

# Complex Number Hard example

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(a) Solve  $z^5 - 1 = 0$

(b) Hence, or otherwise, show that

$$(y+1)^5 - (y-1)^5 = 10 \left( y^2 + \cot^2 \frac{\pi}{5} \right) \left( y^2 + \cot^2 \frac{2\pi}{5} \right)$$

(c) Hence, deduce that

(i)  $\csc^2 \frac{\pi}{5} + \csc^2 \frac{2\pi}{5} = 4$

(ii)  $\tan \frac{\pi}{5} - \tan \frac{2\pi}{5} = -\sqrt{10 - 2\sqrt{5}}$

(a)  $z^5 - 1 = 0$

$$z^5 = \cos 2k\pi + i \sin 2k\pi, k = -2, -1, 0, 1, 2$$

$$z = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}, k = -2, -1, 0, 1, 2$$

$$z = 1, \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}, k = -2, -1, 1, 2$$

(b)  $z^5 - 1$

$$= (z-1) \left( z - \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} \right) \left( z - \cos \frac{2\pi}{5} - i \sin \frac{2\pi}{5} \right) \left( z - \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5} \right) \left( z - \cos \frac{4\pi}{5} - i \sin \frac{4\pi}{5} \right)$$

$$= (z-1) \left( z^2 - 2z \cos \frac{2\pi}{5} + 1 \right) \left( z^2 - 2z \cos \frac{4\pi}{5} + 1 \right) \dots (*)$$

Divide both sides by  $z-1$ :  $\frac{z^5-1}{z-1} = \left( z^2 - 2z \cos \frac{2\pi}{5} + 1 \right) \left( z^2 - 2z \cos \frac{4\pi}{5} + 1 \right)$

$$z^4 + z^3 + z^2 + z + 1 = \left( z^2 - 2z \cos \frac{2\pi}{5} + 1 \right) \left( z^2 - 2z \cos \frac{4\pi}{5} + 1 \right)$$

As  $z \rightarrow 1$ :  $\left( 1 - 2 \cos \frac{2\pi}{5} + 1 \right) \left( 1 - 2 \cos \frac{4\pi}{5} + 1 \right) = 1 + 1 + 1 + 1 + 1$

$$4 \left( 1 - \cos \frac{2\pi}{5} \right) \left( 1 - \cos \frac{4\pi}{5} \right) = 5$$

$$\left( 1 - 1 + 2 \sin^2 \frac{\pi}{5} \right) \left( 1 - 1 + 2 \sin^2 \frac{2\pi}{5} \right) = \frac{5}{4}$$

$$\left( \sin \frac{\pi}{5} \right)^2 \left( \sin \frac{2\pi}{5} \right)^2 = \frac{5}{16} \dots (**)$$

Put  $z = \frac{y+1}{y-1}$  into (\*)

$$\left( \frac{y+1}{y-1} \right)^5 - 1 = \left( \frac{y+1}{y-1} - 1 \right) \left[ \left( \frac{y+1}{y-1} \right)^2 - 2 \left( \frac{y+1}{y-1} \right) \cos \frac{2\pi}{5} + 1 \right] \left[ \left( \frac{y+1}{y-1} \right)^2 - 2 \left( \frac{y+1}{y-1} \right) \cos \frac{4\pi}{5} + 1 \right]$$

Multiply both sides by  $(y-1)^5$

$$(y+1)^5 - (y-1)^5$$

$$= (y+1 - y+1) \left[ (y+1)^2 - 2(y+1)(y-1) \cos \frac{2\pi}{5} + (y-1)^2 \right] \left[ (y+1)^2 - 2(y+1)(y-1) \cos \frac{4\pi}{5} + (y-1)^2 \right]$$

$$= 2 \left[ (y+1)^2 + (y-1)^2 - 2(y^2-1) \cos \frac{2\pi}{5} \right] \left[ (y+1)^2 + (y-1)^2 - 2(y^2-1) \cos \frac{4\pi}{5} \right]$$

$$= 2 \left[ 2y^2 + 2 - 2(y^2-1) \cos \frac{2\pi}{5} \right] \left[ 2y^2 + 2 - 2(y^2-1) \cos \frac{4\pi}{5} \right]$$

$$\begin{aligned}
&= 8 \left[ \left( 1 - \cos \frac{2\pi}{5} \right) y^2 + 1 + \cos \frac{2\pi}{5} \right] \left[ \left( 1 - \cos \frac{4\pi}{5} \right) y^2 + 1 + \cos \frac{4\pi}{5} \right] \\
&= 8 \left[ \left( 1 - 1 + 2 \sin^2 \frac{\pi}{5} \right) y^2 + 1 + 2 \cos^2 \frac{\pi}{5} - 1 \right] \left[ \left( 1 - 1 - 2 \sin^2 \frac{2\pi}{5} \right) y^2 + 1 + 2 \cos^2 \frac{2\pi}{5} - 1 \right] \\
&= 8 \left( 2y^2 \sin^2 \frac{\pi}{5} + 2 \cos^2 \frac{\pi}{5} \right) \left( 2y^2 \sin^2 \frac{2\pi}{5} + 2 \cos^2 \frac{2\pi}{5} \right) \\
&= 32 \sin^2 \frac{\pi}{5} \cdot \sin^2 \frac{2\pi}{5} \left( y^2 + \cot^2 \frac{\pi}{5} \right) \left( y^2 + \cot^2 \frac{2\pi}{5} \right) \\
&= 32 \cdot \frac{5}{16} \cdot \left( y^2 + \cot^2 \frac{\pi}{5} \right) \left( y^2 + \cot^2 \frac{2\pi}{5} \right) \text{ by the result of (**)} \\
&\therefore (y+1)^5 - (y-1)^5 = 10 \left( y^2 + \cot^2 \frac{\pi}{5} \right) \left( y^2 + \cot^2 \frac{2\pi}{5} \right)
\end{aligned}$$

