) = (-2)(-3)(-4)(-a + b) = -1\times 0\times \mathrm{f}(-1) - 1$ $-24\left(-a + \frac{1}{6}\right) = -1 \Rightarrow a = \frac{1}{8}$ $\mathrm{F}(6) = (6-1)(6-2)(6-3)(\frac{1}{8}\cdot 6 + \frac{1}{6}) = 6\times 7\mathrm{f}(6) - 1$ $5\times 4\times 3\times\left(\frac{3}{4} + \frac{1}{6}\right) = 42\mathrm{f}(6) - 1$ $\mathrm{f}(6) = \frac{4}{3}$

G2 求 $\sqrt{2018\times 2012\times 1988\times 1982+8100}$。

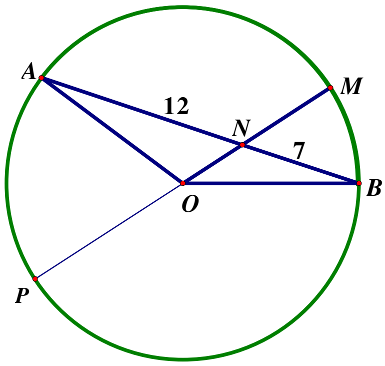
Evaluate $\sqrt{2018\times 2012\times 1988\times 1982+8100}$.

Reference: 2000 FG3.1, 2004 FG3.1, 2012 FI2.3, 2013 HI5

設 $a = 2000$， 則 $\sqrt{2018\times 2012\times 1988\times 1982+8100}$ $=\sqrt{(a+18)\times(a+12)\times(a-12)\times(a-18)+8100}$ $=\sqrt{(a^2-324)\times(a^2-144)+8100}$ $=\sqrt{a^4-468a^2+18^2\times 12^2+90^2}$ $=\sqrt{a^4-468a^2+18^2\times(12^2+5^2)}$ $=\sqrt{a^4-468a^2+18^2\times 13^2}$ $=\sqrt{a^4-468a^2+234^2}$ $=\sqrt{(a^2-234)^2}$ $= 2000^2 - 234$ $= 4000000 - 234$ $= 3999766$	Let $a = 2000$, then $\sqrt{2018\times 2012\times 1988\times 1982+8100}$ $=\sqrt{(a+18)\times(a+12)\times(a-12)\times(a-18)+8100}$ $=\sqrt{(a^2-324)\times(a^2-144)+8100}$ $=\sqrt{a^4-468a^2+18^2\times 12^2+90^2}$ $=\sqrt{a^4-468a^2+18^2\times(12^2+5^2)}$ $=\sqrt{a^4-468a^2+18^2\times 13^2}$ $=\sqrt{a^4-468a^2+234^2}$ $=\sqrt{(a^2-234)^2}$ $= 2000^2 - 234$ $= 4000000 - 234$ $= 3999766$

G3 如圖一所示，OAB 是一個以 O 為圓心的扇形。N 為半徑 OM 與 AB 的交點。已知 $AN = 12$，$BN = 7$ 及 $3ON = 2MN$。求 OM 的長度。

As shown in Figure 1, OAB is a sector with centre O. N is the intersecting point of radius OM and AB. Given that $AN = 12$, $BN = 7$ and $3ON = 2MN$. Find the length of OM.



圖一 Figure 1

將扇形畫至圓形，延長 MO 並交圓形於 P。 設 $ON = 2k$、$MN = 3k$。半徑 $= 5k$ 利用相交弦定理， $PN\times NM = AN\times NB$ $7k\times 3k = 12\times 7$ $k = 2$ $OM = 5k = 10$	Complete the circle. Produce MO to meet the circle again at P. Let $ON = 2k$, $MN = 3k$. The radius $= 5k$ $PN = PO + ON = 5k + 2k = 7k$ By intersecting chords theorem, $PN\times NM = AN\times NB$ $7k\times 3k = 12\times 7$ $k = 2$ $OM = 5k = 10$

