

Individual Events

I1	a	0	I2	a	8	I3	a	8	I4	a	2
	b	-1		b	64		b	2		b	1
	c	7		c	15936		c	10		c	$\frac{3}{44}$
	d	18		d	*5312 see the remark		d	1023		d	$\frac{1945}{3872}$

Group Events

G1	a	-1	G2	A	2	G3	R	0	G4	P	*18 see the remark
	b	3		B	28		S	-1		Q	$\frac{63}{512}$
	c	$\frac{1}{9}$		C	300		T	4		R	377
	d	393		D	11		U	$-2\sqrt{3}$		S	5

Individual Event 1

I1.1 若 a 為 $\frac{1}{(x+2)(x+3)} = \frac{1}{(x+1)(x+4)}$ 的實數解的數量，求 a 的值。

If a is the number of real roots of $\frac{1}{(x+2)(x+3)} = \frac{1}{(x+1)(x+4)}$, determine the value of a .

$$\begin{aligned}(x+1)(x+4) &= (x+2)(x+3) \\ x^2 + 5x + 4 &= x^2 + 5x + 6 \\ 0 &= 2 \\ \text{無解, } a &= 0\end{aligned}$$

$$\begin{aligned}(x+1)(x+4) &= (x+2)(x+3) \\ x^2 + 5x + 4 &= x^2 + 5x + 6 \\ 0 &= 2 \\ \text{No solution, } a &= 0\end{aligned}$$

I1.2 若 x 為實數及 b 為 $-|x-a-9|-|10-x|$ 的最大值，求 b 的值。

If x is a real number and b is the maximum value of $-|x-a-9|-|10-x|$, determine the value of b . (Reference: 2008, HI8, 2010 HG6, 2011 FGS.1, 2012 FG2.3, 2016 FI4.3)

$$\begin{aligned}\text{我們利用三角不等式 } |p| + |q| &\geq |p+q| \\ -|x-9| - |10-x| &= -(|x-9| + |10-x|) \\ &\leq -|x-9+10-x| = -1 \\ \therefore b &= -1\end{aligned}$$

$$\begin{aligned}\text{We use the triangle inequality } |p| + |q| &\geq |p+q| \\ -|x-9| - |10-x| &= -(|x-9| + |10-x|) \\ &\leq -|x-9+10-x| = -1 \\ \therefore b &= -1\end{aligned}$$

I1.3 若實數 x 及 y 滿足 $4x^2 + 4y^2 + 9xy = -119b$ ，求 xy 的最大值 c 。

If real numbers x and y satisfy $4x^2 + 4y^2 + 9xy = -119b$, determine c , the maximum value of xy .

$$\begin{aligned}4x^2 + 4y^2 + 9xy &= 119 \\ 119 &\geq 2\sqrt{(2x)^2 \cdot (2y)^2} + 9xy \quad (\text{A.M.} \geq \text{G.M.}) \\ 119 &\geq 17xy \\ 7 &\geq xy \\ c &= 7\end{aligned}$$

11.4 若正實數 x 滿足方程 $x^2 + \frac{1}{x^2} = c$ ，求 $d = x^3 + \frac{1}{x^3}$ 。

If a positive real number x satisfies $x^2 + \frac{1}{x^2} = c$, determine the value of $d = x^3 + \frac{1}{x^3}$.

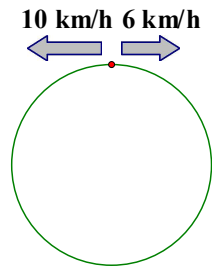
Reference: 1985 FI1.2, 1990 HI12

$x^2 + \frac{1}{x^2} = 7$ $\Rightarrow \left(x + \frac{1}{x}\right)^2 = 9$ $\Rightarrow x + \frac{1}{x} = 3 \text{ or } -3 \text{ (捨去, } \because x > 0)$ $d = x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right)$ $= 3 \times (7 - 1) = 18$	$x^2 + \frac{1}{x^2} = 7$ $\Rightarrow \left(x + \frac{1}{x}\right)^2 = 9$ $\Rightarrow x + \frac{1}{x} = 3 \text{ or } -3 \text{ (rejected, } \because x > 0)$ $d = x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right)$ $= 3 \times (7 - 1) = 18$
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Individual Event 2

- 12.1** 兩個學生於長 1-km 的圓形跑道的起點開始分別以 10 km/h 及 6 km/h 的速率跑沿相反方向跑步。當他們於起點再相遇時便停止跑步。若 a 為他們開始及停止前相互經過的次數，求 a 的值。

Two students run in opposite directions from a starting point of a 1-km circular track at speeds of 10 km/h and 6 km/h, respectively. They stop running when they meet each other at the starting point again. If a is number of times they cross each other after they start and before they stop, determine the value of a .



