Supplementary Exercise on Set Theory

Created by Mr. Francis Hung on 20090415 Last updated: April 22, 2011

Define $A \Delta B = (A \setminus B) \cup (B \setminus A)$, it is called the symmetric difference of A and B.

If A has finite number of elements, $n(A)$ = number of elements in A

1. If A, B and C are finite sets, give an example for which
   (a) $A \cup B = A \cup C$, and $B \neq C$
   (b) $A \cap B = A \cap C$, and $B \neq C$
2. If $A \subset B$ and $A \cap B = \phi$, prove that $A = \phi$
3. Let A, B and C be sets such that
   (i) $A \cup B = A \cup C$, and
   (ii) $A \cap B = A \cap C$,
   Prove that $B = C$
4. Prove that $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subset A$
5. Prove that $\mathrm{P}(E \cap F) = \mathrm{P}(E) \cap \mathrm{P}(F)$
6. (a) Prove that $\mathrm{P}(E) \cup \mathrm{P}(F) \subseteq \mathrm{P}(E \cup F)$
   (b) Give an example to show that equality may not hold.
   (c) Find a necessary and sufficient condition that equality hold.
   (d) Prove your assertion in (c).
7. Is $\mathrm{P}(E \setminus F) = \mathrm{P}(E) \setminus \mathrm{P}(F)$? Prove your assertion.
8. Prove that $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$
9. Prove that $A \Delta B = (A \cup B) \setminus (A \cap B)$
10. If A and B are finite sets, prove that $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
11. If $A \Delta B = A \Delta C$, prove that $B = C$
12. Two sets A and B such that $n(A \cup B) = 75$ and $n(A \cap B) = 17$. If $n(A) - n(B)=22$
    Find (a) $n(B)$,
    (b) $n(B \setminus A)$,
    (c) $n(A \setminus B)$,
    (d) $n(A \Delta B)$
13. 210 students sat for F.5 mock examination for Physics, Chemistry and Biology. 30 failed in Physics, 26 failed in Chemistry and 22 failed in Biology. If 178 passed all three subjects, find the least and the greatest numbers of students who failed in all three subjects.
14. Let S be a non-empty set of subsets of a set E such that the following conditions are satisfied:
    (i) $A_1 \in S, A_2 \in S \Rightarrow A_1 \cup A_2 \in S$
    (ii) $A_1 \in S \Rightarrow E \setminus A_1 \in S$ (i.e. $A_1' \in S$)
    Prove that
    (a) $E \in S$, $\phi \in S$, and
    (b) For any $A_1, A_2 \in S$, $A_1 \cap A_2 \in S$ and $A_1 \setminus A_2 \in S$

