03-04 Individual	**1**	–2009010	**2**	7	**3**	45	**4**	700	**5**	6
	6	12.5	**7**	6	**8**	$-\frac{3}{16}$	**9**	12	**10**	$\frac{19}{4}$

03-04 Group	**1**	2475	**2**	1	**3**	6	**4**	32	**5**	5
	6	500	**7**	34.56	**8**	$\frac{1}{6}$	**9**	10	**10**	$\frac{5}{3}$

Individual Events

I1 Let $A = 1^2 - 2^2 + 3^2 - 4^2 + \cdots + 2003^2 - 2004^2$, find the value of A.

Reference: 1997 HI5, 2002 FG2.3, 2015 FI3.2, 2015 FG4.1

$A = (1^2 - 2^2) + (3^2 - 4^2) + \cdots + (2003^2 - 2004^2)$

$= -3-7-11-\cdots- 4007$, this is an arithmetic series, $a = -3$, $\ell = -4007 = a+(n-1)(-4)$, $n = 1002$

$= -\frac{3+4007}{2}\times 1002 = -2009010$

I2 If $\sqrt[2003]{B} = 2003$, C is the units digit of B, find the value of C.

$B = 2003^{2003}$; $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$; the units digit repeats for every multiples of 4. $2003^{2003} = 2003^{4\times 500+3}$; the units digit is 7; $C = 7$.

I3 If $x + y + z = 10$, $x^2 + y^2 + z^2 = 10$ and $xy + yz + zx = m$, find the value of m.

$(x + y + z)^2 = 10^2 \Rightarrow x^2 + y^2 + z^2 + 2(xy + yz + zx) = 100 \Rightarrow 10 + 2m = 100 \Rightarrow m = 45$

I4 Arrange the natural numbers in the following order. In this arrangement, 9 is in the row 3 and the column 2. If the number 2003 is in the row x and the column y, find the value of xy.

1	2	4	7	11	16	$\cdots$
3	5	8	12	17	$\cdots$	
6	9	13	18	$\cdots$		
10	14	19	$\cdots$			
15	20	$\cdots$				
21	$\cdots$					

Reference: 2003 FI1.4

Consider the integers in the first column of each row: 1, 3, 6, 10, $\cdots$

They are equivalent to 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, $\cdots$

The first integer in the n^{th} row $= 1+ 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

$\frac{n(n+1)}{2} < 2003 \Rightarrow n(n + 1) < 4006$

$\because$ $62\times 63 = 3906$, $63\times 64 = 4032$

$\therefore$ The greatest possible $n = 62$

$3906 \div 2 = 1953$

The 63^{rd} element of the first row = 1954

The 62^{nd} element of the second row = 1955, ... and so on.

$2003 = 1953 + 50$; $63 - 50 + 1 = 14$

The 14^{th} element of the 50^{th} row is 2003; $x = 50$, $y = 14$

$xy = 50\times 14 = 700$

I5 Let $E=\sqrt{12+6\sqrt{3}}+\sqrt{12-6\sqrt{3}}$, find the value of E.

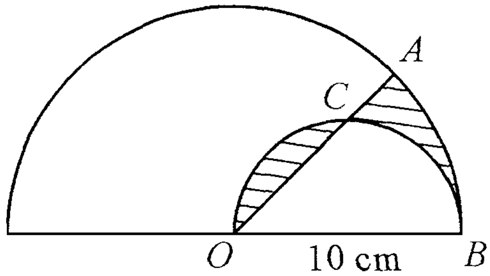
Reference: 1993 FI1.4, 1999 HG3, 2001 FG2.1, 2011 HI7, 2015 FI4.2, 2015 FG3.1

$\sqrt{12+6\sqrt{3}}=\sqrt{9+3+2\sqrt{9\times 3}}=\sqrt{a+b+2\sqrt{ab}}=\sqrt{a}+\sqrt{b}=3+\sqrt{3}$

$\sqrt{12-6\sqrt{3}}=\sqrt{9+3-2\sqrt{9\times 3}}=\sqrt{a+b-2\sqrt{ab}}=\sqrt{a}-\sqrt{b}=3-\sqrt{3}$

$\sqrt{12+6\sqrt{3}}+\sqrt{12-6\sqrt{3}}=3+\sqrt{3}+3-\sqrt{3}=6$

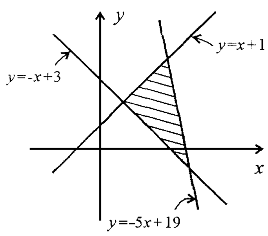
I6 In the figure, O is the centre of the bigger semicircle with radius 10 cm, OB is the diameter of the smaller semicircle and C is the midpoint of arc OB and it lies on the segment OA. Let the area of the shaded region be K cm^2, find the value of K. (Take $\pi = 3$)



Shaded area = area of sector OAB – area of ΔOCB

$= \frac{1}{2}10^2\cdot\frac{\pi}{4}-\frac{1}{2}10\cdot 5 = 12.5$

I7 In the figure, let the shaded area formed by the three straight lines $y = -x + 3$, $y = x + 1$ and $y = -5x + 19$ be R, find the value of R.



