Matrices (Matrix) 矩陣

1 Introduction

Consider the following table

	6A	6B	6C
Male	17	15	10
Felmale	14	16	25

This is an example of matrix.

It is represented by

$$\begin{pmatrix} 17 & 15 & 10 \\ 14 & 16 & 25 \end{pmatrix}$$

or

$$\begin{bmatrix} 17 & 15 & 10 \\ 14 & 16 & 25 \end{bmatrix}$$

2 Definition

A matrix is a <u>rectangle</u> of numbers

Each number are called elements

The order (or dimension) is the number of row × number of columns

eg $\begin{pmatrix} 17 & 15 & 10 \\ 14 & 16 & 25 \end{pmatrix}$

A 2×3 matrix (caution: not 6 matrix)

In general, an $m\times n$ matrix.

$$A = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

row column

3 Different types of matrices.

(1) Real matrix

eg $\begin{pmatrix} 2 & 5 & -3 \\ \sqrt{3} & 0 & \pi \end{pmatrix}$

(2) Row matrix

eg $\begin{pmatrix} 5 & -1 & 0 & z^3 \end{pmatrix}$

(3) column matrix

eg $\begin{pmatrix} i \\ \sqrt{3} \end{pmatrix}$

(4) Square matrix 方陣

(no. of columns = no of rows) . $m=n$

eg $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$(a_{11} \quad a_{22} \quad a_{33})$ is called the diagonal

(4.1) Diagonal matrix, $a_{ij}=0$ for $i \neq j$

eg $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, eg $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Identity matrix

(4.2) Triangular matrix. $a_{ij}=0$ for $i>j$ or $i<j$

eg $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 5 & 3 \end{pmatrix}$ eg $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

lower triangular matrix upper triangular matrix.

eg $\begin{pmatrix} -5 & 0 \\ 0 & 0 \end{pmatrix}$

The following is not a triangular matrix

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

SUPPLEMENTARY ANSWER SHEET

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(4.3) Symmetric matrix , $a_{ij} = a_{ji} \ \forall i,j$

eg $\begin{pmatrix} 1 & 5 \\ 5 & 2 \end{pmatrix}$ eg $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & i \\ 3 & i & 0 \end{pmatrix}$

(4.5) Skew Symmetric matrix $a_{ij} = -a_{ji}$
(Alternate matrix, Asymmetric matrix)

eg (0) eg $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

eg $\begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & i \\ 3 & -i & 0 \end{pmatrix}$

(5) Zero matrix $a_{ij} = 0 \ \forall i,j$.

$$\mathbf{0}_{2\times1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{0}_{3\times2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$0_{m\times n} = m\left\{\underbrace{\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}_{n}\right.$$

$$0_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4 Equality of two matrices.

$A_{m\times n} = B_{p\times q}$ if their dimensions are equal $m = p$ $n = q$
and $a_{ij} = b_{ij} \ \forall i,j$

eg $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & \frac{2}{2} & 3 \\ \frac{4}{2} & -\frac{2}{2} & 0 \end{pmatrix}$

$A = B$

eg $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 5 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 0 & 0 \end{pmatrix}$

$A \neq B$, $B \neq C$, $A \neq C$.

eg find x, y, z, t satisfy the following

$$\begin{pmatrix} x+y & z+3t \\ 2z-t & x-y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 5 & 1 \end{pmatrix}$$

$\therefore$ $x+y=3$ $\quad z+3t=-1$

$2z-t=5$ $\quad x-y=1$

Solving $\quad x=2 \quad y=1$

$\quad z=2 \quad t=-1$

5 Addition of matrices.

Consider the following two tables.

