

89-90 Individual	1	5	2	-2	3	7	4	6	5	(2, 4)
	6	4	7	120	8	-7	9	100	10	109
	11	9	12	0	13	2519	14	2	15	10 days
	16	5	17	$2\sqrt{13}$	18	1 : 2	19	$\frac{9}{20}$	20	58.5

89-90 Group	1	275	2	73	3	2	4	0	5	1783
	6	26	7	$\frac{125}{8} = 15\frac{5}{8}$	8	4 : 1	9	7	10	2π

Individual Events

- I1** Find the value of $\frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2}$.

$$\begin{aligned} & \frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2} \\ &= 3 + \sqrt{8} - (\sqrt{8} + \sqrt{7}) + (\sqrt{7} + \sqrt{6}) - (\sqrt{6} + \sqrt{5}) + (\sqrt{5} + 2) \\ &= 5 \end{aligned}$$

- I2** If $b < 0$ and $2^{2b+4} - 20 \times 2^b + 4 = 0$, find b .

Let $y = 2^b$, then $y^2 = 2^{2b}$, the equation becomes $16y^2 - 20y + 4 = 0$

$$4y^2 - 5y + 1 = 0$$

$$(4y - 1)(y - 1) = 0$$

$$y = 2^b = \frac{1}{4} \text{ or } y = 1$$

$$b = -2 \text{ or } 0$$

$$\because b < 0 \therefore b = -2 \text{ only}$$

- I3** If $f(a) = a - 2$ and $F(a, b) = a + b^2$, find $F(3, f(4))$.

Reference: 1985 FI3.3, 2013 FI3.2, 2015 FI4.3

$$f(4) = 4 - 2 = 2$$

$$F(3, f(4)) = F(3, 2) = 3 + 2^2 = 7$$

- I4** For positive integers a and b , define $a \# b = a^b + b^a$. If $2 \# w = 100$, find the value of w .

Reference: 1999 FI3.1

$$2^w + w^2 = 100 \text{ for positive integer } w.$$

$$\text{By trail and error, } 64 + 36 = 100$$

$$w = 6.$$

- I5** a and b are constants. The straight line $2ax + 3by = 4a + 12b$ passes through a fixed point P whose coordinates do not depend on a and b . Find the coordinates of P .

Reference: 1991 HI6, 1996 HI6

$$2ax + 3by = 4a + 12b \Rightarrow 2a(x - 2) + 3b(y - 4) = 0$$

$$\text{Put } b = 0 \Rightarrow x = 2,$$

$$\text{Put } a = 0 \Rightarrow y = 4$$

$$P(2, 4)$$

- I6** The sines of the angles of a triangle are in the ratio 3 : 4 : 5. If A is the smallest interior angle of the triangle and $\cos A = \frac{x}{5}$, find the value of x .

Reference: 1989 HI10

By Sine rule, $a : b : c = \sin A : \sin B : \sin C = 3 : 4 : 5$

Let $a = 3k$, $b = 4k$, $c = 5k$.

$$a^2 + b^2 = (3k)^2 + (4k)^2 = (5k)^2 = c^2$$

$\therefore \angle C = 90^\circ$ (converse, Pythagoras' theorem)

$$\cos A = \frac{b}{c} = \frac{4}{5}$$

$$\Rightarrow x = 4$$

- I7** If $x + y = 9$, $y + z = 11$ and $z + x = 10$, find the value of xyz .

Reference: 1986 FG10.1, 1989 HI15

$$(1) + (2) - (3): 2y = 10 \Rightarrow y = 5$$

$$(1) + (3) - (2): 2x = 8 \Rightarrow x = 4$$

$$(2) + (3) - (1): 2z = 12 \Rightarrow z = 6$$

$$\Rightarrow xyz = 120$$

- I8** If α, β are the roots of the equation $2x^2 + 4x - 3 = 0$ and α^2, β^2 are the roots of the equation $x^2 + px + q = 0$, find the value of p .