在半小時內，他們分別經過起點 5 次和 3 次。 總跑步距離 = $(5 + 3)\text{km} = 8\text{ km}$ 總相遇次數 = $a = 8$ 他們相遇在 $\frac{5}{8}, \frac{10}{8}, \frac{15}{8}, \frac{20}{8}, \frac{25}{8}, \frac{30}{8}, \frac{35}{8}, \frac{40}{8}$ 。	In half an hour, they will pass the starting point 5 times and 3 times respectively. Total distance travelled = $(5 + 3)\text{km} = 8\text{ km}$ Number of times they meet = $a = 8$ They meet at $\frac{5}{8}, \frac{10}{8}, \frac{15}{8}, \frac{20}{8}, \frac{25}{8}, \frac{30}{8}, \frac{35}{8}, \frac{40}{8}$.
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- 12.2** 袋中有若干粒紅色及藍色的彈珠，紅色彈珠與藍色彈珠的比例為 3 : 1。若加入 a 粒藍色彈珠，紅色彈珠與藍色彈珠的比例則為 2 : 1。求彈珠的總數 b 。

There is a set of red marbles and blue marbles. When a red marbles are added to the set, the ratio of red marbles to the blue marbles is 3 : 1. When a blue marbles are added, the ratio of red marbles to blue marbles becomes 2 : 1. Determine the total number of marbles, b .

假設原本有 $3k$ 粒紅色彈珠及 k 粒藍色彈珠。 $3k : (k + 8) = 2 : 1$ $3k = 2k + 16$ $k = 16$ 彈珠的總數 = $b = 4k = 64$	Let the original number of red marbles and blue marbles be $3k$ and k respectively. $3k : (k + 8) = 2 : 1$ $3k = 2k + 16$ $k = 16$ The total number of marbles = $b = 4k = 64$
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- 12.3** 若 c 為 1 000 000 與一個平方數之最小的相差，其中此平方數為 b 的倍數，求 c 的值。

If c is the smallest difference between 1 000 000 and a square, where the square is a multiple of b , determine the value of c .

假設該平方數為 $64n^2$ 。 $1000 = 8 \times 125$ $1\ 000\ 000 = 64 \times 125^2$ $1\ 000\ 000 - 64n^2 = 64 \times (125^2 - n^2)$ $64 \times (125^2 - 124^2)$ 或 $64 \times (126^2 - 125^2)$ $64 \times (125 + 124)$ 或 $64 \times (126 + 125)$ 最小值 = $c = 64 \times 249 = 15936$	Let the square be $64n^2$. $1000 = 8 \times 125$ $1\ 000\ 000 = 64 \times 125^2$ $1\ 000\ 000 - 64n^2 = 64 \times (125^2 - n^2)$ $64 \times (125^2 - 124^2)$ or $64 \times (126^2 - 125^2)$ $64 \times (125 + 124)$ or $64 \times (126 + 125)$ Minimum value = $c = 64 \times 249 = 15936$
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12.4 於一個月的時間完成建築一個水庫需要 d 個技工或 y 個勞工，當中 $d + y = c$ 。

若挑選 d 個勞工去建築一個同樣的水庫，所需要的時間是挑選 y 個技工的 4 倍，求 d 的值。

The building of a reservoir takes d technicians, or alternatively y labours to complete in a month, where $d + y = c$. If d labours are employed to build the same reservoir, the time taken is 4 times as much as the time taken when y technicians are employed. Determine the value of d .

$d + y = 15936$	$d + y = 15936$
每名技工每天工作量 $= \frac{1}{30d}$.	Amount of work for one technician per day $= \frac{1}{30d}$.
每名勞工每天工作量 $= \frac{1}{30y}$.	Amount of work for one labour per day $= \frac{1}{30y}$.
d 名勞工完成工程所需日數 $= 1 \div \frac{d}{30y} = \frac{30y}{d}$	Days for d labours to finish the job $= 1 \div \frac{d}{30y} = \frac{30y}{d}$
y 名技工完成工程所需日數 $= 1 \div \frac{y}{30d} = \frac{30d}{15936 - d}$	Days for y technicians to finish the job $= 1 \div \frac{y}{30d} = \frac{30d}{15936 - d}$
$\frac{30(15936 - d)}{d} = \frac{4 \times 30d}{15936 - d}$	$\frac{30(15936 - d)}{d} = \frac{4 \times 30d}{15936 - d}$
$(15936 - d)^2 = 4d^2$	$(15936 - d)^2 = 4d^2$
$15936 - d = 2d$	$15936 - d = 2d$
$d = 5312$	$d = 5312$

Remark: The Chinese version and the English version have different meaning.