		Price	Transportation	
Hong Kong: A		20	8	apple
		15	10	orange
		12	9	Mango
Macau: B		14	7	apple
		17	6	orange
		13	5	Mango

Find the total Price and total transportation for each kind of fruit for two places.

$$C = A+B = \begin{pmatrix} 20+14 & 8+7 \\ 15+17 & 10+6 \\ 12+13 & 9+5 \end{pmatrix} = \begin{pmatrix} 34 & 15 \\ 32 & 16 \\ 25 & 14 \end{pmatrix}$$

If the dimensions of A & B are equal

$A+B = (a_{ij} + b_{ij})_{m \times n}$.

eg $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 4 & -2 \end{pmatrix}_{3\times 2}$ $B = \begin{pmatrix} 0 & 2 & 7 \\ 3 & 4 & 0 \end{pmatrix}_{2\times 3}$

$A+B$ is undefined because $3\times 2 \neq 2\times 3$

Some properties of addition

$A+B = B+A$ (commutative)

pf $A+B = (a_{ij}+b_{ij})_{m\times n}$
$= (b_{ij}+a_{ij})_{m\times n}$
$= B+A$

$A+(B+C) = (A+B)+C$ (associative)

pf: $A+(B+C) = A+(b_{ij}+c_{ij})_{m\times n}$
$= (a_{ij}+(b_{ij}+c_{ij}))_{m\times n}$
$= ((a_{ij}+b_{ij})+c_{ij})_{m\times n}$
$= (a_{ij}+b_{ij})_{m\times n} + C$
$= (A+B)+C$

$\underset{m\times n}{A} + \underset{m\times n}{0} = \underset{m\times n}{0} + \underset{m\times n}{A} = A_{m\times n}$ (zero identity)

pf: $A_{m\times n} + \underline{0}_{m\times n} = (a_{ij}+0)_{m\times n}$
$= (0+a_{ij})_{m\times n}$
$= \underline{0}_{m\times n} + A_{m\times n}$
$= (a_{ij})_{m\times n}$
$= A_{m\times n}$

note $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \underline{0}_{2\times 3}$ is meaningless

$\therefore \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

8 Multiplication of matrices.
The following table shows ~~John~~ John buy the fruit in a week.

$$A_{1\times3} = \begin{array}{ccc} (12 & 10 & 15) \\ \text{orange} & \text{apple} & \text{mango} \end{array}$$

The following table shows the prices of each kind fruit in the week

$$B_{3\times1} = \begin{array}{cc} \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} & \begin{array}{l} \text{orange} \\ \text{apple} \\ \text{mango} \end{array} \end{array}$$

Then the expenditure for John in a week to buy fruit is:

$$C = A \times B = (12 \quad 10 \quad 15)\begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} = 152$$

suppose Mary buys fruit in a week according to the table.

$$\begin{array}{ccc} (13 & 14 & 11) \\ \text{orange} & \text{apple} & \text{mango} \end{array}$$

and the price of fruit in a week is now increased to

$$\begin{array}{cc} \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} & \begin{array}{l} \text{orange} \\ \text{apple} \\ \text{mango} \end{array} \end{array}$$

Find the new price and old price of John and mary.

let $D_{2\times3} = \begin{pmatrix} 12 & 10 & 15 \\ 13 & 14 & 11 \end{pmatrix}$.

$$E_{3\times2} = \begin{pmatrix} 6 & 7 \\ 2 & 3 \\ 4 & 5 \end{pmatrix}$$

$$F_{2\times 2} = D \times E$$

$$= \begin{pmatrix} 12 & 10 & 15 \\ 13 & 14 & 11 \end{pmatrix} \begin{pmatrix} 6 & 7 \\ 2 & 3 \\ 4 & 5 \end{pmatrix}$$

$$= \begin{array}{c} \begin{pmatrix} 152 & 189 \\ 150 & 188 \end{pmatrix} \\ \text{old price} \quad \text{new price} \end{array} \begin{array}{l} \text{John} \\ \text{mary.} \\ \end{array}$$

In General $A_{m\times p}$, $B_{p\times n}$.

$$A \times B = C_{m\times n}$$

$$= (C_{ij})_{m\times n} .$$

$$C_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} .$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} .$$

eg $A = (2 \ 1 \ 3)$ $\quad B = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$

$$A \times B = (2 \ 1 \ 3) \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$$

$$= (2\times 6 + 1\times 5 + 3\times 4) = (29)$$

$$B \times A = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} (2 \ 1 \ 3)$$

$$= \begin{pmatrix} 6\times 2 & 6\times 1 & 6\times 3 \\ 5\times 2 & 5\times 1 & 5\times 3 \\ 4\times 2 & 4\times 1 & 4\times 3 \end{pmatrix} = \begin{pmatrix} 12 & 6 & 18 \\ 10 & 5 & 15 \\ 8 & 4 & 12 \end{pmatrix}$$

