Examples on Mathematical induction: Trigonometry

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1. (a) Prove the identity: $\sin(k+1)\theta \sin\frac{\theta}{2} + \sin\frac{k\theta}{2}\sin\frac{(k+1)\theta}{2} = \sin\frac{(k+1)\theta}{2}\sin\frac{(k+2)\theta}{2}$.

(b) Prove that $\sin x + \sin 2x + \cdots + \sin nx = \dfrac{\sin\frac{1}{2}(n+1)x\sin\frac{1}{2}nx}{\sin\frac{1}{2}x}$ for all positive integer n.

(a) L.H.S. $= \sin(k+1)\theta \sin\frac{\theta}{2} + \sin\frac{k\theta}{2}\sin\frac{(k+1)\theta}{2}$

$$= -\frac{1}{2}\left[\cos\left(k+\frac{3}{2}\right)\theta - \cos\left(k+\frac{1}{2}\right)\theta\right] - \frac{1}{2}\left[\cos\left(k+\frac{1}{2}\right)\theta - \cos\left(-\frac{\theta}{2}\right)\right]$$

$$= -\frac{1}{2}\left[\cos\left(k+\frac{3}{2}\right)\theta - \cos\left(\frac{1}{2}\right)\theta\right]$$

$$= \sin\frac{(k+1)\theta}{2}\sin\frac{(k+2)\theta}{2} = \text{R.H.S.}$$

(b) $n = 1$, L.H.S. $= \sin x$, R.H.S. $= \dfrac{\sin\frac{1}{2}(1+1)x\sin\frac{1}{2}x}{\sin\frac{1}{2}x} = \sin x$

$\therefore$ L.H.S. = R.H.S., it is true for $n = 1$.

Suppose $\sin x + \sin 2x + \cdots + \sin kx = \dfrac{\sin\frac{1}{2}(k+1)x\sin\frac{1}{2}kx}{\sin\frac{1}{2}x}$

Add $\sin(k+1)x$ to both sides,

L.H.S. $= \sin x + \sin 2x + \cdots + \sin kx + \sin(k+1)x$

$$= \sin(k+1)x + \frac{\sin\frac{1}{2}(k+1)x\sin\frac{1}{2}kx}{\sin\frac{1}{2}x}$$

$$= \frac{\sin(k+1)x\sin\frac{1}{2}x + \sin\frac{1}{2}(k+1)x\sin\frac{1}{2}kx}{\sin\frac{1}{2}x}$$

$$= \frac{\sin\frac{1}{2}(k+1)x\sin\frac{1}{2}(k+2)x}{\sin\frac{1}{2}x} \quad \text{(by (a))}$$

= R.H.S.

$\therefore$ If it is true for $n = k$ then it is also true for $n = k + 1$.

By the principle of mathematical induction, it is true for all positive integer n.

2. Prove that $\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}$ for all positive integer n .

$n = 1$, L.H.S. $= \frac{1}{2} + \cos x$

R.H.S. $= \frac{\sin\left(1+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x} = \frac{\sin\left(\frac{3}{2}\right)x}{2\sin\frac{1}{2}x}$

$= \frac{3\sin\frac{1}{2}x - 4\sin^3\frac{1}{2}x}{2\sin\frac{1}{2}x}$

$= \frac{3 - 4\sin^2\frac{1}{2}x}{2}$

$= \frac{3 - 2(1-\cos x)}{2}$

$= \frac{1 + 2\cos x}{2}$ = L.H.S.

It is true for $n = 1$.

Suppose $\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos kx = \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}$ for some positive integer k .

When $n = k + 1$,

L.H.S. $= \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos kx + \cos (k+1)x$

$= \cos (k+1)x + \frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}$

$= \frac{2\cos(k+1)x\sin\frac{1}{2}x + \sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}$

$= \frac{\sin\left(k+\frac{3}{2}\right)x - \sin\left(k+\frac{1}{2}\right)x + \sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}$

$= \frac{\sin\left(k+1+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}$ = R.H.S.

$\therefore$ If it is true for $n = k$ then it is also true for $n = k + 1$.

By the principle of mathematical induction, it is true for all positive integer n .

3. **1999 Paper 2 Q12**

(a) Prove, by mathematical induction, that

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}.$$

where $\sin\theta \neq 0$, for all positive integers n.

(b) Using (a) and the substitution $\theta = \frac{\pi}{2} - x$, or otherwise,

show that $\sin x - \sin 3x + \sin 5x = \frac{\sin 6x}{2\cos x}$, where $\cos x \neq 0$.

(a) For $n = 1$, L.H.S. = $\cos\theta$

R.H.S. = $\frac{\sin 2\theta}{2\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$ = L.H.S.

$\therefore$ The statement is true for $n = 1$.

Assume $\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2k-1)\theta = \frac{\sin 2k\theta}{2\sin\theta}$.