Original version: …所需要的時間較挑選 y 個技工的 **多** 4 倍… the time taken is 4 times as much as …

New version: …所需要的時間 **是** 挑選 y 個技工的 4 倍… the time taken is 4 times as much as …

Individual Event 3

I3.1 若 $\{x_0, y_0, z_0\}$ 為以下方程組的解，求 $a = x_0 + y_0 + z_0$ 的值。

If $\{x_0, y_0, z_0\}$ is a solution to the set of simultaneous equations below, determine the value of $a = x_0 + y_0 + z_0$.

$$\begin{cases} 2x - 2y + z = -15 \\ x + 2y + 2z = 18 \\ 2x - y + 2z = -5 \end{cases}$$

$$\begin{bmatrix} 2 & -2 & 1 & -15 \\ 1 & 2 & 2 & 18 \\ 2 & -1 & 2 & -5 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_1 \\ R_3 - R_1 \\ 2R_2 - R_3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 2 & 18 \\ 0 & 1 & 1 & 10 \\ 0 & 5 & 2 & 41 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ 5R_2 - R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & 3 & 9 \end{bmatrix} \begin{matrix} R_2 - \frac{1}{3}R_3 \\ \\ \frac{1}{3}R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x_0 = -2, y_0 = 7, z_0 = 3$$

$$a = -2 + 7 + 3 = 8$$

I3.2 求 $b = \frac{\sqrt{6+2\sqrt{a}} + \sqrt{6-2\sqrt{a}}}{2}$ 的值。

Determine the value of $b = \frac{\sqrt{6+2\sqrt{a}} + \sqrt{6-2\sqrt{a}}}{2}$.

Reference: 2011 HI7, 2013 FI1, 2013 FI3.1, 2015 FG3.1, 2016 FG4.3

$$\begin{aligned} b &= \frac{\sqrt{6+2\sqrt{8}} + \sqrt{6-2\sqrt{8}}}{2} \\ &= \frac{\sqrt{(\sqrt{4})^2 + 2\sqrt{4} \cdot \sqrt{2} + (\sqrt{2})^2} + \sqrt{(\sqrt{4})^2 - 2\sqrt{4} \cdot \sqrt{2} + (\sqrt{2})^2}}{2} \\ &= \frac{\sqrt{(\sqrt{4} + \sqrt{2})^2} + \sqrt{(\sqrt{4} - \sqrt{2})^2}}{2} \\ &= \frac{\sqrt{4} + \sqrt{2} + \sqrt{4} - \sqrt{2}}{2} = 2 \end{aligned}$$

I3.3 若 x 是正整數且 $\log_{10} b^x > 3$ ，求 x 的最小值 c 。

If x is a positive integer and $\log_{10} b^x > 3$, determine c , the minimum value of x .

$$\log_{10} 2^x > 3 = \log_{10} 1000$$

$$2^9 = 512 < 1000 < 1024 = 2^{10}$$

$$c = 10$$

I3.4 若 $f(x) = 2^0 + 2^1 + 2^2 + \dots + 2^{x-2} + 2^{x-1}$ ，求 $d = f(c)$ 的值。

If $f(x) = 2^0 + 2^1 + 2^2 + \dots + 2^{x-2} + 2^{x-1}$, determine the value of $d = f(c)$.

Reference: 2009 FI1.3, 2015 FI1.4

$$d = f(10) = 2^0 + 2^1 + 2^2 + \dots + 2^8 + 2^9$$

$$= 2^{10} - 1$$

$$= 1023$$

Individual Event 4

14.1 若 a 為正整數，求 a 的最大值使得 $ax^2 - (a-3)x + (a-2) = 0$ 有實根。

If a is a positive integer, determine the greatest value of a such that $ax^2 - (a-3)x + (a-2) = 0$ has real root(s).

$\Delta = (a-3)^2 - 4a(a-2) \geq 0$ $a^2 - 6a + 9 - 4a^2 + 8a \geq 0$ $3a^2 - 2a - 9 \leq 0$ Let $3a^2 - 2a - 9 = 0$ $a = \frac{2 \pm \sqrt{2^2 + 4(3)(9)}}{2 \cdot 3}$ $a = \frac{1 \pm \sqrt{28}}{3}$ $a \approx \frac{1 \pm 5.3}{3} = 2.1 \text{ or } -1.4$ $-1.4 \leq a \leq 2.1$ a 的最大整數值 = 2。	$\Delta = (a-3)^2 - 4a(a-2) \geq 0$ $a^2 - 6a + 9 - 4a^2 + 8a \geq 0$ $3a^2 - 2a - 9 \leq 0$ Let $3a^2 - 2a - 9 = 0$ $a = \frac{2 \pm \sqrt{2^2 + 4(3)(9)}}{2 \cdot 3}$ $a = \frac{1 \pm \sqrt{28}}{3}$ $a \approx \frac{1 \pm 5.3}{3} = 2.1 \text{ 或 } -1.4$ $-1.4 \leq a \leq 2.1$ The largest integral value of $a = 2$.
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14.2 若 x 及 y 為實數且 $1 < y < x$ 及 $\log_x y + 3 \log_y x = \frac{13}{a}$ ，求 $b = \frac{x+y^4}{x^2+y^2}$ 的值。

If x and y are real numbers with $1 < y < x$ and $\log_x y + 3 \log_y x = \frac{13}{a}$,

determine the value of $b = \frac{x+y^4}{x^2+y^2}$.