G4 對任意非零實數 x，函數 $f(x)$ 有以下特性：$2f(x)+f\left(\frac{1}{x}\right)=11x+4$。設 S 為所有滿足於 $f(x)=2018$ 的根之和。求 S 之值。

For any non-zero real number x, the function $f(x)$ has the following property: $2f(x)+f\left(\frac{1}{x}\right)=11x+4$. Let S be the sum of all roots satisfying the equation $f(x) = 2018$. Find the value of S. **Reference: 2019 HG5**

$2f(x)+f\left(\frac{1}{x}\right)=11x+4 \quad \cdots (1)$ $2f\left(\frac{1}{x}\right)+f(x)=\frac{11}{x}+4 \quad \cdots (2)$ $2(1)-(2)$: $3f(x)=22x-\frac{11}{x}+4$ $\Rightarrow f(x)=\frac{1}{3}\left(22x-\frac{11}{x}+4\right)$ $f(x)=2018 \Rightarrow \frac{1}{3}\left(22x-\frac{11}{x}+4\right)=2018$ $22x-\frac{11}{x}+4=6054$ $22x^2-6050x-11=0$ $2x^2-550x-1=0$ $S=$ 兩根之和 $=-\frac{b}{a}=275$	$2f(x)+f\left(\frac{1}{x}\right)=11x+4 \quad \cdots (1)$ $2f\left(\frac{1}{x}\right)+f(x)=\frac{11}{x}+4 \quad \cdots (2)$ $2(1)-(2)$: $3f(x)=22x-\frac{11}{x}+4$ $\Rightarrow f(x)=\frac{1}{3}\left(22x-\frac{11}{x}+4\right)$ $f(x)=2018 \Rightarrow \frac{1}{3}\left(22x-\frac{11}{x}+4\right)=2018$ $22x-\frac{11}{x}+4=6054$ $22x^2-6050x-11=0$ $2x^2-550x-1=0$ $S=$ sum of roots $=-\frac{b}{a}=275$

G5 求可滿足下列方程組的 x 的值：$\begin{cases} x^2+9x-10y-220=0 & \cdots(1) \\ y^2-5x+6y-166=0 & \cdots(2) \\ xy=195 & \cdots(3) \end{cases}$.

Find the value of x that satisfy the following system of equations: $\begin{cases} x^2+9x-10y-220=0 \\ y^2-5x+6y-166=0 \\ xy=195 \end{cases}$.

$(1)+(2)-2(3)$: $x^2-2xy+y^2+4x-4y-386+390=0$ $(x-y)^2+4(x-y)+4=0 \Rightarrow (x-y+2)^2=0$ $x=y-2 \cdots\cdots (4)$ 代(4)入(3)：$(y-2)y=195$ $y^2-2y-195=0$ $(y-15)(y+13)=0$ $y=15$ 或 -13 當 $y=15$，$x=13$；當 $y=-13$，$x=-15$ 稽核：代 $x=13$、$y=15$ 入(1)： 左式$=13^2+9(13)-10(15)-220=-84$ (捨去) 代 $x=-15$、$y=-13$ 入 (1)： 左式$=(-15)^2+9(-15)-10(-13)-220=0=$右式 代 $x=-15$、$y=-13$ 入 (2)： 左式$=(-13)^2-5(-15)+6(-13)-166=0=$右式 $\therefore x=-15$	$(1)+(2)-2(3)$: $x^2-2xy+y^2+4x-4y-386+390=0$ $(x-y)^2+4(x-y)+4=0 \Rightarrow (x-y+2)^2=0$ $x=y-2 \cdots\cdots (4)$ Sub. (4) into (3): $(y-2)y=195$ $y^2-2y-195=0$ $(y-15)(y+13)=0$ $y=15$ or -13 When $y=15$, $x=13$; when $y=-13$, $x=-15$ Check: Sub. $x=13$, $y=15$ into (1): L.H.S.$=13^2+9(13)-10(15)-220=-84$ (rejected) Sub. $x=-15$, $y=-13$ into (1): L.H.S.$=(-15)^2+9(-15)-10(-13)-220=0=$ R.H.S. Sub. $x=-15$, $y=-13$ into (2): L.H.S.$=(-13)^2-5(-15)+6(-13)-166=0=$ R.H.S. $\therefore x=-15$

