09-10 Individual	**1**	21	**2**	13	**3**	$\frac{4}{105}$	**4**	4	**5**	–3	**Spare**
	6	1	**7**	$\frac{7}{13}$	**8**	154	**9**	2	**10**	1508	2

09-10 Group	**1**	118	**2**	11	**3**	20	**4**	144	**5**	0.8	**Spare**
	6	250000	**7**	4019	**8**	10105	**9**	$\sqrt{3}$	**10**	20	15

Individual Events

I1 In how many possible ways can 8 identical balls be distributed to 3 distinct boxes so that every box contains at least one ball?

Reference: 2001 HG2, 2006 HI6, 2012 HI2

Align the 8 balls in a row. There are 7 gaps between the 8 balls. Put 2 sticks into two of these gaps, so as to divide the balls into 3 groups.

The following diagrams show one possible division.

○ ○ | ○ ○ ○ ○ ○ | ○

The three boxes contain 2 balls, 5 balls and 1 ball.

The number of ways is equivalent to the number of choosing 2 gaps as sticks from 7 gaps.

The number of ways is $C_2^7 = \frac{7\times 6}{2} = 21$

I2 If α and β are the two real roots of the quadratic equation $x^2 - x - 1 = 0$, find the value of $\alpha^6 + 8\beta$.

Reference 1993 HG2, 2013 HG4

$\alpha + \beta = 1$, $\alpha\beta = -1$

$\alpha^2 = \alpha + 1$

$\alpha^6 = (\alpha^2)^3 = (\alpha + 1)^3 = \alpha^3 + 3\alpha^2 + 3\alpha + 1$

$= \alpha(\alpha^2) + 3(\alpha + 1) + 3\alpha + 1$

$= \alpha(\alpha + 1) + 6\alpha + 4$

$= \alpha^2 + 7\alpha + 4 = (\alpha + 1) + 7\alpha + 4 = 8\alpha + 5$

$\alpha^6 + 8\beta = 8(\alpha + \beta) + 5 = 8 + 5 = 13$

I3 If $a = \frac{1}{5\times 10} + \frac{1}{10\times 15} + \frac{1}{15\times 20} + \cdots + \frac{1}{100\times 105}$, find the value of a. **(Reference: 2015 HG1)**

$$a = \frac{1}{25}\cdot\left(\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{20\times 21}\right) = \frac{1}{25}\cdot\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{20} - \frac{1}{21}\right) = \frac{1}{25}\cdot\left(1 - \frac{1}{21}\right)$$

$$a = \frac{20}{25\cdot 21} = \frac{4}{105}$$

I4 Given that $x + y + z = 3$ and $x^3 + y^3 + z^3 = 3$, where x, y, z are integers.

If $x < 0$, find the value of y.

Let $x = -a$, where $a > 0$, then $y + z = a + 3$ …… (3), $y^3 + z^3 = a^3 + 3$ …… (4)

From (4): $(y + z)^3 - 3yz(y + z) = a^3 + 3$

$\therefore (a + 3)^3 - a^3 - 3 = 3yz(a + 3)$

$$yz = \frac{a^3 + 9a^2 + 27a + 27 - a^3 - 3}{3(a+3)} = \frac{9a^2 + 27a + 24}{3(a+3)} = \frac{3a^2 + 9a + 8}{a+3} = 3a + \frac{8}{a+3} \text{ …… (5)}$$

yz is an integer $\Rightarrow a = 1$ or 5

$\because (y - z)^2 = (y + z)^2 - 4yz$

When $a = 1$, $x = -1$, $y + z = 4$ from (3) and $yz = 5$ from (5)

$\therefore (y - z)^2 = 4^2 - 4\times 5 = -4 < 0$, impossible. Rejected.

When $a = 5$, $y + z = 8$ and $yz = 16$

Solving for y and z gives $x = -5$, $y = 4$, $z = 4$

I5 Given that a, b, c, d are positive integers satisfying $\log_a b = \frac{1}{2}$ and $\log_c d = \frac{3}{4}$.

If $a - c = 9$, find the value of $b - d$.

$a^{\frac{1}{2}} = b$ and $c^{\frac{3}{4}} = d \Rightarrow a = b^2$ and $c = d^{\frac{4}{3}}$

Sub. them into $a - c = 9$.

$b^2 - d^{\frac{4}{3}} = 9$

$\left(b + d^{\frac{2}{3}}\right)\left(b - d^{\frac{2}{3}}\right) = 9$

$b + d^{\frac{2}{3}} = 3$, $b - d^{\frac{2}{3}} = 3$ (no solution, rejected) or $b + d^{\frac{2}{3}} = 9$, $b - d^{\frac{2}{3}} = 1$

$b = 5$, $d^{\frac{2}{3}} = 4 \Rightarrow b = 5$, $d = 8 \Rightarrow b - d = -3$

I6 If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$, where $0 \le x, y \le 1$, find the value of $x^2 + y^2$.

Method 1

Let $x = \sin A$, $y = \sin B$, then $\sqrt{1-y^2} = \cos B$, $\sqrt{1-x^2} = \cos A$

The equation becomes $\sin A \cos B + \cos A \sin B = 1$

$\sin (A + B) = 1$

$A + B = 90° \Rightarrow B = 90° - A$

$x^2 + y^2 = \sin^2 A + \sin^2 B = \sin^2 A + \sin^2(90° - A) = \sin^2 A + \cos^2 A = 1$

Method 2 $x\sqrt{1-y^2} = 1 - y\sqrt{1-x^2}$

$x^2(1 - y^2) = 1 - 2y\sqrt{1-x^2} + y^2(1 - x^2)$

$2y\sqrt{1-x^2} = 1 + y^2 - x^2$

$4y^2(1 - x^2) = y^4 - 2x^2y^2 + x^4 + 2y^2 - 2x^2 + 1$

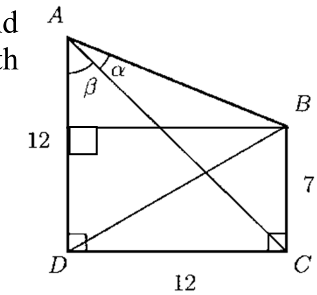
$x^4 + 2x^2y^2 + y^4 - 2y^2 - 2x^2 + 1 = 0$

$(x^2 + y^2)^2 - 2(x^2 + y^2) + 1 = 0$

$(x^2 + y^2 - 1)^2 = 0$

$\Rightarrow x^2 + y^2 = 1$

I7. In figure 1, $ABCD$ is a trapezium. The lengths of segments AD, BC and DC are 12, 7 and 12 respectively. If segments AD and BC are both perpendicular to DC, find the value of $\frac{\sin\alpha}{\sin\beta}$.



Method 1

Draw a perpendicular line from B onto AD.

$\tan\beta = \frac{12}{12} = 1$; $\tan(\alpha + \beta) = \frac{12}{12-7} = \frac{12}{5}$

$\tan\alpha = \tan[(\alpha + \beta) - \beta] = \frac{\tan(\alpha+\beta) - \tan\beta}{1 + \tan(\alpha+\beta)\tan\beta} = \frac{\frac{12}{5} - 1}{1 + \frac{12}{5}} = \frac{12-5}{5+12} = \frac{7}{17}$

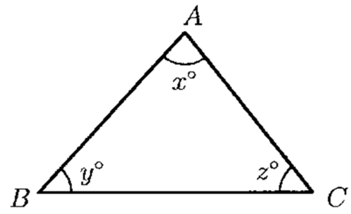
$\sin\alpha = \frac{7}{\sqrt{17^2+7^2}} = \frac{7}{\sqrt{338}} = \frac{7}{13\sqrt{2}}$; $\sin\beta = \frac{1}{\sqrt{2}}$

$\frac{\sin\alpha}{\sin\beta} = \frac{7}{13}$

Method 2 $\angle ACB = \beta$ (alt. $\angle$s, AD // BC)

$\frac{\sin\alpha}{\sin\beta} = \frac{7}{AB} = \frac{7}{13}$ (Sine law on ΔABC)

I8. In Figure 2, ABC is a triangle satisfying $x \geq y \geq z$ and $4x = 7z$. If the maximum value of x is m and the minimum value of x is n, find the value of $m + n$.