$\therefore$ $A \times B \neq B \times A$.

eg $A = \begin{pmatrix} 2 & 1 \\ 4 & 6 \\ 3 & 5 \end{pmatrix}$ $B = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

$$A \times B = \begin{pmatrix} 2 & 1 \\ 4 & 6 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2(-3) + 1(-2) \\ 4(-3) + 6(-2) \\ 3(-3) + 5(-2) \end{pmatrix} = \begin{pmatrix} -6 & -2 \\ -12 & -12 \\ -9 & -10 \end{pmatrix} = \begin{pmatrix} -8 \\ -24 \\ -19 \end{pmatrix}$$

$A_{3\times 2}$ $B_{2\times 1}$

$B_{2\times \underline{1}} \times A_{\underline{3}\times 2}$ is undefined (or meaningless)

because $2 \times \underline{1}$ $\underline{3} \times 2$

not equal

eg Write $2x+3y=4$ as a product of matrices

$$\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \end{pmatrix}$$

eg Write the following system of equations as a product of matrices.

$$\begin{cases} 2x - 3y = 5 \\ 4x + 7y = 20 \end{cases}$$

$$\underbrace{\begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix}}_{\text{coefficient matrix}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 20 \end{pmatrix}$$

augmented matrix : $\begin{pmatrix} 2 & -3 & 5 \\ 4 & 7 & 20 \end{pmatrix}$

9 Some properties of multiplication

(1) Non-Commutative.

ie In general $AB \neq BA$.

eg $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times1+0(-1) & 1\times2+0(3) \\ 2\times1+(-1)(-1) & 2\times2+(-1)\times3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times1+2\times2 & 1\times0+2(-1) \\ -1\times1+3\times2 & -1\times0+3\times(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -2 \\ 5 & -3 \end{pmatrix}$$

$\therefore AB \neq BA$

exercise $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

show that (a) $(A+B)(A-B) \neq A^2 - B^2$

(b) $(A+B)^2 \neq A^2 + 2AB + B^2$

(2) Cancellation law does not hold.

(In real number system $ab = ac$, $a \neq 0$
$\Rightarrow b = c$.

This is called cancellation law)

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eg $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$.

$$AB = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times2+0\times0 & 1\times2+0\times0 \\ 1\times2+0\times0 & 1\times2+0\times0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$AC = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times2+0\times0 & 1\times2+0\times2 \\ 1\times2+0\times0 & 1\times2+0\times2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$\therefore$ $AB = AC$.

now $A \neq \underline{O}_2$

but we cannot cancel A

$\because$ $B \neq C$

In general $AB = AC \not\Rightarrow B = C$

(3) $AB = \underline{O} \not\Rightarrow A = \underline{O}$ or $B = \underline{O}$

eg $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}$

$$AB = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}$$

$$= \begin{pmatrix} 3(-1)+1\times3 & 3\times3+1(-9) \\ 6(-1)+2\times3 & 6\times3+2(-9) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$= \underline{O}$ but $A \neq \underline{O}$, $B \neq \underline{O}$

(4) In real number $\quad X^2 = 1$

$X = \pm 1$

In complex number $\quad z^3 = 1$

$z = 1, \text{cis}120^\circ, \text{cis}240^\circ$

$z^4 = 5, \quad \Rightarrow z = 5^{\frac{1}{4}}, -5^{\frac{1}{4}}, i5^{\frac{1}{4}}, -i \times 5^{\frac{1}{4}}.$

In n-th root of a number has n-values.

This rule <u>does not</u> hold for matrix.

eg $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

$\therefore \sqrt{I} = I, J, K$ or L

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

even more, for $x (\neq 0) \in \mathbb{R}$

$\begin{pmatrix} 0 & x \\ \frac{1}{x} & 0 \end{pmatrix}\begin{pmatrix} 0 & x \\ \frac{1}{x} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$\therefore \sqrt{I}$ has infinitely many values.