G6 已知 $n^4 + 104 = 3^m$，其中 n、m 為正整數。求 n 的最小值。

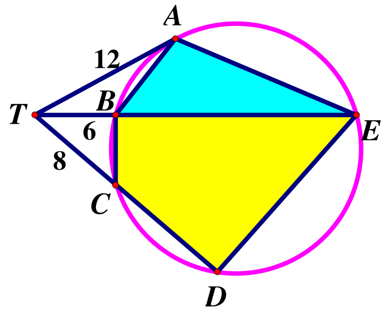
Given that $n^4 + 104 = 3^m$, where n, m are positive integers. Find the least value of n.

Reference: 2013 HI4

$3^3 = 27$，$3^4 = 81$，$3^5 = 243$，$3^6 = 729$ $243 - 104 = 139 \neq n^4$，$729 - 104 = 625 = 5^4$ n 的最小值為 5。	$3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = 729$ $243 - 104 = 139 \neq n^4$, $729 - 104 = 625 = 5^4$ The least value of n is 5.

G7 如圖二所示，A、B、C、D 及 E 為圓上的點。T 是該圓外的一點。TA 是該圓在點 A 的切綫， TBE 及 TCD 為直綫。已知 TBE 是 $\angle ATD$ 的角平分綫、$TA = 12$、$TB = 6$ 及 $TC = 8$。求 ΔABE 與四邊形 $BCDE$ 的面積比。

As shown in Figure 2, A, B, C, D and E are points on the circle. T is a point outside the circle such that TA is a tangent to the circle at A and TBE and TCD are straight lines. It is given that TBE is the angle bisector of $\angle ATD$, $TA = 12$, $TB = 6$ and $TC = 8$. Find the ratio of the area of ΔABE to the area of quadrilateral $BCDE$.