$x = 7k$, $z = 4k$, $x + y + z = 180 \Rightarrow y = 180 - 11k$

$\because x \geq y \geq z \therefore 7k \geq 180 - 11k \geq 4k$

$18k \geq 180$ and $180 \geq 15k$

$12 \geq k \geq 10$

$84 \geq x = 7k \geq 70$

$m = 84$, $n = 70$

$m + n = 154$

I9 Arrange the numbers 1, 2, … , n ($n \geq 3$) in a circle so that adjacent numbers always differ by 1 or 2. Find the number of possible such circular arrangements.

When $n = 3$, there are two possible arrangements: 1, 2, 3 or 1, 3, 2.

When $n = 4$, there are two possible arrangements: 1, 2, 4, 3 or 1, 3, 4, 2.

Deductively, for any $n \geq 3$, there are two possible arrangements:

1, 2, 4, 6, 8, … , largest even integer, largest odd integer, … , 7, 5, 3 or

1, 3, 5, 7, … , largest odd integer, largest even integer, … , 6, 4, 2.

I10 If $\lfloor x \rfloor$ is the largest integer less than or equal to x, find the number of distinct values in the following 2010 numbers: $\left\lfloor \frac{1^2}{2010} \right\rfloor, \left\lfloor \frac{2^2}{2010} \right\rfloor, \ldots, \left\lfloor \frac{2010^2}{2010} \right\rfloor$.

Reference: IMO Preliminary Selection Contest - Hong Kong 2006 Q13.

Let $f(n) = \frac{n^2}{2010}$, where n is an integer from 1 to 2010.

$f(n + 1) - f(n) = \frac{2n+1}{2010}$

$f(n + 1) - f(n) < 1 \Leftrightarrow \frac{2n+1}{2010} < 1 \Leftrightarrow n < 1004.5$

$f(1005) = \frac{1005^2}{2010} = \frac{1005}{2} = 502.5$

$\lfloor f(1) \rfloor = 0, \lfloor f(2) \rfloor = 0, \ldots, \lfloor f(1005) \rfloor = 502$, the sequence contain 503 different integers.

On the other hand, when $n > 1005$, $f(n + 1) - f(n) > 1$

All numbers in the sequence $\lfloor f(1006) \rfloor, \ldots, \lfloor f(2010) \rfloor$ are different, total 1005 numbers

503 + 1005 = 1508. The number of distinct values is 1508.

Spare individual

IS In Figure 3, ABC is an isosceles triangle and P is a point on BC. If $BP^2 + CP^2 : AP^2 = k : 1$, find the value of k.

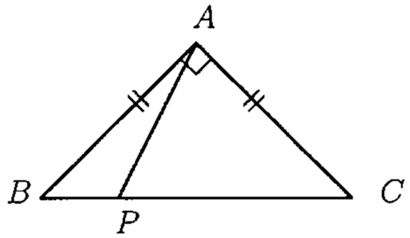


Figure 3

Reference: **2003 FI2.3**

Let $AB = AC = a$, $BC = \sqrt{2}\,a$, $BP = x$, $PC = y$, $AP = t$

Let $\angle APC = \theta$, $\angle APB = 180° - \theta$ (adj. $\angle$s on st. line)

Apply cosine rule on ΔABP and ΔACP

$\cos\theta = \dfrac{t^2+y^2-a^2}{2ty}$ … (1); $-\cos\theta = \dfrac{t^2+x^2-a^2}{2tx}$ … (2)

(1) + (2): $\dfrac{t^2+y^2-a^2}{2ty} + \dfrac{t^2+x^2-a^2}{2tx} = 0$

$x(t^2 + y^2 - a^2) + y(t^2 + x^2 - a^2) = 0$

$t^2(x + y) + xy(x + y) - a^2(x + y) = 0$

$(x + y)(t^2 + xy - a^2) = 0$

$x + y = 0$ (rejected, $\because$ $x > 0$, $y > 0$) or $t^2 + xy - a^2 = 0$

$t^2 + xy = a^2$ … (*)

$BP^2 + CP^2 : AP^2 = x^2 + y^2 : t^2 = [(x + y)^2 - 2xy] : t^2 = [BC^2 - 2xy] : t^2 = (2a^2 - 2xy) : t^2$

$= 2(a^2 - xy) : t^2 = 2t^2 : t^2$ by (*)

$\Rightarrow k = 2$

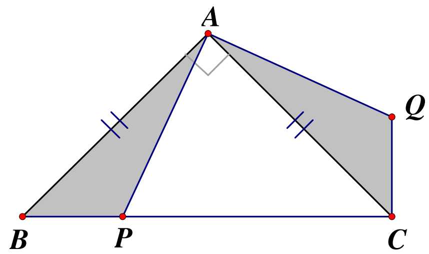
Method 2 (Provided by Chiu Lut Sau Memorial Secondary School Ip Ka Ho)

$\angle ABC = \angle ACB$ (base $\angle$s isosceles triangle)

$= \dfrac{180^\circ - 90^\circ}{2}$ ($\angle$s sum of Δ)

$= 45°$

Rotate AP anticlockwise 90° about the centre at A to AQ.



$AP = AQ$ and $\angle PAQ = 90°$ (property of rotation)

$\angle BAP = 90° - \angle PAC = \angle CAQ$

$AB = AC$ (given)

$\Delta ABP \cong \Delta ACQ$ (S.A.S.)

$\angle ACQ = \angle ABP = 45°$ (corr. $\angle$s $\cong\Delta$s)

$BP = CQ$ (corr. sides $\cong\Delta$s)

$\angle PCQ = \angle ACP + \angle ACQ = 90°$

$BP^2 + CP^2 : AP^2 = (CQ^2 + CP^2) : AP^2$

$= PQ^2 : AP^2$ (Pythagoras' theorem)

$= 1 : \cos^2 45°$

$= 2 : 1$

$k = 2$

Group Events

G1 Given that the six-digit number $503xyz$ is divisible by 7, 9, 11.
Find the minimum value of the three-digit number xyz.
Reference: 2000 FG4.1
There is no common factor for 7, 9, 11 and the L.C.M. of them are 693.
504 is divisible by 7 and 9. 504504 is divisible by 693.
504504 – 693 = 503811, 503811 – 693 = 503118.
The three-digit number is 118.

G2 Find the smallest positive integer n so that $\underbrace{20092009\cdots2009}_{n\text{ copies of }2009}$ is divisible by 11.
Reference: 2008 FI1.2
Sum of odd digits – sum of even digits = multiples of 11
$n(0 + 9) - n(2 + 0) = 11m$, where m is an integer.
$7n = 11m \Rightarrow$ Smallest $n = 11$.

