Individual Events

I1	A	1	I2	α	119999	I3	A	16	I4	α	1
	B	10		β	$\frac{81}{16}=5\frac{1}{16}$		B	399		β	8
	C	42		γ	–4		C	800		γ	0
	D	1260		δ	495		D	4		δ	110

Group Events

G1	X	10	G2	Min	855	G3	s	$\frac{10}{11}$	G4	積	1
$x^5-\frac{1}{x^5}$		±1364	no. opp. 10		8	$ECBF$		$186\sqrt{3}$ see the remark		m	0
Area		2	H.C.F.		9	f(–1)		2	體積		$\frac{\sqrt{2}}{12}$
2023位		7	Probability		0.001215	Area		$12-\frac{25\pi}{9}$	P的位數		34

Individual Event 1

I1.1 若 A 是 2023^{2024} 的個位數，求 A 的值。

If A is the units digit of 2023^{2024}, find the value of A.

Reference: 2014 FI2.4

$3^1\equiv 3$, $3^2\equiv 9$, $3^3\equiv 7$, $3^4\equiv 1$ (mod 10) 這個數字的規律每隔 4 的倍數重複一次。 $2024=4\times 506$ $2023^{2024}\equiv (3^4)^{506}$ (mod 10) $\equiv 1^{506}$ (mod 10) $A=1$	$3^1\equiv 3$, $3^2\equiv 9$, $3^3\equiv 7$, $3^4\equiv 1$ (mod 10) This pattern repeats for every multiples of 4. $2024=4\times 506$ $2023^{2024}\equiv (3^4)^{506}$ (mod 10) $\equiv 1^{506}$ (mod 10) $A=1$

I1.2 若 B 是 336^A 和 528^A 的正公因數的數量，求 B 的值。

If B is the number of positive common factors of 336^A and 528^A, find the value of B.

Reference: 1998 FI1.4

$336=2^4\times 3\times 7$ $528=2^4\times 3\times 11$ 正公因數的數量 $=(4+1)(1+1)=10=B$	$336=2^4\times 3\times 7$ $528=2^4\times 3\times 11$ No. of common factors $=(4+1)(1+1)=10=B$

I1.3 下圖是一個未完成的九宮格，每一格須填入一個正整數使得每一行、每一列和每一對角綫上的三個數字總和相等。求 C 的值。

A 3×3 grid is partially completed as shown below. Fill each square of the grid with a positive integer such that the sum of the three numbers in each row, column and each diagonal are equal. Find the value of C. **Reference: 2019 FG3.3**

C	16	$2B$
4		

C	16	20
4	x	y
z	t	v

設其他未填的空格為 x、y、z、t 及 v 如圖所示。設每一行、列和斜行之和為 s。 $s = C + 16 + 20 \cdots (1)$, $s = 4 + x + y \cdots (2)$ $s = C + 4 + z \cdots (3)$, $s = C + x + v \cdots (4)$ $s = z + t + v \cdots (5)$, $s = 16 + x + t \cdots (6)$ $s = 20 + y + v \cdots (7)$, $s = 20 + x + z \cdots (8)$ $(1) = (4)$ $x + v = 36 \cdots (9)$ $(2) = (7)$: $x = 16 + v \cdots (10)$ 代(10)入(9)：$16 + v + v = 36 \Rightarrow v = 10 \cdots (11)$ $(1) = (3)$：$z + 4 = 16 + 20$ $z = 32 \cdots (12)$ $(2) = (8)$ 及代 (12)：$4 + x + y = 20 + x + 32$ $y = 48 \cdots (13)$ $(2) = (6)$ 及代 (13): $4 + x + 48 = 16 + x + t$ $t = 36 \cdots (14)$ $(4) = (8)$及代(11)，(12): $C + x + 10 = 20 + x + 32$ $C = 42$	Let the other empty cells be x, y, z, t and v as shown. Let the sum be s. $s = C + 16 + 20 \cdots (1)$, $s = 4 + x + y \cdots (2)$ $s = C + 4 + z \cdots (3)$, $s = C + x + v \cdots (4)$ $s = z + t + v \cdots (5)$, $s = 16 + x + t \cdots (6)$ $s = 20 + y + v \cdots (7)$, $s = 20 + x + z \cdots (8)$ $(1) = (4)$ $x + v = 36 \cdots (9)$ $(2) = (7)$: $x = 16 + v \cdots (10)$ Sub. (10) into (9): $16 + v + v = 36 \Rightarrow v = 10 \cdots (11)$ $(1) = (3)$: $z + 4 = 16 + 20$ $z = 32 \cdots (12)$ $(2) = (8)$ and sub. (12): $4 + x + y = 20 + x + 32$ $y = 48 \cdots (13)$ $(2) = (6)$ and sub. (13): $4 + x + 48 = 16 + x + t$ $t = 36 \cdots (14)$ $(4) = (8)$, sub. (11),(12): $C + x + 10 = 20 + x + 32$ $C = 42$

I1.4 有 $\frac{C}{2}$ 對夫婦參加了一個派對，即在派對上共有C人。在這個派對上，沒有人會和同一位客人重複地握手。此外，每位丈夫都會和他妻子以外的所有客人握手，而妻子們不會與其他妻子握手，但會和其他客人握手。D是在這派對上 C 人之間握手的總數，求D的值。

$\frac{C}{2}$ couples are attending a party, which means that there are C people present. At this party, no one will shake hands repeatedly with the same guest. The party also has the condition that each husband will shake hands with every guest except his own wife, and wives will shake hands with every guest except other wives. D represents the total number of handshakes between the C people at the party. Find value of D.

一共有21名丈夫及21名妻子。 每一名丈夫與40名其他客人握手。 每一名妻子與20名其他丈夫握手。 $D = 21\times40 + 21\times20 = 1260$	Altogether there are 21 husbands and 21 wives. For each husband, he will shake hands with 40 other persons. For each wife, she will shake hands with 20 other husbands. $D = 21\times40 + 21\times20 = 1260$

Individual Event 2

I2.1 找出一個能被 11 整除，且各數位之和是 38 的最小正整數 α。

Find the smallest positive integer α that is divisible by 11 and the sum of its digits is equal to 38.

