

Individual Events

I1	<i>a</i>	2	I2	<i>a</i>	136	I3	<i>a</i>	4	I4	<i>a</i>	8	I5	<i>a</i>	20
	<i>b</i>	2		<i>b</i>	-2620		<i>b</i>	24		<i>b</i>	9		<i>b</i>	2
	<i>c</i>	2		<i>c</i>	100		<i>c</i>	50		<i>c</i>	4		<i>c</i>	257
	<i>d</i>	1		<i>d</i>	50		<i>d</i>	500		<i>d</i>	54		<i>d</i>	7

Group Events

G6	<i>p</i>	-2	G7	<i>a</i>	36	G8	<i>m</i>	-2	G9	<i>a</i>	9	G10	<i>a</i>	50
	<i>m</i>	8		<i>b</i>	18		<i>d</i>	3		<i>b</i>	3		<i>b</i>	10
	<i>r</i>	1		<i>c</i>	2		<i>n</i>	96		<i>x</i>	11		<i>c</i>	15
	<i>s</i>	-2		<i>d</i>	6		<i>s</i>	95856		<i>y</i>	10		<i>d</i>	60

Individual Event 1

I1.1 Given that $7^{2x} = 36$ and $7^{-x} = (6)^{-\frac{a}{2}}$, find the value of a .

$$7^x = 6$$

$$\Rightarrow 7^{-x} = (6)^{-\frac{a}{2}} = 6^{-1}$$

$$\Rightarrow a = 2$$

I1.2 Find the value of b if $\log_2 \{ \log_2 [\log_2(2b) + a] + a \} = a$.

$$\log_2 \{ \log_2 [\log_2(2b) + 2] + 2 \} = 2$$

$$\log_2 [\log_2(2b) + 2] + 2 = 2^2 = 4$$

$$\log_2 [\log_2(2b) + 2] = 2$$

$$\log_2(2b) + 2 = 2^2 = 4$$

$$\Rightarrow \log_2(2b) = 2$$

$$2b = 2^2 = 4$$

$$\Rightarrow b = 2$$

I1.3 If c is the total number of positive roots of the equation

$$(x - b)(x - 2)(x + 1) = 3(x - b)(x + 1), \text{ find the value of } c.$$

$$(x - 2)(x - 2)(x + 1) - 3(x - 2)(x + 1) = 0$$

$$(x - 2)(x + 1)[(x - 2) - 3] = 0$$

$$(x - 2)(x + 1)(x - 5) = 0$$

$$x = 2, -1 \text{ or } 5$$

$$\Rightarrow \text{Number of positive roots} = c = 2$$

I1.4 If $\sqrt{3 - 2\sqrt{2}} = \sqrt{c} - \sqrt{d}$, find the value of d .

Reference: 1999 HG3, 2001 FG2.1, 2011 HI7, 2015 FI4.2, 2015 FG3.1

$$\sqrt{3 - 2\sqrt{2}} = \sqrt{1 - 2\sqrt{2} + 2}$$

$$= \sqrt{(\sqrt{1})^2 - 2\sqrt{2} + (\sqrt{2})^2}$$

$$= \sqrt{(\sqrt{2} - 1)^2} = \sqrt{2} - 1$$

$$\Rightarrow d = 1$$

Individual Event 2

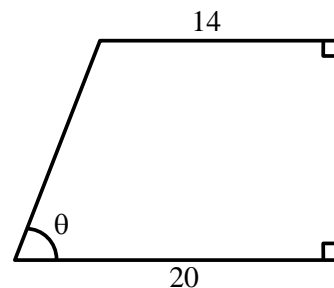
- I2.1** If $\sin \theta = \frac{4}{5}$, find a , the area of the quadrilateral.

