Examples on Mathematical Induction: Fractions

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1. **Mathematics 1977 Paper 1 Q9**

(a) Prove by mathematical induction that for all $n \in \mathbb{N}$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

(b) Hence find the smallest value of n such that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}>\frac{9}{10}.$$

(a) Let $P(n) \equiv$ "$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$", where n is a positive integer.

$n = 1$, L.H.S. $=\frac{1}{1\times 2}=\frac{1}{2}$, R.H.S. $=\frac{1}{1+1}=\frac{1}{2}$

L.H.S. = R.H.S.

$P(1)$ is true.

Suppose that $P(k)$ is true for some positive integer k.

i.e. $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ ……… (*)

When $n = k + 1$,

$$\text{L.H.S.}=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad \text{(by (*))}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{k+2}=\frac{(k+1)}{(k+1)+1}$$

= R.H.S.

If $P(k)$ is true then $P(k + 1)$ is also true

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

(b) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}>\frac{9}{10}$

$\frac{n}{n+1}>\frac{9}{10}$

$10n > 9n + 9$

$n > 9$

The smallest $n = 10$.

2. **HKAL Pure Mathematics 1978 Paper 1 Q3**

(a) Find A and B such that $\dfrac{A}{n-1}+\dfrac{B}{n}=\dfrac{1}{(n-1)n}$.

(b) Use (a) to compute $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)n}$.

(c) The following answer to part (b) was offered by Mr. Wu Lung.

Claim: $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)n}=\dfrac{3}{2}-\dfrac{1}{n}$

Proof: Use mathematical induction on n. For $n = 1$,

R.H.S. $=\dfrac{3}{2}-\dfrac{1}{1}=\dfrac{1}{2}=\dfrac{1}{1\cdot 2}=$ L.H.S.

Suppose it is true for n, then

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n(n+1)}=\frac{3}{2}-\frac{1}{n}+\frac{1}{n(n+1)}$$
$$=\frac{3}{2}-\frac{1}{n+1}$$

Hence it is true for $n + 1$. Q.E.D.

Is it correct? Explain and comment.

(d) Prove your answer to part (b) by mathematical induction.

(a) $\dfrac{A}{n-1}+\dfrac{B}{n}=\dfrac{1}{(n-1)n}$

$An + B(n - 1) \equiv 1$

Put $n = 0$: $B = -1$

Put $n = 1$: $A = 1$

(b) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)n}=\left(1-\dfrac{1}{2}\right)+\left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\cdots+\left(\dfrac{1}{n-1}-\dfrac{1}{n}\right)=1-\dfrac{1}{n}=\dfrac{n-1}{n}$

(c) The claim should be: "$\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)n}=1-\dfrac{1}{n}$ for $n \ge 2$."

The proof by mathematical induction is wrong.

In the 1st and the 2nd line, it is not true for $n = 1$.

When $n = 1$, L.H.S. $=\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)n}=\dfrac{1}{0\cdot 1}$ which is undefined.

The statement should be proved for $n \ge 2$.

(d) Let $P(n)$ be the proposition "$\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)n}=1-\dfrac{1}{n}$ for $n \ge 2$."

When $n = 2$, LHS $=\dfrac{1}{1\cdot 2}=\dfrac{1}{2}$, RHS $=1-\dfrac{1}{2}=\dfrac{1}{2}$

LHS = RHS, $P(2)$ is true.

Suppose $P(k)$ is true.

i.e. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(k-1)k}=1-\dfrac{1}{k}$ for some positive integer $k \ge 2$.

When $n = k + 1$, LHS$=\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(k-1)k}+\dfrac{1}{k(k+1)}=1-\dfrac{1}{k}+\dfrac{1}{k(k+1)}$

$$=1-\frac{k+1-1}{k(k+1)}=1-\frac{k}{k(k+1)}=1-\frac{1}{k+1}=\text{RHS}$$

If $P(k)$ is true, then $P(k + 1)$ is also true for $k \ge 2$.

By the principle of mathematical induction, $P(n)$ is true for all positive integer $n \ge 2$.

3. Prove that $\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+\cdots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$ for $n = 1, 2, 3, \cdots$

4. **1971 香港中文中學會考普通數學課程二試卷二 Q1**
1969 香港中文中學會考高級數學試卷一 Q5(b)

(i) 用歸納法，證明 $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$, $n\in\mathbb{N}$.

(ii) 利用上式，求 $\frac{1}{51\cdot 53}+\frac{1}{53\cdot 55}+\frac{1}{55\cdot 57}+\cdots+\frac{1}{99\cdot 101}$ 之值。

(i) Let $P(n)\equiv$ "$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ for $n = 1, 2, 3, \cdots$"

$n = 1$, L.H.S. $=\frac{1}{1\cdot 3}=\frac{1}{3}$, R.H.S. $=\frac{1}{2(1)+1}=\frac{1}{3}$

It is true for $n = 1$

Suppose $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$ for some positive integer k.

