Hong Kong Mathematics Olympiad 2009-2010
Heat Event (Individual)

除非特別聲明，答案須用數字表達，並化至最簡。　　　　　　　　時限：40 分鐘
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
每題正確答案得一分。Each correct answer will be awarded 1 mark.　　　Time allowed: 40 minutes

1.　把 8 個完全相同的球放入三個不同的盒中，使得每個盒內至少有球一個，
問共有多少個不同的分配方法？
In how many possible ways can 8 identical balls be distributed to 3 distinct boxes so that every box contains at least one ball?

2.　若 α 及 β 為二次方程 $x^2 - x - 1 = 0$ 的兩個實根，求 $\alpha^6 + 8\beta$ 的值。
If α and β are the two real roots of the quadratic equation $x^2 - x - 1 = 0$,
find the value of $\alpha^6 + 8\beta$.

3.　若 $a = \dfrac{1}{5\times10} + \dfrac{1}{10\times15} + \dfrac{1}{15\times20} + \cdots + \dfrac{1}{100\times105}$ ，求 a 的值。
If $a = \dfrac{1}{5\times10} + \dfrac{1}{10\times15} + \dfrac{1}{15\times20} + \cdots + \dfrac{1}{100\times105}$, find the value of a.

4.　已知 $x + y + z = 3$ 及 $x^3 + y^3 + z^3 = 3$，且 x, y, z 為整數。若 $x < 0$，求 y 的值。
Given that $x+y+z = 3$ and $x^3 +y^3 +z^3 = 3$, where x, y, z are integers. If $x < 0$, find the value of y.

5.　已知 a, b, c, d 為正整數，且滿足 $\log_a b = \dfrac{1}{2}$ 及 $\log_c d = \dfrac{3}{4}$。若 $a - c = 9$，求 $b - d$ 的值。
Given that a, b, c, d are positive integers satisfying $\log_a b = \dfrac{1}{2}$ and $\log_c d = \dfrac{3}{4}$.
If $a - c = 9$, find the value of $b - d$.
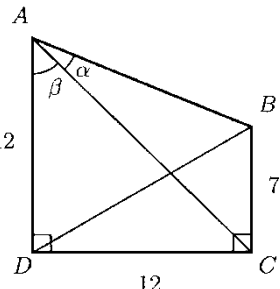
6.　若 $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$，其中 $0 \le x, y \le 1$, 求 $x^2 + y^2$ 的值。
If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$, where $0 \le x, y \le 1$, find the value of $x^2 + y^2$.

7.　在圖一中，$ABCD$ 是一梯形。AD、BC 和 DC 的長分別為 12、7 和 12。若 DC 分別垂直於 AD 及 BC，求 $\dfrac{\sin\alpha}{\sin\beta}$ 的值。
In figure 1, $ABCD$ is a trapezium. The lengths of segments AD, BC and DC are 12, 7 and 12 respectively. If segments AD and BC are both perpendicular to DC, find the value of $\dfrac{\sin\alpha}{\sin\beta}$.

8.　在圖二中，ΔABC 滿足：$x \ge y \ge z$ 及 $4x = 7z$。若 x 的最大值是 m，x 的最小值是 n，求 $m + n$ 的值。
In Figure 2, ABC is a triangle satisfing $x \ge y \ge z$ and $4x = 7z$.
If the maximum value of x is m and the minimum value of x is n, find the value of $m + n$.
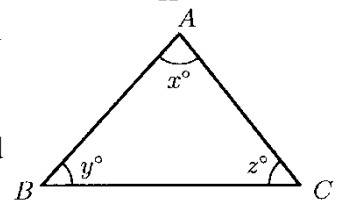
9.　把 $1, 2, \cdots, n$ $(n \ge 3)$ 作環形排列，使得每兩個相鄰的數字相差為 1 或 2。求有多少個此類的環形排列。
Arrange the numbers $1, 2, \cdots, n$ $(n \ge 3)$ in a circle so that adjacent numbers always differ by 1 or 2. Find the number of possible such circular arrangements.