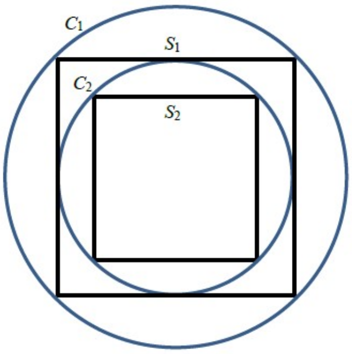
設$\alpha=\overline{a_1b_1a_2b_2\cdots a_nb_n}$. $(a_1+a_2+\cdots+a_n)-(b_1+b_2+\cdots+b_n)=11m\cdots(1)$ $(a_1+a_2+\cdots+a_n)+(b_1+b_2+\cdots+b_n)=38\quad\cdots(2)$ 要使得α為最小，$m=0$。 $a_1+a_2+\cdots+a_n=b_1+b_2+\cdots+b_n=19$ $a_1=1=b_1, a_2=a_3=b_2=b_3=9$ α = 119999	Let $\alpha=\overline{a_1b_1a_2b_2\cdots a_nb_n}$. $(a_1+a_2+\cdots+a_n)-(b_1+b_2+\cdots+b_n)=11m\cdots(1)$ $(a_1+a_2+\cdots+a_n)+(b_1+b_2+\cdots+b_n)=38\quad\cdots(2)$ For the smallest α, $m=0$. $a_1+a_2+\cdots+a_n=b_1+b_2+\cdots+b_n=19$ $a_1=1=b_1, a_2=a_3=b_2=b_3=9$ α = 119999

I2.2 若 α 的最後一位數字是 α′。C_1 是正方形 S_1 的外接圓，它的半徑為 α′，C_2 是正方形 S_1 的內切圓；同時也是正方形 S_2 的外接圓，如此類推。求正方形 S_6 的面積 β。

Let α′ be the last digit of α. A circle C_1 of radius α′ circumscribes a square S_1 which inscribes a circle C_2.

C_2 circumscribes square S_2 and so forth indefinitely.

Find the area β of the square S_6.



設圓形C_1、C_2、…、C_6 的半徑為r_1、r_2、…、r_6。 設正方形S_1、S_2、…、S_6的邊長為x_1、x_2、…、x_6。 $r_1=9,\ 2x_1^2=18^2\Rightarrow x_1=9\sqrt{2}$ $2r_2=9\sqrt{2}\Rightarrow r_2=\frac{9}{\sqrt{2}},\ 2x_2^2=\left(9\sqrt{2}\right)^2\Rightarrow x_2=9$ x_1、x_2、…、x_6 形成一等比數列，公比=$\frac{1}{\sqrt{2}}$。 $x_6=9\sqrt{2}\times\left(\frac{1}{\sqrt{2}}\right)^{6-1}=\frac{9}{\left(\sqrt{2}\right)^4}=\frac{9}{4}$ $\beta=\left(\frac{9}{4}\right)^2=\frac{81}{16}=5\frac{1}{16}$	Let the radii of C_1, C_2, ⋯ , C_6 be r_1, r_2, ⋯ , r_6. Let the sides of S_1, S_2, ⋯ , S_6 be x_1, x_2,⋯, x_6. $r_1=9,\ 2x_1^2=18^2\Rightarrow x_1=9\sqrt{2}$ $2r_2=9\sqrt{2}\Rightarrow r_2=\frac{9}{\sqrt{2}},\ 2x_2^2=\left(9\sqrt{2}\right)^2\Rightarrow x_2=9$ x_1, x_2, ⋯ , x_6 form a geometric sequence with common ratio $\frac{1}{\sqrt{2}}$. $x_6=9\sqrt{2}\times\left(\frac{1}{\sqrt{2}}\right)^{6-1}=\frac{9}{\left(\sqrt{2}\right)^4}=\frac{9}{4}$ $\beta=\left(\frac{9}{4}\right)^2=\frac{81}{16}=5\frac{1}{16}$

I2.3 設 β 的整數部分是 [β]。若 m、n 為整數，方程 $x^3+nx^2+mx+[\beta]=0$ 有三個整數根。假設這三個根不全是正整數，若 $\gamma=n-m$，求 γ 的值。

Let [β] be the integral part of β.

The equation $x^3+nx^2+mx+[\beta]=0$, where m, n are integers, has three integral roots.

Suppose that the roots are not all positive, if $\gamma=n-m$, find the value of γ.

$x^3+nx^2+mx+5=0$ 5 的因子是 $\pm1, \pm5$. ∵這三個根不全是正整數 $\therefore\ x^3+nx^2+mx+5\equiv(x-1)(x-5)(x+1)$ $x^3-5x^2-x+5=0, n=-5, m=-1$ $\gamma=n-m=-5-(-1)=-4$	$x^3+nx^2+mx+5=0$ Factors of 5 are $\pm1, \pm5$. ∵The roots are not all positive $\therefore\ x^3+nx^2+mx+5\equiv(x-1)(x-5)(x+1)$ $x^3-5x^2-x+5=0, n=-5, m=-1$ $\gamma=n-m=-5-(-1)=-4$

I2.4 在 x-y 座標平面上，每一步移動都包含 x 座標和 y 座標分別增加(或減少)1 個單位(即對角綫移動)。若 δ 是由 (0, 0) 開始行走 12 步後到達 (γ, γ) 的方法的數目，求 δ 的值。

On the x-y coordinate plane, a move consists of independently increasing (or decreasing) x-coordinate and y-coordinate by 1 (i.e. moving diagonally). If δ is the number of ways to start from (0, 0), make 12 moves and end at (γ, γ), find the value of δ.

