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試題號數
Question No. ....

本頁積分
Page Total
17a

Vector Space / Linear Space

let n be a positive integer.

$\mathbb{R}^n$ = the set of all ordered n-tuples
$= \{ (a_1, a_2, \dots, a_n) : a_i \in \mathbb{R}, i = 1, 2, \dots, n \}$.

called the n-space, elements in $\mathbb{R}^n$ are called vectors.

$\forall \underline{u}, \underline{v} \in \mathbb{R}^n$ $\quad \underline{u} = (a_1, a_2, \dots, a_n)$, $\underline{v} = (b_1, \dots, b_n)$

$\underline{u} = \underline{v}$ if $a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$
(equality of two vectors)

<u>Scalar multiplication of vector</u>

let $k \in \mathbb{R}$ $\quad \underline{u} = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$.

$k\underline{u} = (ku_1, \dots, ku_n)$ is the scalar multiple of $\underline{u}$

$\underline{v} = (v_1, v_2, \dots, v_n)$

$\underline{u} + \underline{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$
(sum of two vectors)

The zero vector in $\mathbb{R}^n$ is $\underline{0} = (0, 0, \dots, 0)$

<u>Some properties of vector space</u>

let $\underline{u} = (u_1, \dots, u_n)$, $\underline{v} = (v_1, \dots, v_n)$, $\underline{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$, $k, l \in \mathbb{R}$

then [A1] $\quad \underline{u} + \underline{v} = \underline{v} + \underline{u}$

[A2] $\quad [\underline{u} + \underline{v}] + \underline{w} = \underline{u} + [\underline{v} + \underline{w}]$

[A3] $\quad \underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$

[A4] $\quad \underline{u} + (-\underline{u}) = \underline{0}$

[M1] $\quad k(l\,\underline{u}) = (kl)\underline{u}$

[M2] $\quad k(\underline{u} + \underline{v}) = k\underline{u} + k\underline{v}$, $\quad (k+l)\underline{u} = k\underline{u} + l\underline{u}$

[M3] $\quad 1 \cdot \underline{u} = \underline{u}$

Questions a) What geometrical interpertation do you discover when $\mathbb{R}^n = \mathbb{R}^2$ or $\mathbb{R}^3$.

b) If $k\,\underline{u} = \underline{0}$ What conclusion will you get?

For subspace of vector space see supplementary note P.33, 34

A vector $\underline{w}$ is called a linear combination of the vectors $V_1, V_2, \ldots, V_r$ if it can be expressed in the form $W = k_1V_1 + k_2V_2 + \cdots + k_rV_r$, $k_1, k_2, \ldots, k_r \in \mathbb{R}$

eg $u = (1, 2, -1)$ $V = (6, 4, 2)$ Show that $W = (9, 2, 7)$ is a linear combination of u and V and that $W' = (4, -1, 8)$ is not a linear combination of u and V

$$W = k_1 u + k_2 V$$
$$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2)$$
$$(9, 2, 7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

equating components $k_1 + 6k_2 = 9$
$2k_1 + 4k_2 = 2$
$-k_1 + 2k_2 = 7$

Solving yields $k_1 = -3$, $k_2 = 2$ $\therefore W = -3u + 2V$

suppose $w' = k_1 u + k_2 V$
$$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2)$$

equating components
$$\begin{cases} k_1 + 6k_2 = 4 \\ 2k_1 + 4k_2 = -1 \\ -k_1 + 2k_2 = 8 \end{cases}$$

The system is inconsistent $\therefore k_1, k_2$ does not exist. W' is not a linear combination of u and V.

Definition If $v_1, v_2, \ldots, v_r \in V$ (vector space)

$W = \{k_1v_1 + k_2v_2 + \cdots + k_rv_r : k_1, k_2, \ldots, k_r \in \mathbb{R}\}$.

is the span of $\{v_1, \ldots, v_r\}$.

(or the linear space spanned by $\{v_1, \ldots, v_r\}$)

we say $\{v_1, v_2, \ldots, v_r\}$ spans W

we may also write $W = \langle v_1, v_2, \ldots, v_r \rangle$

$= \text{lin}\{v_1, v_2, \ldots, v_r\}$

eg. $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1) \in \mathbb{R}^3$.

then $\{i, j, k\}$ spans $\mathbb{R}^3$ (or $\mathbb{R}^3 = \langle i, j, k \rangle$).

because $\forall (a,b,c) \in \mathbb{R}^3$ $(a,b,c) = ai + bj + ck$.

which is a linear combination of i, j and k.

eg P_n = the vector space of all polynomial of degree $\leq n$

then $\{1, x, x^2, \ldots, x^n\}$ spans P_n

since each polynomial $p \in P_n$ can be written as.

$p = a_0 + a_1x + \cdots + a_nx^n$

which is a linear combination of $1, x, x^2, \ldots, x^n$

eg Determine if $v_1 = (1,1,2)$, $v_2 = (1,0,1)$ and $v_3 = (2,1,3)$ span $\mathbb{R}^3$.