圖二 Figure 2

Reference 2015 HI10

利用相交弦定理， $TB \times TE = TA^2$ $6 \times TE = 12^2$ $TE = 24$ $BE = 24 - 6 = 18$ 設 $\angle ATB = \theta = \angle CTB$ $\frac{S_{\Delta ABT}}{S_{\Delta CTB}} = \frac{\frac{1}{2} TA \cdot TB \sin\theta}{\frac{1}{2} TC \cdot TB \sin\theta} = \frac{12}{8} = \frac{3}{2}$ …… (1) 考慮 ΔABT 及 ΔABE 它們同高不同底 $\frac{S_{\Delta ABE}}{S_{\Delta ABT}} = \frac{BE}{TB} = \frac{18}{6} = 3$ …… (2) 考慮 ΔCTB 及 ΔETD $\angle BTC = \angle DTE$ (公共角) $\angle TBC = \angle TDE$ (圓內接四邊形外角) $\angle TCB = \angle TED$ (圓內接四邊形外角) $\therefore \Delta CTB \sim \Delta ETD$ (等角) $\frac{S_{\Delta ETD}}{S_{\Delta CTB}} = \left(\frac{TE}{TC}\right)^2 = \left(\frac{24}{8}\right)^2 = 9$ $\Rightarrow \frac{S_{BCDE}}{S_{\Delta CTB}} = 9 - 1 = 8$ $\Rightarrow \frac{S_{\Delta CTB}}{S_{BCDE}} = \frac{1}{8}$ …… (3) (1)×(2)×(3): $\frac{\Delta ABE\text{ 的面積}}{BCDE\text{ 的面積}} = \frac{S_{\Delta ABT}}{S_{\Delta CTB}} \times \frac{S_{\Delta ABE}}{S_{\Delta ABT}} \times \frac{S_{\Delta CTB}}{S_{BCDE}}$ $= \frac{3}{2} \times 3 \times \frac{1}{8} = \frac{9}{16}$	By intersecting chords theorem, $TB \times TE = TA^2$ $6 \times TE = 12^2$ $TE = 24$ $BE = 24 - 6 = 18$ Let $\angle ATB = \theta = \angle CTB$ $\frac{S_{\Delta ABT}}{S_{\Delta CTB}} = \frac{\frac{1}{2} TA \cdot TB \sin\theta}{\frac{1}{2} TC \cdot TB \sin\theta} = \frac{12}{8} = \frac{3}{2}$ …… (1) Consider ΔABT and ΔABE They have the same height but different bases. $\frac{S_{\Delta ABE}}{S_{\Delta ABT}} = \frac{BE}{TB} = \frac{18}{6} = 3$ …… (2) Consider ΔCTB and ΔETD $\angle BTC = \angle DTE$ (common $\angle$s) $\angle TBC = \angle TDE$ (ext. $\angle$, cyclic quad.) $\angle TCB = \angle TED$ (ext. $\angle$, cyclic quad.) $\therefore \Delta CTB \sim \Delta ETD$ (equiangular) $\frac{S_{\Delta ETD}}{S_{\Delta CTB}} = \left(\frac{TE}{TC}\right)^2 = \left(\frac{24}{8}\right)^2 = 9$ $\Rightarrow \frac{S_{BCDE}}{S_{\Delta CTB}} = 9 - 1 = 8$ $\Rightarrow \frac{S_{\Delta CTB}}{S_{BCDE}} = \frac{1}{8}$ …… (3) (1)×(2)×(3): $\frac{\text{area of }\Delta ABE}{\text{area of }BCDE} = \frac{S_{\Delta ABT}}{S_{\Delta CTB}} \times \frac{S_{\Delta ABE}}{S_{\Delta ABT}} \times \frac{S_{\Delta CTB}}{S_{BCDE}}$ $= \frac{3}{2} \times 3 \times \frac{1}{8} = \frac{9}{16}$

G8 已知 a、b、c、d、e、f、g 及 h 為正整數，使得 $a>b>c>d>e>f>g>h$ 及 $a+h=b+g=c+f=d+e=35$，問有多少組可行答案 $\{a, b, c, d, e, f, g, h\}$ 存在？

Given that a, b, c, d, e, f, g and h are positive integers such that $a>b>c>d>e>f>g>h$ and $a+h=b+g=c+f=d+e=35$. How many possible solution sets of $\{a, b, c, d, e, f, g, h\}$ exist?

$d+e=35\Rightarrow d>17.5>e$ $\Rightarrow a>b>c>d>17.5>e>f>g>h>0$ $\Rightarrow 17\ge e>f>g>h\ge 1$ 可行的組合 = 由 1 至 17 之中任意選取 4 個不同數字的方法。 選取方法的數量= $C_4^{17}=\frac{17\times16\times15\times14}{1\times2\times3\times4}=2380$	$d+e=35\Rightarrow d>17.5>e$ $\Rightarrow a>b>c>d>17.5>e>f>g>h>0$ $\Rightarrow 17\ge e>f>g>h\ge 1$ It is equivalent to choose 4 distinct integers from 1 to 17 . No. of ways = $C_4^{17}=\frac{17\times16\times15\times14}{1\times2\times3\times4}=2380$

G9 求 $\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}\right)+\left(\frac{2}{3}+\frac{2}{4}+\cdots+\frac{2}{100}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{100}\right)+\cdots+\left(\frac{98}{99}+\frac{98}{100}\right)+\frac{99}{100}$ 的值。

Find the value of $\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}\right)+\left(\frac{2}{3}+\frac{2}{4}+\cdots+\frac{2}{100}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{100}\right)+\cdots+\left(\frac{98}{99}+\frac{98}{100}\right)+\frac{99}{100}$.