When $n = k + 1$,

L.H.S. $=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$

$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$ (By induction assumption)

$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$=\frac{k+1}{2(k+1)+1}=$ R.H.S.

If $P(k)$ is true, then $P(k + 1)$ is also true

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

(ii) $\frac{1}{51\cdot 53}+\frac{1}{53\cdot 55}+\frac{1}{55\cdot 57}+\cdots+\frac{1}{99\cdot 101}$

$=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{99\cdot 101}-\left(\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{49\cdot 51}\right)$

$=\frac{50}{2\cdot 50+1}-\frac{25}{2\cdot 25+1}$

$=\frac{50}{101}-\frac{25}{51}=\frac{25}{5151}$

5. **1987 Paper 2 Q2 2013 M2 Q3**

Prove, by mathematical induction, that

$\frac{1}{1\times 4}+\frac{1}{4\times 7}+\frac{1}{7\times 10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$, for all positive integers n.

Let $P(n)\equiv$ "$\frac{1}{1\times 4}+\frac{1}{4\times 7}+\frac{1}{7\times 10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$", where n is a positive integer.

$n = 1$, L.H.S. $=\frac{1}{1\times 4}=\frac{1}{4}$, R.H.S. $=\frac{1}{3+1}=\frac{1}{4}$

L.H.S. = R.H.S.

$P(1)$ is true.

Suppose that $P(k)$ is true for some positive integer k.

i.e. $\frac{1}{1\times 4}+\frac{1}{4\times 7}+\frac{1}{7\times 10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$ ……… (*)

When $n = k + 1$,

L.H.S. $=\frac{1}{1\times 4}+\frac{1}{4\times 7}+\frac{1}{7\times 10}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}$

$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$ (by (*))

$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}$

$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$

$=\frac{k+1}{3(k+1)+1}$

= R.H.S.

If $P(k)$ is true then $P(k + 1)$ is also true

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

6. Let $\mathrm{f}(n)=\frac{1}{n^2}$, show that $\mathrm{f}(n)-\mathrm{f}(n+1)=\frac{2n+1}{n^2(n+1)^2}$.

Hence show that $\frac{3}{4}+\frac{5}{36}+\cdots+\frac{2n+1}{n^2(n+1)^2}=\frac{n(n+2)}{(n+1)^2}$

7. **1994 Paper 2 Q5**

Prove that $\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\cdots+\frac{2n-1}{2^n}=3-\frac{2n+3}{2^n}$ for any positive integer n .

Let $P(n)\equiv$ "$\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\cdots+\frac{2n-1}{2^n}=3-\frac{2n+3}{2^n}$ for any positive integer n ."

$n = 1$, L.H.S. $=\frac{1}{2}$, R.H.S. $=3-\frac{2+3}{2^1}=\frac{1}{2}$ = L.H.S.

$P(1)$ is true

Suppose $P(k)$ is true for some positive integer k

i.e. $\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\cdots+\frac{2k-1}{2^k}=3-\frac{2k+3}{2^k}$

When $n = k + 1$,

$\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\cdots+\frac{2k-1}{2^k}+\frac{2(k+1)-1}{2^{k+1}}$

$=3-\frac{2k+3}{2^k}+\frac{2k+1}{2^{k+1}}$ (induction assumption)

$=3+\frac{2k+1-2(2k+3)}{2^{k+1}}$

$=3+\frac{-2k-5}{2^{k+1}}=3-\frac{2(k+1)+3}{2^{k+1}}$ = R.H.S.

If it is true for $n = k$, then it is also true for $n = k + 1$

By the principle of mathematical induction, $P(k)$ is true for all positive integer n .

8. Prove that $\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\ldots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}$ for $n = 1, 2, \ldots$

Let P(n) ≡‘ $\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\cdots+\frac{n^2}{(2n-1)\times(2n+1)}=\frac{n(n+1)}{2(2n+1)}$ for all positive integers n.’

L.H.S. $=\frac{1^2}{1\times 3}=\frac{1}{3}$, R.H.S. = $\frac{2}{2\times 3}=\frac{1}{3}$

∴ P(1) is true.

