Exclusive Events verse Independent Events

8.1 The properties of "mutually exclusive events" and "independent events" are **not related**.
i.e. if A, B are two events, we can have:

mutually exclusive	independent
✓	✓
✓	✗
✗	✓
✗	✗

Read the following examples

E.g.1 RE = throw a die, S = {1, 2, 3, 4, 5, 6}

$A = \{1, 2\}$, $B = \{7\}$

$A \cap B = \phi$, $\therefore$ A and B are mutually exclusive events

$P(A \cap B) = P(\phi) = 0$

$P(A) = \frac{1}{3}$

$P(B) = 0$

$\therefore P(A \cap B) = P(A) \times P(B)$, A and B are independent events.

E.g.2 $A = \{1, 2\}$, $C = \{3\}$

then $A \cap C = \phi$, i.e. A, C are mutually exclusive events

$P(A \cap C) = 0$

$P(A) \times P(C) = \frac{1}{3} \times \frac{1}{6} \neq 0$,

$\therefore$ A and C are not independent events

E.g.3 $A = \{2, 3, 5\}$, $B = \{3, 6\}$

$A \cap B = \{3\} \neq \phi$, $\therefore$ A, B are not mutually exclusive events

$P(A \cap B) = \frac{1}{6}$

$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = P(A \cap B)$

$\therefore$ A and B are independent events

E.g.4 $A = \{2, 4, 6\}$, $B = \{2, 3, 5\}$

$A \cap B = \{2\} \neq \phi$, $\therefore$ A, B are not mutually exclusive events

$P(A \cap B) = \frac{1}{6}$

$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq P(A \cap B)$

$\therefore$ A and B are not independent events

8.2 However, under some circumstance, we know **some properties**:

(a) If A, B are both mutually exclusive and both independent, then $A = \phi$ or $B = \phi$

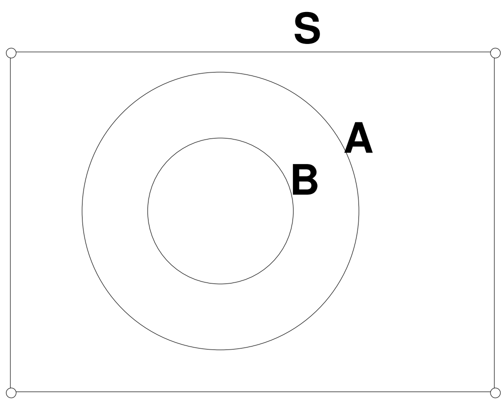
Proof: $P(A \cap B) = P(\phi)$ ($\because$ A, B are mutually exclusive)

$P(A)P(B) = 0$ ($\because$ A, B are independent)

$\therefore P(A) = 0$ or $P(B) = 0$

$A = \phi$ or $B = \phi$

(b) If A, B are non-empty set such that $B \subseteq A \neq S$, then A, B are neither mutually exclusive nor independent.



Proof: $B \subseteq A$

$B \cap A = B \neq \phi$

$\therefore$ A, B are not mutually exclusive.

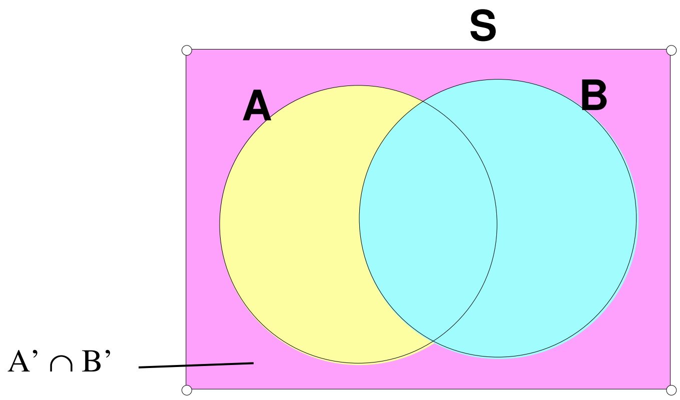
$P(A \cap B) = P(B)$

but $P(A)P(B) \neq P(B)$ (as $P(A) \neq 1$)

$\therefore$ A , B are not independent.

(c) If A, B are independent, then so is A', B' .

