Find all integral solutions to the following equation: $\frac{1}{x}-\frac{1}{y}=\frac{1}{2015}$.

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If (x_1, y_1) is a solution, then $\frac{1}{x_1}-\frac{1}{y_1}=\frac{1}{2015}$, then $\frac{1}{-y_1}-\frac{1}{-x_1}=\frac{1}{2015} \Rightarrow (-y_1, -x_1)$ is a solution.

If $x_1 > 0 > y_1$ is a solution, then $\frac{1}{x_1}-\frac{1}{y_1}=\frac{1}{2015}$ is equivalent to $\frac{1}{x_1}+\frac{1}{-y_1}=\frac{1}{2015}$.

Therefore, $(x_1, -y_1)$ is a solution to $\frac{1}{x}+\frac{1}{y}=\frac{1}{2015}$.

First, we find all positive integral solutions (x, y) of $\frac{1}{x}-\frac{1}{y}=\frac{1}{2015}$ such that $y > x$.

$2015 = 5\times13\times31$, the equation is equivalent to $5\times13\times31(y-x) = xy$

Case 1 Let $x = a$, $y = 2015b$, where a and b are positive integers

$5\times13\times31(2015b-a) = 5\times13\times31ab \Rightarrow 2015b-a = ab$

$b(2015-a) + 2015-a = 5\times13\times31$

$(b+1)(2015-a) = 5\times403$

$b+1 = 5$ and $2015-a = 403$

$b = 4$, $a = 1612$

$x = 1612$, $y = 2015\times4 = 8060$

OR $b+1 = 13$ and $2015-a = 155$

$b = 12$, $a = 1860$

$x = 1860$, $y = 2015\times12 = 24180$

OR $b+1 = 31$ and $2015-a = 65$

$b = 30$, $a = 1950$

$x = 1950$, $y = 2015\times30 = 60450$

OR $b+1 = 65$ and $2015-a = 31$

$b = 64$, $a = 1984$

$x = 1984$, $y = 2015\times64 = 128960$

OR $b+1 = 155$ and $2015-a = 13$

$b = 154$, $a = 2002$

$x = 2002$, $y = 2015\times154 = 310310$

OR $b+1 = 403$ and $2015-a = 5$

$b = 402$, $a = 2010$

$x = 2010$, $y = 2015\times402 = 810030$

OR $b+1 = 2015$ and $2015-a = 1$

$b = 2014$, $a = 2014$

$x = 2014$, $y = 2015\times2014 = 4058210$

Case 2 Let $x = 5a$, $y = 403b$, where a and b are positive integers

$5\times13\times31(403b-5a) = 5\times13\times31ab \Rightarrow 403b-5a = ab$

$b(403-a) + 403\times5-5a = 5\times13\times31$

$(b+5)(403-a) = 13\times155$

$b+5 = 13$ and $403-a = 155$

$b = 8$, $a = 248$

$x = 5\times248 = 1240$, $y = 403\times8 = 3224$

OR $b+5 = 31$ and $403-a = 65$

$b = 26$, $a = 338$

$x = 5\times338 = 1690$, $y = 403\times26 = 10478$

OR $b+5 = 65$ and $403-a = 31$

$b = 60$, $a = 372$

$x = 5\times372 = 1860$, $y = 403\times60 = 24180$

OR $b+5 = 155$ and $403-a = 13$

$\frac{1}{x} - \frac{1}{y} = \frac{1}{2015}$

$b = 150$, $a = 390$

$x = 5\times390 = 1950$, $y = 403\times150 = 60450$

OR $b + 5 = 403$ and $403 - a = 5$

$b = 398$, $a = 398$

$x = 5\times398 = 1990$, $y = 403\times398 = 160394$

OR $b + 5 = 2015$ and $403 - a = 1$

$b = 2010$, $a = 402$

$x = 5\times402 = 2010$, $y = 403\times2010 = 810030$

Case 3 Let $x = 13a$, $y = 155b$, where a and b are positive integers

$5\times13\times31(155b - 13a) = 5\times13\times31ab \Rightarrow 155b - 13a = ab$

