95-96 Individual	**1**	0	**2**	$x^6 - x^3 + 1$	**3**	24	**4**	11	**5**	29
	6	(–2, 2)	**7**	216	**8**	$\frac{229}{99}=2\frac{31}{99}$	**9**	40	**10**	$-(x-y)(y-z)(z-x)$

95-96 Group	**1**	6	**2**	$3\sqrt{3}$ cm	**3**	91	**4**	$\frac{1}{5}$	**5**	$\frac{49}{100}$
	6	432	**7**	400	**8**	$\frac{365}{38}$	**9**	6	**10**	5

Individual Events

I1 Find x if $4^{x-3} = 8^{x-2}$.

$2^{2(x-3)} = 2^{3(x-2)}$

$2(x-3) = 3(x-2)$

$x = 0$

I2 If $f\left(\frac{1+x}{x}\right)=\frac{x^2+1}{x^2}+\frac{1}{x}$, find $f(x^3)$.

Let $y=\frac{1+x}{x}$, then $xy = 1 + x$

$\Rightarrow x=\frac{1}{y-1}$

$f(y)=1+\frac{1}{x}+\frac{1}{x^2}$

$=1+y-1+(y-1)^2$

$= y + y^2 - 2y + 1$

$= y^2 - y + 1$

$f(y^3) = y^6 - y^3 + 1$

$\Rightarrow f(x^3) = x^6 - x^3 + 1$

I3 By considering $n! = n\times(n-1)\times(n-2)\times \dots \times 3\times2\times1$, find the number of trailing zeros of 100!.

Reference: 1990 HG6, 1994 FG7.1, 2004 FG1.1, 2011 HG7, 2012 FI1.4, 2012 FG1.3

When each factor of 5 is multiplied by 2, a trailing zero will appear in the product.

The number of factors of 2 is clearly more than the number of factors of 5 in 100!

It is sufficient to find the number of factors of 5.

5, 10, 15, … , 100; altogether 20 numbers, have at least one factor of 5.

25, 50, 75, 100; altogether 4 numbers, have two factors of 5.

$\therefore$ Total number of factors of 5 is 20 + 4 = 24

There are 24 trailing zeros of 100!

I4 What is the largest integral value n that satisfies the inequality $n^{200} < 5^{300}$?

Reference: 1999 FG5.3, 2008 FI4.3, 2018FG2.4

$n^2 < 5^3 = 125$

$n < \sqrt{125} < \sqrt{144} = 12$

The largest integral $n = 11$

I5 A set of 110 stamps of the denominations of \$0.1, \$3, \$5 worth \$100 in total.
Find the number of \$3 stamps in the set of stamps.

Suppose there are x \$0.1 stamps, y \$3 stamps, $(110 - x - y)$ \$5 stamps; where x and y are integers.

$0.1x + 3y + 5(110 - x - y) = 100$

$x + 30y + 50(110 - x - y) = 1000$

$49x + 20y = 4500$ …… (*)

$49 = 20\times2 + 9$ …… (1) $\Rightarrow 9 = 49 - 20\times2$ …… (1)'

$20 = 9\times2 + 2$ …… (2) $\Rightarrow 2 = 20 - 9\times2$ …… (2)'

$9 = 2\times4 + 1$ …… (3) $\Rightarrow 1 = 9 - 2\times4$ …… (3)'

Sub. (1)' into (2)': $2 = 20 - (49 - 20\times2)\times2$ …… (4)'

$2 = 20\times5 - 49\times2$ …… (5)'

Sub. (1)' and (5)' into (3)': $1 = (49 - 20\times2) - (20\times5 - 49\times2)\times4$

$1 = 49\times9 - 20\times22$

Multiply by 4500: $49\times40500 + 20\times(-99000) = 4500$

$\therefore$ $x = 40500$, $y = -99000$ is a particular solution to (*)

The general solution is $x = 40500 - 20m$, $y = -99000 + 49m$, where m is an integer.