Assume P(k) is true, where k is a positive integer.

i.e. $\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\cdots+\frac{k^2}{(2k-1)\times(2k+1)}=\frac{k(k+1)}{2(2k+1)}$

When $n = k + 1$,

$$\text{L.H.S.}=\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\cdots+\frac{k^2}{(2k-1)\times(2k+1)}+\frac{(k+1)^2}{(2k+1)\times(2k+3)}$$

$$=\frac{k(k+1)}{2(2k+1)}+\frac{(k+1)^2}{(2k+1)(2k+3)}$$

$$=\frac{k(k+1)(2k+3)+2(k+1)^2}{2(2k+1)(2k+3)}$$

$$=\frac{(k+1)[(2k^2+3k)+2(k+1)]}{2(2k+1)(2k+3)}=\frac{(k+1)(2k^2+5k+2)}{2(2k+1)(2k+3)}$$

$$=\frac{(k+1)(2k+1)(k+2)}{2(2k+1)(2k+3)}$$

$$=\frac{(k+1)(k+2)}{2(2k+3)}=\frac{(k+1)(k+1+1)}{2[2(k+1)+1]}=\text{RHS}$$

∴ P(k + 1) is also true.

∴ By mathematical induction, P(n) is true for all positive integers n.

9. Prove by mathematical induction that $\frac{1}{1!\cdot 3}+\frac{1}{2!\cdot 4}+\frac{1}{3!\cdot 5}+\cdots+\frac{1}{n!\cdot(n+2)}=\frac{1}{2}-\frac{1}{(n+2)!}$

Let P(n) be the statement “$\frac{1}{1!\cdot 3}+\frac{1}{2!\cdot 4}+\frac{1}{3!\cdot 5}+\cdots+\frac{1}{n!\cdot(n+2)}=\frac{1}{2}-\frac{1}{(n+2)!}$”

$n = 1$, LHS = $\frac{1}{1!\cdot 3}=\frac{1}{3}$, RHS = $\frac{1}{2}-\frac{1}{3!}=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$ = LHS, P(1) is true.

Suppose P(k) is true for some positive integer k.

i.e. $\frac{1}{1!\cdot 3}+\frac{1}{2!\cdot 4}+\frac{1}{3!\cdot 5}+\cdots+\frac{1}{k!\cdot(k+2)}=\frac{1}{2}-\frac{1}{(k+2)!}$

$n = k + 1$, LHS = $\frac{1}{1!\cdot 3}+\frac{1}{2!\cdot 4}+\frac{1}{3!\cdot 5}+\cdots+\frac{1}{k!\cdot(k+2)}+\frac{1}{(k+1)!\cdot(k+3)}$

$$=\frac{1}{2}-\frac{1}{(k+2)!}+\frac{1}{(k+1)!(k+3)}\quad\text{(by induction assumption)}$$

$$=\frac{1}{2}-\frac{k+3-(k+2)}{(k+2)!(k+3)}$$

$$=\frac{1}{2}-\frac{1}{(k+3)!}=\text{RHS}$$

P(k + 1) is also true when P(k) is true.

By the principle of mathematical induction, P(n) is true for all positive integer n .

10. **2003 Q7**

Prove, by induction, that $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$, for all positive integers n.

$n = 1$, L.H.S. $=\frac{1}{2}$, R.H.S. = $2-\frac{1+2}{2^1}=\frac{1}{2}$ It is true for $n = 1$.

Suppose it is true for $n = k$. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{k}{2^k}=2-\frac{k+2}{2^k}$

Add $\frac{k+1}{2^{k+1}}$ to both sides.

L.H.S. $=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{k}{2^k}+\frac{k+1}{2^{k+1}}=2-\frac{k+2}{2^k}+\frac{k+1}{2^{k+1}}$

$=2-\frac{2(k+2)-(k+1)}{2^{k+1}}$

$=2-\frac{k+3}{2^{k+1}} = 2-\frac{k+1+2}{2^{k+1}}$ = R.H.S.

The statement is also true for $n = k + 1$ if $n = k$ is true.

By M.I., $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$, for all positive integers n.

11. Prove that $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$ for $n = 1, 2, 3, \ldots$

11. Let $P(n) \equiv$ "$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$ for $n = 1, 2, 3, \ldots$"

$n = 1$, L.H.S. $=\frac{1}{1\cdot2\cdot3}=\frac{1}{6}$; R.H.S. $=\frac{1(1+3)}{4(1+1)(1+2)}=\frac{1}{6}$

Suppose $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\cdots+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}$ for some positive integer k .

When $n = k + 1$, L.H.S. $=\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\cdots+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$

$=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$

R.H.S. $=\frac{(k+1)(k+4)}{4(k+2)(k+3)}=\frac{(k+1)(k^2+5k+4)}{4(k+1)(k+2)(k+3)}=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$

$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive n .

