

Individual Events

I1	A	14	I2	A	2	I3	A	18	I4	A	4
	B	27		B	10		B	0		B	$\frac{3}{4}$
	C	116		C	40		C	6		C	$\sqrt{52}$
	D	4		D	144		D	2		D	64

Group Events

G1	P	8010	G2	a	$24\sqrt{7}$	G3	n	14	G4	a	-14
	A	6		b	527		$b - a$	10		x	6
numbers		13696		c	4	gain	$\$ \frac{7400}{3} = \$ 2466 \frac{2}{3}$	equation		$x^2 + 90x + 2 = 0$	
m		1		d	2	Area	1000 cm^2	Area		$4\sqrt{2} - \frac{3\pi}{2}$ see the remark	

Individual Event 1

I1.1 已知 m 和 n 均為正整數。如果 $m + n + mn = 54$ 及 $A = m + n$ ，求 A 的值。

Given that m and n are positive integers. If $m + n + mn = 54$ and $A = m + n$, find the value of A .

Reference: 2006 FG2.4, 2024 FG4.1

$1 + m + n + mn = 55$ $(1 + m)(1 + n) = 55$ $1 + m = 5, 1 + n = 11$ 或 $1 + m = 11, 1 + n = 5$ $m = 4, n = 10$ 或 $m = 10, n = 4$ $A = m + n = 14$	$1 + m + n + mn = 55$ $(1 + m)(1 + n) = 55$ $1 + m = 5, 1 + n = 11$ or $1 + m = 11, 1 + n = 5$ $m = 4, n = 10$ or $m = 10, n = 4$ $A = m + n = 14$
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I1.2 若 $f(a) = a - 2$, $F(a, b) = b^2 + a + A$, $B = F(4, f(5))$ ，求 B 的值。

If $f(a) = a - 2$, $F(a, b) = b^2 + a + A$ and $B = F(4, f(5))$, find the value of B

Reference: 2023 FI4.2

$$f(5) = 5 - 2 = 3$$

$$F(4, f(5)) = F(4, 3) = 3^2 + 4 + 14 = 27$$

$$B = 27$$

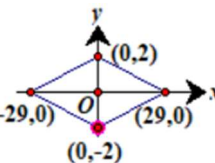
I1.3 在 x - y 座標平面上，由 $(B + 2, 0)$ 、 $(-B - 2, 0)$ 、 $(0, 2)$ 及 $(0, -2)$ 所形成之菱形的面積為 C 平方單位，求 C 的值。

The area of the rhombus on the x - y coordinate plane with vertices $(B + 2, 0)$, $(-B - 2, 0)$, $(0, 2)$ and $(0, -2)$ is C square units. Find the value of C .

Reference: 2023 FI3.3

$$C = \frac{1}{2} [2 - (-2)] \cdot [29 - (-29)]$$

$$= 116$$



I1.4 如果 D 是正整數且 $\left(\frac{C}{4} + 227\right)^{\frac{1}{D}} = D$ ，求 D 的值。

If D is a positive integer such that $\left(\frac{C}{4} + 227\right)^{\frac{1}{D}} = D$, find the value of D .

Reference: 2023 FI3.4

$$\frac{116}{4} + 227 = D^D$$

$$D = 4$$

Individual Event 2

12.1 若 A 是 $2022^{2023^{2024}}$ 的個位數 A 。

A is the units digit of $2022^{2023^{2024}}$. Find the value of A .

$2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 8, 2^4 \equiv 6 \pmod{10}$ 這個數字的規律每隔 4 的倍數重複一次。 $2023 \equiv -1 \pmod{4}, 2023^2 \equiv 1 \pmod{4}$ 這個數字的規律每隔 2 的倍數重複一次。 $2023^{2024} \equiv 1 \pmod{4}$ $2022^{2023^{2024}} \equiv 2022^1 \equiv 2 \pmod{10}$ $A = 2$	$2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 8, 2^4 \equiv 6 \pmod{10}$ This pattern repeats for every multiple of 4. $2023 \equiv -1 \pmod{4}, 2023^2 \equiv 1 \pmod{4}$ This pattern repeats for every multiple of 2. $2023^{2024} \equiv 1 \pmod{4}$ $2022^{2023^{2024}} \equiv 2022^1 \equiv 2 \pmod{10}$ $A = 2$
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12.2 已知 $(x+20)^2 + (y-24)^2$ 的最小值是 B ，當中 x 和 y 是方程 $19x + 13y = A$ 的整數解。求 B 的值。

B is the minimum value of $(x+20)^2 + (y-24)^2$, where x and y are integers that satisfy the equation $19x + 13y = A$. Find the value of B .

$19x + 13y = 2$ $13 \times 6 - 19 \times 4 = 2$ $(-4, 6)$ 為一整數解 一般解： $x = -4 + 13k, y = 6 - 19k$ $(x+20)^2 + (y-24)^2$ $= (-4 + 13k + 20)^2 + (6 - 19k - 24)^2$ $= (13k + 16)^2 + (-19k - 18)^2$ $= 169k^2 + 416k + 256 + 361k^2 + 684k + 324$ $= 530k^2 + 1100k + 580$ $= 530 \left(k^2 + \frac{110}{53}k + \frac{55^2}{53^2} \right) + 580 - \frac{55^2 \times 10}{53}$ $= 530 \left(k + \frac{55}{53} \right)^2 + \frac{490}{53}$ $k = 0$ ，算式 = 580 $k = -1$ ，算式 = 10 $k = -2$ ，算式 = 500 $B = 10$	$19x + 13y = 2$ $13 \times 6 - 19 \times 4 = 2$ $(-4, 6)$ is an integral solution The general solutions: $x = -4 + 13k, y = 6 - 19k$ $(x+20)^2 + (y-24)^2$ $= (-4 + 13k + 20)^2 + (6 - 19k - 24)^2$ $= (13k + 16)^2 + (-19k - 18)^2$ $= 169k^2 + 416k + 256 + 361k^2 + 684k + 324$ $= 530k^2 + 1100k + 580$ $= 530 \left(k^2 + \frac{110}{53}k + \frac{55^2}{53^2} \right) + 580 - \frac{55^2 \times 10}{53}$ $= 530 \left(k + \frac{55}{53} \right)^2 + \frac{490}{53}$ $k = 0$, expression = 580 $k = -1$, expression = 10 $k = -2$, expression = 500 $B = 10$
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12.3 在袋中有若干顆紅色和藍色的彈珠，它們的總數量是 C 。如果加入 B 顆紅色彈珠，紅色和藍色彈珠數量的比例則為 3 : 2；如果加入 B 顆藍色彈珠，紅色和藍色彈珠數量的比例則為 2 : 3。求 C 的值。