Let the height be h .

$$\tan \theta = \frac{4}{3} = \frac{h}{6}$$

$$\Rightarrow h = 8$$

$$\text{Area} = \frac{1}{2}(14 + 20) \cdot 8 = 136$$



- I2.2** If $b = 126^2 - a^2$, find b .

$$b = 126^2 - a^2$$

$$= (126 - 136)(126 + 136) = -2620$$

- I2.3** Dividing $\$(3000 + b)$ in a ratio $5 : 6 : 8$, the smallest part is $\$c$. Find c .

$$\text{Sum of money} = \$(3000 - 2620) = \$380$$

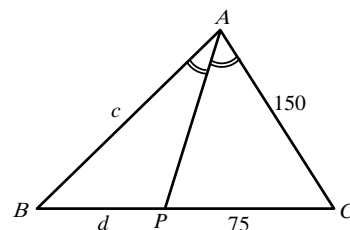
$$c = \frac{5}{5+6+8} \cdot 380 = \frac{5}{19} \cdot 380 = 100$$

- I2.4** In the figure, AP bisects $\angle BAC$. Given that $AB = c$, $BP = d$, $PC = 75$ and $AC = 150$, find d .

Let $\angle BAP = \theta = \angle CAP$, $\angle APC = \alpha$, $\angle BPC = 180^\circ - \alpha$

$$\frac{d}{\sin \theta} = \frac{100}{\sin(180^\circ - \alpha)} \dots (1) \text{ and } \frac{75}{\sin \theta} = \frac{150}{\sin \alpha} \dots (2)$$

$$(1) \div (2) \Rightarrow d = 50$$



Individual Event 3

- I3.1** If a is the remainder when 2614303940317 is divided by 13, find a .

$$2614303930000 = 13 \times 211003030000$$

$$2614303940317 = 13 \times 211003030000 + 1317 = 13 \times 211003030000 + 1313 + 4$$

$$a = 4$$

- I3.2** Let $P(x, b)$ be a point on the straight line $x + y = 30$ such that slope of $OP = a$ (O is the origin). Determine b . (Reference: 1994 FI1.4)

$$x + b = 30$$

$$\Rightarrow x = 30 - b$$

$$m_{OP} = \frac{b}{30 - b} = 4$$

$$\Rightarrow b = 120 - 4b$$

$$\Rightarrow b = 24$$

- I3.3** Two cyclists, initially $(b + 26)$ km apart travelling towards each other with speeds 40 km/h and 60 km/h respectively. A fly flies back and forth between their noses at 100 km/h. If the fly flew c km before crushed between the cyclists, find c .

The velocity of one cyclist relative to the other cyclist is $(40 + 60)$ km/h = 100 km/h.

Distance between the two cyclists = $(24 + 26)$ km = 50 km

$$\text{Time for the two cyclists meet} = \frac{50}{100} \text{ h} = \frac{1}{2} \text{ h}$$

$$\text{The distance the fly flew} = \frac{1}{2} \times 100 \text{ km} = 50 \text{ km}$$

$$\Rightarrow c = 50$$

- I3.4** In the figure, APK and BPH are straight lines.

If d = area of triangle HPK , find d .

$$\angle BAP = \angle KHP = 30^\circ \text{ (given)}$$

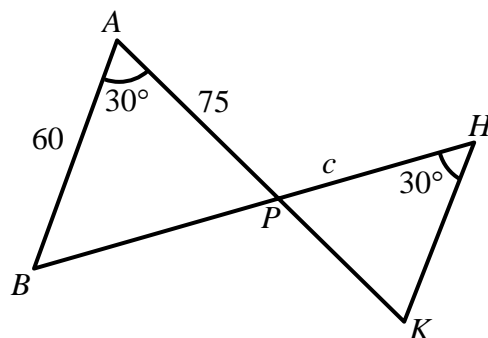
$$\angle APB = \angle KPH \text{ (vert. opp. } \angle \text{s)}$$

$$\triangle ABP \sim \triangle HKP \text{ (equiangular)}$$

$$\frac{HK}{60} = \frac{50}{75}$$

$$\Rightarrow HK = 40$$

$$d = \frac{1}{2} \times 50 \times 40 \cdot \sin 30^\circ = 500$$



Individual Event 4

- I4.1** Given that the means of x and y , y and z , z and x are respectively 5, 9, 10. If a is the mean of x , y , z , find the value of a .

$$\frac{x + y}{2} = 5 \dots (1); \quad \frac{y + z}{2} = 9 \dots (2); \quad \frac{z + x}{2} = 10 \dots (3)$$

$$(1) + (2) + (3): x + y + z = 24$$

$$\Rightarrow a = 8$$

- I4.2** The ratio of two numbers is 5 : a . If 12 is added to each of them, the ratio becomes 3 : 4. If b is the difference of the original numbers and $b > 0$, find the value of b .