G3 In figure 1, ABC is a triangle. D is a point on AC such that $AB = AD$.
If $\angle ABC - \angle ACB = 40°$, find the value of x. **Reference: 1985 FI2.2**

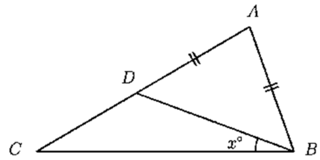
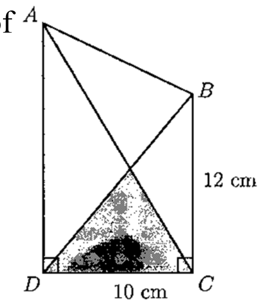


Figure 1

Let $\angle ACB = y°$, then $\angle ABC = y° + 40°$
$\angle BAC = 180° - y° - y° - 40° = 140° - 2y°$ ($\angle$s sum of ΔABC)
$\angle ADB = \angle ABD = \dfrac{180° - (140° - 2y°)}{2} = 20° + y°$ (base $\angle$s isos. Δ)
$x° = \angle CBD = \angle ADB - \angle ACB = 20° + y° - y° = 20°$ (ext. $\angle$ of ΔBCD)
$\Rightarrow x = 20$
Method 2 Let $\angle ACB = y°$
$\angle ADB = x° + y°$ (ext. $\angle$ of ΔBCD)
$\angle ABD = x° + y°$ (base $\angle$s isosceles ΔABD)
$\therefore \angle ABC = x° + x° + y° = 2x° + y°$
$\angle ABC - \angle ACB = 40°$
$2x° + y° - y° = 40°$
$x = 20$

G4 In figure 2, given that the area of the shaded region is 35 cm^2. If the area of the trapezium $ABCD$ is z cm^2, find the value of z.



Reference 1993 HI2, 1997 HG3, 2000 FI2.2, 2002 FI1.3, 2004 HG7, 2013 HG2
Suppose AC and BD intersect at K.
$S_{BCD} = \dfrac{10\times 12}{2} = 60 = S_{CDK} + S_{BCK} = 35 + S_{BCK} \Rightarrow S_{BCK} = 25$
ΔBCK and ΔDCK have the same height but different bases.
$BK : KD = S_{BCK} : S_{DCK} = 25 : 35 = 5 : 7 \Rightarrow BK = 5t, KD = 7t$
$\Delta BCK \sim \Delta DAK$ (equiangular) $\Rightarrow S_{BCK} : S_{DAK} = BK^2 : DK^2 = 7^2 : 5^2 = 49 : 25$
ΔABK and ΔADK have the same height but different bases.
$S_{ABK} : S_{ADK} = BK : KD = 5 : 7 \Rightarrow z = S_{ABCD} = 35 + 25 + 49 + 35 = 144$

G5 Three numbers are drawn from 1, 2, 3, 4, 5, 6.
Find the probability that the numbers drawn contain at least two consecutive numbers.
Method 1
Favourable outcomes = {123, 124, 125, 126, 234, 235, 236, 134, 345, 346, 145, 245, 456, 156, 256, 356}, 16 outcomes
Probability $= \dfrac{16}{C_3^6} = \dfrac{4}{5} = 0.8$

Method 2 Probability = 1 – P(135, 136, 146 or 246) $= 1 - \dfrac{4}{C_3^6} = 0.8$

G6 Find the minimum value of the following function:

$f(x) = |x-1| + |x-2| + \cdots + |x-1000|$, where x is a real number.

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2011 FGS.1, 2012 FG2.3

Method 1

$f(500) = |500-1| + |500-2| + \cdots + |500-1000| = (499 + 498 + \cdots + 1)\times 2 + 500 = 250000$

Let n be an integer, for $1 \le n \le 500$ and $x \le n$,

$|x-n| + |x-(1001-n)| = n - x + 1001 - n - x = 1001 - 2x \ge 1001 - 2n$

$|500-n| + |500-(1001-n)| = 500 - n + 501 - n = 1001 - 2n$

For $1 \le n < x \le 500$, $|x-n| + |x-(1001-n)| = x - n + 1001 - n - x = 1001 - 2n$

$$\text{If } x \le 500, f(x) - f(500) = \sum_{n=1}^{1000}|x-n| - \sum_{n=1}^{1000}|500-n|$$

$$= \left[\sum_{n=1}^{500}|x-n| + |x-(1001-n)|\right] - \sum_{n=1}^{500}\left[|500-n| + |500-(1001-n)|\right]$$

$$\ge \sum_{n=1}^{500}\left[1001-2n-(1001-2n)\right] \ge 0$$

$f(1001-x) = |1001-x-1| + |1001-x-2| + \cdots + |1001-x-1000|$

$= |1000-x| + |999-x| + \cdots + |1-x|$

$= |x-1| + |x-2| + \cdots + |x-1000| = f(x)$

$\therefore$ $f(x) \ge f(500) = 250000$ for all real values of x.

Method 2 We use the following 2 results: (1) $|a-b| = |b-a|$ and (2) $|a| + |b| \ge |a+b|$

$|x-1| + |x-1000| = |x-1| + |1000-x| \ge |999| = 999$

$|x-2| + |x-999| = |x-2| + |999-x| \ge |997| = 997$

…………………………………………………

$|x-500| + |x-501| = |x-500| + |501-x| \ge 1$

Add up these 500 inequalities: $f(x) \ge 1 + 3 + \cdots + 999 = \frac{1}{2}(1+999)\times 500 = 250000$.

G7 Let m, n be positive integers such that $\frac{1}{2010}<\frac{m}{n}<\frac{1}{2009}$. Find the minimum value of n.

Reference: **1996 FG10.3, 2005 HI1**

Method 1

$$\frac{2009}{2010}=1-\frac{1}{2010}>\frac{n-m}{n}>1-\frac{1}{2009}=\frac{2008}{2009}$$

$$1+\frac{1}{2009}=\frac{2010}{2009}<\frac{n}{n-m}<\frac{2009}{2008}=1+\frac{1}{2008}$$

$$\frac{1}{2009}<\frac{n}{n-m}-1=\frac{m}{n-m}<\frac{1}{2008}$$

$$\frac{2008}{2009}=1-\frac{1}{2009}>1-\frac{m}{n-m}=\frac{n-2m}{n-m}>1-\frac{1}{2008}=\frac{2007}{2008}$$

$$1+\frac{1}{2008}=\frac{2009}{2008}<\frac{n-m}{n-2m}<\frac{2008}{2007}=1+\frac{1}{2007}$$

$$\frac{1}{2008}<\frac{n-m}{n-2m}-1=\frac{m}{n-2m}<\frac{1}{2007}$$

Claim: $\frac{1}{2010-a}<\frac{m}{n-am}<\frac{1}{2009-a}$ for $a = 0, 1, 2, \cdots, 2008$.

Proof: Induction on a. When $a = 0, 1, 2$; proved above.

Suppose $\frac{1}{2010-k}<\frac{m}{n-km}<\frac{1}{2009-k}$ for some integer k, where $0 \le k < 2008$

$$\frac{2009-k}{2010-k}=1-\frac{1}{2010-k}>1-\frac{m}{n-km}=\frac{n-(k+1)m}{n-km}>1-\frac{1}{2009-k}=\frac{2008-k}{2009-k}$$

$$1+\frac{1}{2009-k}=\frac{2010-k}{2009-k}<\frac{n-km}{n-(k+1)m}<\frac{2009-k}{2008-k}=1+\frac{1}{2008-k}$$

$$\frac{1}{2009-k}<\frac{n-km}{n-(k+1)m}-1=\frac{m}{n-(k+1)m}<\frac{1}{2008-k}$$

$$\frac{1}{2010-(k+1)}<\frac{m}{n-(k+1)m}<\frac{1}{2009-(k+1)}$$

By MI, the statement is true for $a = 0, 1, 2, \dots, 2008$

Put $a = 2008$: $\frac{1}{2010-2008}<\frac{m}{n-2008m}<\frac{1}{2009-2008}$

$$\frac{1}{2}<\frac{m}{n-2008m}<1$$

The smallest possible n is found by $\frac{m}{n-2008m}=\frac{2}{3}$

$m = 2$, $n - 2008\times 2 = 3$

$\Rightarrow n = 4019$

Method 2 $\frac{1}{2010}<\frac{m}{n}<\frac{1}{2009}\Rightarrow 2010>\frac{n}{m}>2009\Rightarrow 2010m > n > 2009m$

$\because$ m, n are positive integers. We wish to find the least value of n

$\therefore$ It is equivalent to find the least value of m.

