Q1 Problem *ABCD* is a convex quadrilateral such that the diagonals are perpendicular which intersects at *O* and *OA* > *OC* and *OB* > *OD*. To prove *AD* + *BC* > *AB* + *CD*.

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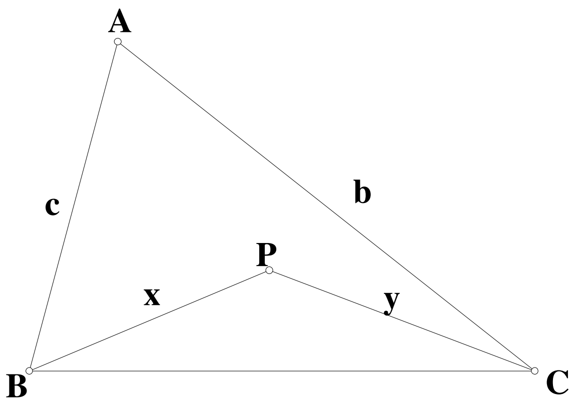
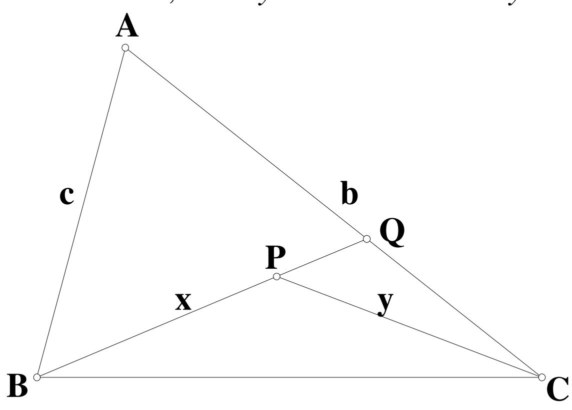
Theorem In ΔABC, $AB = c$, $AC = b$, P is a point <u>inside</u> ΔABC. $BP = x$, $CP = y$. Then $b + c > x + y$.

Proof:





Product BP to Q on AC

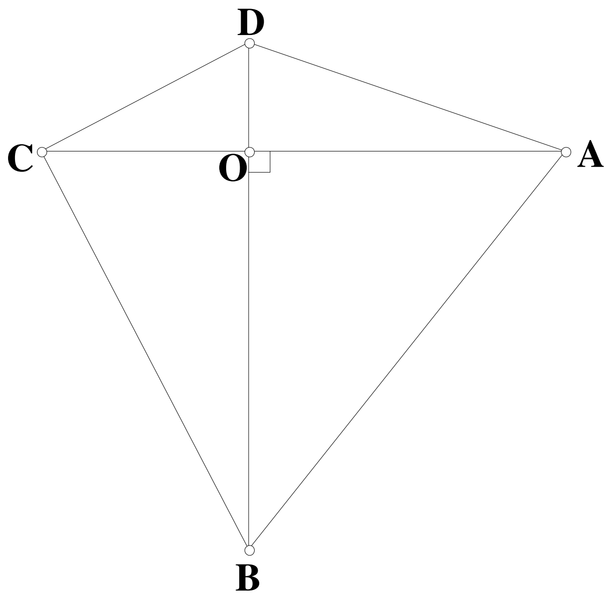
In ΔABQ, $c + AQ > x + PQ$ (Δ inequality) ....(1)

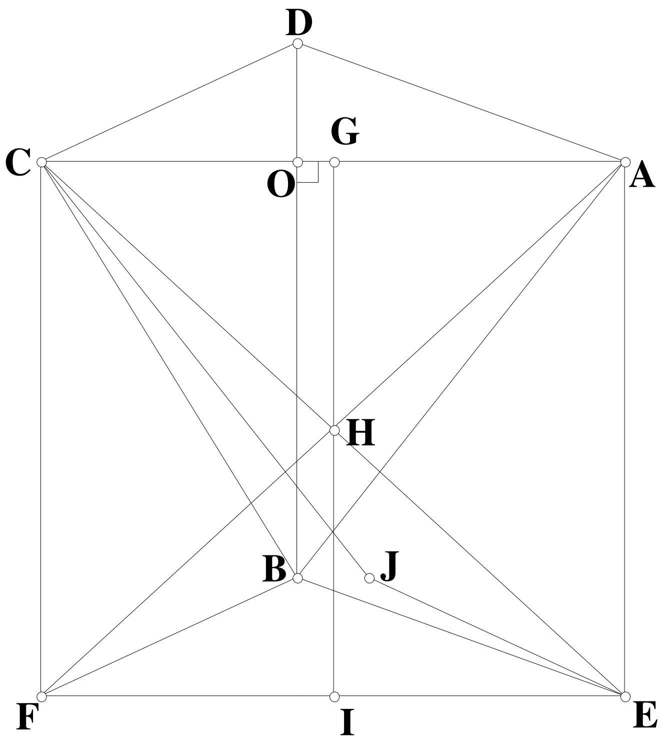
In ΔCPQ, $PQ + QC > y$ (Δ inequality) ....(2)

(1) + (2) $PQ + (AQ + QC) + c > x + y + PQ$

$\therefore$ $b + c > x + y$ The theorem is proved.

