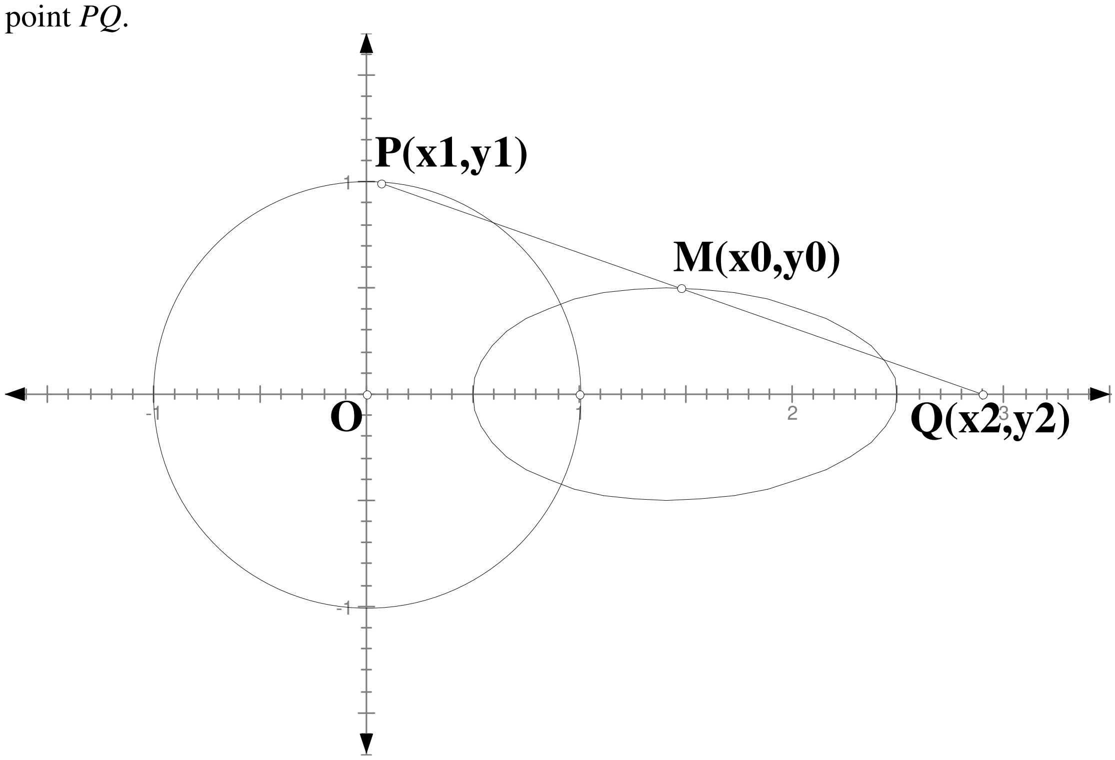
Locus Example 3

Created by Mr. Francis Hung on 20110422 Last updated: August 29, 2021

Example on locus

As shown in the figure, a circle with center at origin and radius = 1 is drawn. $P(x_1,y_1)$ is any point on the circle. $Q(x_2,y_2)$ is a point on the positive x-axis. If $PQ = 2a$ where $a > 1$, find the locus of the mid point PQ.



Method 1

$\because Q$ lies on x-axis, $y_2 = 0$, $x_0 = \dfrac{x_1 + x_2}{2}$, $y_0 = \dfrac{y_1}{2}$

$y_1 = 2y_0$, $x_1 = 2x_0 - x_2$

$\because P(x_1,y_1)$ lies on the circle, $x_1^2 + y_1^2 = 1 \Rightarrow (2x_0 - x_2)^2 + (2y_0)^2 = 1$

$(2x_0 - x_2) = \pm\sqrt{1-(2y_0)^2} \Rightarrow x_2 = 2x_0 \mp \sqrt{1-(2y_0)^2}$ ……….(1)

$\because PQ = 2a$, $MQ = a$, $(x_0 - x_2)^2 + y_0^2 = a^2$ …………..(2)

sub (1) into (2): $\left(x_0 - 2x_0 \pm \sqrt{1-4y_0^2}\right)^2 + (y_0)^2 = a^2$

$x_0^2 \mp 2x_0\sqrt{1-4y_0^2} + 1 - 4y_0^2 + y_0^2 = a^2$

$x_0^2 - 3y_0^2 - a^2 + 1 = \pm 2x_0\sqrt{1-4y_0^2}$

$\left(x_0^2 - 3y_0^2 - a^2 + 1\right)^2 = \left(\pm 2x_0\sqrt{1-4y_0^2}\right)^2$