10.　若 $\lfloor x \rfloor$ 為最大的整數小於或等於 x，求以下 2010 個數中共有多少個不同的值：
$$\left\lfloor \frac{1^2}{2010} \right\rfloor, \left\lfloor \frac{2^2}{2010} \right\rfloor, \dots, \left\lfloor \frac{2010^2}{2010} \right\rfloor 。$$
If $\lfloor x \rfloor$ is the largest integer less than or equal to x, find the number of distinct values in the following 2010 numbers: $\left\lfloor \dfrac{1^2}{2010} \right\rfloor, \left\lfloor \dfrac{2^2}{2010} \right\rfloor, \cdots, \left\lfloor \dfrac{2010^2}{2010} \right\rfloor$.
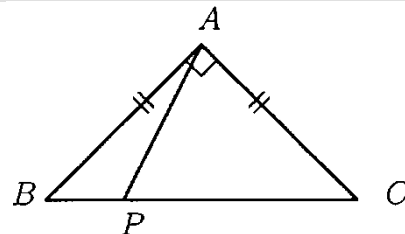
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每題正確答案得一分。Each correct answer will be awarded 1 mark.

1. 在圖三中，ABC 為一等腰三角形及 P 為 BC 上的一點。
    若 $BP^2 + CP^2 : AP^2 = k : 1$，求 k 的值。

    In Figure 3, ABC is an isosceles triangle and P is a point on BC. If $BP^2 + CP^2 : AP^2 = k : 1$, find the value of k.

圖三

Figure 3

除非特別聲明，答案須用數字表達，並化至最簡。　　　　　　　　時限：20 分鐘

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.

每題正確答案得一分。Each correct answer will be awarded 1 mark. Time allowed: 20 minutes

1. 已知六位數 $503xyz$ 可以被 $7, 9, 11$ 整除。求三位數 xyz 的最小值。

   Given that the six-digit number $503xyz$ is divisible by 7, 9, 11. Find the minimum value of the three-digit number xyz.

2. 求最小的正整數 n 使得 $\underbrace{20092009\cdots2009}_{n\text{個}2009}$ 能被 11 整除。

   Find the smallest positive integer n so that $\underbrace{20092009\cdots2009}_{n\text{ copies of }2009}$ is divisible by 11.

3. 在圖一中，ABC 是一三角形。D 是 AC 上的一點，使得 $AB = AD$。若 $\angle ABC - \angle ACB = 40°$，求 x 的值。

   In figure 1, ABC is a triangle. D is a point on AC such that $AB = AD$. If $\angle ABC - \angle ACB = 40°$, find the value of x.
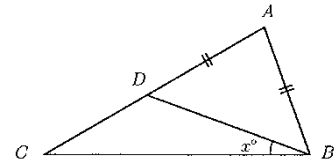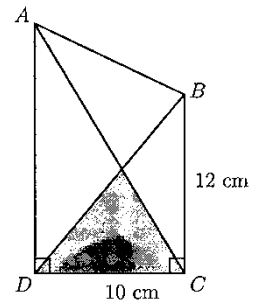

   圖一　Figure 1

4. 在圖二中，已知陰影部分的面積是 35 cm^2。若梯形 $ABCD$ 的面積是 z cm^2，求 z 的值。

   In figure 2, given that the area of the shaded region is 35 cm^2. If the area of the trapezium $ABCD$ is z cm^2, find the value of z.

   圖二　Figure 2

5. 從 $1, 2, 3, 4, 5, 6$ 中抽取三個號碼。求抽得的號碼中包含最少兩個連續數的概率。

   Three numbers are drawn from 1, 2, 3, 4, 5, 6.

   Find the probability that the numbers drawn contain at least two consecutive numbers.

6. 求以下函數的最小值： $f(x) = |x - 1| + |x - 2| + \ldots + |x - 1000|$，其中 x 是一實數。

   Find the minimum value of the following function: $f(x) = |x - 1| + |x - 2| + \ldots + |x - 1000|$, where x is a real number.

7. 設 m, n 為正整數使得 $\dfrac{1}{2010} < \dfrac{m}{n} < \dfrac{1}{2009}$。求 n 的最小值。

   Let m, n be positive integers such that $\dfrac{1}{2010} < \dfrac{m}{n} < \dfrac{1}{2009}$. Find the minimum value of n.

8. 如果正整數 a 的各數位之和等於 7，則 a 稱為「幸運數」。例如 $7, 61, 12310$ 都是「幸運數」。將所有「幸運數」從小到大排成一列 $a_1, a_2, a_3, \ldots$。若 $a_n = 1600$，求 a_{2n} 的值。

   Let a be a positive integer. If the sum of all digits of a is equal to 7, then a is called a "lucky number". For example, 7, 61, 12310 are lucky numbers.

