Individual Events

I1	A	4	**I2**	a	40	**I3**	A	6	**I4**	a	0
	B	*1 see the remark		b	9		B	48		b	9
	C	8		c	1		C	2		c	24
	D	488		d	2		D	3		d	1344

Group Events

G1	s	5	**G2**	u	$\frac{1}{2}$	**G3**	α	$2\sqrt{3}$	**G4**	A	4068289
	w	8		v	122.5		β	594		B	$-\frac{1}{24}$
	q	4		n	9		φ	$-\frac{\sqrt{3}}{2}$		C	$\frac{1}{128}$
	v	0		m	15		γ	9		D	2

Individual Event 1

I1.1 已知 $x^2 = y^2 - 4y$，其中 x 及 y 為整數。求 $A = x + y$ 的最大值。

Given that $x^2 = y^2 - 4y$, where x and y are integers. Determine the largest value of $A = x + y$.

$x^2 = y^2 - 4y + 4 - 4$

$4 = (y - 2)^2 - x^2 = (y + x - 2)(y - x - 2)$

$y + x - 2$	$y - x - 2$	x	y	$x + y$
4	1	非整數	捨去	
2	2	0	4	4
1	4	非整數	捨去	
–1	–4	非整數	捨去	
–2	–2	0	0	0
–4	–1	非整數	捨去	

A的最大值$= 4$

$x^2 = y^2 - 4y + 4 - 4$

$4 = (y - 2)^2 - x^2 = (y + x - 2)(y - x - 2)$

$y + x - 2$	$y - x - 2$	x	y	$x + y$
4	1	non-integer	rejected	
2	2	0	4	4
1	4	non-integer	rejected	
–1	–4	non-integer	rejected	
–2	–2	0	0	0
–4	–1	non-integer	rejected	

The largest value of $A = 4$

I1.2 已知 $y = \sqrt{9A^2 - 12A + 4} \pm \sqrt{A^2 - 4A + 4} \pm \sqrt{A^2 + 6A + 9}$，且 B 是 y 的最小值，求 B 的值。

Given that $y = \sqrt{9A^2 - 12A + 4} \pm \sqrt{A^2 - 4A + 4} \pm \sqrt{A^2 + 6A + 9}$, and B is the least value of y, determine the value of B. (**Reference: 2016 FI4.3**)

$y = \sqrt{(3A-2)^2} \pm \sqrt{(A-2)^2} \pm \sqrt{(A+3)^2}$

$= \sqrt{(3\times 4-2)^2} \pm \sqrt{(4-2)^2} \pm \sqrt{(4+3)^2}$

$= 10 \pm 2 \pm 7$

y 的最小值$= B = 10 - 2 - 7 = 1$

$y = \sqrt{(3A-2)^2} \pm \sqrt{(A-2)^2} \pm \sqrt{(A+3)^2}$

$= \sqrt{(3\times 4-2)^2} \pm \sqrt{(4-2)^2} \pm \sqrt{(4+3)^2}$

$= 10 \pm 2 \pm 7$

The least value of $y = B = 10 - 2 - 7 = 1$

Remark: Original version …y 的最小正數值…the least positive value of y

The value of y must be positive, there is no need to emphasize the word "positive".

I1.3 設 C 為正整數。已知 $144 + (B + 1)^C$ 為平方數，求 C 的值。

Let C be a positive integer. Given that $144 + (B + 1)^C$ is a perfect square, determine the value of C.

$12^2 + 2^C = m^2$，其中 m 為整數

$2^4\cdot(3^2 + 2^{C-4}) = m^2$

$3^2 + 2^{C-4} = n^2$，其中 n 為整數

$2^{C-4} = (n + 3)(n - 3)$

$n + 3 = 2^a$，$n - 3 = 2^b$，$a + b = C - 4$

$6 = 2^a - 2^b = 2^b(2^{a-b} - 1)$

$b = 1$，$2^{a-1} - 1 = 3$，$a = 3$

$C = 8$

$12^2 + 2^C = m^2$, where m is an integer

$2^4\cdot(3^2 + 2^{C-4}) = m^2$

$3^2 + 2^{C-4} = n^2$, where n is an integer

$2^{C-4} = (n + 3)(n - 3)$

$n + 3 = 2^a$, $n - 3 = 2^b$, $a + b = C - 4$

$6 = 2^a - 2^b = 2^b(2^{a-b} - 1)$

$b = 1$, $2^{a-1} - 1 = 3$, $a = 3$

$C = 8$

I1.4 已知 $x+\frac{1}{x}=C$ ，求 $D=x^3+\frac{1}{x^3}$ 的值。

Given that $x+\frac{1}{x}=C$, determine the value of $D=x^3+\frac{1}{x^3}$.

Reference 1991 HI3

$x+\frac{1}{x}=8$

$\left(x+\frac{1}{x}\right)^2=64$

$\Rightarrow x^2+\frac{1}{x^2}=62$

$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$

$= 8\times(62-1) = 488$

Individual Event 2

I2.1 $7778^2 - 2223^2$ 之值的所有數字之和是 a，求 a 的值。

Determine the value of a, where a is the sum of all digits of $7778^2 - 2223^2$.

$7778^2 - 2223^2 = (7778 + 2223)(7778 - 2223)$ $= 10001\times5555$ $= 55555555$ 數位之和 $= a = 5\times8 = 40$	$7778^2 - 2223^2 = (7778 + 2223)(7778 - 2223)$ $= 10001\times5555$ $= 55555555$ Sum of all digits $= a = 5\times8 = 40$

I2.2 若 b 是乘積 $a\times(a-1)\times(a-2)\times\cdots\times2\times1$ 的尾隨零的數量。求 b 的值。

$a\times(a-1)\times(a-2)\times\cdots\times2\times1 = \overline{\cdots*\underbrace{00\cdots0}_{\text{"0" 的數量是}b}}$， * 代表非零數字。

If the number of trailing zeros of the product $a\times(a-1)\times(a-2)\times\cdots\times2\times1$ is b, determine the value of b.

$a\times(a-1)\times(a-2)\times\cdots\times2\times1 = \overline{\cdots*\underbrace{00\cdots0}_{\text{The number of "0" is }b}}$, * represents a non-zero digit.

Reference: 1994 FG7.1, 1996 HI3, 2011 HG7, 2012 FI1.4, 2012 FG1.3

方法一 當每一個因子 5 乘以 2 時，乘積 40!.的末位便出現一個 ‘0’。 40! 當中 2 的因子很明顯比 5 的因子多。 我們只需數一數 5 的因子。 5, 10, 15, … , 40; 共有 8 個數，每個數有最小一個 ‘5’ 的因子。 25 這個數有兩個 5 因子。 5 的因子合共有 $8 + 1 = 9$ 個 $b = 9$	**Method 1** When each factor of 5 is multiplied by 2, a trailing zero will appear in 40! . The number of factors of 2 is clearly more than the number of factors of ‘5’ in 40! It is sufficient to find the number of factors of 5. 5, 10, 15, ⋯ , 40; altogether 8 numbers, each have at least one factor of 5 . The number "25" has two factors of 5 . Total number of factors of 5 is 8 + 1= 9 $b = 9$
方法二 我們可以用以下的連除式找出5的因子的數量: 5 | 40 5 | 8 ⋯ 0 1 ⋯ 3 ∴ 5 的因子合共有 8 + 1 = 9 個 $b = 9$	**Method 2** We can find the total number of factors of 5 by division as follow: 5 | 40 5 | 8 ⋯ 1 1 ⋯ 3 ∴Total no. of factors of 5 is 8 + 1 = 9 $b = 9$

I2.3 若 c 是 $2^{10} - 2^8 + 2^6 - 2^4 + 2^2$ 除以 b 的餘數，求 c 的值。

If c is the remainder when $2^{10} - 2^8 + 2^6 - 2^4 + 2^2$ is divided by b, determine the value of c.

