Individual Events																	
SI	***P***	30	**I1**	***A***	12	**I2**	***P***	5	**I3**	α	45	**I4**	***A***	22	**IS**	***P***	95
	***Q** see the remark	120		***B***	108		***Q***	12		*β see the remark	56		***B***	12		***Q***	329
	R	11		***C** see the remark	280		***R***	3		γ	23		***C***	12		***R** see the remark	6
	***S** see the remark	72		***D***	69		***S***	17		δ	671		***D***	7		***S***	198
Group Events																	
SG	***q***	3	**G1**	tens digit	1	**G2**		2	**G3**	*z*	6	**G4**	$\frac{BD}{CE}$	2	**GS**	***m** see the remark	4
	k	1		***P** see the remark	1031		*K*	2		***r** see the remark	540		***Q***	1		***v***	6
	w	25		***k***	21		**l**	45		***D***	998		***R***	1		α	3
	p	$\frac{3}{2}$		$*S_{\Delta BCD}$ see the remark	32		see the remark	$\frac{1}{4}$		$*F_{2012}(7)$ see the remark	1		x_5	5		***F***	208

Sample Individual Event (2009 Final Individual Event 1)

SI.1 Let a, b, c and d be the roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $P = a^2 + b^2 + c^2 + d^2$, find the value of P.

$x^4 - 15x^2 + 56 = 0 \Rightarrow (x^2 - 7)(x^2 - 8) = 0$

$a = \sqrt{7}$, $b = -\sqrt{7}$, $c = \sqrt{8}$, $d = -\sqrt{8}$

$P = a^2 + b^2 + c^2 + d^2 = 7 + 7 + 8 + 8 = 30$

SI.2 In Figure 1, $AB = AC$ and AB // ED.
If $\angle ABC = P°$ and $\angle ADE = Q°$, find the value of Q.

$\angle ABC = 30° = \angle ACB$ (base $\angle$s isos. Δ)

$\angle BAC = 120°$ ($\angle$s sum of Δ)

$\angle ADE = 120°$ (alt. $\angle$s, AB // ED)

$Q = 120$

Remark: Original question ⋯ AB // DE ⋯.

It is better for AB and ED to be oriented in the same direction.

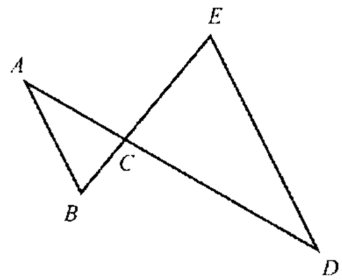


Figure 1

SI.3 Let $F = 1 + 2 + 2^2 + 2^3 + \cdots + 2^Q$ and $R = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of R.

$F = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{120} = \frac{2^{121}-1}{2-1} = 2^{121} - 1$

$R = \sqrt{\frac{\log(1+F)}{\log 2}} = \sqrt{\frac{\log 2^{121}}{\log 2}} = 11$

SI.4 Let f(x) be a function such that $f(n) = (n - 1)\, f(n - 1)$ and $f(1) \neq 0$.

If $S = \frac{f(R)}{(R-1)f(R-3)}$, find the value of S.

$f(n) = (n - 1)\, f(n - 1) = (n - 1)(n - 2)f(n - 2) = \cdots\cdots$

$S = \frac{f(11)}{(11)f(11-3)} = \frac{10\times 9\times 8\times f(8)}{10\times f(8)} = 9\times 8 = 72$

Remark: Original question:

Let f(x) be a function such that $f(n) = (n - 1)\, f(n - 1)$. If $S = \frac{f(R)}{(R-1)f(R-3)}$, find the value of S.

Note that S is undefined when $f(n) = 0$ for some integers n.

Individual Event 1

I1.1 If A is the sum of the squares of the roots of $x^4 + 6x^3 + 12x^2 + 9x + 2$, find the value of A.

Let $f(x) = x^4 + 6x^3 + 12x^2 + 9x + 2$

$f(-1) = 1 - 6 + 12 - 9 + 2 = 0$

$f(-2) = 16 - 48 + 48 - 18 + 2 = 0$

$\therefore$ By factor theorem, $(x^2 + 3x + 2)$ is a factor of $f(x)$.

$x^4 + 6x^3 + 12x^2 + 9x + 2 = (x + 1)(x + 2)(x^2 + 3x + 1)$

The roots are -1, -2 and α, β; where $\alpha + \beta = -3$, $\alpha\beta = 1$

$A = (-1)^2 + (-2)^2 + \alpha^2 + \beta^2 = 5 + (\alpha + \beta)^2 - 2\alpha\beta$

$= 5 + 9 - 2$

$A = 12$

By division,

$$\begin{array}{r|l} & x^2+3x+1 \\ \hline x^2+3x+2 & x^4+6x^3+12x^2+9x+2 \\ & \underline{x^4-3x^3+\ 2x^2} \\ & \quad 3x^3+10x^2+9x \\ & \quad \underline{3x^3+\ 9x^2+6x} \\ & \qquad x^2+3x+2 \\ & \qquad \underline{x^2+3x+2} \end{array}$$

Method 2 By the change of subject, let $y = x^2$, then the equation becomes

$x^4 + 12x^2 + 2 = -x(6x^2 + 9) \Rightarrow y^2 + 12y + 2 = \mp\sqrt{y}(6y + 9)$

$(y^2 + 12y + 2)^2 - y(6y + 9)^2 = 0$

Coefficient of $y^4 = 1$, coefficient of $y^2 = 24 - 36 = -12$

If α, β, δ and γ are the roots of x, then α^2, β^2, δ^2 and γ^2 are the roots of y

$$\alpha^2 + \beta^2 + \delta^2 + \gamma^2 = -\frac{\text{coefficient of } y^3}{\text{coefficient of } y^4} = 12$$

Method 3

Let α, β, δ and γ are the roots of x, then by the relation between roots and coefficients,

$\alpha + \beta + \delta + \gamma = -6 \cdots\cdots (1)$

$\alpha\beta + \alpha\delta + \alpha\gamma + \beta\delta + \beta\gamma + \delta\gamma = 12 \cdots\cdots (2)$

$\alpha^2 + \beta^2 + \delta^2 + \gamma^2 = (\alpha + \beta + \delta + \gamma)^2 - 2(\alpha\beta + \alpha\delta + \alpha\gamma + \beta\delta + \beta\gamma + \delta\gamma)$

$= (-6)^2 - 2(12) = 12$

I1.2 Let x, y, z, w be four consecutive vertices of a regular A-gon. If the length of the line segment xy is 2 and the area of the quadrilateral $xyzw$ is $a + \sqrt{b}$, find the value of $B = 2^a \cdot 3^b$.

Let O be the centre of the regular dodecagon.

Let $Ox = r = Oy = Oz = Ow$

$\angle xOy = \angle yOz = \angle zOw = \frac{360^\circ}{12} = 30^\circ$ ($\angle$s at a point)

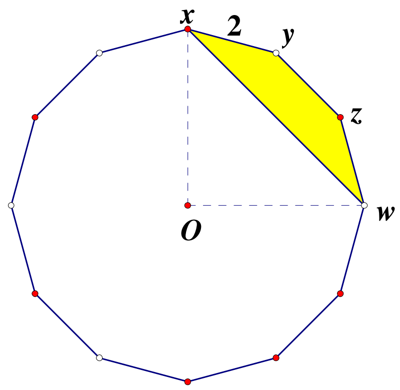
In ΔxOy, $r^2 + r^2 - 2r^2 \cos 30^\circ = 2^2$ (cosine rule)

$(2 - \sqrt{3})r^2 = 4 \Rightarrow r^2 = \frac{4}{2-\sqrt{3}} = 4\left(2+\sqrt{3}\right)$

Area of $xyzw$ = area of $Oxyzw$ – area of ΔOxw

$= 3\times\frac{1}{2}\cdot r^2 \sin 30^\circ - \frac{1}{2} r^2 \sin 90^\circ = \left(\frac{3}{4} - \frac{1}{2}\right)\cdot 4\left(2+\sqrt{3}\right) = 2 + \sqrt{3}$

$a = 2$, $b = 3$, $B = 2^2 \cdot 3^3 = 4\times 27 = 108$



I1.3 If C is the sum of all positive factors of B, including 1 and B itself, find the value of C.

$108 = 2^2 \cdot 3^3$

$C = (1 + 2 + 2^2)\cdot(1 + 3 + 3^2 + 3^3) = 7\times 40 = 280$

Remark: Original version: 若C是B的所有因子之和… If C is the sum of all factors ⋯

Note that if negative factors are also included, then the answer will be different.