There is C marbles in a bag, which are either red or blue. If we add B red marbles to the bag, the ratio of red marbles to the blue marbles becomes 3 : 2. If we add B blue marbles to the bag, the ratio of red marbles to the blue marbles becomes 2 : 3. Find the value of C .

假設原本有 x 顆紅色彈珠。 那麼原本有 $(C-x)$ 顆藍色彈珠。 $(x+10) : (C-x) = 3 : 2 \dots\dots (1)$ $x : (C-x+10) = 2 : 3 \dots\dots (2)$ 由(1)式： $2x + 20 = 3C - 3x$ $5x = 3C - 20 \dots\dots (3)$ 由(2)式： $3x = 2C - 2x + 20$ $5x = 2C + 20 \dots\dots (4)$ $(3) - (4): C = 40$	Suppose there are x red marbles originally. Then there are $(C-x)$ blue marbles originally. $(x+10) : (C-x) = 3 : 2 \dots\dots (1)$ $x : (C-x+10) = 2 : 3 \dots\dots (2)$ From (1): $2x + 20 = 3C - 3x$ $5x = 3C - 20 \dots\dots (3)$ From (2): $3x = 2C - 2x + 20$ $5x = 2C + 20 \dots\dots (4)$ $(3) - (4): C = 40$
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12.4 若 $5\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)=C$ ，求 D 的值。

If $5\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)=C$, find the value of D .

$$5\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)=40$$

$$\left(\sqrt{25+2\sqrt{D}}+\sqrt{25-2\sqrt{D}}\right)^2=64$$

$$25+2\sqrt{D}+2\sqrt{625-4D}+25-2\sqrt{D}=64$$

$$2\sqrt{625-4D}=14$$

$$625-4D=49$$

$$D=144$$

Individual Event 3

13.1 若 x 和 y 為滿足方程 $\frac{1}{x} + \frac{1}{y} = \frac{2}{5}$ 的不同正整數，求 $A = x + y$ 的值。

If x and y are two different positive integers such that $\frac{1}{x} + \frac{1}{y} = \frac{2}{5}$, find the value of $A = x + y$.

$\frac{y+x}{xy} = \frac{2}{5}$ $2xy = 5x + 5y$ $4xy - 10x - 10y + 25 = 25$ $(2x-5)(2y-5) = 25$ $(2x-5, 2y-5) = (1, 25), (5, 5) \text{ 或 } (25, 1)$ $\therefore x, y \text{ 為不同正整數}$ $\therefore (x, y) = (3, 15) \text{ 或 } (15, 3)$ $A = x + y = 18$	$\frac{y+x}{xy} = \frac{2}{5}$ $2xy = 5x + 5y$ $4xy - 10x - 10y + 25 = 25$ $(2x-5)(2y-5) = 25$ $(2x-5, 2y-5) = (1, 25), (5, 5) \text{ or } (25, 1)$ $\therefore x, y \text{ are different positive integers}$ $\therefore (x, y) = (3, 15) \text{ or } (15, 3)$ $A = x + y = 18$
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13.2 若 B 是所有正整數 N 使得 7 整除 $2^N + (19 - A)$ 的數量，求 B 的值。

If B is the number of positive integers N such that $2^N + (19 - A)$ is divisible by 7, find the value of B .

$2^N + (19 - A) = 2^N + (19 - 18) = 2^N + 1$ $2^1 + 1 \equiv 3, 2^2 + 1 \equiv 5, 2^3 + 1 \equiv 2 \pmod{7}$ <p>這個數字的規律每隔 3 的倍數重複一次。</p> $\therefore \text{對於每一個正整數 } N, 2^N + 1 \text{ 不能被 } 7 \text{ 整除}$ $B = 0$	$2^N + (19 - A) = 2^N + (19 - 18) = 2^N + 1$ $2^1 + 1 \equiv 3, 2^2 + 1 \equiv 5, 2^3 + 1 \equiv 2 \pmod{7}$ <p>This pattern repeats for every multiple of 3.</p> $\therefore 2^N + 1 \text{ cannot be divisible by 7 for every positive integer } N$ $B = 0$
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13.3 已知 a 和 b 為滿足方程組 $a^2 - b^2 = 9$ 及 $ab = 3$ 的實數。

若對於正整數 α 和 C ， $a + b = \sqrt{\sqrt{\alpha} + C} - B$ ，求 C 的值。

Given that a and b are real numbers such that $a^2 - b^2 = 9$ and $ab = 3$.

If $a + b = \sqrt{\sqrt{\alpha} + C} - B$ for positive integers α and C , find the value of C .