When $m = 1$, $2010 > n > 2009$, no solution for n.

When $m = 2$, $4020 > n > 4018$

$\Rightarrow n = 4019$

G8 Let a be a positive integer. If the sum of all digits of a is equal to 7, then a is called a "lucky number". For example, 7, 61, 12310 are lucky numbers.
List all lucky numbers in ascending order $a_1, a_2, a_3, \dots$. If $a_n = 1600$, find the value of a_{2n}.

Number of digits	smallest, $\cdots$, largest	Number of lucky numbers	subtotal
1	7	1	1
2	16, 25, $\cdots$, 61, 70	7	7
3	106, 115, $\cdots$, 160	7	
	205, 214, $\cdots$, 250	6	
	304, 313, $\cdots$, 340	5	
	……………	…	
	700	1	28
4	1006, 1015, $\cdots$, 1060	7	
	1105, 1114, $\cdots$, 1150	6	
	1204, $\cdots$, 1240	5	
	…………	…	
	1600	1	$a_{64} = 1600$
	2005, … , 2050	6	
	…………	…	
	2500	1	
	3004, $\cdots$, 3040	5	
	…………	…	
	3400	1	
	4XYZ	4+3+2+1	
	5XYZ	3+2+1	
	6XYZ	2+1	
	7000	1	84
5	100XY	7	
	10105	1	

$a_{128} = 10105$

G9 If $\log_4(x + 2y) + \log_4(x - 2y) = 1$, find the minimum value of $|x| - |y|$.

$(x + 2y)(x - 2y) = 4$

$x^2 - 4y^2 = 4$

$x^2 = 4y^2 + 4$

$T = |x| - |y| = \sqrt{4(y^2 + 1)} - |y|$

$T + |y| = \sqrt{4(y^2 + 1)}$

$T^2 + y^2 + 2|y|T = 4(y^2 + 1)$

$3|y|^2 - 2|y|T + (4 - T^2) = 0$

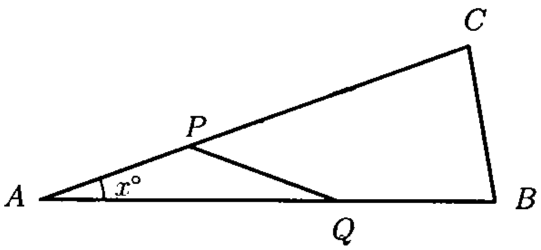
$\Delta = 4[T^2 - (3)(4 - T^2)] \ge 0$

$4T^2 - 12 \ge 0$

$T \ge \sqrt{3}$

The minimum value of $|x| - |y|$ is $\sqrt{3}$.

G10 In Figure 3, in ΔABC, $AB = AC$, $x \le 45$. If P and Q are two points on AC and AB respectively, and $AP = PQ = QB = BC \le AQ$, find the value of x.



Reference:2004 HG9, HKCEE 2002 Q10

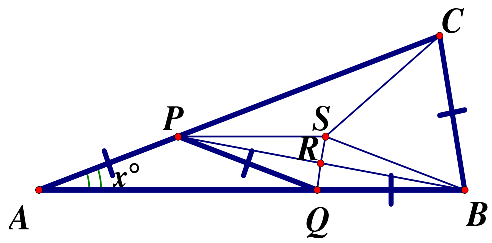
Method 1

Join PB. $\angle AQP = x°$ (base $\angle$s isos. Δ)

$\angle BPQ = \angle PBQ$ (base $\angle$s isos. Δ)

$= \frac{x°}{2}$ (ext. $\angle$ of ΔBPQ)

Let R be the mid point of PB. Join QR and produce its own length to S so that $QR = RS$.



Join PS, BS and CS.

$PQBS$ is a //-gram (diagonals bisect each other)

$\therefore PS = PQ = BQ = BS$ (opp. sides of //-gram)

$\because PS \parallel QB$

$\therefore \angle CPS = x°$ (corr. $\angle$s, $PS \parallel AB$)

$PC = AC - AP = AB - BQ = AQ$

$\therefore \Delta SPC \cong \Delta PAQ$ (S.A.S.)

$\therefore SC = PQ$ (corr. sides, $\cong \Delta$'s)

$\therefore BS = SC = BC$

ΔBCS is an equilateral triangle.

$\angle SBC = \angle SCB = 60°$

$\angle SCP = \angle AQP = x°$ (corr. $\angle$s, $\cong \Delta$'s)

$\angle SBQ = \frac{x°}{2} + \frac{x°}{2} = x°$ (corr. $\angle$s, $\cong \Delta$'s)

In ΔABC, $x° + x° + x° + 60° + 60° = 180°$ ($\angle$ sum of Δ)

$x = 20$

Method 2 Let $AP = PQ = QB = BC = t$, let $AQ = y$

$\angle AQP = x°$ (base $\angle$s isos. Δ)

$AQ = y = 2t \cos x° = y + t - t = AC - AP = CP$

$\angle BPQ = \angle PBQ$ (base $\angle$s isos. Δ)

$= \frac{x°}{2}$ (ext. $\angle$ of ΔBPQ)

$\angle QPC = 2x°$ (ext. $\angle$ of ΔAPQ)

$\angle BPC = \angle QPC - \angle BPQ = 2x° - \frac{x°}{2} = \frac{3x°}{2}$

$\angle ABC = \angle ACB = 90° - \frac{x°}{2}$ ($\angle$ sum of isos. ΔABC)

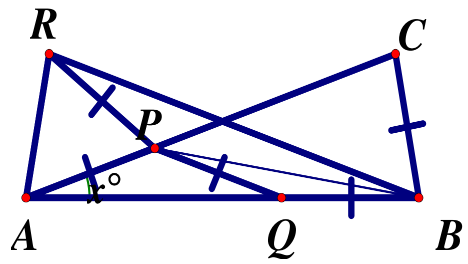
$\angle CBP = \angle ABC - \angle PBQ = 90° - \frac{x°}{2} - \frac{x°}{2} = 90° - x°$

$$\frac{CP}{\sin \angle CBP} = \frac{BC}{\sin \angle BPC}$$ (Sine law on ΔBCP)

$$\frac{2t \cos x°}{\sin(90° - x°)} = \frac{t}{\sin \frac{3x°}{2}}$$

$$\sin \frac{3x°}{2} = \frac{1}{2} \Rightarrow x = 20$$

Method 3 Reflect ΔABP along PB to ΔRPB



$\Delta ABP \cong \Delta RPB$ (by construction)

Join AR, $AB = BR$ (corr. sides, $\cong \Delta$'s)

$\angle AQP = x°$ (base $\angle$s isos. Δ)

$\angle BPQ = \angle PBQ$ (base $\angle$s isos. Δ)

$= \frac{x°}{2}$ (ext. $\angle$ of ΔBPQ)

$\angle PBR = \frac{x°}{2}$ (corr. $\angle$s, $\cong \Delta$'s)

$\angle ABR = \angle ABP + \angle RBP = \frac{x°}{2} + \frac{x°}{2} = x°$

$\therefore \angle ABR = \angle BAC = x°$

$AC = AB$ (given)

$= BR$ (corr. sides, $\cong \Delta$'s)

$\therefore \Delta ABR \cong \Delta BAC$ (S.A.S.)

