The 3 axioms in Probability

3.1 Let E, F be events and S be a non-empty sample space.

$P(E) = \dfrac{n(E)}{n(S)}$

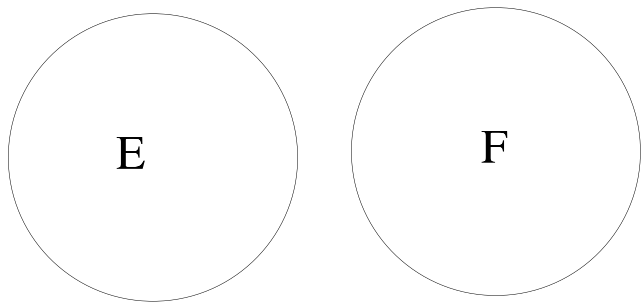
$\because 0 \le n(E) \le n(S)$

$\therefore \dfrac{0}{n(S)} \le \dfrac{n(E)}{n(S)} \le \dfrac{n(S)}{n(S)}$

$0 \le P(E) \le 1$

If $E = S$, $P(S) = \dfrac{n(S)}{n(S)} = 1$

If $E \cap F = \phi$ then $n(E \cup F) = n(E) + n(F)$



$\therefore P(E \cup F) = \dfrac{n(E \cup F)}{n(S)}$

$= \dfrac{n(E) + n(F)}{n(S)}$

$= \dfrac{n(E)}{n(S)} + \dfrac{n(F)}{n(S)}$

$= P(E) + P(F)$

Conclusion

Axiom 1 $0 \le P(E) \le 1$

Axiom 2 $P(S) = 1$, S = sample space

Axiom 3 $P(E \cup F) = P(E) + P(F)$ if $E \cap F = \phi$

Example 1
An unbiased (公平) coin is tossed (擲)3 times. What is the probability of getting
(a) 3 heads;
(b) 2 heads and 1 tail;
(c) at least 1 tail?

S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}, n(S) = 8

(a) E = event of 3 heads = {HHH}, $n(E) = 1$

$P(E) = \dfrac{n(E)}{n(S)} = \dfrac{1}{8}$

(b) F = event of 2 heads and 1 tail = {HHT, HTH, THH}, $n(F) = 3$

$P(E) = \dfrac{3}{8}$

(c) G = event of at least 1 tail

G' = S \ G = event of no tail = E

$\because G \cap E = \phi$

$P(G) = 1 - P(G')$

$= 1 - P(E)$

$= 1 - \dfrac{1}{8} = \dfrac{7}{8}$

3.2 Theorem ϕ = empty set. A, B = arbitrary events

(a) $P(\phi) = 0$

(b) $P(A') = 1 - P(A)$

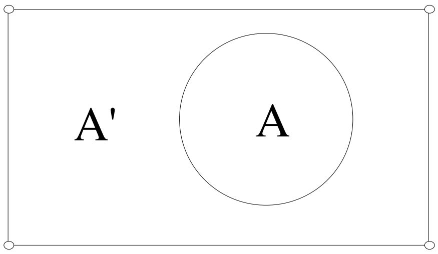
(c) $P(A \cap B') = P(A) - P(A \cap B)$

(d) If $A \subset B$, then $P(A) \le P(B)$

(e) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof: (a) $n(\phi) = 0$, (number of elements in the empty set)

$$P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$$



(b) $A \cap A' = \phi$

$A \cup A' = S$, the sample space.

$P(S) = 1$ (by axiom 2)

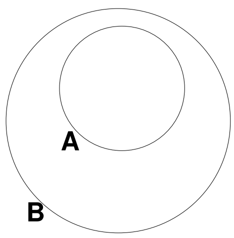
$P(A \cup A') = 1$

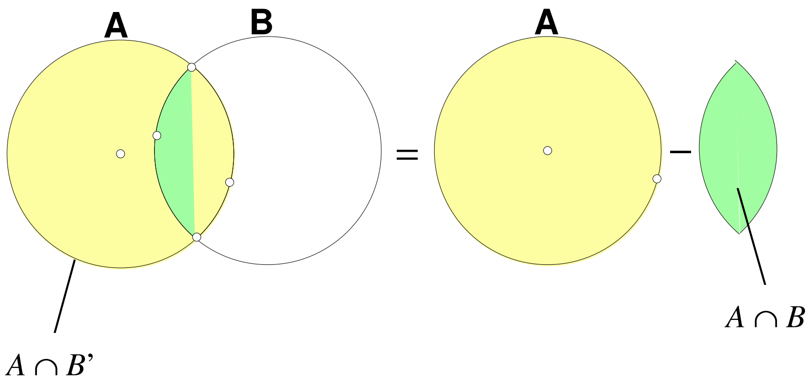
$P(A) + P(A') = 1$ (by axiom 3)