In general $n (>1) \in \mathbb{N}$ $\quad \boxed{A^{\frac{1}{n}} \text{ is meaningless}}.$

$\frac{1}{A}$ is meaningless

(5) Associative law of multiplication

$$A(BC) = (AB)C.$$

pf: $A(BC) = (a_{ij})[(b_{jk})(c_{kl})]$

$$= (a_{ij})\left(\sum_{k=1}^{n} b_{jk}c_{kl}\right)$$

$$= \left(\sum_{j=1}^{m} a_{ij} \sum_{k=1}^{n} b_{jk}c_{kl}\right)$$

$$= \left(\sum_{j=1}^{m}\sum_{k=1}^{n} a_{ij}\, b_{jk}\, c_{kl}\right)$$

$$(AB)\,C = \left(\sum_{j=1}^{m} a_{ij}b_{jk}\right)(c_{kl})$$

$$= \left(\sum_{k=1}^{m}\sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kl}\right)$$

$$= \left(\sum_{j=1}^{m}\sum_{k=1}^{n} a_{ij}\, b_{jk}\, c_{kl}\right)$$

$\therefore A(BC) = (AB)C$.

eg $A = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}$

$$AB = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ -2 & 2 \end{pmatrix}.$$

$$BC = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & -1 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -6 & -5 \\ 4 & 2 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} -5 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -6 & -5 \\ 4 & -2 \end{pmatrix}.$$

$\therefore A(BC) = (AB)C$.

(6) Distributive law of multiplication

$$A(B+C) = AB + AC.$$

$$(A+B)C = AC + BC.$$

we only prove the frist one.

pf: $A(B+C) = (a_{ij})[(b_{jk} + c_{jk})]$.

$$= \left(\sum_{j=1}^{n} a_{ij}(b_{jk}+c_{jk})\right)$$

$$= \left(\sum_{j=1}^{n} a_{ij} b_{jk} + \sum_{j=1}^{n} a_{ij} c_{jk}\right)$$

$$= \left(\sum_{j=1}^{n} a_{ij} b_{jk}\right) + \left(\sum_{j=1}^{n} a_{ij} c_{jk}\right)$$

$$= AB + AC.$$

(7) Let A be an $n \times n$ square matrix

Define $A^0 = I$

for $n \geq 1$ $A^n = A^{n-1} \cdot A$.

eg $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. prove that $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

pf: induction on n.

$n=1$ $A^1 = A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\therefore$ It is true for $n=1$

Suppose $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

$$A^{n+1} = A^n A = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$$

By MI, $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ $\forall n \in \mathbb{N}$

exercise $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ find A^n

General rule of elementary row operations.

1 Multiply a row through by a nonzero constant

2 Interchange two rows.

3 Add a multiple of one row to another row.

The following augmented matrix is in <u>reduced row-echelon form</u>.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

It must have the following properties.

1 If a row does not consist entirely of zeros, then the first non-zero number in the row is a 1 (We call this a leading 1)

2 If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

3 In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

4 Each column that contains a leading 1 has zeros everywhere else.

A matrix having property 1, 2, and 3 is said to be in <u>row-echelon form</u>

eg
$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

reduced row-echelon form.

eg
$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

row-echelon form.

eg
$$\left[\begin{array}{ccccc|c} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right]$$

$$\begin{array}{c} R_3 \to R_1 \\ \sim \\ R_1 \to R_3 \end{array} \left[\begin{array}{ccccc|c} 2 & 4 & -5 & 6 & -5 & -1 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \end{array}\right]$$

$$\begin{array}{c} \frac{1}{2}R_1 \to R_1 \\ \sim \\ R_2 - R_1 \to R_2 \\ -\frac{1}{2}R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & -\frac{5}{2} & 3 & -\frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & -5 & 0 & 17 & 29 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \end{array}\right]$$

$$\begin{array}{c} R_1 + \frac{5}{2}R_3 \to R_1 \\ \sim \\ R_3 \to R_2 \\ 5R_3 + R_2 \to R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -\frac{45}{4} & -\frac{31}{2} \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 \end{array}\right]$$

$$\begin{array}{c} \sim \\ -2R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -\frac{45}{4} & -\frac{31}{2} \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$ ↙ row-echelon form

$$\begin{array}{c} R_1 + \frac{45}{4}R_3 \to R_1 \\ R_2 + \frac{7}{2}R_3 \to R_2 \\ \sim \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

reduced row-echelon form

eg Solve $\begin{cases} -2x_3+7x_5=12 \\ 2x_1+4x_2-10x_3+6x_4+12x_5=28 \\ 2x_1+4x_2-5x_3+6x_4-5x_5=-1 \end{cases}$

