08-09 Individual	**1**	$\frac{2300}{9801}$	**2**	2	**3**	60	**4**	4	**5**	$\frac{113}{4}$ (= 28.25)	**Spare**
	6	$\frac{2007}{2008}$	**7**	$9\sqrt{3}$	**8**	7	**9**	32	**10**	200	$-\frac{3}{2}$

08-09 Group	**1**	105	**2**	$2\sqrt{2}$	**3**	9	**4**	7	**5**	16	**Spare**
	6	$\frac{2}{3}$	**7**	–4	**8**	$\frac{21\sqrt{2}}{5}$	**9**	10	**10**	$\frac{8}{5}$ (= 1.6)	16

Individual Events

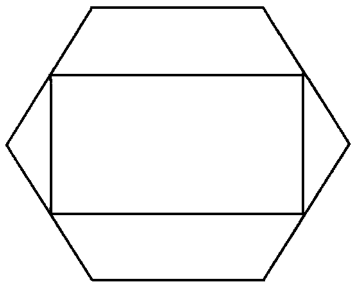
I1 Let $x=0.\dot{2}\dot{3}+0.00\dot{2}\dot{3}+0.0000\dot{2}\dot{3}+0.000000\dot{2}\dot{3}+\cdots$, find the value of x.

Reference: 2000 HI1 Let $x=0.\dot{1}\dot{7}+0.0\dot{1}\dot{7}+0.00\dot{1}\dot{7}+\cdots$

$0.\dot{2}\dot{3}=\frac{23}{99}$; $0.00\dot{2}\dot{3}=\frac{23}{9900}$; $0.0000\dot{2}\dot{3}=\frac{23}{990000}$; $\cdots$

$$x=\frac{23}{99}+\frac{23}{9900}+\frac{23}{990000}+\cdots$$
$$=\frac{23}{99}\left(1+\frac{1}{100}+\frac{1}{10000}+\cdots\right)$$
$$=\frac{23}{99}\cdot\frac{1}{1-\frac{1}{100}}$$
$$=\frac{23}{99}\cdot\frac{100}{99}$$
$$=\frac{2300}{9801}$$

I2 In Figure 1, a regular hexagon and a rectangle are given. The vertices of the rectangle are the midpoints of four sides of the hexagon. If the ratio of the area of the rectangle to the area of the hexagon is 1 : q, find the value of q.



Let one side of the hexagon be $2a$, the height of the rectangle be x, the length be y.

$x = 2a\cos 30° = \sqrt{3}a$

$y = 2a\cos 60° + 2a = 3a$

Ratio of area $=\sqrt{3}a\cdot 3a : 6\times\frac{1}{2}(2a)^2\sin 60^\circ$

$= 3\sqrt{3}:6\sqrt{3}$

$= 1 : 2$

$q = 2$

The following **method** is provided by Mr. Jimmy Pang from Sai Kung Sung Tsun Catholic School (Secondary).

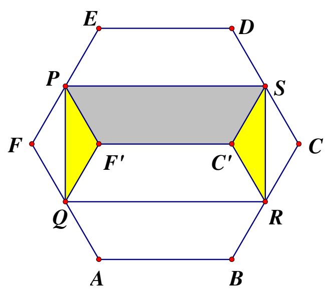
Let the hexagon be $ABCDEF$, the rectangle be $PQRS$ as shown.

Fold ΔFPQ along PQ to $\Delta F'PQ$. Fold ΔCSR along SR to $\Delta C'SR$.

Fold $EDSP$ along PS to $F'C'SP$. Fold $ABRQ$ along QR to $F'C'RQ$.

Then the folded figure covers the rectangle completely.

$\therefore$ Ratio of area = 1 : 2; q = 2



I3 Let $16\sin^4\theta° = 5 + 16\cos^2\theta°$ and $0 \le \theta \le 90$, find the value of θ.

$16\sin^4\theta° = 5 + 16(1 - \sin^2\theta°)$

$16\sin^4\theta° + 16\sin^2\theta° - 21 = 0$

$(4\sin^2\theta° - 3)(4\sin^2\theta° + 7) = 0$

$\sin^2\theta° = \frac{3}{4}$ or $-\frac{7}{4}$ (rejected)

$\sin\theta° = \frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$ (rejected)

$\theta = 60$

I4 Let m be the number of positive factors of gcd(2008, 4518), where gcd(2008, 4518) is the greatest common divisor of 2008 and 4518. Find the value of m.

$2008 = 8\times 251$; $4518 = 2\times 9\times 251$

$\gcd = 2\times 251 = 502$

The positive factors are 1, 2, 251, 502

$m = 4$

I5 Given that $x^2 + (y - 3)^2 = 7$, where x and y are real numbers. If the maximum value of $5y + x^2$ is k, find the value of k. **(Reference: 2003 FG1.2, 2011 HI2)**

$x^2 = 7 - (y - 3)^2$; sub. into $5y + x^2 = 5y + 7 - (y - 3)^2$

$= 5y + 7 - y^2 + 6y - 9$

$= -y^2 + 11y - 2 = -(y^2 - 11y + 5.5^2 - 5.5^2) - 2$

$= -(y - 5.5)^2 + 30.25 - 2 = = -(y - 5.5)^2 + 28.25$

Maximum value $= k = 28.25$

Method 2 $z = 5y + x^2 = -y^2 + 11y - 2$

$y^2 - 11y + (z + 2) = 0$

Discriminant $\Delta \ge 0$ for all real value of y.

$\therefore (-11)^2 - 4(1)(z + 2) \ge 0 \Rightarrow 121 - 4z - 8 \ge 0$

$z \le \frac{113}{4}$; $k = \frac{113}{4}$

I6 Let $f_1(x) = \frac{1}{1-x}$ and $f_n(x) = f_1(f_{n-1}(x))$, where $n = 2, 3, 4, \ldots$. Find the value of $f_{2009}(2008)$.

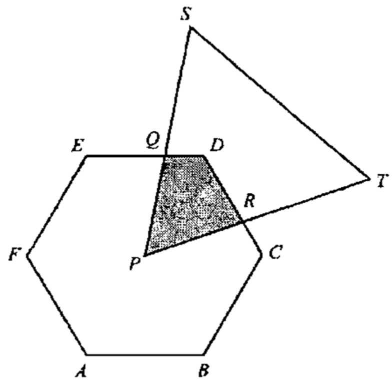
Reference: 1999 FI2.4

$f_2(x) = f_1(f_1(x)) = f_1\left(\frac{1}{1-x}\right) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{1-x-1} = \frac{x-1}{x}$

$f_3(x) = f_1(f_2(x)) = f_1\left(\frac{x-1}{x}\right) = \frac{1}{1-\left(1-\frac{1}{x}\right)} = x$, $\therefore f_{3n}(x) = x$ for all positive integer n.