Intersection points are $A(1, 2)$, $B(3,4)$, $C(4, -1)$.

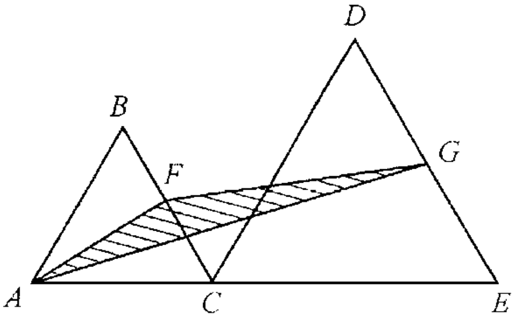
$\angle CAB = 90°$

Area $=\frac{1}{2}\sqrt{8}\sqrt{18}=6$ sq.unit

I8 If $t=\sin^4\frac{\pi}{6}-\cos^2\frac{2\pi}{6}$, find the value of t.

$$t=\sin^4\frac{\pi}{6}-\cos^2\frac{2\pi}{6}$$
$$=\left(\frac{1}{2}\right)^4-\left(\frac{1}{2}\right)^2$$
$$=\frac{1}{16}-\frac{1}{4}$$
$$=-\frac{3}{16}$$

I9 In the figure, C lies on AE, ΔABC and ΔCDE are equilateral triangles, F and G are the mid-points of BC and DE respectively. If the area of ΔABC is 24 cm^2, the area of ΔCDE is 60 cm^2, and the area of ΔAFG is Q cm^2, find the value of Q.



Reference: 2000 FI4.2, 2005 HG7, 2018 HI12

$\angle FAC = \angle GCE = 30°$

AF // CG (corr. $\angle$s eq.)

Area of ΔAFG = Area of ΔACF = 12 cm^2

(They have the same bases AF and the same height)

I10 If α and β are the roots of the quadratic equation $4x^2 - 10x + 3 = 0$ and $k = \alpha^2 + \beta^2$, find the value of k.

$k = \alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(\frac{5}{2}\right)^2 - 2\cdot\left(\frac{3}{4}\right)$

$= \frac{19}{4}$

Group Events

G1 If $x=\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\left(\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)+\cdots+\left(\frac{1}{100}+\frac{2}{100}+\cdots+\frac{99}{100}\right)$,
find the value of x. (**1995 HG3, 1996 FG9.4, 2018 HG9**)

$x=\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\frac{4}{2}+\cdots+\frac{99}{2}=\frac{100}{4}\times 99=2475$

G2 If z is the positive root of the equation $6\times 4^x - 13\times 6^x + 6\times 9^x = 0$, find the value of z.

$(3\cdot 2^x - 2\cdot 3^x)(2\cdot 2^x - 3\cdot 3^x) = 0$

$3\cdot 2^x = 2\cdot 3^x$ or $2\cdot 2^x = 3\cdot 3^x$

$\frac{2^x}{3^x}=\frac{2}{3}$ or $\frac{2^x}{3^x}=\frac{3}{2}=\left(\frac{2}{3}\right)^{-1}$

$x = 1$ or -1 (rejected)

z = positive root = 1

G3 If there are at most k mutually non-congruent isosceles triangles whose perimeter is 25cm and the lengths of the three sides are positive integers when expressed in cm, find the value of k.

Possible triangles are {7,7,11}, {8,8,9}, {9,9,7}, {10,10,5}, {11,11,3}, {12,12,1}

$k = 6$

G4 Given that a, b are positive real numbers satisfying $a^3 = 2004$ and $b^2 = 2004$. If the number of integers x that satisfy the inequality $a < x < b$ is h, find the value of h.

$12^3 = 1728$, $44^2 = 1976$

$a^3 = 2004 \Rightarrow 12 < a < 13$; $b^2 = 2004 \Rightarrow 44 < b < 45$

$a < x < b \Rightarrow 12 < x < 45 \Rightarrow$ number of integral values of $x = 32$

G5 If the sum of R consecutive positive integers is 1000 (where $R > 1$), find the least value of R.