<p>代 $b = \frac{3}{a}$ 入 $a^2 - b^2 = 9$</p> $a^2 - \left(\frac{3}{a}\right)^2 = 9 \Rightarrow a^4 - 9a^2 - 9 = 0$ $\therefore a \text{ 為實數 } \therefore a^2 \geq 0 \Rightarrow a^2 = \frac{9 + \sqrt{117}}{2}$ $b^2 = \frac{9}{a^2} = \frac{9 \times 2}{9 + \sqrt{117}} \cdot \frac{\sqrt{117} - 9}{\sqrt{117} - 9} = \frac{\sqrt{117} - 9}{2}$ $(a + b)^2 = \sqrt{\alpha} + C \text{ (已知)}$ $a^2 + 2ab + b^2 = \sqrt{\alpha} + C$ $\frac{9 + \sqrt{117}}{2} + \frac{\sqrt{117} - 9}{2} + 6 = \sqrt{\alpha} + C$ $\sqrt{117} + 6 = \sqrt{\alpha} + C$ $\therefore \sqrt{117} \text{ 為無理數, } 6 \text{ 及 } C \text{ 為有理數}$ $\therefore C = 6$	<p>Sub. $b = \frac{3}{a}$ into $a^2 - b^2 = 9$</p> $a^2 - \left(\frac{3}{a}\right)^2 = 9 \Rightarrow a^4 - 9a^2 - 9 = 0$ $\therefore a \text{ is real } \therefore a^2 \geq 0 \Rightarrow a^2 = \frac{9 + \sqrt{117}}{2}$ $b^2 = \frac{9}{a^2} = \frac{9 \times 2}{9 + \sqrt{117}} \cdot \frac{\sqrt{117} - 9}{\sqrt{117} - 9} = \frac{\sqrt{117} - 9}{2}$ $(a + b)^2 = \sqrt{\alpha} + C \text{ (given)}$ $a^2 + 2ab + b^2 = \sqrt{\alpha} + C$ $\frac{9 + \sqrt{117}}{2} + \frac{\sqrt{117} - 9}{2} + 6 = \sqrt{\alpha} + C$ $\sqrt{117} + 6 = \sqrt{\alpha} + C$ $\therefore \sqrt{117} \text{ is irrational, } 6 \text{ and } C \text{ are rational}$ $\therefore C = 6$
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I3.4 若 x 為滿足方程 $(\log_a x)^{\log_a x} = x$ 的實數，其中 a 是常數且 $a > 1$ 。求 $D = \frac{C \log_a x}{3a}$ 的值。

If x is real root of the equation $(\log_a x)^{\log_a x} = x$, where a is a constant and $a > 1$,

find the value of $D = \frac{C \log_a x}{3a}$.

$$(\log_a x)^{\log_a x} = x$$

$$\log_a \left[(\log_a x)^{\log_a x} \right] = \log_a x$$

$$\log_a x \log_a (\log_a x) = \log_a x$$

$$\log_a (\log_a x) = 1$$

$$\log_a x = a$$

$$x = a^a$$

$$D = \frac{C \log_a x}{3a} = \frac{6 \log_a a^a}{3a}$$

$$= \frac{2a \log_a a}{a} = 2$$

Individual Event 4

14.1 如果 $A > 1$ 且 $1 + \frac{1}{A} + \frac{1}{A^2} + \frac{1}{A^3} + \cdots = \frac{A}{3}$, 求 A 的值。

If $A > 1$ and $1 + \frac{1}{A} + \frac{1}{A^2} + \frac{1}{A^3} + \cdots = \frac{A}{3}$, find the value of A .

<p>以上是一等比數列的無限項之和， 公比 $= \frac{1}{A}$，其中 $0 < \frac{1}{A} < 1$。</p> $\frac{1}{1 - \frac{1}{A}} = \frac{A}{3}$ $3A = A(A - 1)$ $A^2 - 4A = 0$ $A = 4$	<p>The above is an infinite geometric series with common ratio $= \frac{1}{A}$, where $0 < \frac{1}{A} < 1$.</p> $\frac{1}{1 - \frac{1}{A}} = \frac{A}{3}$ $3A = A(A - 1)$ $A^2 - 4A = 0$ $A = 4$
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14.2 如果 $\frac{1}{A}$ 是二次方程 $x^2 - Bx + \frac{1}{6}B = 0$ 的一個根，求 B 的值。

If $\frac{1}{A}$ is a root of the quadratic equation $x^2 - Bx + \frac{1}{6}B = 0$, find the value of B .

$$\left(\frac{1}{4}\right)^2 - \frac{B}{4} + \frac{1}{6}B = 0$$

$$B = \frac{3}{4}$$

14.3 考慮右圖中的三角形，如果 $\tan \theta = B$ ，其中 $0^\circ < \theta < 90^\circ$ ，求 C 的值。

Consider the triangle in the figure on the right.

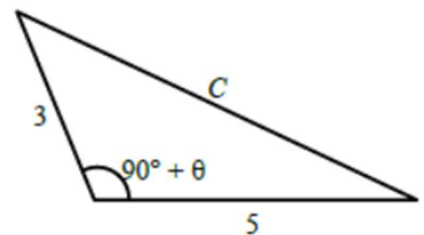
If $\tan \theta = B$, where $0^\circ < \theta < 90^\circ$, find the value of C .

$$\tan \theta = \frac{3}{4}, \sin \theta = \frac{3}{5}, \cos \theta = \frac{4}{5}$$

$$C^2 = 3^2 + 5^2 - 2 \times 3 \times 5 \times \cos(90^\circ + \theta)$$

$$C^2 = 34 + 30 \sin \theta = 34 + 30 \times \frac{3}{5} = 52$$

$$C = \sqrt{52}$$



14.4 設 $d = C^2 - 20$ ，如果 D 滿足方程 $8^D = D^d$ ，求 D 的值。

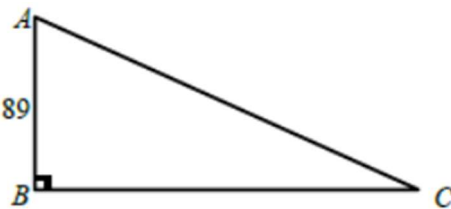
Let $d = C^2 - 20$. If D satisfies the equation $8^D = D^d$, the value of D .