$2^{10} - 2^8 + 2^6 - 2^4 + 2^2 = 1024 - 256 + 64 - 16 + 4$

$= 820 = 9\times91 + 1$

$c = 1$

I2.4 求整數 d，使得對於任何實數 x，$x^{13} + cx + 90$ 可被 $x^2 - x + d$ 整除。

Determine the integral value of d, so that $x^{13} + cx + 90$ is divisible by $x^2 - x + d$ for any real number x. (**Reference 24th Putnam competition 1963 B1**)

$x^{13} + x + 90$ 可被 $x^2 - x + d$ 整除。
若 $d = 0$ 及 $x^{13} + x + 90$ 可被 $x^2 - x$ 整除。
$\Rightarrow x^{13} + x + 90$ 可被 x 整除，不可能。
若 $d < 0$，則 $x^2 - x + d$ 的判別式 $= 1 - 4d > 0$
$\Rightarrow x^2 - x + d$ 有兩個實數根
但是，$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{13} + x + 90\right) = 13x^{12} + 1 > 0\ \forall x$
$\therefore x^{13} + x + 90$ 絕對遞增 $\forall x$
$\Rightarrow x^{13} + x + 90$ 只有一個實數根，矛盾!
$\therefore d > 0$
代 $x = 0$，d 整除 90
代 $x = 1$，d 整除 92
$\therefore d$ 是 90 和 92 的公因數
d 的可能值 $= 1$ 或 2
代 $x = 2$，$d + 2$ 整除 $2^{13} + 92 = 8284$
若 $d = 1$，則 3 整除 8284，錯，捨去
$\therefore d = 2$
事實上，利用短除
$x^{13} + x + 90 = (x^2 - x + 2)\times(x^{11} + x^{10} - x^9 - 3x^8 - x^7 + 5x^6 + 7x^5 - 3x^4 - 17x^3 - 11x^2 + 23x + 45)$

$x^{13} + x + 90$ is divisible by $x^2 - x + d$.
If $d = 0$ and $x^{13} + x + 90$ is divisible by $x^2 - x$
$\Rightarrow x^{13} + x + 90$ is divisible by x, impossible.
If $d < 0$, then Δ of $x^2 - x + d$ is $1 - 4d > 0$
$\Rightarrow x^2 - x + d$ has two real roots
However, $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{13} + x + 90\right) = 13x^{12} + 1 > 0\ \forall x$
$\therefore x^{13} + x + 90$ is strictly increasing $\forall x$
$x^{13} + x + 90$ has only one real root !!!
$\therefore d > 0$
Put $x = 0$, d divides 90
Put $x = 1$, d divides 92
$\therefore d$ is the common factor of 90 and 92
Possible $d = 1$ or 2
Put $x = 2$, $d + 2$ divides $2^{13} + 92 = 8284$
If $d = 1$, then 3 divides 8284, false, rejected
$\therefore d = 2$
In fact, by using synthetic division,
$x^{13} + x + 90 = (x^2 - x + 2)\times(x^{11} + x^{10} - x^9 - 3x^8 - x^7 + 5x^6 + 7x^5 - 3x^4 - 17x^3 - 11x^2 + 23x + 45)$

	x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x	x	1
	1	0	0	0	0	0	0	0	0	0	0	0	1	90
1		1	1	–1	–3	–1	5	7	–3	–17	–11	23	45	
–2			–2	–2	2	6	2	–10	–14	6	34	22	–46	–90
	1	1	–1	–3	–1	5	7	–3	–17	–11	23	45	0	0
	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	1	x	1

商 Quotient

餘數 remainder

Individual Event 3

I3.1 已知 a、b、c 為實數，且 $A = (3a - X)^2 + (3b - X)^2 + (3c - X)^2 + 6$.
若 $X = a + b + c$ 及 $X^2 = a^2 + b^2 + c^2$，求 A 的最小值。
Given that a, b, c are real numbers and $A = (3a - X)^2 + (3b - X)^2 + (3c - X)^2 + 6$.
If $X = a + b + c$ 及 $X^2 = a^2 + b^2 + c^2$, determine the least value of A .

$A = (3a - X)^2 + (3b - X)^2 + (3c - X)^2 + 6$ $= 3X^2 - (6a + 6b + 6c)X + (9a^2 + 9b^2 + 9c^2) + 6$ $= 3X^2 - 6X^2 + 9X^2 + 6$ $= 6(X^2 + 1) \geq 6$ 當 $X^2 = a^2 + b^2 + c^2 = 0$ 時，等式成立。 即 $a = b = c = 0$ A 的最小值$= 6$	$A = (3a - X)^2 + (3b - X)^2 + (3c - X)^2 + 6$ $= 3X^2 - (6a + 6b + 6c)X + (9a^2 + 9b^2 + 9c^2) + 6$ $= 3X^2 - 6X^2 + 9X^2 + 6$ $= 6(X^2 + 1) \geq 6$ Equality holds when $X^2 = a^2 + b^2 + c^2 = 0$ i.e. $a = b = c = 0$ The least value of $A = 6$

I3.2 假設班中有 A 名男同學及 $30 - A$ 名女同學。若男同學的平均體重為 60 kg，女同學的平均體重為 45 kg 及全班同學的平均體重為 B kg，求 B 的值。
Suppose that there are A boys and $30 - A$ girls in a class. If the average weight of the boys is 60 kg, the average weight of the girls is 45 kg, and the average weight of the students in the class is B kg, determine the value of B .

班中有 6 名男同學及 24 名女同學。 平均體重$=\dfrac{60\times 6+45\times 24}{30}= 48$ kg $B = 48$	There are 6 boys and 24 girls. Average weight $=\dfrac{60\times 6+45\times 24}{30}= 48$ kg $B = 48$

I3.3 若 n 是正整數、$a_1 = B$ 及 $a_{n+1} = \begin{cases} \dfrac{a_n}{2} & \text{若 } a_n \text{ 是偶數；} \\ 3a_n + 1 & \text{若 } a_n \text{ 是奇數。} \end{cases}$ 求 $C = a_{2018}$ 的值。

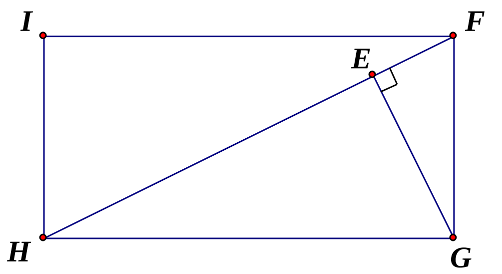
If n is a positive integer $a_1 = B$ and $a_{n+1} = \begin{cases} \dfrac{a_n}{2} & \text{if } a_n \text{ is even;} \\ 3a_n + 1 & \text{if } a_n \text{ is odd.} \end{cases}$

determine the value of $C = a_{2018}$.