Let the smallest positive integer be x. (**Reference: 2006 HG5**)

$x + (x + 1) + \ldots + (x + R - 1) = 1000$

$\frac{R}{2}\times(2x+R-1)=1000$

$R(2x + R - 1) = 2000 \Rightarrow 2x + R - 1 = \frac{2000}{R}$, which is an integer.

Possible R are: 1,2,4,5,8,10,16,20,25,40,50,80,100,125,250,400,500,1000,2000.

When $R = 4m + 2$, where m is an integer.

$(4m + 2)(2x + 4m + 1) = 2000 \Rightarrow (2m + 1)(2x + 4m + 1) = 1000$

L.H.S. is odd, R.H.S. is even $\therefore$ reject 2, 10, 50, 250.

When $R = 4m$, where m is an integer.

$4m(2x + 4m - 1) = 2000 \Rightarrow m(2x + 4m - 1) = 500 = 4\times 125 \Rightarrow m$ is a multiple of 4

$\therefore$ R = multiple of 16 $\Rightarrow$ reject 4, 8, 20, 40, 100, 500, 1000

$2x + R - 1 = \frac{2000}{R} > R - 1 \Rightarrow 2000 > R(R - 1) \Rightarrow \sqrt{2000} > R - 1 \Rightarrow 45 > R$

The possible values of R are 1, 5, 16, 25.

When $R = 1$, $1(2x) = 2000 \Rightarrow x = 1000$

When $R = 5$, $5(2x + 4) = 2000 \Rightarrow x = 198$

When $R = 16$, $16(2x + 15) = 2000 \Rightarrow x = 55$

When $R = 25$, $25(2x + 24) = 2000 \Rightarrow x = 28$

The least value of $R > 1$ is 5, $x = 198$.

$198 + 199 + 200 + 201 + 202 = 1000$

G6 If a, b and c are positive integers such that $abc + ab + bc + ac + a + b + c = 2003$, find the least value of abc.

Reference: 2018 FI4.3

$(a + 1)(b + 1)(c + 1) = 2004 = 2^2 \times 3 \times 167$

abc is the least when the difference between a, b and c are the greatest.

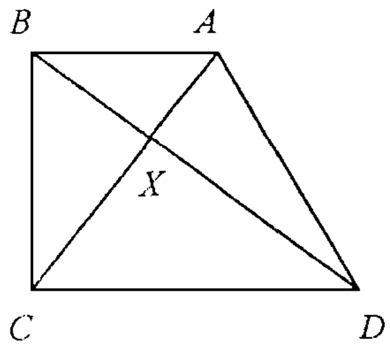
$a + 1 = 2$, $b + 1 = 2$, $c + 1 = 501$

$a = 1$, $b = 1$, $c = 500$

$abc = 500$

G7 In the figure, $ABCD$ is a trapezium, the segments AB and CD are both perpendicular to BC and the diagonals AC and BD intersect at X. If $AB = 9$ cm, $BC = 12$ cm and $CD = 16$ cm, and the area of ΔBXC is W cm^2, find the value of W.

Reference: 1993 HI2, 1997 HG3, 2000 FI2.2, 2002 FI1.3, 2010HG4, 2013 HG2



$\Delta ABX \sim \Delta CDX$

$AX : CX = AB : CD = 9 : 16$

$S_{\Delta ABX} : S_{\Delta CDX} = 9^2 : 16^2 = 81 : 256$

Let $S_{\Delta ABX} = 81y$, $S_{\Delta CDX} = 256y$

Let $AX = 9t$, $CX = 16t$ ($\because$ $\Delta ABX \sim \Delta CDX$)

ΔABX and ΔBCX have the same height.

$S_{\Delta BCX} = S_{\Delta ABX} \times \dfrac{16t}{9t} = 81y \times \dfrac{16}{9} = 144y$

$S_{\Delta ABC} = S_{\Delta ABX} + S_{\Delta BCX}$

$\dfrac{9\times 12}{2} = 81y + 144y$

$y = \dfrac{6}{25}$

$\Rightarrow S_{\Delta BCX} = 144y = 144\times\dfrac{6}{25} = 34.56$

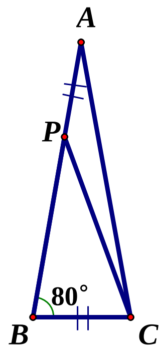
G8 Let $y = \log_{1400}\sqrt{2} + \log_{1400}\sqrt[3]{5} + \log_{1400}\sqrt[6]{7}$, find the value of y.