$d = 52 - 20 = 32$ $8^D = D^{32}$ $2^{3D} = D^{32} \dots\dots (1)$ 設 $D = 2^x$ 比較(1)式的指數： $3 \times 2^x = 32x \dots\dots (2)$ 利用嘗試錯誤法，代 $x = 3$ 入(2)式： $\text{LHS} \neq \text{RHS}$ 代 $x = 6$ 入(2)式： $\text{LHS} = \text{RHS}$ $x = 6$ $D = 2^6 = 64$	$d = 52 - 20 = 32$ $8^D = D^{32}$ $2^{3D} = D^{32} \dots\dots (1)$ Let $D = 2^x$ Compare the indices of (1): $3 \times 2^x = 32x \dots\dots (2)$ By trial and error, put $x = 3$ in (2): $\text{LHS} \neq \text{RHS}$ Put $x = 6$ in (2): $\text{LHS} = \text{RHS}$ $x = 6$ $D = 2^6 = 64$
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Group Event 1

G1.1 若直角三角形 ABC 所有邊長均為正整數，且 $AB = 89$ ，求三角形 ABC 的周界 P 。

Find the perimeter P of the right-angled triangle ABC if all the side lengths are positive integers and $AB = 89$.



<p>設 $BC = a, AC = b$ $a^2 + 89^2 = b^2$ $(b + a)(b - a) = 89^2$ $\therefore 89$ 為質數及 $b > a$ 皆為正整數 $\therefore (b + a, b - a) = (7921, 1)$ $b = 3961, a = 3960$ $P = 3961 + 3960 + 89 = 8010$</p>	<p>Let $BC = a, AC = b$ $a^2 + 89^2 = b^2$ $(b + a)(b - a) = 89^2$ $\therefore 89$ is a prime and $b > a$ are positive integers $\therefore (b + a, b - a) = (7921, 1)$ $b = 3961, a = 3960$ $P = 3961 + 3960 + 89 = 8010$</p>
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G1.2 若 A 是 $8888^{20242024}$ 的個位數。求 A 的值。

If A is the units digit of $8888^{20242024}$. Find the value of A .

<p>$8^1 \equiv 8, 8^2 \equiv 4, 8^3 \equiv 2, 8^4 \equiv 6 \pmod{10}$ 這個數字的規律每隔 4 的倍數重複一次。 $20242024 \equiv 0 \pmod{4}$ $8888^{20242024} \equiv 8^4 \equiv 6 \pmod{10}$ $A = 6$</p>	<p>$8^1 \equiv 8, 8^2 \equiv 4, 8^3 \equiv 2, 8^4 \equiv 6 \pmod{10}$ This pattern repeats for every multiple of 4. $20242024 \equiv 0 \pmod{4}$ $8888^{20242024} \equiv 8^4 \equiv 6 \pmod{10}$ $A = 6$</p>
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G1.3 有多少個 5 位數包含最少 1 個「1」和最少 1 個「3」？

How many 5-digit numbers contain at least one "1" and at least one "3"?

<p>方法一 5 位數一共有：$9 \times 10^4 = 90000$ 個 沒有「1」和沒有「3」的 5 位數， 一共有 $7 \times 8^4 = 28672$ 個 只有一個「1」和沒有「3」的 5 位數， 一共有 $1 \times 8^4 + 7 \times 4 \times 8^3 = 18432$ 個 只有一個「3」和沒有「1」的 5 位數，有 18432 個 只有兩個「1」和沒有「3」的 5 位數， 一共有 $= 4 \times 8^3 + 7 \times C_2^4 \times 8^2 = 4736$ 個 只有兩個「3」和沒有「1」的 5 位數，有 4736 個 只有三個「1」和沒有「3」的 5 位數， 一共有 $= 7 \times 4 \times 8 + C_2^4 \times 8^2 = 608$ 個 只有三個「3」和沒有「1」的 5 位數，有 608 個 四個「1」和沒有「3」的 5 位數，有 $7 + 4 \times 8 = 39$ 個 四個「3」和沒有「1」的 5 位數，有 39 個 「11111」有 1 個；「33333」有 1 個 總數 $= 90000 - 28672 - (18432 + 4736 + 608 + 39 + 1) \times 2$ $= 13696$ 個</p> <p>方法二 只有一個「1」和一個「3」的 5 位數， 一共有 $= 4 \times 8^3 \times 2 + 7 \times P_2^4 \times 8^2 = 9472$ 個 兩個「1」和一個「3」的 5 位數， 共有 $= 7 \times 8 \times C_2^4 \times 2 + P_2^4 \times 8^2 + C_2^4 \times 8^2 = 1824$ 個 兩個「3」和一個「1」的 5 位數，有 1824 個 兩個「1」和兩個「3」的 5 位數， 共有 $= 7 \times C_2^4 + C_2^4 \times 2 \times 8 + C_2^4 \times 2 \times 8 = 234$ 個</p>	<p>Method 1 No. of 5-digit numbers $= 9 \times 10^4 = 90000$ If there are no '1' and no '3', numbers $= 7 \times 8^4 = 28672$ If there is only one '1' but no '3', numbers $= 1 \times 8^4 + 7 \times 4 \times 8^3 = 18432$ If there is only one '3' but no '1', numbers $= 18432$ If there are 2 '1's but no '3', numbers $= 4 \times 8^3 + 7 \times C_2^4 \times 8^2 = 4736$ If there are 2 '3's but no '1', numbers $= 4736$ If there are 3 '1's but no '3', numbers $= 7 \times 4 \times 8 + C_2^4 \times 8^2 = 608$ If there are 3 '3's but no '1', numbers $= 608$ If there are 4 '1's but no '3', numbers $= 7 + 4 \times 8 = 39$ If there are 4 '3's but no '1', numbers $= 39$ '11111', number = 1; '33333', number = 1 Total $= 90000 - 28672 - (18432 + 4736 + 608 + 39 + 1) \times 2$ $= 13696$</p> <p>Method 2 If there is only one '1' and one '3', numbers $= 4 \times 8^3 \times 2 + 7 \times P_2^4 \times 8^2 = 9472$ If there is 2 '1's but only one '3', numbers $= 7 \times 8 \times C_2^4 \times 2 + P_2^4 \times 8^2 + C_2^4 \times 8^2 = 1824$ If there is 2 '3's but only one '1', numbers $= 1824$ If there are 2 '3's and 2 '1's, numbers $= 7 \times C_2^4 + C_2^4 \times 2 \times 8 + C_2^4 \times 2 \times 8 = 234$ If there are 3 '1's and one '3',</p>
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<p>三個‘1’和一個‘3’的 5 位數， 共有 $=C_2^4 \times 2 \times 8 + 4 \times 8 + 7 \times 4 = 156$ 個 三個‘3’和一個‘1’的 5 位數，有 156 個 三個‘1’和兩個‘3’的 5 位數，共有 $=C_2^4 + 4 = 10$ 個 三個‘3’和兩個‘1’的 5 位數，有 10 個 四個‘1’和一個‘3’的 5 位數，共有 $=1 + 4 = 5$ 個 四個‘3’和一個‘1’的 5 位數，有 5 個 總數 $= 9472 + 1824 \times 2 + 156 \times 2 + 234 + 30 = 13696$ 個</p>	<p>numbers $= C_2^4 \times 2 \times 8 + 4 \times 8 + 7 \times 4 = 156$ If there are 3‘3’s and one ‘1’, numbers $= 156$ If there are 3‘1’s and 2‘3’s, numbers $= C_2^4 + 4 = 10$ If there are 3‘3’s and 2‘1’s, numbers $= 10$ If there are 4‘1’s and 1‘3’, numbers $= 1 + 4 = 5$ If there are 4‘3’s and 1‘1’, numbers $= 5$ Total $= 9472 + 1824 \times 2 + 156 \times 2 + 234 + 30 = 13696$</p>
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G1.4 設有 m 對正整數 a 和 b ，使 $a^4 + 4b^4$ 為質數，求 m 的值。