$a_1 = 48,\ a_2 = 24,\ a_3 = 12,\ a_4 = 6,\ a_5 = 3$ $a_6 = 3\times3 + 1 = 10,\ a_7 = 5,\ a_8 = 3\times5 + 1 = 16$ $a_9 = 8,\ a_{10} = 4,\ a_{11} = 2,\ a_{12} = 1,$ $a_{13} = 3\times1 + 1 = 4,\ a_{14} = 2,\ a_{15} = 1, \cdots$ $a_{3k} = 1$ 由 $k = 4, 5, 6, \cdots$ $a_{3k+1} = 4$，$a_{3k+2} = 2$ 由 $k = 4, 5, 6, \cdots$ $2018 = 3\times672 + 2$ $a_{2018} = 2$	$a_1 = 48,\ a_2 = 24,\ a_3 = 12,\ a_4 = 6,\ a_5 = 3$ $a_6 = 3\times3 + 1 = 10,\ a_7 = 5,\ a_8 = 3\times5 + 1 = 16$ $a_9 = 8,\ a_{10} = 4,\ a_{11} = 2,\ a_{12} = 1,$ $a_{13} = 3\times1 + 1 = 4,\ a_{14} = 2,\ a_{15} = 1, \cdots$ $a_{3k} = 1$ for $k = 4, 5, 6, \cdots$ $a_{3k+1} = 4$，$a_{3k+2} = 2$ for $k = 4, 5, 6, \cdots$ $2018 = 3\times672 + 2$ $a_{2018} = 2$

I3.4 長方形 $FGHI$ 被直綫 FH 分為兩個直角三角形。三角形 ΔFGH 被直綫 EG 分為另外兩個直角三角形。若 FH：$FG = C$：1 及三角形 ΔEGH 與三角形 ΔFEG 的面積比為 D：1，求 D 的值。

Suppose that a rectangle $FGHI$ is divided into two right-angled triangles by line FH. The triangle ΔFGH is then divided into two right-angled triangles by line EG. If the ratio of lengths $FH : FG$ is $C : 1$ and the ratio of the areas of ΔEGH to ΔFEG is $D : 1$, determine the value of D.



設$\angle GFH = \theta$ $FH : FG = 2 : 1 \Rightarrow \cos\theta = \frac{1}{2}$ $\theta = 60°$ $\angle GHE = 30°$ (三角形內角和) $EF = EG \div \tan 60° = \frac{EG}{\sqrt{3}}$ $EH = EG \div \tan 30° = \sqrt{3}EG$ $S_{\Delta EGH} : S_{\Delta FEG} = \frac{1}{2}EG \times EH : \frac{1}{2}EG \times EF$ $= EH : EF$ $= \sqrt{3}EG : \frac{EG}{\sqrt{3}} = 3 : 1$ $D = 3$	Let $\angle GFH = \theta$ $FH : FG = 2 : 1 \Rightarrow \cos\theta = \frac{1}{2}$ $\theta = 60°$ $\angle GHE = 30°$ ($\angle$ sum of Δ) $EF = EG \div \tan 60° = \frac{EG}{\sqrt{3}}$ $EH = EG \div \tan 30° = \sqrt{3}EG$ $S_{\Delta EGH} : S_{\Delta FEG} = \frac{1}{2}EG \times EH : \frac{1}{2}EG \times EF$ $= EH : EF$ $= \sqrt{3}EG : \frac{EG}{\sqrt{3}} = 3 : 1$ $D = 3$

Individual Event 4

I4.1 若 a 為 $(1^{2018}+2^{2018}+3^{2018}+4^{2018})\div 5$ 的餘數，求 a 的值。

If a is the remainder of $(1^{2018}+2^{2018}+3^{2018}+4^{2018})\div 5$, determine the value of a.

$1^{2018}\equiv 1 \pmod{10}$ $2^1=2,\ 2^2=4,\ 2^3=8,\ 2^4=16,\ 2^5=32$ $2018=4\times504+2,\ 2^{2018}\equiv 4 \pmod{10}$ $3^1=3,\ 3^2=9,\ 3^3=27,\ 3^4=81,\ 3^5=243$ $3^{2018}\equiv 9 \pmod{10}$ $4^1=4,\ 4^2=16,\ 4^3=64$ $2018=2\times1009$ $4^{2018}\equiv 6 \pmod{10}$ $1^{2018}+2^{2018}+3^{2018}+4^{2018}\equiv 1+4+9+6\equiv 0$ 餘數$=a=0$	$1^{2018}\equiv 1 \pmod{10}$ $2^1=2,\ 2^2=4,\ 2^3=8,\ 2^4=16,\ 2^5=32$ $2018=4\times504+2,\ 2^{2018}\equiv 4 \pmod{10}$ $3^1=3,\ 3^2=9,\ 3^3=27,\ 3^4=81,\ 3^5=243$ $3^{2018}\equiv 9 \pmod{10}$ $4^1=4,\ 4^2=16,\ 4^3=64$ $2018=2\times1009$ $4^{2018}\equiv 6 \pmod{10}$ $1^{2018}+2^{2018}+3^{2018}+4^{2018}\equiv 1+4+9+6\equiv 0$ The remainder $=a=0$

I4.2 若 x、y 為正整數及 b 為 x、y 組合的數量使得它們的乘積 $x\times y=\overline{1aa}$，求 b 的值。

If x, y are positive integers numbers and b is the number of groups of x, y such that the product $x\times y=\overline{1aa}$, determine the value of b.

$xy=100$

$(x, y)=(1, 100), (2, 50), (4, 25), (5, 20), (10, 10), (20, 5), (25, 4), (50, 2), (100, 1)$

$b=9$

I4.3 若對於正整數 $x>y>z$，$xyz+xy+xz+yz+x+y+z+1=30b+87$。求 $c=x+y+z$ 的值。

If $xyz+xy+xz+yz+x+y+z+1=30b+87$ for positive integers $x>y>z$, determine the value of $c=x+y+z$.

Reference: 2004 HG6

$(x+1)(y+1)(z+1)=30\times9+87=357$

$(x+1)(y+1)(z+1)=17\times7\times3$

$x=16,\ y=6,\ z=2$

$c=16+6+2=24$

I4.4 若某長方形的面積為 d cm^2，它能被邊長為 $\frac{c}{3}$ cm 的正方形階磚密鋪，若該長方形亦能被闊度為 $\frac{c}{2}$ cm、長度為 7 cm 的長方形階磚密鋪，求 d 的最小值。

Let d cm^2 be the area of a rectangle that can be tessellated by square tiles with sides length of $\frac{c}{3}$ cm. If the rectangle can also be tessellated by rectangular tiles with width of $\frac{c}{2}$ cm and length of 7 cm, determine the least value of d.

$\frac{c}{3}=8,\ \frac{c}{2}=12$ 假設長方形的長、度為 $8p$ cm$\times 8q$ cm，其中 p 及 q 為正整數。 $8p=12r \cdots (1),\ 8q=7s \cdots (2)$ 其中 r 及 s 為正整數。 由(1)式， $2p=3r$ 最小值為 $s=8$、$q=7$、$r=2$、$p=3$ $d=8p\times8q=64\times3\times7=1344$	$\frac{c}{3}=8,\ \frac{c}{2}=12$ Let the dimensions of the rectangle be $8p$ cm$\times 8q$ cm, where p and q are positive integers. $8p=12r \cdots (1),\ 8q=7s \cdots (2)$ where r and s are positive integers From (1), $2p=3r$ For minimum values $s=8,\ q=7,\ r=2,\ p=3$ $d=8p\times8q=64\times3\times7=1344$

Remark: original question …求 d 的最小正數值。 …the least positive value of d.

d must be positive, there is no need to emphasize the word "positive".