由(0, 0) 移動12步到達(–4, –4)。 可能的步法 $=-1-1-1-1+(1-1)+(1-1)+(1-1)+(1-1)$ 及以上項目的不同掉動。 我們將8個‘–1’放在12個位置上及4個‘1’放在其餘位置上。 $\delta=C_4^{12}=\frac{12\times 11\times 10\times 9}{1\times 2\times 3\times 4}=495$	From (0, 0) moves 12 steps to (–4, –4). Possible movements $=-1-1-1-1+(1-1)+(1-1)+(1-1)+(1-1)$ and different arrangements of these terms We put 8 ‘–1’ into 12 positions and 4‘1’ into the remaining positions. $\delta=C_4^{12}=\frac{12\times 11\times 10\times 9}{1\times 2\times 3\times 4}=495$

Individual Event 3

I3.1 已知 m 和 n 均為正整數。如果 $m+n+mn=76$ 及 $A=m+n$，求 A 的值。

Given that m and n are positive integers. If $m+n+mn=76$ and $A=m+n$, find the value of A.

Reference: 2023 HI2

$m+n+mn=76 \Rightarrow m+n+mn+1=77$ $(m+1)(n+1)=7\times 11$ $(m+1, n+1)=(1, 77), (7, 11), (11, 7)$或$(77, 1)$ $(m, n)=(0, 76), (6, 10), (10, 6)$或$(76, 0)$ $\because$ m 和 n 均為正整數 $\therefore$ $(m, n)=(6, 10)$ or $(10, 6)$ $A=m+n=16$	$m+n+mn=76 \Rightarrow m+n+mn+1=77$ $(m+1)(n+1)=7\times 11$ $(m+1, n+1)=(1, 77), (7, 11), (11, 7)$ or $(77, 1)$ $(m, n)=(0, 76), (6, 10), (10, 6)$ or $(76, 0)$ $\because$ m, n are positive integer $\therefore$ $(m, n)=(6, 10)$ or $(10, 6)$ $A=m+n=16$

I3.2 如果 $B=\sqrt{(401)^2-100A}$，求 B 的值。

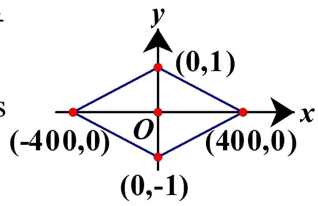
If $B=\sqrt{(401)^2-100A}$, find the value of B.

$$B=\sqrt{(401)^2-100\times 16}$$
$$=\sqrt{401^2-40^2}=\sqrt{(401+40)(401-40)}$$
$$=\sqrt{441\times 361}=\sqrt{21^2\times 19^2}$$
$$=21\times 19=(20+1)(20-1)=20^2-1^2=399$$

I3.3 在 x-y 座標平面上，由 $(B+1, 0)$、$(-B-1, 0)$、$(0, 1)$ 及 $(0, -1)$所形成之菱形的面積為 C 平方單位，求 C 的值。

The area of the rhombus on the x-y coordinate plane with vertices $(B+1, 0)$, $(-B-1, 0)$, $(0, 1)$ and $(0, -1)$ is C square units.

Find the value of C. **Reference: 2024 FI1.3**



$$C=\frac{1}{2}[1-(-1)]\cdot[400-(-400)]$$
$$=800$$

I3.4 如果 D 是正整數且 $\left(\frac{C}{4}+56\right)^{\frac{1}{D}}=D$，求 D 的值。

If D is a positive integer such that $\left(\frac{C}{4}+56\right)^{\frac{1}{D}}=D$, find the value of D.

Reference: 2024 FI1.4

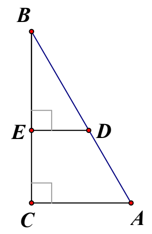
$$\left(\frac{800}{4}+56\right)^{\frac{1}{D}}=D$$
$$(256)^{\frac{1}{D}}=D$$
$$4^4=D^D$$
$$D=4$$

Individual Event 4

I4.1 在三角形ABC中，$\angle C = 90°$，$DE \perp BC$，$BE = AC$，$BD = \frac{1}{2}$ 及 $DE + BC = 1$。如果 $\alpha = 4ED$，求 α 的值。

In triangle ABC, $\angle C = 90°$, $DE \perp BC$, $BE = AC$, $BD = \frac{1}{2}$ and $DE + BC = 1$.

If $\alpha = 4ED$, find the value of α. **Reference: 2015 FI3.4**



Let $ED = x$, $BE = y$	Let $ED = x$, $BE = y$
Then $AC = y$, $BC = 1 - x$	Then $AC = y$, $BC = 1 - x$
$\Delta BED \sim \Delta BCA$ (equiangular)	$\Delta BED \sim \Delta BCA$ (equiangular)
$\frac{ED}{BE} = \frac{AC}{BC}$ (cor. sides, ~Δs)	$\frac{ED}{BE} = \frac{AC}{BC}$ (cor. sides, ~Δs)
$\frac{x}{y} = \frac{y}{1-x} \Rightarrow y^2 = x(1 - x)$	$\frac{x}{y} = \frac{y}{1-x} \Rightarrow y^2 = x(1 - x)$
$BE^2 + ED^2 = BD^2$ (Pythagoras' theorem)	$BE^2 + ED^2 = BD^2$ (Pythagoras' theorem)
$y^2 + x^2 = \frac{1}{4} \Rightarrow x(1 - x) + x^2 = \frac{1}{4} \Rightarrow x = ED = \frac{1}{4}$	$y^2 + x^2 = \frac{1}{4} \Rightarrow x(1 - x) + x^2 = \frac{1}{4} \Rightarrow x = ED = \frac{1}{4}$
$\delta = 4ED = 1$	$\delta = 4ED = 1$

I4.2 若 $f(a) = a - 2$，且 $F(a, b) = b^2 + a + \alpha$ 及 $\beta = F(3, f(4))$，求 β 的值。

If $f(a) = a - 2$, $F(a, b) = b^2 + a + \alpha$ and $\beta = F(3, f(4))$, find the value of β.