$f_{2009}(2008) = f_2(f_{2007}(2008)) = f_2(f_{3(669)}(2008)) = f_2(2008) = \frac{2008-1}{2008} = \frac{2007}{2008}$

I7 In Figure 2, $ABCDEF$ is a regular hexagon centred at the point P. ΔPST is an equilateral triangle. It is given that $AB = 6$ cm, $QD = 2$ cm and $PT = 12$ cm. If the area of the common part of the hexagon and triangle is c cm^2, find the value of c.



Reference: 2016 FI2.3

Join PD and CP. ΔCDP is an equilateral triangle, side = 6 cm

$\angle QPR + \angle QDR = 60° + 120° = 180°$

$\therefore$ $DQPR$ is a cyclic quadrilateral (opp. sides supp.)

$\angle PQD = \angle PRC$ (ext. $\angle$, cyclic quad.)

$\angle DPQ = 60° - \angle DPR = \angle CPR$

$CP = DP$ sides of an equilateral triangle

$\therefore$ $\Delta CPR \cong \Delta DPQ$ (AAS)

Shaded area = area of ΔPDQ + area of ΔPDR

= area of ΔCPR + area of ΔPDR

= area of ΔCPD

$= \frac{1}{2} \cdot 6^2 \cdot \sin 60^\circ = 9\sqrt{3}$ cm^2; $c = 9\sqrt{3}$

I8 Find the unit digit of 7^{2009}.

Reference: 2006 HI9: Given that the units digit of 7^{2006} is C, ...

$7^1 = 7$, $7^2 \equiv 9 \pmod{10}$, $7^3 \equiv 3 \pmod{10}$, $7^4 \equiv 1$ (mode 10)

$7^{2009} = (7^4)^{502} \times 7$

$\equiv 7 \bmod 10$

I9 Given that a and b are integers. Let $a - 7b = 2$ and $\log_{2b} a = 2$, find the value of $a \times b$.

$(2b)^2 = a$

$\Rightarrow 4b^2 = 7b + 2$

$\Rightarrow 4b^2 - 7b - 2 = 0$

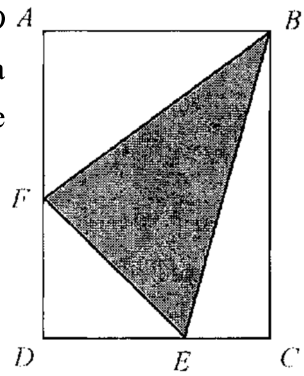
$\Rightarrow (b - 2)(4b + 1) = 0$

$b = 2$ or $-\frac{1}{4}$ (rejected)

$a = 7b + 2 = 16$

$a \times b = 32$

I10 In Figure 3, $ABCD$ is a rectangle. Points E and F lie on CD and AD respectively, such that $AF = 8$ cm and $EC = 5$ cm. Given that the area of the shaded region is 80 cm^2. Let the area of the rectangle $ABCD$ be g cm^2, find the value of g.



Let $DF = x$ cm, $DE = y$ cm

$$g = \frac{1}{2}xy + \frac{1}{2}(8 + x) \cdot 5 + \frac{1}{2}(y + 5) \cdot 8 + 80$$

$$= \frac{1}{2}(xy + 5x + 8y + 40 + 200)$$

$$= \frac{1}{2}[(x + 8)(y + 5) + 200] = \frac{1}{2}g + 100$$

$\Rightarrow g = 200$

IS Given that a is a negative real number. If $\dfrac{1}{a+\frac{1}{a+2}}=2$, find the value of a.

$1=2a+\dfrac{2}{a+2}$

$\Rightarrow 1-2a=\dfrac{2}{a+2}$

$\Rightarrow 2-3a-2a^2=2$

$\Rightarrow a=-\dfrac{3}{2}$

Group Events

G1 If a is a positive integer and $\dfrac{1}{a(a+1)}+\dfrac{1}{(a+1)(a+2)}+\cdots+\dfrac{1}{2008\times 2009}=\dfrac{272}{30135}$, find the value of a.

$\left(\dfrac{1}{a}-\dfrac{1}{a+1}\right)+\left(\dfrac{1}{a+1}-\dfrac{1}{a+2}\right)+\cdots+\left(\dfrac{1}{2008}-\dfrac{1}{2009}\right)=\dfrac{272}{30135}$

$\dfrac{1}{a}-\dfrac{1}{2009}=\dfrac{272}{30135}$

$\Rightarrow \dfrac{1}{a}=\dfrac{272}{2009\times 15}+\dfrac{1}{2009}$

$=\dfrac{272+15}{2009\times 15}=\dfrac{287}{3\times 5\times 7^2\times 41}$

$=\dfrac{1}{15\times 7}=\dfrac{1}{15\times 7}$

$a=105$

G2 Let $x=1+\sqrt{2}$, find the value of $x^5-2x^4+3x^3-4x^2-10x-6$.

Reference: 1993 HI9, 2000 HG1, 2001 FG2.1, 2007 HG3

Method 1

$x-1=\sqrt{2}$

$(x-1)^2=2 \Rightarrow x^2-2x-1=0$

By division, $x^5-2x^4+3x^3-4x^2-10x-6=(x^2-2x-1)(x^3+4x+4)+2x-2$

$=2x-2=2(1+\sqrt{2})-2=2\sqrt{2}$

Method 2 Divide $x^5-2x^4+3x^3-4x^2-10x-6$ by $(x-1)$ successively:

$$\begin{array}{r|rrrrrr}
1 & 1 & -2 & 3 & -4 & -10 & -6 \\
 & & 1 & -1 & 2 & -2 & -12 \\ \hline
1 & 1 & -1 & 2 & -2 & -12 & \vert\ -18 \\
 & & 1 & 0 & 2 & 0 & \\ \hline
1 & 1 & 0 & 2 & 0 & \vert\ -12 & \\
 & & 1 & 1 & 3 & & \\ \hline
1 & 1 & 1 & 3 & \vert\ 3 & & \\
 & & 1 & 2 & & & \\ \hline
1 & 1 & 2 & \vert\ 5 & & & \\
 & & 1 & & & & \\ \hline
 & 1 & \vert\ 3 & & & &
\end{array}$$

Let $y=x-1$, the expression in terms of y becomes

$y^5+3y^4+5y^3+3y^2-12y-18$

$=\sqrt{2}^5+3(\sqrt{2})^4+5(\sqrt{2})^3+3(\sqrt{2})^2-12\sqrt{2}-18$

$=4\sqrt{2}+12+10\sqrt{2}+6-12\sqrt{2}-18$

$=2\sqrt{2}$

G3 Given that p and q are integers. If $\frac{2}{p}+\frac{1}{q}=1$, find the maximum value of $p\times q$.