Let the two numbers be $5k$, $8k$.

$$\frac{5k + 12}{8k + 12} = \frac{3}{4}$$

$$\Rightarrow 20k + 48 = 24k + 36$$

$$\Rightarrow 4k = 12$$

$$\Rightarrow k = 3$$

$$5k = 15, 8k = 24$$

$$b = 24 - 15 = 9$$

- I4.3** $PQRS$ is a rectangle. If c is the radius of the smaller circle, find the value of c .

Let the centres of the two circles be C and D , with radius 9 and c respectively.

Suppose the circles touch each other at E .

Further, assume that the circle with centre at C touches SR , PS , PQ at I , J and G respectively. Let the circle with centre at D touches PQ , QR at K and H respectively.

Join CI , CJ , CG , CE , DF , DK , DH .

$CI \perp SR$, $CJ \perp PS$, $CG \perp PQ$, $DK \perp PQ$, $DH \perp PR$ (tangent \perp radius)

$DK \parallel HQ$ (corr. \angle s eq.)

$\angle FDK = 90^\circ$ (corr. \angle s, $DK \parallel HQ$)

$DFGK$ is a rectangle (3 angles $= 90^\circ$)

$\therefore \angle DFG = 90^\circ$ (\angle s sum of polygon)

$\angle DFC = 90^\circ$ (adj. \angle s on st. line)

C , E , D are collinear (\because the two circles touch each other at E)

$CI = CJ = CG = CE = 9$ (radii of the circle with centre at C)

$DH = DK = DE = c$ (radii of the circle with centre at D)

$CD = c + 9$

$FG = DK = c$ (opp. sides of rectangle $DFGK$)

$CF = 9 - c$

$FD = GK$ (opp. sides of rectangle $DFGK$)

$= PD - PG - KQ$

$= 25 - 9 - c$ (opp. sides of rectangle)

$= 16 - c$

$CF^2 + DF^2 = CD^2$ (Pythagoras' theorem)

$$(9 - c)^2 + (16 - c)^2 = (9 + c)^2$$

$$81 - 18c + c^2 + 256 - 32c + c^2 = 81 + 18c + c^2$$

$$c^2 - 68c + 256 = 0$$

$$(c - 4)(c - 64) = 0$$

$c = 4$ or 64 (> 18 , rejected)

- I4.4** $ABCD$ is a rectangle and CEF is an equilateral triangle, $\angle ABD = 6c^\circ$, find the value of d .

Reference: HKCEE MC 1982 Q51

$\angle ABD = 24^\circ$ (given)

$\angle CAB = 24^\circ$ (diagonals of rectangle)

$\angle AEB = 132^\circ$ (\angle s sum of Δ)

$\angle CED = 132^\circ$ (vert. opp. \angle s)

$\angle CEF = 60^\circ$ (\angle of an equilateral triangle)

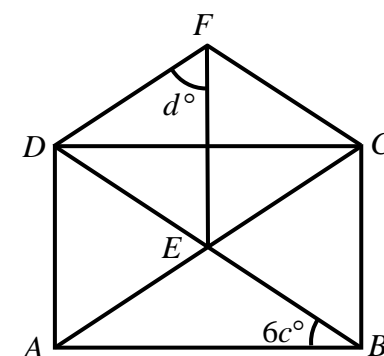
$\angle DEF = 132^\circ - 60^\circ = 72^\circ$

$ED = EC = EF$ (diagonals of rectangle, sides of equilateral Δ)

$\therefore \Delta DEF$ is isosceles (2 sides equal)

$\angle EFD = \angle EDF$ (base \angle s isos. Δ)

$d = (180 - 72) \div 2 = 54$ (\angle s sum of isos. Δ)



Individual Event 5

- 15.1** Two opposite sides of a rectangle are increased by 50% while the other two are decreased by 20%. If the area of the rectangle is increased by $a\%$, find a .