Reference: 1990 HI3 2013 FI3.2 2015 FI4.3

$f(4) = 4 - 2 = 2$

$F(3, f(4)) = F(3, 2) = 2^2 + 3 + 1 = 8$

$\beta = 8$

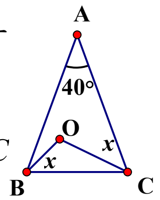
I4.3 如果方程組 $\begin{cases} x^2 - 3xy + 2y^2 - z^2 = 31 \\ -x^2 + 6yz + 2z^2 = 44 \\ x^2 + xy + \beta \cdot z^2 = 100 \end{cases}$ 整數解的數量是 γ，求 γ 的值。

If the system of equations $\begin{cases} x^2 - 3xy + 2y^2 - z^2 = 31 \\ -x^2 + 6yz + 2z^2 = 44 \\ x^2 + xy + \beta \cdot z^2 = 100 \end{cases}$ has γ sets of integral solutions, find the value of γ.

(1) + (2) + (3): $(x^2 - 2xy + y^2) + (y^2 + 6yz + 9z^2) = 175 \Rightarrow (x - y)^2 + (y + 3z)^2 = 175$

設 $a = (x - y)$、$b = (y + 3z)$，則 a、b 為整數	Let $a = (x - y)$, $b = (y + 3z)$, then a, b are integers
$a^2 + b^2 = 175$	$a^2 + b^2 = 175$
若 a、b 均為雙數，則 L.H.S. = 雙數 ≠ R.H.S.	If both a, b are even, then L.H.S. = even ≠ R.H.S.
若 a、b 均為單數，設 $a = 2m + 1$，$b = 2n + 1$	If both a, b are odd, let $a = 2m + 1$, $b = 2n + 1$
$4m^2 + 4m + 1 + 4n^2 + 4n + 1 = 175$	$4m^2 + 4m + 1 + 4n^2 + 4n + 1 = 175$
$4(m^2 + m + n^2 + n) = 173$	$4(m^2 + m + n^2 + n) = 173$
L.H.S. = 4 的倍數 ≠ R.H.S.	L.H.S. = multiple of 4 ≠ R.H.S.
若 $a = 2m + 1$，$b = 2n$	If $a = 2m + 1$, $b = 2n$
$4m^2 + 4m + 1 + 4n^2 = 175$	$4m^2 + 4m + 1 + 4n^2 = 175$
$4(m^2 + m + n^2) = 174$	$4(m^2 + m + n^2) = 174$
L.H.S. = 4 的倍數 ≠ R.H.S.	L.H.S. = multiple of 4 ≠ R.H.S.
若 $a = 2m$，$b = 2n + 1$	If $a = 2m$, $b = 2n + 1$
$4m^2 + 4n^2 + 4n + 1 = 175$	$4m^2 + 4n^2 + 4n + 1 = 175$
$4(m^2 + n^2 + n) = 174$	$4(m^2 + n^2 + n) = 174$
L.H.S. = 4 的倍數 ≠ R.H.S.	L.H.S. = multiple of 4 ≠ R.H.S.
以上所有情況，均沒有整數解。$\therefore \gamma = 0$	In all cases, there are no integral solutions $\therefore \gamma = 0$

I4.4 在三角形ABC中，$AB = AC$，$\angle A = 40° + \gamma°$。點 O 在三角形ABC內且 $\angle OBC = \angle OCA$。如果 $\angle BOC = \delta°$，求 δ 的值。



In a triangle ABC, $AB = AC$, $\angle A = 40° + \gamma°$. Point O is inside the triangle ABC with $\angle OBC = \angle OCA$. If $\angle BOC = \delta°$, the value of δ .

$\angle A = 40°$ $\angle ABC = \angle ACB$ (等腰三角形底角) $= \frac{180° - 40°}{2}$ (三角形內角和) $= 70°$ Let $\angle OBC = \angle OCA = x$ $\angle BCO = 70° - x$ 在ΔBOC 中， $x + \delta° + (70° - x) = 180°$ (三角形內角和) $\delta = 110$	$\angle A = 40°$ $\angle ABC = \angle ACB$ (base $\angle$s, isos. Δ) $= \frac{180° - 40°}{2}$ ($\angle$ sum of Δ) $= 70°$ Let $\angle OBC = \angle OCA = x$ $\angle BCO = 70° - x$ In ΔBOC, $x + \delta° + (70° - x) = 180°$ ($\angle$ sum of Δ) $\delta = 110$

Group Event 1

G1.1 有 100 個燈泡，編號從 1 到 100。班上有 100 名學生。每個學生輪流按下燈泡開關，情序如下：第一個學生按下編號為 1 及其倍數的燈泡開關，第二個學生按下編號為2及其倍數的燈泡開關，以此類推。每個學生只出來一次。如果燈泡亮著，按下開關後就會熄滅，反之亦然。一開始所有燈泡都是熄滅的。X 代表在第 100 個學生按下開關後，燈泡亮著的數量。求 X 的值。

There are 100 light bulbs labeled from 1 to 100, and there are 100 students in the class. Each student takes a turn to press the switch buttons of the light bulbs with a label that is a multiple of their assigned number. For example, the first student presses the switch buttons of the light bulb with label 1 and all of its multiples, the second student presses the switch buttons of the light bulb with label 2 and all of its multiples, and so on. Each student will only come out once, and if a light bulb is on, it becomes off after being pressed, and vice versa. All the light bulbs are off at the beginning. X is the number of light bulbs that are on after the 100th student presses. Find the value of X. **Reference: IMO (HK Preliminary Selection Contest) 2000 Q8**

若某編號的燈泡有 n 個正因子(包括 1 及該編號)，則它會被按下 n 次。除了平方數之外，其餘的正整數均有雙數數目的正因子。因此，編號為 1、4、9、16、25、35、49、64、81 及 100 的燈泡最後仍亮著。在第 100 個學生按下開關後，一共有 10 個燈泡亮著。$X = 10$	A light bulb was switched n times if it has n factors (including 1 and itself). As factors of a number occur in pairs, numbers have even numbers of factors except perfect squares. Therefore, only light bulbs numbered 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100 are still ‘on’. There are 10 light bulbs will be ‘on’ after tthe 100th student presses. $X = 10$

G1.2 已知 $x+\frac{1}{x}=2\sqrt{5}$。求 $x^5-\frac{1}{x^5}$ 的值。

Given that $x+\frac{1}{x}=2\sqrt{5}$. Find the value of $x^5-\frac{1}{x^5}$. **Reference: 2015 FG4.3**

$$\left(x+\frac{1}{x}\right)^2=x^2+2+\frac{1}{x^2}=20\Rightarrow x^2+\frac{1}{x^2}=18$$