Reference 2008 HI3 $\cdots\frac{1}{x}+\frac{1}{y}=\frac{1}{15}$. If $35 < y_0 < 50$ and $x_0 + y_0 = z_0 \ldots$

$2q + p = pq$

$pq - p - (2q - 2) = 2$

$(p - 2)(q - 1) = 2$

$(p - 2, q - 1) = (1, 2), (2, 1), (-1, -2), (-2, -1)$

$(p, q) = (3, 3), (4, 2), (1, -1), (0, 0)$

$\Rightarrow$ maximum $p\times q = 3\times 3 = 9$

G4 Given that $0 \le x \le 180$. If the equation $\cos 7x° = \cos 5x°$ has r distinct roots, find the value of r.

$7x = 5x + 360n$	or	$7x = 360 - 5x + 360n$
$x = 180n$	or	$x = 30(n + 1)$
$n = 0, x = 0$	or	30
$n = 1, x = 180$	or	60
$n = 2$		$x = 90$
$n = 3$		$x = 120$
$n = 4$		$x = 150$

$r = 7$

G5 Let x, y and z be positive integers and satisfy $\sqrt{z-\sqrt{28}}=\sqrt{x}-\sqrt{y}$. Find the value of $x + y + z$.

$z-\sqrt{28}=\left(\sqrt{x}-\sqrt{y}\right)^2$

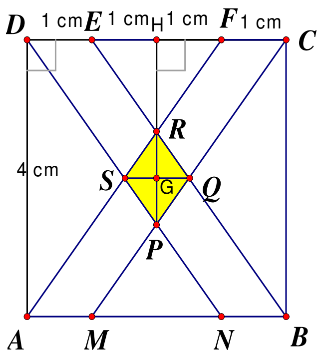
$\Rightarrow z-2\sqrt{7}=x+y-2\sqrt{xy}$

$\Rightarrow x + y = z; xy = 7$

$\therefore x = 7, y = 1$ and $z = 1 + 7 = 8$

$x + y + z = 16$

G6 In Figure 1, $ABCD$ is a square and $AM = NB = DE = FC = 1$ cm and $MN = 2$ cm. Let the area of quadrilateral $PQRS$ be c cm^2, find the value of c. (**Reference: 2017 HG6**)



$DF = 3$ cm, $AD = CD = 4$ cm, $AF=\sqrt{3^2+4^2}$ cm $= 5$ cm $= DN$

$\Delta ADF \cong \Delta NFD$ (SAS) $\Rightarrow \angle AFD = \angle NDF$ (corr. $\angle$s $\cong \Delta$'s)

$\therefore DS = FS$ (sides opp. eq. $\angle$s)

But $\Delta DSF \cong \Delta NSA$ (ASA) $\Rightarrow AS = SF = \frac{1}{2}AF = \frac{5}{2}$ cm

Let H be the mid point of EF. $EH = HF = 1$ cm

Suppose SQ intersects PR at G.

It is easy to show that $\Delta ADF \sim \Delta RHF \sim \Delta RGS$

$\frac{FR}{AF}=\frac{FH}{AD}\Rightarrow\frac{FR}{5\text{ cm}}=\frac{1}{3}\Rightarrow FR=\frac{5}{3}$ cm (ratio of sides, ~Δ's)

$SR = SF - RF = \frac{5}{2}\text{ cm}-\frac{5}{3}\text{ cm}=\frac{5}{6}$ cm

$\frac{\text{Area }\Delta RGS}{\text{Area }\Delta ADF}=\left(\frac{SR}{AF}\right)^2\Rightarrow\frac{\text{Area }\Delta RGS}{\frac{1}{2}\cdot 3\cdot 4\text{cm}^2}=\left(\frac{\frac{5}{6}}{5}\right)^2\Rightarrow$ Area $\Delta RSG=\frac{1}{6}$ cm^2

Area of $PQRS$ = 4 area of $\Delta RSG = \frac{2}{3}$ cm^2, $c = \frac{2}{3}$

Method 2 Let $\angle RFH = \theta = \angle PCH = \angle CMB$ (corr. $\angle$s and alt. $\angle$s, //-lines)

$\tan\theta = \frac{HR}{1\text{ cm}} = \frac{HP}{2\text{ cm}} = \frac{4}{3}$

$\Rightarrow HR = \frac{4}{3}$ cm; $HP = \frac{8}{3}$ cm

$PR = HP - HR = \left(\frac{8}{3} - \frac{4}{3}\right)$ cm $= \frac{4}{3}$ cm

Further, $FCQS$ is a //-gram.

$\therefore SQ = FC = 1$ cm (opp. sides, //-gram)

Area of $PQRS = \frac{1}{2} PR \cdot SQ = \frac{1}{2}\cdot\frac{4}{3}\times 1\text{ cm}^2$

$c = \frac{2}{3}$

G7 Given that x is a real number and satisfies $2^{2x+8} + 1 = 32\times 2^x$. Find the value of x.

Let $y = 2^x$, $y^2 = 2^{2x}$

Then the equation becomes: $2^8\cdot y^2 + 1 = 32y$

$256y^2 - 32y + 1 = 0$

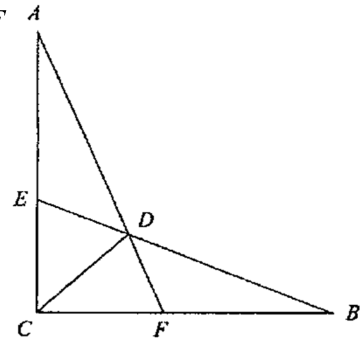
$\Rightarrow (16y - 1)^2 = 0$

$\Rightarrow y = \frac{1}{16}$

$\Rightarrow 2^x = 2^{-4}$

$x = -4$

G8 In Figure 2, $\angle ABC$ is a right angle, $AC = BC = 14$ cm and $CE = CF = 6$ cm. If $CD = d$ cm, find the value of d.



Reference: 2005 HI5, 2012 HI10

$\angle ACD = \angle BCD = 45°$

We find the area of $ACBD$ in two different ways.

$S_{ACBD} = S_{ACD} + S_{BCD} = S_{ACF} + S_{BCE} - S_{CDE} - S_{CDF}$

$2\times\frac{1}{2}\cdot 14d\sin 45^\circ = 2\times\frac{1}{2}\cdot 6\times 14 - 2\times\frac{1}{2}\cdot 6d\sin 45^\circ$

$20d\cdot\frac{1}{\sqrt{2}} = 84$

$\Rightarrow d = \frac{21\sqrt{2}}{5}$

Method 2 Set up a coordinate system with BC as x-axis, CA as y-axis, and C as the origin.