From the above operations, we get

$$\begin{aligned} x_1+2x_2+3x_4 &= 7 \\ x_3 &= 1 \\ x_5 &= 2. \end{aligned}$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7-2s-3t \\ s \\ 1 \\ t \\ 2 \end{pmatrix}, s,t \in \mathbb{R}$$

$$= \begin{pmatrix} 7 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

exercise (1) Solve

$$\begin{cases} x_1+x_2-2x_3+x_4+3x_5=1 \\ 3x_1+2x_2-4x_3-3x_4-9x_5=3. \\ 2x_1-x_2+2x_3+2x_4+6x_5=2. \\ 6x_1+2x_2-4x_3=6. \\ 2x_2-4x_3-6x_4-18x_5=0. \end{cases}$$

(2) How many solutions if a homogeneous system of linear equations with more unknowns than equations.

(1) − (2) $AX_1 - AX_2 = \underline{0}$

$A(X_1 - X_2) = \underline{0}$.

let $X_0 = X_1 - X_2$ (non-zero)

$\forall k \in \mathbb{R}$ $A(X_1 + kX_0) = AX_1 + A(kX_0)$

$= B + k(AX_0)$

$= B + k\underline{0}$

$= B + \underline{0}$

$= B$.

again $X_1 + kX_0$ is a solution $\forall k \in \mathbb{R}$

$\therefore$ $AX = B$ has infinitely many solutions.

12 if $AB = BA = I$.

we say B the inverse of A

and denote $B = A^{-1}$ (not $\frac{1}{A}$)

(because $A^{-1}C \neq CA^{-1}$)

Theorem A^{-1} is unique

pf: suppose $AC = CA = \underline{I}$

$C = CI = C(AB)$

$= (CA)B$

$= IB$

$= B$

eg Find the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$A^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ $AA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

solving $p = \frac{d}{ad-bc} = \frac{d}{|A|}$ $r = \frac{-c}{|A|}$

$q = \frac{-b}{|A|}$ $s = \frac{a}{|A|}$

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Let $n \in \mathbb{N}$

Define $A^{-0} = I$

$A^{-n} = (A^n)^{-1}$

Claim $A^{-n} = (A^{-1})^n$

pf: induction on n

$n=0, 1$ obviously true

suppose $A^{-k} = (A^{-1})^k$.

$$\begin{aligned}(A^{-1})^{k+1} A^{k+1} &= (A^{-1})^k A^{-1} A\, A^k. \\ &= (A^{-1})^k I A^k \\ &= (A^{-1})^k A^k \\ &= A^{-k} A^k \quad (\text{by induction assumption}) \\ &= I \quad (\text{similarly } A^{k+1}(A^{-1})^{k+1} = I)\end{aligned}$$

$\therefore (A^{-1})^{k+1} = A^{-(k+1)}$

$r, s \in \mathbb{Z}$ $\quad A^r A^s = A^{r+s}$ $\quad (A^r)^s = A^{rs}$

The transpose of a matrix

$A = (a_{ij})_{m\times n}$ $\quad A^t = A' = (a_{ji})_{n\times m}$

eg $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}_{3\times 2}$ $\quad A^t = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{2\times 3}$.

eg $B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}_{2\times 2}$ $\quad B^t = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}_{2\times 2}$.

If A is symmetric $A^t = A$.

pf: $A^t = (a_{ji})_{n\times n} = (a_{ij})_{n\times n} = A$

eg $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & i \\ 3 & i & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & i \\ 3 & i & 0 \end{pmatrix}$.