Let the length and width be x and y respectively.

$$1.5x \times 0.8y = 1.2xy$$

$$\Rightarrow a = 20$$

- 15.2** Let $f(x) = x^3 - 20x^2 + x - a$ and $g(x) = x^4 + 3x^2 + 2$. If $h(x)$ is the highest common factor of $f(x)$ and $g(x)$, find $b = h(1)$.

Reference: 1992 HI5, 2001 FI1.2, 2011 FI3.2

$$f(x) = x^3 - 20x^2 + x - 20 = (x^2 + 1)(x - 20)$$

$$g(x) = x^4 + 3x^2 + 2 = (x^2 + 1)(x^2 + 2)$$

$$h(x) = \text{H.C.F.} = x^2 + 1$$

$$b = h(1) = 2$$

- 15.3** It is known that $b^{16} - 1$ has four distinct prime factors, determine the largest one, denoted by c

$$2^{16} - 1 = (2 - 1)(2 + 1)(2^2 + 1)(2^4 + 1)(2^8 + 1) = 3 \times 5 \times 17 \times 257$$

$$c = 257$$

- 15.4** When c is represented in binary scale, there are d '0's. Find d .

$$257_{(x)} = 256 + 1$$

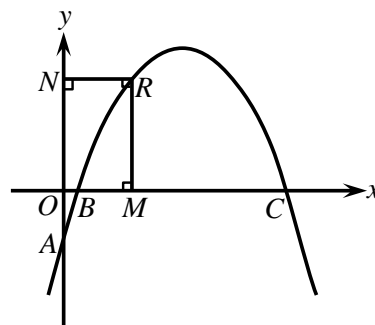
$$= 2^8 + 1$$

$$= 100000001_{(ii)}$$

$$d = 7$$

Group Event 6

The following shows the graph of $y = px^2 + 5x + p$. $A = (0, -2)$, $B = \left(\frac{1}{2}, 0\right)$, $C = (2, 0)$, $O = (0, 0)$.



G6.1 Find the value of p .

$$y = p\left(x - \frac{1}{2}\right)(x - 2)$$

$$\text{It passes through } A(0, -2): -2 = p\left(-\frac{1}{2}\right)(-2).$$

$$p = -2$$

G6.2 If $\frac{9}{m}$ is the maximum value of y , find the value of m .

$$y = -2x^2 + 5x - 2$$

$$\frac{9}{m} = \frac{4(-2)(-2) - 5^2}{4(-2)}$$

$$\Rightarrow m = 8$$

G6.3 Let R be a point on the curve such that $OMRN$ is a square. If r is the x -coordinate of R , find the value of r .

$$R(r, r) \text{ lies on } y = -2x^2 + 5x - 2$$

$$r = -2r^2 + 5r - 2$$

$$2r^2 - 4r + 2 = 0$$

$$\Rightarrow r = 1$$

G6.4 A straight line with slope $= -2$ passes through the origin cutting the curve at two points E and

F . If $\frac{7}{s}$ is the y -coordinate of the midpoint of EF , find the value of s .

$$\text{Sub. } y = -2x \text{ into } y = -2x^2 + 5x - 2$$

$$-2x = -2x^2 + 5x - 2$$

$$2x^2 - 7x + 2 = 0$$

$$\text{Let } E = (x_1, y_1), F = (x_2, y_2).$$

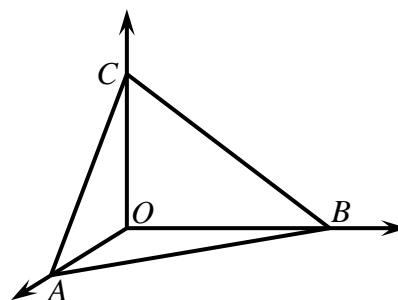
$$x_1 + x_2 = \frac{7}{2}$$

$$\frac{7}{s} = \frac{y_1 + y_2}{2} = \frac{-2x_1 - 2x_2}{2} = -(x_1 + x_2) = \frac{7}{-2}$$

$$s = -2$$

Group Event 7

$OABC$ is a tetrahedron with OA , OB and OC being mutually perpendicular. Given that $OA = OB = OC = 6x$.



