Answers: (1999-00 HKMO Final Events)

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Individual Events

SI	***P***	6	**I1**	***P***	25	**I2**	***P***	16	**I3**	***P***	1	**I4**	***P***	2	**I5**	***P***	2
	Q	7		***Q***	8		***Q***	81		***Q***	2		***Q***	12		***Q***	1
	R	2		***R***	72		***R***	1		***R***	3996		***R***	12		***R***	1
	S	9902		***S***	6		***S***	333332		***S***	666		***S***	2		***S***	0

Group Events

SG	***a***	1	**G1**	***a***	243	**G2**	***a***	9025	**G3**	***a***	3994001	**G4**	***a***	504	**G5**	***a***	729000
	b	15		***b***	25		***b***	9		***b***	5		***b***	3		***b***	12
	c	80		***c***	4		***c***	6		***c***	3		***c***	60		***c***	26
	d	1		***d***	3		***d***	–40		***d***	38		***d***	48		***d***	3

Sample Individual Event (1999 Individual Event 3)

SI.1 For all integers m and n, $m \otimes n$ is defined as $m \otimes n = m^n + n^m$. If $2 \otimes P = 100$, find the value of P.

$2^P + P^2 = 100$

$64 + 36 = 2^6 + 6^2 = 100$, $P = 6$

SI.2 If $\sqrt[3]{13Q+6P+1} - \sqrt[3]{13Q-6P-1} = \sqrt[3]{2}$, where $Q > 0$, find the value of Q.

$\left(\sqrt[3]{13Q+37} - \sqrt[3]{13Q-37}\right)^3 = 2$

$13Q + 37 - 3\sqrt[3]{(13Q+37)^2}\sqrt[3]{13Q-37} + 3\sqrt[3]{(13Q-37)^2}\sqrt[3]{13Q+37} - (13Q - 37) = 2$

$24 = \sqrt[3]{(13Q)^2 - 37^2}\sqrt[3]{13Q+37} - \sqrt[3]{(13Q)^2 - 37^2}\sqrt[3]{13Q-37}$

$24 = \sqrt[3]{(13Q)^2 - 37^2}\ \sqrt[3]{2}$; $\quad (\because \sqrt[3]{13Q+37} - \sqrt[3]{13Q-37} = \sqrt[3]{2})$

$13824 = [(13Q)^2 - 1369] \times 2$

$6912 + 1369 = 169\, Q^2$

$Q^2 = 49 \Rightarrow Q = 7$

Method 2 $\sqrt[3]{13b+37} - \sqrt[3]{13b-37} = \sqrt[3]{2}$,

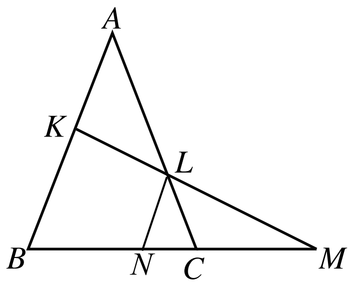
We look for the difference of multiples of $\sqrt[3]{2}$

$\sqrt[3]{8\times 2} - \sqrt[3]{2} = \sqrt[3]{2} \Rightarrow 13b + 37 = 16$, $13b - 37 = 2$, no solution

$\sqrt[3]{27\times 2} - \sqrt[3]{8\times 2} = \sqrt[3]{2} \Rightarrow 13b + 37 = 54$, $13b - 37 = 16$, no solution

$\sqrt[3]{64\times 2} - \sqrt[3]{27\times 2} = \sqrt[3]{2} \Rightarrow 13b + 37 = 128$, $13b - 37 = 54 \Rightarrow b = 7$

SI.3 In figure 1, $AB = AC$ and $KL = LM$. If $LC = Q - 6$ cm and $KB = R$ cm, find the value of R.



Draw LN // AB on BM.

$BN = NM$ intercept theorem

$\angle LNC = \angle ABC = \angle LCN$ (corr. $\angle$s, AB // LN, base $\angle$s, isos. Δ)

$LN = LC = Q - 6$ cm $= 1$ cm (sides opp. eq. $\angle$s)

R cm $= KB = 2\, LN = 2$ cm (mid point theorem)

SI.4 The sequence $\{a_n\}$ is defined as $a_1 = R$, $a_{n+1} = a_n + 2n$ $(n \geq 1)$. If $a_{100} = S$, find the value of S.

$a_1 = 2$, $a_2 = 2 + 2$, $a_3 = 2 + 2 + 4$, …

$a_{100} = 2 + 2 + 4 + \ldots + 198$

$= 2 + \frac{1}{2}(2 + 198) \cdot 99 = 9902 = S$

Individual Event 1

I1.1 Let $[x]$ represents the integral part of the decimal number x.

Given that $[3.126] + [3.126+\frac{1}{8}]+[3.126+\frac{2}{8}]+\ldots+[3.126+\frac{7}{8}] = P$, find the value of P.

$P = [3.126] + [3.126+\frac{1}{8}]+[3.126+\frac{2}{8}]+\ldots+[3.126+\frac{7}{8}]$

$= 3 + 3 + 3 + 3 + 3 + 3 + 3 + 4 = 25$

I1.2 Let $a + b + c = 0$. Given that $\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab} = P - 3Q$, find the value of Q.