See the Venn diagram:



It is clear that $(A \cup B)' = A' \cap B'$

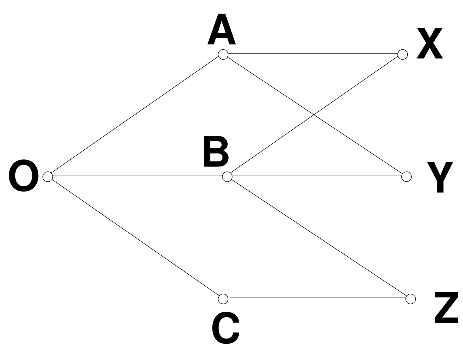
$$\begin{aligned} P(A' \cap B') &= P((A \cup B)') \\ &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= [1 - P(A)][1 - P(B)] \\ &= P(A')\,P(B') \end{aligned}$$

$\therefore$ A', B' are independent.

8.3 Non-equiprobable space

Consider the following examples:

E.g.5 A man starts from O to walk in the following diagram:



At each joint (路口), he has an equal chance of choosing each path, find the probability that he will go to X.

Method 1

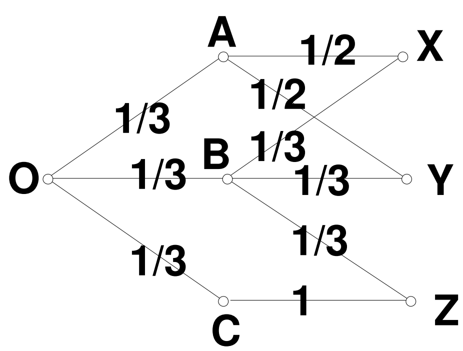
S = {AX, AY, BX, BY, BZ, CZ}

$$P(X) = \frac{2}{6} = \frac{1}{3}$$

Method 2

$$P(X) = \frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{3}$$

$$= \frac{5}{18}$$



Method 2 is correct and **method 1** is wrong because the outcomes in S are not equiprobable.

E.g.6 There are 40 candidates waiting for interview in two rooms A and B. Initially, there are 23 candidates in room A and 17 candidates in room B. 4 candidates in Room A are of ages 40 or more, whereas only 1 candidate in Room B is of age 40 or more. Since room A is too crowded, one of the candidates in room A is chosen at random and moved to room B. Then, a candidate is randomly selected from room B as the first interviewee. It is known that only one candidate passes the interview and the candidate is of age 40 or above.

(a) Find the probability that the first interviewee passes the interview.

(b) Find the probability that the first interviewee passes the interview and he is originally from room A .

(c) Given that the first interviewee passes the interview, find the probability that he/she is originally from room A .

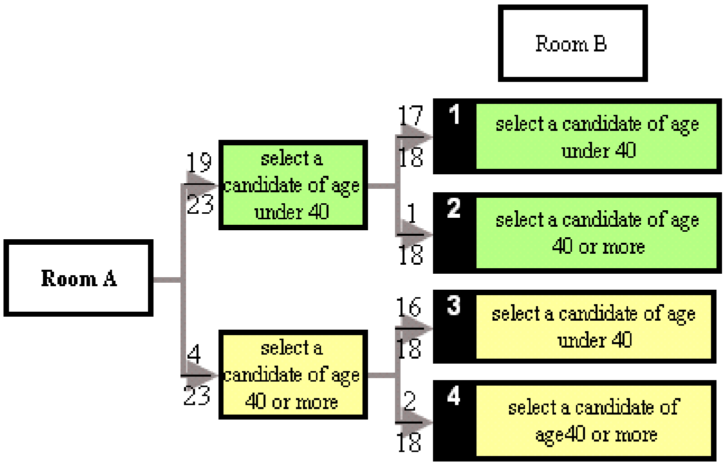
Age	Room A	Room B
less than 40	19	16
40 or above	4	1

(a) From the tree diagram on the right,

required probability

$=\frac{19}{23}\times\frac{1}{18}+\frac{4}{23}\times\frac{2}{18}$

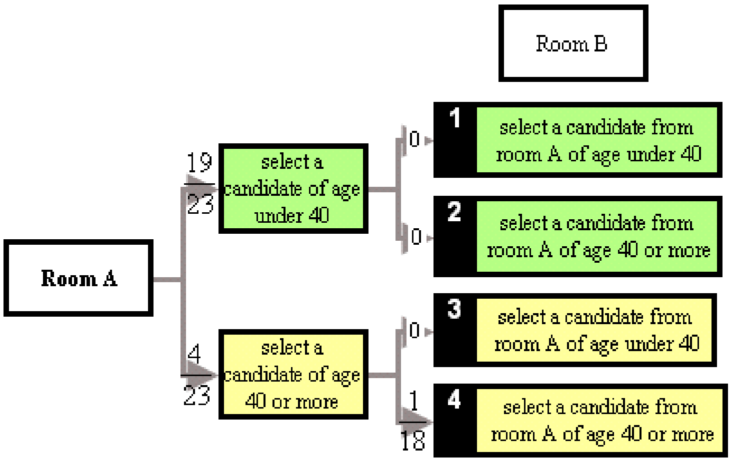
$=\frac{3}{46}$



(b) From the tree diagram on the right, required probability

$=\frac{4}{23}\times\frac{1}{18}$

$=\frac{2}{207}$



(c) P(the candidate is from A | he is the first one passed the interview of age 40 or more)