$b(155 - a) + 13\times155 - 13a = 5\times13\times31$

$(b + 13)(155 - a) = 31\times65$

$b + 13 = 31$ and $155 - a = 65$

$b = 18$, $a = 90$

$x = 13\times90 = 1170$, $y = 155\times18 = 2790$

OR $b + 13 = 65$ and $155 - a = 31$

$b = 52$, $a = 124$

$x = 13\times124 = 1612$, $y = 155\times52 = 8060$

OR $b + 13 = 155$ and $155 - a = 13$

$b = 142$, $a = 142$

$x = 13\times142 = 1846$, $y = 155\times142 = 22010$

OR $b + 13 = 403$ and $155 - a = 5$

$b = 390$, $a = 150$

$x = 13\times150 = 1950$, $y = 155\times390 = 60450$

OR $b + 13 = 2015$ and $155 - a = 1$

$b = 2002$, $a = 154$

$x = 13\times154 = 2002$, $y = 155\times2002 = 310310$

Case 4 Let $x = 31a$, $y = 65b$, where a and b are positive integers

$5\times13\times31(65b - 31a) = 5\times13\times31ab \Rightarrow 65b - 31a = ab$

$b(65 - a) + 31\times65 - 31a = 5\times13\times31$

$(b + 31)(65 - a) = 65\times31$

$b + 31 = 65$ and $65 - a = 31$

$b = 34$, $a = 34$

$x = 31\times34 = 1054$, $y = 65\times34 = 2210$

OR $b + 31 = 155$ and $65 - a = 13$

$b = 124$, $a = 52$

$x = 31\times52 = 1612$, $y = 65\times124 = 8060$

OR $b + 31 = 403$ and $65 - a = 5$

$b = 372$, $a = 60$

$x = 31\times60 = 1860$, $y = 65\times372 = 24180$

OR $b + 31 = 2015$ and $65 - a = 1$

$b = 1984$, $a = 64$

$x = 31\times64 = 1984$, $y = 65\times1984 = 128960$

Case 5 Let $x = 65a$, $y = 31b$, where a and b are positive integers

$5\times13\times31(31b - 65a) = 5\times13\times31ab \Rightarrow 31b - 65a = ab$

$b(31 - a) + 65\times31 - 65a = 5\times13\times31$

$(b + 65)(31 - a) = 155\times13$

$b + 65 = 155$ and $31 - a = 13$

$b = 90$, $a = 18$

$x = 65\times18 = 1170$, $y = 31\times90 = 2790$

OR $b + 65 = 403$ and $31 - a = 5$

$\frac{1}{x}-\frac{1}{y}=\frac{1}{2015}$

$b = 338$, $a = 26$

$x = 65\times26 = 1690$, $y = 31\times338 = 10478$

OR $b + 65 = 2015$ and $31 - a = 1$

$b = 1950$, $a = 30$

$x = 65\times30 = 1950$, $y = 31\times1950 = 60450$

Case 6 Let $x = 155a$, $y = 13b$, where a and b are positive integers

$5\times13\times31(13b - 155a) = 5\times13\times31ab \Rightarrow 13b - 155a = ab$