$a = -b - c$ $\quad \frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}$

$=\frac{(b+c)^2}{2b^2+5bc+2c^2}+\frac{b^2}{2b^2-bc-c^2}+\frac{c^2}{2c^2-bc-b^2}$

$=\frac{a^2}{(2b+c)(b+2c)}+\frac{b^2}{(2b+c)(b-c)}+\frac{c^2}{(b+2c)(c-b)}$

$=\frac{(b+c)^2(b-c)+b^2(b+2c)-c^2(2b+c)}{(2b+c)(b+2c)(b-c)}$

$=\frac{(b+c)^2(b-c)+b^3-c^3+2bc(b-c)}{(2b+c)(b+2c)(b-c)}$

$=\frac{(b-c)(b^2+2bc+c^2+b^2+bc+c^2+2bc)}{(2b+c)(b+2c)(b-c)}$

$=\frac{(2b^2+5bc+2c^2)}{(2b+c)(b+2c)} = 1 = 25 - 3Q \Rightarrow Q = 8$

Method 2

$\because \frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab} = 25-3Q$

$\therefore$ The above is an identity which holds for all values of a, b and c, provided that $a+b+c = 0$

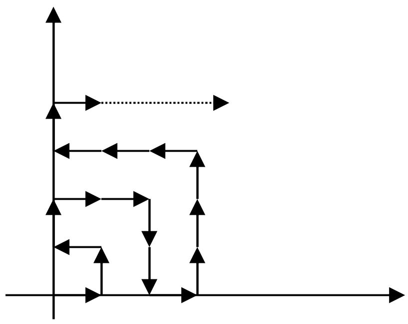
Let $a = 0$, $b = 1$, $c = -1$, then

$0+\frac{1}{2}+\frac{1}{2} = 25 - 3Q$.

$Q = 8$

I1.3 In the first quadrant of the rectangular co-ordinate plane, all integral points are numbered as follows,

point (0, 0) is numbered as 1,
point (1, 0) is numbered as 2,
point (1, 1) is numbered as 3,
point (0, 1) is numbered as 4,
point (0, 2) is numbered as 5,
point (1, 2) is numbered as 6,
point (2, 2) is numbered as 7,
point (2, 1) is numbered as 8,
……………………………



Given that point $(Q-1, Q)$ is numbered as R, find the value of R.

point (0, 1) is numbered as $4 = 2^2$
point (2, 0) is numbered as $9 = 3^2$
point (0, 3) is numbered as $16 = 4^2$
point (4, 0) is numbered as $25 = 5^2$
……………………………
point (0, 7) is numbered as $64 = 8^2$
point (0, 8) is numbered as 65, point (1, 8) is numbered as 66, point (2, 8) is numbered as 67
……………………………
$(Q-1, Q) = (7, 8)$ is numbered as 72

I1.4 When $x + y = 4$, the minimum value of $3x^2 + y^2$ is $\frac{R}{S}$, find the value of S.

$3x^2 + y^2 = 3x^2 + (4 - x)^2 = 4x^2 - 8x + 16 = 4(x - 1)^2 + 12$, min $= 12 = \frac{72}{S}$; $S = 6$

Individual Event 2

I2.1 If $\log_2(\log_4 P) = \log_4(\log_2 P)$ and $P \neq 1$, find the value of P.

$$\frac{\log(\log_4 P)}{\log 2} = \frac{\log(\log_2 P)}{\log 4}$$

$$\frac{\log(\log_4 P)}{\log 2} = \frac{\log(\log_2 P)}{2\log 2}$$

$2\log(\log_4 P) = \log(\log_2 P)$

$\Rightarrow \log(\log_4 P)^2 = \log(\log_2 P)$

$(\log_4 P)^2 = \log_2 P$

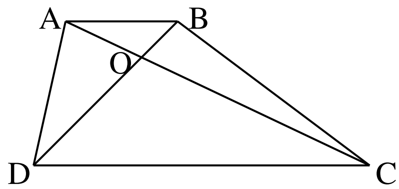
$$\left(\frac{\log P}{\log 4}\right)^2 = \frac{\log P}{\log 2}$$

$P \neq 1$, $\log P \neq 0 \Rightarrow \dfrac{\log P}{(2\log 2)^2} = \dfrac{1}{\log 2}$

$\log P = 4\log 2 = \log 16$

$P = 16$

I2.2 In the trapezium $ABCD$, AB // DC. AC and BD intersect at O. The areas of triangles AOB and COD are P and 25 respectively. Given that the area of the trapezium is Q, find the value of Q.



Reference 1993 HI2, 1997 HG3, 2002 FI1.3, 2004 HG7, 2010HG4, 2013 HG2

$\Delta AOB \sim \Delta COD$ (equiangular)

$$\frac{\text{area of }\Delta\text{AOB}}{\text{area of }\Delta\text{COD}} = \left(\frac{OA}{OC}\right)^2;\ \frac{16}{25} = \left(\frac{OA}{OC}\right)^2$$

$OA : OC = 4 : 5$

$\dfrac{\text{area of }\Delta\text{AOB}}{\text{area of }\Delta\text{BOC}} = \dfrac{4}{5}$ (the two triangles have the same height, but different bases.)