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If A is skew-symmetric $A^t = -A$

pf: $A^t = (a_{ji})_{n\times n}$
$= (-a_{ij})_{n\times n}$
$= -(a_{ij})_{n\times n}$
$= -A$

eg $\begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & i \\ 3 & -i & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & -i \\ -3 & i & 0 \end{pmatrix}$

$= -\begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & i \\ 3 & i & 0 \end{pmatrix}$

Properties of the Transpose Operation

(i) $(A^t)^t = A$

pf $(A^t)^t = (a_{ji})^t = (a_{ij}) = A$

(ii) $(A+B)^t = A^t + B^t$

pf: $(A+B)^t = (a_{ij}+b_{ij})^t$
$= (a_{ji}+b_{ji})$
$= (a_{ji}) + (b_{ji})$
$= A^t + B^t$

(iii) $(kA)^t = kA^t$, $k \in \mathbb{R}$

pf $(kA)^t = (ka_{ij})^t$
$= (ka_{ji})$
$= k(a_{ji})$
$= kA^t$

(iv) $(AB)^t = B^t A^t$

pf: $(AB)^t = \left(\sum_{k=1}^{p} a_{ik} b_{kj}\right)^t_{m\times n}$ A_{mp}, B_{pn}

$= \left(\sum_{k=1}^{p} b_{ki} a_{jk}\right)_{n\times m}$

$B^t A^t = (b_{ki})_{n\times p} (a_{jk})_{p\times m}$

$= \left(\sum_{k=1}^{p} b_{ki} a_{jk} \right)_{n\times m}$.

$\therefore (AB)^t = B^t A^t$

Similarly $(ABC)^t = C^t B^t A^t$.

Inductively $(A_1 \cdots A_n)^t = A_n^t A_{n-1}^t \cdots A_1^t$

example if A is any matrix, show that AA^t and A^tA are both symmetric.

pf: $(AA^t)^t = (A^t)^t A^t = AA^t$

$\therefore AA^t$ is symmetric.

$(A^tA)^t = A^t (A^t)^t = A^t A$

$\therefore AA^t$ is also symmetric

example show that any square matrix A can be expressed in a unique way as a sum of a symmetric matrix and a skew-symmetric matrix

pf $S = \frac{1}{2}(A + A^t)$

$T = \frac{1}{2}(A - A^t)$

$S + T = A$

$S^t = \frac{1}{2}(A + A^t)^t$

$= \frac{1}{2}(A^t + A) = S$ $\therefore S$ is symmetric.

$T^t = \frac{1}{2}(A - A^t)^t$

$= \frac{1}{2}(A^t - A) = -T$ $\therefore T$ is skew-symmetric

exercise (1) Show that if A is symmetric, then BAB^t is also symmetric

(2) if A and B are symmetric $n\times n$ matrices. show that $AB - BA$ is skew-symmetric.

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If A^{-1} exist. $(A^{-1})^{-1} = A$

$(A^t)^{-1}$ also exists. and $(A^t)^{-1} = (A^{-1})^t$

pf $(A^{-1})A = A A^{-1} = I$.

$\therefore (A^{-1})^{-1} = A$

$(A^{-1} A)^t = (A A^{-1})^t = I^t$

$A^t (A^{-1})^t = (A^{-1})^t A^t = I$.

$\therefore (A^t)^{-1} = (A^{-1})^t$.

If A^{-1} and B^{-1} exist and A, B are both $n \times n$ matrices then $(AB)^{-1}$ exists and $(AB)^{-1} = B^{-1}A^{-1}$

pf: $AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A I A^{-1}$

$= A A^{-1}$

$= I$.

$B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1} I B = B^{-1}B = I$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$.

Similarly $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ (same as transpose).

Inductively $(A_1 A_2 \cdots A_m)^{-1} = A_m^{-1} \cdots A_2^{-1} A_1^{-1}$

example. If A and B are invertible $n \times n$ matrices show that $A^{-1} + B^{-1} = A^{-1}(A+B)B^{-1}$

If $A+B$ is also invertible find $(A^{-1}+B^{-1})^{-1}$

pf: $A^{-1}(A+B)B^{-1} = (I + A^{-1}B)B^{-1}$

$= B^{-1} + A^{-1} = A^{-1} + B^{-1}$ //

$(A^{-1}+B^{-1})^{-1} = (A^{-1}(A+B)B^{-1})^{-1} = B(A+B)^{-1}A$

exercise let A, B be $n \times n$ matrices such that $I - AB$ is invertible, Show that $I - BA$ is invertible and $(I-BA)^{-1} = I + B(I-AB)^{-1}A$

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Not all matrix has an inverse.

eg
$$A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_2 - 2R_1 \to R_2 \\ \sim \\ R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

$$R_2 + R_3 \to R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

Consisting of a row of zeros

A is <u>not</u> invertible.