I1.4 If $C! = 10^D k$, where D and k are integers such that k is not divisible by 10, find the value of D.

Reference: 1990 HG6, 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FG1.3

Method 1

When each factor of 5 is multiplied by 2, a trailing zero will appear in $n!$.

The number of factors of 2 is clearly more than the number of factors of 5 in 280!

It is sufficient to find the number of factors of 5.

5, 10, 15, … , 280; altogether 56 numbers, each have at least one factor of 5.

25, 50, 75, … , 275; altogether 11 numbers, each have at least two factors of 5.

125, 250; altogether 2 numbers, each have at least three factors of 5.

$\therefore$ Total number of factors of 5 is 56 + 11 + 2 = 69

$D = 69$

Method 2

We can find the total number of factors of 5 by division as follow:

$$\begin{array}{r|l}
5 & 2\ 8\ 0 \\ \hline
5 & 5\ 6 \\ \hline
5 & 1\ 1\ \ldots\ 1 \\ \hline
 & \ \ 2\ \ldots\ 1
\end{array}$$

$\therefore$Total no. of factors of 5 is $56 + 11 + 2 = 69$

$D = 69$

Individual Event 2

I2.1 If the product of the real roots of the equation $x^2 + 9x + 13 = 2\sqrt{x^2 + 9x + 21}$ is P, find the value of P.

Let $y = x^2 + 9x$, then the equation becomes $y + 13 = 2\sqrt{y + 21}$

$(y + 13)^2 = 4y + 84$

$y^2 + 26y + 169 - 4y - 84 = 0$

$y^2 + 22y + 85 = 0$

$(y + 17)(y + 5) = 0$

$y = -17$ or $y = -5$

Check put $y = -17$ into the original equation: $-17 + 13 = 2\sqrt{-17 + 21}$

LHS < 0, RHS > 0, rejected

Put $y = -5$ into the original equation: LHS $= -5 + 13 = 2\sqrt{-5 + 21}$ = RHS, accepted

$x^2 + 9x = -5$

$x^2 + 9x + 5 = 0$

Product of real roots = 5

Method 2

Let $y = \sqrt{x^2 + 9x + 21} \ge 0$

Then the equation becomes $y^2 - 8 = 2y \Rightarrow y^2 - 2y - 8 = 0$

$(y - 4)(y + 2) = 0 \Rightarrow y = 4$ or -2 (rejected)

$\Rightarrow x^2 + 9x + 21 = 16$

$x^2 + 9x + 5 = 0$

$\Delta = 9^2 - 4(5) > 0$

Product of real roots = 5

I2.2 If $f(x) = \dfrac{25^x}{25^x + P}$ and $Q = f\left(\dfrac{1}{25}\right) + f\left(\dfrac{2}{25}\right) + \cdots + f\left(\dfrac{24}{25}\right)$, find the value of Q.

Reference: 2004 FG4.1, 2011 HG5

$$f(x) + f(1-x) = \frac{25^x}{25^x + 5} + \frac{25^{1-x}}{25^{1-x} + 5} = \frac{25 + 5\cdot 25^x + 25 + 5\cdot 25^{1-x}}{25 + 5\cdot 25^{1-x} + 5\cdot 25^x + 25} = 1$$

$$Q = f\left(\frac{1}{25}\right) + f\left(\frac{2}{25}\right) + \cdots + f\left(\frac{24}{25}\right)$$

$$= f\left(\frac{1}{25}\right) + f\left(\frac{24}{25}\right) + f\left(\frac{2}{25}\right) + f\left(\frac{23}{25}\right) + \cdots + f\left(\frac{12}{25}\right) + f\left(\frac{13}{25}\right) = 12$$

I2.3 If $X = \sqrt{(100)(102)(103)(105) + (Q-3)}$ is an integer and R is the units digit of X, find the value of R.

Reference: 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1

Let $y = 102.5$, then

$(100)(102)(103)(105) + (12 - 3)$

$= (y - 2.5)(y - 0.5)(y + 0.5)(y + 2.5) + 9$

$= (y^2 - 6.25)(y^2 - 0.25) + 9$

$= y^4 - 6.5y^2 + \frac{25}{16} + 9 = y^4 - 6.5y^2 + \frac{169}{16}$

$= \left(y^2 - \frac{13}{4}\right)^2 = \left(102.5^2 - \frac{13}{4}\right)^2 = \left(\frac{205^2}{4} - \frac{13}{4}\right)^2$

$X = \frac{42025^2 - 13}{4} = 10503$

R = the units digit of $X = 3$

Method 2 $X = \sqrt{(100)(102)(103)(105)+9} = \sqrt{(100)(100+5)(100+2)(100+3)+9}$

$= \sqrt{(100^2+500)(100^2+500+6)+9} = \sqrt{(100^2+500)^2 + 6(100^2+500)+9} = (100^2 + 500) + 3$

R = the units digit of $X = 3$

I2.4 If S is the sum of the last 3 digits (hundreds, tens, units) of the product of the positive roots of $\sqrt{2012} \cdot x^{\log_{2012} x} = x^R$, find the value of S.

$\log_{2012}\left(\sqrt{2012} \cdot x^{\log_{2012} x}\right) = \log_{2012} x^3$

$\frac{1}{2} + (\log_{2012} x)^2 = 3\log_{2012} x$

Let $y = \log_{2012} x$, then $2y^2 - 6y + 1 = 0$

$y = \log_{2012} x = \frac{3 \pm \sqrt{7}}{2}$

$\Rightarrow x = 2012^{\frac{3+\sqrt{7}}{2}}$ or $2012^{\frac{3-\sqrt{7}}{2}}$

Product of positive roots $= 2012^{\frac{3+\sqrt{7}}{2}} \times 2012^{\frac{3-\sqrt{7}}{2}}$

$= 2012^3$

$\equiv 12^3 \pmod{1000}$

$= 1728 \pmod{1000}$

S = sum of the last 3 digits = 7 + 2 + 8 = 17

Individual Event 3

I3.1 In Figure 1, a rectangle is sub-divided into 3 identical squares of side length 1 .

If $\alpha° = \angle ABD + \angle ACD$, find the value of α .

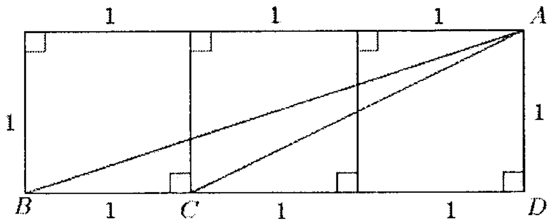


Figure 1

Method 1 (compound angle)

$\tan\angle ABD = \frac{1}{3}$, $\tan\angle ACD = \frac{1}{2}$

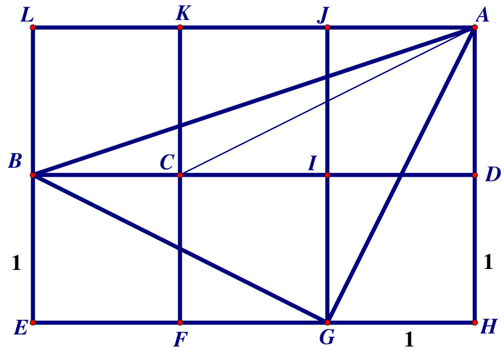
$0° < \angle ABD, \angle ACD < 45°$

$\therefore 0° < \angle ABD + \angle ACD < 90°$

$$\tan\alpha° = \frac{\tan\angle ABD + \tan\angle ACD}{1 - \tan\angle ABD\cdot\tan\angle ACD} = \frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}} = 1 > 0$$

$\alpha = 45$

Method 2 (congruent triangles)



Draw 3 more identical squares *BCFE*, *CIGH*, *IDHG* of length 1 as shown in the figure. *ALBD*, *DBEH* are identical rectangles. Join *BG*, *AG*.

$BE = GH = AD$ (sides of a square)

$EG = HA = CD$ (sides of 2 squares)

$\angle BEG = 90° = \angle GHA = \angle ADC$ (angle of a square)

$\Delta BEG \cong \Delta GHA \cong \Delta ADC$ (SAS)

Let $\angle BGE = \theta = \angle GAH$ (corr. $\angle$s $\cong \Delta$'s)

$\angle AGH = 90° - \theta$ ($\angle$s sum of Δ)

$\angle AGB = 180° - \angle AGH - \angle BGE$ (adj. $\angle$s on st. line)

$= 180° - \theta - (90° - \theta) = 90°$

$BG = AG$ (corr. sides $\cong \Delta$'s)

$\angle ABG = \angle BAG$ (base $\angle$s isos. Δ)

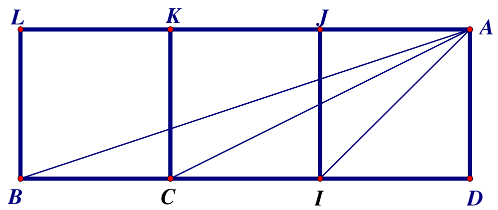
$= \frac{180° - 90°}{2} = 45°$ ($\angle$s sum of Δ)

$\alpha° = \angle ABD + \angle ACD$

$= \angle ABD + \angle GBI$ (corr. $\angle$s, $\cong\Delta$'s)

$= 45° \Rightarrow \alpha = 45$

Method 3 (similar triangles)



Join *AI*.