Reference 1995 HG3, 1996 FG9.4, 2004 HG1

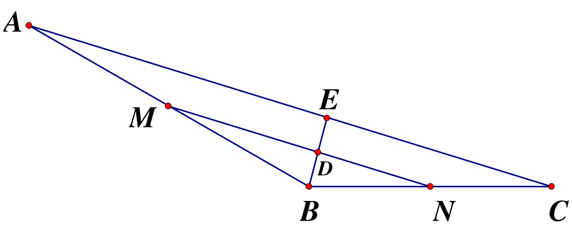
$$\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}\right)+\left(\frac{2}{3}+\frac{2}{4}+\cdots+\frac{2}{100}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{100}\right)+\cdots+\left(\frac{98}{99}+\frac{98}{100}\right)+\frac{99}{100}$$

$$=\frac{1}{2}+\frac{\frac{3\times2}{2}}{3}+\frac{\frac{4\times3}{2}}{4}+\frac{\frac{5\times4}{2}}{5}+\cdots+\frac{\frac{100\times99}{2}}{100}=\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\frac{4}{2}+\cdots+\frac{99}{2}=\frac{1}{2}(1+2+3+4+\cdots+99)$$

$$=\frac{1}{2}\times\frac{1}{2}\cdot100\cdot99=2475$$

G10 如圖三所示，ABC 是一個三角形，其中 $AB=40$、$BC=30$ 及 $\angle ABC=150°$。M 及 N 分別為 AB 及 BC 的中點。$\angle ABC$ 的角平分綫分別相交 MN 及 AC 於 D 及 E。求 $AMDE$ 的面積。

As shown in Figure 3, ABC is a triangle with $AB=40$, $BC=30$ and $\angle ABC=150°$. M and N are the mid-points of AB and BC respectively. The angle bisector of $\angle ABC$ intersects MN and AC at D and E respectively. Find the area of quadrilateral $AMDE$.

