Elementary Set Theory

1.1 Definition of **sets** (集合)

A set is a collection of objects.

e.g.1 The set of 2, 3, 4 is a total of three objects.

Denote the objects in the set as follows: {objects}

e.g.2 A = {2, dog, aeroplane, Mary}, a set of 4 objects.

The objects inside the set are called **elements** (元素).

e.g.3 A = {2, dog, aeroplane, Mary}, the elements are 2, dog, aeroplane, Mary.

e.g.4 B = {8, A, 8}, the elements are A, 8

Note that (i) identical elements count once.

(ii) order (次序) is not important.

A set containing nothing is called the **empty set** (空集); it is denoted by { } or ϕ.

If a is an element of a set X, we write $a \in X$ (read a is an element of X).

If a is not an element of a set X, we write $a \notin X$ (read a is not an element of X).

e.g.5 X = {any integers}, then $1 \in X$, but $1.5 \notin X$

1.2 Equality of sets

Two set are **equal** if they have the same elements.

Two sets are **unequal** if some elements are different.

e.g.6 $A = \{1, 3, 5, 7, 9\}$, B = {positive odd integers less than ten}, then $A = B$.

C = $\{1, 2, 5, 7, 9\}$, then $A \neq C$

$D = \{3, 5, 9, 7, 9\}$, then $D \neq A$

1.3 Number of elements in a set

Let A be a set. If A has a finite number of elements, then $n(A)$ = number of (different) elements in A.

e.g.7 $A = \{ 2, 1, \text{dog}\}$, $n(A) = 3$

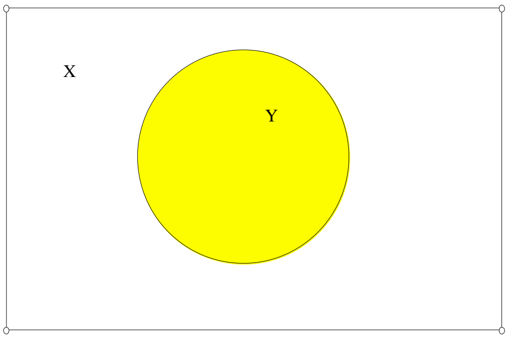
$B = \{7, 7, 1, 7\}$, $n(B) = 2$

ϕ = empty set, a set containing no element. $n(\phi) = 0$

$\mathbb{Z}$ = {any integer}, $n(\mathbb{Z})$ is undefined, since it has infinite number of elements.

1.4 Subsets (子集)

Let X be a set. A set Y is a subset of X if every element in Y is inside X.



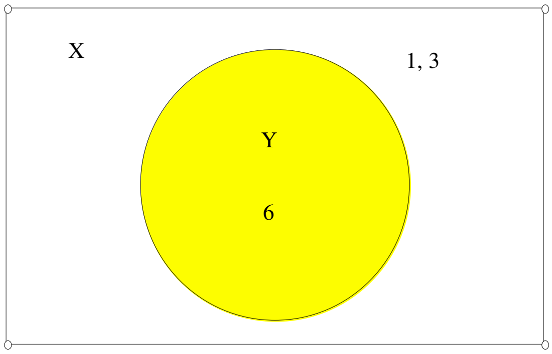
Y is a subset of X.

The diagram is called a Venn diagram.

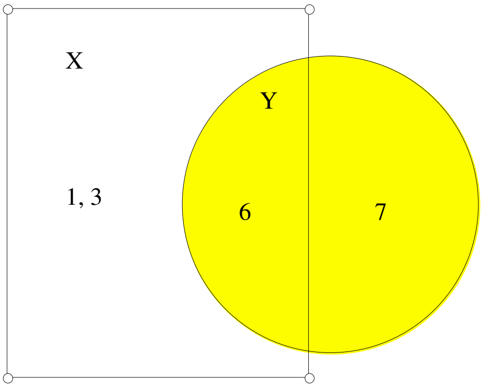
We write $Y \subset X$.

Otherwise we write $Y \not\subset X$.

e.g.8 $X = \{1, 3, 6, 6\}$, $Y = \{6, 6, 6\}$
then $Y \subset X$



e.g.9 $X = \{1, 3, 6\}$, $Y = \{6, 7\}$
then $Y \not\subset X$

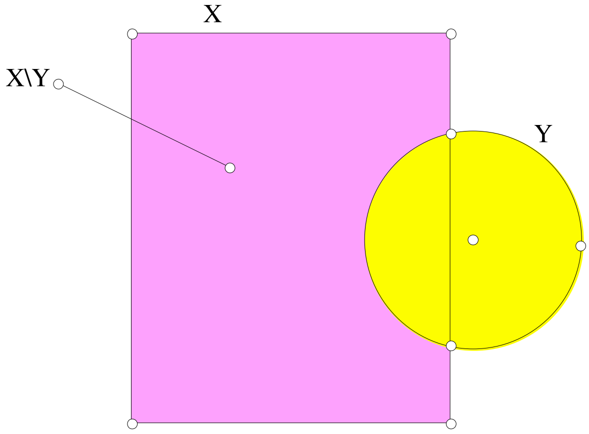


e.g.10 $X = \{1, 3, 6\}$, ϕ = empty set, a set containing no element.
then $\phi \subset X$