   List all lucky numbers in ascending order $a_1, a_2, a_3, \ldots$. If $a_n = 1600$, find the value of a_{2n}.

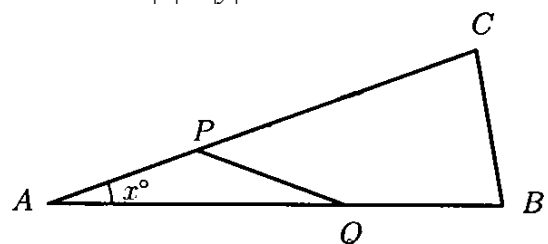9. 若 $\log_4(x + 2y) + \log_4(x - 2y) = 1$，求 $|x| - |y|$ 的最小值。

   If $\log_4(x + 2y) + \log_4(x - 2y) = 1$, find the minimum value of $|x| - |y|$.

10. 在圖三中，ΔABC 滿足 $AB = AC$ 且 $x \le 45$。若 P 和 Q 分別是 AC 及 AB 上的兩點，且 $AP = PQ = QB = BC \le AQ$，求 x 的值。

    In Figure 3, in ΔABC, $AB = AC$, $x \le 45$. If P and Q are two points on AC and AB respectively, and $AP = PQ = QB = BC \le AQ$, find the value of x.

    圖三　Figure 3

*** 試卷完 End of Paper ***

1. 在圖四中，$ABCD$ 為一矩形。設 E 及 F 分別為 DC 及 AB 上的點，並使得 $AFCE$ 為一菱形。若 $AB = 16$ 及 $BC = 12$，求 EF 的值。

   In Figure 4, $ABCD$ is a rectangle. Let E and F be two points on DC and AB respectively, so that $AFCE$ is a rhombus.
   If $AB = 16$ and $BC = 12$, find the value of EF.
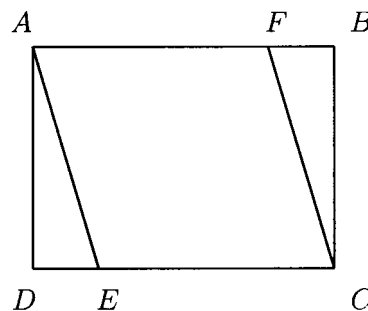
   圖四
   Figure 4

每隊必須列出詳細所有步驟(包括作圖步驟)。	時限：15 分鐘
All working (including geometric drawing) must be clearly shown.	
此部份不計分。This part does not carries any marks.	Time allowed: 15 minutes

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第一題 Question No. 1

圖一所示為長度 1 單位的綫段 *AB*。試構作一長度為 $\sqrt{7}$ 單位的綫段。

Figure 1 shows a line segment *AB* of length 1 unit. Construct a line segment of length $\sqrt{7}$ units.

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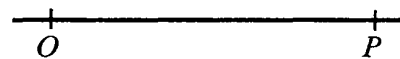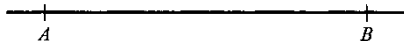|	
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第二題 Question No. 2

已知 Δ*ABC* 是等邊三角形。*P*、*Q* 及 *R* 為分別位於綫段 *AB*、*BC* 及 *CA* 上的相異點，
且 *OP* ⊥ *AB*，*OQ* ⊥ *BC*，*OR* ⊥ *CA* 及 *OP* = *OQ* = *OR*。圖二展示綫段 *OP*。試構作 Δ*ABC*。

Given that Δ*ABC* is equilateral. *P*, *Q* and *R* are distinct points lying on the lines *AB*, *BC* and *CA* such that *OP* ⊥ *AB*, *OQ* ⊥ *BC*, *OR* ⊥ *CA* and *OP* = *OQ* = *OR*. Figure 2 shows the line segment *OP*. Construct Δ*ABC*.

$$\overset{O}{|}\rule[0.5ex]{4cm}{0.4pt}\overset{P}{|}$$

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第三題 Question No. 3

圖三所示為一綫段 *AB*。試構作三角形 *ABC* 使 *AC* : *BC* = 3: 2 及 ∠*ACB* = 60°。

Figure 3 shows a line segment *AB*. Construct a triangle *ABC* such that *AC*:*BC* =3: 2 and ∠*ACB*=60°

A ———————————————————————— *B*

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