Equation of AF in intercept form: $\frac{x}{6} + \frac{y}{14} = 1 \quad \cdots (1)$

Equation of CD is $y = x \cdots (2)$

Sub. (2) into (1): $\frac{x}{6} + \frac{y}{14} = 1$

$\Rightarrow x = \frac{21}{5}$

$\therefore d = \frac{21}{5}\sin 45° = \frac{21\sqrt{2}}{5}$

G9 If there are 6 different values of real number x that satisfies $||x^2 - 6x - 16| - 10| = f$, find the value of f. **(Reference: 2002 FG4.3, 2005 FG4.2, 2012 FG4.2, 2017 FG1.2)**

Remark: The original solution is wrong. Thanks for Mr. Ng Ka Lok's (from EDB) comment.

Absolute values must be non-negative, $f \ge 0 \cdots$(*)

$|x^2 - 6x - 16| - 10 = \pm f$

$|x^2 - 6x - 16| = 10 + f$ or $|x^2 - 6x - 16| = 10 - f \cdots$ (**)

$x^2 - 6x - 16 = 10 + f$, $x^2 - 6x - 16 = 10 - f$, $x^2 - 6x - 16 = -10 + f$, $x^2 - 6x - 16 = -10 - f$

$x^2 - 6x - 26 - f = 0\ldots(1)$, $x^2 - 6x - 26 + f = 0\ldots(2)$, $x^2 - 6x - 6 - f = 0\ldots(3)$, $x^2 - 6x - 6 + f = 0\cdots(4)$

The respective discriminants are: $\Delta_1 = 4(35 + f)$, $\Delta_2 = 4(35 - f)$, $\Delta_3 = 4(15 + f)$, $\Delta_4 = 4(15 - f)$

Consider the different values of f (≥ 0) by table:

	$f = 0$	$0 < f < 15$	$f = 15$	$15 < f < 35$	$f = 35$	$35 < f$
Δ_1	+	+	+	+	+	+
Δ_2	+	+	+	+	0	–
Δ_3	+	+	+	+	+	+
Δ_4	+	+	0	–	–	–

Consider the following cases: (I) the equation has 6 real roots and 2 complex roots

From the table, the range of possible values of f is $15 < f < 35$.

Sub. $15 < f < 35$ into (**):$|x^2 - 6x - 16| = 10 + f$ or $|x^2 - 6x - 16| = 10 - f$ (no solution, RHS<0)

$\Rightarrow x^2 - 6x - 16 = 10 + f$ or $x^2 - 6x - 16 = -10 - f$

$\Rightarrow x^2 - 6x - 26 - f = 0\ldots(1)$ or $x^2 - 6x - 6 + f = 0\ldots(4)$

$\because$ For $15 < f < 35$, $\Delta_1 > 0$, $\Delta_4 < 0$

$\therefore$ The equation has only 2 real roots,contradicting to our assumption that there are 6 real roots

(II) The equation has 8 real roots. 6 of these roots are distinct and 2 other roots are the same as the first 6 roots.

If (1) $\equiv$ (2), then $-26 - f = -26 + f \Rightarrow f = 0$, if (3) $\equiv$ (4), then $-6 - f = -6 + f \Rightarrow f = 0$

If (1) $\equiv$ (3), then $-26 - f = -6 - f \Rightarrow$ no solution, if (2)$\equiv$ (4), then $-26 + f = -6 + f \Rightarrow$ no solution

If (1) $\equiv$ (4), then $-26 - f = -6 + f \Rightarrow f = -10$ (contradict with (*), rejected)

If (2) $\equiv$ (3), then $-26 + f = -6 - f \Rightarrow f = 10$

If $f = 0$ the equations are $x^2 - 6x - 26 = 0$ and $x^2 - 6x - 6 = 0$, which have only 4 roots,rejected

If $f = 10$, the equations are $x^2 - 6x - 36 = 0$, $x^2 - 6x - 16 = 0$ and $x^2 - 6x - 4 = 0$, which have 6 distinct real roots, accepted.

Method 2

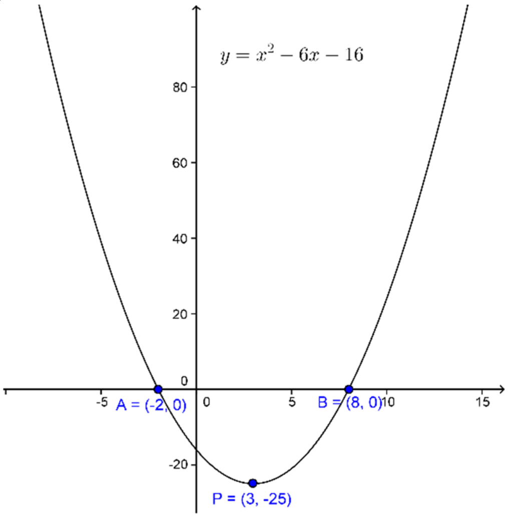
The following method is provided by Mr. Jimmy Pang from Sai Kung Sung Tsun Catholic School (Secondary).

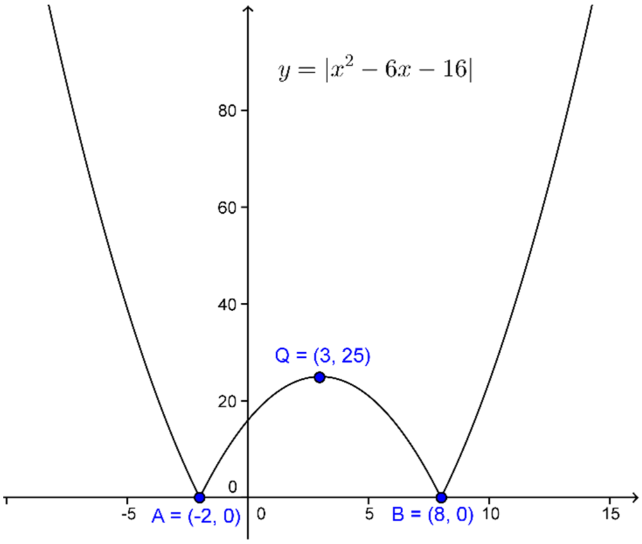
First sketch the graph $y = ||x^2 - 6x - 16| - 10|$.