Group Event 1

G1.1 瑪莉和小明在中文科、英文科及數學科獲得的分數為 s 或 t，及 $s > t > 0$。若瑪莉於中文科的分數比小明的高以及小明於英文的分數比瑪莉的高，而瑪莉和小明的總分分別為 12 分和 9 分。求 s 的值。

Suppose that Mary and Ming obtained a score of either s or t in each of the subjects: Chinese, English and Mathematics, where $s > t > 0$. It is known that Mary did better in Chinese but Ming did better in English. Mary's and Ming's total scores are 12 and 9 respectively. Determine the value of s.

根據已知資料，

	中文科	英文科	數學科	總分
瑪莉	s	t	s	12
小明	t	s	t	9

$2s + t = 12$ …… (1)

$2t + s = 9$ …… (2)

$2(1) – (2)$: $3s = 15 \Rightarrow s = 5$

According to the given information,

	Chinese	English	Mathematics	Total
Mary	s	t	s	12
Ming	t	s	t	9

$2s + t = 12$ …… (1)

$2t + s = 9$ …… (2)

$2(1) – (2)$: $3s = 15 \Rightarrow s = 5$

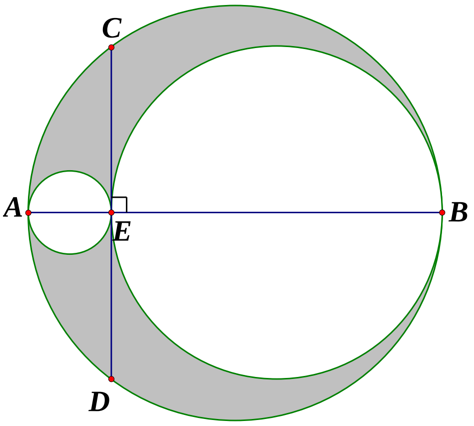
Remark: Original question: … 分數為 s 或 t 的整數… an integral score of either s or t

s and t can be solved without the information of integral value.

G1.2 已知兩圓的直徑為 AE 及 BE，內接於直徑為 AB 的圓中。若 $CE \perp AB$，$AB = 10$，$CE = 4$ 及陰影部份總面積為 $w\pi$，求 w 的值。

Given that the two circles, one with diameter AE and the other with diameter BE, are inscribed by a larger circle with diameter AB. If $CE \perp AB$ with $AB = 10$ and $CE = 4$, and the total area of the shaded regions is $w\pi$, determine the value of w.

Reference: 1990 HG10



將 CE 延長交圓形於 D。

假設三個以 AE、BE 及 AB 為直徑凡圓形的半徑分別為 a、b 及 c。

$2a + 2b = 2c = 10$

$c = 5$ 及 $a + b = 5$ …… (1)

利用相交弦定理，$2a \times 2b = 4^2$

$ab = 4$ …… (2)

陰影面積 $= \pi c^2 – \pi a^2 – \pi b^2$

$= \pi[5^2 – (a^2 + b^2)]$

$= \pi[5^2 – (a + b)^2 + 2ab]$

$= \pi(5^2 – 5^2 + 2\times4)$ 由(1)及(2)所得

$= 8\pi$

$w = 8$

Produce CE to meet the circle again at D.

Let the radii of the 3 circles with diameters AE, BE and AB are a, b and c respectively.

$2a + 2b = 2c = 10$

$c = 5$ and $a + b = 5$ …… (1)

By intersecting chords theorem, $2a\times2b = 4^2$

$ab = 4$ …… (2)

Shaded area $= \pi c^2 – \pi a^2 – \pi b^2$

$= \pi[5^2 – (a^2 + b^2)]$

$= \pi[5^2 – (a + b)^2 + 2ab]$

$= \pi(5^2 – 5^2 + 2\times4)$ by (1) and (2)

$= 8\pi$

$w = 8$

G1.3 設 m 及 r 為非負整數。若 $f(7m + r) = r$，求 $q = f(2^{2018})$ 的值。

Let m and r be non-negative integers.

If $f(7m + r) = r$, determine the value of $q = f(2^{2018})$.

我們找出 $2^{2018} \div 7$ 的餘數。 $2 \div 7 \cdots\cdots 2$ $2^2 \div 7 \cdots\cdots 4$ $2^3 \div 7 \cdots\cdots 1$ $2^4 \div 7 \cdots\cdots 2$ 餘數出現的規律就是每隔 3 的倍數重複一次。 $2018 = 3\times672 + 2$，$2^{2018} \div 7$ 的餘數是 4。 $q = f(2^{2018}) = f(7m + 4) = 4$	We find the remainder of $2^{2018} \div 7$. $2 \div 7 \cdots\cdots 2$ $2^2 \div 7 \cdots\cdots 4$ $2^3 \div 7 \cdots\cdots 1$ $2^4 \div 7 \cdots\cdots 2$ The pattern of the remainders repeats for every multiple of 3. $2018 = 3\times672 + 2$, the remainder of $2^{2018} \div 7$ is 4 . $q = f(2^{2018}) = f(7m + 4) = 4$

G1.4 在五進制中，若 v 為 $234234_5 \div 234_5$ 的餘數，求 v 的值。

In base 5 system, if v is the remainder of $234234_5 \div 234_5$, determine the value of v .

Reference: 2019 FI3.4

$234234_5 = (234\times1000 + 234)_5$ $= (234\times1001)_5$ 餘數 $v = 0$	$234234_5 = (234\times1000 + 234)_5$ $= (234\times1001)_5$ The remainder is $v = 0$

Group Event 2

G2.1 已知 $\frac{1-2^{-1/u}}{2^{-1/u}-2^{-2/u}}=4$，求 u 的值。Given that $\frac{1-2^{-1/u}}{2^{-1/u}-2^{-2/u}}=4$, determine the value of u.

Reference: 2018 HI5

$$\frac{\left(1-\frac{1}{2^{\frac{1}{u}}}\right)}{\left(\frac{1}{2^{\frac{1}{u}}}-\frac{1}{2^{\frac{2}{u}}}\right)}\cdot\frac{2^{\frac{2}{u}}}{2^{\frac{2}{u}}}=4$$

$$\frac{2^{\frac{1}{u}}\left(2^{\frac{1}{u}}-1\right)}{2^{\frac{1}{u}}-1}=4$$

$$2^{\frac{1}{u}}=2^2$$

$$u=\frac{1}{2}$$

G2.2 已知 $b\ge 1$、$a-12b=15$ 及 x 是實數，求 $v=\frac{(x-a)^2}{2b}+5x$ 的最小值。

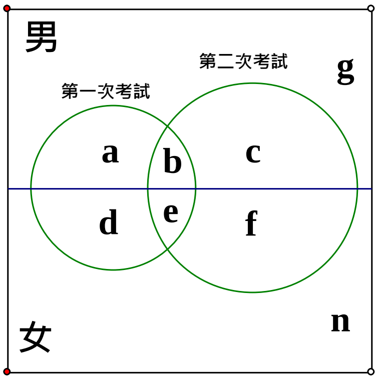
Given that $b \ge 1$, $a - 12b = 15$ and x is a real number,

determine the least value of $v=\frac{(x-a)^2}{2b}+5x$.

