Locus Example 2

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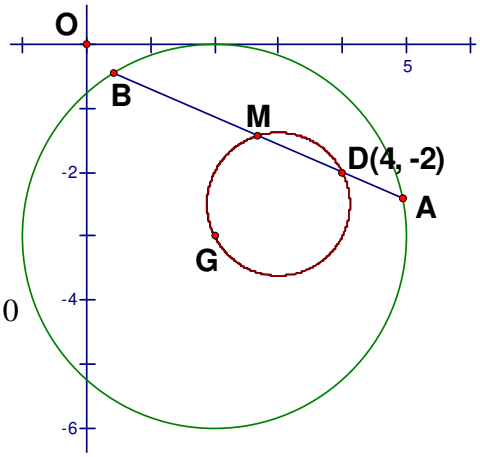
Given the circle C: $x^2 + y^2 - 4x + 6y + 4 = 0$ and a point $D(4, -2)$ lies inside the circle.

A variable chord passes through D meets the circle at A and B. Find the locus of the mid-point M.

Let the slope of AB be m.



AB: $\dfrac{y+2}{x-4} = m$ .....(1)

$y = mx - 4m - 2$ .....(2)

Sub. into C: $x^2 + (mx - 4m - 2)^2 - 4x + 6(mx - 4m - 2) + 4 = 0$

$x^2 + m^2x^2 - 2m(4m + 2)x + (4m + 2)^2 - 4x + 6mx - 6(4m + 2) + 4 = 0$

$(1 + m^2)x^2 + (-8m^2 + 2m - 4)x + (16m^2 + 16m + 4 - 24m - 12 + 4) = 0$

$(1 + m^2)x^2 - 2(4m^2 - m + 2)x + (16m^2 - 8m - 4) = 0$ .....(3)

If $A(x_1, y_1)$, $B(x_2, y_2)$, then x_1, x_2 are the solution of (3)

$x_1 + x_2$ = sum of roots in (3) $= \dfrac{2(4m^2 - m + 2)}{1 + m^2}$

If $M(x_0, y_0)$, $x_0 = \dfrac{x_1 + x_2}{2} = \dfrac{4m^2 - m + 2}{1 + m^2} = \dfrac{4m^2 + 4 - m - 2}{1 + m^2} = 4 - \dfrac{m + 2}{1 + m^2}$

$$x_0 = 4 - \frac{m+2}{1+m^2} \text{ .....(4)}$$

Sub. (1) into (4): $x = 4 - \dfrac{\dfrac{y+2}{x-4} + 2}{1 + \left(\dfrac{y+2}{x-4}\right)^2}$

$x - 4 = -\dfrac{(y+2)(x-4) + 2(x-4)^2}{(x-4)^2 + (y+2)^2}$

$1 = -\dfrac{y + 2 + 2(x-4)}{x^2 + y^2 - 8x + 4y + 20}$

$x^2 + y^2 - 8x + 4y + 20 = -2x - y + 6$

$x^2 + y^2 - 6x + 5y + 14 = 0$

Method 2

Let $M = (x, y)$, G = centre = $(2, -3)$

$GM \perp AB$ (line joining centre to mid point chord $\perp$ chord)

$\therefore m_{GM} \cdot m_{MD} = -1$

$\dfrac{y+3}{x-2} \cdot \dfrac{y+2}{x-4} = -1$

$(x - 2)(x - 4) + (y + 3)(y + 2) = 0$

$x^2 + y^2 - 6x + 5y + 14 = 0$