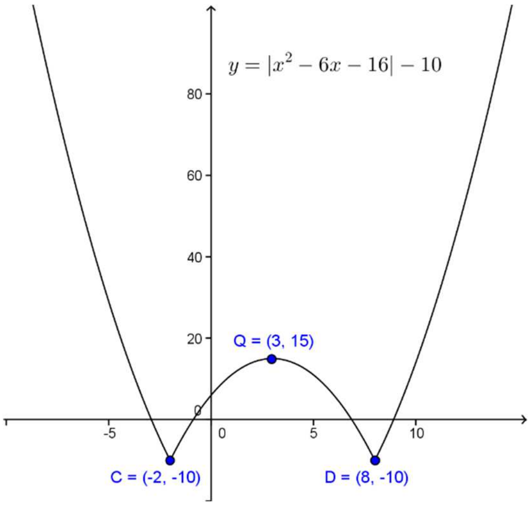
From the graph, draw different horizontal lines.

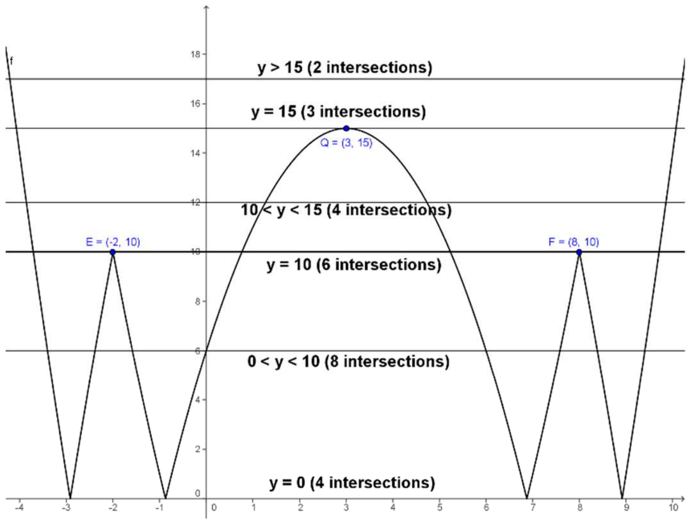
The line $y = 10$ cuts $y = ||x^2 - 6x - 16| - 10|$ at 6 different points.

$\therefore f = 10$

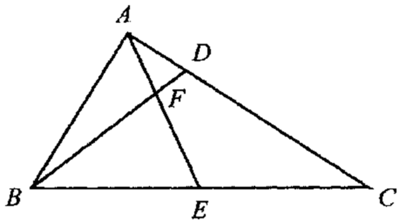








G10 In Figure 3, ABC is a triangle, E is the midpoint of BC, F is a point on AE where $AE = 3AF$. The extension segment of BF meets AC at D. Given that the area of ΔABC is 48 cm^2. Let the area of ΔAFD be g cm^2, find the value of g.

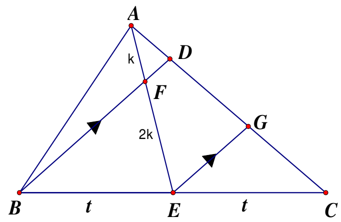


From E, draw a line EG // BD which cuts AC at G.

$AE = 3AF \Rightarrow AF : FE = 1 : 2$; let $AE = k$, $FE = 2k$

E is the midpoint of $BC \Rightarrow BE = EC = t$

$S_{ABE} = S_{ACE} = \frac{1}{2}\cdot 48\text{ cm}^2 = 24\text{ cm}^2$ (same base, same height)



$AD : DG = AF : FE = 1 : 2$ (theorem of equal ratio)

$DG : GC = BE : EC = 1 : 1$ (theorem of equal ratio)

$\therefore AD : DG : GC = 1 : 2 : 2$

$S_{AEG} : S_{CEG} = 3 : 2$ (same height, ratio of base = 3 : 2)

$$S_{AEG} = 24\times\frac{3}{2+3}\text{cm}^2 = \frac{72}{5}\text{cm}^2$$

$$\Delta ADF \sim \Delta AGE \Rightarrow S_{ADF} = \frac{1}{9}S_{AEG} = \frac{1}{9}\cdot\frac{72}{5}\text{cm}^2 = \frac{8}{5}\text{cm}^2;\ g = \frac{8}{5}$$

GS Let R be the remainder of 588^{2009} divided by 97. Find the value of R.

Reference: 2008 FIS.4 … the remainder of 588^{2008} divided by 97 $\cdots$

$588^{2009} = (97\times 6 + 6)^{2009}$

$= (97\times 6)^{2009} + {}_{2009}C_1\cdot(97\times 6)^{2009}\times 6 + \ldots + 6^{2009} = 97m + 6^{2009}$, where m is an integer.

Note that $2^5\times 3 = 96 = 97 - 1 \equiv -1 \pmod{97}$; $2\times 3^5 = 486 = 97\times 5 + 1 \equiv 1 \pmod{97}$;

$\therefore 6^6 = (2^5\times 3)\times(2\times 3^5) \equiv -1 \pmod{97}$

$6^{2009} = (6^6)^{334}\times 6^5 \equiv (-1)^{334}\times 2^4\times(2\times 3^5) \equiv 16 \pmod{97}$; $d = 16$

HKMO 2009 Geometrical Construction Sample Paper solution

1. 在下列三角形中，試作出一點使它與該三角形各邊的距離相等。

Reference: 2012 HC2, 2014 HC1, 2019 HC3

First, we find the **locus** of a point D which is equidistance to a given angle $\angle BAC$. (Figure 1)

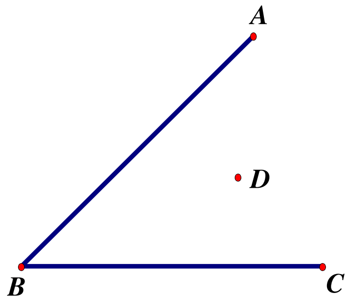


Figure 1

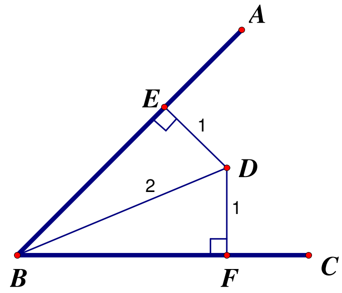


Figure 2

(1) From D, let E and F be the feet of perpendiculars onto AB and BC respectively. (Fig. 2)

(2) Join BD.

$BD = BD$ (common side)

$\angle BED = \angle BFD = 90°$ (By construction)

$DE = DF$ (given that D is equidistance to AB and BC)

$\therefore \Delta BDE \cong \Delta BDF$ (RHS)

$\angle DBE = \angle DBF$ (corr. $\angle$s $\cong \Delta$s)

$\therefore$ D lies on the **angle bisector** of $\angle ABC$.

$\therefore$ In ΔABC, if P is equidistance to ABC, then P must lie on the **intersection of the three angle bisectors**. (i.e. the incentre of ΔABC.)