$AR = BC$ (corr. sides, $\cong \Delta$'s)

$= AP = PR$ (given)

ΔAPR is an equilateral triangle. (3 sides equal)

$\angle PAR = 60°$ ($\angle$ of an equilateral triangle)

$\angle BAR = 60° + x°$

$\angle ABC = 90° - \frac{x°}{2}$ ($\angle$ sum of isos. ΔABC)

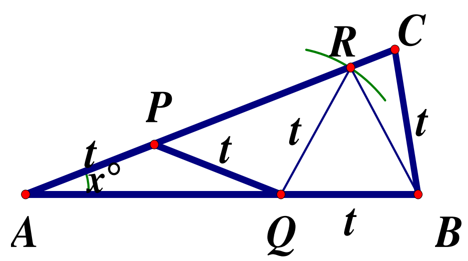
$\angle ABC = \angle BAR$ (corr. $\angle$s, $\cong \Delta$'s)

$60° + x° = 90° - \frac{x°}{2}$

$x = 20$

Method 4 Let $AP = PQ = QB = BC = t$

Use Q as centre, QP as radius to draw an arc, cutting AC at R. $QR = QP = t$ (radius of the arc)



$\angle AQP = x°$ (base $\angle$s isos. Δ)

$\angle QPR = 2x°$ (ext. $\angle$ of ΔAPQ)

$\angle QRP = 2x°$ (base $\angle$s isos. Δ)

$\angle BQR = 3x°$ (ext. $\angle$ of ΔAQR)

$\angle QBR = 90° - \frac{3x°}{2}$ ($\angle$ sum of isos. ΔQBR)

$\angle BRC = 90° - \frac{3x°}{2} + x° = 90° - \frac{x°}{2}$ (ext. $\angle$ of ΔABR)

$\angle ABC = \angle ACB = 90° - \frac{x°}{2} = \angle BRC$ ($\angle$ sum of isos. ΔABC)

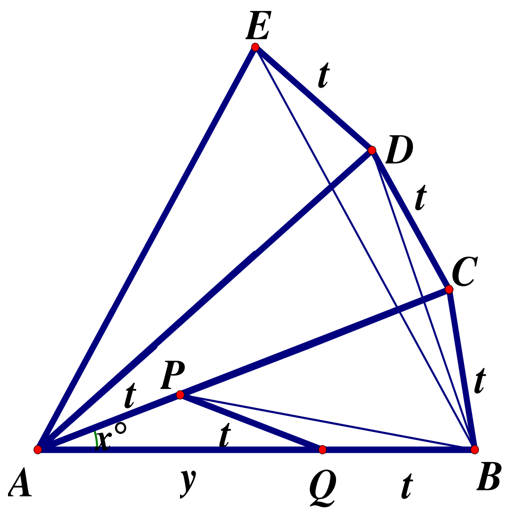
$\therefore BR = BC = t$ (sides opp. eq. $\angle$s)

ΔBQR is an equilateral triangle. (3 sides equal)

$\angle BQR = 3x° = 60°$

$x = 20$

Method 5 Let $AP = PQ = QB = BC = t$, $AQ = y$



$\angle AQP = x°$ (base $\angle$s isos. Δ)

$\angle BPQ = \angle PBQ$ (base $\angle$s isos. Δ)

$= \frac{x°}{2}$ (ext. $\angle$ of ΔBPQ)

$\angle QPC = 2x°$ (ext. $\angle$ of ΔAPQ)

$\angle BPC = \angle QPC - \angle BPQ = 2x° - \frac{x°}{2} = \frac{3x°}{2}$

$\angle ABC = \angle ACB = 90° - \frac{x°}{2}$ ($\angle$s sum of ΔABC)

As shown, construct two triangles so that

$\Delta ABC \cong \Delta ACD \cong \Delta ADE$

Join BE, BD, BP.

$AP = BC = t$, $PQ = CD = t$ (corr. sides $\cong \Delta$'s)

$\angle BCD = 2\times\angle ACB = 180° - x° = \angle BQP$ (adj. $\angle$s on st. line)

$\therefore \Delta BCD \cong \Delta BQP$ (S.A.S.)

$BD = BP$ …… (1) (corr. sides $\cong \Delta$'s)

$\angle CBD = \angle QBP = \frac{x°}{2}$; $\angle BDC = \angle BPQ = \frac{x°}{2}$ (corr. $\angle$s $\cong \Delta$'s)

$\angle BDE = \angle ADE + \angle ADC - \angle BDC$

$= 90° - \frac{x°}{2} + 90° - \frac{x°}{2} - \frac{x°}{2}$

$= 180° - \frac{3x°}{2}$

$= 180° - \angle BPC$

$= \angle APB$ (adj. $\angle$s on st. line)

$\therefore \angle BDE = \angle APB$ …… (2)

$AP = DE$ …… (3) (by construction, corr. sides $\cong \Delta$'s)

By (1), (2) and (3), $\Delta BDE \cong \Delta BPA$ (S.A.S.)

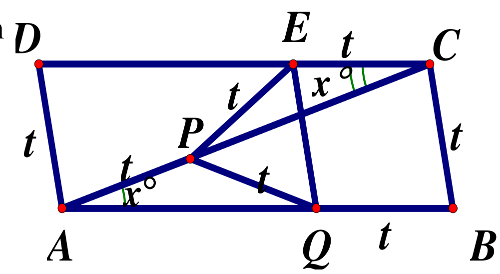
$\therefore BE = AB = y + t = AE$ (corr. sides $\cong \Delta$'s)

$\therefore \Delta ABE$ is an equilateral triangle

$\angle BAE = x° + x° + x° = 60°$ (angle of an equilateral triangle)

$x = 20$

Method 6 The method is provided by Ms. Li Wai Man



Construct another identical triangle ACD so that

$\angle ACD = x°$, $CE = t = EP = PA = AD$

$CD = AB$ and $AD = BC$

$\therefore ABCD$ is a parallelogram (opp. sides equal)

$CE = t = QB$ and CE // BQ (property of //-gram)

$\therefore BCEQ$ is a parallelogram (opp. sides equal and //)

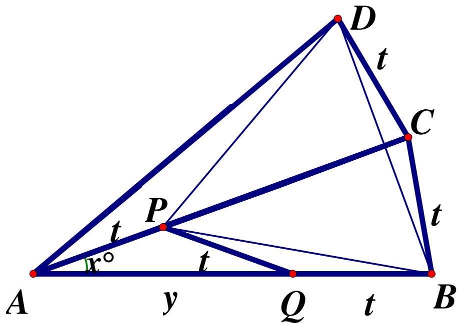
$\therefore EQ = t = PQ = EQ$ (property of //-gram)

ΔPQE is an equilateral triangle

$\angle QPE = x° + 2x° = 60°$

$x = 20$

Method 7 Let $AP = PQ = QB = BC = t$, $AQ = y$



$\angle AQP = x°$ (base $\angle$s isos. Δ)

$\angle BPQ = \angle PBQ$ (base $\angle$s isos. Δ)

$= \dfrac{x^\circ}{2}$ (ext. $\angle$ of ΔBPQ)

$\angle QPC = 2x°$ (ext. $\angle$ of ΔAPQ)

$\angle BPC = \angle QPC - \angle BPQ = 2x° - \dfrac{x^\circ}{2} = \dfrac{3x^\circ}{2}$

$\angle ABC = \angle ACB = 90° - \dfrac{x^\circ}{2}$ ($\angle$ sum of ΔABC)

As shown, reflect ΔABC along AC to ΔADC

$\Delta ABC \cong \Delta ADD$

Join BD, BP, PD.

$AP = BC = t$, $PQ = CD = t$ (corr. sides $\cong$ Δ's)

$\angle BCD = 2\times\angle ACB = 180° - x° = \angle BQP$ (adj. $\angle$s on st. line)

$\therefore$ $\Delta BCD \cong \Delta BQP$ (S.A.S.)