$b(13 - a) + 155\times13 - 155a = 5\times13\times31$

$(b + 155)(13 - a) = 403\times5$

$b + 155 = 403$ and $13 - a = 5$

$b = 248$, $a = 8$

$x = 155\times8 = 1240$, $y = 13\times248 = 3224$

OR $b + 155 = 2015$ and $13 - a = 1$

$b = 1860$, $a = 12$

$x = 155\times12 = 1860$, $y = 13\times1860 = 24180$

Case 7 Let $x = 403a$, $y = 5b$, where a and b are positive integers

$5\times13\times31(5b - 403a) = 5\times13\times31ab \Rightarrow 5b - 403a = ab$

$b(5 - a) + 403\times5 - 403a = 5\times13\times31$

$(b + 403)(5 - a) = 2015\times1$

$b + 403 = 2015$ and $5 - a = 1$

$b = 1612$, $a = 4$

$x = 403\times4 = 1612$, $y = 5\times1612 = 8060$

Next, we find all positive integral solutions (x, y) of $\frac{1}{x}+\frac{1}{y}=\frac{1}{2015}$. So. $(x, -y)$ is a solution to $\frac{1}{x}-\frac{1}{y}=\frac{1}{2015}$. Also, $(y, -x)$ is another solution. The equation is equivalent to $2015(y + x) = xy$.