1. If A, B and C are finite sets, give an example for which
   (*a*) $A \cup B = A \cup C$, and $B \neq C$
   $A = \{a, b, c\}$, $B = \{b, c\}$, $C = \{a, b\}$
   $A \cup B = A = A \cup C$, but $B \neq C$.
   (*b*) $A \cap B = A \cap C$, and $B \neq C$
   $A = \{1, 2, 3\}$, $B = \{1, 4\}$, $C = \{1, 5\}$
   $A \cap B = \{1\} = A \cap C$, but $B \neq C$
2. If $A \subset B$ and $A \cap B = \phi$, prove that $A = \phi$
   $A \subset B \Rightarrow A \cap B = A$
   $\Rightarrow A = \phi$
3. Let A, B and C be sets such that
   (*i*) $A \cup B = A \cup C$, and
   (*ii*) $A \cap B = A \cap C$,
   Prove that $B = C$
   $A \cup B = A \cup C$
   $(A \cup B) \cap C = (A \cup C) \cap C$
   $(A \cap C) \cup (B \cap C) = C$
   $(A \cap B) \cup (B \cap C) = C$
   $B \cap (A \cup C) = C$
   $B \cap (A \cup B) = C$
   $\therefore B = C$
4. Prove that $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subset A$
   ($\Leftarrow$) $C \subset$ A given
   $C = A \cap C$
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   $= (A \cap B) \cup C$
   ($\Rightarrow$) $(A \cap B) \cup C = A \cap (B \cup C)$ given
   $C \subset (A \cap B) \cup C = A \cap (B \cup C) \subset A$
   $\therefore C \subset A$
5. Prove that $\mathrm{P}(E) \cap \mathrm{P}(F) = \mathrm{P}(E \cap F)$
   $A \in \mathrm{P}(E) \cap \mathrm{P}(F) \Leftrightarrow A \in \mathrm{P}(E) \wedge A \in \mathrm{P}(F)$
   $\Leftrightarrow A \subset E \wedge A \subset F$
   $\Leftrightarrow A \subset E \cap F$
   $\Leftrightarrow A \in \mathrm{P}(E \cap F)$
6. (*a*) Prove that $\mathrm{P}(E) \cup \mathrm{P}(F) \subseteq \mathrm{P}(E \cup F)$
   $A \in \mathrm{P}(E) \cup \mathrm{P}(F) \Rightarrow A \in \mathrm{P}(E) \vee A \in \mathrm{P}(F)$
   $\Rightarrow A \subset E \vee A \subset F$
   $\Rightarrow A \subset E \cup F$
   $\Rightarrow A \in \mathrm{P}(E \cup F)$
   (*b*) Give an example to show that equality may not hold.
   Let $E = \{1\}$, $F = \{2\}$
   $P(E) = \{\phi, \{1\}\}$, $P(F) = \{\phi, \{2\}\}$
   $P(E \cup F) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$
   $P(E) \cup P(F) = \{\phi, \{1\}, \{2\}\}$
   $\therefore \mathrm{P}(E) \cup \mathrm{P}(F) \neq \mathrm{P}(E \cup F)$
   (*c*) Find a necessary and sufficient condition that equality hold.
   The condition for equality holds is "$E \subset F$ or $F \subset E$"

(d) Prove your assertion in (c).

To show that $P(E) \cup P(F) = P(E \cup F)$ if and only if $E \subset F$ or $F \subset E$

($\Rightarrow$) Suppose $E \not\subset F \wedge F \not\subset E$

$\Rightarrow \exists\, x \in E \setminus F \wedge \exists\, y \in F \setminus E$

$\Rightarrow \{x, y\} \subset E \cup F$

$\Rightarrow \{x, y\} \in P(E \cup F)$

clearly $\{x, y\} \notin P(E) \wedge \{x, y\} \notin P(F)$

$\Rightarrow \{x, y\} \notin P(E) \cup P(F)$

$\Rightarrow P(E) \cup P(F) \neq P(E \cup F)$

($\Leftarrow$) $E \subset F \Rightarrow P(E) \subset P(F)$

$\Rightarrow P(F) \cup P(E) = P(F)$

$\Rightarrow P(E \cup F) = P(F) = P(E) \cup P(F)$

$F \subset E \Rightarrow P(F) \subset P(E)$

$\Rightarrow P(F) \cup P(E) = P(E)$

$\Rightarrow P(E \cup F) = P(E) = P(E) \cup P(F)$

7. Is $P(E \setminus F) = P(E) \setminus P(F)$? Prove your assertion.

False. Let $E = \{1\}$, $F = \phi$

$E \setminus F = E = \{1\}$

$P(E \setminus F) = P(E) = \{\{1\}, \phi\}$

$P(E) \setminus P(F) = \{\{1\}, \phi\} \setminus \{\phi\} = \{\{1\}\}$

But $P(E \setminus F) \neq P(E) \setminus P(F)$

8. Prove that $A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)$

LHS $= A \cap [(B \cap \overline{C}) \cup (C \cap \overline{B})]$

$= (A \cap B \cap \overline{C}) \cup (A \cap C \cap \overline{B})$

RHS $= [(A \cap B) \cap \overline{A \cap C}] \cup [\overline{A \cap B} \cap (A \cap C)]$

$= [(A \cap B) \cap (\overline{A} \cup \overline{C})] \cup [(\overline{A} \cup \overline{B}) \cap (A \cap C)]$