$$y = \frac{\log\sqrt{2}\times\sqrt[3]{5}\times\sqrt[6]{7}}{\log 1400} = \frac{\frac{1}{2}\log 2+\frac{1}{3}\log 5+\frac{1}{6}\log 7}{\log 1400} = \frac{3\log 2+2\log 5+\log 7}{6\log 1400}$$

$$y = \frac{\log 8+\log 25+\log 7}{6\log 1400} = \frac{\log(2\times 4\times 25\times 7)}{6\log 1400} = \frac{\log 1400}{6\log 1400} = \frac{1}{6}$$

G9 In the figure, ΔABC is an isosceles triangle with $AB = AC$ and $\angle ABC = 80°$. If P is a point on the AB such that $AP = BC$, $\angle ACP = k°$, find the value of k.

Reference:《數學教育》第八期（一九九九年六月）, 2010 HG10



$\angle ACB = 80° = \angle ACB$ (base $\angle$s isos. Δ)

$\angle BAC = 20°$ ($\angle$s sum of Δ)

$\angle BPC = (20 + k)°$ (ext. $\angle$ of ΔAPC)

$\dfrac{AP}{\sin k°} = \dfrac{CP}{\sin 20°}$ …… (1) (sine rule on ΔACP)

$\dfrac{BC}{\sin(20+k)°} = \dfrac{CP}{\sin 80°}$ …… (2) (sine rule on ΔBCP)

(1) ÷ (2): $\dfrac{\sin(20+k)°}{\sin k°} = \dfrac{\sin 80°}{\sin 20°} = \dfrac{\cos 10°}{2\sin 10° \cos 10°} = \dfrac{1}{2\sin 10°}$

$2 \sin(20 + k)° \sin 10° = \sin k°$

$\cos(10 + k)° - \cos(30 + k)° = \sin k°$

$\cos(10 + k)° = \sin(60 - k)° + \sin k°$

$\cos(10 + k)° = 2 \sin 30° \cos(30 - k)°$

$\cos(10 + k)° = \cos(30 - k)°$

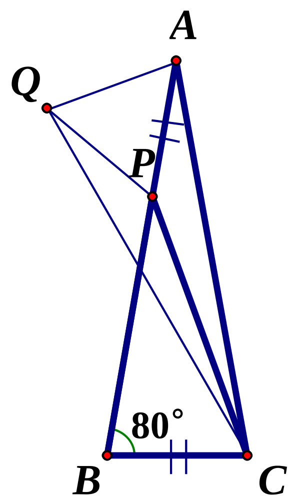
$10 + k = 30 - k$

$k = 10$

Method 2

Rotate A 60° in anti-clockwise direction about P as shown.

ΔAPQ is an equilateral triangle. Join QC.



$\angle ACB = 80° = \angle ACB$ (base $\angle$s isos. Δ)

$\angle BAC = 20°$ ($\angle$ sum of Δ)

$\angle QAP = 60° = \angle AQP$ ($\angle$s of an equilateral Δ)

$\angle QAC = 60° + 20° = 80° = \angle ACB$

$\because QA = AP = BC$ (given)

$AC = AC$ (common)

$\therefore\ \Delta ACB \cong \Delta CAQ$ (S.A.S.)

$\therefore\ \angle AQC = \angle ABC = 80°$ (corr. $\angle$s $\cong \Delta$'s)

$\angle CQP = 80° - 60° = 20° = \angle CAP$

$CP = CP$ (common)

$AP = QP$ (by construction)

$\Delta APC \cong \Delta QPC$ (S.S.S.)

$\therefore\ \angle ACQ = \angle BAC = 20°$ (corr. $\angle$s $\Delta ACB \cong \Delta CAQ$)

$\angle ACP = \angle QCP = 10°$ (corr. $\angle$s $\Delta APC \cong \Delta QPC$)

G10 Suppose $P(a, b)$ is a point on the straight line $x - y + 1 = 0$ such that the sum of the distance between P and the point $A(1,0)$ and the distance between P and the point $B(3,0)$ is the least, find the value of $a + b$.

Regard $x - y + 1 = 0$ as mirror.

$C(-1,2)$ is the mirror image of $A(1,0)$.

Sum of distance is the least

$\Rightarrow P(a, b)$ lies on BC.

$P(a, b)$ lies on $x - y + 1 = 0$

$\Rightarrow b = a + 1$

$m_{PB} = m_{BC}$

$$\frac{a+1}{a-3} = \frac{2}{-4}$$

$-2a - 2 = a - 3$

$$a = \frac{1}{3},\quad b = \frac{4}{3}$$

$$a + b = \frac{5}{3}$$