設 $t = \log_x y$, 則 $\log_y x = \frac{1}{t}$ 原式變成: $t + \frac{3}{t} = \frac{13}{2}$ $2t^2 - 13t + 6 = 0$ $(2t-1)(t-6) = 0$ $t = \frac{1}{2}$ 或 $t = 6$ $\therefore 1 < y < x \therefore \log_x y < 1$ $\log_x y = \frac{1}{2}$ $y = \sqrt{x}$ $b = \frac{x+y^4}{x^2+y^2} = \frac{x+x^2}{x^2+x} = 1$	Let $t = \log_x y$, then $\log_y x = \frac{1}{t}$ The equation becomes: $t + \frac{3}{t} = \frac{13}{2}$ $2t^2 - 13t + 6 = 0$ $(2t-1)(t-6) = 0$ $t = \frac{1}{2}$ or $t = 6$ $\therefore 1 < y < x \therefore \log_x y < 1$ $\log_x y = \frac{1}{2}$ $y = \sqrt{x}$ $b = \frac{x+y^4}{x^2+y^2} = \frac{x+x^2}{x^2+x} = 1$
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14.3 一個袋中有紅球 $b+2$ 個，白球 $b+3$ 個及藍球 $b+4$ 個，從袋中隨機抽出 3 個並不重新放進袋中。求三個抽出的球都是相同顏色的概率 c 的值。

A bag contains $b+2$ red balls, $b+3$ white balls and $b+4$ blue balls. Three balls are randomly drawn from the bag without replacement. Determine the value of the probability, c , that the 3 balls are of the same colours.

紅球 = 3、白球 = 4、藍球 = 5 $P(\text{同色}) = \frac{3 \times 2 \times 1 + 4 \times 3 \times 2 + 5 \times 4 \times 3}{12 \times 11 \times 10}$ $= \frac{3}{44}$	Red balls = 3, White balls = 4, Blue balls = 5 $P(\text{same colour}) = \frac{3 \times 2 \times 1 + 4 \times 3 \times 2 + 5 \times 4 \times 3}{12 \times 11 \times 10}$ $= \frac{3}{44}$
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14.4 若 $\cos 2\theta = c$ ，求 $d = \sin^4 \theta + \cos^4 \theta$ 的值。

If $\cos 2\theta = c$, determine the value of $d = \sin^4 \theta + \cos^4 \theta$.

$$d = (\cos^2 \theta + \sin^2 \theta)^2 - 2 \sin^2 \theta \cos^2 \theta$$

$$d = 1 - 0.5 \sin^2 2\theta$$

$$d = 1 - \frac{1}{2}(1 - \cos^2 2\theta)$$

$$d = \frac{1}{2} \left(1 + \frac{3^2}{44^2} \right)$$

$$d = \frac{1936 + 9}{2 \cdot 1936}$$

$$d = \frac{1945}{3872}$$

Group Event 1

G1.1 若實數 x 、 y 及 z 滿足 $x + \frac{1}{y} = -1$ ， $y + \frac{1}{z} = -2$ 及 $z + \frac{1}{x} = -5$ 。求 $a = \frac{1}{xyz}$ 的值。

If real numbers x , y and z satisfy $x + \frac{1}{y} = -1$, $y + \frac{1}{z} = -2$ and $z + \frac{1}{x} = -5$. Determine the value of a

$= \frac{1}{xyz}$. (Reference: 2008 FG2.4, 2010 FG2.2)

$(1) \times (2) \times (3) - (1) - (2) - (3)$:

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} - \left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) = -10 + 8 \Rightarrow xyz + \frac{1}{xyz} = -2$$

$$(xyz + 1)^2 = 0$$

$$a = \frac{1}{xyz} = -1$$

G1.2 若 $|x - |2x - 1|| = \frac{1}{2}$ 為實數方程，求實根數量 b 的值。

If $|x - |2x - 1|| = \frac{1}{2}$ is a real equation, determine the value of b , the number of real solutions of the equation.