Case 8 Let $x = a$, $y = 2015b$, where a and b are positive integers

$5\times13\times31(2015b + a) = 5\times13\times31ab \Rightarrow 2015b + a = ab$

$5\times13\times31 = 2015 - 2015b + a(b - 1)$

$(b - 1)(a - 2015) = 1\times2015$

$b - 1 = 1$ and $a - 2015 = 2015$

$b = 2$, $a = 4030$

$x = 4030$, $y = 2015\times2 = 4030$

OR $b - 1 = 5$ and $a - 2015 = 403$

$b = 6$, $a = 2418$

$x = 2418$, $y = 2015\times6 = 12090$

OR $b - 1 = 13$ and $a - 2015 = 155$

$b = 14$, $a = 2170$

$x = 2170$, $y = 2015\times14 = 28210$

OR $b - 1 = 31$ and $a - 2015 = 65$

$b = 32$, $a = 2080$

$x = 2080$, $y = 2015\times32 = 64480$

OR $b - 1 = 65$ and $a - 2015 = 31$

$b = 66$, $a = 2046$

$x = 2046$, $y = 2015\times66 = 132990$

OR $b - 1 = 155$ and $a - 2015 = 13$

$b = 156$, $a = 2028$

$x = 2028$, $y = 2015\times156 = 314340$

OR $b - 1 = 403$ and $a - 2015 = 5$

$b = 404$, $a = 2020$

$\frac{1}{x} - \frac{1}{y} = \frac{1}{2015}$

$x = 2020$, $y = 2015\times404 = 814060$

OR $b - 1 = 2015$ and $a - 2015 = 1$

$b = 2016$, $a = 2016$

$x = 2016$, $y = 2015\times2016 = 4062240$

Case 9 Let $x = 5a$, $y = 403b$, where a and b are positive integers

$5\times13\times31(403b + 5a) = 5\times13\times31ab \Rightarrow 403b + 5a = ab$

$5\times13\times31 = 2015 - 403b + a(b - 5)$

$(b - 5)(a - 403) = 1\times2015$

$b - 5 = 1$ and $a - 403 = 2015$

$b = 6$, $a = 2418$

$x = 5\times2418 = 12090$, $y = 403\times6 = 2418$

OR $b - 5 = 5$ and $a - 403 = 403$

$b = 10$, $a = 806$

$x = 5\times806 = 4030$, $y = 403\times10 = 4030$

OR $b - 5 = 13$ and $a - 403 = 155$

$b = 18$, $a = 558$

$x = 5\times558 = 2790$, $y = 403\times18 = 7254$

OR $b - 5 = 31$ and $a - 403 = 65$

$b = 36$, $a = 468$

$x = 5\times468 = 2340$, $y = 403\times36 = 14508$

OR $b - 5 = 65$ and $a - 403 = 31$

$b = 70$, $a = 434$

$x = 5\times434 = 2170$, $y = 403\times70 = 28210$

OR $b - 5 = 155$ and $a - 403 = 13$

$b = 160$, $a = 416$

$x = 5\times416 = 2080$, $y = 403\times160 = 64480$

OR $b - 5 = 403$ and $a - 403 = 5$

$b = 408$, $a = 408$

$x = 5\times408 = 2040$, $y = 403\times408 = 164424$

OR $b - 5 = 2015$ and $a - 403 = 1$

$b = 2020$, $a = 404$

$x = 5\times404 = 2020$, $y = 403\times2020 = 814060$

Case 10 Let $x = 13a$, $y = 155b$, where a and b are positive integers

$5\times13\times31(155b + 13a) = 5\times13\times31ab \Rightarrow 155b + 13a = ab$