$\because$ The three angle bisectors must concurrent at one point

$\therefore$ We need to find the intersection of any two angle bisectors.

The construction is as follows:

(1) Draw the bisector of $\angle ABC$.

(2) Draw the bisector of $\angle ACB$.

P is the intersection of the two angle bisectors.

(3) Join AP.

(4) Let R, S, T be the feet of perpendiculars from P onto BC, AC and AB respectively.

$\Delta BPR \cong \Delta BPT$ (A.A.S.)

$\Delta CPR \cong \Delta CPS$ (A.A.S.)

$PT = PR = PS$ (Corr. sides, $\cong \Delta$s)

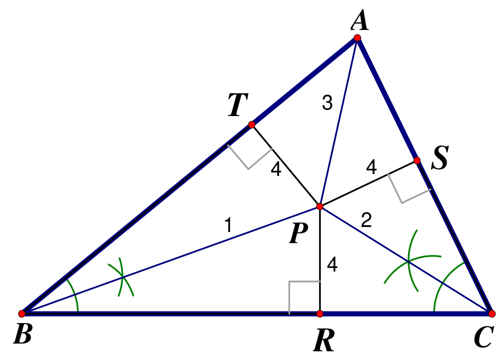
$\Delta APT \cong \Delta APS$ (R.H.S.)

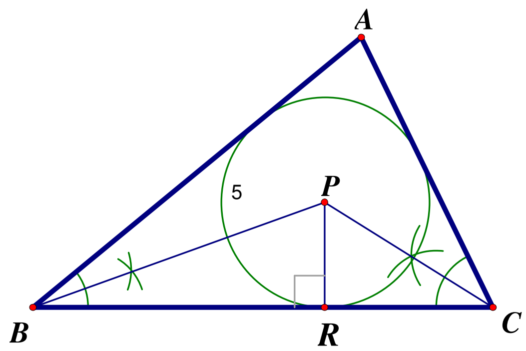
$\angle PAT = \angle PAS$ (Corr. $\angle$s, $\cong \Delta$s)

$\therefore$ AP is the angle bisector of $\angle BAC$.

The 3 angle bisectors are concurrent at one point.

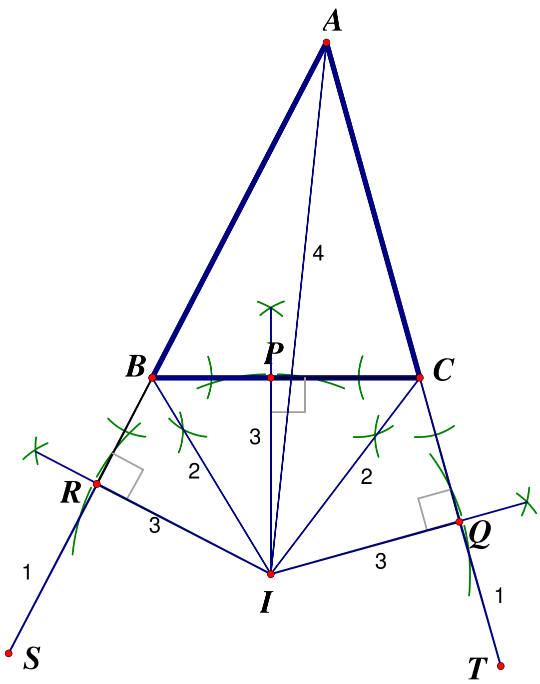
(5) Using P as centre, PR as radius to draw a circle. This circle touches ΔABC internally





at R, S, and T. It is called the **inscribed circle**.

Note that the 3 **ex-centres** are equidistance from AB, BC, CA. I shall draw one ex-centre as demonstration.



(1) Extend AB and AC to S and T respectively.

(2) Draw the exterior angle bisectors of $\angle ABC$ and $\angle ACB$ respectively. The two exterior angle bisectors intersect at I.

(3) Let P, Q, R be the feet of perpendiculars drawn from I onto to BC, AC and AB respectively.

(4) Join AI.

(5) $\Delta IBP \cong \Delta IBR$ (AAS); $\Delta ICP \cong \Delta ICQ$ (AAS)

$\therefore IP = IQ = IR$ (corr. sides, $\cong \Delta$'s)

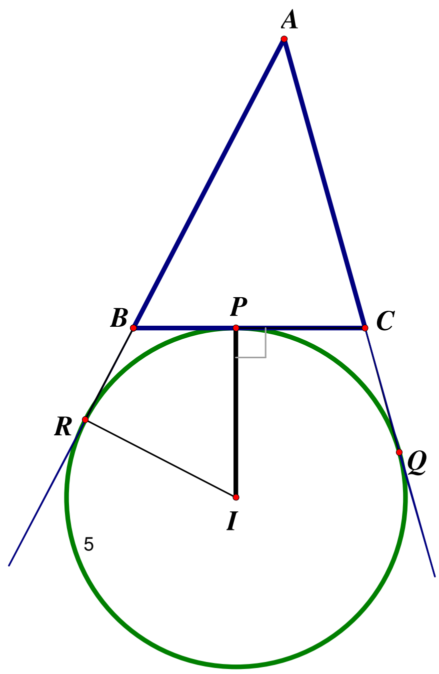
$\therefore I$ is equidistance from AB, BC and AC.

$\Delta IAR \cong \Delta IAQ$ (R.H.S.)

$\angle IAR = \angle IAQ$ (corr. sides, $\cong \Delta$s)

$\therefore IA$ is the interior angle bisector of $\angle BAC$

Two exterior angle bisectors and one interior angle bisector of a triangle are concurrent.



Use I as centre, IP as radius to draw a circle. This circle touches triangle ABC externally. It is called the **escribed circle or ex-circle**.

2. 已知一直綫 L，及兩點 P、Q 位於 L 的同一方。試在 L 上作一點 T 使得 PT 及 QT 的長度之和最小。(Figure 1)

Figure 1

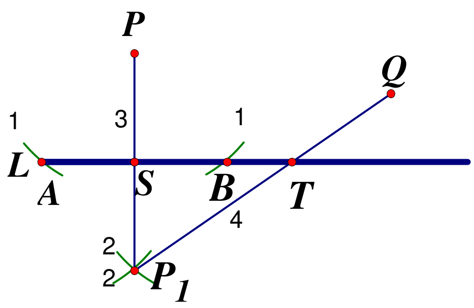


Figure 2

Let A be one of the end point of L nearer to P.

(1) Use P as centre, PA as radius to draw an arc, which intersects L at A and B.

(2) Use A as centre, AP as radius to draw an arc. Use B as centre, BP as radius to draw an arc. The two arcs intersect at P_1.

(3) Join PP_1 which intersects L at S.