$AI = \sqrt{2}$ (Pythagoras' theorem)

$\frac{BI}{AI} = \frac{2}{\sqrt{2}} = \sqrt{2}$, $\frac{AI}{CI} = \frac{\sqrt{2}}{1} = \sqrt{2}$

$\angle AIB = \angle CIA$ (common $\angle$)

$\Delta AIB \sim \Delta CIA$ (2 sides proportional, included $\angle$)

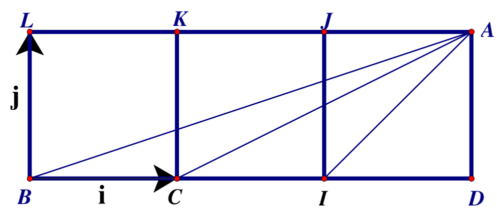
$\therefore \angle ACD = \angle BAI$ (corr. $\angle$s, $\sim\Delta$'s)

$\alpha° = \angle ABD + \angle ACD = \angle ABI + \angle BAI$

$= \angle AID$ (ext. $\angle$ of Δ)

$= 45°$ (diagonal of a square) $\Rightarrow \alpha = 45$

Method 4 (vector dot product)



Define a rectangular system with $\overrightarrow{BC} = \mathbf{i}$, $\overrightarrow{BL} = \mathbf{j}$.

$\overrightarrow{AB} = -3\mathbf{i} - \mathbf{j}$, $\overrightarrow{AC} = -2\mathbf{i} - \mathbf{j}$, $\overrightarrow{AI} = -\mathbf{i} - \mathbf{j}$.

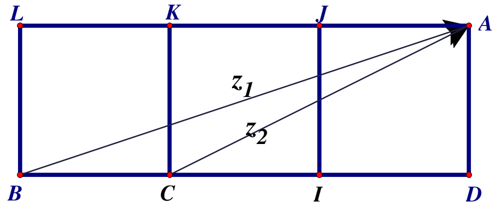
$\overrightarrow{AB}\cdot\overrightarrow{AI} = |\overrightarrow{AB}||\overrightarrow{AI}|\cos\angle BAI$

$$\cos\angle BAI = \frac{(-3)(-1)+(-1)(-1)}{\sqrt{(-3)^2+(-1)^2}\cdot\sqrt{(-1)^2+(-1)^2}} = \frac{2}{\sqrt{5}}$$

$\cos\angle ACD = \frac{2}{\sqrt{5}} \Rightarrow \angle BAI = \angle ACD$

$\alpha° = \angle ABD + \angle ACD = \angle ABI + \angle BAI = \angle AID$ (ext. $\angle$ of Δ) $\Rightarrow \alpha = 45$

Method 5 (complex number)



Define the Argand diagram with B as the origin, BD as the real axis, BL as the imaginary axis.

Let the complex numbers represented by AB and AC be z_1 and z_2 respectively.

$z_1 = 3 + i$, $z_2 = 2 + i$

$z_1 \cdot z_2 = (3 + i)(2 + i) = 6 - 1 + (2 + 3)i = 5 + 5i$

$\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$

$\alpha° = \angle ABD + \angle ACD = \tan^{-1}\frac{5}{5} = 45°$

$\alpha = 45$

I3.2 Let ABC be an acute-angled triangle. If $\sin A = \frac{36}{\alpha}$, $\sin B = \frac{12}{13}$ and $\sin C = \frac{\beta}{y}$, find the value of β, where β and y are in the lowest terms.

(Reference: 2003 FG2.4)

$\sin A = \frac{36}{45} = \frac{4}{5} \Rightarrow \cos A = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$

$\sin B = \frac{12}{13} \Rightarrow \cos B = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \frac{5}{13}$

$\sin C = \sin(180° - (A + B)) = \sin(A + B) = \sin A \cos B + \cos A \sin B$

$= \frac{4}{5} \cdot \frac{5}{13} + \frac{3}{5} \cdot \frac{12}{13} = \frac{56}{65}$

$\beta = 56$

Remark The original question is: Let ABC be a triangle. ……

Case 1 If all angles are acute, then $\beta = 56$ (done above)

Case 2 If $\angle A$ is obtuse, then $\cos A = -\frac{3}{5}$

$\cos B = \frac{5}{13}$, $\sin C = \frac{4}{5} \cdot \frac{5}{13} - \frac{3}{5} \cdot \frac{12}{13} = -\frac{16}{65} \Rightarrow C > 180°$ or $C < 0°$ (rejected)

Case 3 If $\angle B$ is obtuse, then $\cos B = -\frac{5}{13}$

$\cos A = \frac{3}{5}$, $\sin C = \frac{4}{5} \cdot \left(-\frac{5}{13}\right) + \frac{3}{5} \cdot \frac{12}{13} = \frac{16}{65}$

$\beta = 16$

There are two possible values of β, of which $\beta = 16$ could not be carried forward.

I3.3 In Figure 2, a circle at centre O has three points on its circumference, A, B and C. There are line segments OA, OB, AC and BC, where OA is parallel to BC. If D is the intersection of OB and AC with $\angle BDC = (2\beta - 1)°$ and $\angle ACB = \gamma°$, find the value of γ.

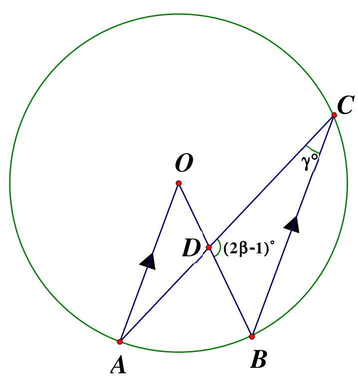


Figure 2

$\angle AOB = 2\gamma°$ ($\angle$ at centre twice $\angle$ at circumference)

$\angle OBC = 2\gamma°$ (alt. $\angle$, OA // CB)

$\gamma° + 2\gamma° + (2\beta - 1)° = 180°$ ($\angle$s sum of Δ)

$3\gamma + 111 = 180$

$\gamma = 23$

I3.4 In the expansion of $(ax + b)^{2012}$, where a and b are relatively prime positive integers. If the coefficients of x^{γ} and $x^{\gamma+1}$ are equal, find the value of $\delta = a + b$.

Coefficient of $x^{23} = C_{23}^{2012} \cdot a^{23} \cdot b^{1989}$; coefficient of $x^{24} = C_{24}^{2012} \cdot a^{24} \cdot b^{1988}$

$C_{23}^{2012} \cdot a^{23} \cdot b^{1989} = C_{24}^{2012} \cdot a^{24} \cdot b^{1988}$

$b = \frac{C_{24}^{2012}}{C_{23}^{2012}} \cdot a$

$b = \frac{2012 - 24 + 1}{24} \cdot a$

$24b = 1989a$

$8b = 663a$

$\because$ a and b are relatively prime integers

$\therefore$ $a = 8$, $b = 663$

$\delta = 8 + 663 = 671$

Individual Event 4

I4.1 If A is a positive integer such that $\frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(A+1)(A+3)}=\frac{12}{25}$, find the value of A.

$\frac{1}{(n+1)(n+3)}=\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)$ for $n\ge 0$

$\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{A+1}-\frac{1}{A+3}\right)=\frac{12}{25}$

$1-\frac{1}{A+3}=\frac{24}{25}$

$\frac{1}{A+3}=\frac{1}{25}$

$A = 22$

I4.2 If x and y be positive integers such that $x > y > 1$ and $xy = x + y + A$.

Let $B=\frac{x}{y}$, find the value of B.

Reference: 1987 FG10.4, 2002 HG9

$xy = x + y + 22$

$xy - x - y + 1 = 23$

$(x - 1)(y - 1) = 23$

$\because$ 23 is a prime number and $x > y > 1$

$\therefore x - 1 = 23, y - 1 = 1$

$x = 24$ and $y = 2$

$B=\frac{24}{2}=12$

I4.3 Let f be a function satisfying the following conditions:

(i) f(n) is an integer for every positive integer n;

(ii) f(2) = 2;

(iii) f(mn) = f(m)f(n) for all positive integers m and n and

(iv) f(m) > f(n) if $m > n$.