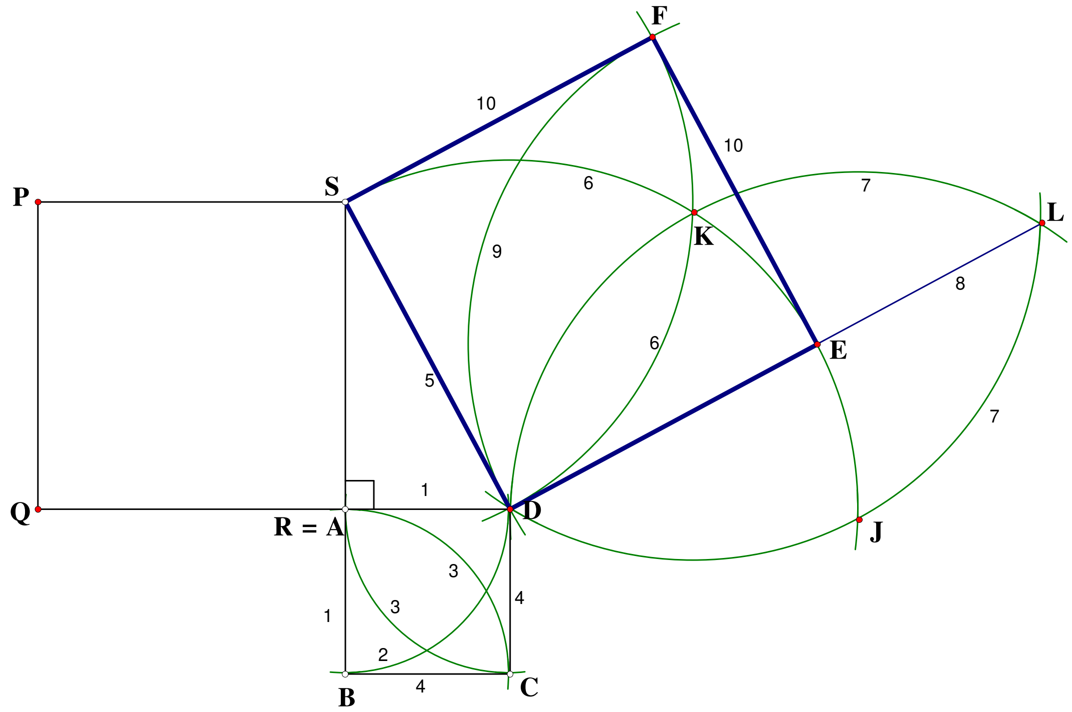


圖三 Figure 3

$MN // AC$ (中點定理) Let $BD=x$, $BE=2x$ (截綫定理) $S_{\Delta BMN}=\frac{1}{2}\cdot20\times15\sin150^\circ=75$ $S_{\Delta BAC}=\frac{1}{2}\cdot40\times30\sin150^\circ=300$ $S_{\Delta BDN}:S_{\Delta BDM}=\frac{1}{2}\cdot15x\sin75^\circ:\frac{1}{2}\cdot20x\sin75^\circ=3:4$ $S_{\Delta BDM}=75\times\frac{4}{7}=\frac{300}{7}$ 同理，$S_{\Delta BEA}=300\times\frac{4}{7}=\frac{1200}{7}$ $S_{AMDE}=\frac{1200}{7}-\frac{300}{7}=\frac{900}{7}=128\frac{4}{7}$	$MN // AC$ (mid-point theorem) Let $BD=x$, $BE=2x$ (intercept theorem) $S_{\Delta BMN}=\frac{1}{2}\cdot20\times15\sin150^\circ=75$ $S_{\Delta BAC}=\frac{1}{2}\cdot40\times30\sin150^\circ=300$ $S_{\Delta BDN}:S_{\Delta BDM}=\frac{1}{2}\cdot15x\sin75^\circ:\frac{1}{2}\cdot20x\sin75^\circ=3:4$ $S_{\Delta BDM}=75\times\frac{4}{7}=\frac{300}{7}$ Similarly, $S_{\Delta BEA}=300\times\frac{4}{7}=\frac{1200}{7}$ $S_{AMDE}=\frac{1200}{7}-\frac{300}{7}=\frac{900}{7}=128\frac{4}{7}$

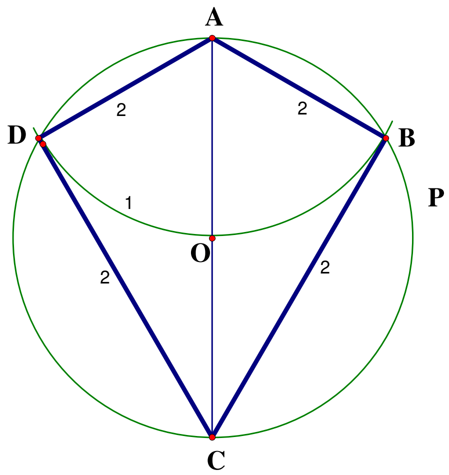
Geometrical Construction

1. 求作一個正方形使得其面積等於下圖的兩個正方形 *ABCD* 及 *PQRS* 的面積之和。
   Construct a square whose area is equal to the sum of the areas of the squares *ABCD* and *PQRS* as shown below.
   Reference: 2015 HC3