$$\begin{aligned}
\text{(c) (i)} \quad &10 \left( y^2 + \cot^2 \frac{\pi}{5} \right) \left( y^2 + \cot^2 \frac{2\pi}{5} \right) = (y^5 + 5y^4 + 10y^3 + 10y^2 + 5y + 1) - (y^5 - 5y^4 + 10y^3 - 10y^2 + 5y - 1) \\
&10 \left( y^2 + \cot^2 \frac{\pi}{5} \right) \left( y^2 + \cot^2 \frac{2\pi}{5} \right) = 10y^4 + 20y^2 + 2
\end{aligned}$$

Compare the coefficient of  $y^2$ :  $10 \left( \cot^2 \frac{\pi}{5} + \cot^2 \frac{2\pi}{5} \right) = 20$

$$\cot^2 \frac{\pi}{5} + \cot^2 \frac{2\pi}{5} = 2 \dots\dots (1)$$

$$1 + \cot^2 \frac{\pi}{5} + 1 + \cot^2 \frac{2\pi}{5} = 4$$

$$\csc^2 \frac{\pi}{5} + \csc^2 \frac{2\pi}{5} = 4$$

(ii) Compare the constant term:  $10 \cot^2 \frac{\pi}{5} \cot^2 \frac{2\pi}{5} = 2$

$$\cot \frac{\pi}{5} \cot \frac{2\pi}{5} = \frac{1}{\sqrt{5}} \quad (\because 0 < \frac{\pi}{5}, \frac{2\pi}{5} < \frac{\pi}{2} \therefore 0 < \cot \frac{\pi}{5}, \cot \frac{2\pi}{5})$$

$$\tan \frac{\pi}{5} \tan \frac{2\pi}{5} = \sqrt{5} \dots\dots (2)$$

By (1):  $\frac{1}{\tan^2 \frac{\pi}{5}} + \frac{1}{\tan^2 \frac{2\pi}{5}} = 2$

$$\tan^2 \frac{\pi}{5} + \tan^2 \frac{2\pi}{5} = 2 \tan^2 \frac{\pi}{5} \tan^2 \frac{2\pi}{5}$$

$$\tan^2 \frac{\pi}{5} + \tan^2 \frac{2\pi}{5} = 2(\sqrt{5})^2 \text{ by the result of (2)}$$

$$\tan^2 \frac{\pi}{5} + \tan^2 \frac{2\pi}{5} = 10 \dots\dots (3)$$

$$\left( \tan \frac{\pi}{5} - \tan \frac{2\pi}{5} \right)^2 = \tan^2 \frac{\pi}{5} + \tan^2 \frac{2\pi}{5} - 2 \tan \frac{\pi}{5} \tan \frac{2\pi}{5}$$

$$\left( \tan \frac{\pi}{5} - \tan \frac{2\pi}{5} \right)^2 = 10 - 2\sqrt{5} \text{ by (3) and (2)}$$

$$\tan \frac{\pi}{5} - \tan \frac{2\pi}{5} = -\sqrt{10 - 2\sqrt{5}} \quad (\because \tan \frac{\pi}{5} < \tan \frac{2\pi}{5})$$

**Generalisation**

$$(a) \quad z^{2m+1} - 1 = 0 \Rightarrow z = 1, \operatorname{cis} \frac{2k\pi}{2m+1}, k = -m, -m+1, \dots, -1, 1, \dots, m-1 \text{ (where } m \in \mathbf{N})$$

$$\begin{aligned} (b) \quad z^{2m+1} - 1 &= \prod_{k=-m}^m \left( z - \operatorname{cis} \frac{2k\pi}{2m+1} \right) \\ &= (z-1) \prod_{k=-m}^{-1} \left( z - \operatorname{cis} \frac{2k\pi}{2m+1} \right) \cdot \prod_{k=1}^m \left( z - \operatorname{cis} \frac{2k\pi}{2m+1} \right) \\ &= (z-1) \cdot \prod_{k=1}^m \left[ \left( z - \cos \frac{2k\pi}{2m+1} - i \sin \frac{2k\pi}{2m+1} \right) \left( z - \cos \frac{2k\pi}{2m+1} + i \sin \frac{2k\pi}{2m+1} \right) \right] \\ &= (z-1) \cdot \prod_{k=1}^m \left( z^2 - 2z \cos \frac{2k\pi}{2m+1} + 1 \right) \quad \dots\dots (*) \end{aligned}$$

$$\prod_{k=1}^m \left( z^2 - 2z \cos \frac{2k\pi}{2m+1} + 1 \right) = \frac{z^{2m+1} - 1}{z - 1} = z^{2m} + z^{2m-1} + \dots + 1$$

$$\text{As } z \rightarrow 1: \prod_{k=1}^m \left( 2 - 2 \cos \frac{2k\pi}{2m+1} \right) = 2m + 1$$

$$2^m \cdot \prod_{k=1}^m \left( 1 - \cos \frac{2k\pi}{2m+1} \right) = 2m + 1$$

$$\prod_{k=1}^m \left( 1 - 1 + 2 \sin^2 \frac{k\pi}{2m+1} \right) = \frac{2m+1}{2^m}$$

$$2^m \cdot \prod_{k=1}^m \left( \sin^2 \frac{k\pi}{2m+1} \right) = \frac{2m+1}{2^m}$$

$$\left[ \prod_{k=1}^m \left( \sin \frac{k\pi}{2m+1} \right) \right]^2 = \frac{2m+1}{2^{2m}} \quad \dots\dots (**)$$

$$\text{Put } z = \frac{y+1}{y-1} \text{ into } (*)$$

$$\left( \frac{y+1}{y-1} \right)^{2m+1} - 1 = \left( \frac{y+1}{y-1} - 1 \right) \prod_{k=1}^m \left[ \left( \frac{y+1}{y-1} \right)^2 - 2 \cdot \frac{y+1}{y-1} \cdot \cos \frac{2k\pi}{2m+1} + 1 \right]$$