$$\alpha + \beta = -2$$

$$\alpha\beta = -\frac{3}{2}$$

$$\begin{aligned} p &= -(\alpha^2 + \beta^2) = -(\alpha + \beta)^2 + 2\alpha\beta \\ &= -(-2)^2 - 3 = -7 \end{aligned}$$

- I9** If $x^{\log_{10} x} = \frac{x^3}{100}$ and $x > 10$, find the value of x . **Reference: 2015 FI4.4, 2023 FG4.1**

Take log on both sides, $\log x \cdot \log x = 3 \log x - \log 100$

$$(\log x)^2 - 3 \log x + 2 = 0$$

$$(\log x - 1)(\log x - 2) = 0$$

$$\log x = 1 \text{ or } \log x = 2$$

$$x = 10 \text{ or } 100$$

$$\because x > 10 \therefore x = 100 \text{ only}$$

- I10** Given that $a_0 = 1$, $a_1 = 3$ and $a_n^2 - a_{n-1}a_{n+1} = (-1)^n$ for positive integers n . Find a_4 .

$$\text{Put } n = 1, \quad a_1^2 - a_0a_2 = (-1)^1 \Rightarrow 3^2 - a_2 = -1 \Rightarrow a_2 = 10$$

$$\text{Put } n = 2, \quad a_2^2 - a_1a_3 = (-1)^2 \Rightarrow 10^2 - 3a_3 = 1 \Rightarrow a_3 = 33$$

$$\text{Put } n = 3, \quad a_3^2 - a_2a_4 = (-1)^3 \Rightarrow 33^2 - 10a_4 = -1 \Rightarrow a_4 = 109$$

- I11** Find the units digit of 2137^{754} .

Reference 1991 HG1

$$7^1 = 7, 7^2 = 49, 7^3 = 343, 7^4 = 2401$$

The pattern of units digit repeats for every multiples of 4.

$$2137^{754} \equiv (7^4)^{188} \cdot 7^2 \equiv 9 \pmod{10}$$

The units digit is 9.

I12 If $\left(r + \frac{1}{r}\right)^2 = 3$, find $r^3 + \frac{1}{r^3}$.

Reference: 1985 FI1.2, 2017 FI1.4

$$r + \frac{1}{r} = \pm\sqrt{3}$$

$$r^2 + \frac{1}{r^2} = \left(r + \frac{1}{r}\right)^2 - 2 = 3 - 2 = 1$$

$$\begin{aligned} r^3 + \frac{1}{r^3} &= \left(r + \frac{1}{r}\right)\left(r^2 - 1 + \frac{1}{r^2}\right) \\ &= \pm\sqrt{3}(1-1) = 0 \end{aligned}$$

- I13** A positive integer N , when divided by 10, 9, 8, 7, 6, 5, 4, 3 and 2, leaves remainders 9, 8, 7, 6, 5, 4, 3, 2 and 1 respectively. Find the least value of N .

Reference: 1985 FG7.2, 2013FG4.3

$N + 1$ is divisible by 10, 9, 8, 7, 6, 5, 4, 3, 2.

The L.C.M. of 2, 3, 4, 5, 6, 7, 8, 9, 10 is 2520.

$\therefore N = 2520k - 1$, where k is an integer.

The least positive integral of $N = 2520 - 1 = 2519$

I14 If $\frac{1}{A} = \frac{\cos 45^\circ \sin 70^\circ \cos 60^\circ \tan 40^\circ}{\cos 340^\circ \sin 135^\circ \tan 220^\circ}$, find the value of A .

Reference: 1989 HI14

$$\begin{aligned} \frac{1}{A} &= \frac{\cos 45^\circ \cos 20^\circ \cos 60^\circ \tan 40^\circ}{\cos 20^\circ \cos 45^\circ \tan 40^\circ} \\ &= \cos 60^\circ = \frac{1}{2} \end{aligned}$$

$$A = 2$$

- I15** If 10 men can make 20 tables in 5 days, how many days are required to make 60 tables by 15 men?

$$1 \text{ man can make } \frac{20}{10 \times 5} = \frac{2}{5} \text{ table in 1 day.}$$

$$15 \text{ men can make } \frac{2}{5} \times 15 = 6 \text{ tables in one day.}$$

