Examples on Mathematical Induction: Miscellaneous

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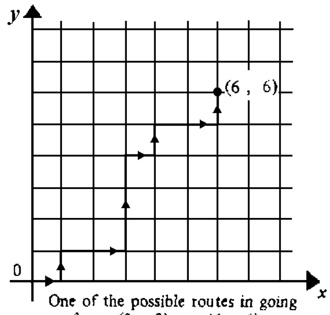
Reference: Essential Additional Mathematics p.45-47

1. Let $f(x + y) = f(x) + f(y)$ for all x, y, prove that $f(p) = pf(1)$ for all rational numbers p.
2. **1974 (Syllabus B) Paper 2 Q12**

The lines in the above figure represent the streets of a city. A boy wants to go from (0, 0) to (x, y), where x, y are non-negative integers, moving in the positive directions of x and y only.

Let the number of his possible routes be denoted by $f(x, y)$.



One of the possible routes in going from (0 , 0) to (6 , 6).

(i) Evaluate $f(k, 1)$.

(ii) Show that $f(k + 1, 2) = f(k, 2) + f(k + 1, 1)$.

(iii) By means of mathematical induction or otherwise, prove that

$f(m, 2) = \frac{1}{2}(m^2 + 3m + 2)$ for all non-negative integral values of m.

(i) Possible routes $(0,0)\to(0,1)\to(k,1)$ or $(0,0)\to(1,0)\to(1,1)\to(1,k)$ or $(0,0)\to(2,0)\to(2,1)\to(2,k)$ or $\cdots$ or $(0,0)\to(k,0)\to(k,1)$

$f(k, 1) = k + 1$

(ii) The final position = $(k+1,2)$

The position of one step just before reaching the final position may be at $(k, 2)$ or $(k +1, 1)$.

$\therefore f(k + 1, 2) = f(k, 2) + f(k + 1, 1)$

(iii) Let $P(m) \equiv$ "$f(m, 2) = \frac{1}{2}(m^2 + 3m + 2)$ for all non-negative integral values of m."

$m = 0$, $f(0, 2) = 1$ (The only possible route $(0, 0) \to (0,2)$

R.H.S. $= \frac{1}{2}(m^2 + 3m + 2) = \frac{1}{2}(0^2 + 3\times 0 + 2) = 1 =$ L.H.S.

$\therefore P(0)$ is true

Suppose $P(k)$ is true

i.e. $f(k, 2) = \frac{1}{2}(k^2 + 3k + 2)$ for some non-negative integral values of k.

When $m = k + 1$, by (ii),

$$\begin{aligned} f(k + 1, 2) &= f(k, 2) + f(k + 1, 1) \\ &= \frac{1}{2}(k^2 + 3k + 2) + (k + 2) \text{ (by induction assumption and (i))} \\ &= \frac{1}{2}(k^2 + 3k + 2 + 2k + 4) \\ &= \frac{1}{2}(k^2 + 5k + 6) \end{aligned}$$

$$\begin{aligned} \text{R.H.S.} &= \frac{1}{2}[(k + 1)^2 + 3(k + 1) + 2] \\ &= \frac{1}{2}(k^2 + 2k + 1 + 3k + 3 + 2) \\ &= \frac{1}{2}(k^2 + 5k + 6) = \text{L.H.S.} \end{aligned}$$

If $P(k)$ is true then $P(k + 1)$ is also true

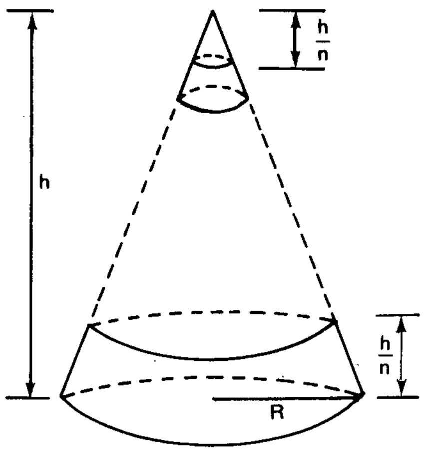
By the principle of mathematical induction, $P(m)$ is true for all non-negative integer m .

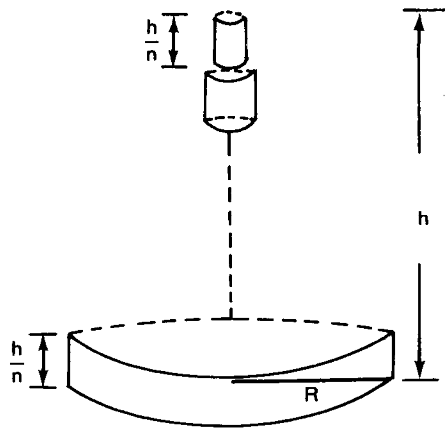
3. (a) Prove, by mathematical induction, that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all positive integers n.

(b) Tom was asked to calculate the volume of right circular cone of height h and base radius R. He knows the formula for the volume of a right cylinder, but not the one for the volume of a circular cone. Therefore, he used the following approximation method.

He divided the cone into n portions, each of height $\dfrac{h}{n}$, by planes parallel to the base. Then, he calculated the volume of each portion as if it was a circular cylinder with height $\dfrac{h}{n}$. In each case, he took the larger of the two plane surfaces as the base of the cylinder. Finally, he added up the volumes of all the portions, and obtained

$V_n = \dfrac{1}{6}\pi R^2 h\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right)$.