$\forall$ $b = (b_1, b_2, b_3) \in \mathbb{R}^3$

suppose $(b_1, b_2, b_3) = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3)$

$$\Rightarrow \begin{cases} k_1 + k_2 + 2k_3 = b_1 \\ k_1 + k_3 = b_2 \\ 2k_1 + k_2 + 3k_3 = b_3 \end{cases}$$

Coefficient matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}$

but $|A| = 0$ $\therefore$ A^{-1} does not exist.

試題號數
Question No. ....

本頁積分
Page Total 18b

$\therefore$ the matrix equation $A\underline{k} = b$ does not always has a solution.

$\therefore$ The system is inconsistent for non-zero vectors $b \in \mathbb{R}^3$.

$\therefore$ Not every vector $b \in \mathbb{R}^3$ can be expressed as a linear combination of v_1, v_2, v_3

$\therefore$ $\{v_1, v_2, v_3\}$ does not span $\mathbb{R}^3$

Theorem If $v_1, v_2, \ldots, v_r$ are vectors in V then

a) $W = \langle v_1, v_2, \ldots, v_r \rangle$ is a subspace of V.

b) W is the smallest subspace of V that contains $v_1, v_2, \ldots, v_r$ in the sense that every other subspace of V that contains $v_1, v_2, \ldots, v_r$ must contain W.

pf: a) let $u, v \in W$, $\alpha, \beta \in \mathbb{R}$

then $u = c_1 v_1 + c_2 v_2 + \ldots + c_r v_r$

$v = k_1 v_1 + k_2 v_2 + \ldots + k_r v_r$

$\alpha u + \beta v = (\alpha c_1 + \beta k_1) v_1 + (\alpha c_2 + \beta k_2) v_2 + \ldots + (\alpha c_r + \beta k_r) v_r \in W$

since it is a linear combination of $v_1, v_2, \ldots, v_r$

b) $\forall i = 1, 2, \ldots, r$ $v_i = 0v_1 + 0v_2 + \ldots + 1v_i + \ldots + 0v_r$

linear combination of $v_1, \ldots, v_r$

$\therefore$ $v_1, v_2, \ldots, v_r \in W$ $\forall i = 1, 2, \ldots, r$.

let W' be any other subspace that contains $v_1, v_2, \ldots, v_r$

by the definition of subspace.

$\forall u = c_1 v_1 + c_2 v_2 + \ldots + c_r v_r \in W$

$\Rightarrow u \in W'$ ($\because$ $v_1, v_2, \ldots, v_r \in W'$ and W' is a subspace).

If $S = \{v_1, v_2, \ldots, v_r\}$ is the set of vectors, then then the vector equation $k_1 v_1 + k_2 v_2 + \ldots + k_r v_r = 0$.
has at least one solution, namely

$$k_1 = 0, \quad k_2 = 0, \ldots, k_r = 0$$

If this is the only solution, then S is called <u>linear independent set</u>
If there are other solutions, then S is called a <u>linear dependent set.</u>

eg $v_1 = (2, -1, 0, 3)$, $v_2 = (1, 2, 5, -1)$, $v_3 = (7, -1, 5, 8)$.
$S = \{v_1, v_2, v_3\}$.
then S is linear dependent.
because $3v_1 + v_2 - v_3 = 0$

eg P_2 = vector space of real polynomial degree ≤ 2.
$p_1 = 1 - x$, $p_2 = 5 + 3x - 2x^2$, $p_3 = 1 + 3x - x^2$.
$S = \{p_1, p_2, p_3\}$ is a <u>linear dependent set</u> in P_2.
because $3p_1 - p_2 + 2p_3 = 0$

eg $i = (1, 0, 0)$, $j = (0, 1, 0)$ and $k = (0, 0, 1) \in \mathbb{R}^3$.
suppose $k_1 i + k_2 j + k_3 k = \underline{0}$ (vector equation)

$$k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0)$$
$$(k_1, k_2, k_3) = (0, 0, 0)$$
$$\therefore \quad k_1 = 0 \quad k_2 = 0, \quad k_3 = 0$$

$\therefore$ $S = \{i, j, k\}$ is linear independent

eg $v_1 = (1, -2, 3)$ $v_2 = (5, 6, -1)$ $v_3 = (3, 2, 1)$
suppose $k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$.

$$(k_1 + 5k_2 + 3k_3, -2k_1 + 6k_2 + 2k_3, 3k_1 - k_2 + k_3) = (0, 0, 0)$$

$$\begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases}$$

Solving, we get $k_1 = -\frac{1}{2}t$
$k_2 = -\frac{1}{2}t$, $t \in \mathbb{R}$
$k_3 = t$.

$\therefore$ V_1, V_2 and V_3 form a linear dependent set

Characterization of independent vectors in $\mathbb{R}^3$,

let $V_1, V_2, \ldots, V_n$ be vectors in $\mathbb{R}^3$

a) For $n=1$.