$x > 0$ and $y > 0$ and $110 - x - y > 0$

$40500 - 20m > 0$ and $-99000 + 49m > 0$ and $110 - (40500 - 20m) - (-99000 + 49m) > 0$

$2025 > m > 2020\frac{20}{49}$ and $58610 > 29m$

$2025 > m > 2020\frac{20}{49}$ and $2021\frac{1}{29} > m$

$\therefore$ $m = 2021$; $y = -99000 + 49\times2021 = 29$

Method 2 (*) can be written as $49x = 20(225 - y)$

$\because$ 49 and 20 are relatively prime

$\therefore$ x is divisible by 20 and $225 - y$ is divisible by 49

x	$225 - y$	y	$110 - x - y$
20	49	176	no solution
40	98	127	no solution
60	147	78	no solution
80	196	29	1

$\therefore$ $y = 29$

I6 For any value of m, a straight line $y = mx + 2m + 2$ passes through a fixed point P. Find the coordinates of P. **Reference: 1990 HI5, 1991 HI6**

Put $m = 0$, $y = 2$

Put $m = 1$ and $y = 2 \Rightarrow x = -2$

The coordinates of P is $(-2, 2)$.

I7 How many 3-digit numbers can be made from the figures 4, 5, 6, 7, 8, 9 when repetitions are allowed?

The number of 3-digit numbers $= 6^3 = 216$

I8 Express $2.\dot{3}\dot{1}$ as a fraction.

Let $a = 2.\dot{3}\dot{1}$

$100a = 231.\dot{3}\dot{1}$

$100a - a = 229$

$a = \frac{229}{99} = 2\frac{31}{99}$

I9 If x and y are positive integers and $x - y = 5$, find the least value of $x^2 - y^2 + 5$.

$x - y = 5 \Rightarrow x = 5 + y$ …… (1)

Sub. (1) into $x^2 - y^2 + 5 = (5 + y)^2 - y^2 + 5$

$= 30 + 10y$

$y \geq 1 \Rightarrow 30 + 10y \geq 40$

$\therefore$ The least value of $x^2 - y^2 + 5$ is 40.

I10 Factorize $x^2(y - z) + y^2(z - x) + z^2(x - y)$.

Let $f(x, y, z) = x^2(y - z) + y^2(z - x) + z^2(x - y)$

$f(x, y, z) = f(y, z, x) = f(z, x, y)$

$f(x, y, z)$ is a cyclic expression of order 3.

$f(x, x, z) = x^2(x - z) + x^2(z - x) + z^2(x - x) = 0$

$\therefore$ $(x - y)$ is a factor

By symmetry, $(y - z)$ and $(z - x)$ are factors

$f(x, y, z) = k(x - y)(y - z)(z - x)$

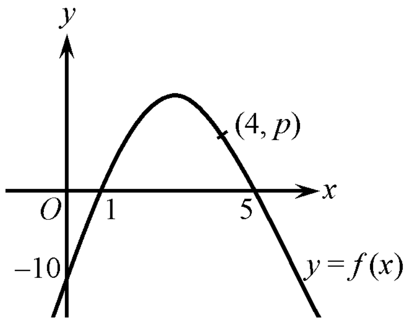
Compare the coefficients of x^2y: $-k = 1$

$\Rightarrow k = -1$

$\therefore$ $x^2(y - z) + y^2(z - x) + z^2(x - y) = -(x - y)(y - z)(z - x)$

Group Events

G1 In the figure, the quadratic curve $y = f(x)$ cuts the x-axis at the two points (1, 0) and (5, 0) and the y-axis at the point (0, –10). Find the value of p.



$f(x) = k(x - 1)(x - 5)$

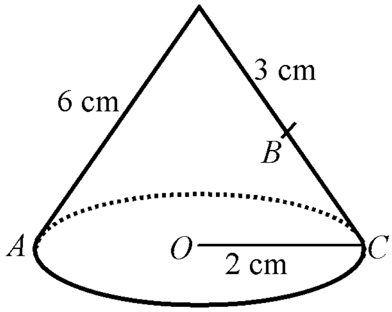
It passes through (0, –10) $\Rightarrow f(0) = -10$

$-10 = k(0 - 1)(0 - 5) \Rightarrow k = -2$

$f(x) = -2(x - 1)(x - 5)$

$f(4) = p = -2(4 - 1)(4 - 5) = 6$

G2 In the figure, O is the centre of the base circle of a cone and the points A, B, C and O lie in the same plane. An ant walks from A to B on the surface of the cone. Find the length of the shortest path from A to B.