12. **Additional Mathematics 1974 (中文版) Syllabus A Paper 1 Q4**

(a) 試用數學歸納法證明

$$\frac{1}{1\cdot 3\cdot 5}+\frac{1}{3\cdot 5\cdot 7}+\frac{1}{5\cdot 7\cdot 9}+\cdots+\frac{1}{(2n-1)(2n+1)(2n+3)}\equiv\frac{1}{4}\left[\frac{1}{3}-\frac{1}{(2n+1)(2n+3)}\right]$$

(a) Let $P(n) \equiv$ "$\frac{1}{1\cdot 3\cdot 5}+\frac{1}{3\cdot 5\cdot 7}+\frac{1}{5\cdot 7\cdot 9}+\cdots+\frac{1}{(2n-1)(2n+1)(2n+3)}\equiv\frac{1}{4}\left[\frac{1}{3}-\frac{1}{(2n+1)(2n+3)}\right]$"

$n = 1$, L.H.S. $=\frac{1}{1\cdot 3\cdot 5}=\frac{1}{15}$

R.H.S. $=\frac{1}{4}\left[\frac{1}{3}-\frac{1}{(2+1)(2+3)}\right]=\frac{1}{4}\times\left(\frac{1}{3}-\frac{1}{15}\right)=\frac{1}{15}$

L.H.S. = R.H.S.

$P(1)$ is true

Suppose $P(k)$ is true for some positive integer k .

i.e. $\frac{1}{1\cdot 3\cdot 5}+\frac{1}{3\cdot 5\cdot 7}+\frac{1}{5\cdot 7\cdot 9}+\cdots+\frac{1}{(2k-1)(2k+1)(2k+3)}=\frac{1}{4}\left[\frac{1}{3}-\frac{1}{(2k+1)(2k+3)}\right]$

Add $\frac{1}{(2k+1)(2k+3)(2k+5)}$ to both sides.

$$\begin{aligned}
\text{L.H.S.} &= \frac{1}{1\cdot 3\cdot 5}+\frac{1}{3\cdot 5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)(2k+5)}\\
&=\frac{1}{4}\left[\frac{1}{3}-\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{(2k+1)(2k+3)(2k+5)}\\
&=\frac{1}{4}\cdot\frac{1}{3}-\frac{1}{4(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)(2k+5)}\\
&=\frac{1}{4}\cdot\frac{1}{3}-\frac{(2k+5)-4}{4(2k+1)(2k+3)(2k+5)}\\
&=\frac{1}{4}\cdot\frac{1}{3}-\frac{(2k+1)}{4(2k+1)(2k+3)(2k+5)}\\
&=\frac{1}{4}\cdot\frac{1}{3}-\frac{1}{4(2k+3)(2k+5)}\\
&=\frac{1}{4}\left[\frac{1}{3}-\frac{1}{(2k+3)(2k+5)}\right]=\text{R.H.S.}
\end{aligned}$$

If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .

13. Prove that $\frac{1}{1\cdot3\cdot5}+\frac{2}{3\cdot5\cdot7}+\frac{3}{5\cdot7\cdot9}+\cdots+\frac{n}{(2n-1)(2n+1)(2n+3)}=\frac{n(n+1)}{2(2n+1)(2n+3)}$ for $n = 1, 2, 3, \cdots$

Let $P(n) \equiv$ "$\frac{1}{1\cdot3\cdot5}+\frac{2}{3\cdot5\cdot7}+\frac{3}{5\cdot7\cdot9}+\cdots+\frac{n}{(2n-1)(2n+1)(2n+3)}=\frac{n(n+1)}{2(2n+1)(2n+3)}$ for $n = 1, 2, 3, \cdots$"

$n = 1$, L.H.S. $=\frac{1}{1\cdot3\cdot5}=\frac{1}{15}$; R.H.S. $=\frac{1(1+1)}{2(2+1)(2+3)}=\frac{1}{15}$. $P(1)$ is true.

Suppose $\frac{1}{1\cdot3\cdot5}+\frac{2}{3\cdot5\cdot7}+\frac{3}{5\cdot7\cdot9}+\cdots+\frac{k}{(2k-1)(2k+1)(2k+3)}=\frac{k(k+1)}{2(2k+1)(2k+3)}$ for some k.

$n = k + 1$,

$$\text{L.H.S.} =\frac{1}{1\cdot3\cdot5}+\frac{2}{3\cdot5\cdot7}+\frac{3}{5\cdot7\cdot9}+\cdots+\frac{k}{(2k-1)(2k+1)(2k+3)}+\frac{k+1}{(2k+1)(2k+3)(2k+5)}$$

$$=\frac{k(k+1)}{2(2k+1)(2k+3)}+\frac{k+1}{(2k+1)(2k+3)(2k+5)}$$

$$=\frac{k(k+1)(2k+5)+2(k+1)}{2(2k+1)(2k+3)(2k+5)}$$

$$=\frac{(k+1)(2k^2+5k+2)}{2(2k+1)(2k+3)(2k+5)}$$

$$=\frac{(k+1)(2k+1)(k+2)}{2(2k+1)(2k+3)(2k+5)}=\frac{(k+1)(k+2)}{2(2k+3)(2k+5)}=\text{R.H.S.}$$

$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive n .