Let m be the number of possible pairs of positive integers a and b for which $a^4 + 4b^4$ is a prime number. Find the value of m .

<p>$a^4 + 4b^4 = a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2$ $= (a^2 + 2b^2)^2 - (2ab)^2$ $= (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$ $a^2 + 2ab + 2b^2 = p$，一個質數及 $a^2 - 2ab + 2b^2 = 1$ $(a - b)^2 + b^2 = 1$ $a = b$ 及 $b = 1$ 或 $a - b = 1$ 及 $b = 0$ $(a, b) = (1, 1)$ 或 $(1, 0)$ 代以上答案入 $a^2 + 2ab + 2b^2 = p$ $1 + 2 + 2 = p$ (接受) $1 + 0 + 0 = p$ 不是質數 (捨去) $\therefore (a, b) = (1, 1)$ $m = 1$</p>	<p>$a^4 + 4b^4 = a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2$ $= (a^2 + 2b^2)^2 - (2ab)^2$ $= (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$ $a^2 + 2ab + 2b^2 = p$，a prime number and $a^2 - 2ab + 2b^2 = 1$ $(a - b)^2 + b^2 = 1$ Either $a = b$ and $b = 1$ or $a - b = 1$ and $b = 0$ $(a, b) = (1, 1)$ or $(1, 0)$ Sub. the solutions into $a^2 + 2ab + 2b^2 = p$ $1 + 2 + 2 = p$ (accepted) $1 + 0 + 0 = p$，not a prime, (rejected) $\therefore (a, b) = (1, 1)$ $m = 1$</p>
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Group Event 2

G2.1 設 $x > 0$ 。已知 $x - \frac{1}{x} = \sqrt{3}$ 且 $a = x^5 + x^3 + x + \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^5}$ ，求 a 的值。

Let $x > 0$. Given that $x - \frac{1}{x} = \sqrt{3}$ and $a = x^5 + x^3 + x + \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^5}$, find the value of a .

$$x - \frac{1}{x} = \sqrt{3} \Rightarrow x^2 - 2 + \frac{1}{x^2} = 3$$

$$x^2 + 2 + \frac{1}{x^2} = 7 \Rightarrow \left(x + \frac{1}{x}\right)^2 = 7$$

$$x + \frac{1}{x} = \sqrt{7}$$

$$x^2 + \frac{1}{x^2} = 5$$

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right) = \sqrt{7}(5 - 1) = 4\sqrt{7}$$

$$\left(x^2 + \frac{1}{x^2}\right)\left(x^3 + \frac{1}{x^3}\right) = (5)(4\sqrt{7}) \Rightarrow x^5 + \frac{1}{x^5} + x + \frac{1}{x} = 20\sqrt{7} \Rightarrow x^5 + \frac{1}{x^5} = 19\sqrt{7}$$

$$a = \left(x^5 + \frac{1}{x^5}\right) + \left(x^3 + \frac{1}{x^3}\right) + \left(x + \frac{1}{x}\right) = 19\sqrt{7} + 4\sqrt{7} + \sqrt{7} = 24\sqrt{7}$$

G2.2 用首2024個正整數：1、2、3、4、5、6、...、2024，造出一個新整數：123456789101112...2024。若 b 是這個整數裡「0」的數量，求 b 的值。

Using the first 2024 positive integers: 1, 2, 3, 4, 5, 6, ..., 2024, a new integer is formed as 123456789101112...2024. If b is the number of "0" in this integer, find the value of b .