Area of $\Delta BOC = 16\times\dfrac{5}{4} = 20$

Similarly, area of $\Delta AOD = 20$

Q = the area of the trapezium = 16 + 25 + 20 + 20 = 81

I2.3 When 1999^Q is divided by 7, the remainder is R. Find the value of R.

$1999^{81} = (7\times285+4)^{81}$

$= 7m + 4^{81}$

$= 7m + (4^3)^{27}$

$= 7m + (7\times9+1)^{27}$

$= 7m + 7n + 1$, where m and n are integers

$R = 1$

I2.4 If $111111111111 - 222222 = (R + S)^2$ and $S > 0$, find the value of S.

Reference: 1995 FG7.4

$111111111111 - 222222 = (1 + S)^2$

$111111(1000001 - 2) = (1 + S)^2$

$111111\times999999 = (1 + S)^2$

$3^2 \times 111111^2 = (1 + S)^2$

$1 + S = 333333$

$S = 333332$

Individual Event 3

I3.1 Given that the units digit of $1+2+3+\cdots+1997+1998+1999+1998+1997+\cdots+3+2+1$ is P, find the value of P.

$1+2+3+\cdots+1997+1998+1999+1998+1997+\cdots+3+2+1$

$= 2(1 + 2 + \cdots + 1998) + 1999$

$= (1 + 1998)\times 1998 + 1999$

P = units digit = 1

I3.2 Given that $x+\dfrac{1}{x}=P$. If $x^6+\dfrac{1}{x^6}=Q$, find the value of Q.

$x+\dfrac{1}{x}=1$

$\left(x+\dfrac{1}{x}\right)^2=1$

$\Rightarrow x^2+\dfrac{1}{x^2}=-1$

$\left(x^2+\dfrac{1}{x^2}\right)^3=-1$

$\Rightarrow x^6+\dfrac{1}{x^6}+3\left(x^2+\dfrac{1}{x^2}\right)=-1$

$\Rightarrow x^6+\dfrac{1}{x^6}=2$

$\therefore Q = 2$

I3.3 Given that $\dfrac{\mathrm{Q}}{\sqrt{\mathrm{Q}}+\sqrt{2\mathrm{Q}}}+\dfrac{\mathrm{Q}}{\sqrt{2\mathrm{Q}}+\sqrt{3\mathrm{Q}}}+\cdots+\dfrac{\mathrm{Q}}{\sqrt{1998\mathrm{Q}}+\sqrt{1999\mathrm{Q}}}=\dfrac{\mathrm{R}}{\sqrt{\mathrm{Q}}+\sqrt{1999\mathrm{Q}}}$, find the value of R.

$\dfrac{2}{\sqrt{2}+\sqrt{4}}+\dfrac{2}{\sqrt{4}+\sqrt{6}}+\cdots+\dfrac{2}{\sqrt{3996}+\sqrt{3998}}=\dfrac{\mathrm{R}}{\sqrt{2}+\sqrt{3998}}$

$2\left(\dfrac{\sqrt{4}-\sqrt{2}}{4-2}+\dfrac{\sqrt{6}-\sqrt{4}}{6-4}+\cdots+\dfrac{\sqrt{3998}-\sqrt{3996}}{3998-3996}\right)=\dfrac{\mathrm{R}}{\sqrt{2}+\sqrt{3998}}$

$\sqrt{3998}-\sqrt{2}=\dfrac{R}{\sqrt{3998}+\sqrt{2}}$

$R=\left(\sqrt{3998}-\sqrt{2}\right)\left(\sqrt{3998}+\sqrt{2}\right)=3996$

I3.4 Let $f(0) = 0$; $f(n) = f(n - 1) + 3$ when $n = 1, 2, 3, 4, \ldots$. If $2 f(S) = R$, find the value of S.

$f(1) = 0 + 3 = 3$, $f(2) = 3 + 3 = 3\times 2$, $f(3) = 3\times 3$, $\ldots$, $f(n) = 3n$

$R = 3996 = 2 f(S) = 2\times 3S$

$S = 666$

Individual Event 4

I4.1 Suppose $a+\frac{1}{a+1}=b+\frac{1}{b-1}-2$, where $a \neq -1$, $b \neq 1$, and $a - b + 2 \neq 0$.

Given that $ab - a + b = P$, find the value of P.

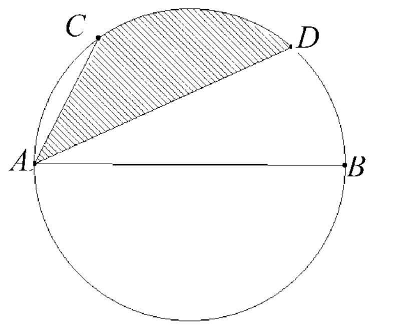
$a-b+2+\frac{1}{a+1}-\frac{1}{b-1}=0$

$(a-b+2)[1-\frac{1}{(a+1)(b-1)}]=0$

$\Rightarrow ab + b - a - 2 = 0$

$P = 2$

I4.2 In the following figure, AB is a diameter of the circle. C and D divide the arc AB into three equal parts. The shaded area is P.