13. Prove that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$ for $n = 1, 2, 3, \cdots$

Let $P(n) \equiv$ "$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$ for $n = 1, 2, 3, \cdots$"

$n = 1$, L.H.S. = $1-\frac{1}{2}=\frac{1}{2}$, R.H.S. $=\frac{1}{2}$. $P(1)$ is true.

Suppose $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}=\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}$ for some positive integer k.

When $n = k + 1$,

$$\text{L.H.S.} =1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2}$$

$$=\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2}$$

$$=\frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}=\text{R.H.S.}$$

$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive n .

14. Let $S_k = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{k-1}\left(\frac{1}{k}\right)$; $T_k = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}$, where k and n are a positive integer.

(a) Express S_{2n} in terms of n .

(b) Prove, by mathematical induction, that $S_{2n} = T_n$ for all positive integers n .

(a) $S_{2n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{2n-1}\left(\frac{1}{2n}\right)$

$S_{2n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}$

(b) Let $P(n) \equiv$ "$S_{2n} = T_n$ for all positive integers n ."

When $n = 1$, $S_2 = 1-\frac{1}{2}=\frac{1}{2}$, $T_1 = \frac{1}{2}$; $P(1)$ is true.

Suppose $P(k)$ is true.

i.e. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k} = \frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{k+k}$

Add $\frac{1}{2k+1}-\frac{1}{2k+2}$ to both sides.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2}$$

$$=\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{k+k}+\frac{1}{2k+1}-\frac{1}{2k+2}$$

$$=\frac{2}{2k+2}+\frac{1}{k+2}+\cdots+\frac{1}{k+k}+\frac{1}{2k+1}-\frac{1}{2k+2}$$

$$=\frac{1}{k+2}+\cdots+\frac{1}{k+k}+\frac{1}{2k+1}+\frac{1}{2k+2}$$

If $P(k)$ is true, then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n .

15. Prove that $1+\frac{2^2}{1+2\cdot 2^2}+\frac{3^2}{(1+2\cdot 2^2)(1+2\cdot 3^2)}+\cdots+\text{n}^{\text{th}}\text{ term}=\frac{1}{2}\left(3-\frac{\text{n}^{\text{th}}\text{ term}}{\text{n}^2}\right)$ for n = 1, 2, ⋯

Let $P(n)$≡"$1+\frac{2^2}{1+2\cdot 2^2}+\frac{3^2}{(1+2\cdot 2^2)(1+2\cdot 3^2)}+\cdots+\text{n}^{\text{th}}\text{ term}=\frac{1}{2}\left(3-\frac{\text{n}^{\text{th}}\text{ term}}{\text{n}^2}\right)$ for n = 1, 2, ⋯"

n = 1, L.H.S. =1, R.H.S. $=\frac{1}{2}\left(3-\frac{1}{1^2}\right)$= 1, P(1) is true.

Note that the n^{th} term is $\frac{k^2}{(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)}$

Assume $1+\frac{2^2}{1+2\cdot 2^2}+\frac{3^2}{(1+2\cdot 2^2)(1+2\cdot 3^2)}+\cdots+\frac{k^2}{(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)}=\frac{1}{2}\left[3-\frac{1}{(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)}\right]$

$n = k + 1$,

L.H.S. $=1+\frac{2^2}{1+2\cdot 2^2}+\frac{3^2}{(1+2\cdot 2^2)(1+2\cdot 3^2)}+\cdots+\frac{k^2}{(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)}+\frac{(k+1)^2}{(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)\left[1+2\cdot(k+1)^2\right]}$

$=\frac{1}{2}\left[3-\frac{1}{(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)}\right]+\frac{(k+1)^2}{(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)\left[1+2\cdot(k+1)^2\right]}$

$=\frac{3}{2}+\frac{(k+1)^2}{(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)\left[1+2\cdot(k+1)^2\right]}-\frac{1}{2(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)}$

$=\frac{3}{2}+\frac{2(k+1)^2-\left[1+2\cdot(k+1)^2\right]}{2(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)\left[1+2\cdot(k+1)^2\right]}$

$=\frac{1}{2}\left\{3-\frac{1}{(1+2\cdot 2^2)(1+2\cdot 3^2)\cdots(1+2\cdot k^2)\left[1+2\cdot(k+1)^2\right]}\right\}$= R.H.S.

∴ If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive n .