$\underline{x}_1$ is independent iff $\underline{x}_1 \neq \underline{0}$.

eg $k(1,2,3) = (0,0,0) \Rightarrow k=0$ only one solution

$\therefore$ $(1,2,3)$ is independent.

b) For $n=2$,

$\underline{x}_1, \underline{x}_2$ are independent iff $\underline{x}_1, \underline{x}_2$ non-zero and non-parallel.

For $\underline{x}_1 = (a_1, a_2)$, $\underline{x}_2 = (b_1, b_2) \in \mathbb{R}^2$;

$\underline{x}_1, \underline{x}_2$ are independent iff $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$.

iff every $\underline{x} \in \mathbb{R}^2$ can be expressed as a linear combination of $\underline{x}_1$ and $\underline{x}_2$.

pf: $k_1 \underline{x}_1 + k_2 \underline{x}_2 = \underline{0}$

$$\Leftrightarrow \begin{cases} k_1 a_1 + k_2 b_1 = 0 \\ k_1 a_2 + k_2 b_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

it has a non-zero solution $\Leftrightarrow \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$.

$\therefore$ $\underline{x}_1, \underline{x}_2$ are independent iff $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

c) For $n=3$,

$\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3$ are independent iff $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3$ are non-zero and no one of them lie on the plane of the other two.

For $\underset{\sim}{x}_1 = (a_1, a_2, a_3)$, $\underset{\sim}{x}_2 = (b_1, b_2, b_3)$, $\underset{\sim}{x}_3 = (c_1, c_2, c_3) \in R$,

$\therefore$ $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3$ are independent iff $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

iff every $\underset{\sim}{x} \in \mathbb{R}^3$ can be expressed as a linear combination of $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ and $\underset{\sim}{x}_3$.

pf: Suppose $k_1\underset{\sim}{x}_1 + k_2\underset{\sim}{x}_2 + k_3\underset{\sim}{x}_3 = \underset{\sim}{0}$ (vector equation)

$$\therefore \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The system has a non-zero solution iff $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

d) For $n \geq 4$.

Always dependent.

pf: $\underset{\sim}{x}_1 = (a_1, a_2, a_3)$ $\underset{\sim}{x}_2 = (b_1, b_2, b_3)$, $\underset{\sim}{x}_3 = (c_1, c_2, c_3)$

$\underset{\sim}{x}_4 = (d_1, d_2, d_3) \in \mathbb{R}^3$

Suppose $k_1\underset{\sim}{x}_1 + k_2\underset{\sim}{x}_2 + k_3\underset{\sim}{x}_3 + k_4\underset{\sim}{x}_4 = \underset{\sim}{0}$

$$\therefore \begin{cases} k_1 a_1 + k_2 b_1 + k_3 c_1 + k_4 d_1 = 0 \\ k_1 a_2 + k_2 b_2 + k_3 c_2 + k_4 d_2 = 0 \\ k_1 a_3 + k_2 b_3 + k_3 c_3 + k_4 d_3 = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since the number of equations is less than the number of variables $\therefore$ it has a non-zero solution

Definition If V is any vector space $S=\{v_1, v_2, \dots, v_r\}$ is a finite set of vectors in V, then S is called a <u>basis</u> of V if

(i) S is linear independent.

(ii) S spans V.

eg $e_1=(1,0,0,\dots,0), e_2=(0,1,0,\dots,0), \dots, e_n=(0,0,\dots,0,1)$

$S=\{e_1, e_2, \dots, e_n\}$ is linear independent.

and $\forall\, v=(v_1, v_2, \dots, v_n) \in \mathbb{R}^n$

$$v = v_1e_1+v_2e_2+\dots+v_ne_n \in \langle S\rangle$$

$\therefore$ S spans $\mathbb{R}^n$.

$\therefore$ S is a <u>standard basis</u> for $\mathbb{R}^n$

eg $v_1=(1,2,1)$, $v_2=(2,9,0)$, and $v_3=(3,3,4)$

$S=\{v_1, v_2, v_3\}$ Show that S is a basis for $\mathbb{R}^3$.

pf: $\forall\ b=(b_1,b_2,b_3) \in \mathbb{R}^3$

suppose $b = k_1v_1+k_2v_2+k_3v_3$

$$\begin{aligned} k_1+2k_2+3k_3 &= b_1 \\ 2k_1+9k_2+3k_3 &= b_2 \\ k_1 \qquad +4k_3 &= b_3 \end{aligned}$$

coefficient matrix
$$A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{pmatrix} \qquad |A| = -1 \neq 0$$

$\therefore$ the system has unique solution

$\therefore$ b can be expressed as a linear combination of v_1, v_2, v_3, namely $\underline{b} = k_1\underline{v_1}+k_2v_2+k_3v_3$.

Suppose $k_1v_1+k_2v_2+k_3v_3=0$

giving the same coefficient matrix A.

As $|A|\neq 0$ $\therefore$ $k_1=0$, $k_2=0$ and $k_3=0$ is the only solution $\therefore$ S is linear independent and spans $\mathbb{R}^3$