Multiply both sides by  $(y-1)^{2m+1}$

$$\begin{aligned} (y+1)^{2m+1} - (y-1)^{2m+1} &= (y+1 - y+1) \prod_{k=1}^m \left[ (y+1)^2 - 2 \cdot (y^2 - 1) \cdot \cos \frac{2k\pi}{2m+1} + (y-1)^2 \right] \\ &= 2 \cdot \prod_{k=1}^m \left[ 2y^2 + 2 - 2 \cdot (y^2 - 1) \cdot \cos \frac{2k\pi}{2m+1} \right] \\ &= 2^{m+1} \cdot \prod_{k=1}^m \left[ y^2 + 1 - (y^2 - 1) \cdot \cos \frac{2k\pi}{2m+1} \right] \\ &= 2^{m+1} \cdot \prod_{k=1}^m \left[ \left( 1 - \cos \frac{2k\pi}{2m+1} \right) y^2 + 1 + \cos \frac{2k\pi}{2m+1} \right] \\ &= 2^{m+1} \cdot \prod_{k=1}^m \left[ \left( 1 - 1 + 2 \sin^2 \frac{k\pi}{2m+1} \right) y^2 + 1 + 2 \cos^2 \frac{k\pi}{2m+1} - 1 \right] \\ &= 2^{2m+1} \cdot \prod_{k=1}^m \left( y^2 \sin^2 \frac{k\pi}{2m+1} + \cos^2 \frac{k\pi}{2m+1} \right) \\ &= 2^{2m+1} \cdot \prod_{k=1}^m \left[ \sin^2 \frac{k\pi}{2m+1} \left( y^2 + \cot^2 \frac{k\pi}{2m+1} \right) \right] \end{aligned}$$

$$\begin{aligned}
&= 2^{2m+1} \cdot \prod_{k=1}^m \left( \sin^2 \frac{k\pi}{2m+1} \right) \cdot \prod_{k=1}^m \left( y^2 + \cot^2 \frac{k\pi}{2m+1} \right) \\
&= 2^{2m+1} \cdot \frac{2m+1}{2^{2m}} \cdot \prod_{k=1}^m \left( y^2 + \cot^2 \frac{k\pi}{2m+1} \right) \quad \text{by the result of (**)}
\end{aligned}$$

$$\therefore (y+1)^{2m+1} - (y-1)^{2m+1} = (4m+2) \cdot \prod_{k=1}^m \left( y^2 + \cot^2 \frac{k\pi}{2m+1} \right)$$

$$\begin{aligned}
\text{(c) (i)} \quad (4m+2) \cdot \prod_{k=1}^m \left( y^2 + \cot^2 \frac{k\pi}{2m+1} \right) &= \sum_{k=0}^{2m+1} C_k^{2m+1} y^{2m+1-k} - \sum_{k=0}^{2m+1} (-1)^k C_k^{2m+1} y^{2m+1-k} \\
(4m+2) \cdot \prod_{k=1}^m \left( y^2 + \cot^2 \frac{k\pi}{2m+1} \right) &= 2 \cdot \sum_{k=0}^m C_{2k+1}^{2m+1} y^{2m-2k} \\
(2m+1) \cdot \prod_{k=1}^m \left( y^2 + \cot^2 \frac{k\pi}{2m+1} \right) &= \sum_{k=0}^m C_{2k+1}^{2m+1} y^{2(m-k)} = C_1^{2m+1} y^{2m} + C_3^{2m+1} y^{2m-2} + \dots + C_{2m+1}^{2m+1}
\end{aligned}$$

$$\text{Compare coefficient of } y^{2m-2}: (2m+1) \cdot \prod_{k=1}^m \cot^2 \frac{k\pi}{2m+1} = C_3^{2m+1}$$

$$\sum_{k=1}^m \cot^2 \frac{k\pi}{2m+1} = \frac{1}{2m+1} \cdot \frac{2m+1}{1} \times \frac{2m}{2} \times \frac{2m-1}{3} = \frac{m(2m-1)}{3} \quad \dots\dots (1)$$

$$\sum_{k=1}^m \left( 1 + \cot^2 \frac{k\pi}{2m+1} \right) = m + \frac{m(2m-1)}{3} = \frac{2m(m+1)}{3}$$

$$\sum_{k=1}^m \csc^2 \frac{k\pi}{2m+1} = \frac{2m(m+1)}{3} \quad \dots\dots (2)$$

$$\text{(ii) By (c)(i), } (2m+1) \cdot \prod_{k=1}^m \left( y^2 + \cot^2 \frac{k\pi}{2m+1} \right) = \sum_{k=0}^m C_{2k+1}^{2m+1} y^{2(m-k)}$$