If C = f B), find the value of C.

Reference: 2003 HI1

$2 = f(2) > f(1) > 0 \Rightarrow f(1) = 1$

$f(4) = f(2\times2) = f(2)f(2) = 4$

$4 = f(4) > f(3) > f(2) = 2$

$\Rightarrow f(3) = 3$

$C = f(12) = f(4\times3) = f(4)f(3) = 4\times3 = 12$

I4.4 Let D be the sum of the last three digits of 2401×7^C (in the denary system).

Find the value of D.

$2401\times7^C = 7^4\times7^{12} = 7^{16} = (7^2)^8 = 49^8 = (50 - 1)^8$

$= 50^8 - 8\times50^7 + \cdots - 56\times50^3 + 28\times50^2 - 8\times50 + 1$

$\equiv 28\times2500 - 400 + 1 \pmod{1000}$

$\equiv -399 \equiv 601 \pmod{1000}$

$D = 6 + 0 + 1 = 7$

Method 2 $2401\times7^C = 7^4\times7^{12} = 7^{16}$

$7^4 = 2401$

$7^8 = (2400 + 1)^2 = 5760000 + 4800 + 1 \equiv 4801 \pmod{1000}$

$7^{16} \equiv (4800 + 1)^2 \equiv 48^2\times10000 + 9600 + 1 \equiv 9601 \pmod{1000}$

$D = 6 + 0 + 1 = 7$

Individual Spare (2011 Final Group Spare Event)

IS.1 Let P be the number of triangles whose side lengths are integers less than or equal to 9. Find the value of P .

The sides must satisfy triangle inequality. i.e. $a + b > c$.

Possible order triples are (1, 1, 1), (2, 2, 2), $\cdots$, (9, 9, 9) ,

(2, 2, 1), (2, 2, 3), (3, 3, 1), (3, 3, 2), (3, 3, 4), (3, 3, 5) ,

(4, 4, 1), (4, 4, 2), (4, 4, 3), (4, 4, 5), (4, 4, 6), (4, 4, 7) ,

(5, 5, 1), $\cdots$, (5, 5, 4), (5, 5, 6), (5, 5, 7), (5, 5, 8), (5, 5, 9) ,

(6, 6, 1), $\cdots$, (6, 6, 9) (except (6, 6, 6))

(7, 7, 1), $\cdots$, (7, 7, 9) (except (7, 7, 7))

(8, 8, 1), $\cdots$, (8, 8, 9) (except (8, 8, 8))

(9, 9, 1), $\cdots$, (9, 9, 8)

(2, 3, 4), (2, 4, 5), (2, 5, 6), (2, 6, 7), (2, 7, 8), (2, 8, 9) ,

(3, 4, 5), (3, 4, 6), (3, 5, 6), (3, 5, 7), (3, 6, 7), (3, 6, 8), (3, 7, 8), (3, 7, 9), (3, 8, 9) ,

(4, 5, 6), (4, 5, 7), (4, 5, 8), (4, 6, 7), (4, 6, 8), (4, 6, 9), (4, 7, 8), (4, 7, 9), (4, 8, 9) ,

(5, 6, 7), (5, 6, 8), (5, 6, 9), (5, 7, 8), (5, 7, 9), (5, 8, 9), (6, 7, 8), (6, 7, 9), (6, 8, 9), (7, 8, 9) .

Total number of triangles = 9 + 6 + 6 + 8×5 + 6 + 9 + 9 + 6 + 4 = 95

Method 2 First we find the number of order triples.

Case 1 All numbers are the same: (1, 1, 1), $\cdots$, (9, 9, 9).

Case 2 Two of them are the same, the third is different: (1, 1, 2), $\cdots$, (9, 9, 1)

There are $C_1^9 \times C_1^8 = 72$ possible triples.

Case 3 All numbers are different. There are $C_3^9 = 84$ possible triples.

$\therefore$ Total 9 + 72 + 84 = **165** possible triples.

Next we find the number of triples which **cannot form a triangle**, i.e. $a + b \le c$.

Possible triples are (1, 1, 2), $\cdots$ (1, 1, 9) (8 triples)

(1, 2, 3), $\cdots$, (1, 2, 9) (7 triples)

(1, 3, 4), $\cdots$, (1, 3, 9) (6 triples)

(1, 4, 5), $\cdots$, (1, 4, 9) (5 triples)

(1, 5, 6), $\cdots$, (1, 5, 9) (4 triples)

(1, 6, 7), (1, 6, 8), (1, 6, 9), (1, 7, 8), (1, 7, 9), (1, 8, 9) ,

(2, 2, 4), $\cdots$, (2, 2, 9) (6 triples)

(2, 3, 5), $\cdots$, (2, 3, 9) (5 triples)

(2, 4, 6), $\cdots$, (2, 4, 9) (4 triples)

(2, 5, 7), (2, 5, 8), (2, 5, 9), (2, 6, 8), (2, 6, 9), (2, 7, 9) ,

(3, 3, 6), $\cdots$, (3, 3, 9) (4 triples)

(3, 4, 7), (3, 4, 8), (3, 4, 9), (3, 5, 8), (3, 5, 9), (3, 6, 9), (4, 4, 8), (4, 4, 9), (4, 5, 9) .

Total number of triples which cannot form a triangle

= (8 + 7 + $\cdots$ + 1) + (6 + 5 + $\cdots$ + 1) + (4 + 3 + 2 + 1) + (2 + 1) = 36 + 21 + 10 + 3 = **70**

$\therefore$ Number of triangles = **165 – 70 = 95**

IS.2 Let $Q = \log_{128} 2^3 + \log_{128} 2^5 + \log_{128} 2^7 + \cdots + \log_{128} 2^P$. Find the value of Q.

$Q = 3 \log_{128} 2 + 5 \log_{128} 2 + 7 \log_{128} 2 + \cdots + 95 \log_{128} 2$

$= (3 + 5 + \ldots + 95) \log_{128} 2 = \frac{3+95}{2}\times 47\times \log_{128} 2 = \log_{128} 2^{2303} = \log_{128} (2^7)^{329} = 329$

IS.3 Consider the line $12x - 4y + (Q - 305) = 0$. If the area of the triangle formed by the x-axis, the y-axis and this line is R square units, what is the value of R ?

$12x - 4y + 24 = 0 \Rightarrow$ Height = 6, base = 2; area $R = \frac{1}{2}\cdot 6\cdot 2 = 6$

Remark: the original question is … $12x - 4y + Q = 0$ ….

The answer is very difficult to carry forward to next question.

IS.4 If $x + \frac{1}{x} = R$ and $x^3 + \frac{1}{x^3} = S$, find the value of S.

$S = \left(x+\frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right) = R\left[\left(x+\frac{1}{x}\right)^2 - 3\right] = R^3 - 3R = 216 - 3(6) = 198$

Sample Group Event (2009 Final Group Event 1)

SG.1 Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

When $a = 1$, possible $b = 2$

When $a = 2$, possible $b = 2$ or 3

$\therefore q = 3$

SG.2 Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

When $x > 0$: $x^2 - 4 = 3x \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x + 1)(x - 4) = 0 \Rightarrow x = 4$

When $x < 0$: $-x^2 - 4 = -3x \Rightarrow x^2 - 3x + 4 = 0$; $D = 9 - 16 < 0 \Rightarrow$ no real roots.

$k = 1$ (There is only one real root.)

SG.3 Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x - y = 7$. If $w = x + y$, find the value of w.

The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$

Sub. $y = \frac{144}{x}$ into $x - y = 7$: $x - \frac{144}{x} = 7 \Rightarrow x^2 - 7x - 144 = 0 \Rightarrow (x + 9)(x - 16) = 0$

$x = -9$ or 16; when $x = -9$, $y = -16$ (rejected $\because \sqrt{x}$ is undefined); when $x = 16$; $y = 9$

$w = 16 + 9 = 25$

Method 2 The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$ …… (1)

$\because x - y = 7$ and $x + y = w$

$\therefore x = \frac{w+7}{2}$, $y = \frac{w-7}{2}$

Sub. these equations into (1): $\left(\frac{w+7}{2}\right)\left(\frac{w-7}{2}\right) = 144$

$w^2 - 49 = 576 \Rightarrow w = \pm 25$

$\because$ From the given equation $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$, we know that both $x > 0$ and $y > 0$

$\therefore w = x + y = 25$ only

SG.4 Given that x and y are real numbers and $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$.