They can make 60 tables in 10 days

- I16** In figure 1, the exterior angles of the triangle are in the ratio $x' : y' : z' = 4 : 5 : 6$ and the interior angles are in the ratio $x : y : z = a : b : 3$. Find the value of b .

$$\text{Let } x' = 4k, y' = 5k, z' = 6k$$

$$4k + 5k + 6k = 360^\circ \text{ (sum of ext. } \angle \text{ of polygon)}$$

$$15k = 360^\circ$$

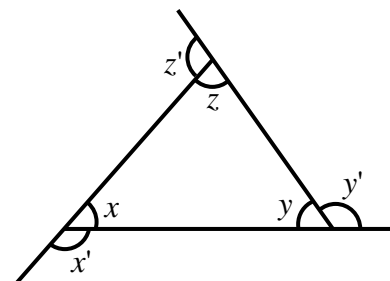
$$\Rightarrow k = 24$$

$$x' = 96^\circ, y' = 120^\circ, z' = 144^\circ$$

$$x = 84^\circ, y = 60^\circ, z = 36^\circ \text{ (adj. } \angle \text{ s on st. line)}$$

$$x : y : z = 7 : 5 : 3$$

$$\Rightarrow b = 5$$



(Figure 1)

- I17** In $\triangle ABC$, $\angle C = 90^\circ$ and D, E are the mid-points of BC and CA respectively. If $AD = 7$ and $BE = 4$, find the length of AB .
(See figure 2.)

Let $BD = x = DC$, $AE = y = EC$

$$x^2 + (2y)^2 = 7^2 \dots\dots (1)$$

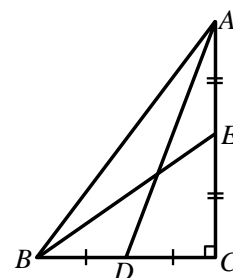
$$(2x)^2 + y^2 = 4^2 \dots\dots (2)$$

$$4(1) - (2): 15y^2 = 180 \Rightarrow y^2 = 12$$

$$4(2) - (1): 15x^2 = 15 \Rightarrow x^2 = 1$$

$$AB^2 = (2x)^2 + (2y)^2 = 4 + 48$$

$$\Rightarrow AB = \sqrt{52} = 2\sqrt{13}$$



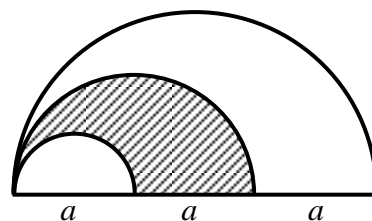
(Figure 2)

- I18** Figure 3 shows 3 semi-circles of diameters a , $2a$ and $3a$ respectively. Find the ratio of the area of the shaded part to that of the unshaded part.

$$\text{Area of the shaded part} = \frac{\pi}{2} \cdot a^2 - \frac{\pi}{2} \cdot \left(\frac{a}{2}\right)^2 = \frac{3\pi}{8} a^2$$

$$\text{Area of the unshaded part} = \frac{\pi}{2} \cdot \left(\frac{3a}{2}\right)^2 - \frac{3\pi}{8} \cdot a^2 = \frac{6\pi}{8} \cdot a^2$$

$$\text{The ratio} = 3 : 6 = 1 : 2$$



(Figure 3)

- I19** Find the value of $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{19 \times 20}$.

$$\begin{aligned} & \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots + \frac{1}{19 \times 20} \\ &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{19} - \frac{1}{20}\right) \\ &= \frac{1}{2} - \frac{1}{20} = \frac{9}{20} \end{aligned}$$

- I20** In figure 4, $\angle C = 90^\circ$, $AD = DB$ and DE is perpendicular to AB . If $AB = 20$ and $AC = 12$, find the area of the quadrilateral $ADEC$.