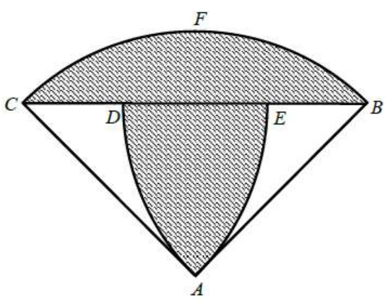
$$\left(x^2+\frac{1}{x^2}\right)^2=x^4+2+\frac{1}{x^4}=324\Rightarrow x^4+\frac{1}{x^4}=322$$

$$\left(x-\frac{1}{x}\right)^2=x^2-2+\frac{1}{x^2}=18-2=16\Rightarrow x-\frac{1}{x}=\pm4$$

$$x^5-\frac{1}{x^5}=\left(x-\frac{1}{x}\right)\left(x^4+x^2+1+\frac{1}{x^2}+\frac{1}{x^4}\right)$$

$$=\pm4\times(322+18+1)=\pm1364$$

G1.3 右圖中，ABC 是一個等腰三角形，其中 $\angle A=90°$ 及 $AB=2$。圖中有三個弧，它們分別是弧 BFC、弧 AD 和弧 AE。弧 BFC 是以 A 為圓心、AB 為半徑畫出的。弧 AD 是以 B 為圓心、AB 為半徑畫出的。弧 AE 是以 C 為圓心、AC 為半徑畫出的。求這個圖形的陰影面積。（取 $\pi=3$）



In the above figure, ABC is an isosceles triangle, where $\angle A=90°$ and $AB=2$. The figure includes three arcs: arc BFC, arc AD, and arc AE. Arc BFC has a radius of AB and is drawn from centre A. Arc AD is drawn from centre B with radius AB, while arc AE is drawn from centre C with radius AC. Find the area of this shaded region. (Take $\pi=3$)

陰影面積 $=2\left(\frac{\pi}{4}\cdot2^2-\frac{1}{2}\cdot2\times2\right)$ $=2$	Shaded area $=2\left(\frac{\pi}{4}\cdot2^2-\frac{1}{2}\cdot2\times2\right)$ $=2$

G1.4 使用正整數序列1、2、3、4、5、6 等等，通過將它們連接起來形成一個新的整數：123456789101112131415161718 ⋯ 這個整數的最左邊的數位被定義為第一個數位。問在第 2023 數位是 0 至 9 的哪一個數？

Using the sequence of positive integers 1, 2, 3, 4, 5, 6, and so on, a new integer is formed by concatenating them: 123456789101112131415161718⋯ The leftmost digit in this integer is defined as first position. What is the digit at position 2023?

Reference: 1999 HG5

$1, 2, \cdots ,9$	9 位	9 digits
$10, 11, \cdots , 99$	$90\times 2 = 180$ 位	$90\times 2 = 180$ digits
$100, 101, \cdots , 999$	$900\times 3 = 2700$ 位	$900\times 3 = 2700$ digits

$2023 - 9 - 180 = 1834 = 3\times 611 + 1$

由 100 開始，第 611 個整數是 710。 第 612 個整數是 711。 第 2023 位是 7。	Starting from 100, the 611$^{\text{th}}$ integer is 710. The 612$^{\text{th}}$ integer is 711. The 2023$^{\text{th}}$ digit is 7.

Group Event 2

G2.1 假如 x 和 y 都是正整數且它們的和是 15，找出 x^3+y^3 的最小值。

Find the minimum value of x^3+y^3 if x and y are two positive integers whose sum is 15.

$x^3+(15-x)^3$ $=(x+15-x)[x^2-x(15-x)+(15-x)^2]$ $=15(3x^2-45x+225)$ $=45(x^2-15x+75)$ $=45[(x-7.5)^2+18.75]$ ∵ x 是正整數 ∴ 當 $x=7$ 或 8 時，該算式達到最小 最小值 $=7^3+8^3=855$	$x^3+(15-x)^3$ $=(x+15-x)[x^2-x(15-x)+(15-x)^2]$ $=15(3x^2-45x+225)$ $=45(x^2-15x+75)$ $=45[(x-7.5)^2+18.75]$ ∵ x is a positive integer ∴ The minimum value is attained when $x=7$ or 8 Minimum value $=7^3+8^3=855$

G2.2 有一顆骰子，它的六個面上分別寫上數字 6 至 11。現投擲這顆骰子兩次，第一次得知四個側面的數字和是 36，第二次的數字和是 33。請問數字 10 的對面是甚麼數字？

A cubic dice has faces marked with numbers from 6 to 11. The dice was rolled twice. At the first time, the sum of the numbers on the four lateral faces was 36.

At the second time, the sum was 33. What number is on the face opposite to the one with the number 10?

將骰子的點數寫成 5+1,5+2,5+3,5+4,5+5,5+6 36 = 5×4 +16, 33 = 5×4 + 13 16 = 2+3+5+6 = 1+4+5+6 13 = 1+2+4+6 = 1+3+4+5 情況 1: 2+3+5+6 及 1+2+4+6 情況 2: 2+3+5+6 及 1+3+4+5 情況 3: 1+4+5+6 及 1+2+4+6 不可能 情況 4: 1+4+5+6 及 1+3+4+5 不可能 5+5 對面的點數為 3+5 = 8	Regard the numbers as 5+1,5+2,5+3,5+4,5+5,5+6 36 = 5×4 +16, 33 = 5×4 + 13 16 = 2+3+5+6 = 1+4+5+6 13 = 1+2+4+6 = 1+3+4+5 Case 1: 2+3+5+6 and 1+2+4+6 Case 2: 2+3+5+6 and 1+3+4+5 Case 3: 1+4+5+6 and 1+2+4+6 impossible Case 4: 1+4+5+6 and 1+3+4+5 impossible The number opposite to 5+5 is 3+5 = 8

G2.3 找出 $10^{12}+809$ 和 $10^{10}+8$ 的最大公因數。

Find the greatest common divisor of $10^{12}+809$ and $10^{10}+8$.