$$\text{Compare the constant term, } (2m+1) \cdot \prod_{k=1}^m \cot^2 \frac{k\pi}{2m+1} = C_{2m+1}^{2m+1}$$

$$\left( \prod_{k=1}^m \cot \frac{k\pi}{2m+1} \right)^2 = \frac{1}{2m+1}$$

$$\prod_{k=1}^m \tan \frac{k\pi}{2m+1} = \sqrt{2m+1} \quad \dots\dots (3) \quad \left( \because 0 < \frac{k\pi}{2m+1} < \frac{\pi}{2} \therefore 0 < \tan \frac{k\pi}{2m+1} \right)$$

$$\text{By (**), } \left[ \prod_{k=1}^m \left( \sin \frac{k\pi}{2m+1} \right) \right]^2 = \frac{2m+1}{2^{2m}}$$

$$\prod_{k=1}^m \sin \frac{k\pi}{2m+1} = \frac{\sqrt{2m+1}}{2^m} \quad \dots\dots (4)$$

$$(4) \div (3): \prod_{k=1}^m \cos \frac{k\pi}{2m+1} = \frac{1}{2^m} \quad \dots\dots (5)$$

$$\log(2m+1) + \sum_{k=1}^m \log \left( y^2 + \cot^2 \frac{k\pi}{2m+1} \right) = \log \left[ \sum_{p=0}^m C_{2p+1}^{2m+1} y^{2(m-p)} \right]$$

Differentiate w.r.t.  $y$ :

$$\begin{aligned}
\sum_{k=1}^m \frac{2y}{y^2 + \cot^2 \frac{k\pi}{2m+1}} &= \frac{\sum_{k=0}^{m-1} (2m-2k) C_{2k+1}^{2m+1} y^{2m-2k-1}}{\sum_{k=0}^m C_{2k+1}^{2m+1} y^{2(m-k)}} = \frac{2m C_1^{2m+1} y^{2m-1} + (2m-2) C_3^{2m+1} y^{2m-3} + \dots + 2 C_{2m-1}^{2m+1} \cdot y}{C_1^{2m+1} y^{2m} + C_3^{2m+1} y^{2m-2} + \dots + C_{2m+1}^{2m+1}}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^m \frac{2y \tan^2 \frac{k\pi}{2m+1}}{1 - \left(-y^2 \tan^2 \frac{k\pi}{2m+1}\right)} &= 2 \sum_{k=1}^m \left[ y \tan^2 \frac{k\pi}{2m+1} \cdot \sum_{j=0}^{\infty} \left(-y^2 \tan^2 \frac{k\pi}{2m+1}\right)^j \right] \\
&= 2 \sum_{k=1}^m \left[ \sum_{j=0}^{\infty} (-1)^j \cdot \tan^{2j+2} \frac{k\pi}{2m+1} \cdot y^{2j+1} \right] \\
&= 2 \sum_{j=0}^{\infty} (-1)^j \cdot \left( \sum_{k=1}^m \tan^{2j+2} \frac{k\pi}{2m+1} \right) \cdot y^{2j+1} \\
\text{RHS} &= \frac{\sum_{k=0}^{m-1} (2m-2k) C_{2k+1}^{2m+1} y^{2m-2k-1}}{C_1^{2m+1} y^{2m} + C_3^{2m+1} y^{2m-2} + \dots + C_{2m-1}^{2m+1} y^2 + C_{2m+1}^{2m+1}} \\
&= \frac{2 \sum_{k=0}^{m-1} (m-k) C_{2k+1}^{2m+1} y^{2m-2k-1}}{1 - \left( - \sum_{p=0}^{m-1} C_{2p+1}^{2m+1} y^{2(m-p)} \right)} \\
&= 2 \sum_{k=0}^{m-1} (m-k) C_{2k+1}^{2m+1} y^{2m-2k-1} \cdot \sum_{q=0}^{\infty} \left[ - \sum_{p=0}^{m-1} C_{2p+1}^{2m+1} y^{2(m-p)} \right]^q \\
&= \sum_{j=0}^{\infty} (-1)^j \cdot \left( \sum_{k=1}^m \tan^{2j+2} \frac{k\pi}{2m+1} \right) \cdot y^{2j+1} = \sum_{k=0}^{m-1} (m-k) C_{2k+1}^{2m+1} y^{2m-2k-1} \cdot \sum_{q=0}^{\infty} \left[ - \sum_{p=0}^{m-1} C_{2p+1}^{2m+1} y^{2(m-p)} \right]^q
\end{aligned}$$

Compare coefficients of  $y$ :  $\sum_{k=1}^m \tan^2 \frac{k\pi}{2m+1} = C_{2m-1}^{2m+1} = m(2m+1) \dots \dots (6)$

Compare coefficients of  $y^3$ :

$$\begin{aligned}
- \sum_{k=1}^m \tan^4 \frac{k\pi}{2m+1} &= - \left( C_{2m-1}^{2m+1} \right)^2 + 2 C_{2m-3}^{2m+1} \\
&= - m^2 (2m+1)^2 + 2 \cdot \frac{(2m+1)(2m)(2m-1)(2m-2)}{1 \cdot 2 \cdot 3 \cdot 4} \\
&= - m^2 (2m+1)^2 + \frac{1}{3} m(2m+1)(m-1)(2m-1) \\
&= \frac{1}{3} m(2m+1)(2m^2 - 3m + 1 - 6m^2 - 3m) \\
&= - \frac{1}{3} m(2m+1)(4m^2 + 6m - 1) \\
\sum_{k=1}^m \tan^4 \frac{k\pi}{2m+1} &= \frac{1}{3} m(2m+1)(4m^2 + 6m - 1) \dots \dots (7)
\end{aligned}$$