	‘0’ 的數目	Number of ‘0’s
1, 2, ..., 9, 10, 11, ..., 99	9	9
100, 211, ..., 999	$(11 + 9) \times 9$	$(11 + 9) \times 9$
1000, 1001, ..., 1009	21	21
1010, 1011, ..., 1099	$90 + 9$	$90 + 9$
1100, 1101, ..., 1999	$(11 + 9) \times 9$	$(11 + 9) \times 9$
2000, 2001, ..., 2009	21	21
2010, 2011, ..., 2019	11	11
2020, 2021, 2022, 2023, 2024	6	6

‘0’ 的總數 Total number of ‘0’ = $9 + (11 + 9) \times 9 + 21 + 90 + 9 + (11 + 9) \times 9 + 21 + 11 + 6 = 527$

G2.3 c 是 $2024^2 - 2023^2$ 的正因數的數量。求 c 的值。

c is the number of positive factors of $2024^2 - 2023^2$.

$2024^2 - 2023^2 = (2024 + 2023)(2024 - 2023)$ $= 4047 = 3 \times 1349$ ，1349為質數 正因數為 1、3、1349、4047。 $c = 4$	$2024^2 - 2023^2 = (2024 + 2023)(2024 - 2023)$ $= 4047 = 3 \times 1349$, 1349 is a prime The positive factors are 1, 3, 1349, 4047. $c = 4$
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G2.4 假設「0」、「1」、「2」、…及「6」分別為星期日、星期一、星期二、…和星期六，

今日是星期一，若 $20^{24^{2024}}$ 天後的那一天是星期幾之代號為「 d 」，求 d 的值。

Let “0”, “1”, “2”, … and “6” represent Sunday, Monday, Tuesday, … and Saturday respectively. Today is Monday. If “ d ” represents the day of week that comes after $20^{24^{2024}}$ days. Find the value of d .

$20 = 7 \times 2 + 6 \Rightarrow 20 \equiv -1 \pmod{7}$ $20^2 \equiv (-1)^2 \equiv 1 \pmod{7}$ 這個數字的規律每隔 2 的倍數重複一次。 24^{2024} 是一個偶數。 $20^{24^{2024}} \equiv 1 \pmod{7}$ $20^{24^{2024}}$ 日後為星期二。 $d = 2$	$20 = 7 \times 2 + 6 \Rightarrow 20 \equiv -1 \pmod{7}$ $20^2 \equiv (-1)^2 \equiv 1 \pmod{7}$ This pattern repeats for every multiple of 2. 24^{2024} is an even number. $20^{24^{2024}} \equiv 1 \pmod{7}$ The day after $20^{24^{2024}}$ days is a Tuesday. $d = 2$
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Group Event 3

G3.1 試找出最小的正整數 n 使得 $2^{10} + 2^{13} + 2^n$ 成為一個完全平方數。

Find the smallest positive integer n such that $2^{10} + 2^{13} + 2^n$ is a perfect square number.

$$2^{10} + 2^{13} + 2^n = m^2$$

$$2^{10}(1 + 8) + 2^n = m^2$$

$$2^n = m^2 - (2^5 \times 3)^2$$

$$2^a \times 2^b = (m + 96)(m - 96)$$

$$m + 96 = 2^a, m - 96 = 2^b; a, b \in \mathbb{Z}^+$$

$$192 = 2^a - 2^b, a > b \in \mathbb{Z}^+$$

$$2^6 \times 3 = 2^b(2^{a-b} - 1)$$

$$b = 6, 2^{a-b} - 1 = 3 \Rightarrow 2^{a-b} = 4 \Rightarrow a = 8$$

$$2^n = 2^8 \times 2^6 = 2^{14}$$

$$n = 14$$

G3.2 設 $a^2 + b^2 + 6a - 14b + 58 = 0$ 。求 $b - a$ 的值。

Suppose $a^2 + b^2 + 6a - 14b + 58 = 0$. Find the value of $b - a$.

$$a^2 + 2a(3) + 3^2 + b^2 - 2b(7) + 7^2 = 0$$

$$(a + 3)^2 + (b - 7)^2 = 0$$

$$a = -3, b = 7$$

$$b - a = 7 - (-3) = 10$$

G3.3 在正方形土地的某一個角落裡埋著一個裝有\$8,000 的箱子。在一次比賽中，你和另一個叫「倒霉先生」的人一起挖箱子。倒霉先生有一個特點：他總是做出錯誤的選擇。你贏了擲骰子先選。你選了一個角落，倒霉先生選了另一個角落。在你準備開始時，你發現倒霉先生沒有找到箱子。遊戲規則允許你換另一個角落，但要罰\$200。計算換角落的期望收益。

There was a chest containing \$8,000 buried in one of the corners of a square piece of land. In a contest, you and another man called “Mr. Badluck” were digging for the chest. Mr. Badluck had one peculiarity: he always made the wrong choice. You won the toss and chose first. You picked a corner, and Mr. Badluck picked another. Before you started, you observed that Mr. Badluck found no chest. The rules of the game allowed you to make a switch to another corner, but with a penalty of \$200. Calculate the expected gain from making the switch in dollars.

將正方形的四角命名為 A 、 B 、 C 及 D 。	Label the four corners as A , B , C and D .
若不轉換： $P(A) = P(B) = P(C) = P(D) = \frac{1}{4}$	Without switching, $P(A) = P(B) = P(C) = P(D) = \frac{1}{4}$
期望收益 = $\$ \frac{1}{4} \times 8000 = \2000	Expected gain = $\$ \frac{1}{4} \times 8000 = \2000
假設你已選擇了 A 及倒霉先生選擇了 B ；而 B 沒有箱子。如果你轉換，那麼轉換的條件概率為 $P(C B \text{ 沒有箱子}) = P(D B \text{ 沒有箱子}) = \frac{1}{3}$	Suppose you had already chosen A and Mr. Badluck had chosen B ; while B doesn't contain the chest. If you switch, then the conditional probability $P(C B \text{ is not}) = P(D B \text{ is not}) = \frac{1}{3}$
期望收益 = $\$(\frac{1}{3} \times 8000 - 200) = \$ \frac{7400}{3}$ 。	Expected gain = $\$(\frac{1}{3} \times 8000 - 200) = \$ \frac{7400}{3}$ 。

