

Divisible by 11

Created by Mr. Francis Hung on 20110422

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Given a 5-digits integer $x = \overline{abcde}$. If $a + c + e - (b + d) = 11k$, where k is an integer, prove that x is divisible by 11.

$$\begin{aligned}\text{Proof: } x &= 10000a + 1000b + 100c + 10d + e \\ &= 9999a + a + 1001b - b + 99c + c + 11d - d + e \\ &= 11(909a + 91b + 9c + d) + a - b + c - d + e \\ &= 11(900a + 100b + 9c + d) + 11k, \text{ which is divisible by 11.}\end{aligned}$$

HKHLE General Mathematics 1976 Q8(b)

Prove that, for any positive integer n , the integer $10^n + (-1)^{n-1}$ is divisible by 11.

Hence deduce a necessary and sufficient condition for an integer to be divisible by 11 by considering only the sum and difference of the digits of the integer.

Induction on n .

$n = 1$, $10^1 + (-1)^0 = 11$ which is obviously divisible by 11.

Suppose $10^k + (-1)^{k-1} = 11m$, where m is an integer, for some positive integer k .

$$\begin{aligned}10^{k+1} + (-1)^k &= 10(10^k) + (-1)^k = 10[11m - (-1)^{k-1}] + (-1)^k = 110m + (-1)^k(1 + 10) = 11[10m + (-1)^k] \\ &\text{which is divisible by 11.}\end{aligned}$$

So, by M.I., $10^n + (-1)^{n-1}$ is divisible by 11 for any positive integer n .

The necessary and sufficient condition is: Let S_1 be the sum of all odd digits of an integer N , S_2 be the sum of all even digits of N . $S_1 - S_2$ is divisible by 11 if and only if N is divisible by 11.

Proof: Let $N = a_n \times 10^n + a_{n-1} \times 10^{n-1} + \dots + a_1 \times 10 + a_0$, where $0 \leq a_r \leq 9$ and a_r are integers, $0 \leq r \leq n$.

$$S_1 - S_2 = (-1)^{n-1}a_n + (-1)^{n-2}a_{n-1} + \dots + a_1 - a_0$$

If N is divisible by 11, $a_n \times 10^n + a_{n-1} \times 10^{n-1} + \dots + a_1 \times 10 + a_0 = 11m$, where m is an integer.

$$\begin{aligned}N &= 11m = a_n \times 10^n + a_{n-1} \times 10^{n-1} + \dots + a_1 \times 10 + a_0 \\ &= a_n \times [10^n + (-1)^{n-1}] + a_{n-1} \times [10^{n-1} + (-1)^{n-2}] + \dots + a_1 \times [10 + 1] + a_0(1 - 1) \\ &\quad - [(-1)^{n-1}a_n + (-1)^{n-2}a_{n-1} + \dots + a_1 - a_0] \\ &= a_n \times 11k_n + a_{n-1} \times 11k_{n-1} + \dots + a_1 \times 11k_1 + a_0 \times 11k_0 + S_1 - S_2, \text{ where } k_r \text{ are integers, } 0 \leq r \leq n \\ \Rightarrow S_1 - S_2 &= 11m - [a_n \times 11k_n + a_{n-1} \times 11k_{n-1} + \dots + a_1 \times 11k_1 + a_0 \times 11k_0] \\ &= 11[m - (a_n k_n + a_{n-1} k_{n-1} + \dots + a_1 k_1 + a_0 k_0)], \text{ which is divisible by 11.}\end{aligned}$$

If $S_1 - S_2$ is divisible by 11, then $(-1)^{n-1}a_n + (-1)^{n-2}a_{n-1} + \dots + a_1 - a_0 = 11m$, where m is an integer.

$$\begin{aligned}N &= a_n \times 10^n + a_{n-1} \times 10^{n-1} + \dots + a_1 \times 10 + a_0 \\ &= a_n \times [10^n + (-1)^{n-1}] + a_{n-1} \times [10^{n-1} + (-1)^{n-2}] + \dots + a_1 \times [10 + 1] + a_0(1 - 1) \\ &\quad - [(-1)^{n-1}a_n + (-1)^{n-2}a_{n-1} + \dots + a_1 - a_0] \\ &= a_n \times 11k_n + a_{n-1} \times 11k_{n-1} + \dots + a_1 \times 11k_1 + a_0 \times 11k_0 - (S_1 - S_2) \\ &= 11(a_n k_n + a_{n-1} k_{n-1} + \dots + a_1 k_1 + a_0 k_0) - 11m, \text{ which is divisible by 11.}\end{aligned}$$

The problem is solved.