(d)  $\tan \theta > \theta > \sin \theta \Rightarrow \cot^2 \theta < \frac{1}{\theta^2} < \csc^2 \theta$

$$\cot^2 \frac{k\pi}{2m+1} < \left( \frac{2m+1}{k\pi} \right)^2 < \csc^2 \frac{k\pi}{2m+1}$$

$$\sum_{k=1}^n \cot^2 \frac{k\pi}{2m+1} < \sum_{k=1}^n \left( \frac{2m+1}{k\pi} \right)^2 < \sum_{k=1}^n \csc^2 \frac{k\pi}{2m+1}$$

$$\frac{m(2m-1)}{3} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^m \frac{1}{k^2} < \frac{2m(m+1)}{3} \text{ by the result of (1) and (2).}$$

$$\begin{aligned}\frac{\pi^2}{6} \left( \frac{2m}{2m+1} \right) \left( \frac{2m-1}{2m+1} \right) &< \sum_{k=1}^m \frac{1}{k^2} < \frac{\pi^2}{6} \left( \frac{2m}{2m+1} \right) \left( \frac{2m+2}{2m+1} \right) \\ \frac{\pi^2}{6} \lim_{m \rightarrow \infty} \left( \frac{2m}{2m+1} \right) \left( \frac{2m-1}{2m+1} \right) &\leq \lim_{m \rightarrow \infty} \sum_{k=1}^m \frac{1}{k^2} \leq \frac{\pi^2}{6} \lim_{m \rightarrow \infty} \left( \frac{2m}{2m+1} \right) \left( \frac{2m+2}{2m+1} \right) \\ \frac{\pi^2}{6} &= \frac{\pi^2}{6} \lim_{m \rightarrow \infty} \left( \frac{1}{1 + \frac{1}{2m}} \right) \left( \frac{1 - \frac{1}{2m}}{1 + \frac{1}{2m}} \right) \leq \lim_{m \rightarrow \infty} \sum_{k=1}^m \frac{1}{k^2} \leq \frac{\pi^2}{6} \lim_{m \rightarrow \infty} \left( \frac{1}{1 + \frac{1}{2m}} \right) \left( \frac{1 + \frac{2}{2m}}{1 + \frac{1}{2m}} \right) = \frac{\pi^2}{6}\end{aligned}$$

By squeezing principle,  $\lim_{m \rightarrow \infty} \sum_{k=1}^m \frac{1}{k^2} = \frac{\pi^2}{6}$  ..... (8)

$$\text{By (c)(i), } (2m+1) \cdot \prod_{k=1}^m \left( y^2 + \cot^2 \frac{k\pi}{2m+1} \right) = \sum_{k=0}^m C_{2k+1}^{2m+1} y^{2(m-k)}$$

Compare coefficient of  $y^{2(m-2)}$ :  $(2m+1) \cdot \sum_{p < q} \cot^2 \frac{p\pi}{2m+1} \cot^2 \frac{q\pi}{2m+1} = C_5^{2m+1}$

$$\sum_{p < q} \cot^2 \frac{p\pi}{2m+1} \cot^2 \frac{q\pi}{2m+1} = \frac{1}{2m+1} \cdot \frac{(2m+1)(2m)(2m-1)(2m-2)(2m-3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$\sum_{p < q} \cot^2 \frac{p\pi}{2m+1} \cot^2 \frac{q\pi}{2m+1} = \frac{1}{30} \cdot (m-1)m(2m-3)(2m-1) \text{ ..... (9)}$$

$$\left( \sum_{k=1}^m \cot^2 \frac{k\pi}{2m+1} \right)^2 = \sum_{k=1}^m \cot^4 \frac{k\pi}{2m+1} + 2 \sum_{p < q} \cot^2 \frac{p\pi}{2m+1} \cot^2 \frac{q\pi}{2m+1}$$

$$\left[ \frac{m(2m-1)}{3} \right]^2 = \sum_{k=1}^m \cot^4 \frac{k\pi}{2m+1} + \frac{1}{15} \cdot (m-1)m(2m-3)(2m-1) \text{ by (1) and (9)}$$

$$\begin{aligned}\sum_{k=1}^m \cot^4 \frac{k\pi}{2m+1} &= \frac{m^2(2m-1)^2}{9} - \frac{1}{15} \cdot (m-1)m(2m-3)(2m-1) \\ &= \frac{1}{45} \cdot m(2m-1)[5m(2m-1) - 3(m-1)(2m-3)] \\ &= \frac{1}{45} \cdot m(2m-1)(10m^2 - 5m - 6m^2 + 15m - 9)\end{aligned}$$

$$\sum_{k=1}^m \cot^4 \frac{k\pi}{2m+1} = \frac{1}{45} \cdot m(2m-1)(4m^2 + 10m - 9) \text{ ..... (10)}$$

$$\sum_{k=1}^m \left( \csc^2 \frac{k\pi}{2m+1} - 1 \right)^2 = \frac{1}{45} \cdot m(2m-1)(4m^2 + 10m - 9)$$

$$\sum_{k=1}^m \csc^4 \frac{k\pi}{2m+1} - 2 \sum_{k=1}^m \csc^2 \frac{k\pi}{2m+1} + m = \frac{1}{45} \cdot m(2m-1)(4m^2 + 10m - 9)$$