If the area of the circle is Q, find the value of Q.

Reference: 2004 HI9, 2005 HG7, 2018 HI12

Let O be the centre.

Area of ΔACD = area of ΔOCD

(same base, same height) and $\angle COD = 60°$

Shaded area = area of sector COD = 2

$\therefore$ area of the circle = $6 \times 2 = 12$

I4.3 Given that there are R odd numbers in the digits of the product of the two Q-digit numbers $1111\cdots11$ and $9999\cdots99$, find the value of R.

Reference: 2015 FI1.2

Note that $99 \times 11 = 1089$; $999\times111 = 110889$.

Deductively, $999999999999 \times 111111111111 = 111111111110888888888889$

$R = 12$ odd numbers in the digits.

I4.4 Let $a_1, a_2, \cdots, a_R$ be positive integers such that $a_1 < a_2 < a_3 < \cdots < a_{R-1} < a_R$. Given that the sum of these R integers is 90 and the maximum value of a_1 is S, find the value of S.

$a_1 + a_2 + \cdots + a_{12} = 90$

$a_1 + (a_1 + 1) + (a_1 + 2) + \cdots + (a_1 + 11) \leq 90$

$12a_1 + 55 \leq 90$

$a_1 \leq 2.9167$

S = maximum value of $a_1 = 2$

Individual Event 5

I5.1 If $\left(\frac{1\times2\times4+2\times4\times8+3\times6\times12+\cdots+1999\times3998\times7996}{1^3+2^3+3^3+\cdots+1999^3}\right)^{\frac{1}{3}}=P$, find the value of P.

Reference: 2015 FG1.1

$$P=\left(\frac{1\times2\times4+2\times4\times8+3\times6\times12+\cdots+1999\times3998\times7996}{1^3+2^3+3^3+\cdots+1999^3}\right)^{\frac{1}{3}}$$

$$=\left[\frac{1\times2\times4\left(1^3+2^3+3^3+\cdots+1999^3\right)}{1^3+2^3+3^3+\cdots+1999^3}\right]^{\frac{1}{3}}$$

$$=8^{\frac{1}{3}}=2$$

I5.2 If $(x-P)(x-2Q)-1=0$ has two integral roots, find the value of Q.

Reference: 2001 FI2.1, 2010 FI2.2, 2011 FI3.1, 2013 HG1

$(x-2)(x-2Q)-1=0$

$x^2-2(1+Q)x+4Q-1=0$

Two integral roots $\Rightarrow \Delta$ is perfect square

$\Delta=4[(1+Q)^2-(4Q-1)]$

$=4(Q^2-2Q+2)$

$=4(Q-1)^2+4$

It is a perfect square $\Rightarrow Q-1=0$, $Q=1$

Method 2

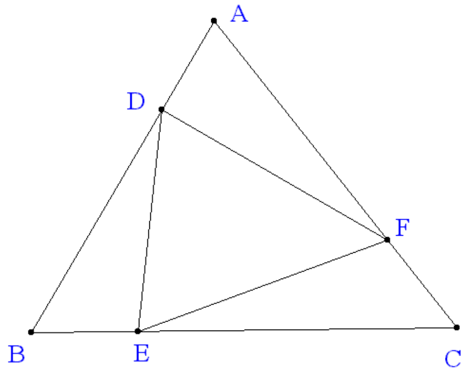
$(x-2)(x-2Q)=1$

$(x-2=1$ and $x-2Q=1)$ or $(x-2=-1$ and $x-2Q=-1)$

$(x=3$ and $Q=1)$ or $(x=1$ and $Q=1)$

$\therefore Q=1$

I5.3 Given that the area of the ΔABC is $3Q$; D, E and F are the points on AB, BC and CA respectively such that $AD=\frac{1}{3}AB$, $BE=\frac{1}{3}BC$, $CF=\frac{1}{3}CA$. If the area of ΔDEF is R, find the value of R. (**Reference: 1993 FG9.2**)



$R=3-$ area $\Delta ADF-$ area $\Delta BDE-$ area ΔCEF

$=3-\left(\frac{1}{2}AD\cdot AF\sin A+\frac{1}{2}BE\cdot BD\sin B+\frac{1}{2}CE\cdot CF\sin C\right)$

$=3-\frac{1}{2}\left(\frac{c}{3}\cdot\frac{2b}{3}\sin A+\frac{2c}{3}\cdot\frac{a}{3}\sin B+\frac{2a}{3}\cdot\frac{b}{3}\sin C\right)$

$=3-\frac{2}{9}\left(\frac{1}{2}\cdot bc\sin A+\frac{1}{2}\cdot ac\sin B+\frac{1}{2}\cdot ab\sin C\right)$

$=3-\frac{2}{9}(3\times\text{area of }\Delta\text{ABC})$

$=3-\frac{2}{9}\times9=1$

I5.4 Given that $(Rx^2-x+1)^{1999}\equiv a_0+a_1x+a_2x^2+\cdots+a_{3998}x^{3998}$.

If $S=a_0+a_1+a_2+\cdots+a_{3997}$, find the value of S.