(4) Join P_1Q which intersects L at T。

$AP = AP_1$	(same radii)
$AB = AB$	(common side)
$BP = BP_1$	(same radii)
$\Delta APB \cong \Delta AP_1B$	(S.S.S.)
$\angle PBA = \angle P_1BA$	(corr. sides $\cong \Delta$s)
$BS = BS$	(common side)
$\Delta PBS \cong \Delta P_1BS$	(S.A.S.)
$\angle BSP = \angle BSP_1$	(corr. sides $\cong \Delta$s)
$= 90°$	(adj. $\angle$s on st. line)
$SP = SP_1$	(corr. sides $\cong \Delta$s)
$ST = ST$	(common side)
$\Delta PST \cong \Delta P_1ST$	(S.A.S.)
$PT = P_1T$	(corr. sides $\cong \Delta$s)

$PT + QT = P_1T + QT$

It is known that $P_1T + QT$ is a minimum when P_1, T, Q are collinear.

$\therefore$ T is the required point.

3. 試繪畫一固定圓的切綫，且該切綫通過一固定點 P。(註：P 在該圓形外。)

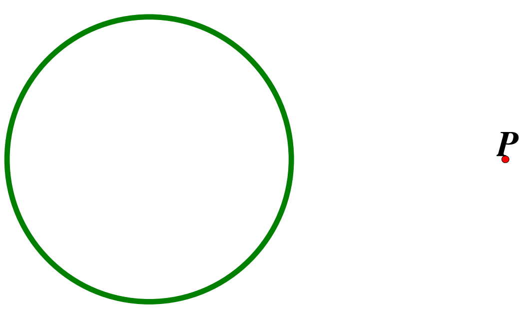


Figure 1

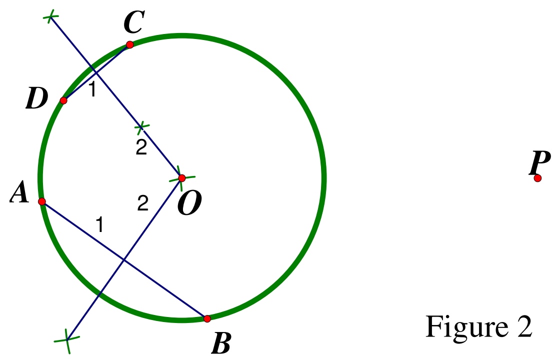


Figure 2

First, we locate the centre of the circle. (Figure 2)

(1) Let AB and CD be two non-parallel chords.

(2) Draw the perpendicular bisectors of AB and CD respectively which intersect at O.

(3) Join OP.

(4) Draw the perpendicular bisector of OP, K is the mid-point of OP.

(5) Using K as centre, KP as radius to draw a circle, which intersects the given circle at M and N.

(6) Join OM, ON, MP, NP.

$\angle ONP = 90° = \angle OMP$ ($\angle$ in semi-circle)

PM, PN are the required tangents.

(converse, tangent $\perp$ radius)

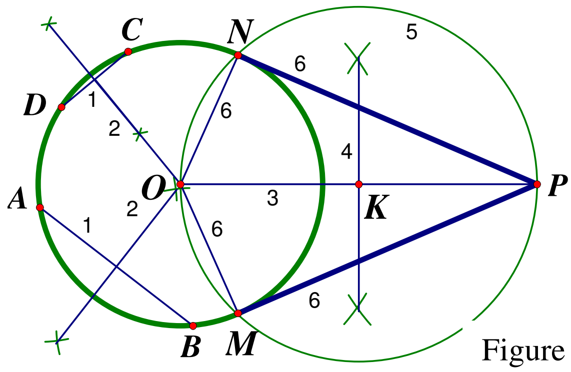


Figure 3

Method 2

(1) From P, draw a line segment cutting the circle at R and Q. (Q lies between P and R.)

(2) Draw the perpendicular bisector of PR, which intersects PR at O.

(3) Use O as centre, $OP = OR$ as radius to draw a semi-circle PTR.

(4) From Q (the point between PR), draw a line QT perpendicular to PR, cutting the semi-circle at T.

(5) Join PT.

(6) Join TR.

(7) Use P as centre, PT as radius to draw an arc, cutting the given circle at M and N.

(8) Join PM, PN.

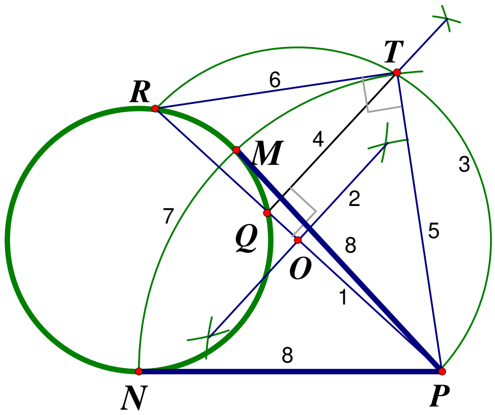


Figure 4

$\angle PTR = 90°$ ($\angle$ in semi-circle)

$\Delta PQT \sim \Delta PTR$ (equiangular)

$\dfrac{PQ}{PT} = \dfrac{PT}{PR}$ (ratio of sides, ~Δs)

$\therefore PQ \cdot PR = PT^2$

By (7), $PQ \cdot PR = PT^2 = PM^2 = PN^2$

$\therefore$ PM and PN are the tangents from external point. (converse, intersecting chords theorem)

Geometrical construction

1. In Figure 1, A, B and C are three fixed non-collinear points. Construct a circle passing through the given three points.

The 3 perpendicular bisectors of AB, BC and AC are concurrent at the circumcentre O. It is sufficient to locate the circumcentre by any two perpendicular bisectors.

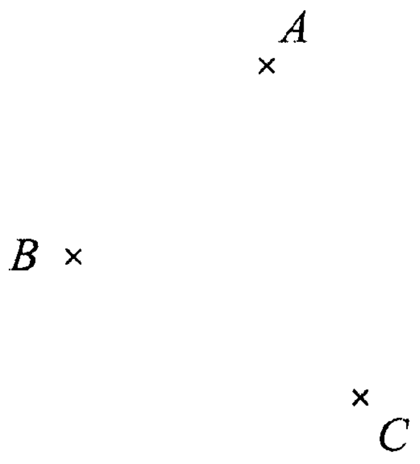


Figure 1

(1) Join AB, AC and BC.

(2) Draw the perpendicular bisector of AB.
M is the mid-point of AB.

(3) Draw the perpendicular bisector of AC.
N is the mid-point of AC.
The two perpendicular bisectors intersect at O.

(4) Use O as centre, OA as radius, draw a circle.

$\Delta AOM \cong \Delta BOM$ (S.A.S.)