$10^{12}+809=9\times111111111201$ $10^{10}+8=9\times1388888889$ $10^{12}+809$ 和 $10^{10}+8$的其中一個公因數是9。 $10^{12}+809=100(10^{10}+8)+9$ $1\times(10^{12}+809)-100(10^{10}+8)=9$ 利用輾轉相除法的逆定理 最大公因數 $=9$	$10^{12}+809=9\times111111111201$ $10^{10}+8=9\times1388888889$ 9 is a common factor of $10^{12}+809$ and $10^{10}+8$ $10^{12}+809=100(10^{10}+8)+9$ $1\times(10^{12}+809)-100(10^{10}+8)=9$ By the converse of Euclean algorithm, The H.C.F. = 9

G2.4 在直角坐標平面上，香港的坐標是 (0, 0)，颱風是 (4, –2)。假設颱風向西（左）移動時，概率為 0.1，和向北（上）移動時，概率為 0.9，而且只能在移動一個單位距離後才可改變方向，請問這個颱風遇到香港的概率是多少？（答案需準確至四位有效數字。）

Hong Kong is located at (0, 0) of a grid map and a typhoon is at (4, –2). Suppose the typhoon will only move to the west (left) with a probability of 0.1 or to the north (up) with a probability of 0.9 , and may only change course after moving one unit distance. What is the probability that it will hit Hong Kong? (Give your answer in 4 significant figures.)

設 L = 颱風向西移動一格 設 L = 颱風向北移動一格 以下為這個颱風遇到香港的可能路徑：LLLLUU及LLLLUU的不同排列。. 概率$=C^6_2\cdot0.1^4\times0.9^2=0.001215$	Let L = the typhoon moves to the west one unit Let U = the typhoon moves to the north one unit One possible way to hit Hong Kong: LLLLUU. and different arrangements of LLLLUU Probability $=C^6_2\cdot0.1^4\times0.9^2=0.001215$

Group Event 3

G3.1 設 a_n 為序列且 $a_n=\dfrac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}}$。如果 $s=a_1+a_2+a_3+\cdots+a_{120}$，求 s 的值。

Let a_n be a sequence such that $\dfrac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}}$.

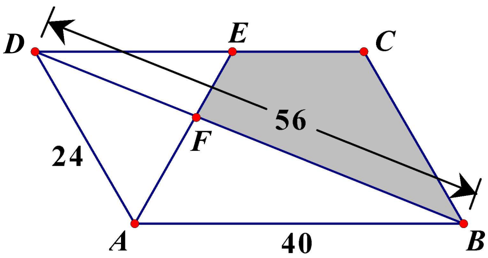
Find the value of s where $s=a_1+a_2+a_3+\cdots+a_{120}$.

$$\begin{aligned}\frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}}&=\frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}}\cdot\frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{(n+1)\sqrt{n}-n\sqrt{n+1}}\\&=\frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{(n+1)^2n-n^2(n+1)}=\frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{n(n+1)}\\&=\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\end{aligned}$$

$$\begin{aligned}s&=a_1+a_2+a_3+\cdots+a_{120}\\&=\left(1-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\cdots+\left(\frac{1}{\sqrt{120}}-\frac{1}{\sqrt{121}}\right)\\&=1-\frac{1}{\sqrt{121}}=1-\frac{1}{11}=\frac{10}{11}\end{aligned}$$

G3.2 設 $ABCD$ 為平行四邊形且 AB = 40，AD = 24 及 $DB = 56$。$\angle DAB$ 的角平分線與 DC 相交於 E 點，且對角線 DB 與 AE 相交於 F 點。求四邊形 $ECBF$ 的面積。

Let $ABCD$ be a parallelogram with $AB = 40$, $AD = 24$ and $DB = 56$. The angle bisector of $\angle DAB$ meets side DC at the point E, and the diagonal DB meets AE at the point F. Find the area of the quadrilateral $ECBF$.



$\cos\angle BAD=\dfrac{24^2+40^2-56^2}{2\times24\times40}=-\dfrac{1}{2}$, $\angle BAD=120°$

$\angle BAE=\angle DAE=60°$

利用角平分綫定理, $BF : DF = AB : AD$

設 $BF = 40k$，$DF = 24k$

$BF + DF = BD \Rightarrow 40k + 24k = 56 \Rightarrow 8k = 7$

$BF = 5(8k) = 35$, $DF = 3(8k) = 21$

在 ΔABD 中，$\cos\angle ABD=\dfrac{40^2+56^2-24^2}{2\times40\times56}=\dfrac{13}{14}$

在 ΔABF 中，$AF^2 = 40^2 + 35^2 - 2\times40\times35\cos\angle ABD$

$= 40^2 + 35^2 - 2\times40\times35\times\dfrac{13}{14}$

$= 225$

$\therefore AF = 15$

$\Delta ABF \sim \Delta EFD$ (等角)

$\dfrac{EF}{15}=\dfrac{21}{35}$ (相似三角形對應邊) $\Rightarrow EF = 9$

$S_{ECBF} = 2S_{\Delta ABD} - S_{\Delta ADE} - S_{\Delta ABF}$

$=24\cdot40\sin120°-\dfrac{1}{2}\cdot24\cdot24\sin60°-\dfrac{1}{2}\cdot15\cdot40\sin60°=186\sqrt{3}$

$\cos\angle BAD=\dfrac{24^2+40^2-56^2}{2\times24\times40}=-\dfrac{1}{2}$, $\angle BAD=120°$

$\angle BAE=\angle DAE=60°$

By angle bisector theorem, $BF : DF = AB : AD$

Let $BF = 40k$, $DF = 24k$

$BF + DF = BD \Rightarrow 40k + 24k = 56 \Rightarrow 8k = 7$

$BF = 5(8k) = 35$, $DF = 3(8k) = 21$

In ΔABD, $\cos\angle ABD=\dfrac{40^2+56^2-24^2}{2\times40\times56}=\dfrac{13}{14}$