$$\begin{aligned}\sum_{k=1}^m \csc^4 \frac{k\pi}{2m+1} &= 2 \sum_{k=1}^m \csc^2 \frac{k\pi}{2m+1} + \frac{1}{45} \cdot m(2m-1)(4m^2 + 10m - 9) - m \\ &= 2 \cdot \frac{2m(m+1)}{3} + \frac{1}{45} \cdot m(2m-1)(4m^2 + 10m - 9) - m \text{ by (2)} \\ &= \frac{m}{45} \cdot (60m + 60 + 8m^3 + 16m^2 - 28m + 9 - 45) \\ &= \frac{m}{45} \cdot (8m^3 + 16m^2 + 32m + 24)\end{aligned}$$

$$\sum_{k=1}^m \csc^4 \frac{k\pi}{2m+1} = \frac{8m}{45} \cdot (m^3 + 2m^2 + 4m + 3) = \frac{8m}{45} \cdot (m+1)(m^2 + m + 3) \text{ ..... (11)}$$

Similarly,  $\tan \theta > \theta > \sin \theta \Rightarrow \cot^4 \theta < \frac{1}{\theta^4} < \csc^4 \theta$

$$\cot^4 \frac{k\pi}{2m+1} < \left( \frac{2m+1}{k\pi} \right)^4 < \csc^4 \frac{k\pi}{2m+1}$$

$$\sum_{k=1}^n \cot^4 \frac{k\pi}{2m+1} < \sum_{k=1}^n \left( \frac{2m+1}{k\pi} \right)^4 < \sum_{k=1}^n \csc^4 \frac{k\pi}{2m+1}$$

$$\frac{1}{45} \cdot m(2m-1)(4m^2+10m-9) < \frac{(2m+1)^4}{\pi^4} \sum_{k=1}^m \frac{1}{k^4} < \frac{8m}{45} \cdot (m+1)(m^2+m+3) \text{ by (10) and (11).}$$

$$\frac{\pi^4}{90} \left( \frac{2m}{2m+1} \right) \left( \frac{2m-1}{2m+1} \right) \left[ \frac{4m^2+10m-9}{(2m+1)^2} \right] < \sum_{k=1}^m \frac{1}{k^4} < \frac{\pi^4}{90} \left( \frac{2m}{2m+1} \right) \left( \frac{2m+2}{2m+1} \right) \left[ \frac{4m^2+4m+12}{(2m+1)^2} \right]$$

Take limit  $m \rightarrow \infty$ ,  $\lim_{m \rightarrow \infty} \sum_{k=1}^m \frac{1}{k^4} = \frac{\pi^4}{90} \dots\dots (12)$

**Variation**

$$(a) \quad z^{2m} - 1 = 0 \Rightarrow z = 1, -1, \operatorname{cis} \frac{2k\pi}{2m} \quad (\text{i.e. } \operatorname{cis} \frac{k\pi}{m}), k = -m+1, \dots, -1, 1, \dots, m-1 \quad (\text{where } m \in \mathbf{N})$$

$$(b) \quad z^{2m} - 1 = (z+1)(z-1) \prod_{k=1}^{m-1} \left( z - \cos \frac{k\pi}{m} + i \sin \frac{k\pi}{m} \right) \left( z - \cos \frac{k\pi}{m} - i \sin \frac{k\pi}{m} \right) \\ = (z^2 - 1) \cdot \prod_{k=1}^{m-1} \left( z^2 - 2z \cos \frac{k\pi}{m} + 1 \right) \dots\dots (*)$$

$$\prod_{k=1}^{m-1} \left( z^2 - 2z \cos \frac{k\pi}{m} + 1 \right) = \frac{z^{2m} - 1}{z^2 - 1} = z^{2m-2} + z^{2m-4} + \dots + 1$$

$$\text{As } z \rightarrow 1: \prod_{k=1}^{m-1} \left( 2 - 2 \cos \frac{k\pi}{m} \right) = m$$

$$2^{m-1} \cdot \prod_{k=1}^{m-1} \left( 1 - \cos \frac{k\pi}{m} \right) = m$$

$$\prod_{k=1}^{m-1} \left( 1 - 1 + 2 \sin^2 \frac{k\pi}{2m} \right) = \frac{m}{2^{m-1}}$$

$$2^{m-1} \cdot \prod_{k=1}^{m-1} \left( \sin^2 \frac{k\pi}{2m} \right) = \frac{m}{2^{m-1}}$$

$$\left[ \prod_{k=1}^{m-1} \left( \sin \frac{k\pi}{2m} \right) \right]^2 = \frac{m}{2^{2(m-1)}} \dots\dots (**)$$

$$\text{Put } z = \frac{y+1}{y-1} \text{ into } (*)$$

$$\left( \frac{y+1}{y-1} \right)^{2m} - 1 = \left[ \left( \frac{y+1}{y-1} \right)^2 - 1 \right] \prod_{k=1}^{m-1} \left[ \left( \frac{y+1}{y-1} \right)^2 - 2 \cdot \frac{y+1}{y-1} \cdot \cos \frac{k\pi}{m} + 1 \right]$$