$BD = BP$ …… (1) (corr. sides $\cong$ Δ's)

$BP = PD$ (corr. sides $\cong$ Δ's)

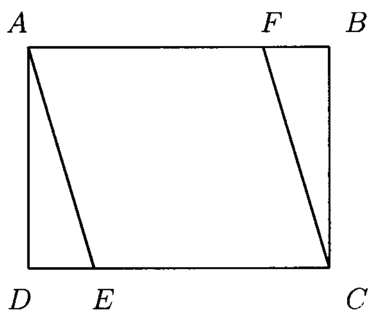
$\therefore$ ΔBDP is an equilateral triangle.

$\angle BPD = 2\angle BPC = 2\times\dfrac{3x^\circ}{2} = 60°$

$x = 20$

Spare Group

GS In Figure 4, $ABCD$ is a rectangle. Let E and F be two points on DC and AB respectively, so that $AFCE$ is a rhombus. If $AB = 16$ and $BC = 12$, find the value of EF.



Let $AF = FC = CE = EA = t$

$DE = 16 - t = BF$

In ΔADE, $12^2 + (16 - t)^2 = t^2$ (Pythagoras' Theorem)

$144 + 256 - 32t + t^2 = t^2$

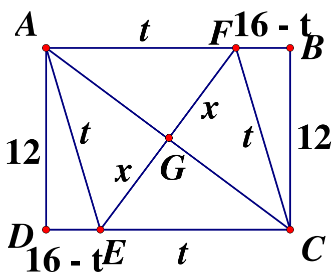
$32t = 400$

$t = 12.5$

In ΔACD, $AC^2 = 12^2 + 16^2$ (Pythagoras' Theorem)

$AC = 20$

G = centre of rectangle = centre of the rhombus



$AG = GC = 10$ (Diagonal of a rectangle)

Let $EG = x = FG$ (Diagonal of a rhombus)

In ΔAEG, $x^2 + AG^2 = t^2$ (Pythagoras' Theorem)

$x^2 + 10^2 = 12.5^2$

$x = 7.5$

$EF = 2x = 15$

Geometrical Construction

1. Figure 1 shows a line segment AB of length 1 unit. Construct a line segment of length $\sqrt{7}$ units.

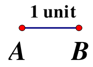


Figure 1

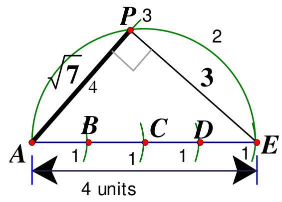
Method 1

(1) Extend AB. Use a pair of compasses to mark the points C, D, E so that $AB = BC = CD = DE$. $AE = 4$ units.

(2) Use C as centre, $CA = CE$ as radius to draw a semi-circle.

(3) Use E as centre, EB as radius (3 units) to draw an arc, which intersects the semi-circle at P.

(4) Join AP.



$\angle APC = 90°$ ($\angle$ in semi-circle)

$AP = \sqrt{4^2 - 3^2} = \sqrt{7}$ (Pythagoras' Theorem)

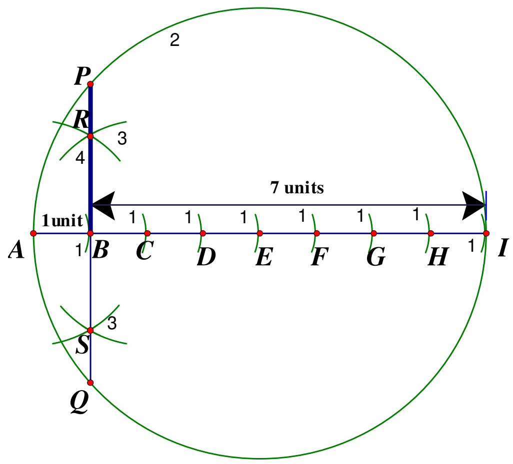
Method 2

(1) Extend AB. Use a pair of compasses to mark the points C, D, E, F, G, H, I so that $AB = BC = CD = DE = EF = FG = GH = HI$.
$BI = 7$ units.

(2) Use E as centre, $EA = EI$ (4 units) as radius to draw a semi-circle.

(3) Use A as centre, AC as radius to draw an arc; use C as centre, CA as radius to draw an arc. The two arcs intersect at R and S.

(4) Join RS and extend it to cut the circle at P and Q. respectively



Then $PB = \sqrt{7}$ units.

Proof: $PB = BQ$ ($\perp$ from centre bisect chord)

$AB \times BC = PB \times BQ$ (intersection chords theorem)

$1 \times 7 = PB^2$

$PB = \sqrt{7}$ units

2. Given that ΔABC is equilateral. P, Q and R are distinct points lying on the lines AB, BC and CA such that $OP \perp AB$, $OQ \perp BC$, $OR \perp CA$ and $OP = OQ = OR$. Figure 2 shows the line segment OP. Construct ΔABC.

Figure 2

Construction steps

(1) Use O as centre, OP as radius to construct an arc; use P as centre, PO as radius to construct another arc. The two arcs intersect at H and I. ΔOPH and ΔOPI are equilateral.

(2) Use H as centre, HP as radius to construct an arc; use P as centre, PH as radius to construct another arc. The two arcs intersect at O and J. ΔPHJ is equilateral.

(3) Use I as centre, IP as radius to construct an arc; use P as centre, PI as radius to construct another arc. The two arcs intersect at O and K. ΔPIK is equilateral.

(4) Use H as centre, HJ as radius to construct an arc; use J as centre, JH as radius to construct another arc. The two arcs intersect at P and B. ΔBHJ is equilateral.

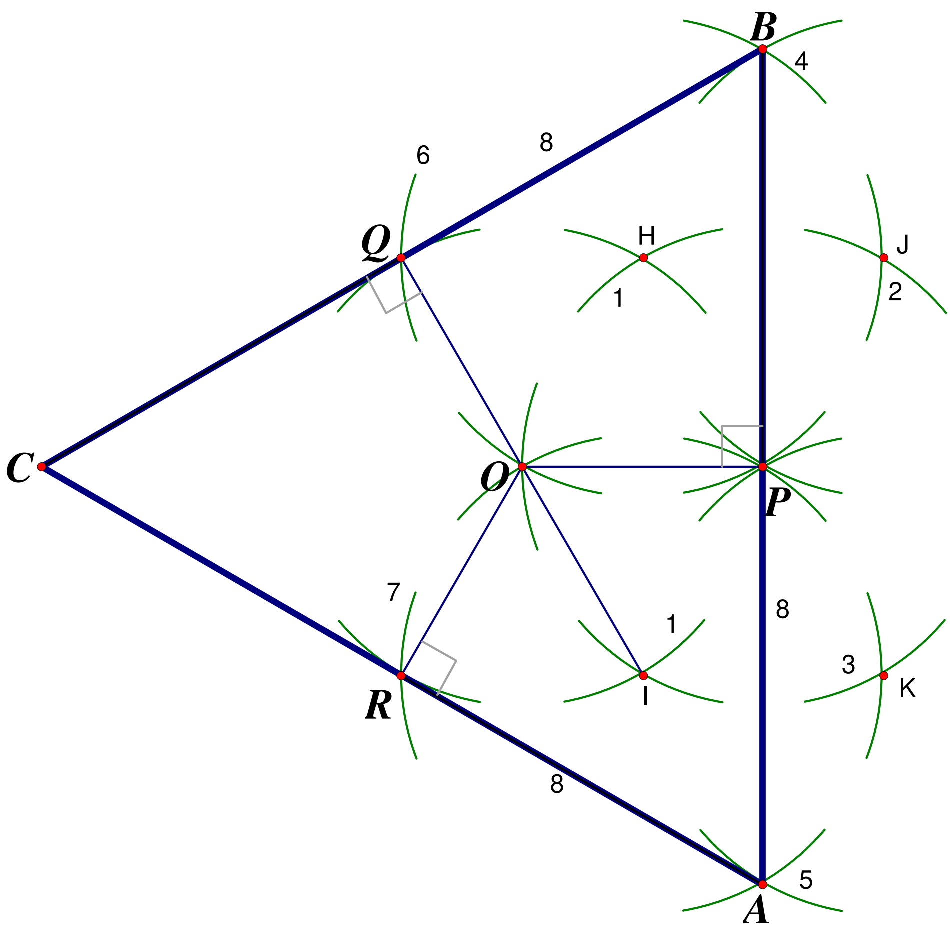
(5) Use I as centre, IK as radius to construct an arc; use K as centre, KI as radius to construct another arc. The two arcs intersect at P and A. ΔAIJ is equilateral.
BP is the angle bisector of $\angle HPJ$. $AB \perp OP$.