$\Delta AON \cong \Delta CON$ (S.A.S.)

$OB = OA = OC$ (corr. sides, $\cong \Delta$s)

$\therefore$ The circle pass through A, B and C.

Let L be the mid-point of BC. Join OL.

$\Delta BOL \cong \Delta COL$ (S.S.S.)

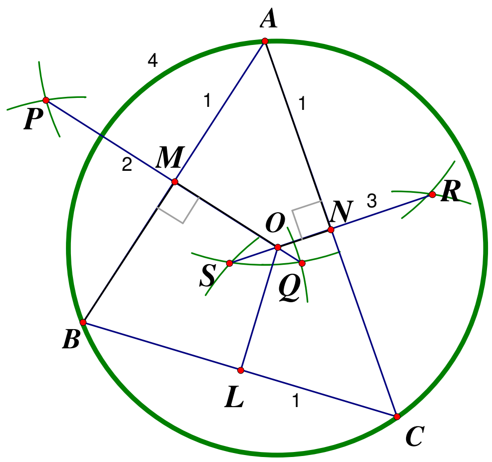
$\angle BLO = \angle CLO$ (corr. $\angle$s, $\cong \Delta$s)

$\angle BLO + \angle CLO = 180°$ (adj. $\angle$s on st. line)

$\therefore \angle BLO = \angle CLO = 90°$

$\therefore$ OL is the perpendicular bisector of BC.

$\therefore$ The 3 perpendicular bisectors of AB, BC and AC are concurrent at a point O.



2. In Figure 2, AB is the base of a triangle ABC, and the length of BD is the sum of the lengths of BC and CA. Given that $\angle ABC = 60°$, construct the triangle ABC.

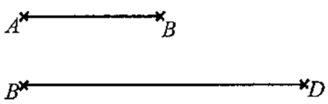


Figure 2

(1) Use A as centre, AB as radius to draw an arc, use B as centre, BA as radius to draw another arc. The two arcs intersect at P. ABP is an equilateral triangle. $\angle ABP = 60°$

(2) With B as centre, BD as radius, draw an arc, cutting BP produced at D.

(3) Join AD.

(4) Draw the perpendicular bisector of AD which cuts BD at C. N is the mid point of AD.

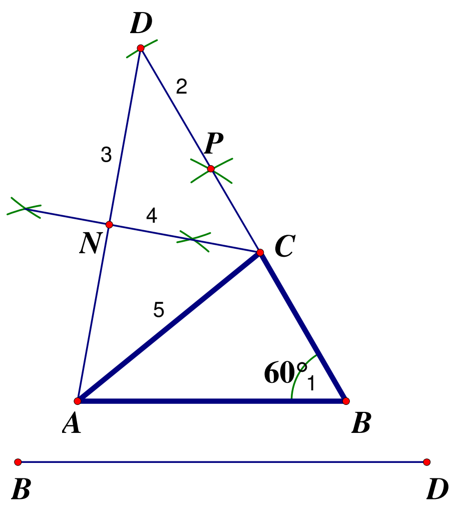
(5) Join AC.

$\Delta ANC \cong \Delta DNC$ (S.A.S.)

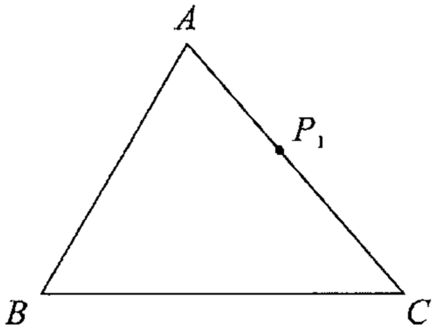
$\therefore AC = DC$ (corr. sides, $\cong \Delta$s)

$AC + CB = BC + CD$

ΔABC is the required triangle.



3. In Figure 3, ΔABC is an acute angle triangle. P_1 is a point on AC. Construct a triangle P_1XY such that X is a point on AB, Y is a point on BC and the perimeter of ΔP_1XY is the least.



(1) Use A as centre, AC as radius to draw an arc.

(2) Use B as centre, BC as radius to draw another arc. These two arcs intersect at C'.

$\Delta ABC \cong \Delta ABC'$ (S.S.S.)

$\angle BAC = \angle BAC'$ (corr. $\angle$s, $\cong \Delta$s)

(3) Use C as centre, CA as radius to draw an arc

(4) Use B as centre, BA as radius to draw another arc. These two arcs intersect at A'.

$\Delta ABC \cong \Delta A'BC$ (S.S.S.)

$\angle ACB = \angle A'CB$ (corr. $\angle$s, $\cong \Delta$s)

(5) Use A as centre, AP_1 as radius to draw an arc, which cuts AC' at P.

$AX = AX$ (common side)

$AP = AP_1$ (radii)

$\Delta APX \cong \Delta AP_1X$ (S.A.S.)

$PX = P_1X$ (corr. sides, $\cong \Delta$s)

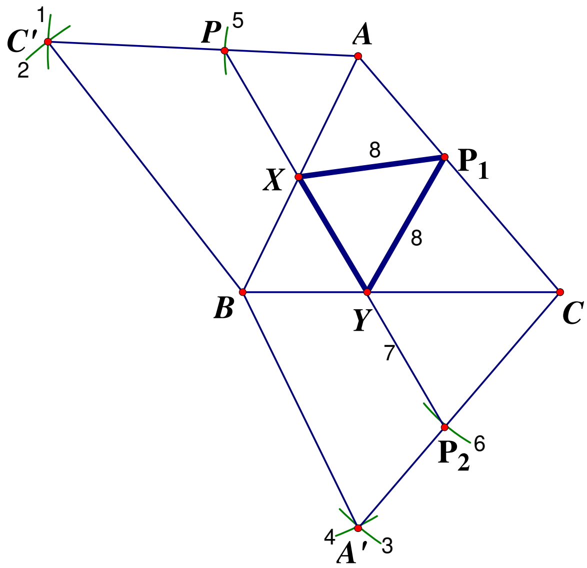
(6) Use C as centre, CP_1 as radius to draw an arc, which cuts CA' at P_2.

$CY = CY$ (common side)

$CP_1 = CP_2$ (radii)

$\Delta CP_1Y \cong \Delta CP_2Y$ (S.A.S.)

$P_1Y = P_2Y$ (corr. sides, $\cong \Delta$s)



(7) Join PP_2, which cuts AB at X and BC at Y.

(8) Join P_1X, YP_2.

The position of P and P_2 are fixed irrespective the size and the shape of ΔP_1XY.

Perimeter of $\Delta P_1XY = PX + XY + YP_2$

$PX + XY + YP_2$ is the minimum when P, X, Y, P_2 are collinear.

ΔP_1XY is the required triangle.