G7.1 If the volume of $OABC$ is ax^3 , find a .

$$ax^3 = \frac{1}{3} \cdot \frac{1}{2} (6x)^2 \cdot (6x) = 36x^3$$

$$\Rightarrow a = 36$$

G7.2 If the area of $\triangle ABC$ is $b\sqrt{3}x^2$, find b .

$$AB = BC = AC = \sqrt{(6x)^2 + (6x)^2} = 6x\sqrt{2}$$

$\triangle ABC$ is equilateral

$$\angle BAC = 60^\circ$$

$$\text{Area of } \triangle ABC = b\sqrt{3}x^2 = \frac{1}{2} (6x\sqrt{2})^2 \sin 60^\circ = 18\sqrt{3}x^2$$

$$b = 18$$

G7.3 If the distance from O to $\triangle ABC$ is $c\sqrt{3}x$, find c .

By finding the volume of $OABC$ in two different ways.

$$\frac{1}{3} \cdot 18\sqrt{3}x^2 \times (c\sqrt{3}x) = 36x^3$$

$$c = 2$$

G7.4 If θ is the angle of depression from C to the midpoint of AB and $\sin \theta = \frac{\sqrt{d}}{3}$, find d .

$$\frac{1}{3} \cdot 18\sqrt{3}x^2 \times (c\sqrt{3}x) = 36x^3$$

Let the midpoint of AB be M .

$$OC = 6x, \quad \frac{OM \times AB}{2} = \frac{OA \times OB}{2}$$

$$\Rightarrow 6x\sqrt{2} \cdot OM = (6x)^2$$

$$\Rightarrow OM = 3\sqrt{2}x$$

$$CM = \sqrt{OM^2 + OC^2}$$

$$= \sqrt{(3\sqrt{2}x)^2 + (6x)^2}$$

$$= 3\sqrt{6}x$$

$$\sin \theta = \frac{\sqrt{d}}{3} = \frac{OC}{CM}$$

$$= \frac{6x}{3\sqrt{6}x} = \frac{\sqrt{6}}{3}$$

$$d = 6$$

Group Event 8

Given that the equation $x^2 + (m + 1)x - 2 = 0$ has 2 integral roots $(\alpha + 1)$ and $(\beta + 1)$ with $\alpha < \beta$ and $m \neq 0$. Let $d = \beta - \alpha$.

G8.1 Find the value of m .

$$(\alpha + 1)(\beta + 1) = -2$$

$$\Rightarrow \alpha + 1 = -1, \beta + 1 = 2 \text{ or } \alpha + 1 = -2, \beta + 1 = 1$$

$$\Rightarrow (\alpha, \beta) = (-2, 1), (-3, 0)$$

$$\text{When } (\alpha, \beta) = (-3, 0), \text{ sum of roots} = (\alpha + 1) + (\beta + 1) = -(m + 1) \Rightarrow m = 0 \text{ (rejected)}$$

$$\text{When } (\alpha, \beta) = (-2, 1), \text{ sum of roots} = (\alpha + 1) + (\beta + 1) = -(m + 1) \Rightarrow m = -2$$

G8.2 Find the value of d .

$$d = \beta - \alpha = 1 - (-2) = 3$$

Let n be the total number of integers between 1 and 2000 such that each of them gives a remainder of 1 when it is divided by 3 or 7. **Reference: 1994 FG8.1-2, 1998 HI6, 2015 FI3.1**

G8.3 Find the value of n .

These numbers give a remainder of 1 when it is divided by 21.

They are 1, $21 + 1$, $21 \times 2 + 1$, ..., $21 \times 95 + 1$ ($= 1996$)

$$n = 96$$

G8.4 If s is the sum of all these n integers, find the value of s .