$(x^2-x+1)^{1999}\equiv a_0+a_1x+a_2x^2+\cdots+a_{3998}x^{3998}$

Compare coefficients of x^{3998} on both sides, $a_{3998}=1$

Put $x=1$, $1^{1999}=a_0+a_1+a_2+\cdots+a_{3998}$

$S=a_0+a_1+a_2+\cdots+a_{3997}$

$=\left(a_0+a_1+a_2+\cdots+a_{3998}\right)-a_{3998}$

$= 1 - 1 = 0$

Sample Group Event (1999 Final Group Event 1)

SG.1 Let $x * y = x + y - xy$, where x, y are real numbers. If $a = 1 * (0 * 1)$, find the value of a.

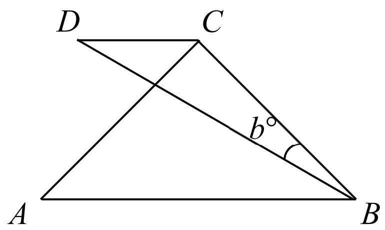
$0 * 1 = 0 + 1 - 0 = 1$

$a = 1 * (0 * 1)$

$= 1 * 1$

$= 1 + 1 - 1 = 1$

SG.2 In figure 1, AB is parallel to DC, $\angle ACB$ is a right angle, $AC = CB$ and $AB = BD$. If $\angle CBD = b°$, find the value of b.



ΔABC is a right angled isosceles triangle.

$\angle BAC = 45°$ ($\angle$s sum of Δ, base $\angle$s isos. Δ)

$\angle ACD = 45°$ (alt. $\angle$s, AB // DC)

$\angle BCD = 135°$

Apply sine law on ΔBCD,

$$\frac{BD}{\sin 135°} = \frac{BC}{\sin D}$$

$AB\sqrt{2} = \dfrac{AB \sin 45°}{\sin D}$, given that $AB = BD$

$\sin D = \dfrac{1}{2}$; $D = 30°$

$\angle CBD = 180° - 135° - 30° = 15°$ ($\angle$s sum of ΔBCD)

$b = 15$

SG.3 Let x, y be non-zero real numbers. If x is 250% of y and $2y$ is c% of x, find the value of c.

$x = 2.5y$ ....(1)

$2y = \dfrac{c}{100} \cdot x$ ....(2)

sub. (1) into (2): $2y = \dfrac{c}{100} \cdot 2.5y$

$c = 80$

SG.4 If $\log_p x = 2$, $\log_q x = 3$, $\log_r x = 6$ and $\log_{pqr} x = d$, find the value of d.

$$\frac{\log x}{\log p} = 2;\ \frac{\log x}{\log q} = 3;\ \frac{\log x}{\log r} = 6$$

$$\frac{\log p}{\log x} = \frac{1}{2};\ \frac{\log q}{\log x} = \frac{1}{3};\ \frac{\log r}{\log x} = \frac{1}{6}$$

$$\frac{\log p}{\log x} + \frac{\log q}{\log x} + \frac{\log r}{\log x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

$$\frac{\log pqr}{\log x} = 1$$

$$\frac{\log x}{\log pqr} = 1$$

$d = \log_{pqr} x = 1$

Group Event 1

G1.1 Given that when 81849, 106392 and 124374 are divided by an integer n, the remainders are equal. If a is the maximum value of n, find a.

Reference: 2016 FI4.2

$81849 = pn + k$ …… (1)

$106392 = qn + k$ …… (2)

$124374 = rn + k$ …… (3)

(2) – (1): $24543 = (q - p)n$ …… (4)

(3) – (2): $17982 = (r - q)n$ …… (5)

(4): $243 \times 101 = (q - p)n$

(5): $243 \times 74 = (r - q)n$

a = maximum value of $n = 243$

G1.2 Let $x = \dfrac{1-\sqrt{3}}{1+\sqrt{3}}$ and $y = \dfrac{1+\sqrt{3}}{1-\sqrt{3}}$. If $b = 2x^2 - 3xy + 2y^2$, find the value of b.

Reference: 2019 FG1.4

$b = 2x^2 - 3xy + 2y^2 = 2x^2 - 4xy + 2y^2 + xy = 2(x - y)^2 + xy$

$$= 2\left(\frac{1-\sqrt{3}}{1+\sqrt{3}} - \frac{1+\sqrt{3}}{1-\sqrt{3}}\right)^2 + \frac{1-\sqrt{3}}{1+\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$= 2\left[\frac{\left(1-\sqrt{3}\right)^2 - \left(1+\sqrt{3}\right)^2}{1-3}\right]^2 + 1$$

$$= 2\left(\frac{-4\sqrt{3}}{-2}\right)^2 + 1 = 25$$

G1.3 Given that c is a positive number. If there is only one straight line which passes through point A$(1, c)$ and meets the curve C: $x^2 + y^2 - 2x - 2y - 7 = 0$ at only one point, find the value of c.

The curve is a circle.

There is only one straight line which passes through point A and meets the curve at only one point $\Rightarrow$ the straight line is a tangent and the point $A(1, c)$ lies on the circle.

(otherwise two tangents can be drawn if A lies outside the circle)

Put $x = 1$, $y = c$ into the circle.