Let the vertex of the cone be V.

If we cut the curved surface of the cone along OA, a sector $VACA'$ is formed with C as the mid-point of $\overset{\frown}{AA'}$.

Let $\angle AVB = \theta$ (in degree), then $\angle A'VB = \theta$

$\overset{\frown}{ACA'}$ = circumference of the base = 4π cm

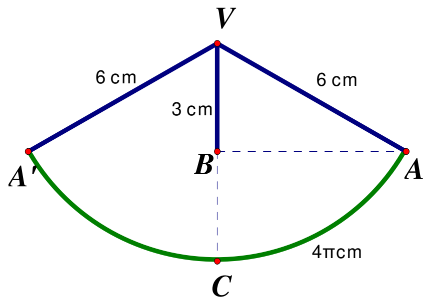
$$2\pi \cdot 6 \cdot \frac{2\theta}{360} = 4\pi$$

$\theta = 60°$

The shortest distance = the line segment AB

By cosine rule, $AB^2 = [3^2 + 6^2 - 2(3)(6)\cos 60°]$ cm^2

$AB = 3\sqrt{3}$ cm



G3 When a sum of \$7020, in the form of ten-dollar notes, is divided equally among x persons, \$650 remains. When this sum \$650 is changed to five-dollar coins and then divided equally among the x persons, \$195 remains. Find x.

Suppose each person gets a pieces of ten-dollar notes and b five-dollar coins.

$7020 = 10ax + 650$ …… (1), a, x are positive integers and $x > 65$

$650 = 5bx + 195$ …… (2), b, x are positive integers and $x > 195 \div 5 = 39$

From (1): $637 = ax$ …… (3)

From (2): $91 = bx$ …… (4)

From (3): $637 = 7\times 91 = 7\times 7\times 13$

$\therefore$ The only positive factor of 91 is 91 which is greater than 65.

$\therefore x = 91$

G4 In a shooting competition, according to statistics, A misses one in every 5 shoots, B misses one in every 4 shoots and C misses one in every 3 shoots. Find the probability of obtaining successful shoots by A, B but not C.

Probability $= \frac{4}{5}\times\frac{3}{4}\times\frac{1}{3} = \frac{1}{5}$

G5 Given that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, find the value of $\frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{99\times 100}$.

$$\frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{99\times 100} = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right) = \frac{1}{2} - \frac{1}{100} = \frac{49}{100}$$

G6 If 3 is added to a 3-digit number A, the sum of the digits of the new number is $\frac{1}{3}$ of the value of the sum of digits of the original number A. Find the sum of all such possible numbers A.

Let $A = 100a + 10b + c$

New number $= 100a + 10b + c + 3$

If $c \le 6$, $a + b + c + 3 = \frac{1}{3}(a + b + c)$

$\Rightarrow 2(a + b + c) + 9 = 0$, no solution

If new number $= 100a + 10(b + 1) + c + 3 - 10 = 100a + 10(b + 1) + c - 7$, $c \ge 7$

$a + b + 1 + c - 7 = \frac{1}{3}(a + b + c)$

$\Rightarrow 2(a + b + c) - 18 = 0$

$a + b + c = 9$

a	b	c	A
1	0	8	108
1	1	7	117
2	0	7	207

If $a \le 8$, $b = 9$ and $c \ge 7$, new number $= 100(a + 1) + c - 7$

$a + 1 + c - 7 = \frac{1}{3}(a + b + c)$

$\Rightarrow 2(a + c) - 18 = 9$

$\Rightarrow 2(a + c) = 27$, no solution

If $a = 9$, $b = 9$ and $c \ge 7$, new number $= 1000 + c - 7$

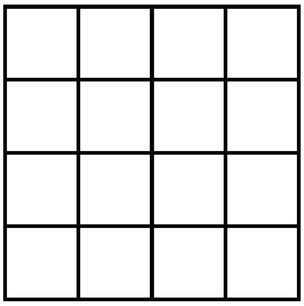
$1 + c - 7 = \frac{1}{3}(a + b + c)$

$\Rightarrow 3c - 18 = 18 + c$

$\Rightarrow c = 18$, rejected

Sum of all possible $A = 108 + 117 + 207 = 432$

G7 In the figure, the side of each smaller square is 1 unit long. Find the sum of the area of all possible rectangles (squares included) that can be formed in the figure.