16. Prove that $\frac{1}{3\cdot7\cdot11}+\frac{1}{7\cdot11\cdot15}+\frac{1}{11\cdot15\cdot19}+\cdots+\frac{1}{(4n-1)(4n+3)(4n+7)}=\frac{1}{8}\left[\frac{1}{3\cdot7}-\frac{1}{(4n+3)(4n+7)}\right]$ for $n=1,2,\cdots$

$n = 1$, L.H.S. $=\frac{1}{3\cdot7\cdot11}=\frac{1}{231}$

R.H.S. $=\frac{1}{8}\left[\frac{1}{3\cdot7}-\frac{1}{(4+3)(4+7)}\right]=\frac{1}{231}$

It is true for $n = 1$

Suppose $\frac{1}{3\cdot7\cdot11}+\frac{1}{7\cdot11\cdot15}+\frac{1}{11\cdot15\cdot19}+\cdots+\frac{1}{(4k-1)(4k+3)(4k+7)}=\frac{1}{8}\left[\frac{1}{3\cdot7}-\frac{1}{(4k+3)(4k+7)}\right]$

for some positive integer $k = 1, 2, \cdots$

$n = k + 1$,

L.H.S. $=\frac{1}{3\cdot7\cdot11}+\frac{1}{7\cdot11\cdot15}+\frac{1}{11\cdot15\cdot19}+\cdots+\frac{1}{(4k-1)(4k+3)(4k+7)}+\frac{1}{(4k+3)(4k+7)(4k+11)}$

$=\frac{1}{8}\left[\frac{1}{3\cdot7}-\frac{1}{(4k+3)(4k+7)}\right]+\frac{1}{(4k+3)(4k+7)(4k+11)}$ by induction assumption

$=\frac{1}{8\times3\times7}-\frac{1}{8(4k+3)(4k+7)}+\frac{1}{(4k+3)(4k+7)(4k+11)}$

$=\frac{1}{8\times3\times7}+\frac{8-(4k+11)}{8(4k+3)(4k+7)(4k+11)}$

$=\frac{1}{8\times3\times7}-\frac{4k+3}{8(4k+3)(4k+7)(4k+11)}=\frac{1}{8}\left[\frac{1}{3\cdot7}-\frac{1}{(4k+7)(4k+11)}\right]$ = R.H.S.

If P(k) is true then P(k + 1) is also true.

By the principle of mathematical induction, P(n) is true for all positive integer n .

17. Prove that $\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)(n+3)}=\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$ for $n = 1, 2, \cdots$

Hence, or otherwise, find the sum of $\frac{1}{3\cdot 4\cdot 5\cdot 6}+\frac{1}{4\cdot 5\cdot 6\cdot 7}+\cdots+\frac{1}{99\cdot 100\cdot 101\cdot 102}$.

Let $P(n) \equiv$ "$\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)(n+3)}=\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$ for $n = 1, 2, \cdots$"

When $n = 1$, L.H.S. $=\frac{1}{1\cdot 2\cdot 3\cdot 4}=\frac{1}{24}$

R.H.S. $=\frac{1}{18}-\frac{1}{3(2)(3)(4)}=\frac{1}{18}-\frac{1}{72}=\frac{1}{24}=$ L.H.S.

$\therefore$ P(1) is true

Assume P(k) is true for some positive integers k .

i.e. $\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\cdots+\frac{1}{k(k+1)(k+2)(k+3)}=\frac{1}{18}-\frac{1}{3(k+1)(k+2)(k+3)}$

When $n = k + 1$,

L.H.S. $=\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\cdots+\frac{1}{k(k+1)(k+2)(k+3)}+\frac{1}{(k+1)(k+2)(k+3)(k+4)}$

$=\frac{1}{18}-\frac{1}{3(k+1)(k+2)(k+3)}+\frac{1}{(k+1)(k+2)(k+3)(k+4)}$ (induction assumption)

$=\frac{1}{18}-\frac{k+4-3}{3(k+1)(k+2)(k+3)(k+4)}$

$=\frac{1}{18}-\frac{k+1}{3(k+1)(k+2)(k+3)(k+4)}$

$=\frac{1}{18}-\frac{1}{3(k+2)(k+3)(k+4)}=$ R.H.S.