Reference: 2012 FG4.3, 2005 FG4.2, 2009 HG9, 2012 FG4.2

$x - 2x - 1 = \frac{1}{2}$ 或 $x - 2x - 1 = -\frac{1}{2}$	$x - 2x - 1 = \frac{1}{2}$ or $x - 2x - 1 = -\frac{1}{2}$
$x - \frac{1}{2} = 2x - 1 $ 或 $x + \frac{1}{2} = 2x - 1 $	$x - \frac{1}{2} = 2x - 1 $ or $x + \frac{1}{2} = 2x - 1 $
$2x - 1 = \pm \left(x - \frac{1}{2} \right)$ 或 $2x - 1 = \pm \left(x + \frac{1}{2} \right)$	$2x - 1 = \pm \left(x - \frac{1}{2} \right)$ or $2x - 1 = \pm \left(x + \frac{1}{2} \right)$
$x = \frac{1}{2}$ 或 $x = \frac{3}{2}, \frac{1}{6}$	$x = \frac{1}{2}$ or $x = \frac{3}{2}, \frac{1}{6}$
$b = \text{實根數量} = 3$	$b = \text{number of real solution} = 3$

G1.3 若實數 x 及 y 滿足 $xy > 0$ 及 $x + y = 3$, 求 $\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)$ 的最大值 c 。

If real numbers x and y satisfy $xy > 0$ and $x + y = 3$,

find c , the maximum value of $\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right)$.

$\because xy > 0, x + y = 3$ $\therefore x > 0, y > 0$ $0 < xy \leq \left(\frac{x+y}{2}\right)^2 = \frac{9}{4}$ (A.M. \geq G.M.) $0 < xy \leq 1$ 或 $1 \leq xy \leq \frac{9}{4}$ $\frac{1}{xy} \geq 1$ 或 $\frac{4}{9} \leq \frac{1}{xy} \leq 1 \Rightarrow \frac{4}{9} \leq \frac{1}{xy}$ $-\frac{4}{9} \geq -\frac{1}{xy}$ $\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right) = \frac{xy - (x+y) + 1}{xy} = 1 - \frac{2}{xy} \leq \frac{1}{9} = c$	$\because xy > 0, x + y = 3$ $\therefore x > 0, y > 0$ $0 < xy \leq \left(\frac{x+y}{2}\right)^2 = \frac{9}{4}$ (A.M. \geq G.M.) $0 < xy \leq 1$ or $1 \leq xy \leq \frac{9}{4}$ $\frac{1}{xy} \geq 1$ or $\frac{4}{9} \leq \frac{1}{xy} \leq 1 \Rightarrow \frac{4}{9} \leq \frac{1}{xy}$ $-\frac{4}{9} \geq -\frac{1}{xy}$ $\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right) = \frac{xy - (x+y) + 1}{xy} = 1 - \frac{2}{xy} \leq \frac{1}{9} = c$
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G1.4 若實數 x 滿足 $x - \frac{1}{x} = 3$, 求 $d = x^5 - \frac{1}{x^5}$ 的值。

If a real number x satisfies $x - \frac{1}{x} = 3$, determine the value of $d = x^5 - \frac{1}{x^5}$.

$$\left(x - \frac{1}{x}\right)^2 = 3^2 = 9$$

$$x^2 + \frac{1}{x^2} = 11$$

$$\left(x^2 + \frac{1}{x^2}\right)^2 = 11^2 = 121$$

$$x^4 + \frac{1}{x^4} = 119$$

$$d = x^5 - \frac{1}{x^5} = \left(x - \frac{1}{x}\right)\left(x^4 + x^2 + 1 + \frac{1}{x^2} + \frac{1}{x^4}\right)$$

$$d = 3 \times (119 + 11 + 1)$$

$$d = 393$$

Group Event 2**G2.1** 在六進制中，若 A 為 $12345_6 \div 13_6$ 的餘數，求 A 的值。In base-6 system, if $12345_6 \div 13_6$ has remainder A , determine the value of A .

$$12345_6 = 6^4 + 2 \times 6^3 + 3 \times 6^2 + 4 \times 6 + 5 = 1296 + 2 \times 216 + 108 + 24 + 5 = 1865_{10}$$

$$13_6 = 9_{10}$$

$$1865 \div 9, A = 2$$

G2.2 立方體的任意兩個頂點可相連成一線段。若 B 為最多所能夠相連成的直線的數量，求 B 的值。Any two vertices in a cube can form a line segment. If B is the greatest number of line segments thus formed, determine the value of B .

立方體有 8 個頂點。 從中選取兩點形成一線段。 $B = C_2^8 = 28$	There are 8 vertices in a cube. Select any two vertices to form a line segment. $B = C_2^8 = 28$
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G2.3 若實數 x 、 y 及 z 滿足 $(x + y + z) = 30$ 及 $C = x^2 + y^2 + z^2$ ，求 C 的最小值。If real numbers x , y and z satisfy $(x + y + z) = 30$ and $C = x^2 + y^2 + z^2$, determine the least value of C .