$v=\frac{(x-a)^2}{2b}+5x$ $=\frac{(x-12b-15)^2}{2b}+5x$ $=\frac{x^2-2(12b+15)x+(12b+15)^2+10bx}{2b}$ $=\frac{x^2-2(7b+15)x+(12b+15)^2}{2b}$ $=\frac{\left[x-(7b+15)\right]^2+(12b+15)^2-(7b+15)^2}{2b}$ $=\frac{\left[x-(7b+15)\right]^2+5b(19b+30)}{2b}$ $=\frac{\left[x-(7b+15)\right]^2}{2b}+\frac{5}{2}(19b+30)$ $\ge 0+\frac{5}{2}(19\times 1+30)=\frac{245}{2}=122.5$ v 的最小值= 122.5	$v=\frac{(x-a)^2}{2b}+5x$ $=\frac{(x-12b-15)^2}{2b}+5x$ $=\frac{x^2-2(12b+15)x+(12b+15)^2+10bx}{2b}$ $=\frac{x^2-2(7b+15)x+(12b+15)^2}{2b}$ $=\frac{\left[x-(7b+15)\right]^2+(12b+15)^2-(7b+15)^2}{2b}$ $=\frac{\left[x-(7b+15)\right]^2+5b(19b+30)}{2b}$ $=\frac{\left[x-(7b+15)\right]^2}{2b}+\frac{5}{2}(19b+30)$ $\ge 0+\frac{5}{2}(19\times 1+30)=\frac{245}{2}=122.5$ The least value of v = 122.5

G2.3 若班中有 20 位男同學及 15 位女同學參加兩次考試。已知 8 位同學在第一次考試中不合格，12 位同學在第二次考試中不合格，及 6 位同學於兩次考試均不合格。若 5 位男同學在第一次考試中不合格，7 位男同學在第二次考試中不合格，4 位男同學兩次考試均不合格及 n 位女同學兩次考試均合格，求 n 的值。

Suppose that there were 20 boys and 15 girls in a class taking two examinations. Given that 8 students failed in the first examinations, 12 students failed in the second examinations, and 6 students failed in both examinations. If 5 boys failed in the first examinations, 7 boys failed in the second examinations, 4 boys failed in both examinations, and n girls passed in both examinations, determine the value of n .



如上溫氏圖所示，左圓表示在第一次考試中不合格的同學。右圓表示在第二次考試中不合格的同學。兩圓重疊部分(b 及 e)表示在兩次考試中皆不合格的同學。g 及 n 在兩圓以外，表示在兩次考試中皆合格的同學。上半部份 (a, b, c, g) 表示男同學。下半部份 (d, e, f, n) 表示女同學。

根據已給資料，

$b + e = 6 \cdots (1)$

$a + d + (b + e) = 8 \Rightarrow a + d = 2 \cdots (2)$

$c + f + (b + e) = 12 \Rightarrow c + f = 6 \cdots (3)$

$b = 4 \cdots (4)$

$a + b = 5 \Rightarrow a = 1 \cdots (5)$

$b + c = 7 \Rightarrow c = 3 \cdots (6)$

代 (5) 入 (2): $1 + d = 2 \Rightarrow d = 1 \cdots (7)$

代 (4) 入 (1): $4 + e = 6 \Rightarrow e = 2 \cdots (8)$

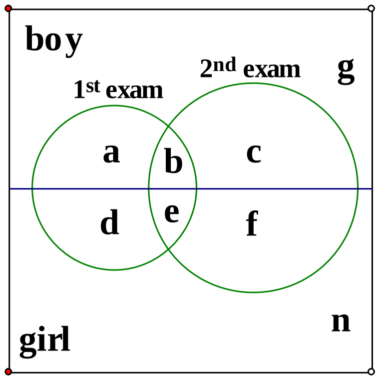
代 (6) 入 (3): $3 + f = 6 \Rightarrow f = 3 \quad \cdots (9)$

女同學數目 $d + e + f + n = 15 \cdots (10)$

代 (7)、(8)、(9) 入 (10): $1 + 2 + 3 + n = 15$

$n = 9$

9 名女同學在兩次考試中皆合格。



As shown in the Venn diagram, the left circle represents the students failed in the first examination. The right circle represents the students failed in the second examination. The overlapping part of the two circles (b and e) represents the students failing in both examinations. g and n outside the circles represent students passed in both examinations. The upper part (a, b, c, g) represents boys. The lower part (d, e, f, n) represents girls.

According to the given information,

$b + e = 6 \cdots (1)$

$a + d + (b + e) = 8 \Rightarrow a + d = 2 \cdots (2)$

$c + f + (b + e) = 12 \Rightarrow c + f = 6 \cdots (3)$

$b = 4 \cdots (4)$

$a + b = 5 \Rightarrow a = 1 \cdots (5)$

$b + c = 7 \Rightarrow c = 3 \cdots (6)$

Sub. (5) into (2): $1 + d = 2 \Rightarrow d = 1 \cdots (7)$

Sub. (4) into (1): $4 + e = 6 \Rightarrow e = 2 \cdots (8)$

Sub. (6) into (3): $3 + f = 6 \Rightarrow f = 3 \quad \cdots (9)$

Number of girls: $d + e + f + n = 15 \cdots (10)$

Sub. (7), (8), (9) into (10): $1 + 2 + 3 + n = 15$

$n = 9$

9 girls passed in both examinations.

G2.4 求最小正整數 m，使得 $m^{200} > 6^{300}$。

Determine the least positive integer m such that $m^{200} > 6^{300}$.

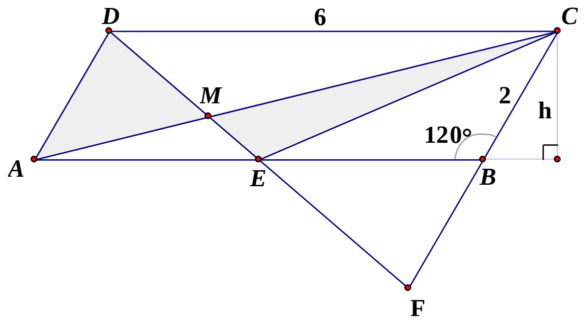
Reference: 1996 HI4, 1999 FG5.3, 2008 FI4.3

$(m^2)^{100} > (6^3)^{100}$ $m^2 > 216$ $m > \sqrt{216} > \sqrt{196} = 14$ m的最小正整數= 15	$(m^2)^{100} > (6^3)^{100}$ $m^2 > 216$ $m > \sqrt{216} > \sqrt{196} = 14$ The least positive integer $m = 15$

Group Event 3

G3.1 AC 是平行四邊形 $ABCD$ 的對角綫，$CD = 6$，$BC = 2$ 及 $\angle ABC = 120°$。若 E 是 AB 的中點，AC 與 DE 相交於 M 及陰影部分的總面積是α，求 α 的值。

$ABCD$ is a parallelogram with diagonal AC, $CD = 6$, $BC = 2$, and $\angle ABC = 120°$. If E is the midpoint of AB, AC and DE intersect at M, and the total area of the shaded regions in α, determine the value of α.