Let $p = |x| + |y|$, find the value of p.

Reference: 2006 FI4.2 $\cdots$ $y^2 + 4y + 4 + \sqrt{x + y + k} = 0$. If $r = |xy|$, $\cdots$

Both $\left|x - \frac{1}{2}\right|$ and $\sqrt{y^2 - 1}$ are non-negative numbers.

The sum of two non-negative numbers = 0 means each of them is zero

$x = \frac{1}{2}$, $y = \pm 1$; $p = \frac{1}{2} + 1 = \frac{3}{2}$

Group Event 1

G1.1 Calculate the tens digit of 2011^{2011} .

$2011^{2011} \equiv (10 + 1)^{2011} \mod 100$

$\equiv 10^{2011} + \cdots + 2011\times10 + 1$ (binomial theorem)

$\equiv 11 \mod 100$

The tens digit is 1 .

G1.2 Let $a_1, a_2, a_3, \cdots$ be an arithmetic sequence with common difference 1 and $a_1 + a_2 + a_3 + \cdots + a_{100} = 2012$. If $P = a_2 + a_4 + a_6 + \cdots + a_{100}$, find the value of P .

Let $a_1 = a$ then $\frac{100(2a+99)}{2} = 2012$

$2a + 99 = \frac{1006}{25}$

$2a = \frac{1006 - 99\times25}{25} = -\frac{1469}{25}$

$P = a_2 + a_4 + a_6 + \cdots + a_{100} = \frac{50(a+1+a+99)}{2} = 25\times(2a + 100) = 25\times\left(100 - \frac{1469}{25}\right)$

$P = 2500 - 1469 = 1031$

Method 2 $P = a_2 + a_4 + a_6 + \cdots + a_{100}$

$\underline{Q = a_1 + a_3 + a_5 + \cdots + a_{99}}$

$P - Q = 1 + 1 + 1 + \cdots + 1$ (50 terms) $= 50$

But since $P + Q = a_1 + a_2 + a_3 + \cdots + a_{100} = 2012$

$\therefore P = \frac{2012+50}{2} = 1031$

Remark: the original question …等差級數… , … arithmetic progression …

The phrases are changed to … 等差數列 … and … arithmetic sequence … according to the mathematics syllabus since 1999 .

G1.3 If 90! is divisible by 10^k, where k is a positive integer, find the greatest possible value of k .

Reference: 1990 HG6, 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FI1.4

Method 1

When each factor of 5 is multiplied by 2, a trailing zero will appear in n!.

The number of factors of 2 is clearly more than the number of factors of 5 in 280!

It is sufficient to find the number of factors of 5.

5, 10, 15, ⋯ , 90; altogether 18 numbers, each have at least one factor of 5.

25, 50, 75, altogether 3 numbers, each have at least two factors of 5.

$\therefore$ Total number of factors of 5 is 18 + 3 = 21

$k = 21$

Method 2

We can find the total number of factors of 5 by division as follow:

$$\begin{array}{r|l} 5 & 90 \\ \hline 5 & 18 \\ \hline & 3 \ldots 3 \end{array}$$

No. of factors of 5 is 18+3

$k = 21$

G1.4 In Figure 1, ΔABC is a right-angled triangle with $AB \perp BC$.
If $AB = BC$, D is a point such that $AD \perp BD$ with $AD = 5$ and $BD = 8$, find the value of the area of ΔBCD.

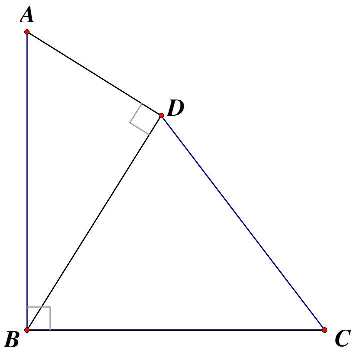


Figure 1

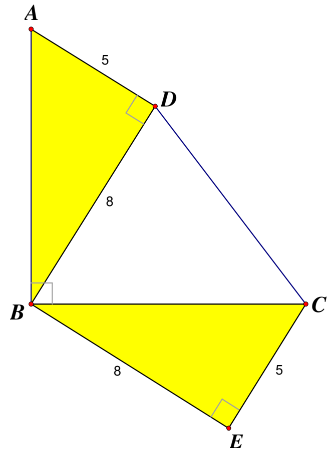
$AB = BC = \sqrt{5^2 + 8^2} = \sqrt{89}$ (Pythagoras' theorem)

Let $\angle ABD = \theta$, then $\cos\theta = \dfrac{8}{\sqrt{89}}$

$\angle CBD = 90° - \theta$, $\sin\angle CBD = \dfrac{8}{\sqrt{89}}$

$S_{\Delta BCD} = \dfrac{1}{2} BD \cdot BC \sin\angle CBD = \dfrac{1}{2}\cdot 8 \cdot \sqrt{89} \cdot \dfrac{8}{\sqrt{89}} = 32$

Method 2



Rotate ΔABD about the centre at B in clockwise direction through $90°$ to ΔCBE.

Then $\Delta ABD \cong \Delta CBE$

$\angle CEB = \angle ADB = 90°$ (corr. $\angle$s $\cong \Delta$s)

$CE = 5$, $BE = 8$ (corr. sides $\cong \Delta$s)

$\angle DBE = \angle DBC + \angle CBE$

$= \angle DBC + \angle ABD$ (corr. $\angle$s $\cong \Delta$s)

$= 90°$

$\angle DBE + \angle CEB = 180°$

BD // CE (int. $\angle$s supp.)

$\therefore$ $BDCE$ is a right-angled trapezium.

Area of ΔBCD = area of trapezium $BDCE$ – area of ΔBCE

$= \dfrac{1}{2}(8 + 5)\times 8 - \dfrac{1}{2}\cdot 8\times 5$

$= 32$

Remark: the original question "⋯ right-angle triangle ⋯ "

It should be changed to right-angled triangle. Furthermore, the condition $AD \perp BD$ is not specified.

Group Event 2

G2.1 Find the value of $2\times\tan 1°\times\tan 2°\times \tan 3°\times \cdots \times \tan 87°\times \tan 88°\times \tan 89°$.

Similar question: 2008 FI1.1

$\tan\theta\times\tan(90° - \theta) = 1$ for $\theta = 1°, 2°, \cdots . 44°$ and $\tan 45° = 1$

$2\times\tan 1°\times\tan 2°\times \tan 3°\times \cdots \times \tan 87°\times \tan 88°\times \tan 89° = 2$

G2.2 If there are K integers that satisfy the equation $(x^2 - 3x + 2)^2 - 3(x^2 - 3x) - 4 = 0$, find the value of K.

$(x^2 - 3x + 2)^2 - 3(x^2 - 3x) - 4 = 0$

$(x^2 - 3x)^2 + 4(x^2 - 3x) - 3(x^2 - 3x) = 0$

$(x^2 - 3x)^2 + (x^2 - 3x) = 0$

$(x^2 - 3x)(x^2 - 3x + 1) = 0$

$x = 0, 3$ or $\dfrac{3\pm\sqrt{5}}{2}$

K = number of integral roots = 2

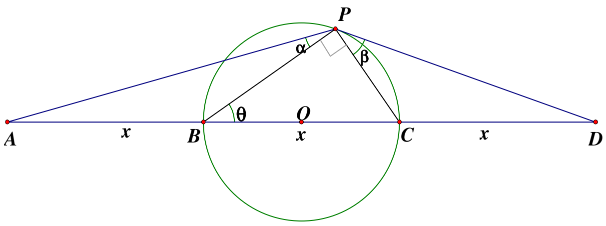
G2.3 If ℓ is the minimum value of $|x - 2| + |x - 47|$, find the value of ℓ.

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2011 FGS.1

Using the triangle inequality: $|a| + |b| \ge |a + b|$

$|x - 2| + |x - 47| = |x - 2| + |47 - x| \ge |x - 2 + 47 - x| = 45 \Rightarrow \ell = 45$

G2.4 In Figure 1, P, B and C are points on a circle with centre O and diameter BC. If A, B, C, D are collinear such that $AB = BC = CD$, $\alpha = \angle APB$ and $\beta = \angle CPD$, find the value of $(\tan\alpha)(\tan\beta)$.



Let $AB = x = BC = CD$, $\angle CBP = \theta$.