$P(A') = 1 - P(A)$

(c) $n(A \cap B') = n(A) - n(A \cap B)$

$P(A \cap B') = P(A) - P(A \cap B)$

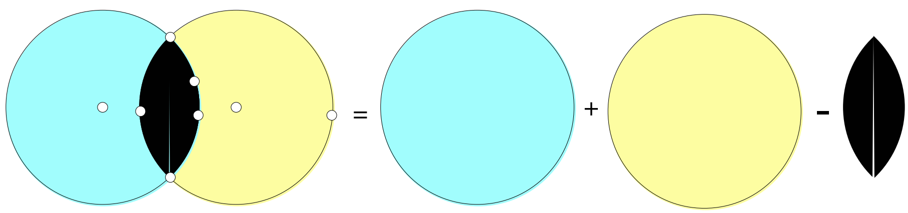




(d) If $A \subset B$, then $n(A) \le n(B)$

$$P(A) = \frac{n(A)}{n(S)} \le \frac{n(B)}{n(S)} = P(B)$$

(e)



$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$

3.3 Miscellaneous Examples

Example 2 A number is picked at random from the set $\{1, 2, 3, \cdots, 70\}$.

Let A = event of multiple of 2, B be the event of multiple of 3.

Find $P(A)$, $P(B)$, $P(A \cap B)$ and $P(A \cup B)$.

Solution: $A = \{2, 4, 6, \cdots, 70\}$, $n(A)$ = ____

$B = \{3,$ ________________ $\}$, $n(B)$ = _____

$A \cap B$ = {multiple of 6} = { ______________________ }, $n(A \cap B)$ = ________

$P(A)$ = __________; $P(B)$ = ____________; $P(A \cap B)$ = ____________

By theorem (e), $P(A \cup B)$ = ______+ ______ − _______ = _______

[Ans. $\frac{1}{2}, \frac{23}{70}, \frac{11}{70}, \frac{47}{70}$]

Example 3 In a class of 45 students, 25 students failed in Mathematics, 20 students failed in Physics and 10 students failed in both subjects. A student is selected at random.

(a) How many students passed in both subjects ?

What is the probability that the selected student passed in both subjects ?

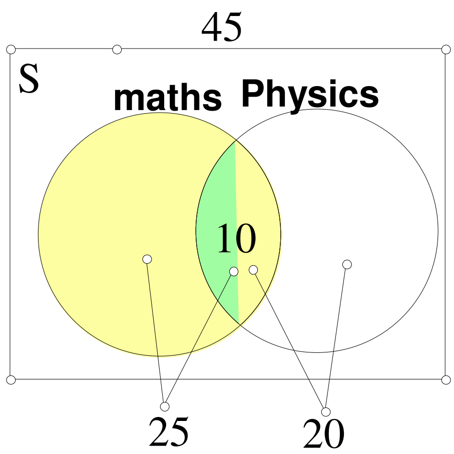
(b) What is the probability that the selected student failed in Mathematics but passing in Physics ?

Solution (a) S = {the class}, $n(S) = 45$

A = {students failed in Maths}, $n(A) = 25$

B = {students failed in Physics}, $n(B) = 20$

$A \cap B$ ={students failed in both subjects}, $n(A \cap B) = 10$



$P(A \cup B)$

= P(the selected student failed in Maths or Physics)

= $P(A) + P(B) - P(A \cap B)$ (theorem (e))

= ______________________________

P(passed in both subjects)

= $1 - P(A \cup B)$ (theorem (b))

= ______________________

(b) Number of students failed in Maths but passed in Physics = $n(A \cap B')$

$\therefore$ $P(A \cap B') = P(A) - P(A \cap B)$ (theorem (c))

= ___________________

[Ans. 10, $\frac{2}{9}, \frac{1}{3}$]