考慮 $t^2 - 2xt + x^2 = (t - x)^2 \cdots (1)$ $t^2 - 2yt + y^2 = (t - y)^2 \cdots (2)$ $t^2 - 2zt + z^2 = (t - z)^2 \cdots (3)$ $(1) + (2) + (3):$ $\text{L.H.S.} = 3t^2 - 2(x + y + z)t + (x^2 + y^2 + z^2)$ 此函數必為非負 $\Delta = 4(x + y + z)^2 - 4(3)(x^2 + y^2 + z^2) \leq 0$ $(1^2 + 1^2 + 1^2)(x^2 + y^2 + z^2) \geq (x + y + z)^2$ $3C \geq 30^2$ $C \geq 300$ C 的最小值 = 300	Consider $t^2 - 2xt + x^2 = (t - x)^2 \cdots (1)$ $t^2 - 2yt + y^2 = (t - y)^2 \cdots (2)$ $t^2 - 2zt + z^2 = (t - z)^2 \cdots (3)$ $(1) + (2) + (3):$ $\text{L.H.S.} = 3t^2 - 2(x + y + z)t + (x^2 + y^2 + z^2)$ The function is always non-negative $\Delta = 4(x + y + z)^2 - 4(3)(x^2 + y^2 + z^2) \leq 0$ $(1^2 + 1^2 + 1^2)(x^2 + y^2 + z^2) \geq (x + y + z)^2$ $3C \geq 30^2$ $C \geq 300$ The minimum value of $C = 300$
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G2.4 已知 $D = (x - 1)^3 + 3$ 。當 $-3 \leq x \leq 3$ ，求 D 的最大值。**Given that** $D = (x - 1)^3 + 3$. Determine the greatest value of D for $-3 \leq x \leq 3$.

$-4 \leq x - 1 \leq 2$ $-64 \leq (x - 1)^3 \leq 8$ $-61 \leq (x - 1)^3 + 3 \leq 11$ D 的最大值 = 11	$-4 \leq x - 1 \leq 2$ $-64 \leq (x - 1)^3 \leq 8$ $-61 \leq (x - 1)^3 + 3 \leq 11$ The greatest value of $D = 11$
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Group Event 3

G3.1 設 a 、 b 及 c 為整數且 $1 < a < b < c$ 。若 $(ab-1)(bc-1)(ac-1)$ 可被 abc 整除，求 $ab+bc+ac-1$ 除以 abc 所得之餘數 R 的值。

Let a , b and c be integers with $1 < a < b < c$. If $(ab-1)(bc-1)(ac-1)$ is divisible by abc , determine the value of the remainder R when $ab+bc+ac-1$ is divided by abc .

$(ab-1)(bc-1)(ac-1)$ $= (abc)^2 - abc(a+b+c) + (ab+bc+ca) - 1$ 它可被 abc 整除。 $\therefore ab+bc+ca-1 = mabc$, m 為整數。 餘數 $R = 0$	$(ab-1)(bc-1)(ac-1)$ $= (abc)^2 - abc(a+b+c) + (ab+bc+ca) - 1$ It is divisible by abc . $\therefore ab+bc+ca-1 = mabc$, m is an integer The remainder $R = 0$
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G3.2 若 $0 < x < 1$ ，求 $S = \left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}} + \frac{1-x}{\sqrt{1-x^2}+x-1} \right) \cdot \left(\sqrt{\frac{1}{x^2}-1} - \frac{1}{x} \right)$ 的值。

If $0 < x < 1$, determine the value of $S = \left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}} + \frac{1-x}{\sqrt{1-x^2}+x-1} \right) \cdot \left(\sqrt{\frac{1}{x^2}-1} - \frac{1}{x} \right)$.

Reference: 2016 FI3.3

$$\begin{aligned}
 S &= \left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}} + \frac{1-x}{\sqrt{1-x^2}+x-1} \right) \times \left(\sqrt{\frac{1}{x^2}-1} - \frac{1}{x} \right) \\
 &= \left\{ \frac{\sqrt{1+x} \cdot (\sqrt{1+x} + \sqrt{1-x})}{(1+x) - (1-x)} + \frac{(1-x) \cdot [\sqrt{1-x^2} - (x-1)]}{(1-x^2) - (x-1)^2} \right\} \times \left(\sqrt{\frac{1-x^2}{x^2}} - \frac{1}{x} \right) \\
 &= \left\{ \frac{1+x+\sqrt{1-x^2}}{2x} + \frac{(1-x) \cdot [\sqrt{1-x^2} + (1-x)]}{(1-x^2) - (1-2x+x^2)} \right\} \times \left(\frac{\sqrt{1-x^2}}{x} - \frac{1}{x} \right) \\
 &= \left\{ \frac{1+x+\sqrt{1-x^2}}{2x} + \frac{(1-x) \cdot [\sqrt{1-x^2} + (1-x)]}{2x(1-x)} \right\} \times \left(\frac{\sqrt{1-x^2}-1}{x} \right) \\
 &= \left[\frac{1+x+\sqrt{1-x^2}}{2x} + \frac{\sqrt{1-x^2} + (1-x)}{2x} \right] \times \left(\frac{\sqrt{1-x^2}-1}{x} \right) \\
 &= \left(\frac{2+2\sqrt{1-x^2}}{2x} \right) \times \left(\frac{\sqrt{1-x^2}-1}{x} \right) = \left(\frac{1+\sqrt{1-x^2}}{x} \right) \times \left(\frac{\sqrt{1-x^2}-1}{x} \right) \\
 &= \frac{(1-x^2)-1}{x^2} = -1
 \end{aligned}$$

Remark: You may substitute $x = 0.5$ directly to find the value of c .