Determinant of a <u>square</u> $n \times n$ matrix $(n \le 3)$

a) $M = (a)$ $\quad |M| = \det M = a$

b) $M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}$ $\quad |M| = \det M = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$$= a_{11}a_{22} - a_{21}a_{12}$$

if $|N| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$ then $|MN| = |M||N|$

pf: $\left|\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right| = \left|\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}\right|$

$= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{21}b_{11} + a_{22}b_{21})(a_{11}b_{12} + a_{12}b_{22})$

$= \cancel{a_{11}a_{21}b_{11}b_{12}} + a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} + \cancel{a_{12}a_{22}b_{21}b_{22}}$
$\quad - \cancel{a_{11}a_{21}b_{11}b_{12}} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} - \cancel{a_{12}a_{22}b_{21}b_{22}}$

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$= a_{11}a_{22}(b_{11}b_{22}-b_{12}b_{21})+a_{12}a_{21}(b_{12}b_{21}-b_{11}b_{22})$

$=(a_{11}a_{22}-a_{21}a_{12})(b_{11}b_{22}-b_{12}b_{21})$

$=\det M \det N$

c) $|M| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

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$|M| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$
$\quad - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$

if N is also a 3×3 square matrix.

$|MN| = |M||N|$

pf: difficult, omit

Elementary row / column operations of determinant.
(see the past note yourself!)

Definition An $n\times n$ matrix A is <u>non-singular</u> if $|A| \neq 0$

Theorem A is invertible (ie A^{-1} exist) if and only if $|A| \neq 0$

pf: We have proved for the case $n=2$ (on P10b).
now we are going to prove for $n=3$

let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

let $C_{ij} = (-1)^{i+j} \det$ (matrix by deleting the ith row jth column of A)
called the $(i,j)^{th}$ cofactor of A

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eg $C_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

$C_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -(a_{11}a_{32} - a_{31}a_{12})$

$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}a_{33} - a_{32}a_{23})$

Then we have the <u>cofactor expansion of det A</u>

$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} \quad i=1,2,3$ (row)
$= a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} \quad j=1,2,3$ (column)

Futhermore $a_{i1}C_{j1} + a_{i2}C_{j2} + a_{i3}C_{j3} = 0$ for $i \neq j$
$a_{1i}C_{1j} + a_{2i}C_{2j} + a_{3i}C_{3j} = 0$ for $i \neq j$.

Define $C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$

where C_{ij} is the (i,j)th cofactor of A.

C is called the matrix of cofactors from A

C^t is called the adjoint of A.

denote $C^t = \mathrm{adj}(A)$

We show first that $A\,\mathrm{adj}(A) = \det(A) I$

Consider $A\,\mathrm{adj}(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

$= \begin{pmatrix} a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13} & a_{11}C_{21}+a_{12}C_{22}+a_{13}C_{23} & a_{11}C_{31}+a_{12}C_{32}+a_{13}C_{33} \\ a_{21}C_{11}+a_{22}C_{12}+a_{23}C_{13} & a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23} & a_{21}C_{31}+a_{22}C_{32}+a_{23}C_{33} \\ a_{31}C_{11}+a_{32}C_{12}+a_{33}C_{13} & a_{31}C_{21}+a_{32}C_{22}+a_{33}C_{23} & a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33} \end{pmatrix}$

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$$A\,\text{adj}(A) = \begin{pmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{pmatrix} = \det(A)\, I$$

$\therefore$ if $\det(A) \neq 0$ $\quad A\left(\dfrac{\text{adj}(A)}{\det(A)}\right) = I$.