$5\times13\times31 = 2015 - 155b + a(b - 13)$

$(b - 13)(a - 155) = 1\times2015$

$b - 13 = 1$ and $a - 155 = 2015$

$b = 14$, $a = 2170$

$x = 13\times2170 = 28210$, $y = 155\times14 = 2170$

OR $b - 13 = 5$ and $a - 155 = 403$

$b = 18$, $a = 558$

$x = 13\times558 = 7254$, $y = 155\times18 = 2790$

OR $b - 13 = 13$ and $a - 155 = 155$

$b = 26$, $a = 310$

$x = 13\times310 = 4030$, $y = 155\times26 = 4030$

OR $b - 13 = 31$ and $a - 155 = 65$

$b = 44$, $a = 220$

$x = 13\times220 = 2860$, $y = 155\times44 = 6820$

OR $b - 13 = 65$ and $a - 155 = 31$

$b = 78$, $a = 186$

$x = 13\times186 = 2418$, $y = 155\times78 = 12090$

$\frac{1}{x}-\frac{1}{y}=\frac{1}{2015}$

OR $b-13=155$ and $a-155=13$

$b=168$, $a=168$

$x=13\times168=2184$, $y=155\times168=26040$

OR $b-13=403$ and $a-155=5$

$b=416$, $a=160$

$x=13\times160=2080$, $y=155\times416=64480$

OR $b-13=2015$ and $a-155=1$

$b=2028$, $a=156$

$x=13\times156=2028$, $y=155\times2028=314340$

Case 11 Let $x = 31a$, $y = 65b$, where a and b are positive integers

$5\times13\times31(65b+31a)=5\times13\times31ab \Rightarrow 65b+31a=ab$

$5\times13\times31=2015-65b+a(b-31)$

$(b-31)(a-65)=1\times2015$

$b-31=1$ and $a-65=2015$

$b=32$, $a=2080$

$x=31\times2080=64480$, $y=65\times32=2080$

OR $b-31=5$ and $a-65=403$

$b=36$, $a=468$

$x=31\times468=14508$, $y=65\times36=2340$

OR $b-31=13$ and $a-65=155$

$b=44$, $a=220$

$x=31\times220=6820$, $y=65\times44=2860$

OR $b-31=31$ and $a-65=65$

$b=62$, $a=130$

$x=31\times130=4030$, $y=65\times62=4030$

OR $b-31=65$ and $a-65=31$

$b=96$, $a=96$

$x=31\times96=2976$, $y=65\times96=6240$

OR $b-31=155$ and $a-65=13$

$b=186$, $a=78$

$x=31\times78=2418$, $y=65\times186=12090$

OR $b-31=403$ and $a-65=5$

$b=434$, $a=70$

$x=31\times70=2170$, $y=65\times434=28210$

OR $b-31=2015$ and $a-65=1$

$b=2046$, $a=66$

$x=31\times66=2046$, $y=65\times2046=132990$

Case 12 Let $x = 65a$, $y = 31b$, where a and b are positive integers

$5\times13\times31(31b+65a)=5\times13\times31ab \Rightarrow 31b+65a=ab$

$5\times13\times31=2015-31b+a(b-65)$

$(b-65)(a-31)=1\times2015$

$b-65=1$ and $a-31=2015$

$b=66$, $a=2046$

$x=65\times2046=132990$, $y=31\times66=2046$

OR $b-65=5$ and $a-31=403$

$b=70$, $a=434$

$x=65\times434=28210$, $y=31\times70=2170$

OR $b-65=13$ and $a-31=155$

$b=78$ $a=186$

$x=65\times186=12090$, $y=31\times78=2418$

OR $b-65=31$ and $a-31=65$

$\frac{1}{x} - \frac{1}{y} = \frac{1}{2015}$

$b = 96$, $a = 96$

$x = 65\times96 = 6240$, $y = 31\times96 = 2976$

OR $b - 65 = 65$ and $a - 31 = 31$

$b = 130$, $a = 62$

$x = 65\times62 = 4030$, $y = 31\times130 = 4030$

OR $b - 65 = 155$ and $a - 31 = 13$

$b = 220$, $a = 44$

$x = 65\times44 = 2860$, $y = 31\times220 = 6820$

OR $b - 65 = 403$ and $a - 31 = 5$

$b = 468$, $a = 36$

$x = 65\times36 = 2340$, $y = 31\times468 = 14508$

OR $b - 65 = 2015$ and $a - 31 = 1$

$b = 2080$, $a = 32$

$x = 65\times32 = 2080$, $y = 31\times2080 = 64480$

Case 13 Let $x = 155a$, $y = 13b$, where a and b are positive integers

$5\times13\times31(13b + 155a) = 5\times13\times31ab \Rightarrow 13b + 155a = ab$