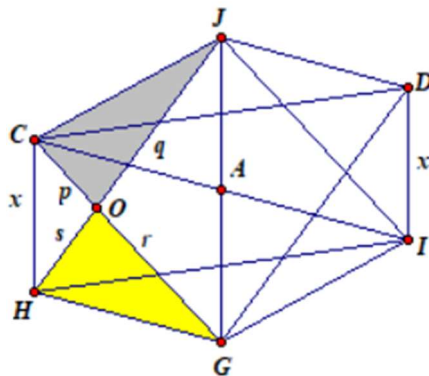
G3.4 一個凸六邊形有以下性質：

- (i) 由任意頂點與相鄰兩個頂點組成的三角形的面積都是 $1,000 \text{ cm}^2$ ；及
(ii) $CH = DI$ 。
求六邊形的面積。

A convex hexagon has the following property:

- (i) all the triangles formed from any vertex with the two adjacent vertices have an area of $1,000 \text{ cm}^2$; and
(ii) $CH = DI$.

Find the area of the hexagon.



假設 CG 與 JH 相交於 O ， CI 與 DH 相交於 A 。
假設 $OC = p$ 、 $OJ = q$ 、 $OG = r$ 、 $OH = s$ 。
 $S_{\triangle CHI} = S_{\triangle CHG} \Rightarrow S_{\triangle OCH} + S_{\triangle OCJ} = S_{\triangle OCH} + S_{\triangle OHG}$
 $S_{\triangle OCJ} = S_{\triangle OHG} \Rightarrow \frac{1}{2} \cdot pq \sin \angle COJ = \frac{1}{2} \cdot rs \sin \angle GOH$
 $\therefore \angle COJ = \angle GOH$ (對頂角)
 $\therefore pq = rs \Rightarrow \frac{OJ}{OH} = \frac{OC}{OG}$
 $\angle JOG = \angle COH$ (對頂角)
 $\triangle COH \sim \triangle GOJ$ (兩邊成比例，一夾角相等)
 $\angle CHO = \angle GJO$ (相似三角形對應角)
 $\therefore CH \parallel JG$ (交錯角相等)
 $\therefore CH \parallel DI \parallel JG, CJ \parallel HD \parallel GI, JD \parallel CI \parallel HG$
 $CDHI$ 是一個平行四邊形 (對邊相等且平行)
 $ACJD$ 是一個平行四邊形 (由兩組平行邊組成)
 $AHGI$ 是一個平行四邊形 (由兩組平行邊組成)
 $S_{\triangle AHI} = S_{\triangle AGHI} = 1000 \text{ cm}^2$
 $S_{\triangle ACD} = S_{\triangle JCD} = 1000 \text{ cm}^2$
 $ACHG$ 是一個平行四邊形 (由兩組平行邊組成)
 $ADGI$ 是一個平行四邊形 (由兩組平行邊組成)
 $S_{\triangle ACH} = S_{\triangle ACGH} = 1000 \text{ cm}^2$
 $S_{\triangle ADI} = S_{\triangle AGDI} = 1000 \text{ cm}^2$
六邊形的面積 $= 1000 \times 6 \text{ cm}^2 = 6000 \text{ cm}^2$

Suppose CG meets JH at O , CI meets DH at A .
Let $OC = p$, $OJ = q$, $OG = r$, $OH = s$.
 $S_{\triangle CHI} = S_{\triangle CHG} \Rightarrow S_{\triangle OCH} + S_{\triangle OCJ} = S_{\triangle OCH} + S_{\triangle OHG}$
 $S_{\triangle OCJ} = S_{\triangle OHG} \Rightarrow \frac{1}{2} \cdot pq \sin \angle COJ = \frac{1}{2} \cdot rs \sin \angle GOH$
 $\therefore \angle COJ = \angle GOH$ (vert. opp. \angle s)
 $\therefore pq = rs \Rightarrow \frac{OJ}{OH} = \frac{OC}{OG}$
 $\angle JOG = \angle COH$ (vert. opp. \angle s)
 $\triangle COH \sim \triangle GOJ$ (ratio of 2 sides, inc. \angle)
 $\angle CHO = \angle GJO$ (corr. \angle s, $\sim \Delta$ s)
 $\therefore CH \parallel JG$ (alt. \angle s eq.)
 $\therefore CH \parallel DI \parallel JG, CJ \parallel HD \parallel GI, JD \parallel CI \parallel HG$
 $CDHI$ is a // -gram (opp. sides eq. and //)
 $ACJD$ is a // -gram (formed by 2 pairs of // lines)
 $AHGI$ is a // -gram (formed by 2 pairs of // lines)
 $S_{\triangle AHI} = S_{\triangle AGHI} = 1000 \text{ cm}^2$
 $S_{\triangle ACD} = S_{\triangle JCD} = 1000 \text{ cm}^2$
 $ACHG$ is a // -gram (formed by 2 pairs of // lines)
 $ADGI$ is a // -gram (formed by 2 pairs of // lines)
 $S_{\triangle ACH} = S_{\triangle ACGH} = 1000 \text{ cm}^2$
 $S_{\triangle ADI} = S_{\triangle AGDI} = 1000 \text{ cm}^2$
Area of hexagon $= 1000 \times 6 \text{ cm}^2 = 6000 \text{ cm}^2$

Group Event 4

G4.1 設 a, b 為非零整數，且滿足方程 $a - ab + b = 18$ 。求 $a + b$ 的值。

Let a, b be non-zero integers satisfying the equation $a - ab + b = 18$. Find the value of $a + b$.