$BD = 10$, $BC = 16$ (Pythagoras' theorem)

$\triangle BDE \sim \triangle BCA$ (equiangular)

$BD : DE : BE = 16 : 12 : 20$ (ratio of sides, $\sim \Delta$'s)

$DE = 7.5$, $BE = 12.5$

$CE = 16 - 12.5 = 3.5$

$$S_{ADEC} = \frac{1}{2} \cdot 10 \cdot 7.5 + \frac{1}{2} \cdot 12 \cdot 3.5 = 58.5$$

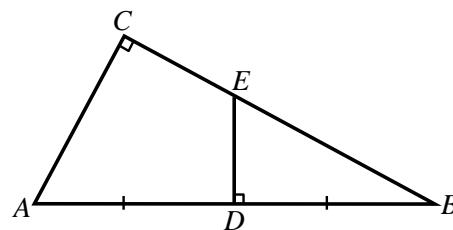
Method 2

$BD = 10$, $BC = 16$ (Pythagoras' theorem)

$\triangle BDE \sim \triangle BCA$ (equiangular)

$$S_{\triangle BDE} = \left(\frac{BD}{BC}\right)^2 \cdot S_{\triangle ABC} = \left(\frac{10}{16}\right)^2 \cdot \frac{1}{2} \cdot 12 \cdot 16 = 37.5$$

$$S_{ADEC} = \frac{1}{2} \cdot 12 \cdot 16 - 37.5 = 58.5$$



(Figure 4)

Group Events

G1 If $\frac{1}{a} + \frac{1}{b} = 5$ and $\frac{1}{a^2} + \frac{1}{b^2} = 13$, find the value of $\frac{1}{a^5} + \frac{1}{b^5}$.

$$(1)^2 - (2): \frac{2}{ab} = 12$$

$$\Rightarrow ab = \frac{1}{6} \dots\dots (4)$$

$$\text{From (1): } (a+b) \cdot \frac{1}{ab} = 5 \dots\dots (5)$$

$$\text{Sub. (4) into (5): } 6(a+b) = 5$$

$$\Rightarrow a+b = \frac{5}{6} \dots\dots (6)$$

From (4) and (6), a and b are roots of $6t^2 - 5t + 1 = 0$

$$(2t-1)(3t-1) = 0$$

$$\Rightarrow t = \frac{1}{2} \text{ or } \frac{1}{3}$$

$$\frac{1}{a^5} + \frac{1}{b^5} = 2^5 + 3^5$$

$$= 32 + 243 = 275$$

G2 There are N pupils in a class.

When they are divided into groups of 4, 1 pupil is left behind.

When they are divided into groups of 5, 3 pupils are left behind.

When they are divided into groups of 7, 3 pupils are left behind.

Find the least value of N .

Reference: 1992 HG4

$$N = 4p + 1 \dots\dots (1), p \text{ is an integer}$$

$$N = 5q + 3 \dots\dots (2), q \text{ is an integer}$$

$$N = 7r + 3 \dots\dots (3), r \text{ is an integer}$$

$$(3) - (2): 7r = 5q$$

$$r = 5k, q = 7k, \text{ where } k \text{ is an integer}$$

$$N = 35k + 3 = 4p + 1$$

$$4p - 35k = 2$$

By trial and error,

$$p = 18, k = 2 \text{ is a solution}$$

$$N = 73$$

- G3** The coordinates of A , B , C and D are $(10, 1)$, $(1, 7)$, $(-2, 1)$ and $(1, 3)$ respectively. AB and CD meet at P . Find the value of $\frac{AP}{PB}$.

Reference: 1989 HG5

$$\text{Equation of } AB: \frac{y-1}{x-10} = \frac{1-7}{10-1}$$

$$\Rightarrow 2x + 3y - 23 = 0 \dots\dots (1)$$

$$\text{Equation of } CD: \frac{y-1}{x+2} = \frac{3-1}{1+2}$$

$$\Rightarrow 2x - 3y + 7 = 0 \dots\dots (2)$$

$$(1) + (2): 4x - 16 = 0$$

$$\Rightarrow x = 4$$

$$(1) - (2): 6y - 30 = 0$$

$$\Rightarrow y = 5$$

$$\text{Let } \frac{AP}{PB} = r$$

$$4 = \frac{10+r}{1+r}$$

$$\Rightarrow 4 + 4r = 10 + r$$

$$\Rightarrow r = 2$$

- G4** Find the remainder when $2^{1989} + 1$ is divided by 3.

$$2^{1989} + 1 = (3 - 1)^{1989} + 1 = 3m - 1 + 1, \text{ binomial theorem, } m \text{ is an integer.}$$

The remainder is 0.