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2k-1)\theta + \cos(2k+1)\theta$$

$$= \frac{\sin 2k\theta}{2\sin\theta} + \cos(2k+1)\theta$$

$$= \frac{\sin 2k\theta + \sin(2k+2)\theta - \sin 2k\theta}{2\sin\theta}$$

$$= \frac{\sin 2(k+1)\theta}{2\sin\theta}$$

The statement is also true for $n = k + 1$ if it is true for $n = k$.

By the principle of induction, the statement is true for all positive integer n.

(b) Put $\theta = \frac{\pi}{2} - x$, $n = 3$.

$$\cos\left(\frac{\pi}{2} - x\right) + \cos 3\left(\frac{\pi}{2} - x\right) + \cos 5\left(\frac{\pi}{2} - x\right) = \frac{\sin 2\times 3\left(\frac{\pi}{2} - x\right)}{2\sin\left(\frac{\pi}{2} - x\right)}$$

$$\sin x - \sin 3x + \sin 5x = \frac{\sin 6x}{2\cos x}$$

4. Prove that $\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\cdots+\frac{1}{2^n}\tan\frac{x}{2^n}=\frac{1}{2^n}\cot\frac{x}{2^n}-\cot x$ where $x\neq m\pi$ for $n = 1,2, \cdots$

Note that $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}\to \cot x=\frac{1-\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}=\frac{1}{2}\cot\frac{x}{2}-\frac{1}{2}\tan\frac{x}{2}$ ……(*)

$n = 1$, L.H.S. $=\frac{1}{2}\tan\frac{x}{2}$

R.H.S. $=\frac{1}{2}\cot\frac{x}{2}-\cot x$

= L.H.S. by (*)

It is true for $n = 1$.

Suppose $\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\cdots+\frac{1}{2^k}\tan\frac{x}{2^k}=\frac{1}{2^k}\cot\frac{x}{2^k}-\cot x$

When $n = k + 1$, L.H.S. $=\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\cdots+\frac{1}{2^k}\tan\frac{x}{2^k}+\frac{1}{2^{k+1}}\tan\frac{x}{2^{k+1}}$

$$=\frac{1}{2^{k+1}}\tan\frac{x}{2^{k+1}}+\frac{1}{2^k}\cot\frac{x}{2^k}-\cot x$$

$$=\frac{1}{2^k}\cdot\left(\frac{1}{2}\tan\frac{x}{2^{k+1}}\right)+\frac{1}{2^k}\cot\frac{x}{2^k}-\cot x$$

$$=\frac{1}{2^k}\cdot\left(\frac{1}{2}\cot\frac{x}{2^{k+1}}-\cot\frac{x}{2^k}\right)+\frac{1}{2^k}\cot\frac{x}{2^k}-\cot x\text{, by (*)}$$

$$=\frac{1}{2^{k+1}}\cot\frac{x}{2^{k+1}}-\cot x=\text{R.H.S.}$$

It is also true for $n = k + 1$

By the principle of mathematical induction, the formula is true for all positive integer n .

5. Prove that $\sum_{r=1}^{n}\cot^{-1}(2r^2)=\tan^{-1}(2n+1)-\frac{1}{4}\pi=\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)$ for all positive integer n.

$\cot\theta=\frac{1}{\tan\theta}\Rightarrow\cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)$

$\therefore\ \sum_{r=1}^{n}\cot^{-1}(2r^2)=\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)$

$\tan^{-1}(2n+1)-\frac{1}{4}\pi=\tan^{-1}(2n+1)-\tan^{-1}1=\tan^{-1}\frac{2n+1-1}{1+2n+1}=\tan^{-1}\frac{n}{n+1}$

Let $P(n)\equiv$ “$\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)=\tan^{-1}\frac{n}{n+1}$.” for all positive integers n .

$n = 1$, $\sum_{r=1}^{1}\tan^{-1}\left(\frac{1}{2r^2}\right)=\tan^{-1}\left(\frac{1}{2}\right)=$ R.H.S.

Suppose $\sum_{r=1}^{k}\tan^{-1}\left(\frac{1}{2r^2}\right)=\tan^{-1}\frac{k}{k+1}$

$n = k + 1$, $\sum_{r=1}^{k+1}\tan^{-1}\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{k}\tan^{-1}\left(\frac{1}{2r^2}\right)+\tan^{-1}\frac{1}{2(k+1)^2}=\tan^{-1}\frac{k}{k+1}+\tan^{-1}\frac{1}{2(k+1)^2}$

$$=\tan^{-1}\left(\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\frac{k}{(k+1)}\cdot\frac{1}{2(k+1)^2}}\right)=\tan^{-1}\frac{(k+1)(2k^2+2k+1)}{2k^3+6k^2+5k+2}=\tan^{-1}\frac{k+1}{k+2}$$

It is also true for $n = k + 1$. By M.I., the formula is true for all positive integer n .