Multiply both sides by  $(y-1)^{2m}$

$$(y+1)^{2m} - (y-1)^{2m} = (y^2 + 2y + 1 - y^2 + 2y - 1) \prod_{k=1}^{m-1} \left[ (y+1)^2 - 2 \cdot (y^2 - 1) \cdot \cos \frac{k\pi}{m} + (y-1)^2 \right] \\ = 4y \cdot \prod_{k=1}^{m-1} \left[ 2y^2 + 2 - 2 \cdot (y^2 - 1) \cdot \cos \frac{k\pi}{m} \right] \\ = 2^{m+1} y \cdot \prod_{k=1}^{m-1} \left[ y^2 + 1 - (y^2 - 1) \cdot \cos \frac{k\pi}{m} \right] \\ = 2^{m+1} y \cdot \prod_{k=1}^{m-1} \left[ \left( 1 - \cos \frac{k\pi}{m} \right) y^2 + 1 + \cos \frac{k\pi}{m} \right] \\ = 2^{m+1} y \cdot \prod_{k=1}^{m-1} \left[ \left( 1 - 1 + 2 \sin^2 \frac{k\pi}{2m} \right) y^2 + 1 + 2 \cos^2 \frac{k\pi}{2m} - 1 \right] \\ = 2^{2m} y \cdot \prod_{k=1}^{m-1} \left( y^2 \sin^2 \frac{k\pi}{2m} + \cos^2 \frac{k\pi}{2m} \right) = 2^{2m} y \cdot \prod_{k=1}^{m-1} \left[ \sin^2 \frac{k\pi}{2m} \left( y^2 + \cot^2 \frac{k\pi}{2m} \right) \right] \\ = 2^{2m} y \cdot \prod_{k=1}^{m-1} \left( \sin^2 \frac{k\pi}{2m} \right) \cdot \prod_{k=1}^{m-1} \left( y^2 + \cot^2 \frac{k\pi}{2m} \right) \\ = 2^{2m} y \cdot \frac{m}{2^{2(m-1)}} \cdot \prod_{k=1}^{m-1} \left( y^2 + \cot^2 \frac{k\pi}{2m} \right) \text{ by the result of } (**) \\ \therefore (y+1)^{2m} - (y-1)^{2m} = 4my \cdot \prod_{k=1}^{m-1} \left( y^2 + \cot^2 \frac{k\pi}{2m} \right)$$



$$\begin{aligned}
(c) \quad (i) \quad & 4my \cdot \prod_{k=1}^{m-1} \left( y^2 + \cot^2 \frac{k\pi}{2m} \right) = \sum_{k=0}^{2m} C_k^{2m} y^{2m-k} - \sum_{k=0}^{2m} (-1)^k C_k^{2m} y^{2m-k} \\
& 4my \cdot \prod_{k=1}^{m-1} \left( y^2 + \cot^2 \frac{k\pi}{2m} \right) = 2 \cdot \sum_{k=0}^{m-1} C_{2k+1}^{2m} y^{2m-2k-1} \\
& 2m \prod_{k=1}^{m-1} \left( y^2 + \cot^2 \frac{k\pi}{2m} \right) = \sum_{k=0}^{m-1} C_{2k+1}^{2m} y^{2m-2k-2} = C_1^{2m} y^{2m-2} + C_3^{2m} y^{2m-4} + \dots + C_{2m-1}^{2m}
\end{aligned}$$

Compare coefficient of  $y^{2m-4}$ :  $2m \cdot \sum_{k=1}^{m-1} \cot^2 \frac{k\pi}{2m} = C_3^{2m}$

$$\sum_{k=1}^{m-1} \cot^2 \frac{k\pi}{2m} = \frac{1}{2m} \cdot \frac{2m}{1} \times \frac{2m-1}{2} \times \frac{2m-2}{3} = \frac{(m-1)(2m-1)}{3} \dots\dots (1)$$

$$\sum_{k=1}^{m-1} \left( 1 + \cot^2 \frac{k\pi}{2m} \right) = m-1 + \frac{(m-1)(2m-1)}{3} = \frac{2(m-1)(m+1)}{3}$$

$$\sum_{k=1}^{m-1} \csc^2 \frac{k\pi}{2m} = \frac{2(m-1)(m+1)}{3} \dots\dots (2)$$

(ii) By (c)(i),  $2m \prod_{k=1}^{m-1} \left( y^2 + \cot^2 \frac{k\pi}{2m} \right) = \sum_{k=0}^{m-1} C_{2k+1}^{2m} y^{2m-2k-2}$

Compare the constant term,  $2m \cdot \prod_{k=1}^{m-1} \cot^2 \frac{k\pi}{2m} = C_{2m-1}^{2m}$

$$\left( \prod_{k=1}^{m-1} \cot \frac{k\pi}{2m} \right)^2 = 1$$

$$\prod_{k=1}^{m-1} \tan \frac{k\pi}{2m} = 1 \dots\dots (3) \quad (\because 0 < \frac{k\pi}{2m} < \frac{\pi}{2} \therefore 0 < \tan \frac{k\pi}{2m})$$

By (\*\*),  $\left[ \prod_{k=1}^{m-1} \left( \sin \frac{k\pi}{2m} \right) \right]^2 = \frac{m}{2^{2(m-1)}}$

$$\prod_{k=1}^{m-1} \sin \frac{k\pi}{2m} = \frac{\sqrt{m}}{2^{m-1}} \dots\dots (4)$$

$$(4) \div (3): \prod_{k=1}^{m-1} \cos \frac{k\pi}{2m} = \frac{\sqrt{m}}{2^{m-1}} \dots\dots (5)$$

$$\log 2m + \sum_{k=1}^{m-1} \log \left( y^2 + \cot^2 \frac{k\pi}{2m} \right) = \log \left( \sum_{k=0}^{m-1} C_{2k+1}^{2m} y^{2m-2k-2} \right)$$