Method 2

$$2^1 + 1 = 3 \equiv 0 \pmod{3}, 2^2 + 1 = 5 \equiv 2 \pmod{3}, 2^3 + 1 \equiv 0 \pmod{3}, 2^4 + 1 \equiv 2 \pmod{3}$$

The pattern of the remainder repeats for every multiples of 2.

$$2^{1989} + 1 \equiv 2^1 + 1 \equiv 0 \pmod{3}$$

$$\Rightarrow \text{the remainder} = 0$$

- G5** Euler was born and died between 1700 A.D. and 1800 A.D. He was $n + 9$ years old in n^3 A.D. and died at the age of 76. Find the year in which Euler died.

Suppose he was born in x years after 1700 A.D.

$$1700 + x + n + 9 - 1 = n^3 \dots\dots (1)$$

$$11^3 = 1331, 12^3 = 1728, 13^3 > 1800$$

$$\therefore n = 12, x = 1728 - 1700 - 12 - 9 + 1 = 8$$

$$1700 + x + 76 - 1 = 1783$$

$$\Rightarrow \text{He was died in A.D. 1783.}$$

G6 Let $N!$ denotes the product of the first N natural numbers, i.e. $N! = 1 \times 2 \times 3 \times \dots \times N$.

If k is a positive integer such that $30! = 2^k \times$ an odd integer, find k .

Reference: 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FI1.4, 2012 FG1.3

2, 4, 6, 8, ..., 30 each has at least one factor of 2. Subtotal = 15

4, 8, ..., 28 each has at least 2 factors of 2. Subtotal = 7

8, 16, 24 each has at least 3 factors of 2. Subtotal = 3

16 has 4 factors of 2. Subtotal = 1

Total number of factors of 2 = $15 + 7 + 3 + 1 = 26$

G7 The graph of the parabola $y = x^2 - 4x - \frac{9}{4}$ cuts the x -axis at A and B (figure 1). If C is the vertex of the parabola, find the area of $\triangle ABC$.

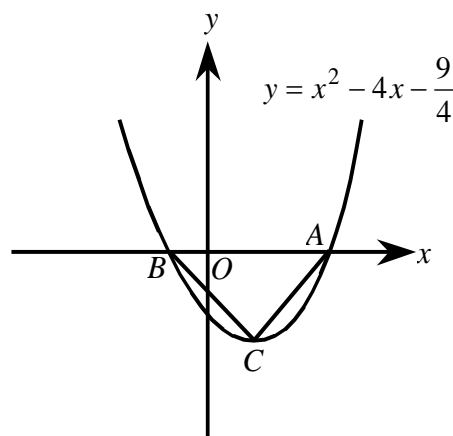
Let the roots be α, β , where $\alpha > \beta$.

$AB = \alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

$$= \sqrt{4^2 + 4 \cdot \frac{9}{4}} = 5$$

$$\text{Minimum} = \frac{4ac - b^2}{4a} = \frac{4\left(-\frac{9}{4}\right) - (-4)^2}{4} = -\frac{25}{4}$$

$$\text{Area of } \triangle ABC = \frac{1}{2} \times \frac{25}{4} \times 5 = \frac{125}{8}$$



(Figure 1)

G8 In figure 2, $FE \parallel BC$ and $ED \parallel AB$. If $AF : FB = 1 : 4$, find the ratio of area of $\triangle EDC$: area of $\triangle DEF$.

Reference: 1989 HI17

$BDEF$ is a parallelogram formed by 2 pairs of parallel lines

$\triangle DEF \cong \triangle FBD$ (A.S.A.)

