Numbers divisible by 7

Created by Mr. Francis Hung on 20151228 Last updated: 19 January 2016

Let a be a positive integer. If $a = 10x + y$, where y is the unit digit of a.

The necessary and sufficient condition for a to be divisible by 7 is $(x - 2y)$ is divisible by 7.

($\Rightarrow$) $a = 7m$, where m is a positive integer

$10x + y = 7m$

$7(x + y) + 3(x - 2y) = 7m$

$3(x - 2y) = 7(m - x - y)$

3 and 7 are relatively prime

$\therefore$ $x - 2y$ is divisible by 7

($\Leftarrow$) $x - 2y = 7k$, where k is an integer

$a = 10x + y$

$= 7(x + y) + 3(x - 2y)$

$= 7(x + y) + 3\times 7k$

$= 7(x + y + 3k)$

$\therefore$ a is divisible by 7

Example 1 Determine whether 98 is divisible by 7.

$98 = 10\times 9 + 8$

$x = 9$, $y = 8$

$x - 2y = 9 - 2\times 8 = -7$, which is divisible by 7

$\therefore$ 98 is divisible by 7

Example 2 Determine whether 12899 is divisible by 7.

$x = 1289$, $y = 9$

$x - 2y = 1289 - 2\times 9 = 1271$

Let $x_1 = 127$, $y_1 = 1$

$x_1 - 2y_1 = 127 - 2\times 1 = 125$

Let $x_2 = 12$, $y_2 = 5$

$x_1 - 2y_1 = 12 - 10 = 2$, which is not divisible by 7

$\therefore$ 12899 is not divisible by 7

Numbers divisible by 7, 11, 13

Let a be a positive integer. If $a=1000x+y$, where y is the number formed by the last three digits of a.
$1001 = 7\times11\times13 = 77\times13 = 7\times143 = 91\times11$. The positive factors are 1, 7, 11, 13, 77, 91, 143, 1001.
Let p and q be two positive factors of 1001 such that $pq = 1001$
The necessary and sufficient condition for a to be divisible by p is $(y - x)$ is divisible by p.

($\Rightarrow$) $a = pm$, where m is a positive integer

$1000x + y = pm$

$1001x + (y - x) = pm$

$pqx + (y - x) = pm$

$(y - x) = p(m - qx)$

$\therefore$ $y - x$ is divisible by p

($\Leftarrow$) $y - x = ps$, where s is an integer

$a = 1000x + y$

$= 1001x + (y - x)$

$= pqx + ps$

$= p(qx - s)$

$\therefore$ a is divisible by p

In other words, the remainder when $1000x + y$ is divided by p is $y - x$.

Example 3 Find the remainder when 123456789 is divided by 143.
We separate the first six-digits of 123456789 as $x_1 = 123$, $y_1 = 456$
$y_1 - x_1 = 456 - 123 = 333 \equiv 47 \pmod{143}$
Consider 47789. Let $x_2 = 47$, $y_2 = 789$
$y_2 - x_2 = 789 - 47 = 27 \pmod{143}$
The remainder is 27.

Example 4 Let n be a positive integer and $a = \underbrace{\overline{20152015\cdots2015}}_{n \text{ copies of } 2015}$. Find the least possible value of n such that a is divisible by 7.
Consider the first six-digit of a, $x_1 = 20152$, $y_1 = 015$
$y_1 - x_1 = 15 - 20152 = -20137$, $137 - 20 = 117 = 7\times16 + 5$, $y_1 - x_1 \equiv 2 \pmod 7$
$x_2 = 220152$, $y_2 = 015$, $y_2 - x_2 = 15 - 220152 = -220137$, $220 - 137 = 83 = 7\times11+6$, $220152 \equiv 6 \pmod 7$
$x_3 = 620152$, $y_3 = 015$, $y_3 - x_3 = 15 - 60152 = -60137$, $137 - 60 = 77 = 7\times11$, $620152 \equiv 0 \pmod 7$
The least value of $n = 3$.