$\therefore$ If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive n.

$$\frac{1}{3\cdot 4\cdot 5\cdot 6}+\frac{1}{4\cdot 5\cdot 6\cdot 7}+\cdots+\frac{1}{99\cdot 100\cdot 101\cdot 102}$$

$$=\frac{1}{18}-\frac{1}{3(100)(101)(102)}-\frac{1}{18}+\frac{1}{3(3)(4)(5)}$$

$$=-\frac{1}{3(3)(4)(5)(5)(101)(34)}+\frac{1}{3(3)(4)(5)}=\frac{-1+5\times 101\times 34}{3(3)(4)(5)(5)(101)(34)}=\frac{17169}{3090600}=\frac{5723}{1030200}$$

18. **2005 Q8**

Prove, by mathematical induction, that $\frac{1\times 2}{2\times 3}+\frac{2\times 2^2}{3\times 4}+\frac{3\times 2^3}{4\times 5}+\cdots+\frac{n\times 2^n}{(n+1)\times(n+2)}=\frac{2^{n+1}}{n+2}-1$ for all positive integers n.

$n = 1$, L.H.S. = $\frac{1\times 2}{2\times 3}=\frac{1}{3}$, R.H.S. $=\frac{2^2}{3}-1=\frac{1}{3}$

It is true for $n = 1$

Suppose it is true for $n = k$.

i.e. $\frac{1\times 2}{2\times 3}+\frac{2\times 2^2}{3\times 4}+\frac{3\times 2^3}{4\times 5}+\cdots+\frac{k\times 2^k}{(k+1)\times(k+2)}=\frac{2^{k+1}}{k+2}-1$

Add $\frac{(k+1)\times 2^{k+1}}{(k+2)\times(k+3)}$ to both sides.

$$\frac{1\times 2}{2\times 3}+\frac{2\times 2^2}{3\times 4}+\frac{3\times 2^3}{4\times 5}+\cdots+\frac{k\times 2^k}{(k+1)\times(k+2)}+\frac{(k+1)\times 2^{k+1}}{(k+2)\times(k+3)}$$

$$=\frac{(k+1)\times 2^{k+1}}{(k+2)\times(k+3)}+\frac{2^{k+1}}{k+2}-1$$

$$=\frac{(k+1)\times 2^{k+1}+(k+3)\times 2^{k+1}}{(k+2)\times(k+3)}-1$$

$$=\frac{(2k+4)\times 2^{k+1}}{(k+2)\times(k+3)}-1$$

$$=\frac{(k+2)\times 2^{k+2}}{(k+2)\times(k+3)}-1$$

$$=\frac{2^{k+2}}{k+3}-1=\text{R.H.S.}$$

It is also true for $n = k + 1$

By the principle of mathematical induction, it is true for all positive integer n .

19. **2019 Q5**

(a) Using mathematical induction, prove that $\sum_{k=n}^{2n}\frac{1}{k(k+1)}=\frac{n+1}{n(2n+1)}$ for all positive integer n.

(b) Using (a), evaluate $\sum_{k=50}^{200}\frac{1}{k(k+1)}$.

(a) Let P(n) ≡"$\sum_{k=n}^{2n}\frac{1}{k(k+1)}=\frac{n+1}{n(2n+1)}$ for all positive integer n."

$n = 1$, L.H.S. $=\sum_{k=1}^{2}\frac{1}{k(k+1)}=\frac{1}{1\times2}+\frac{1}{2\times3}=\frac{2}{3}$

R.H.S. $=\frac{1+1}{1(2+1)}=\frac{2}{3}$

$\therefore$ L.H.S. = R.H.S.

P(1) is true

Suppose P(m) is true for some positive integer m.

i.e. $\sum_{k=m}^{2m}\frac{1}{k(k+1)}=\frac{m+1}{m(2m+1)}$ …… (*)

When $n = m + 1$,

$$\text{L.H.S.} = \sum_{k=m+1}^{2(m+1)}\frac{1}{k(k+1)}$$

$$=\sum_{k=m}^{2m}\frac{1}{k(k+1)}+\frac{1}{(2m+1)(2m+2)}+\frac{1}{(2m+2)(2m+3)}-\frac{1}{m(m+1)}$$

$$=\frac{m+1}{m(2m+1)}+\frac{1}{2(2m+1)(m+1)}+\frac{1}{2(m+1)(2m+3)}-\frac{1}{m(m+1)} \quad \text{by (*)}$$

$$=\frac{2(m+1)^2(2m+3)+m(2m+3)+m(2m+1)-2(2m+1)(2m+3)}{2m(m+1)(2m+1)(2m+3)}$$

$$=\frac{2(m^2+2m+1-2m-1)(2m+3)+m(4m+4)}{2m(m+1)(2m+1)(2m+3)}$$

$$=\frac{m^2(2m+3)+2m(m+1)}{m(m+1)(2m+1)(2m+3)}$$

$$=\frac{m(2m+3)+2(m+1)}{(m+1)(2m+1)(2m+3)}$$

$$=\frac{2m^2+5m+2}{(m+1)(2m+1)(2m+3)}=\frac{(2m+1)(m+2)}{(m+1)(2m+1)(2m+3)}$$

$$=\frac{m+1+1}{(m+1)[2(m+1)+1]}=\text{R.H.S.}$$

If P(m) is true then P(m + 1) is also true.