In ΔABF, $AF^2 = 40^2 + 35^2 - 2\times40\times35\cos\angle ABD$

$= 40^2 + 35^2 - 2\times40\times35\times\dfrac{13}{14}$

$= 225$

$\therefore AF = 15$

$\Delta ABF \sim \Delta EFD$ (equiangular)

$\dfrac{EF}{15}=\dfrac{21}{35}$ (corr. sides, ~Δs) $\Rightarrow EF = 9$

$S_{ECBF} = 2S_{\Delta ABD} - S_{\Delta ADE} - S_{\Delta ABF}$

$=24\cdot40\sin120°-\dfrac{1}{2}\cdot24\cdot24\sin60°-\dfrac{1}{2}\cdot15\cdot40\sin60°=186\sqrt{3}$

Remark: The original question is: Let $ABCD$ be a parallelogram with $AB = 48$, $AD = 36$ and $DB = 56$. The angle bisector of $\angle DAB$ meets side DC at the point E, and the diagonal DB meets AE at the point F. If the area of $ABCD$ is 560 square units, find the area of the quadrilateral $ECBF$.
Using Heron's formula, area of ΔABD = 856 sq. units > 560 sq. unit = area of $ABCD$, impossible

G3.3 設 f(x)為函數並滿足 $\mathrm{f}(x)+\mathrm{f}\left(-\frac{1}{x-1}\right)=\frac{2x}{3}+\frac{5}{3}+\mathrm{f}\left(1-\frac{1}{x}\right)$，$x \neq 0$、1。求 f(–1)的值。

Let f(x) be a function such that $\mathrm{f}(x)+\mathrm{f}\left(-\frac{1}{x-1}\right)=\frac{2x}{3}+\frac{5}{3}+\mathrm{f}\left(1-\frac{1}{x}\right)$, $x \neq 0, 1$.

Find the value of f(–1).

$x=-1$: $\mathrm{f}(-1)+\mathrm{f}\left(\frac{1}{2}\right)=-\frac{2}{3}+\frac{5}{3}+\mathrm{f}(2) \Rightarrow \mathrm{f}(-1)+\mathrm{f}\left(\frac{1}{2}\right)=1+\mathrm{f}(2)\cdots\cdots(1)$

$x=\frac{1}{2}$: $\mathrm{f}\left(\frac{1}{2}\right)+\mathrm{f}(2)=\frac{1}{3}+\frac{5}{3}+\mathrm{f}(-1) \Rightarrow \mathrm{f}(-1)-\mathrm{f}\left(\frac{1}{2}\right)=-2+\mathrm{f}(2)\cdots\cdots(2)$

(1) + (2): $2\mathrm{f}(-1) = -1 + 2\mathrm{f}(2) \Rightarrow \mathrm{f}(2)=\mathrm{f}(-1)+\frac{1}{2}\cdots\cdots(3)$

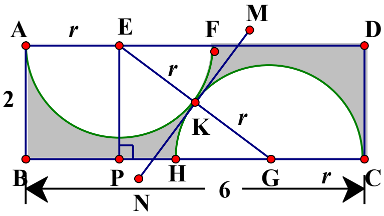
(1) – (2): $2\mathrm{f}\left(\frac{1}{2}\right)=3 \Rightarrow \mathrm{f}\left(\frac{1}{2}\right)=\frac{3}{2}$

$x=2$: $\mathrm{f}(2)+\mathrm{f}(-1)=\frac{4}{3}+\frac{5}{3}+\mathrm{f}\left(\frac{1}{2}\right) \Rightarrow \mathrm{f}(2)+\mathrm{f}(-1)=3+\frac{3}{2}=\frac{9}{2}\cdots\cdots(4)$

Sub. (3) into (4) 代(3)入(4): $\mathrm{f}(-1)+\frac{1}{2}+\mathrm{f}(-1)=\frac{9}{2} \Rightarrow \mathrm{f}(-1) = 2$

G3.4 右圖中，$ABCD$ 是一個長方形。兩個半圓形完全相等且它們彼此相切。如果 $AB = 2$ 及 $BC = 6$，求圖中陰影面積（答案以 π 表示）。

In the following figure, $ABCD$ is a rectangle. The two semi-circles are identical and they are tangent to each other. If $AB = 2$ and $BC = 6$, find the area of the shaded part in terms of π.



假設兩半圓相切於K。 設該兩半圓AKF 及 CKH (半徑= r)的圓心分別為 E 及 G 如圖。 過 K 作公切綫 MN。 $EK \perp MN$, $GK \perp MN$ (切綫⊥半徑) $\therefore E$、K、G 共綫 $EG = 2r$ 作$EP \perp BC$ 如圖。 $EP = 2$，$BP = AE = CG = r$，$PG = 6 - 2r$ 在ΔEPG中，$EP^2 + PG^2 = EG^2$ (畢氏定理) $2^2 + (6 - 2r)^2 = (2r)^2$ $1 + (3 - r)^2 = r^2$ $r=\frac{5}{3}$ 陰影面積$=2\times 6-\pi\left(\frac{5}{3}\right)^2$ $=12-\frac{25\pi}{9}$	Suppose the semi-circles touches each other at K. Let E and G be the centres of the semi-circles AKF and CKH (radii = r) as shown. Draw the common tangent MN at K. $EK \perp MN$, $GK \perp MN$ (tangent ⊥ radii) $\therefore E, K, G$ are collinear $EG = 2r$ Draw $EP \perp BC$ as shown. $EP = 2$, $BP = AE = CG = r$, $PG = 6 - 2r$ In ΔEPG, $EP^2 + PG^2 = EG^2$ (Pyth. theorem) $2^2 + (6 - 2r)^2 = (2r)^2$ $1 + (3 - r)^2 = r^2$ $r=\frac{5}{3}$ Shaded area $=2\times 6-\pi\left(\frac{5}{3}\right)^2$ $=12-\frac{25\pi}{9}$

Group Event 4

G4.1 求方程 $x^{\log_{10} x} = 10$ 所有實根的積。

Find the product of all the real roots of the equation $x^{\log_{10} x} = 10$.