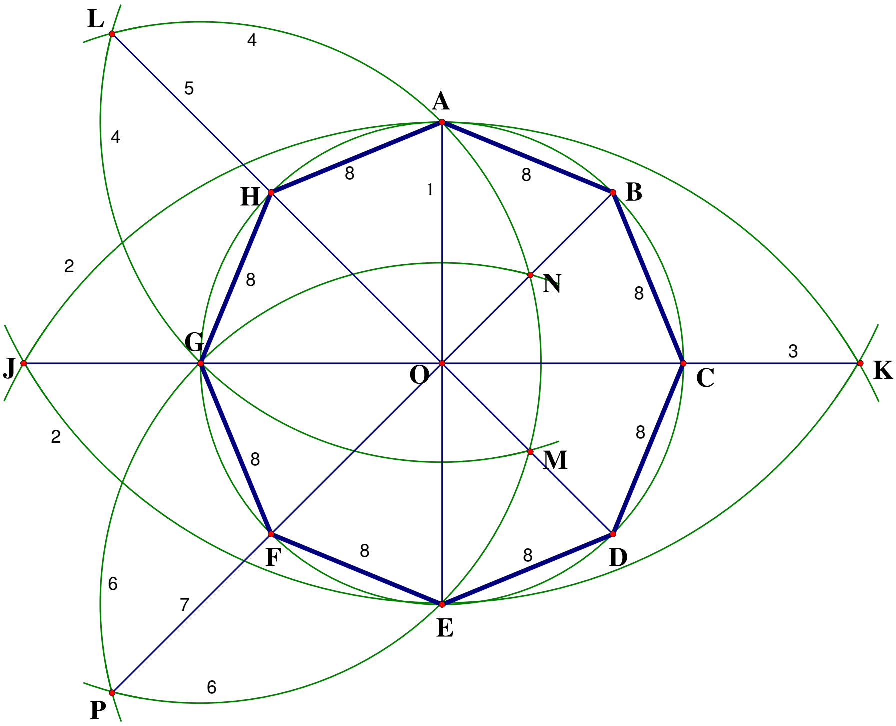
作圖方法如下：	Construction steps:
(1) 將較大的正方形 *PQRS* 抄至上圖，延長 *QR* 及 *SR*。	(1) Copy the larger square *PQRS* as shown, produce *QR* and *SR*.
(2) 以 *R* 為圓心，較小的正方形 *ABCD* 的邊長 *AD* 為半徑作一弧，分別交 *QR* 及 *SR* 的延長綫於 *D* 及 *B*，重新命名 *R* 為 *A*。	(2) Use *R* as centre, the length of side *AD* of the smaller square *ABCD* as radius to draw an arc, intersecting *QR* and *SR* produced at *D* and *B* respectively, rename *R* as *A*.
(3) 以 *B* 為圓心，*BA* 為半徑作一弧；以 *D* 為圓心，*DA* 為半徑作一弧。兩弧相交於 *A* 及 *C*。	(3) Use *B* as centre, *BA* as radius to draw an arc; use *D* as centre, *DA* as radius to draw an arc. The two arcs intersect at *A* and *C*.
(4) 連接 *BC* 及 *DC*，*ABCD* 為較小的正方形，且 $RS \perp AD$。	(4) Join *BC* and *DC*, *ABCD* is the smaller square and $RS \perp AD$.
(5) 連接 *SD*。	(5) Join *SD*.
(6) 以 *D* 為圓心，*DS* 為半徑作一弧；以 *S* 為圓心，*SD* 為半徑作一弧。兩弧相交於 *K*。	(6) Use *D* as centre, *DS* as radius to draw an arc; use *S* as centre, *SD* as radius to draw an arc. The two arcs intersect at *K*.
(7) 以 *K* 為圓心，*KD* 為半徑作一弧，交步驟(6)的弧於 *D* 及 *J*。以 *J* 為圓心，*JD* 為半徑作一弧，交剛才的弧於 *D* 及 *L*。	(7) Use *K* as centre, *KD* as radius to draw an arc, intersecting the arc in step (6) at *D* and *J*. Use *J* as centre, *JD* as radius to draw an arc. The two arcs intersect at *D* and *L*.
(8) 連接 *DL*，交步驟(6)的弧於 *E*。	(8) Join *DL*, intersecting the arc in step (6) at *E*.
(9) 以 *E* 為圓心，*ED* 為半徑作一弧，交步驟(6)的弧於 *D* 及 *F*。	(9) Use *E* as centre, *ED* as radius to draw an arc, intersecting the arc in step (6) at *D* and *F*.
(10) 連接 *EF* 及 *SF*。	(10) Join *EF* and *SF*.
DEFS 便是所須的正方形，證明從略。	*DEFS* is the required square, proof omitted.