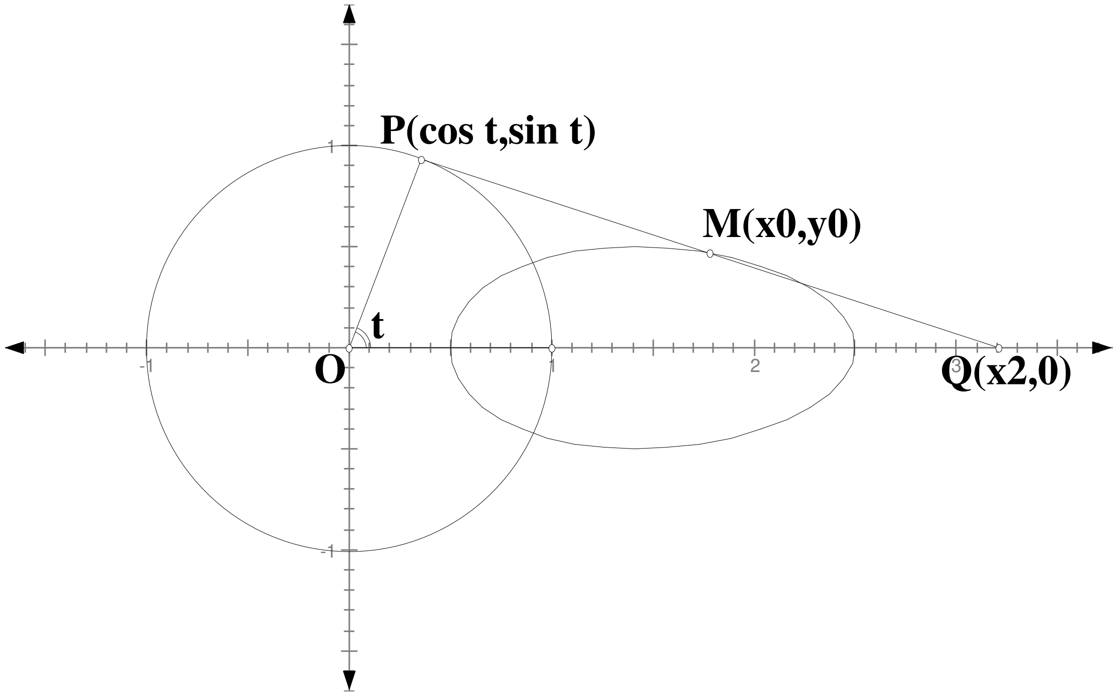
$x_0^4 + 9y_0^4 + a^4 + 1 - 6x_0^2y_0^2 + 2x_0^2 - 2a^2x_0^2 - 6y_0^2 + 6a^2y_0^2 - 2a^2 = 4x_0^2 - 16x_0^2y_0^2$

$x_0^4 + 9y_0^4 + 10x_0^2y_0^2 - 2(1+a^2)x_0^2 + 6(a^2-1)y_0^2 + 1 - 2a^2 + a^4 = 0$

The locus is:

$x^4 + 9y^4 + 10x^2y^2 - 2(1+a^2)x^2 + 6(a^2-1)y^2 + (1-a^2)^2 = 0$

Method 2



Suppose OP makes an angle t with the positive x-axis in anti-clockwise direction.
Then $P = (\cos t, \sin t)$

$\because Q$ lies on x-axis, $y_2 = 0$, $x_0 = \dfrac{\cos t + x_2}{2}$, $y_0 = \dfrac{\sin t}{2}$

$x_2 = 2x_0 - \cos t$ …………(1)

$\sin t = 2\ y_0$

$\cos t = \pm\sqrt{1-4y_0^2}$ …………(2)

By cosine formula on ΔOPQ, $PQ^2 = OP^2 + OQ^2 - 2\ OP\cdot OQ \cos t$

$(2a)^2 = 1^2 + x_2^2 - 2\ x_2 \cos t$ …………(3)

sub (1) into (3): $4a^2 = 1 + (2x_0 - \cos t)^2 - 2(2x_0 - \cos t) \cos t$

$4a^2 = 1 + 4\ x_0^2 - 4x_0 \cos t + \cos^2 t - 4x_0 \cos t + 2 \cos^2 t$

$4a^2 = 1 + 4\ x_0^2 - 8x_0 \cos t + 3 \cos^2 t$ …………(4)

sub (2) into (4)

$4a^2 = 1 + 4\ x_0^2 - 8x_0\ (\pm\sqrt{1-4y_0^2}\ ) + 3\ (\pm\sqrt{1-4y_0^2}\ )^2$

$4a^2 = 1 + 4\ x_0^2 - 8x_0\ (\pm\sqrt{1-4y_0^2}\ ) + 3\ (1 - 4\ y_0^2)$

$4a^2 = 4 + 4\ x_0^2 - 12\ y_0^2 - 8x_0\ (\pm\sqrt{1-4y_0^2}\ )$

$x_0^2 - 3y_0^2 - a^2 + 1 = \pm 2x_0\sqrt{1-4y_0^2}$

$\left(x_0^2 - 3y_0^2 - a^2 + 1\right)^2 = \left(\pm 2x_0\sqrt{1-4y_0^2}\right)^2$

$x_0^4 + 9y_0^4 + a^4 + 1 - 6x_0^2y_0^2 + 2x_0^2 - 2a^2x_0^2 - 6y_0^2 + 6a^2y_0^2 - 2a^2 = 4x_0^2 - 16x_0^2y_0^2$

$x_0^4 + 9y_0^4 + 10x_0^2y_0^2 - 2\left(1 + a^2\right)x_0^2 + 6\left(a^2 - 1\right)y_0^2 + 1 - 2a^2 + a^4 = 0$

The locus is:

$x^4 + 9y^4 + 10x^2y^2 - 2\left(1 + a^2\right)x^2 + 6\left(a^2 - 1\right)y^2 + \left(1 - a^2\right)^2 = 0$