$1 + c^2 - 2 - 2c - 7 = 0$

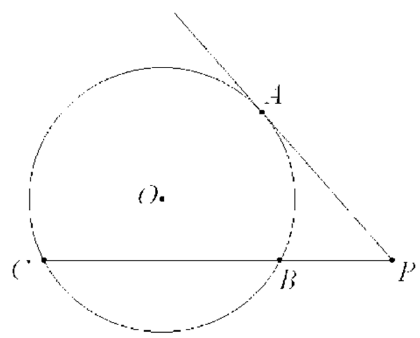
$c^2 - 2c - 8 = 0$

$(c - 4)(c + 2) = 0$

$c = 4$ or $c = -2$ (rejected)

G1.4 In Figure 1, PA touches the circle with centre O at A.

If $PA = 6$, $BC = 9$, $PB = d$, find the value of d.



It is easy to show that $\Delta PAB \sim \Delta PCA$

$\dfrac{PA}{PB} = \dfrac{PC}{PA}$ (ratio of sides, ~Δ's)

$\dfrac{6}{d} = \dfrac{9+d}{6}$

$36 = 9d + d^2$

$d^2 + 9d - 36 = 0$

$(d - 3)(d + 12) = 0$

$d = 3$ or -12 (rejected)

Group Event 2

G2.1 If 191 is the difference of two consecutive perfect squares, find the value of the smallest square number, a.

Let $a = t^2$, the larger perfect square is $(t+1)^2$

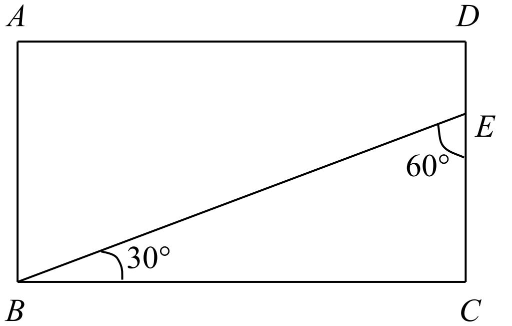
$(t + 1)^2 - t^2 = 191$

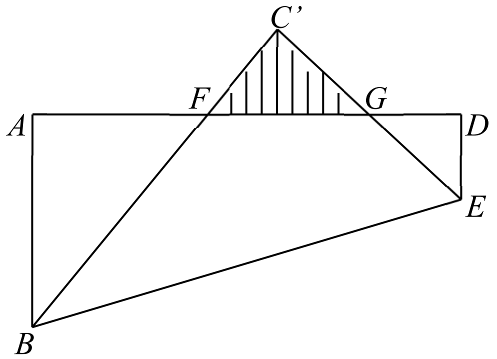
$2t + 1 = 191$

$t = 95$

$a = 95^2 = 9025$

G2.2 In Figure 2(a), $ABCD$ is a rectangle. DE:EC = 1: 5, and $DE = 12^{\frac{1}{4}}$. ΔBCE is folded along the side BE. If b is the area of the shaded part as shown in Figure 2(b), find the value of b.





Let $DE = t$, then $CE = 5t$. Suppose BC' intersects AD at F, C'E intersects AD at G.

$BC = BC' = AD = 5t \tan 60° = 5\sqrt{3}\, t$

$\angle C'ED = 60°$, $\angle ABC' = 30°$, $\angle C'FG = 60°$, $\angle C'GF = 30°$

$AF = 6t \tan 30° = 2\sqrt{3}\, t$, $DG = t \tan 60° = \sqrt{3}\, t$

$FG = 5\sqrt{3}\, t - 2\sqrt{3}\, t - \sqrt{3}\, t = 2\sqrt{3}\, t$

$C'F = 2\sqrt{3}\, t \cos 60° = \sqrt{3}\, t$, $C'G = 2\sqrt{3}\, t \cos 30° = 3t$

Area of $\Delta C'FG = \frac{1}{2}\sqrt{3}t \times 3t = \frac{3\sqrt{3}}{2}t^2 = \frac{3\sqrt{3}}{2}\sqrt{12} = 9$

G2.3 Let the curve $y = x^2 - 7x + 12$ intersect the x-axis at points A and B, and intersect the y-axis at C. If c is the area of ΔABC, find the value of c.

$x^2 - 7x + 12 = (x - 3)(x - 4)$

The x-intercepts of 3, 4.

Let $x = 0$, $y = 12$

$c = \frac{1}{2}(4-3)\cdot 12 = 6$ sq. units

G2.4 Let $f(x) = 41x^2 - 4x + 4$ and $g(x) = -2x^2 + x$. If d is the smallest value of k such that $f(x) + kg(x) = 0$ has a single root, find d.

$41x^2 - 4x + 4 + k(-2x^2 + x) = 0$

$(41 - 2k)x^2 + (k - 4)x + 4 = 0$

It has a single root $\Rightarrow \Delta = 0$ or $41 - 2k = 0$

$(k - 4)^2 - 4(41 - 2k)(4) = 0$ or $k = \frac{41}{2}$

$k^2 - 8 + 16 - 16\times 41 + 32k = 0$ or $k = \frac{41}{2}$

$k^2 + 24k - 640 = 0$ or $k = \frac{41}{2}$

$k = 16$ or -40 or $\frac{41}{2}$, d = the smallest value of $k = -40$

Group Event 3

G3.1 Let $a=\sqrt{1997\times1998\times1999\times2000+1}$, find the value of a.