Differentiate w.r.t.  $y$ :

$$\sum_{k=1}^{m-1} \frac{2y}{y^2 + \cot^2 \frac{k\pi}{2m}} = \frac{\sum_{k=0}^{m-2} (2m-2k-2) C_{2k+1}^{2m} y^{2m-2k-3}}{\sum_{p=0}^{m-1} C_{2p+1}^{2m} y^{2m-2p-2}}$$

$$\sum_{k=1}^{m-1} \frac{2y}{y^2 + \cot^2 \frac{k\pi}{2m}} = \frac{(2m-2) C_1^{2m} y^{2m-3} + (2m-4) C_3^{2m} y^{2m-5} + \dots + 2 C_{2m-3}^{2m} \cdot y}{C_1^{2m} y^{2m-2} + C_3^{2m} y^{2m-4} + \dots + C_{2m-1}^{2m}}$$

$$\begin{aligned}
\text{L.H.S.} &= \sum_{k=1}^{m-1} \frac{2y}{y^2 + \cot^2 \frac{k\pi}{2m}} = \sum_{k=1}^{m-1} \frac{2y \tan^2 \frac{k\pi}{2m}}{1 - \left( -y^2 \tan^2 \frac{k\pi}{2m} \right)} = 2 \sum_{k=1}^{m-1} \left[ y \tan^2 \frac{k\pi}{2m} \cdot \sum_{j=0}^{\infty} \left( -y^2 \tan^2 \frac{k\pi}{2m} \right)^j \right] \\
&= 2 \sum_{k=1}^{m-1} \left[ \sum_{j=0}^{\infty} (-1)^j \cdot \tan^{2j+2} \frac{k\pi}{2m} \cdot y^{2j+1} \right] = 2 \sum_{j=0}^{\infty} (-1)^j \cdot \left( \sum_{k=1}^{m-1} \tan^{2j+2} \frac{k\pi}{2m} \right) \cdot y^{2j+1}
\end{aligned}$$

$$\begin{aligned}
\text{R.H.S.} &= \frac{(2m-2)C_1^{2m}y^{2m-3} + (2m-4)C_3^{2m}y^{2m-5} + \cdots + 2C_{2m-3}^{2m} \cdot y}{C_1^{2m}y^{2m-2} + C_3^{2m}y^{2m-4} + \cdots + C_{2m-1}^{2m}} \\
&= \frac{2 \sum_{k=0}^{m-2} (m-k-1)C_{2k+1}^{2m}y^{2m-2k-3}}{2m \left[ 1 - \left( -\frac{1}{2m} \sum_{p=0}^{m-2} C_{2p+1}^{2m}y^{2m-2p-2} \right) \right]} \\
&= \frac{1}{m} \sum_{k=0}^{m-2} (m-k-1)C_{2k+1}^{2m}y^{2m-2k-3} \cdot \sum_{q=0}^{\infty} \left[ -\frac{1}{2m} \sum_{p=0}^{m-2} C_{2p+1}^{2m}y^{2m-2p-2} \right]^q \\
2 \sum_{j=0}^{\infty} (-1)^j \cdot \left( \sum_{k=1}^{m-1} \tan^{2j+2} \frac{k\pi}{2m} \right) \cdot y^{2j+1} &= \frac{1}{m} \sum_{k=0}^{m-2} (m-k-1)C_{2k+1}^{2m}y^{2m-2k-3} \cdot \sum_{q=0}^{\infty} \left[ -\frac{1}{2m} \sum_{p=0}^{m-2} C_{2p+1}^{2m}y^{2m-2p-2} \right]^q
\end{aligned}$$

Compare coefficients of  $y$ :  $2 \sum_{k=1}^{m-1} \tan^2 \frac{k\pi}{2m} = \frac{1}{m} C_{2m-3}^{2m} = \frac{1}{m} \cdot \frac{(2m)(2m-1)(2m-2)}{1 \cdot 2 \cdot 3}$

$$\sum_{k=1}^{m-1} \tan^2 \frac{k\pi}{2m} = \frac{(m-1)(2m-1)}{3} \quad \dots\dots (6)$$

In particular, put  $m = 90$ .

$$\tan^2 1^\circ + \tan^2 2^\circ + \cdots + \tan^2 89^\circ = \frac{(90-1)(180-1)}{3} = \frac{15931}{3} = 5310.333$$

Compare coefficients of  $y^3$ :

$$\begin{aligned}
-2 \sum_{k=1}^{m-1} \tan^4 \frac{k\pi}{2m} &= \frac{1}{m} \left[ -\frac{1}{2m} (C_{2m-3}^{2m})^2 + 2C_{2m-5}^{2m} \right] \\
\sum_{k=1}^{m-1} \tan^4 \frac{k\pi}{2m} &= \frac{1}{2m} \cdot \left\{ \frac{1}{2m} \left[ \frac{(2m)(2m-1)(2m-2)}{1 \cdot 2 \cdot 3} \right]^2 - 2 \cdot \frac{(2m)(2m-1)(2m-2)(2m-3)(2m-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \right\} \\
&= \frac{1}{2m} \cdot \left[ \frac{(2m)(m-1)^2(2m-1)^2}{9} - \frac{(2m)(2m-1)(m-1)(2m-3)(m-2)}{15} \right] \\
&= \frac{(m-1)(2m-1)}{45} \cdot [5(m-1)(2m-1) - 3(2m-3)(m-2)] \\
&= \frac{(m-1)(2m-1)}{45} \cdot [5(2m^2 - 3m + 1) - 3(2m^2 - 7m + 6)] \\
\sum_{k=1}^m \tan^4 \frac{k\pi}{2m} &= \frac{(m-1)(2m-1)}{45} \cdot (4m^2 + 6m - 13) \quad \dots\dots (7)
\end{aligned}$$