Let $S_{\triangle DEF} = x = S_{\triangle FBD}$ (where S stands for the area)

$\triangle AEF \sim \triangle ACB$ ($\because FE \parallel BC$, equiangular)

$$\frac{S_{\triangle AEF}}{S_{\triangle ACB}} = \left(\frac{1}{1+4}\right)^2 = \frac{1}{25} \quad \dots\dots (1)$$

$\therefore AE : EC = AF : FB = 1 : 4$ (theorem of equal ratio)

$\because DE \parallel AB$

$\therefore AE : EC = BD : DC = 1 : 4$ (theorem of equal ratio)

$\triangle CDE \sim \triangle CBA$ ($\because DE \parallel BA$, equiangular)

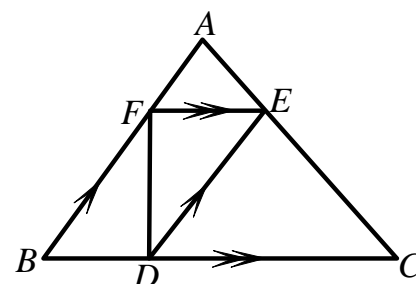
$$\frac{S_{\triangle CDE}}{S_{\triangle CBA}} = \left(\frac{4}{1+4}\right)^2 = \frac{16}{25} \quad \dots\dots (2)$$

Compare (1) and (2) $S_{\triangle AEF} = k$, $S_{\triangle CDE} = 16k$, $S_{\triangle ABC} = 25k$

$$k + 16k + x + x = 25k$$

$$x = 4k$$

$$\Rightarrow \text{area of } \triangle DEF : \text{area of } \triangle ABC = 16 : 4 = 4 : 1$$



(Figure 2)

- G9** In the attached multiplication (figure 3), the letters O, L, Y, M, P, I, A and D represent different integers ranging from 1 to 9. Find the integer $\times \frac{D}{O O O O O O O O O}$ represented by A .

$$1^2 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 16, 5^2 = 25, 6^2 = 36, 7^2 = 49, 8^2 = 64, 9^2 = 81$$

Possible $(D, O) = (2, 4), (3, 9), (4, 6), (7, 9), (8, 4), (9, 1)$

When $D = 2, O = 4, (OLYMPIAD) = 444444444 \div 2 = 222222222$ rejected

When $D = 3, O = 9, (OLYMPIAD) = 999999999 \div 3 = 333333333$ rejected

When $D = 4, O = 6, (OLYMPIAD) = 666666666 \div 4 = 166666666.5$ rejected

When $D = 7, O = 9, (OLYMPIAD) = 999999999 \div 7 = 142857142.7$ rejected

When $D = 8, O = 4, (OLYMPIAD) = 444444444 \div 8 = 55555555.5$ rejected

When $D = 9, O = 1, (OLYMPIAD) = 111111111 \div 9 = 12345679$

$A = 7$

- G10** Three circles, with centres A, B and C respectively, touch one another as shown in figure 4. If A, B and C are collinear and PQ is a common tangent to the two smaller circles, where $PQ = 4$, find the area of the shaded part in terms of π .

Reference: 2018FG1.2

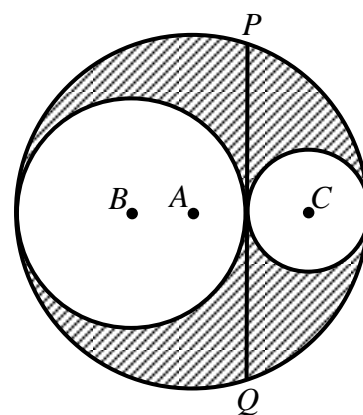
Let the radii of the 3 circles with centres A, B and C be a, b, c .

$$2b + 2c = 2a \Rightarrow a = b + c \dots\dots (1)$$

By intersecting chords theorem, $2c \times 2b = 2^2$

$$bc = 1 \dots\dots (2)$$

$$\begin{aligned} \text{Shaded area} &= \pi a^2 - \pi b^2 - \pi c^2 \\ &= \pi[a^2 - (b^2 + c^2)] \\ &= \pi[a^2 - (b + c)^2 + 2bc] \\ &= \pi(a^2 - a^2 + 2) \text{ by (1) and (2)} \\ &= 2\pi \end{aligned}$$



(Figure 4)