$= (A \cap B \cap \overline{A}) \cup (A \cap B \cap \overline{C}) \cup (\overline{A} \cap A \cap C) \cup (\overline{B} \cap A \cap C)$

$= (A \cap B \cap \overline{C}) \cup (A \cap C \cap \overline{B})$

$\therefore$ LHS = RHS

9. Prove that $A \,\Delta\, B = (A \cup B) \setminus (A \cap B)$

$A \,\Delta\, B = (A \cap \overline{B}) \cup (B \cap \overline{A})$

$= [(A \cap \overline{B}) \cup B] \cap [(A \cap \overline{B}) \cup \overline{A}]$

$= [(A \cup B) \cap (\overline{B} \cup B) \cap (A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})$

$= (A \cup B) \cap (\overline{B} \cup \overline{A})$

$= (A \cup B) \cap (\overline{A \cap B})$

$= (A \cup B) \setminus (A \cap B)$

10. If A and B are finite sets, prove that $n(A \,\Delta\, B) = n(A) + n(B) - 2n(A \cap B)$

We shall make use of the formula: if $X \cap Y = \phi$, then $n(X \cup Y) = n(X) + n(Y)$

Let $X = A \setminus B$, $Y = B \setminus A$

$X \cap Y = A \cap \overline{B} \cap B \cap \overline{A} = \phi$

$n(A \,\Delta\, B) = n(A \setminus B \cup B \setminus A)$

$= n(A \setminus B) + n(B \setminus A)$

$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

11. If $A \Delta B = A \Delta C$, prove that $B = C$

We first prove that $(X \Delta Y) \Delta Z = X \Delta (Y \Delta Z)$

$(X \Delta Y) \Delta Z = \{[(X \cap \overline{Y}) \cup (\overline{X} \cap Y)] \cap \overline{Z}\} \cup \{\overline{[(X \cap \overline{Y}) \cup (\overline{X} \cap Y)]} \cap Z\}$

$= (X \cap \overline{Y} \cap \overline{Z}) \cup (\overline{X} \cap Y \cap \overline{Z}) \cup [(\overline{X} \cup Y) \cap (X \cup \overline{Y}) \cap Z]$

$= (X \cap \overline{Y} \cap \overline{Z}) \cup (\overline{X} \cap Y \cap \overline{Z}) \cup \{[(\overline{X} \cup Y) \cap X] \cup [(\overline{X} \cup Y) \cap \overline{Y}] \cap Z\}$

$= (X \cap \overline{Y} \cap \overline{Z}) \cup (\overline{X} \cap Y \cap \overline{Z}) \cup \{[(\overline{X} \cap X) \cup (X \cap Y) \cup (\overline{X} \cap \overline{Y}) \cup (Y \cap \overline{Y})] \cap Z\}$

$= (X \cap \overline{Y} \cap \overline{Z}) \cup (\overline{X} \cap Y \cap \overline{Z}) \cup \{[(X \cap Y) \cup (\overline{X} \cap \overline{Y})] \cap Z\}$

$= (X \cap \overline{Y} \cap \overline{Z}) \cup (\overline{X} \cap Y \cap \overline{Z}) \cup (X \cap Y \cap Z) \cup (\overline{X} \cap \overline{Y} \cap Z)$

$X \Delta (Y \Delta Z) = (Y \Delta Z) \Delta X$

$= (Y \cap \overline{Z} \cap \overline{X}) \cup (\overline{Y} \cap Z \cap \overline{X}) \cup (X \cap Y \cap Z) \cup (\overline{Y} \cap \overline{Z} \cap X)$

$\therefore (X \Delta Y) \Delta Z = X \Delta (Y \Delta Z)$

Now $A \Delta B = A \Delta C$ given

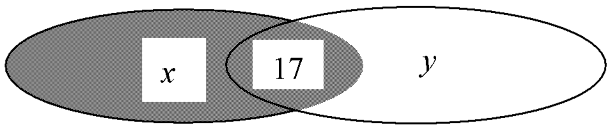
$A \Delta (A \Delta B) = A \Delta (A \Delta C)$