Reference 1998 HG5, 2016 HI14

$AB = 6$，$AE = 3$，假設平行四邊形的高為 h。 $S_{ABCD} = 6\times 2 \sin 120° = 6\sqrt{3}$ $h = 2 \sin 120° = \sqrt{3}$ $S_{\Delta ACD} = \frac{1}{2}\cdot 6h = 3\sqrt{3}$ $S_{\Delta ACE} = \frac{1}{2}\cdot 3h = \frac{3\sqrt{3}}{2}$ 將 DE 及 CB 延長並相交於 F。 $AE = EB$ (E 是 AB 的中點) $\angle AED = \angle BEF$ (對頂角) $\angle EAD = \angle EBF$ (交錯角，AD // CF) $\Delta ADE \cong \Delta BFE$ (A.S.A.) $BF = AD = 2$ (全等三角形的對應角) $CF = CB + BF = 2 + 2 = 4$ $\angle AMD = \angle CMF$ (對頂角) $\angle ADM = \angle CFM$ (交錯角，AD // CF) $\angle DAM = \angle FCM$ (交錯角，AD // CF) $\Delta ADM \sim \Delta FCM$ (等角) $AM : MC = AD : CF$ (相似三角形的對應邊) $AM : MC = 2 : 4 = 1 : 2$ ΔADM 與 ΔCDM 同高不同底 $S_{\Delta ADM} = \frac{1}{1+2}\cdot S_{\Delta ACD} = \frac{1}{3}\cdot 3\sqrt{3} = \sqrt{3}$ ΔAEM 與 ΔCEM 同高不同底 $S_{\Delta CEM} = \frac{2}{1+2}\cdot S_{\Delta ACE} = \frac{2}{3}\cdot\frac{3\sqrt{3}}{2} = \sqrt{3}$ 陰影面積 $\alpha = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$	$AB = 6$, $AE = 3$, let the height of the //-gram be h. $S_{ABCD} = 6\times 2 \sin 120° = 6\sqrt{3}$ $h = 2 \sin 120° = \sqrt{3}$ $S_{\Delta ACD} = \frac{1}{2}\cdot 6h = 3\sqrt{3}$ $S_{\Delta ACE} = \frac{1}{2}\cdot 3h = \frac{3\sqrt{3}}{2}$ Produce DE and CB to meet at F. $AE = EB$ (E is the mid-point of AB) $\angle AED = \angle BEF$ (vert. opp. $\angle$s) $\angle EAD = \angle EBF$ (alt. $\angle$s, AD // CF) $\Delta ADE \cong \Delta BFE$ (A.S.A.) $BF = AD = 2$ (corr. $\angle$s, $\cong \Delta$s) $CF = CB + BF = 2 + 2 = 4$ $\angle AMD = \angle CMF$ (vert. opp. $\angle$s) $\angle ADM = \angle CFM$ (alt. $\angle$s, AD // CF) $\angle DAM = \angle FCM$ (alt. $\angle$s, AD // CF) $\Delta ADM \sim \Delta FCM$ (equiangular) $AM : MC = AD : CF$ (corr. sides, ~Δs) $AM : MC = 2 : 4 = 1 : 2$ ΔADM and ΔCDM have the same height but different bases $S_{\Delta ADM} = \frac{1}{1+2}\cdot S_{\Delta ACD} = \frac{1}{3}\cdot 3\sqrt{3} = \sqrt{3}$ ΔAEM and ΔCEM have the same height but different bases $S_{\Delta CEM} = \frac{2}{1+2}\cdot S_{\Delta ACE} = \frac{2}{3}\cdot\frac{3\sqrt{3}}{2} = \sqrt{3}$ Shaded area $\alpha = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$

G3.2 設 β為三位正整數且能被 11 整除，且其商相等於其值的各數字之和的三倍，求 β 的值。
If β is a 3-digit positive integer that is divisible by 11 and whose quotient when divided by 11 is 3 times the sum of its digits, determine the value of β .

假設該數 $\beta = 100a + 10b + c$，其中 a、b 及 c 為 0 至 9 之間的整數及 $a \neq 0$。 它能被11整除 $\Rightarrow a + c - b = 11k$，$k = 0$ 或 1 $b = a + c - 11k$ …… (1) 代 (1) 入 $\beta = 100a + 10(a + c - 11k) + c$ $= 110a + 11c - 110k$ $= 11(10a + c - 10k)$ $Q =$ 商 $= 10a + c - 10k$ …… (2) 由已知資料， $Q = 3(a + b + c)$ …… (3) 代 (1) 及 (2) 入 (3): $10a + c - 10k = 3(2a + 2c - 11k)$ $4a + 23k = 5c$ 當 $k = 0$，$4a = 5c \Rightarrow a = 5$，$c = 4$，$b = a + c = 9$ $\beta = 594$ 當 $k = 1$，$4a + 23 = 5c$ 當 $c = 5$、6、8 或 9，對 a 沒有整數解 當 $c = 7$，$a = 3$，$b = a + c - 11k < 0$，捨去	Let the integer be $\beta = 100a + 10b + c$, where a, b and c are integers from 0, 1, ⋯ , 9 and $a \neq 0$. It is divisible by 11 $\Rightarrow a + c - b = 11k$, $k = 0$ or 1 $b = a + c - 11k$ …… (1) Sub. (1) into $\beta = 100a + 10(a + c - 11k) + c$ $= 110a + 11c - 110k$ $= 11(10a + c - 10k)$ $Q =$ quotient $= 10a + c - 10k$ …… (2) According to the given information, $Q = 3(a + b + c)$ …… (3) Sub. (1) and (2) into (3): $10a + c - 10k = 3(2a + 2c - 11k)$ $4a + 23k = 5c$ When $k = 0$, $4a = 5c \Rightarrow a = 5$, $c = 4$, $b = a + c = 9$ $\beta = 594$ When $k = 1$, $4a + 23 = 5c$ When $c = 5, 6, 8$ or 9, no integral solution for a When $c = 7$, $a = 3$, $b = a + c - 11k < 0$, rejected

Remark: original question: ⋯ β 的最大值 ⋯ largest value of β

β is uniquely found. There is no need to emphasize the word "largest".

G3.3 求 φ 的最大實數值，使不等式 $\sqrt{1-\varphi}-\sqrt{1+\varphi}\geq 1$ 成立。

Determine the largest real value of φ such that the inequality $\sqrt{1-\varphi}-\sqrt{1+\varphi}\geq 1$ holds.

為使表達式成立，$1 - \varphi \geq 0$ 及 $1 + \varphi \geq 0$ $\Rightarrow -1 \leq \varphi \leq 1$ $\sqrt{1-\varphi}\geq 1+\sqrt{1+\varphi} > 0$ 當 $\varphi \geq 0$，則 $1 > \sqrt{1-\varphi}\geq 1+\sqrt{1+\varphi} > 1$ 矛盾 $\therefore -1 \leq \varphi < 0$ $\left(\sqrt{1-\varphi}\right)^2 \geq \left(1+\sqrt{1+\varphi}\right)^2$ $1-\varphi \geq 1+2\sqrt{1+\varphi}+1+\varphi$ $-2\varphi-1 \geq 2\sqrt{1+\varphi} \geq 0$ $\left(-2\varphi-1\right)^2 \geq \left(2\sqrt{1+\varphi}\right)^2$ $4\varphi^2 + 4\varphi + 1 \geq 4(1 + \varphi)$ $4\varphi^2 \geq 3$ $-1 \leq \varphi \leq -\frac{\sqrt{3}}{2}$ φ 的最大實數值 $= -\frac{\sqrt{3}}{2}$	In order that the expression is well defined, $1 - \varphi \geq 0$ and $1 + \varphi \geq 0 \Rightarrow -1 \leq \varphi \leq 1$ $\sqrt{1-\varphi}\geq 1+\sqrt{1+\varphi} > 0$ If $\varphi \geq 0$, then $1 > \sqrt{1-\varphi}\geq 1+\sqrt{1+\varphi} > 1$!!! $\therefore -1 \leq \varphi < 0$ $\left(\sqrt{1-\varphi}\right)^2 \geq \left(1+\sqrt{1+\varphi}\right)^2$ $1-\varphi \geq 1+2\sqrt{1+\varphi}+1+\varphi$ $-2\varphi-1 \geq 2\sqrt{1+\varphi} \geq 0$ $\left(-2\varphi-1\right)^2 \geq \left(2\sqrt{1+\varphi}\right)^2$ $4\varphi^2 + 4\varphi + 1 \geq 4(1 + \varphi)$ $4\varphi^2 \geq 3$ $-1 \leq \varphi \leq -\frac{\sqrt{3}}{2}$ The largest value of $\varphi = -\frac{\sqrt{3}}{2}$.