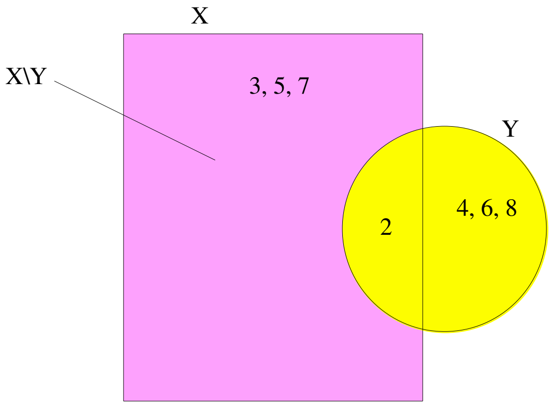
e.g.11 $X = \{1, 3, 6\}$, list out all possible subsets of X
ϕ, a subset containing nothing.
$\{1\}, \{3\}, \{6\}$, subsets containing 1 element
$\{1, 3\}, \{3, 6\}, \{1, 6\}$ subsets containing 2 elements
$\{1, 3, 6\} = X$, a subset containing 3 elements.
note that (i) $\phi \subset X$, (ii) $X \subset X$

1.5 Complement of a set (補集)

Let X, Y be sets. The complement of Y relative to X is the subset of X excluding all the elements of Y. It is written as $X \setminus Y$. (read X except Y). The following figure is an illustration.

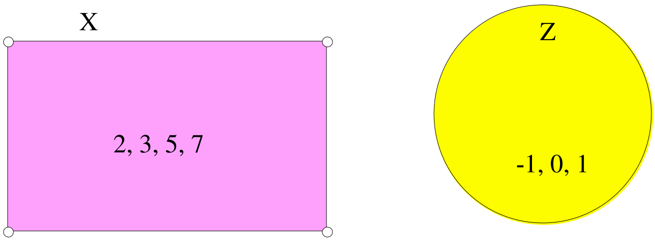


e.g.12 $X = \{2, 3, 5, 7\}$, $Y = \{2, 4, 6, 8\}$
$X \setminus Y = \{3, 5, 7\}$



e.g.13 $X = \{2, 3, 5, 7\}$, $Z = \{-1, 0, 1\}$

$X \setminus Z = \{2, 3, 5, 7\}$



Note that (i) $X \setminus \phi = X$

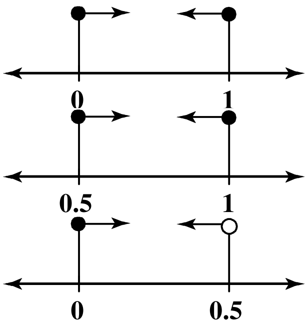
(ii) If X and Y has no common element, then $X \setminus Y = X$

In particular, if Y is a subset of X, then $X \setminus Y$ is denoted by $\overline{Y}$ or Y'.

e.g.14 $X = \{0 \le \text{any number} \le 1\}$

$Y = \{\frac{1}{2} \le \text{any number} \le 1\}$

$\overline{Y} = \{0 \le \text{any number} < \frac{1}{2}\}$



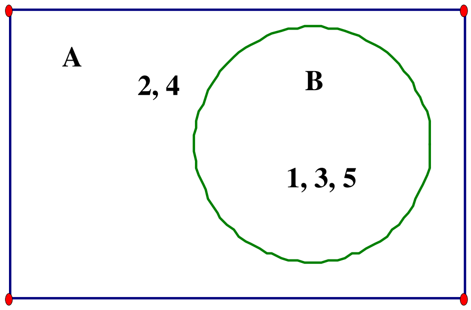
e.g.15 $A = \{1, 2, 3, 4, 5\}$

$B = \{1, 3, 5\}$

$B' = \{2, 4\}$

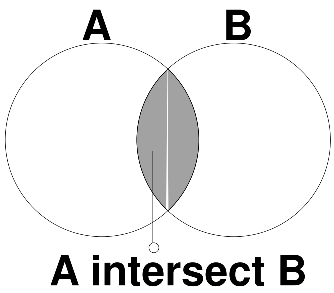
Note that $X' = X \setminus X = \phi$ (the empty set)

and $\{\ \ \}' = \phi' = X$



1.6 Intersection of Sets (交集)

Let A, B be 2 sets. The intersection of A, B is the set containing the <u>common elements</u> of A and B.