Number of rectangles with sides $1 \times 1 = 16$

Number of rectangles with sides $1 \times 2 = 2 \times 3 \times 4 = 24$

Number of rectangles with sides $1 \times 3 = 2 \times 2 \times 4 = 16$

Number of rectangles with sides $1 \times 4 = 2 \times 4 = 8$

Number of rectangles with sides $2 \times 2 = 3 \times 3 = 9$

Number of rectangles with sides $2 \times 3 = 2 \times 2 \times 3 = 12$

Number of rectangles with sides $2 \times 4 = 2 \times 3 = 6$

Number of rectangles with sides $3 \times 3 = 2 \times 2 = 4$

Number of rectangles with sides $3 \times 4 = 2 \times 2 = 4$

Number of rectangles with sides $4 \times 4 = 1$

Sum of areas $= 16 + 24{\times}2 + 16{\times}3 + 8{\times}4 + 9{\times}4 + 12{\times}6 + 6{\times}8 + 4{\times}9 + 4{\times}12 + 16 = 400$

G8 If prime numbers a, b are the roots of the quadratic equation $x^2 - 21x + t = 0$, find the value of $\left(\frac{b}{a}+\frac{a}{b}\right)$.

Reference: 1996FG7.1, 2001 FG4.4, 2005 FG1.2, 2012 HI6

$a + b = 21$; $ab = t$

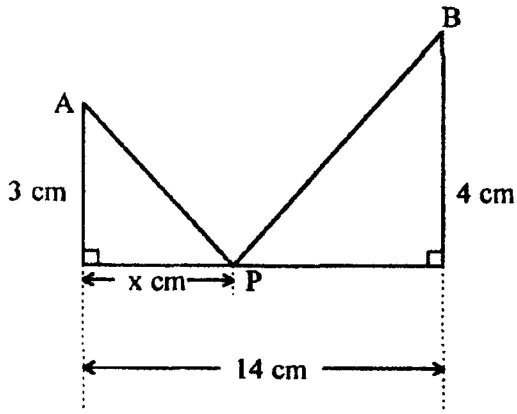
$\because$ a, b are prime numbers and 21 is odd

$\therefore$ $a = 2$, $b = 19$

$$\left(\frac{b}{a}+\frac{a}{b}\right)=\frac{19}{2}+\frac{2}{19}$$

$$=\frac{365}{38}$$

G9 Find the value of x such that the length of the path APB in the figure is the smallest.



Reference: 1983 FG8.1, 1991 HG9, 1993 HI1

Let the straight line segment DPE be as shown in the figure.

Reflect A along DE to A'.

Then $\Delta ADP \cong \Delta A'DP$

$APB = A'P + PB$

It is the shortest when A', P, B are collinear.

In this case, $\angle A'PD = \angle BPE$ (vert. opp. $\angle$s)

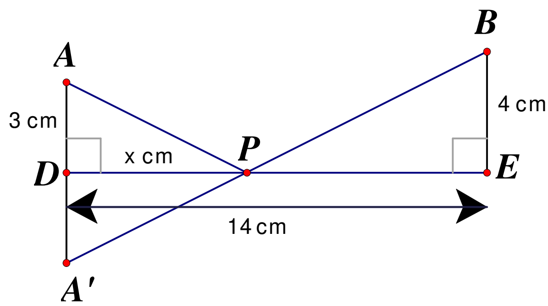
$\Delta A'PD \sim \Delta BPE$ (equiangular)

$DP : A'D = PE : BE$ (ratio of sides, ~ Δ's)

$$\frac{x}{3}=\frac{14-x}{4}$$

$4x = 42 - 3x$

$x = 6$



G10 Find the units digit of the sum $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 123456789^2$.

Reference 2012 HI1

Sum of units digits from 1^2 to 10^2

$\equiv 0 + 1 + 4 + 9 + 6 + 5 + 6 + 9 + 4 + 1 \pmod{10}$

$\equiv 5 \pmod{10}$

$\therefore$ Required units digit

= units digit of 12345679×5

$\equiv 5 \pmod{10}$