Reference: 2024 F11.1

$a - ab + b - 1 = 17$	$a - ab + b - 1 = 17$
$(1 - a)(b - 1) = 17$	$(1 - a)(b - 1) = 17$
$(1 - a, b - 1) = (1, 17)$ 或 $(-1, -17)$	$(1 - a, b - 1) = (1, 17)$ or $(-1, -17)$
$(a, b) = (0, 18)$ (捨去) 或 $(2, -16)$	$(a, b) = (0, 18)$ (rejected) or $(2, -16)$
$a + b = -14$	$a + b = -14$

G4.2 設 x 為一正整數，且滿足 $x(x+1)(x+2)(x+3) = 3024$ 。求 x 的值。

Let x be a positive integer satisfying $x(x+1)(x+2)(x+3) = 3024$. Find the value of x .

$x(x+1)(x+2)(x+3) = 3024$	$x(x+1)(x+2)(x+3) = 3024$
$(x^2 + 3x)(x^2 + 3x + 2) = 3024$	$(x^2 + 3x)(x^2 + 3x + 2) = 3024$
$(x^2 + 3x)^2 + 2(x^2 + 3x) + 1 = 3025$	$(x^2 + 3x)^2 + 2(x^2 + 3x) + 1 = 3025$
$(x^2 + 3x + 1)^2 = 55^2$	$(x^2 + 3x + 1)^2 = 55^2$
$x^2 + 3x + 1 = 55$ 或 $x^2 + 3x + 1 = -55$	$x^2 + 3x + 1 = 55$ or $x^2 + 3x + 1 = -55$
$x^2 + 3x - 54 = 0$ 或 $x^2 + 3x + 56 = 0$	$x^2 + 3x - 54 = 0$ or $x^2 + 3x + 56 = 0$
$(x - 6)(x + 9) = 0$ 或沒有實數解	$(x - 6)(x + 9) = 0$ or no real solution
$\therefore x > 0 \therefore x = 6$	$\therefore x > 0 \therefore x = 6$

G4.3 設 α, β 為二次方程 $x^2 + 6x + 2 = 0$ 的兩個根，

求以 $\frac{\alpha^2}{\beta}$ 和 $\frac{\beta^2}{\alpha}$ 為根及 x^2 的係數為 1 的二次方程。

Let α, β be the two roots of the quadratic equation $x^2 + 6x + 2 = 0$.

Find the quadratic equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$, and coefficient of x^2 is 1.

$$\alpha + \beta = -6, \alpha\beta = 2$$

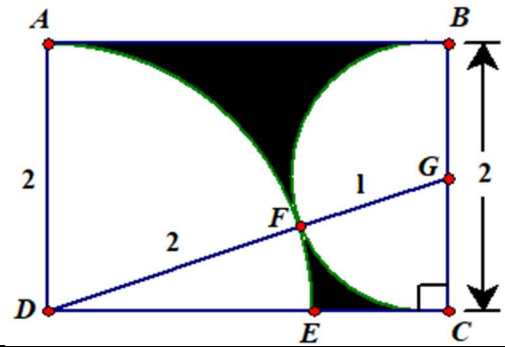
$$\begin{aligned} \alpha^3 + \beta^3 &= (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) \\ &= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta] \\ &= -6[(-6)^2 - 3(2)] = -180 \end{aligned}$$

$$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{-180}{2} = -90$$

$$\frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha} = \alpha\beta = 2$$

$$x^2 + 90x + 2 = 0$$

G4.4 右圖空白部分由一個四分一圓和一個半圓互相外切組成。 $ABCD$ 是一個長方形。求陰影部分的面積。
The unshaded part in the diagram on the right is made up of a quarter-circle and a semi-circle which touch each other externally. $ABCD$ is a rectangle.
Find the area of the shaded part.



假設該四分一圓和半圓互相外切於 F 及 G 為半圓的圓心。那麼 D 、 F 、 G 共線。

$$DG = DF + FG = 2 + 1 = 3$$

$$CD^2 + CG^2 = DG^2 \text{ (畢氏定理)}$$

$$CG = \sqrt{3^2 - 1^2} = 2\sqrt{2}$$

$$\text{陰影面積} = 2\sqrt{2} \times 2 - \frac{1}{4} \cdot \pi(2)^2 - \frac{1}{2} \cdot \pi(1)^2$$

$$= 4\sqrt{2} - \frac{3\pi}{2}$$

Suppose the quarter-circle and the semi-circle touch each other at F and G is the centre of the semi-circle. Then D , F , G are collinear.

$$DG = DF + FG = 2 + 1 = 3$$

$$CD^2 + CG^2 = DG^2 \text{ (Pythagoras' theorem)}$$

$$CG = \sqrt{3^2 - 1^2} = 2\sqrt{2}$$

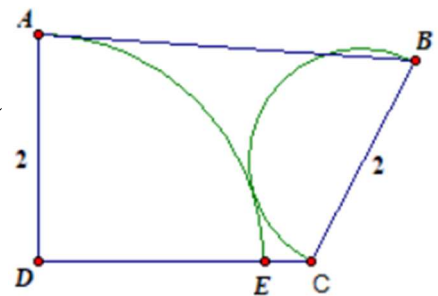
$$\text{Shaded area} = 2\sqrt{2} \times 2 - \frac{1}{4} \cdot \pi(2)^2 - \frac{1}{2} \cdot \pi(1)^2$$

$$= 4\sqrt{2} - \frac{3\pi}{2}$$

Remark: the original question was:

右圖空白部分由一個四分一圓和一個半圓組成，
求陰影部分的面積。

The unshaded part in the diagram on the right is made up of a quarter-circle and a semi-circle. Find the area of the shaded part.



If $ABCD$ is not a rectangle, then it is impossible to find CD and hence the area of the shaded part. Furthermore, the fact that the quarter-circle and the semi-circle touch each other externally must be specified.