Problem *ABCD* is a convex quadrilateral such that the diagonals are perpendicular which intersects at *O* and $OA > OC$ and $OB > OD$. To prove $AD + BC > AB + CD$.





Step 1 Translate ΔADC in the direction DB to ΔEBF. Join AE, CE.

Then AE // DB // CF and $EA \perp AC$ and $FC \perp AC$ and $AE = CF$. $AEFC$ is a rectangle.

Join AF and CE. The diagonals AF and CE bisect each other at H.

Step 2 Draw a line GHI // DB. Reflect ΔABF along the line GHI to ΔCJE.

Then $AD = BE$, $AB = CJ$, $CD = BF = EJ$.

$\because$ $OA > OC$, $OB > OD$ (given) and GHI bisects the rectangle $AEFC$.

$\therefore O$ and B lie on the same side of GHI. Also B and D lie on the opposite sides of AF.

$\therefore$ B is a point inside ΔEFH.

Apply Pythagoras' Theorem on ΔADO and ΔCDO.

$DO^2 = AD^2 - OA^2 = CD^2 - OC^2 \Rightarrow AD^2 - CD^2 = OA^2 - OC^2$

$\because$ $OA > OC$ $\therefore AD > CD$

Consider ΔBEF, by the property of translation, $BE = AD$ and $BF = CD$

$\because AD > CD \Rightarrow BE > BF \Rightarrow \angle BFE > \angle BEF$ ....(*)

By the property of diagonals of a rectangle, $\angle HFE = \angle HEF$

$\angle HFB = \angle HFE - \angle BFE < \angle HEF - \angle BEF = \angle HEB$ by (*)

By the property of reflection, $\angle HFB = \angle HEJ \Rightarrow \angle HEB > \angle HEJ$ ....(1)

In a similar manner, we can prove that

(i) H lies inside ΔABC

(ii) $AB > AC \Rightarrow \angle BCO > \angle BAO$

(iii) Use the property that $\angle HAG = \angle HCG$ and (ii) to deduce that $\angle BCH > \angle BAH$

(iv) Use the property of reflection to deduce that $\angle BCH > \angle JCH$ ....(2)

Combine (1) and (2) to conclude that J is a point <u>inside</u> ΔCBE.

By the theorem at the beginning, $CB + BE > CJ + JE$

By the property of reflection again, we conclude that $AD + BC > AB + CD$.

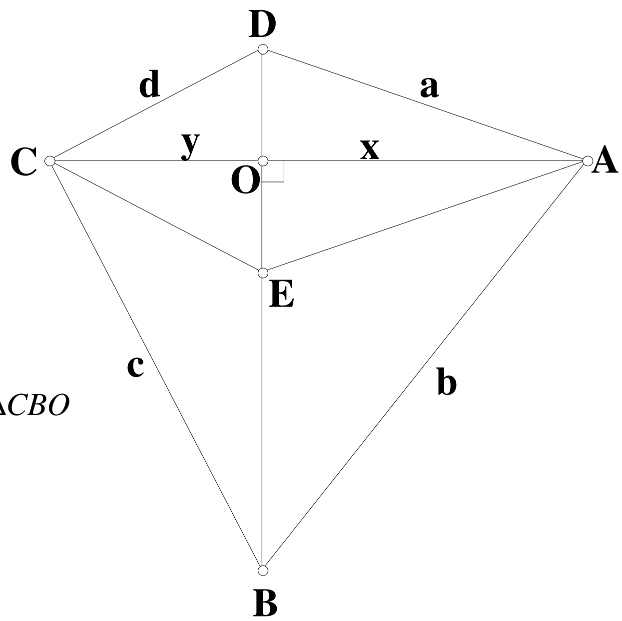
The result is proved.

Method 2

Let the letters a, b, c, d, x, y be as shown.

Reflect ΔACD along the line AC to ΔACE.

$\because OD < OB$ $\therefore OB > OE \Rightarrow E$ lies <u>inside</u> ΔABC.



By the theorem at the beginning, $b + c > a + d$

$b - a > d - c$

$b + c > a + d$ ....(1)

Apply Pythagoras' Theorem on ΔAEO, ΔCEO, ΔABO, ΔCBO

$x^2 - y^2 = a^2 - d^2 = b^2 - c^2$

$(a + d)(a - d) = (b + c)(b - c)$

$\frac{b+c}{a+d} = \frac{a-d}{b-c}$ ....(2)

In (1) $\frac{b+c}{a+d} > 1 \Rightarrow$ (2) $\frac{a-d}{b-c} > 1$

$a - d > b - c$

$a + c > b + d$

$AD + BC > AB + CD$. The result is proved.