(6) Use O as centre, OH as radius to construct an arc; use H as centre, HO as radius to construct another arc. The two arcs intersect at P and Q. ΔOHQ is equilateral.

(7) Use O as centre, OI as radius to construct an arc; use I as centre, IO as radius to construct another arc. The two arcs intersect at P and R. ΔOIR is equilateral.

(8) Join AB, AR produced and BQ produced to meet at C.

Then ΔABC is the required equilateral triangle.



3. Figure 3 shows a line segment AB. Construct a triangle ABC such that $AC : BC = 3 : 2$ and $\angle ACB = 60°$.

Method 1

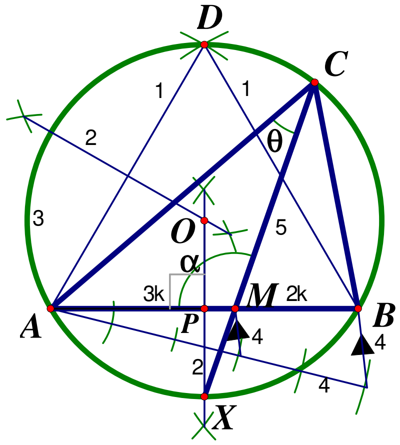
Step 1 Construct an equilateral triangle ABD.

Step 2 Construct the perpendicular bisectors of AB and AD respectively to intersect at the circumcentre O.

Step 3 Use O as centre, OA as radius to draw the circumscribed circle ABD.

Step 4 Locate M on AB so that $AM : MB = 3 : 2$ (intercept theorem)

Step 5 The perpendicular bisector of AB intersect the minor arc AB at X and AB at P. Produce XM to meet the circle again at C. Let $\angle ACM = \theta$, $\angle AMC = \alpha$.



$\Delta APX \cong \Delta BPX$ (S.A.S.)

$AX = BX$ (corr. sides $\cong$ Δ's)

$\angle ACX = \angle BCX = \theta$ (eq. chords eq. angles)

$\angle AMC = \alpha$, $\angle BMC = 180° - \alpha$ (adj. ∠s on st. line)

$3k : \sin\theta = AC : \sin\alpha$ …… (1) (sine rule on ΔACM)

$2k : \sin\theta = BC : \sin(180° - \alpha)$ …… (2) (ΔBCM)

Use the fact that $\sin(180° - \alpha) = \sin\alpha$;

(1) ÷ (2): $3 : 2 = AC : BC$

$\angle ACB = \angle ADB = 60°$ (∠s in the same segment)

ΔABC is the required triangle.

Method 2

Step 1 Use A as centre, AB as radius to draw an arc PBH.

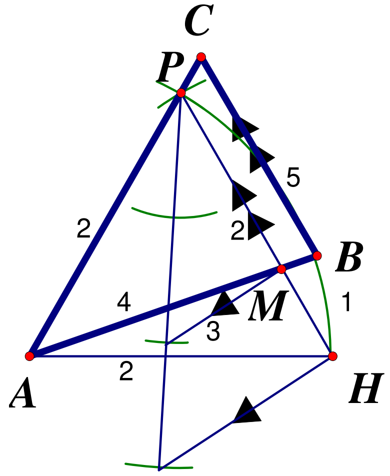
Step 2 Draw an equilateral triangle AHP (H is any point on the arc) $\angle APH = 60°$

Step 3 Locate M on PH so that $PM = \frac{2}{3}PH$

(intercept theorem)

Step 4 Produce AM to meet the arc at B.

Step 5 Draw a line BC // PH to meet AP produced at C.



$\angle ACB = 60°$ (corr. ∠s, PH // CB)

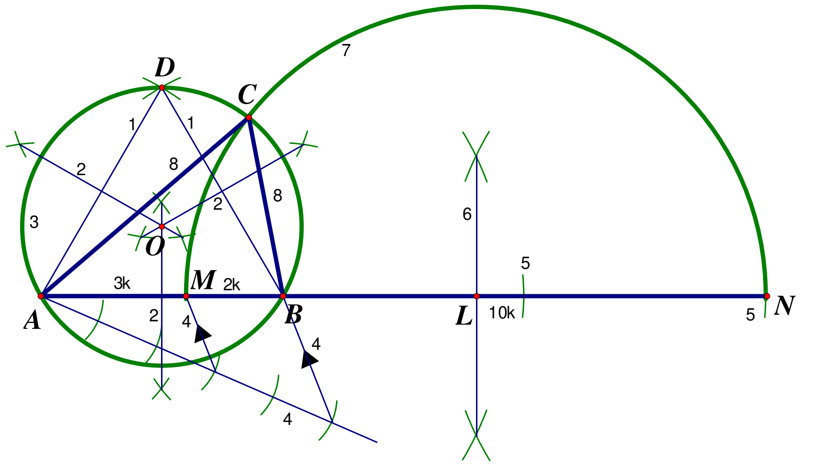
$\Delta ABC \sim \Delta AMP$ (equiangular)

$AC : CB = AP : PM$ (ratio of sides, ~Δ's)

$= 1 : \frac{2}{3} = 3 : 2$

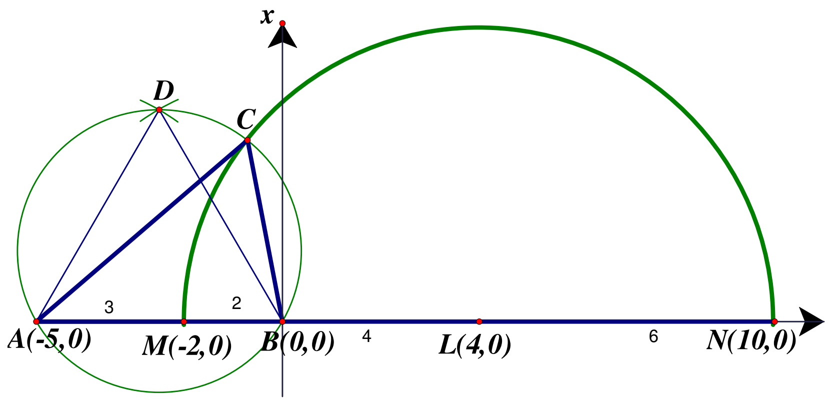
ΔABC is the required triangle.

Method 3 (Provided by Mr. Lee Chun Yu, James from St. Paul's Co-educational College)



Step 1 Construct an equilateral triangle ABD.
Step 2 Construct the perpendicular bisectors of AB, BD and AD respectively to intersect at the circumcentre O.
Step 3 Use O as centre, OA as radius to draw the circumscribed circle ABD.
Step 4 Locate M on AB so that $AM : MB = 3 : 2$ (intercept theorem)
Step 5 Produce AB to N so that $BN = 2AB$.
Let $AM = 3k$, $MB = 2k$, $BN = 10k$, then $AN : NB = 15k : -10k = 3 : -2$ (signed distance)
N divides AB externally in the ratio $3 : -2$.
Step 6 Construct the perpendicular bisectors of MN to locate the mid-point L.
Step 7 Use L as centre, LM as radius to draw a semi-circle MCN which intersects the circle ABD at C.
Step 8 Join AC and BC, then ΔABC is the required triangle.