Reference: 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2004 FG3.1, 2012 FI2.3

Let t = 1998.5, then 1997 = t – 1.5, 1998 = t – 0.5, 1999 = t + 0.5, 2000 = t + 1.5

$$\sqrt{1997\times1998\times1999\times2000+1}=\sqrt{(t-1.5)\times(t-0.5)\times(t+0.5)\times(t+1.5)+1}$$

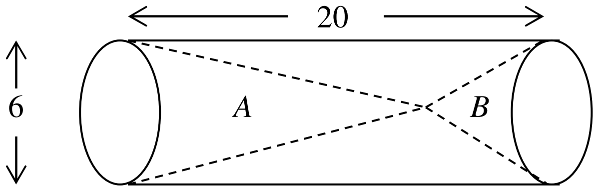
$$=\sqrt{\left(t^2-2.25\right)\times\left(t^2-0.25\right)+1}=\sqrt{\left(t^2-\frac{9}{4}\right)\times\left(t^2-\frac{1}{4}\right)+1}$$

$$=\sqrt{t^4-\frac{10}{4}t^2+\frac{25}{16}}=\sqrt{\left(t^2-\frac{5}{4}\right)^2}=t^2-1.25$$

$= 1998.5^2 - 1.25 = (2000 - 1.5)^2 - 1.25$

$= 4000000 - 6000 + 2.25 - 1.25 = 3994001$

G3.2 In Figure 3, A and B are two cones inside a cylindrical tube with length of 20 and diameter of 6. If the volumes of A and B are in the ratio 3:1 and b is the height of the cone B, find the value of b.



$$\frac{1}{3}\pi\cdot3^2(20-b):\frac{1}{3}\pi\cdot3^2b=3:1$$

$20 - b = 3b$

$b = 5$

G3.3 If c is the largest slope of the tangents from the point $A\left(\frac{\sqrt{10}}{2},\frac{\sqrt{10}}{2}\right)$ to the circle C: $x^2+y^2=1$, find the value of c.

Let the equation of tangent be $y-\frac{\sqrt{10}}{2}=c\left(x-\frac{\sqrt{10}}{2}\right)$

$$cx-y+\frac{\sqrt{10}}{2}(1-c)=0$$

Distance form centre (0, 0) to the straight line = radius

$$\left|\frac{0-0+\frac{\sqrt{10}}{2}(1-c)}{\sqrt{c^2+(-1)^2}}\right|=1$$

$$\frac{5}{2}(1-c)^2=c^2+1$$

$5 - 10c + 5c^2 = 2c^2 + 2$

$3c^2 - 10c + 3 = 0$

$(3c - 1)(c - 3) = 0$

$c=\frac{1}{3}$ or 3. The largest slope = 3.

G3.4 P is a point located at the origin of the coordinate plane. When a dice is thrown and the number n shown is even, P moves to the right by n. If n is odd, P moves upward by n. Find the value of d, the total number of tossing sequences for P to move to the point (4, 4).

Possible combinations of the die:

2,2,1,1,1,1. There are ${}_6C_2$ permutations, i.e. 15.

4,1,1,1,1. There are ${}_5C_1$ permutations, i.e. 5.

2,2,1,3. There are ${}_4C_2\times2$ permutations, i.e. 12.

4,1,3. There are 3! permutations, i.e. 6.

Total number of possible ways = 15 + 5 + 12 + 6 = 38.

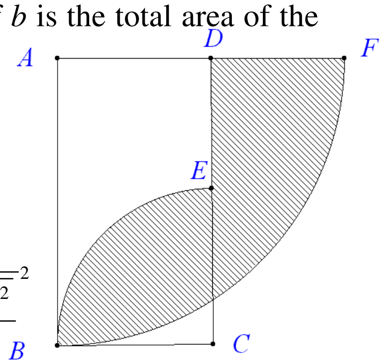
Group Event 4

G4.1 Let a be a 3-digit number. If the 6-digit number formed by putting a at the end of the number 504 is divisible by 7, 9, and 11, find the value of a.

Reference: 2010 HG1

Note that 504 is divisible by 7 and 9. We look for a 3-digit number which is a multiple of 63 and that 504000 + a is divisible by 11. 504504 satisfied the condition.

G4.2 In Figure 4, $ABCD$ is a rectangle with $AB=\sqrt{\dfrac{8+\sqrt{64-\pi^2}}{\pi}}$ and $BC=\sqrt{\dfrac{8-\sqrt{64-\pi^2}}{\pi}}$. BE and BF are the arcs of circles with centres at C and A respectively. If b is the total area of the shaded parts, find the value of b.