G3.3 求方程 $x^4 + (x-4)^4 = 544$ 的實根之和 T 的值。

Determine the value of T , the sum of real roots of $x^4 + (x-4)^4 = 544$.

Reference: 2014 FG4.4

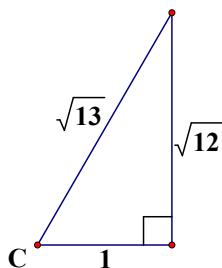
設 $t = 2 - x$ ，則 $x = t + 2$ ， $x - 4 = t - 2$ 方程變成： $(t+2)^4 + (t-2)^4 = 544$ $2[t^4 + 6(2)^2t^2 + 2^4] = 544$ $t^4 + 24t^2 - 256 = 0$ $(t^2 + 32)(t^2 - 8) = 0$ $t^2 = -32$ (捨去) or $t^2 = 8$ $x = 2 \pm 2\sqrt{2}$ $T = \text{實根之和} = 4$	Let $t = 2 - x$, then $x = t + 2$, $x - 4 = t - 2$ The equation becomes: $(t+2)^4 + (t-2)^4 = 544$ $2[t^4 + 6(2)^2t^2 + 2^4] = 544$ $t^4 + 24t^2 - 256 = 0$ $(t^2 + 32)(t^2 - 8) = 0$ $t^2 = -32$ (rejected) or $t^2 = 8$ $x = 2 \pm 2\sqrt{2}$ $T = \text{Sum of real roots} = 4$
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G3.4 在三角形 ABC 中， $BC = a$ ， $\angle ABC = \frac{\pi}{3}$ 及面積為 $\sqrt{3}a^2$ 。求 $U = \tan(\angle ACB)$ 的值。

In triangle ABC , $BC = a$, $\angle ABC = \frac{\pi}{3}$ and its area is $\sqrt{3}a^2$.

Determine the value of $U = \tan(\angle ACB)$.

<p>設 $AB = c$</p> $\frac{1}{2}ac \sin \frac{\pi}{3} = \sqrt{3}a^2$ $c = 4a$ $BC^2 = a^2 + (4a)^2 - 2a \cdot 4a \cos \frac{\pi}{3}$ $BC = \sqrt{13}a$ $\cos \angle ACB = \frac{a^2 + (\sqrt{13}a)^2 - (4a)^2}{2 \cdot a \cdot \sqrt{13}a} = -\frac{1}{\sqrt{13}}$ $U = \tan \angle ACB = -\frac{\sqrt{13-1}}{1} = -2\sqrt{3}$	<p>Let $AB = c$</p> $\frac{1}{2}ac \sin \frac{\pi}{3} = \sqrt{3}a^2$ $c = 4a$ $BC^2 = a^2 + (4a)^2 - 2a \cdot 4a \cos \frac{\pi}{3}$ $BC = \sqrt{13}a$ $\cos \angle ACB = \frac{a^2 + (\sqrt{13}a)^2 - (4a)^2}{2 \cdot a \cdot \sqrt{13}a} = -\frac{1}{\sqrt{13}}$ $U = \tan \angle ACB = -\frac{\sqrt{13-1}}{1} = -2\sqrt{3}$
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Group Event 4

G4.1 製作某玩具，需要先倒模，後上色。甲先生每日可以為 3 件玩具倒模，或為 15 件玩具上色；乙先生每日則可以為 5 件玩具倒模，或為 15 件玩具上色。各人每日只能倒模或上色，而不能同做兩事。若甲先生和乙先生合作，求最小多少日 P 才可以製作 120 件玩具。

To make a specific toy, it must be first moulded and then painted. Mr. A can mould 3 pieces of toys or paint 15 pieces of toys in one day, whereas Mr. B can mould 5 pieces or paint 15 pieces of toys in one day. Each of them can either mould or paint toys in one day, but not both. If Mr. A and Mr. B work together, determine the least number of days P to make 120 toys.