$5\times13\times31 = 2015 - 13b + a(b - 155)$

$(b - 155)(a - 13) = 1\times2015$

$b - 155 = 1$ and $a - 13 = 2015$

$b = 156$, $a = 2028$

$x = 155\times2028 = 314340$, $y = 13\times156 = 2028$

OR $b - 155 = 5$ and $a - 13 = 403$

$b = 160$, $a = 416$

$x = 155\times416 = 64480$, $y = 13\times160 = 2080$

OR $b - 155 = 13$ and $a - 13 = 155$

$b = 168$ $a = 168$

$x = 155\times168 = 26040$, $y = 13\times168 = 2184$

OR $b - 155 = 31$ and $a - 13 = 65$

$b = 186$, $a = 78$

$x = 155\times78 = 12090$, $y = 13\times186 = 2418$

OR $b - 155 = 65$ and $a - 13 = 31$

$b = 220$, $a = 44$

$x = 155\times44 = 6820$, $y = 13\times220 = 2860$

OR $b - 155 = 155$ and $a - 13 = 13$

$b = 310$, $a = 26$

$x = 155\times26 = 4030$, $y = 13\times310 = 4030$

OR $b - 155 = 403$ and $a - 13 = 5$

$b = 558$, $a = 18$

$x = 155\times18 = 2790$, $y = 13\times558 = 7254$

OR $b - 155 = 2015$ and $a - 13 = 1$

$b = 2170$, $a = 14$

$x = 155\times14 = 2170$, $y = 13\times2170 = 28210$

Case 14 Let $x = 403a$, $y = 5b$, where a and b are positive integers

$5\times13\times31(5b + 403a) = 5\times13\times31ab \Rightarrow 5b + 403a = ab$

$5\times13\times31 = 2015 - 5b + a(b - 403)$

$(b - 403)(a - 5) = 1\times2015$

$b - 403 = 1$ and $a - 5 = 2015$

$b = 404$, $a = 2020$

$x = 403\times2020 = 814060$, $y = 5\times404 = 2020$

OR $b - 403 = 5$ and $a - 5 = 403$

$b = 408$, $a = 408$

$\frac{1}{x} - \frac{1}{y} = \frac{1}{2015}$

$x = 403\times408 = 164424$, $y = 5\times408 = 2040$

OR $b - 403 = 13$ and $a - 5 = 155$

$b = 416$ $a = 160$

$x = 403\times160 = 64480$, $y = 5\times416 = 2080$

OR $b - 403 = 31$ and $a - 5 = 65$

$b = 434$, $a = 70$

$x = 403\times70 = 28210$, $y = 5\times434 = 2170$

OR $b - 403 = 65$ and $a - 5 = 31$

$b = 468$, $a = 36$

$x = 403\times36 = 14508$, $y = 5\times468 = 2340$

OR $b - 403 = 155$ and $a - 5 = 13$

$b = 558$, $a = 18$

$x = 403\times18 = 7254$, $y = 5\times558 = 2790$

OR $b - 403 = 403$ and $a - 5 = 5$

$b = 806$, $a = 10$

$x = 403\times10 = 4030$, $y = 5\times403 = 4030$

OR $b - 403 = 2015$ and $a - 5 = 1$

$b = 2418$, $a = 6$

$x = 403\times6 = 2418$, $y = 5\times2418 = 12090$

Case 15 Let $x = 2015a$, $y = b$, where a and b are positive integers

$5\times13\times31(b + 2015a) = 5\times13\times31ab \Rightarrow b + 2015a = ab$

$5\times13\times31 = 2015 - b + a(b - 2015)$

$(b - 2015)(a - 1) = 1\times2015$

$b - 2015 = 1$ and $a - 1 = 2015$

$b = 2016$, $a = 2016$

$x = 2015\times2016 = 4062240$, $y = 2016$

OR $b - 2015 = 5$ and $a - 1 = 403$

$b = 2020$, $a = 404$

$x = 2015\times404 = 814060$, $y = 2020$

OR $b - 2015 = 13$ and $a - 1 = 155$

$b = 2028$ $a = 156$

$x = 2015\times156 = 314340$, $y = 2028$

OR $b - 2015 = 31$ and $a - 1 = 65$

$b = 2046$, $a = 66$

$x = 2015\times66 = 132990$, $y = 2046$

OR $b - 2015 = 65$ and $a - 1 = 31$

$b = 2080$, $a = 32$

$x = 2015\times32 = 64480$, $y = 2080$

OR $b - 2015 = 155$ and $a - 1 = 13$

$b = 2170$, $a = 14$

$x = 2015\times14 = 28210$, $y = 2170$

OR $b - 2015 = 403$ and $a - 1 = 5$

$b = 2418$, $a = 6$

$x = 2015\times6 = 12090$, $y = 2418$

OR $b - 2015 = 2015$ and $a - 1 = 1$

$b = 4030$, $a = 2$

$x = 2015\times2 = 4030$, $y = 4030$

$$\frac{1}{x}-\frac{1}{y}=\frac{1}{2015}$$

$\therefore$ The solutions are (x, y) = (1054, 2210), (1170, 2790), (1240, 3224), (1612, 8060), (1690, 10478), (1846, 22010), (1860, 24180), (1950, 60450), (1984, 128960), (1990, 160394), (2002, 310310), (2010, 810030), (2014, 4058210).

OR (–2210, –1054), (–2790, –1170), (–3224, –1240), (–8060, –1612), (–10478, –1690), (–22010, –1846), (–24180, –1860), (–60450, –1950), (–128960, –1984), (–160394, –1990), (–310310 , –2002), (–810030, –2010), (–4058210, –2014).

OR

(2016, –4062240), (2020, –814060), (2040, –164424), (2046, –132990), (2028, –314340), (2080, –64480), (2170, –28210), (2184, –26040), (2340, –14508), (2418, –12090), (2790, –7254), (2860, –6820), (2976, –6240), (4030, –4030), (6240, –2976), (6820, –2860)

OR

(7254, –2790), (12090, –2418), (14508, –2340), (26040, –2184), (28210, –2170), (64480, –2080), (132990, –2046), (164424, –2040), (314340, –2028), (814060, –2020), (4062240, –2016).