$AB = AF$, $BC = CE$

Shaded area = sector ABF – rectangle $ABCD$ + sector BCE

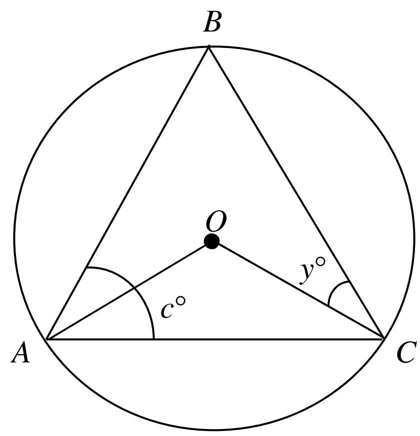
$$=\frac{\pi}{4}AB^2 - AB\cdot BC + \frac{\pi}{4}BC^2$$

$$=\frac{\pi}{4}\sqrt{\frac{8+\sqrt{64-\pi^2}}{\pi}}^2 - \sqrt{\frac{8+\sqrt{64-\pi^2}}{\pi}}\cdot\sqrt{\frac{8-\sqrt{64-\pi^2}}{\pi}} + \frac{\pi}{4}\sqrt{\frac{8-\sqrt{64-\pi^2}}{\pi}}^2$$

$$=\frac{\pi}{4}\left(\frac{8+\sqrt{64-\pi^2}}{\pi}+\frac{8-\sqrt{64-\pi^2}}{\pi}\right)-\sqrt{\frac{64-\left(64-\pi^2\right)}{\pi^2}}$$

$$=\frac{\pi}{4}\left(\frac{16}{\pi}\right)-\sqrt{\frac{\pi^2}{\pi^2}}=4-1=3=b$$

G4.3 In Figure 5, O is the centre of the circle and $c° = 2y°$. Find the value of c.



$\angle BOC = 2c°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$y + y + 2c = 180$ ($\angle$s sum of ΔOBC)

$2y + 2c = 180$

$c + 2c = 180$

$c = 60$

G4.4 A, B, C, D, E, F, G are seven people sitting around a circular table. If d is the total number of ways that B and G must sit next to C, find the value of d.

Reference: 1998 FI5.3, 2011 FI1.4

If B, C, G are neighbours, we can consider these persons bound together as one person. So, there are 5 persons sitting around a round table. The number of ways should be 5!. Since it is a round table, every seat can be counted as the first one. That is, $ABCDE$ is the same as $BCDEA$, $CDEAB$, $DEABC$, $EABCD$. Therefore every 5 arrangements are the same. The number of arrangement should be 5! ÷ 5 = 4! = 24. But B and G can exchange their seats. ∴Total number of arrangements = 24 × 2 = 48.

Group Event 5

G5.1 If a is the smallest cubic number divisible by 810, find the value of a.

Reference: 2002 HI2

$810 = 2\times 3^4 \times 5$

$a = 2^3 \times 3^6 \times 5^3 = 729000$

G5.2 Let b be the maximum of the function $y = |x^2 - 4| - 6x$ (where $-2 \le x \le 5$), find the value of b.

When $-2 \le x \le 2$, $y = 4 - x^2 - 6x = -(x + 3)^2 + 13$

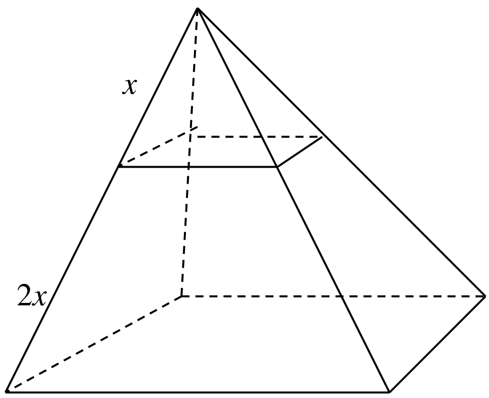
Maximum value occurs at $x = -2$, $y = -(-2 + 3)^2 + 13 = 12$

When $2 \le x \le 5$, $y = x^2 - 4 - 6x = (x - 3)^2 - 13$

Maximum value occurs at $x = 5$, $y = -9$

Combing the two cases, $b = 12$

G5.3 In Figure 6, a square-based pyramid is cut into two shapes by a cut running parallel to the base and made $\frac{2}{3}$ of the way up. Let $1 : c$ be the ratio of the volume of the small pyramid to that of the truncated base, find the value of c.



Reference: 2001 HG5

The two pyramids are similar.

$$\frac{\text{volume of the small pyramid}}{\text{volume of the big pyramid}} = \left(\frac{x}{3x}\right)^3 = \frac{1}{27}$$

$c = 27 - 1 = 26$

G5.4 If $\cos^6 \theta + \sin^6 \theta = 0.4$ and $d = 2 + 5 \cos^2 \theta \sin^2 \theta$, find the value of d.

$(\cos^2 \theta + \sin^2 \theta)(\cos^4 \theta - \sin^2 \theta \cos^2 \theta + \cos^4 \theta) = 0.4$

$\cos^4 \theta + 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta - 3 \sin^2 \theta \cos^2 \theta = 0.4$

$(\cos^2 \theta + \sin^2 \theta)^2 - 3 \sin^2 \theta \cos^2 \theta = 0.4$

$1 - 0.4 = 3 \sin^2 \theta \cos^2 \theta$

$\sin^2 \theta \cos^2 \theta = 0.2$

$d = 2 + 5 \cos^2 \theta \sin^2 \theta = 2 + 5\times 0.2 = 3$