G3.4 設 θ 及 γ 為正整數，當中 $\theta<\gamma$。若 $\frac{\theta+\gamma}{2}:\sqrt{\theta\gamma}=13:12$，求 γ 的最小值。

Suppose that θ and γ are positive integers, where $\theta<\gamma$.

If $\frac{\theta+\gamma}{2}:\sqrt{\theta\gamma}=13:12$, determine the least value of γ.

$6(\theta+\gamma)=13\sqrt{\theta\gamma}$ $36(\theta+\gamma)^2=169\theta\gamma$ $36\theta^2-97\theta\gamma+36\gamma^2=0$ $(4\theta-9\gamma)(9\theta-4\gamma)=0$ $\theta:\gamma=9:4$ or $4:9$ $\because$ θ 及 γ 為正整數及 $\theta<\gamma$ $\therefore$ $\theta:\gamma=4:9$ γ 的最小值= 9 (當 $\theta=4$)	$6(\theta+\gamma)=13\sqrt{\theta\gamma}$ $36(\theta+\gamma)^2=169\theta\gamma$ $36\theta^2-97\theta\gamma+36\gamma^2=0$ $(4\theta-9\gamma)(9\theta-4\gamma)=0$ $\theta:\gamma=9:4$ or $4:9$ $\because$ θ and γ are positive integers and $\theta<\gamma$ $\therefore$ $\theta:\gamma=4:9$ The least value of $\gamma=9$ (when $\theta=4$)

Group Event 4

G4.1 設 $X=\sqrt{2018-\sqrt{A}}$ 是正整數，求 A 的最大值。

Let $X=\sqrt{2018-\sqrt{A}}$ be a positive integer. Determine the largest value of A.

Reference: 2016 HI3, 2019 FG2.1

$45=\sqrt{2025}>\sqrt{2018-\sqrt{A}}$ $X=\sqrt{2018-\sqrt{A}}=1$、2、⋯、43 或 44 $2018-\sqrt{A}=1^2$、2^2、⋯ 、43^2 或 44^2 $\sqrt{A}=2018-1^2$、$2018-2^2$、⋯、或$2018-44^2$ $A=(2018-1^2)^2$、$(2018-2^2)^2$、⋯、或$(2018-44^2)^2$ A 的最大值$=2017^2=4068289$	$45=\sqrt{2025}>\sqrt{2018-\sqrt{A}}$ $X=\sqrt{2018-\sqrt{A}}=1, 2, \cdots, 43$ or 44 $2018-\sqrt{A}=1^2, 2^2, \cdots, 43^2$ or 44^2 $\sqrt{A}=2018-1^2, 2018-2^2, \cdots$, or $2018-44^2$ $A=(2018-1^2)^2, (2018-2^2)^2, \cdots$, or $(2018-44^2)^2$ The largest value of $A=2017^2=4068289$

G4.2 求方程 $(12x-1)(6x-1)(4x-1)(3x-1)=5$ 的所有實根之乘積 B 的值。

Determine the value of B, the product of all real roots of $(12x-1)(6x-1)(4x-1)(3x-1)=5$

方法一 方程式兩邊乘以24: $(12x-1)(12x-2)(12x-3)(12x-4)=120$ 設 $a=12x-1$，則方程式可寫成： $a(a-1)(a-2)(a-3)=120$ $(a^2-3a)(a^2-3a+2)=120$ $(a^2-3a)^2+2(a^2-3a)-120=0$ $(a^2-3a-10)(a^2-3a+12)=0$ $a=-2$ 或 5 或 沒有解 $12x-1=-2$ 或 $12x-1=5$ $x=-\frac{1}{12}$ 或 $\frac{1}{2}$ 所有實根之乘積$=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$	**Method 1** Multiply both sides by 24 : $(12x-1)(12x-2)(12x-3)(12x-4)=120$ Let $a=12x-1$, then the equation becomes: $a(a-1)(a-2)(a-3)=120$ $(a^2-3a)(a^2-3a+2)=120$ $(a^2-3a)^2+2(a^2-3a)-120=0$ $(a^2-3a-10)(a^2-3a+12)=0$ $a=-2$ or 5 or no solution $12x-1=-2$ or $12x-1=5$ $x=-\frac{1}{12}$ or $\frac{1}{2}$ Product of roots $=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$
方法二 將方程重新排列: $(12x-1)(3x-1)(6x-1)(4x-1)=5$ $(36x^2-15x+1)(24x^2-10x+1)=5$ 設 $t=12x^2-5x$，則方程式可寫成： $(3t+1)(2t+1)=5$ $6t^2+5t-4=0$ $(2t-1)(3t+4)=0$ $(24x^2-10x-1)(36x^2-15x+4)=0$ $(2x-1)(12x+1)=0$ 或 沒有解 $x=\frac{1}{2}$ 或 $-\frac{1}{12}$ 所有實根之乘積$=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$	**Method 2** Rearrange the equation as $(12x-1)(3x-1)(6x-1)(4x-1)=5$ $(36x^2-15x+1)(24x^2-10x+1)=5$ Let $t=12x^2-5x$, then the equation becomes: $(3t+1)(2t+1)=5$ $6t^2+5t-4=0$ $(2t-1)(3t+4)=0$ $(24x^2-10x-1)(36x^2-15x+4)=0$ $(2x-1)(12x+1)=0$ or no solution $x=\frac{1}{2}$ or $-\frac{1}{12}$ Product of roots $=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$

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方法三

該方程式可寫成：

$\left(x-\frac{1}{12}\right)\left(x-\frac{1}{6}\right)\left(x-\frac{1}{4}\right)\left(x-\frac{1}{3}\right)=\frac{5}{12\times 6\times 4\times 3}$

設 $y=\frac{1}{4}\left(x-\frac{1}{12}+x-\frac{1}{6}+x-\frac{1}{4}+x-\frac{1}{3}\right)$

所以，$y=x-\frac{5}{24}\Rightarrow x=y+\frac{5}{24}$

該方程式可寫成：

$\left(y+\frac{5}{24}-\frac{1}{12}\right)\left(y+\frac{5}{24}-\frac{1}{6}\right)\left(y+\frac{5}{24}-\frac{1}{4}\right)\left(y+\frac{5}{24}-\frac{1}{3}\right)=\frac{5}{864}$

$\left(y+\frac{3}{24}\right)\left(y+\frac{1}{24}\right)\left(y-\frac{1}{24}\right)\left(y-\frac{3}{24}\right)=\frac{5}{864}$

$\left(y^2-\frac{1}{24^2}\right)\left(y^2-\frac{3^2}{24^2}\right)=\frac{5}{864}$

設 $t=y^2-\frac{5}{24^2}\Rightarrow y^2=t+\frac{5}{24^2}$

該方程式可寫成：

$\left(t+\frac{4}{24^2}\right)\left(t-\frac{4}{24^2}\right)=\frac{5}{864}\Rightarrow\ t^2-\frac{16}{24^4}=\frac{5}{864}$

$t^2=\frac{1}{12^4}+\frac{5}{12^2\times 6}\times\frac{2}{2}\times\frac{12}{12}=\frac{121}{144^2}$

$t=\frac{11}{144}$ 或 $-\frac{11}{144}$

$y^2=\frac{11}{144}+\frac{5}{24^2}$ 或 $-\frac{11}{144}+\frac{5}{24^2}=\frac{49}{24^2}$ 或 $-\frac{39}{24^2}$