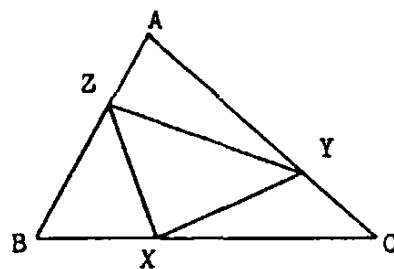
$$s = 1 + 22 + 43 + \dots + 1996 = \frac{1}{2}(1 + 1996) \cdot 96 = 95856$$

Group Event 9

BC , CA , AB are divided respectively by the points X , Y , Z in the ratio $1 : 2$. Let

area of $\triangle AZY$: area of $\triangle ABC = 2 : a$ and

area of $\triangle AZY$: area of $\triangle XYZ = 2 : b$.



G9.1 Find the value of a .

area of $\triangle AZY = \frac{2}{3}$ area of $\triangle ACZ$ (same height)

$$= \frac{2}{3} \times \frac{1}{3} \text{ area of } \triangle ABC \text{ (same height)}$$

$$\Rightarrow a = 9$$

G9.2 Find the value of b .

Reference: 2000 FI5.3

Similarly, area of $\triangle BZX = \frac{2}{9}$ area of $\triangle ABC$; area of $\triangle CXY = \frac{2}{9}$ area of $\triangle ABC$

area of $\triangle XYZ = \text{area of } \triangle ABC - \text{area of } \triangle AZY - \text{area of } \triangle BZX - \text{area of } \triangle CXY$

$$= \frac{1}{3} \text{ area of } \triangle ABC$$

$$2 : b = \text{area of } \triangle AZY : \text{area of } \triangle XYZ = \frac{2}{9} : \frac{1}{3}$$

$$\Rightarrow b = 3$$

A die is thrown 2 times. Let $\frac{x}{36}$ be the probability that the sum of numbers obtained is 7 or 8 and

$\frac{y}{36}$ be the probability that the difference of numbers obtained is 1.

G9.3 Find the value of x .

Favourable outcomes are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2)

$$P(7 \text{ or } 8) = \frac{x}{36}$$

$$\Rightarrow x = 11$$

G9.4 Find the value of y .

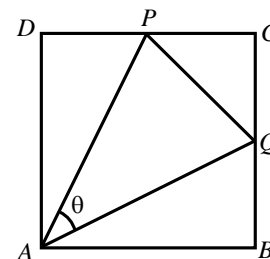
Favourable outcomes are (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 5), (5, 4), (4, 3), (3, 2), (2, 1).

$$P(\text{difference is } 1) = \frac{y}{36}$$

$$\Rightarrow y = 10$$

Group Event 10

$ABCD$ is a square of side length $20\sqrt{5}x$. P , Q are midpoints of DC and BC respectively.



G10.1 If $AP = ax$, find a .

$$\begin{aligned} AP &= \sqrt{AD^2 + DP^2} \\ &= \sqrt{(20\sqrt{5}x)^2 + (10\sqrt{5}x)^2} = 50x \\ \Rightarrow a &= 50 \end{aligned}$$

G10.2 If $PQ = b\sqrt{10}x$, find b .

$$\begin{aligned} PQ &= \sqrt{CP^2 + CQ^2} = 10\sqrt{10}x \\ \Rightarrow b &= 10 \end{aligned}$$

G10.3 If the distance from A to PQ is $c\sqrt{10}x$, find c .

$$\begin{aligned} c\sqrt{10}x &= AC - \text{distance from } C \text{ to } PQ \\ &= 20\sqrt{5}x \cdot \sqrt{2} - 10\sqrt{5}x \cdot \left(\frac{1}{\sqrt{2}}\right) \\ &= 15\sqrt{10}x \\ \Rightarrow c &= 15 \end{aligned}$$

G10.4 If $\sin \theta = \frac{d}{100}$, find d .

$$\begin{aligned} \text{Area of } \triangle APQ &= \frac{1}{2} \cdot AP \cdot AQ \sin \theta = \frac{1}{2} \cdot PQ \cdot (c\sqrt{10}x) \\ \Leftrightarrow \frac{1}{2} \cdot (50x)^2 \sin \theta &= \frac{1}{2} \cdot 10\sqrt{10}x \cdot 15\sqrt{10}x \\ \sin \theta &= \frac{d}{100} = \frac{3}{5} \\ \Rightarrow d &= 60 \end{aligned}$$