$\angle BPC = 90°$ ($\angle$ in semi circle), $\angle BCP = 90° - \theta$ ($\angle$s sum of Δ)

$BP = x\cos\theta$, $CP = x\sin\theta$

$\angle BAP = \theta - \alpha$, $\angle CDP = 90° - \theta - \beta$ (ext. $\angle$ of Δ)

$\dfrac{x}{\sin\alpha} = \dfrac{BP}{\sin\angle BAP}$ (sine rule on ΔABP); $\dfrac{x}{\sin\beta} = \dfrac{CP}{\sin\angle CDP}$ (sine rule on ΔCDP)

$\dfrac{x}{\sin\alpha} = \dfrac{x\cos\theta}{\sin(\theta-\alpha)}$; $\dfrac{x}{\sin\beta} = \dfrac{x\sin\theta}{\cos(\theta+\beta)}$

$\sin\theta\cos\alpha - \cos\theta\sin\alpha = \cos\theta\sin\alpha$; $\cos\theta\cos\beta - \sin\theta\sin\beta = \sin\theta\sin\beta$

$\sin\theta\cos\alpha = 2\cos\theta\sin\alpha$; $\cos\theta\cos\beta = 2\sin\theta\sin\beta$

$\tan\alpha = \dfrac{\tan\theta}{2}$; $\tan\beta = \dfrac{1}{2\tan\theta}$

$(\tan\alpha)(\tan\beta) = \dfrac{\tan\theta}{2}\cdot\dfrac{1}{2\tan\theta} = \dfrac{1}{4}$

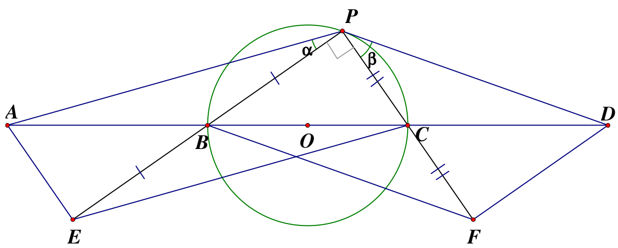
Method 2 $\angle BPC = 90°$ ($\angle$ in semi circle),

Produce PB to E so that $PB = BE$.

Produce PC to F so that $PC = CF$.

$\because AB = BC = CD$ (given)

$\therefore APCE$, $BPDF$ are //-grams (diagonals bisect each other)



$\angle PEC = \alpha$ (alt. $\angle$s, $AP//EC$)

$\angle PFB = \beta$ (alt. $\angle$s, $PD//BF$)

In ΔEPC, $\tan\alpha = \dfrac{PC}{PE} = \dfrac{PC}{2PB}$

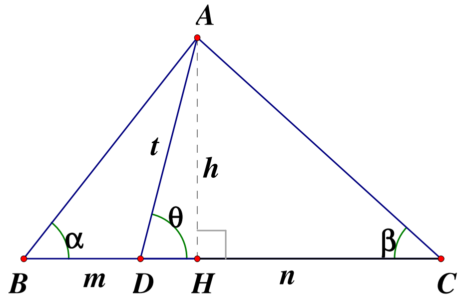
In ΔBPF, $\tan\beta = \dfrac{PB}{PF} = \dfrac{PB}{2PC}$

$(\tan\alpha)(\tan\beta)=\frac{PC}{2PB}\cdot\frac{PB}{2PC}=\frac{1}{4}$

Method 3 Lemma A Given a triangle ABC. D is a point on BC such that $BD : DC = m : n$, $AD = t$.

$\angle ABD = \alpha$, $\angle ADC = \theta < 90°$, $\angle ACD = \beta$.

Then $n \cot\alpha - m \cot\beta = (m + n)\cot\theta$



Proof: Let H be the projection of A on BC.

$AH = h$, $DH = h\cot\theta$.

$BH = \cot\alpha$, $CH = h\cot\beta$

$$\frac{m}{n}=\frac{BD}{DC}=\frac{BH-DH}{CH+HD}=\frac{h\cot\alpha-h\cot\theta}{h\cot\beta+h\cot\theta}$$

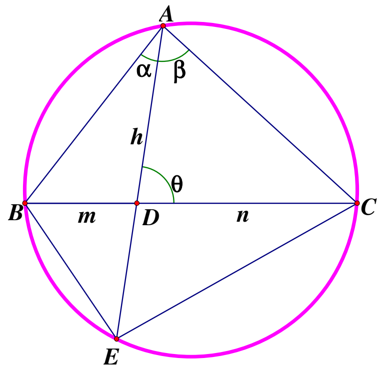
$m(\cot\beta + \cot\theta) = n(\cot\alpha - \cot\theta)$

$\therefore n\cot\alpha - m\cot\beta = (m + n)\cot\theta$

Lemma B Given a triangle ABC. D is a point on BC such that $BD : DC = m : n$, $AD = t$.

$\angle BAD = \alpha$, $\angle ADC = \theta < 90°$, $\angle CAD = \beta$.

Then $m\cot\alpha - n\cot\beta = (m + n)\cot\theta$



Proof: Draw the circumscribed circle ABC.

Produce AD to cut the circle again at E.

$\angle BCE = \alpha$, $\angle CBE = \beta$ ($\angle$s in the same seg.)

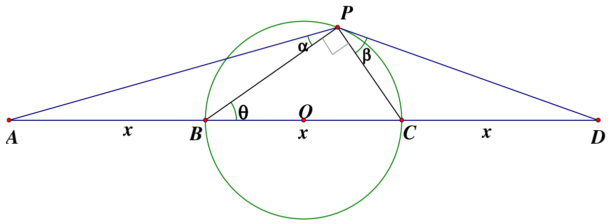
$\angle BDE = \theta < 90°$ (vert. opp. $\angle$s)

Apply **Lemma A** on ΔBEC.

$\therefore m\cot\alpha - n\cot\beta = (m + n)\cot\theta$

Now return to our original problem

$\angle BPC = 90°$ ($\angle$ in semi circle)



Apply **Lemma B** to ΔAPC:

$x\cot\alpha - x\cot 90° = (x + x)\cot\theta$

$\cot\alpha = 2\cot\theta$ …… (1)

Apply **Lemma B** to ΔBPC, $\angle BPC = 90° - \theta$

$x\cot\beta - x\cot 90° = (x + x)\cot(90° - \theta)$

$\cot\beta = 2\tan\theta$ …… (2)

(1)×(2): $\cot\alpha\cot\beta = 2\cot\theta \times 2\tan\theta = 4$

$\therefore (\tan\alpha)(\tan\beta)=\frac{1}{4}$

Remark: the original question 圓有直徑 BC，圓心在 O，P、B 及 C 皆為圓周上的點。若 $AB = BC = CD$ 及 AD 為一綫段 … $AB = BC = CD$ and AD is a line segment …

Both versions are not smooth and clear. The new version is as follow:

BC 是圓的直徑，圓心為 O，P、B 及 C 皆為圓周上的點。若 A、B、C 及 D 共綫且 $AB = BC = CD$ … If A, B, C, D are collinear such that $AB = BC = CD$ …

Group Event 3

G3.1 Let $x=\dfrac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, $y=\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$ and $192z = x^4 + y^4 + (x + y)^4$, find the value of z.

$x+y=\dfrac{\left(\sqrt{7}+\sqrt{3}\right)^2+\left(\sqrt{7}-\sqrt{3}\right)^2}{7-3}=\dfrac{2(7+3)}{4}=5$; $xy = 1$

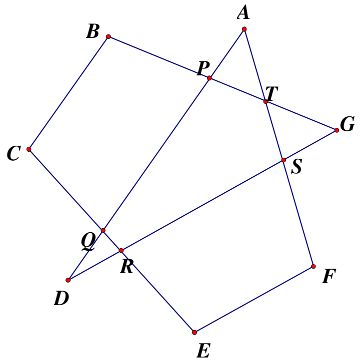
$x^2 + y^2 = (x + y)^2 - 2xy = 5^2 - 2 = 23$

$x^4 + y^4 = (x^2 + y^2)^2 - 2(xy)^2 = 23^2 - 2 = 527$

$192z = 527 + 5^4 = 527 + 625 = 1152$

$z = 6$

G3.2 In Figure 1, AD, DG, GB, BC, CE, EF and FA are line segments. If $\angle FAD + \angle GBC + \angle BCE + \angle ADG + \angle CEF + \angle AFE + \angle DGB = r°$, find the value of r.



Reference: 1992 HI13, 2000 HI5

In the figure, let P, Q, R, S, T be as shown.

