Problem on 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.

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Find the only 10-digit number such that

1. The number is formed by the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 and 0 each once.
2. For $n = 1$ to 10, the integer formed by first n numbers is divisible by n.

Solution:

We denote the integer formed by the digits a, b and c as $\underline{abc}$.

Let the number be $X = \underline{abcdefghij}$.

First of all, let us discover more properties of the digits.

1. Since X is divisible by 10, the last digit must be zero, i.e. $j = 0$. And the number $\underline{abcde}$ is divisible by 5, meaning that its last digit, e, is either 5 or 0. However, the digit j already holds 0. Hence $e = 5$.
2. Since the number $\underline{ab}$ is divisible by 2, b should be even. Similarly, the digits d, f, h and j are all even. The five even numbers are occupied, meaning that the digits a, c, e, g and i are odd.
3. Since the number $\underline{abcd}$ is divisible by 4, the number $\underline{cd}$ should also be divisible by 4. Similarly, since the number $\underline{abcdefgh}$ is divisible by 8, the number $\underline{fgh}$ should also be divisible by 8.
4. Observe that the number $\underline{abc}$ is divisible by 3. Hence $a + b + c$ is also divisible by 3. The number $\underline{abcdef}$ is divisible by 6, implying that it is divisible by 3. Hence $a + b + c + d + e + f$ is divisible by 3. Similarly, $a + b + c + d + e + f + g + h + i$ is divisible by 3. $d + e + f = (a + b + c + d + e + f) - (a + b + c)$; $g + h + i = (a + b + c + d + e + f + g + h + i) - (a + b + c + d + e + f)$. Both $d + e + f$ and $g + h + i$ are also divisible by 3.
5. For any three numbers x, y and z, if $x + y + z$ is divisible by 3, there are only two possibilities:
   (A) All the three numbers have the same remainder when divided by 3.
   (B) They all have different remainders when divided by 3.

Now we can find out the digits more easily.

6. Consider the number $\underline{def}$. Since $d + e + f$ is divisible by 3, we now consider the above two cases (A) and (B)
   (A) d, e and f have the same remainder when divided by 3. $e = 5$. Hence there are only two possibilities: $d = 8, f = 2$ or $d = 2, f = 8$.
      (a) If $d = 8$ and $f = 2$ ($X = \underline{abc852ghi0}$), the digit h can only be 4 or 6 (since it is even). g can be 1, 3, 7 or 9. If $h = 6$, $\underline{fgh}$ becomes $\underline{2g6}$. But notice that the number $\underline{fgh}$ is divisible by 8. Simple checking reveals that $g = 1$ or $g = 9$. But g cannot be 1. Why? Remember that the number $\underline{ghi}$ is divisible by 3. If $g = 1$, $\underline{ghi}$ becomes $\underline{16i}$ and i must be 2, 5 or 8, which are occupied by f, e and d respectively! If $g = 9$, $\underline{ghi}$ becomes $\underline{96i}$, and $i = 3$. X becomes $\underline{a4c8529630}$. Sadly, the numbers 1478529 and 7418529 are not divisible by 7. If $h = 4$, $\underline{fgh}$ becomes $\underline{2g4}$. However, none of the numbers 214, 234, 274 or 294 is divisible by 8.

(b) If $d = 2$ and $f = 8$ ($X = \underline{abc258ghi0}$), again the digit h can only be 4 or 6. If $h = 6$, $\underline{fgh}$ becomes $\underline{8g6}$. Checking reveals that $g = 1$ or 9. If $g = 1$, $\underline{ghi} = \underline{16i}$ and it must be divisible by 3, which force i to be 2 or 8. But 2 is already occupied by 2 and 8 is occupied by f. $\therefore$ $g = 9$ and $\underline{ghi} = 96i$ which is divisible by 3, so $i = 3$. $X = \underline{a4c2589630}$. But neither of the numbers 1472589 or 7412589 is not divisible by 7. If $h = 4$, $\underline{fgh}$ becomes $\underline{8g4}$. However, none of the numbers 814, 834, 874 or 894 is divisible by 8.

(B) e (= 5) has remainder 2 when divided by 3. Therefore, one of the digits d and f has remainder 1 when divided by 3 while the other is divisible by 3. Notice that both d and f are even. If one of them is divisible by 3, it must be 6 (in the numbers 1 to 10, only 6 is divisible by both 2 and 3). And $d + e + f$ is divisible by 3. If one of the digits d and f is 6, the other number (which is even) is 2, 4 or 8. But both 6 + 5 + 2 = 13 and 6 + 5 + 8 = 19 are not divisible by 3. Only 6 + 5 + 4 = 15 is divisible by 3. Hence the number is 4. So we conclude that the digits are 4 and 6.

7. We now decide which digit is 6 and which is 4. First, let's suppose $d = 4$ and $f = 6$. c is an odd integer. Remember that the number $\underline{cd}$ is divisible by 4. However, none of the numbers 34, 74 or 94 is divisible by 4. So there is no number for poor c. We conclude that $d = 6$ and $f = 4$. $X = \underline{abc654ghi0}$.

8. Notice that h is even. Hence $h = 2$ or $h = 8$. If $h = 8$, the number $\underline{fgh}$ becomes $\underline{4g8}$. g is an odd integer. Unfortunately, none of the numbers 418, 438, 478 or 498 is divisible by 8. This forces $h = 2$ and hence $b = 8$.

9. $h = 2$, $\underline{fgh}$ becomes $\underline{4g2}$. Only 432 and 472 are divisible by 8. So g = 3 or 7. Remember that the number $\underline{ghi}$ is divisible by 3. Now $\underline{ghi}$ becomes either $\underline{32i}$ or $\underline{72i}$. If $g = 3$, then $i = 1$ or 7. The remaining numbers go to a and c, and we get 4 sets of X = 1896543270, 9816543270, 9876543210 and 7896543210.. We still have to check the final condition: the number $\underline{abcdefg}$ is divisible by 7. Sadly, none of the numbers is not divisible by 7. So $g = 7$.

10. Finally, $g = 7$, then $i = 3$ or 9. We get another 4 sets of X: 1836547290, 3816547290, 1896547230 or 9816547230. By checking, only the number 3816547 is divisible by 7. So we get the only X: 3816547290. Pretty easy, isn't it?

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