By the principal of mathematical induction, P(n) is true for all positive integer n.

Method 2 Note that $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$; $\frac{m+1}{m(2m+1)}\equiv\frac{A}{m}+\frac{B}{2m+1}$

$m + 1 \equiv A(2m + 1) + Bm$

$A = 1$, $B = -1$

When $n = m + 1$,

$$\text{L.H.S.} = \sum_{k=m+1}^{2(m+1)} \frac{1}{k(k+1)}$$

$$= \sum_{k=m}^{2m} \frac{1}{k(k+1)} + \frac{1}{(2m+1)(2m+2)} + \frac{1}{(2m+2)(2m+3)} - \frac{1}{m(m+1)}$$

$$= \frac{m+1}{m(2m+1)} + \frac{1}{(2m+1)(2m+2)} + \frac{1}{(2m+2)(2m+3)} - \frac{1}{m(m+1)} \quad \text{by } (*)$$

$$= \frac{1}{m} - \frac{1}{2m+1} + \frac{1}{2m+1} - \frac{1}{2m+2} + \frac{1}{2m+2} - \frac{1}{2m+3} - \frac{1}{m} + \frac{1}{m+1}$$

$$= -\frac{1}{2m+3} + \frac{1}{m+1}$$

$$= \frac{m+2}{(m+1)(2m+3)} = \frac{m+1+1}{(m+1)\left[2(m+1)+1\right]} = \text{R.H.S.}$$

(b) $$\sum_{k=50}^{200} \frac{1}{k(k+1)} = \sum_{k=50}^{100} \frac{1}{k(k+1)} + \sum_{k=100}^{200} \frac{1}{k(k+1)} - \frac{1}{100\times 101}$$

$$= \frac{50+1}{50(2\times 50+1)} + \frac{100+1}{100(2\times 100+1)} - \frac{1}{100\times 101} \quad \text{by (a)}$$

$$= \frac{51}{50\times 101} + \frac{101}{100\times 201} - \frac{1}{100\times 101}$$

$$= \frac{151}{10050}$$

20. **1968 香港中文中學會考高級數學試卷一 Q3(b)**

(b) 寫出下列級數 $\frac{5}{1\times 2\times 3}+\frac{6}{2\times 3\times 4}+\frac{7}{3\times 4\times 5}+\cdots$ 之第 n 項，並以數學歸納法證明首 n 項之和為 $\frac{n(3n+7)}{2(n+1)(n+2)}$ 。

(b) $\frac{5}{1\times 2\times 3}+\frac{6}{2\times 3\times 4}+\frac{7}{3\times 4\times 5}+\cdots$, the nth term $=\frac{n+4}{n(n+1)(n+2)}$.

Let $P(n) \equiv$ "$\frac{5}{1\times 2\times 3}+\frac{6}{2\times 3\times 4}+\frac{7}{3\times 4\times 5}+\cdots+\frac{n+4}{n(n+1)(n+2)}=\frac{n(3n+7)}{2(n+1)(n+2)}$, $n = 1, 2, \cdots$"

$n = 1$, L.H.S. $=\frac{5}{1\times 2\times 3}=\frac{5}{6}$, R.H.S. $=\frac{1(3+7)}{2(1+1)(1+2)}=\frac{5}{6}$, $P(1)$ is true

Suppose $\frac{5}{1\times 2\times 3}+\frac{6}{2\times 3\times 4}+\frac{7}{3\times 4\times 5}+\cdots+\frac{k+4}{k(k+1)(k+2)}=\frac{k(3k+7)}{2(k+1)(k+2)}$ for some $k \geq 1$

$n = k + 1$,

$$\text{L.H.S.}=\frac{5}{1\times 2\times 3}+\frac{6}{2\times 3\times 4}+\cdots+\frac{k+4}{k(k+1)(k+2)}+\frac{k+5}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(3k+7)}{2(k+1)(k+2)}+\frac{k+5}{(k+1)(k+2)(k+3)} \quad \text{induction assumption}$$

$$=\frac{(k^2+3k)(3k+7)}{2(k+1)(k+2)(k+3)}+\frac{2(k+5)}{2(k+1)(k+2)(k+3)}$$

$$=\frac{3k^3+16k^2+23k+10}{2(k+1)(k+2)(k+3)}=\frac{(k+1)(k+1)(3k+10)}{2(k+1)(k+2)(k+3)}$$

$$\text{R.H.S.}=\frac{(k+1)(3k+10)}{2(k+2)(k+3)}=\text{L.H.S.}$$

If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all positive integer n .