$\angle ATP + \angle BPQ + \angle DQR + \angle ERS + \angle GST = 360°$ …… (1)

$\angle APT + \angle CQP + \angle DRQ + \angle FSR + \angle GTS = 360°$ …… (2)

(sum of ext. $\angle$ of polygon)

$\angle FAD = 180° - (\angle ATP + \angle APT)$ ($\angle$s sum of Δ)

$\angle GBC + \angle BCE = 360° - (\angle BPQ + \angle CQP)$ ($\angle$s sum of polygon)

$\angle ADG = 180° - (\angle DQR + \angle DRQ)$ ($\angle$s sum of Δ)

$\angle CEF + \angle AFE = 360° - (\angle ERS + \angle FSR)$ ($\angle$s sum of polygon)

$\angle DGB = 180° - (\angle GST + \angle GST)$ ($\angle$s sum of Δ)

Add these 5 equations up and make use of equations (1) and (2):

$r° = 180°\times 7 - 2\times 360° \Rightarrow r = 540$

Remark: The original question $\angle FAD + \angle GBC + \angle BCE + \angle ADG + \angle CEF + \angle DGB = r°$ $\angle AFE$ is missing, the original question is wrong.

中文版題目："…及 *FA* 都是綫段。…" is changed into "…及 *FA* 都是直綫綫段。…"

G3.3 Let k be positive integer and f(k) a function that if $\dfrac{k-1}{k}=0.k_1k_2k_3\ldots\ldots$, then $f(k)=\overline{k_1k_2k_3}$, for example, f(3) = 666 because $\dfrac{3-1}{3}=0.666\ldots\ldots$, find the value of D = f(f(f(f(f(112))))).

$0.99=1-\dfrac{1}{100}<\dfrac{112-1}{112}=1-\dfrac{1}{112}<1 \Rightarrow f(112)=\overline{99k_3}$

$0.998=1-\dfrac{1}{500}<\dfrac{\overline{99k_3}-1}{\overline{99k_3}}=1-\dfrac{1}{\overline{99k_3}}<1-\dfrac{1}{1000}=0.999 \Rightarrow f(f(112))=998$

$\Rightarrow f(f(f(112))) = 998 \Rightarrow D = f(f(f(f(f(112))))) = 998$

G3.4 If F_n is an integral valued function defined recursively by $F_n(k) = F_1(F_{n-1}(k))$ for $n \ge 2$ where $F_1(k)$ is the sum of squares of the digits of k, find the value of $F_{2012}(7)$.

$F_1(7) = 7^2 = 49$

$F_2(7) = F_1(F_1(7)) = F_1(49) = 4^2 + 9^2 = 97$

$F_3(7) = 9^2 + 7^2 = 130$

$F_4(7) = 1^2 + 3^2 + 0^2 = 10$

$F_5(7) = 1$

$F_{2012}(7) = 1$

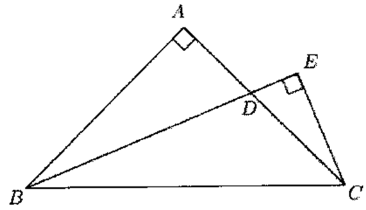
Remark: the original question If f is an integer valued function ⋯

The recursive function is defined for F_n, not f .

Group Event 4

G4.1 In figure 1, ABC and EBC are two right-angled triangles, $\angle BAC = \angle BEC = 90°$, $AB = AC$ and EDB is the angle bisector of $\angle ABC$.

Find the value of $\frac{BD}{CE}$.



ΔABC is a right-angled isosceles triangle.

$\angle ABC = \angle ACB = 45°$ ($\angle$s sum of isos. Δ)

$\angle ABD = \angle CBD = 22.5°$ ($\angle$ bisector)

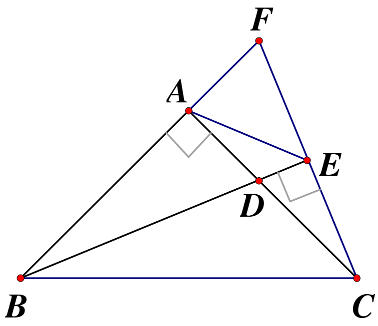
Let $BC = x$

$AB = x\cos 45° = \frac{\sqrt{2}x}{2}$; $CE = x\sin 22.5°$

$BD = AB \div \cos 22.5° = \frac{\sqrt{2}x}{2\cos 22.5°}$

$\frac{BD}{CE} = \frac{\sqrt{2}x}{2\cos 22.5° \cdot x\sin 22.5°} = \frac{\sqrt{2}}{\cos 45°} = 2$

Method 2 Produce CE and BA to meet at F.



$AB = AC \Rightarrow \angle ABC = \angle ACB = 45°$

$\angle BAC = \angle BEC = 90°$ (given)

$\Rightarrow ABCE$ is a cyclic quad. (converse, $\angle$s in the same seg.)

$\angle ABD = \angle CBD = 22.5°$ ($\angle$ bisector)

$\angle ACF = 22.5°$ ($\angle$s in the same seg.)

$\angle CAE = \angle CBE = 22.5°$ ($\angle$s in the same seg.)

$CE = AE$ … (1) (sides, opp. eq. $\angle$s)

$\angle EAF = 180° - 90° - 22.5° = 67.5°$ (adj. $\angle$s on st. line)

$\angle AFE = 180° - 90° - 22.5° = 67.5°$ ($\angle$s of ΔACF)

$AE = EF$ … (2) (sides, opp. eq. $\angle$s)

By (1) and (2), $CF = 2CE$ … (3)

$\Delta ACF \cong \Delta ABD$ (A.S.A.)

$BD = CF$ (corr. sides, $\cong \Delta$'s)

$\frac{BD}{CE} = \frac{CF}{CE} = \frac{2CE}{CE} = 2$ by (3)

G4.2 If $Q > 0$ and satisfies $\left|3Q - |1-2Q|\right| = 2$, find the value of Q.

Reference: 2002 FG4.3, 2005 FG4.2, 2009 HG9, 2017 FG1.2

$\left|3Q - |1-2Q|\right| = 2$

$3Q - |1 - 2Q| = 2$ or $3Q - |1 - 2Q| = -2$

$3Q - 2 = |1 - 2Q|$ or $3Q + 2 = |1 - 2Q|$

$3Q - 2 = 1 - 2Q$ or $3Q - 2 = 2Q - 1$ or $3Q + 2 = 1 - 2Q$ or $3Q + 2 = 2Q - 1$

$Q = \frac{3}{5}$ or 1 or $-\frac{1}{5}$ (rejected) or -3 (rejected)

Check: when $Q = \frac{3}{5}$, LHS $= \left|\frac{9}{5} - \left|1 - \frac{6}{5}\right|\right| = \frac{8}{5} \neq 2$, rejected

When $Q = 1$, LHS $= |3 - |1 - 2|| = 2 =$ RHS accepted

$\therefore Q = 1$

G4.3 Let $xyzt = 1$. If $R = \frac{1}{1+x+xy+xyz} + \frac{1}{1+y+yz+yzt} + \frac{1}{1+z+zt+ztx} + \frac{1}{1+t+tx+txy}$, find the value of R.

$$\frac{1}{1+x+xy+xyz} = \frac{1}{1+x+xy+\frac{1}{t}} = \frac{t}{1+t+tx+txy}$$

$$\frac{1}{1+y+yz+yzt} = \frac{1}{1+y+\frac{1}{tx}+\frac{t}{tx}} = \frac{tx}{1+t+tx+txy}$$

$$\frac{1}{1+z+zt+ztx} = \frac{1}{1+\frac{1}{txy}+\frac{t}{txy}+\frac{tx}{txy}} = \frac{txy}{1+t+tx+txy}$$

$$R = \frac{t}{1+t+tx+txy} + \frac{tx}{1+t+tx+txy} + \frac{txy}{1+t+tx+txy} + \frac{1}{1+t+tx+txy} = 1$$

G4.4 If x_1, x_2, x_3, x_4 and x_5 are positive integers that satisfy $x_1 + x_2 + x_3 + x_4 + x_5 = x_1x_2x_3x_4x_5$, that is the sum is the product, find the maximum value of x_5.

The expression is symmetric. We may assume that $1 \le x_1 \le x_2 \le x_3 \le x_4 \le x_5$.

If $x_1 = x_2 = x_3 = x_4 = 1$, then $1 + 1 + 1 + 1 + x_5 = 1\times1\times1\times1\times x_5 \Rightarrow$ no solution

$\therefore x_1x_2x_3x_4 - 1 \ne 0$

$x_1 + x_2 + x_3 + x_4 = (x_1x_2x_3x_4 - 1)x_5$

$x_5 = \frac{x_1+x_2+x_3+x_4}{x_1x_2x_3x_4-1}$

When x_5 attains the maximum value, the denominator must be 1, i.e. $x_1x_2x_3x_4 = 2$

$\because 1 \le x_1 \le x_2 \le x_3 \le x_4$ $\therefore x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 2$, max. $x_5 = \frac{1+1+1+2}{2-1} = 5$

Method 2 We begin from the lowest integer.