2. 已知 AC 是一條通過一個以 O 作圓心的綫段，如下圖所示。求作一個鳶形 $ABCD$ 使得 $\angle BAD = 2\times\angle BCD$ 及 B、D 分別位於圓 APC 上。
Given that AC is a line segment passing through the centre O of a circle, as shown in the figure below. Construct a kite $ABCD$ such that $\angle BAD = 2\times\angle BCD$ and B, D lies on the circle APC.
Remark: There is a typing mistake in the Chinese old version: $\angle BAC = 2\times\angle BDC$.



作圖方法如下：	Construction steps:
(1) 以 A 為圓心，AO 為半徑作一弧，分別交圓於 B 及 D。	(1) Use A as centre, AO as radius to construct an arc, intersecting the circle at B and D respectively.
(2) 連接 AB、BC、CD 及 DA。	(2) Join AB, BC, CD and DA.
$ABCD$ 便是所須鳶形，作圖完畢。	$ABCD$ is the required kite, construction complete.
證明如下：	Proof:
$AD = AO = OD$ (圓的半徑)	$AD = AO = OD$ (radii)
ΔAOD 為等邊三角形。	ΔAOD is an equiangular triangle
同理，ΔAOB 亦為等邊三角形。	Similarly, ΔAOB is also an equiangular triangle.
$\angle DAO = \angle BAO = 60°$ (等邊三角形的性質)	$\angle DAO = \angle BAO = 60°$ (Property of equilateral Δ)
$\angle BAD = 120°$	$\angle BAD = 120°$
$\angle BCD = 60°$ (圓內接四邊形對角)	$\angle BCD = 60°$ (opp. $\angle$s, cyclic quad.)
$\therefore \angle BAD = 2\times\angle BCD$	$\therefore \angle BAD = 2\times\angle BCD$
易證 $\Delta ABC \cong \Delta ADC$ (S.A.S.)	It is easy to show that $\Delta ABC \cong \Delta ADC$ (S.A.S.)
$\therefore AB = AD$ 及 $BC = DC$ (全等三角形對應邊)	$\therefore AB = AD$ 及 $BC = DC$ (corr. sides, $\cong\Delta$'s)
$ABCD$ 是一個鳶形。	$ABCD$ is a kite

3. 作一個以 O 為圓心的圓上的正八邊形 $ABCDEFGH$。
Construct a regular octagon $ABCDEFGH$ on a circle with centre O .



作圖方法如下： (1) 作直徑 AOE。 (2) 以 A 為圓心，AE 為半徑作一弧；以 E 為圓心，EA 為半徑作一弧；兩弧相交於 J 及 K。 (3) 連接 JK，交圓於 G 及 C。 (4) 以 A 為圓心，AG 為半徑作一弧；以 G 為圓心，GA 為半徑作一弧；兩弧相交於 L 及 M。 (5) 連接並延長 LM，交圓於 H 及 D。 (6) 以 G 為圓心，GE 為半徑作一弧；以 E 為圓心，EG 為半徑作一弧；兩弧相交於 P 及 N。 (7) 連接並延長 PN，交圓於 F 及 B。 (8) 連接 AB、BC、CD、DE、EF、FG、GH 及 HA。 $ABCDEFGH$ 便是所須的正八邊形，證明從略。	Construction steps: (1) Construct a diameter AOE. (2) Use A as centre, AE as radius to draw an arc; use E as centre, EA as radius to draw another arc; the two arcs intersect at J and K. (3) Join JK, intersecting the circle at G and C. (4) Use A as centre, AG as radius to draw an arc; use G as centre, GA as radius to draw another arc; the two arcs intersect at L and M. (5) Join and produce LM, intersecting the circle at H and D. (6) Use G as centre, GE as radius to draw an arc; use E as centre, EG as radius to draw another arc; the two arcs intersect at P and N. (7) Join and produce PN, intersecting the circle at F and B. (8) Join AB, BC, CD, DE, EF, FG, GH and HA. $ABCDEFGH$ is the required regular octagon, proof omitted.