$=\dfrac{\text{P(the first interviewee passes the interview and he is originally from room A)}}{\text{P(the first interviewee passes the interview)}}$

$=\dfrac{\text{answer in (b)}}{\text{answer in (a)}}$

$=\frac{2}{207}\div\frac{3}{46}$

$=\frac{4}{27}$

Remark: Although there are altogether 5 person whose ages are 40 or above, the answer in (c) is not $\frac{4}{5}$ because the outcomes are not equal probable.

8.4 Let A, B be mutually exclusive events, B, C be mutually exclusive events. Are A, C mutually exclusive?

The answer is 'NO' as shown by the following example:

E.g.7 RE = throw a die, S = {1, 2, 3, 4, 5, 6}

A = {1, 2}, B = {3}, C = {2, 4}

$A \cap B = \phi$, $\therefore$ A, B are mutually exclusive events.

$B \cap C = \phi$, $\therefore$ B, C are mutually exclusive events.

but $A \cap C = \{2\} \neq \phi$, $\therefore$ A, C are not mutually exclusive events.

A, B, C are mutually exclusive events if $A \cap B = \phi$, $B \cap C = \phi$, and $C \cap A = \phi$.

In this case, $P(A \cup B \cup C) = P(A) + P(B) + P(C)$.

Can you define the condition that A, B, C, D are mutually exclusive?

8.5 Independent Events

If A, B and C are mutually independent events, then the following 4 conditions must be satisfied:

(i) $P(A \cap B) = P(A)\,P(B)$

(ii) $P(A \cap C) = P(A)\,P(C)$

(iii) $P(B \cap C) = P(B)\,P(C)$

(iv) $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

We are going to prove that if A and B are independent, A and C are also independent, then B and C may not be independent.

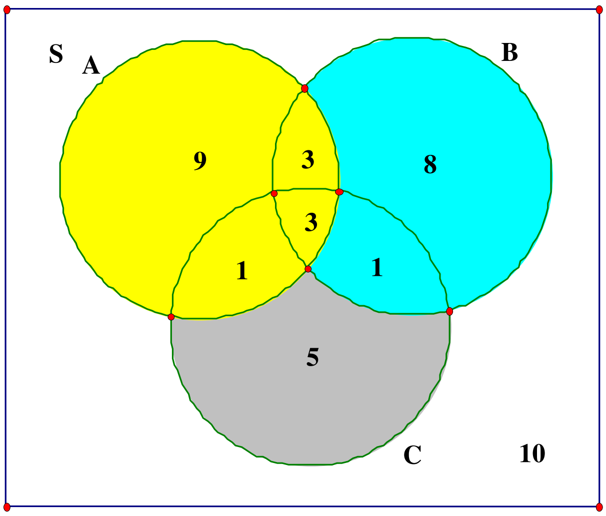
E.g.8 Suppose in an equiprobable space which contains 40 equiprobable outcomes. $n(S) = 40$

A, B, C are 3 events such that

$n(A) = 16$, $n(B) = 15$, $n(C) = 10$, $n(A \cap B) = 6$,

$n(B \cap C) = 4$, $n(A \cap C) = 4$, $n(A \cap B \cap C) = 3$;

as shown in the diagram:



$$P(A) = \frac{16}{40} = \frac{2}{5};\ P(B) = \frac{15}{40} = \frac{3}{8}$$

$$P(C) = \frac{10}{40} = \frac{1}{4}$$

$$P(A \cap B) = \frac{6}{40} = \frac{3}{20}$$

$$\text{while } P(A) \times P(B) = \frac{2}{5} \times \frac{3}{8} = \frac{3}{20}$$

$\therefore\ P(A \cap B) = P(A)\,P(B)$

$\therefore A$ and B are independent.

$$\text{Similarly, } P(A \cap C) = \frac{4}{40} = \frac{1}{10}$$

$$\text{while } P(A) \times P(C) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$$

$\therefore A$ and C are independent.

$$\text{But } P(B \cap C) = \frac{4}{40} = \frac{1}{10}, \text{ while } P(B) \times P(C) = \frac{3}{8} \times \frac{1}{4} = \frac{3}{32}$$