Reference: 1990 HI9, 2015 FI4.4

$x^{\log_{10} x} = 10$ $(\log_{10} x)(\log_{10} x) = 1$ $\log_{10} x = \pm 1$ $x = 10$ 或 $\frac{1}{10}$ 實根的積= 1	$x^{\log_{10} x} = 10$ $(\log_{10} x)(\log_{10} x) = 1$ $\log_{10} x = \pm 1$ $x = 10$ or $\frac{1}{10}$ Product of roots = 1

G4.2 設 p 為質數及 m 為整數。如果 $p(p+m)+2p=(m+2)^3$，求 m 的最大值。

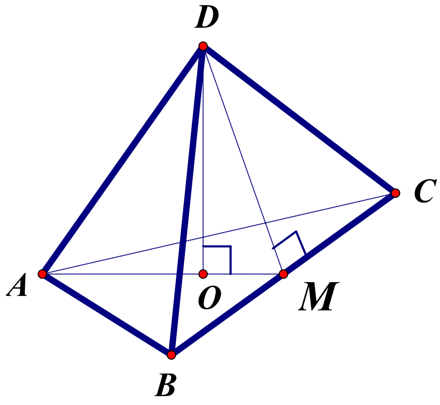
Let p be a prime and m be an integer.

If $p(p+m)+2p=(m+2)^3$, find the greatest possible value of m.

Reference: 2015 FG3.2

$p(p+m+2)=(m+2)^3$ 若 m 為雙及 p 為單， 則 單×(單 + 雙 + 2) = (雙+ 2)^3 $\Rightarrow$ L.H.S. $\neq$ R.H.S. 矛盾 若 m 為單及 p 為單， 則單×(單 + 單 + 2) = (單+ 2)^3 $\Rightarrow$ L.H.S. $\neq$ R.H.S. 矛盾 任何情況，p 必為雙。 $\because$ 唯一的為雙數且同時是質數是 2 $\therefore p = 2$ $2(m+4)=(m+2)^3$ …… (*) L.H.S.為雙 $\Rightarrow (m+2)^3$ 為雙 $\Rightarrow m+2$ 為雙 $\Rightarrow$ R.H.S. 為 8 的倍數$\Rightarrow$ L.H.S. 為 8 的倍數 $\Rightarrow m+4=4n$，其中 n 為整數 $\Rightarrow m+2=4n-2$ 代 $m+2=4n-2$ 入方程(*): $2(4n)=(4n-2)^3$ $n=(2n-1)^3$ 對於 $n>1$，$n<(2n-1)^3$；對於 $n<1$，$n>(2n-1)^3$ $\Rightarrow n=1, m=0$	$p(p+m+2)=(m+2)^3$ If m is even and p is odd, then odd×(odd + even + 2) = (even + 2)^3 $\Rightarrow$ L.H.S. $\neq$ R.H.S. !!! If m is odd and p is odd, then odd×(odd + odd + 2) = (odd + 2)^3 $\Rightarrow$ L.H.S. $\neq$ R.H.S. !!! In all cases, p must be even. $\because$ the only even prime is 2 $\therefore p = 2$ $2(m+4)=(m+2)^3$ …… (*) L.H.S. is even $\Rightarrow (m+2)^3$ is even $\Rightarrow m+2$ is even $\Rightarrow$ R.H.S. is a multiple of 8 $\Rightarrow$ L.H.S. is a multiple of 8 $\Rightarrow m+4=4n$, where n is an integer $\Rightarrow m+2=4n-2$ Put $m+2=4n-2$ into the equation (*): $2(4n)=(4n-2)^3$ $n=(2n-1)^3$ For $n>1$, $n<(2n-1)^3$; for $n<1$, $n>(2n-1)^3$ $\Rightarrow n=1, m=0$

G4.3 如果正四面體的邊長是 1，求該正四面體的體積。
If the length of one side of a regular tetrahedron is 1, find the volume of such tetrahedron.
Reference: 2008 HG3



如圖，設該四面體為 $ABCD$。 設 M 為 BC 的中點。$(BM = MC = 0.5)$ $\Delta ABM \cong \Delta ACM$ (S.S.S.) $AM = \frac{\sqrt{3}}{2}$ (畢氏定理) O 為 ΔABC 的重心。$AO = \frac{2}{3}AM = \frac{\sqrt{3}}{3}$ $DO =$ 四面體的高 $= \sqrt{AD^2 - AO^2} = \sqrt{\frac{2}{3}}$ 體積 $= \frac{1}{3}\cdot\frac{1}{2}\cdot 1^2 \sin 60^\circ \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{12}$	Let the tetrahedron be $ABCD$ as shown above. Let M be the mid point of BC. $(BM = MC = 0.5)$ $\Delta ABM \cong \Delta ACM$ (S.S.S.) $AM = \frac{\sqrt{3}}{2}$ (Pythagoras' Theorem) O is the centriod of ΔABC. $AO = \frac{2}{3}AM = \frac{\sqrt{3}}{3}$ $DO =$ height of tetrahedron $= \sqrt{AD^2 - AO^2} = \sqrt{\frac{2}{3}}$ Volume $= \frac{1}{3}\cdot\frac{1}{2}\cdot 1^2 \sin 60^\circ \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{12}$

G4.4 設 P 為 3659893456789325678 和 342973489379256 的乘積。求 P 的位數。
Let P be the product of 3659893456789325678 and 342973489379256.
Find the number of digits of P .
Reference: 2013 FG4.1, 2015 FG1.3

$3659893456789325678 = 3.7\times 10^{18}$ (兩位有效數字) $342973489379256 = 3.4\times 10^{14}$ (兩位有效數字) $P \approx 3.7\times 10^{18}\times 3.4\times 10^{14} = 12.58\times 10^{32} = 1.258\times 10^{33}$ P 的位數為34。	$3659893456789325678 = 3.7\times 10^{18}$ (cor. to 2 s.f.) $342973489379256 = 3.4\times 10^{14}$ (cor. to 2 s.f.) $P \approx 3.7\times 10^{18}\times 3.4\times 10^{14} = 12.58\times 10^{32} = 1.258\times 10^{33}$ The number of digits is 34.