$\therefore$ A^{-1} exist and equal to $\dfrac{1}{|A|}\text{adj}(A)$.

if A^{-1} exist. $\quad A A^{-1} = I$.

$$|A A^{-1}| = |I|$$
$$|A|\,|A^{-1}| = 1$$
$$\therefore |A| \neq 0$$

eg Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 3 & 1 \end{pmatrix}$

and hence solve $\begin{cases} 2x + y + 4z = 2 \\ x + 2z = 3 \\ 2x + 3y + z = -6 \end{cases}$

$$[C_{ij}] = \begin{bmatrix} \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -6 & 3 & 3 \\ 11 & -6 & -4 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\text{adj}\,A = (C_{ij})^t = \begin{pmatrix} -6 & 11 & 2 \\ 3 & -6 & 0 \\ 3 & -4 & -1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -1 \times \begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} + (-1) \times 2 \times \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$$
$$= 11 - 8 = 3$$

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$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{3}\begin{bmatrix} -6 & 11 & 2 \\ 3 & -6 & 0 \\ 3 & -4 & -1 \end{bmatrix} = \begin{pmatrix} -2 & \frac{11}{3} & \frac{2}{3} \\ 1 & -2 & 0 \\ 1 & -\frac{4}{3} & -\frac{1}{3} \end{pmatrix}$$

Second part $AX = H$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $H = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$

$A^{-1}(AX) = A^{-1}H$

$$(A^{-1}A)X = A^{-1}H$$
$$IX = A^{-1}H$$
$$X = A^{-1}H$$
$$= \begin{pmatrix} -2 & \frac{4}{3} & \frac{2}{3} \\ 1 & -2 & 0 \\ 1 & -\frac{4}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}$$

$\therefore$ $x = 3$ $y = -4$ $z = 0$.

Exercise. Find the inverse of $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$ and hence

solve $\begin{cases} x - y + z = 1 \\ x + y + 2z = 0 \\ 2x - y + 3z = 2 \end{cases}$

Finding an inverse is always time-consuming. You can use a programmable calculator to help you or any other methods such as:

(A) Matrix equation (Hamilton - Cayley theorem).

let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any 2×2 matrix.

then $B^2 - (a+d)B + (\det B) I = \underline{0}$ (the zero matrix)

pf: $B^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$

$-(a+d)B = \begin{pmatrix} -a^2-ad & -ab-bd \\ -ac-cd & -ad-d^2 \end{pmatrix}$

$\det(B) I = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

+)

$B^2 - (a+d)B + \det(B)I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \underline{0}$

It follows that every 2×2 matrix satisfies a quadratic equation

eg $B = \begin{pmatrix} -2 & -3 \\ 4 & 5 \end{pmatrix}$ It follows that $B^2 - 3B + 2I = 0$

$B(B - 3I) = -2I$.

$\therefore B[-\frac{1}{2}(B-3I)] = I$

$\therefore B^{-1} = -\frac{1}{2}(B-3I) = -\frac{1}{2}\left(\begin{pmatrix} -2 & -3 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right)$

$= -\frac{1}{2}\begin{pmatrix} -5 & -3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ -2 & -1 \end{pmatrix}$

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Finding the nth power ($n \in \mathbb{N}$) is also very easy from Cayley-Hamilton theorem.

eg $B^2 - 3B + 2I = 0$, (given, B is a 2×2 matrix)

find B^{101}

Solution $x^{101} = (x^2 - 3x + 2)Q(x) + ax + b$

by remainder theorem.

$x^{101} = (x-1)(x-2)Q(x) + ax + b$

$x = 1 \quad 1 = a + b$

$x = 2 \quad 2^{101} = 2a + b$

$a = 2^{101} - 1$

$b = 2 - 2^{101}$

$\therefore \quad B^{101} = (B^2 - 3B + 2I)Q(B) + (2^{101}-1)B + (2-2^{101})I$

$=$

$= (2^{101}-1)B + (2-2^{101})I$

Can you find B^{-101} in terms of B and I?

exercise (1) If $A = \begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix}$ find an matrix equation

and hence find $(A^{-1})^3$

(2) If $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -2 & -1 \\ 2 & 3 & 2 \end{pmatrix}$ show that $A^3 - A = A^2 - I$.

Prove by induction that $\forall n \geq 3 \quad A^n - A^{n-2} = A^2 - I$.

Hence find A^{100}