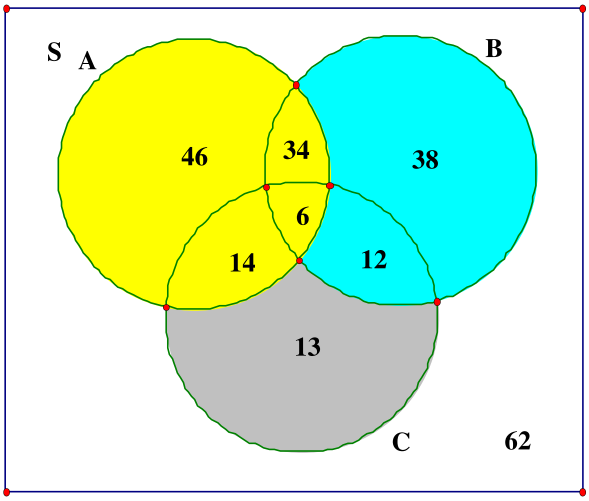
$\therefore\ P(B \cap C) \neq P(B)\,P(C)$

$\therefore B$ and C are not independent.

E.g.9 In the diagram, let S be an equiprobable space containing 225 elements. A, B, C are events such that $n(A) = 100$, $n(B) = 90$, $n(C) = 45$,

$n(A \cap B) = 40$, $n(B \cap C) = 18$, $n(A \cap C) = 20$,

$n(A \cap B \cap C) = 6$.



$P(A) = \frac{100}{225} = \frac{4}{9}$; $P(B) = \frac{90}{225} = \frac{2}{5}$

$P(C) = \frac{45}{225} = \frac{1}{5}$;

$P(A \cap B) = \frac{40}{225} = \frac{8}{45} = \frac{4}{9} \times \frac{2}{5} = P(A)\ P(B)$

$P(B \cap C) = \frac{18}{225} = \frac{2}{25} = \frac{2}{5} \times \frac{1}{5} = P(B)\ P(C)$

$P(C \cap A) = \frac{20}{225} = \frac{4}{45} = \frac{1}{5} \times \frac{4}{9} = P(C)\ P(A)$

yet $P(A \cap B \cap C) = \frac{6}{225} = \frac{2}{75}$

$P(A)P(B)\ P(C) = \frac{4}{9} \times \frac{2}{5} \times \frac{1}{5} = \frac{8}{225} \neq P(A \cap B \cap C)$

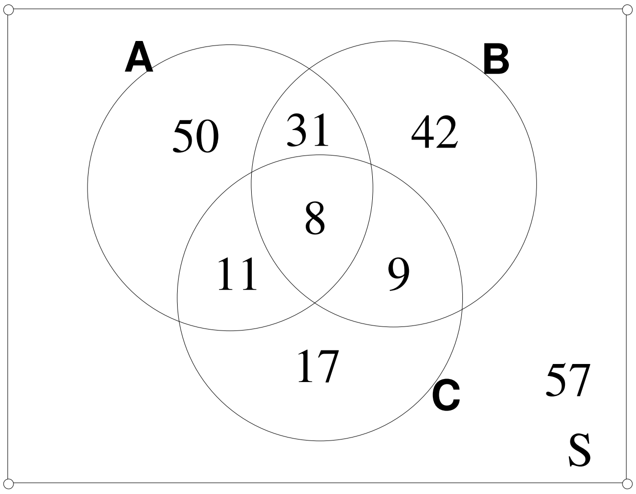
$\therefore$ A, B and C are not independent.

E.g.10 In the figure, $n(S) = 225$

$n(A) = 100$, $n(B) = 90$, $n(C) = 45$

$n(A \cap B) = 39$, $n(B \cap C) = 17$, $n(A \cap C) = 19$

$n(A \cap B \cap C) = 8$



$P(A) \times P(B) = \frac{100}{225} \times \frac{90}{225} = \frac{8}{45} \neq \frac{39}{225} = P(A \cap B)$

$P(B) \times P(C) = \frac{90}{225} \times \frac{45}{225} = \frac{2}{25} \neq \frac{17}{225} = P(B \cap C)$

$P(A) \times P(C) = \frac{100}{225} \times \frac{45}{225} = \frac{4}{45} \neq \frac{19}{225} = P(A \cap C)$

Yet $P(A \cap B \cap C) = \frac{8}{225}$

$P(A) \times P(B) \times P(C) = \frac{100}{225} \times \frac{90}{225} \times \frac{45}{225} = \frac{8}{225}$

$\therefore$ Even though $P(A \cap B \cap C) = P(A)\ P(B)\ P(C)$, it is not sufficient to guarantee that A, B and C are independent events.

Exercise 1 However, in the following case. Events A, B, C are independent. Check it!