Proof: Method 3.1



For ease of reference, assume $AM = 3$, $MB = 2$
Introduce a rectangular co-ordinate system with B as the origin, MN as the x-axis.
The coordinates of A, M, B, L, N are $(-5, 0)$, $(-2, 0)$, $(0, 0)$, $(4, 0)$ and $(10, 0)$ respectively.
Equation of circle MCN: $(x + 2)(x - 10) + y^2 = 0 \Rightarrow y^2 = 20 + 8x - x^2$ …… (1)
Let $C = (x, y)$.

$CA = \sqrt{(x+5)^2 + y^2} = \sqrt{x^2 + 10x + 25 + 20 + 8x - x^2} = \sqrt{18x + 45} = 3\sqrt{2x+5}$ by (1)

$CB = \sqrt{x^2 + y^2} = \sqrt{x^2 + 20 + 8x - x^2} = \sqrt{8x + 20} = 2\sqrt{2x+5}$ by (2)

$\dfrac{CA}{CB} = \dfrac{3\sqrt{2x+5}}{2\sqrt{2x+5}} = \dfrac{3}{2}$

$\angle ACB = \angle ADB = 60°$ ($\angle$s in the same segment)
ΔABC is the required triangle.

Proof: **(method 3.2)**

$MN = 12k$

$ML = LN = 6k$

$BL = 4k$

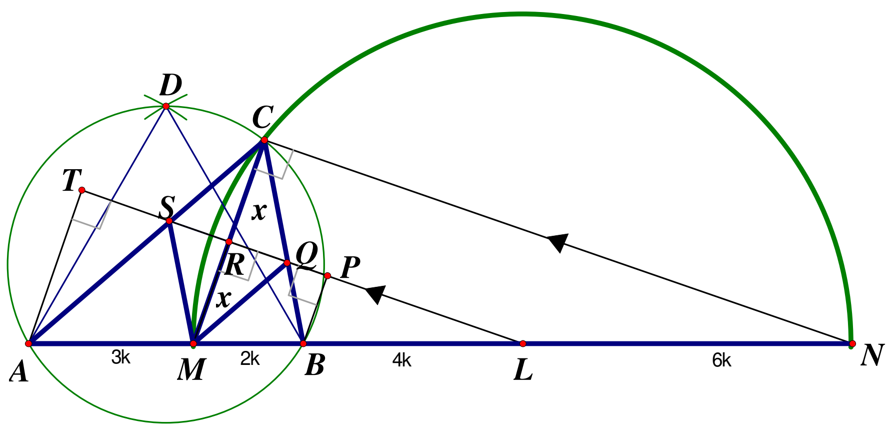
Join CM, CN.

Draw TL // CN

TL intersects AC, MC and BC at S, R and Q respectively.

$\angle MCN = 90°$

$\angle$ in semi-circle



T and P are the feet of perpendiculars from A and B onto TL respectively.

$\angle MRL = 90°$ (corr. $\angle$s TL//CN)

Let $CR = x = RM$ ($\perp$ from centre bisects chord)

$\Delta CSR \cong \Delta MSR$ (S.A.S.) and $\Delta CQR \cong \Delta MQR$ (S.A.S.)

$\therefore$ $CS = MS$ and $CQ = MQ$ ……(*) (corr. sides, $\cong$ Δs)

$\Delta LMR \sim \Delta LAT$ (AT // MR, equiangular)

$AT : MR = AL : ML$ (ratio of sides, ~Δs)

$AT = \frac{9k}{6k} \cdot x = 1.5x$

$\Delta ATS \sim \Delta CRS$ (AT // CR, equiangular)

$AS : SC = AT : CR$ (ratio of sides, ~Δs)

$= 1.5x : x$

$= 3 : 2$ …… (1)

$\Delta LMR \sim \Delta LBP$ (BP // MR, equiangular)

$BP : MR = BL : ML$ (ratio of sides, ~Δs)

$BP = \frac{4k}{6k} \cdot x = \frac{2x}{3}$

$\Delta BPQ \sim \Delta CRQ$ (PB // CR, equiangular)

$BQ : QC = BP : CR$ (ratio of sides, ~Δs)

$= \frac{2x}{3} : x$

$= 2 : 3$ …… (2)

By (1): $AS : SC = 3 : 2 = AM : MB$

$\therefore$ SM // CB (converse, theorem of equal ratio)

By (2): $BQ : QC = 2 : 3 = BM : MA$

$\therefore$ AC // MQ (converse, theorem of equal ratio)

$\therefore$ $CSMQ$ is a parallelogram formed by 2 pairs of parallel lines

By (*), $CS = MS$ and $CQ = MQ$

$\therefore$ $CSMQ$ is a rhombus

Let $\angle SCM = \theta = \angle QCM$ (Property of a rhombus)

Let $\angle AMC = \alpha$, $\angle BMC = 180° - \alpha$ (adj. $\angle$s on st. line)

$3k : \sin\theta = AC : \sin\alpha$ …… (3) (sine rule on ΔACM)

$2k : \sin\theta = BC : \sin(180° - \alpha)$ …… (4) (sine rule on ΔBCM)

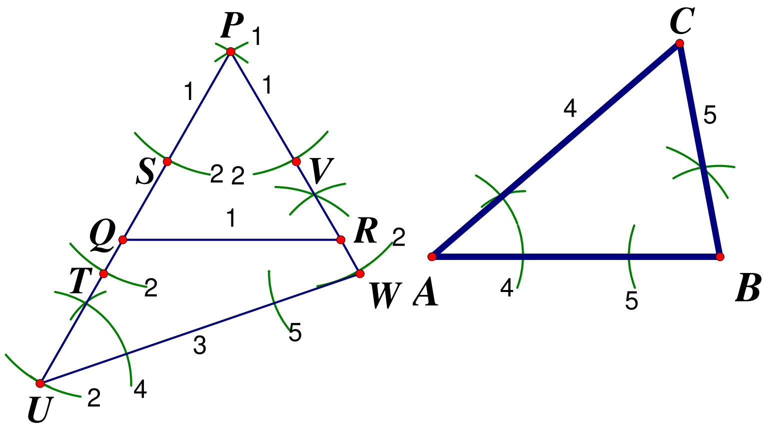
Use the fact that $\sin(180° - \alpha) = \sin\alpha$;

(3) ÷ (4): $3 : 2 = AC : BC$

$\angle ACB = \angle ADB = 60°$ ($\angle$s in the same segment)

ΔABC is the required triangle.

Method 4 (Provided by Chiu Lut Sau Memorial Secondary School Ip Ka Ho)



Step 1 Construct an equilateral triangle *PQR*. (*QR* is any length)

Step 2 Produce *PQ* and *PR* longer. On *PQ* produced and *PW* produced, mark the points *S*, *T*, *U*, *V* and *W* such that $PS = ST = TU = PV = VW$, where *PS* is any distance.

Step 3 Join *UW*.

Step 4 Copy $\angle PUW$ to $\angle BAC$.

Step 5 Copy $\angle PWU$ to $\angle ABC$. *AC* and *BC* intersect at *C*.

ΔABC is the required triangle.

Proof: By step 1, $\angle QPR = 60°$ (Property of equilateral triangle)

By step 2, $PU : PW = 3 : 2$

By step 4 and step 5, $\angle PUW = \angle BAC$ and $\angle PWU = \angle ABC$

$\Delta PUW \sim \Delta CAB$ (equiangular)

$AC : BC = PU : PW = 3 :2$ (corr. sides, ~Δs)

The proof is completed.