$(A \Delta A) \Delta B = (A \Delta A) \Delta C$

$\phi \Delta B = \phi \Delta C$

$B = C$

12. Two sets A and B such that $n(A \cup B) = 75$ and $n(A \cap B) = 17$. If $n(A) - n(B) = 22$



$n(A \cup B) = 75 \Rightarrow x + y + 17 = 75$

$n(A \cap B) = 17$

$n(A) - n(B) = 22 \Rightarrow (x + 17) - (y - 17) = 22$

$$\begin{cases} x + y = 58 \\ x - y = 22 \end{cases} \Rightarrow \begin{cases} x = 40 \\ y = 18 \end{cases}$$

(*a*) $n(B) = y + 17 = 35$

(*b*) $n(B \setminus A) = y = 18$

(*c*) $n(A \setminus B) = x = 40$

(*d*) $n(A \Delta B) = x + y = 58$

13. 210 students sat for F.5 mock examination for Physics, Chemistry and Biology. 30 failed in Physics, 26 failed in Chemistry and 22 failed in Biology. If 178 passed all three subjects, find the least and the greatest numbers of students who failed in all three subjects.

$\because$ 22 failed in Biology.

$\therefore$ Greatest number of students failed in three subjects = 22

Let P = set of students who passed Physics.

C = set of students who passed Chemistry.

B = set of students who passed Biology.

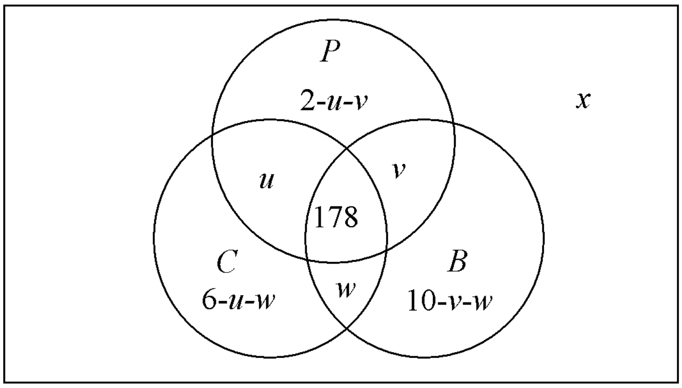
Draw a Venn diagram and let

u = number of students who passed in Physics and Chemistry only

v = number of students who passed in Physics and Biology only

w = number of students who passed in Biology and Chemistry only

x = number of students who failed in all three subjects.



$210 = 180 + (6 - u) + (10 - v - w) + x$

$x = 14 + (u + v + w) \geq 14$

The least number of students who failed in all three subjects is 14.

14. Let S be a non-empty set of subsets of a set E such that the following conditions are satisfied:

(i) $A_1 \in S, A_2 \in S \Rightarrow A_1 \cup A_2 \in S$

(ii) $A_1 \in S \Rightarrow E \setminus A_1 \in S$ (i.e. $A_1' \in S$)

Prove that

(a) $E \in S$, $\phi \in S$,

$S \neq \phi \Rightarrow \exists\, A \subset E$ such that $A \in S$

$\because A \subset E \Rightarrow E \setminus A \subset E$ by (ii)

$\therefore A \cup (E \setminus A) \in S$ by (i)

$\Rightarrow E \in S$

$E \setminus E \in S$ by (ii)

$\Rightarrow \phi \in S$

(b) For any $A_1, A_2 \in S$, $A_1 \cap A_2 \in S$ and $A_1 \setminus A_2 \in S$

$A_1, A_2 \in S \Rightarrow A_1', A_2' \in S$ by (ii)

$\Rightarrow A_1' \cup A_2' \in S$ by (i)

$\Rightarrow (A_1' \cup A_2')' \in S$ by (ii)

$\Rightarrow A_1 \cap A_2 \in S$ De Morgan's law

$A_1, A_2 \in S \Rightarrow A_1 \in S, A_2' \in S$

$\Rightarrow A_1 \cap A_2' \in S$ proved above

i.e. $A_1 \setminus A_2 \in S$