we denote A intersect B as $A \cap B$

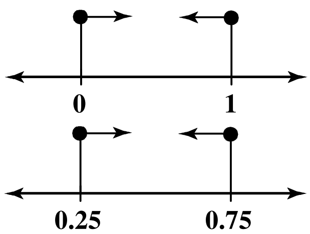
e.g.16 $A = \{-1, 0, 1, 2\}$, $B = \{0, 2, 3\}$

$A \cap B = \{0, 2\} = B$

$A \cap A = A$

e.g.17 $A = \{0 \le x \le 1\}$

$B = \{\frac{1}{4} \le x \le \frac{3}{4}\}$



$A \cap B = B$

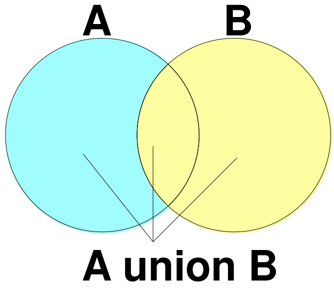
$C = \{\text{dog, cat, pig}\}$

$C \cap A = \phi$, a set containing nothing.

$\phi \cap A = \phi$; $\phi \cap \phi = \phi$.

1.7 Union of Sets (和集)

Let A, B be 2 sets. The union of A, B is the set containing <u>all</u> elements of A and B.



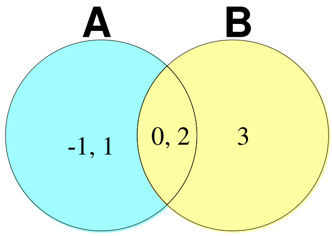
we denote A union B as $A \cup B$

e.g.18 $A = \{-1, 0, 1, 2\}$, $B = \{0, 2, 3\}$

$A \cup B = \{0, 2, 3, -1, 1\}$

$B \cup B = B$

$A \cup \phi = \phi \cup A = A$

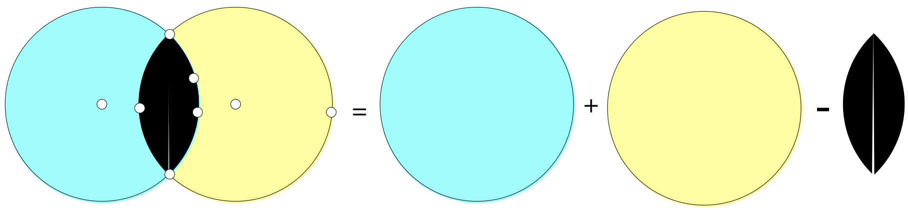


e.g.19 $A = \{1\}$, $B = \{2\}$

$A \cup B = \{1, 2\}$

e.g.20 $\phi \cup \phi = \phi$

Number of elements in $A \cup B$.



$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

e.g.21 For the set of integers $\{1, 2, 3, 4, ... , 70\}$

(i) How many of its elements are divisible by 2 or 3?

(ii) How many are not divisible by 2 nor 3?

Solution: $X = \{1, 2, 3, 4, ... , 70\}$,

A = {numbers divisible by 2} = $\{2, 4, 6, ..., 70\}$, $n(A)$ = ________

B = {numbers divisible by 3} = , $n(B)$ = ________

$A \cap B$ = {numbers divisible by 2 and 3} = {numbers divisible by ____ }

= {6, 12, ... , _____ }, $n(A \cap B)$ = ________

$A \cup B$ = {numbers divisible by 2 or 3}

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

= _______ + ________ – ________

= _______

$X \setminus (A \cup B)$ = {numbers not divisible by 2 nor 3}

$n(X \setminus (A \cup B))$ = 70 – _______ = _______

1.8 Algebra of Sets

Example 22

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, $C = \{3, 4, 5, 6\}$

Find $A \cup (B \cap C)$, $(A \cup B) \cap (A \cup C)$

$B \cap C = \{2, 4, 6, 8\} \cap \{3, 4, 5, 6\} = \{$ __________________ $\}$

$A \cup (B \cap C) = \{1, 2, 3, 4\} \cup \{$ __________________ $\} = \{$ __________________ $\}$

$A \cup B = \{1, 2, 3, 4\} \cup \{2, 4, 6, 8\} = \{$ __________________ $\}$

$A \cup C = \{1, 2, 3, 4\} \cup \{3, 4, 5, 6\} = \{$ __________________ $\}$

$(A \cup B) \cap (A \cup C) = \{$ __________________ $\} \cap \{$ __________________ $\} = \{$ __________________ $\}$

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Exercise 1 Find $A \cap (B \cup C)$.

Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Example 23

$U \setminus (A \cap B) = \{$ __________________ $\}$

$(U \setminus A) \cup (U \setminus B) = \{$ __________________ $\}$

$\therefore U \setminus (A \cap B) = (U \setminus A) \cup (U \setminus B)$

Exercise 2 Prove that $U \setminus (A \cup B) = (U \setminus A) \cap (U \setminus B)$