<p>A倒模的速度比B慢，而A上色的速度和B一樣。因此，為了要使得製作120件玩具的日數最短，B所有時間皆被指派去倒模。假設A花了x日倒模，y日上色。</p> <p>$3x + 5(x + y) = 120 \Rightarrow 8x + 5y = 120 \dots\dots (1)$</p> <p>$15y = 120 \Rightarrow y = 8 \dots\dots (2)$</p> <p>代 (2) 入 (1): $8x + 40 = 120$</p> <p>$x = 10$</p> <p>最小日數 $P = 18$</p>	<p>The speed of making moulds for A is slower than B while the speed of painting for A is the same as B. So, in order to minimize the number of days to make 120 toys, all time of B is allocated for moulding. Let A uses x days for moulding and y days for painting.</p> <p>$3x + 5(x + y) = 120 \Rightarrow 8x + 5y = 120 \dots\dots (1)$</p> <p>$15y = 120 \Rightarrow y = 8 \dots\dots (2)$</p> <p>Sub. (2) into (1): $8x + 40 = 120$</p> <p>$x = 10$</p> <p>The least number of days $P = 18$</p>
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Remark: The following sentence is missing in the Chinese version:

各人每日只能倒模或上色，而不能同做兩事。

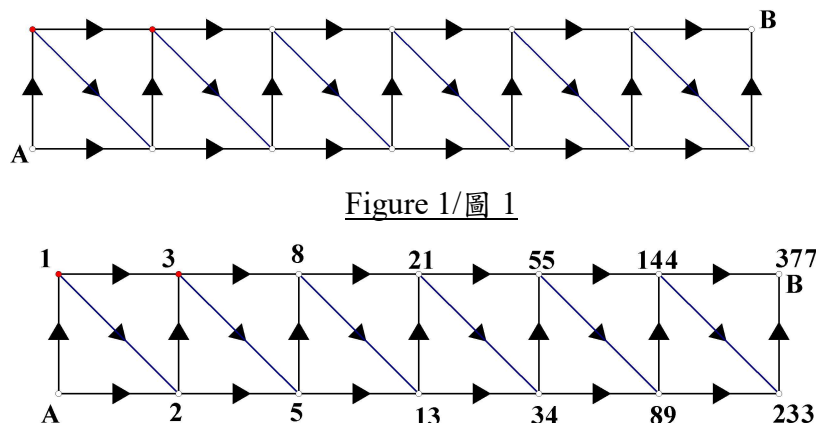
G4.2 在一個射鴨子遊戲中一男孩射了 10 發子彈，該男孩每發子彈射中鴨子的概率為 0.5。求他於最後一發子彈射中第六隻鴨子的概率 Q 。

In a duck shooting game, a boy fires 10 shots. The probability of him shooting down a duck with a shot is 0.5. Determine the probability Q of him shooting down the 6th duck at the last shot.

<p>P(最後一發子彈射中第六隻鴨子)</p> <p>$= P(\text{頭9發射中5隻, 第10發射中1隻})$</p> <p>$= \frac{C_5^9}{2^9} \cdot \frac{1}{2} = \frac{63}{512}$</p>	<p>P(shoot down the 6th duck at the last shot)</p> <p>$= P(1-9 \text{ shots } 5 \text{ ducks, last shot } 6\text{th duck})$</p> <p>$= \frac{C_5^9}{2^9} \cdot \frac{1}{2} = \frac{63}{512}$</p>
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G4.3 如圖 1，求按箭咀方向由 A 往 B 的路線總數 R 。

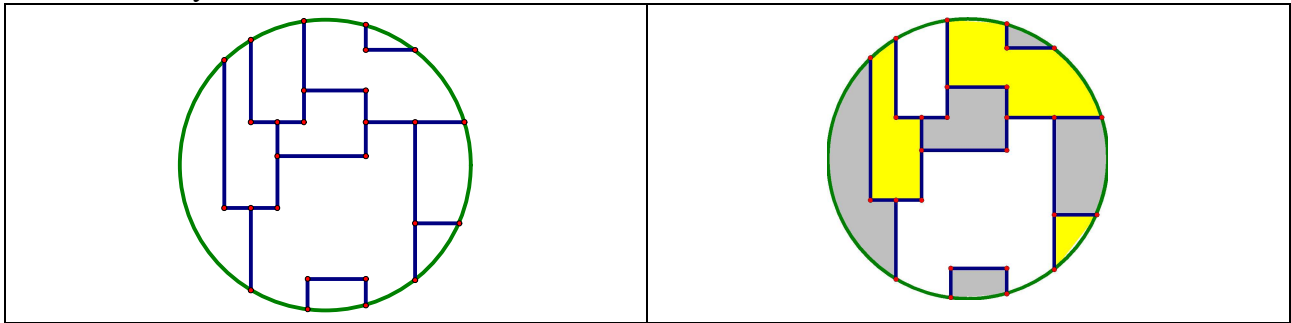
As shown in Figure 1 below, determine the number of ways R getting from point A to B with the direction indicated by the arrows.



$$R = 377$$

G4.4 如果用 3 款顏料替下圖中所有區域著色，並且相鄰的區域不可用相同顏料。求同一款顏料最多可用作上色的區域數目 S 。

To shade all the regions inside the following circular map using 3 colours, for which adjacent regions must not be in the same colour. Determine the maximum number S of regions being shaded by the same colour.



$S = 5$