$y=\frac{7}{24}$ 或 $-\frac{7}{24}$

$x=\frac{7}{24}+\frac{5}{24}$ 或 $-\frac{7}{24}+\frac{5}{24}=\frac{1}{2}$ 或 $-\frac{1}{12}$

B =所有實根之乘積$=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$

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Method 3

The equation can be written as:

$\left(x-\frac{1}{12}\right)\left(x-\frac{1}{6}\right)\left(x-\frac{1}{4}\right)\left(x-\frac{1}{3}\right)=\frac{5}{12\times 6\times 4\times 3}$

Let $y=\frac{1}{4}\left(x-\frac{1}{12}+x-\frac{1}{6}+x-\frac{1}{4}+x-\frac{1}{3}\right)$

So, $y=x-\frac{5}{24}\Rightarrow x=y+\frac{5}{24}$

Then the equation becomes:

$\left(y+\frac{5}{24}-\frac{1}{12}\right)\left(y+\frac{5}{24}-\frac{1}{6}\right)\left(y+\frac{5}{24}-\frac{1}{4}\right)\left(y+\frac{5}{24}-\frac{1}{3}\right)=\frac{5}{864}$

$\left(y+\frac{3}{24}\right)\left(y+\frac{1}{24}\right)\left(y-\frac{1}{24}\right)\left(y-\frac{3}{24}\right)=\frac{5}{864}$

$\left(y^2-\frac{1}{24^2}\right)\left(y^2-\frac{3^2}{24^2}\right)=\frac{5}{864}$

Let $t=y^2-\frac{5}{24^2}\Rightarrow y^2=t+\frac{5}{24^2}$

The equation becomes:

$\left(t+\frac{4}{24^2}\right)\left(t-\frac{4}{24^2}\right)=\frac{5}{864}\Rightarrow\ t^2-\frac{16}{24^4}=\frac{5}{864}$

$t^2=\frac{1}{12^4}+\frac{5}{12^2\times 6}\times\frac{2}{2}\times\frac{12}{12}=\frac{121}{144^2}$

$t=\frac{11}{144}$ or $-\frac{11}{144}$

$y^2=\frac{11}{144}+\frac{5}{24^2}$ or $-\frac{11}{144}+\frac{5}{24^2}=\frac{49}{24^2}$ or $-\frac{39}{24^2}$

$y=\frac{7}{24}$ or $-\frac{7}{24}$

$x=\frac{7}{24}+\frac{5}{24}$ or $-\frac{7}{24}+\frac{5}{24}=\frac{1}{2}$ or $-\frac{1}{12}$

B = product of all real roots $=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$

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G4.3 求 $C=\cos\frac{\pi}{15}\times\cos\frac{2\pi}{15}\times\cos\frac{3\pi}{15}\times\cos\frac{4\pi}{15}\times\cos\frac{5\pi}{15}\times\cos\frac{6\pi}{15}\times\cos\frac{7\pi}{15}$ 的值。

Determine the value of $C=\cos\frac{\pi}{15}\times\cos\frac{2\pi}{15}\times\cos\frac{3\pi}{15}\times\cos\frac{4\pi}{15}\times\cos\frac{5\pi}{15}\times\cos\frac{6\pi}{15}\times\cos\frac{7\pi}{15}$.

設 Let $\alpha=\frac{\pi}{5}$, $\beta=\frac{\pi}{15}$

$$\cos\frac{7\pi}{15}=-\cos\left(\pi-\frac{7\pi}{15}\right)=-\cos\frac{8\pi}{15}=-\cos 8\beta$$

$$\cos\frac{3\pi}{15}=\cos\frac{\pi}{5}=\cos\alpha$$

$$\cos\frac{6\pi}{15}=\cos\frac{2\pi}{5}=\cos 2\alpha$$

$$\cos\frac{5\pi}{15}=\cos\frac{\pi}{3}=\frac{1}{2}$$

$$\cos\frac{\pi}{15}\times\cos\frac{2\pi}{15}\times\cos\frac{3\pi}{15}\times\cos\frac{4\pi}{15}\times\cos\frac{5\pi}{15}\times\cos\frac{6\pi}{15}\times\cos\frac{7\pi}{15}$$

$$=\cos\beta\times\cos 2\beta\times\cos\alpha\times\cos 4\beta\times\frac{1}{2}\times\cos 2\alpha\times(-\cos 8\beta)$$

$$=-\frac{1}{2}\times\cos\alpha\times\cos 2\alpha\times\cos\beta\times\cos 2\beta\times\cos 4\beta\times\cos 8\beta$$

$$=-\frac{1}{2}\times\frac{2\sin\alpha\cos\alpha\times\cos 2\alpha}{2\sin\alpha}\times\frac{2\sin\beta\cos\beta\times\cos 2\beta\times\cos 4\beta\times\cos 8\beta}{2\sin\beta}$$

$$=-\frac{1}{2}\times\frac{2\sin 2\alpha\times\cos 2\alpha}{4\sin\alpha}\times\frac{2\sin 2\beta\times\cos 2\beta\times\cos 4\beta\times\cos 8\beta}{4\sin\beta}$$

$$=-\frac{1}{2}\times\frac{\sin 4\alpha}{4\sin\alpha}\times\frac{2\sin 4\beta\times\cos 4\beta\times\cos 8\beta}{8\sin\beta}$$

$$=-\frac{1}{2}\times\frac{\sin\frac{4\pi}{5}}{4\sin\frac{\pi}{5}}\times\frac{2\sin 8\beta\times\cos 8\beta}{16\sin\beta}$$

$$=-\frac{1}{2}\times\frac{\sin\left(\pi-\frac{\pi}{5}\right)}{4\sin\frac{\pi}{5}}\times\frac{\sin 16\beta}{16\sin\beta}=-\frac{1}{2}\times\frac{\sin\frac{\pi}{5}}{4\sin\frac{\pi}{5}}\times\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}$$

$$=-\frac{1}{128}\times\frac{\sin\left(\pi+\frac{\pi}{15}\right)}{\sin\frac{\pi}{15}}=-\frac{1}{128}\times\frac{-\sin\frac{\pi}{15}}{\sin\frac{\pi}{15}}=\frac{1}{128}$$

$$C=\frac{1}{128}$$

G4.4 設 r、s 及 t 是正實數，且 $r^2 + s^2 + t^2 = rs + st + rt$。若 $r = 1$，求 $D = s + t$ 的值。

Let r, s and t be positive real numbers with $r^2 + s^2 + t^2 = rs + st + rt$.

If $r = 1$, determine the value of $D = s + t$.

Reference: 2005 FI4.1

$r^2 + s^2 + t^2 = rs + st + rt$ $2r^2 + 2s^2 + 2t^2 = 2rs + 2st + 2rt$ $(r^2 - 2rs + s^2) + (s^2 - 2st + t^2) + (t^2 - 2tr + r^2) = 0$ $(r - s)^2 + (s - t)^2 + (t - r)^2 = 0$ 3個非負數之和= 0 每個非負數= 0 $r = s$，$s = t$ 及 $t = r$ 當 $r = 1$，$s = t = 1$ $D = 1 + 1 = 2$	$r^2 + s^2 + t^2 = rs + st + rt$ $2r^2 + 2s^2 + 2t^2 = 2rs + 2st + 2rt$ $(r^2 - 2rs + s^2) + (s^2 - 2st + t^2) + (t^2 - 2tr + r^2) = 0$ $(r - s)^2 + (s - t)^2 + (t - r)^2 = 0$ sum of 3 non-negative numbers = 0 Each non-negative number = 0 $r = s$, $s = t$ and $t = r$ When $r = 1$, $s = t = 1$ $D = 1 + 1 = 2$