Case 1 Let $x_1 = x_2 = x_3 = x_4 = 1$, then $1 + 1 + 1 + 1 + x_5 = 1\times1\times1\times1\times x_5 \Rightarrow$ no solution

Case 2 Let $x_1 = x_2 = x_3 = 1$ and $x_4 > 1$, then $3 + x_4 + x_5 = x_4x_5 \Rightarrow x_5 = \frac{x_4+3}{x_4-1}$

When $x_4 = 2$, $x_5 = 5$; when $x_4 = 3$, $x_5 = 3$

When $x_4 = 4$, no integral solution for x_5

When $x_4 = 5$, $x_5 = 2$, contradicting that $1 \le x_1 \le x_2 \le x_3 \le x_4 \le x_5$.

When $x_4 > 5$, then $x_5 < x_4$, which is a contradiction

Case 3 Let $x_1 = x_2 = 1$ and $x_3 > 1$, then $2 + x_3 + x_4 + x_5 = x_3x_4x_5$

When $x_3 = 2$, $4 + x_4 + x_5 = 2x_4x_5 \Rightarrow x_5 = \frac{x_4+4}{2x_4-1} > 1 \Rightarrow x_4 + 4 > 2x_4 - 1 \Rightarrow x_4 < 5$

When $x_4 = 2$, $x_5 = 2$

When $x_4 = 3, 4$, no integral solution for x_5

Case 4 $1 = x_1 < x_2 \le x_3 \le x_4 \le x_5$, then $x_5 = \frac{1+x_2+x_3+x_4}{x_2x_3x_4-1} < \frac{1+3x_4}{4x_4-1}$

When $x_2 = x_3 = x_4 = 2$, $x_5 = 1 < x_4$, contradiction

When $2 \le x_2 = x_3 < x_4$

$1 + 3x_4 < 4x_4 - 1$

$\frac{1+3x_4}{4x_4-1} < 1 \Rightarrow x_5 < 1$, contradiction $\therefore$ There is no integral solution for x_5.

Case 5 $2 \le x_1 \le x_2 \le x_3 \le x_4 \le x_5$, then $x_5 = \frac{x_1+x_2+x_3+x_4}{x_1x_2x_3x_4-1} < \frac{4x_4}{8x_4-1}$

$1 < 4x_4$

$4x_4 < 8x_4 - 1$

$\frac{4x_4}{8x_4-1} < 1 \Rightarrow x_5 < 1$, contradiction $\therefore$ There is no integral solution for x_5.

Conclusion: The solution set for $(x_1, x_2, x_3, x_4, x_5)$ is $\{(1, 1, 1, 2, 5), (1, 1, 1, 3, 3), (1,1,2,2,2)\}$.

Maximum for $x_5 = 5$

Group Spare (2011 Final Group Spare Event)

GS.1 Let α and β be the real roots of $y^2 - 6y + 5 = 0$. Let m be the minimum value of $|x - \alpha| + |x - \beta|$ over all real values of x . Find the value of m .

Remark: there is a typing mistake in the English version. $\cdots$ minimum value a of $\cdots$

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2012 FG2.3

$\alpha = 1$, $\beta = 5$

If $x < 1$, $|x - \alpha| + |x - \beta| = 1 - x + 5 - x = 6 - 2x > 4$

If $1 \le x \le 5$, $|x - \alpha| + |x - \beta| = x - 1 + 5 - x = 4$

If $x > 5$, $|x - \alpha| + |x - \beta| = x - 1 + x - 5 = 2x - 6 > 4$

m = min. of $|x - \alpha| + |x - \beta| = 4$

Method 2 Using the triangle inequality: $|a| + |b| \ge |a + b|$

$|x - \alpha| + |x - \beta| \ge |x - 1 + 5 - x| = 4 \Rightarrow m = 4$

GS.2 Let α, β, γ be real numbers satisfying $\alpha + \beta + \gamma = 2$ and $\alpha\beta\gamma = 4$. Let v be the minimum value of $|\alpha| + |\beta| + |\gamma|$. Find the value of v .

If at least one of $\alpha, \beta, \gamma = 0$, then $\alpha\beta\gamma \ne 4 \Rightarrow \alpha, \beta, \gamma \ne 0$

If $\alpha, \beta, \gamma > 0$, then

$$\frac{\alpha+\beta+\gamma}{3} \ge \sqrt[3]{\alpha\beta\gamma} \quad \text{(A.M. } \ge \text{ G.M.)}$$

$$\frac{2}{3} \ge \sqrt[3]{4}$$

$2^3 \ge 27\times4 = 108$, which is a contradiction

If $\beta < 0$, in order that $\alpha\beta\gamma = 4 > 0$, WLOG let $\gamma < 0$, $\alpha > 0$

$\alpha = 2 - \beta - \gamma > 2$

$|\alpha| + |\beta| + |\gamma| = \alpha - (\beta + \gamma) = 2 + 2(-\beta - \gamma) \ge 2 + 4\sqrt{(-\beta)(-\gamma)}$, equality holds when $\beta = \gamma$

$4 = (2 - 2\beta)\beta^2$

$\beta^3 - \beta^2 + 2 = 0$

$(\beta + 1)(\beta^2 - 2\beta + 2) = 0$

$\beta = -1$ (For the 2nd equation, $\Delta = -4 < 0$, no real solution)

$\gamma = -1$, $\alpha = 4$

$|\alpha| + |\beta| + |\gamma| = 1 + 1 + 4 = 6$

v = min. of $|\alpha| + |\beta| + |\gamma| = 6$

GS.3 Let $y = |x + 1| - 2|x| + |x - 2|$ and $-1 \le x \le 2$. Let α be the maximum value of y . Find the value of α .

$y = x + 1 - 2|x| + 2 - x = 3 - 2|x|$

$0 \le |x| \le 2 \Rightarrow 3 \ge 3 - 2|x| \ge -1$

$\alpha = 3$

GS.4 Let F be the number of integral solutions of $x^2 + y^2 + z^2 + w^2 = 3(x + y + z + w)$.
Find the value of F.

$(x, y, z, w) = (0, 0, 0, 0)$ is a trivial solution.

$x^2 + y^2 + z^2 + w^2 - 3(x + y + z + w) = 0$

$$\left(x^2-3x+\frac{9}{4}\right)+\left(y^2-3y+\frac{9}{4}\right)+\left(z^2-3z+\frac{9}{4}\right)+\left(w^2-3w+\frac{9}{4}\right)=9$$

$$\left(x-\frac{3}{2}\right)^2+\left(y-\frac{3}{2}\right)^2+\left(z-\frac{3}{2}\right)^2+\left(w-\frac{3}{2}\right)^2=9$$

$(2x - 3)^2 + (2y - 3)^2 + (2z - 3)^2 + (2w - 3)^2 = 36$

Let $a = 2x - 3$, $b = 2y - 3$, $c = 2z - 3$, $d = 2w - 3$, the equation becomes $a^2 + b^2 + c^2 + d^2 = 36$

For integral solutions of (x, y, z, w), (a, d, c, d) must be odd integers.

In addition, the permutation of (a, b, c, d) is also a solution. (e.g. (b, d, c, a) is a solution)

$\because$ a, b, c, d are odd integers and $a^2 + b^2 + c^2 + d^2 \geq 0$

If one of the four unknowns, say, $a > 6$, then L.H.S. > 36, so L.H.S. $\neq$ R.H.S.

$\therefore$ $a , b, c, d = \pm1, \pm3, \pm5$

When $a = \pm5$, then $25 + b^2 + c^2 + d^2 = 36 \Rightarrow b^2 + c^2 + d^2 = 11$

The only integral solution to this equation is $b = \pm3$, $c = \pm1 = d$ or its permutations.

When the largest (in magnitude) of the 4 unknowns, say, a is ±3, then $9 + b^2 + c^2 + d^2 = 36$
$\Rightarrow b^2 + c^2 + d^2 = 27$, the only solution is $b = \pm3$, $c = \pm3$, $d = \pm3$ or its permutations.

$\therefore$ The integral solutions are $(a, b, c, d) = (5, 3, 1, 1)$ and its permutations $\cdots (1) \times P_2^4 = 12$

$(3, 3, 3, 3) \cdots (2) \times 1$

If (a, b, c, d) is a solution, then $(\pm a, \pm b, \pm c, \pm d)$ are also solutions.

There are 16 solutions with different signs for $(\pm a, \pm b, \pm c, \pm d)$.

$\therefore$ $F = (12 + 1)\times16$
$= 208$