(i) Prove the equality for Tom.

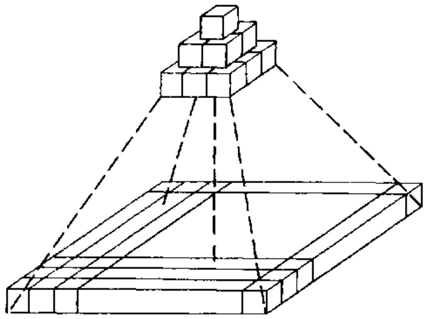
(ii) Find $\lim_{n\to\infty} V_n$ and check whether $\lim_{n\to\infty} V_n$ is the same as the volume of the right circular cone.

4. A man borrows \$1000 and has to pay 10% interest per year on the money that he owes. At the end of each year he pays back a fixed sum of \$$x$. He continues to pay until he has repaid the debt, together with the interest, exactly at the end of n years.
   (a) If \$$P_k$ is the amount owing at the end of the k^{th} year after the payment of \$$x$ has been made(i.e. $P_0 = 1000$ and $P_n = 0$), show that
   (i) $P_1 = 1.1P_0 - x$,
   (ii) $P_{k+1} = 1.1P_k - x$.
   Hence, prove by mathematical induction that $P_k = (1.1)^k (P_0 - 10x) + 10x$.
   (b) If he repays the sum and interest in 7 years exactly, how much does he pay each year, to the nearest dollar?
5. Prove the following statement is false: "$n^2 + n + 17$ is a prime, where n is a positive integer."
6. Prove the following statement is false: "$n^2 + n + 41$ is a prime, where n is a positive integer."
7. Prove the following statement is false: "$n^2 + n + 72491$ is a prime, where n is a positive integer."
8. Prove the following statement is false: "2^{2^n+1} is a prime, where n is a positive integer."
   (Hint: $2^{2^5+1} = 641\times 6700417$)
9. Prove that the following statement is false:
   "$1^3 + 2^3 + \cdots + n^3 = \left[\frac{1}{2}n(n+1)\right]^2 + m$ for all positive integer n." For any constant m.
10. Prove the following statement is false:
   "$2 + 6 + 10 + \cdots + 2(2n - 1) = 2n^2 + 2$, where n is a positive integer."
   (Hint: assume $P(k)$ is true, prove that $P(k + 1)$ is also true, but $P(1)$ is false.)
11. Consider the following identities:
   $x - 1 = x - 1$
   $x^2 - 1 = (x - 1)(x + 1)$
   $x^3 - 1 = (x - 1)(x^2 + x + 1)$
   $x^4 - 1 = (x - 1)(x + 1)(x^2 + 1)$
   $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$
   $x^6 - 1 = (x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1)$
   ……………………………………………
   Let $P(n) \equiv$ "The coefficients of all irreducible real factors of $x^n - 1$ are either 0, 1, or –1 ."
   The statement is false because one of the irreducible factor of $x^{105} - 1$ is
   $x^{48} + x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} - x^{40} - x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} - x^9 - x^8 - 2x^7 - x^6 - x^5 + x^2 + x + 1$.
   Here the coefficient of x^{41} and x^7 are –2, not 0, 1 or –1.
12. If a plane contains n lines such that (a) no two of them are parallel and (b) no three of them are concurrent; prove that it is divided into $1+\frac{1}{2}n(n+1)$ regions by these n lines.
13. In the expansion of $(x_1 + x_2 + \cdots + x_n)^\ell$, if $\ell_1, \ell_2, \cdots, \ell_n \geq 0$ and $\ell_1 + \ell_2 + \cdots + \ell_n = \ell$, prove by M.I. that the coefficient of $x_1^{\ell_1}\cdot x_2^{\ell_2}\cdots x_n^{\ell_n}$ is $\frac{\ell!}{\ell_1!\ell_2!\cdots\ell_n!}$.
14. (Similar to HKAL 1984 Paper 1 Q4, 1988 Paper 1 Q4)
   Let $R(n)$ be the number of non-negative integral solution in $x + 2y = n$. (e.g. $R(5) = 3$)
   Prove that $R(n + 2) = R(n) + 1$
   Prove by MI that $R(n) = \frac{1}{2}(n+1)+\frac{1}{4}\left[1+(-1)^4\right]$. (Hint: prove odd and even separately.)
15. Prove the angle sum of a n–sided polygon is $(n - 2)\pi$.

16. **1981 Paper 1 Q11**

(a) Prove, by mathematical induction, that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

(b) Identical cubical bricks are piled up in layers to form a pyramid-like solid with a square base of side x metres as shown in Figure 2. The side of the bottom layer consists of n bricks whereas each side of the square layer immediately above has $n - 1$ bricks, and so on. There is only one brick in the top layer.



(i) Find the volume of the rth layer counting from the top. Hence find the volume of the solid.

(ii) Using the results of (a) and (b)(i), show that the volume of the solid is always greater than that of a pyramid of the same height, standing on the same base. When n is very large, what value will the difference in volume be close to?

(a) Let $P(n) \equiv$ "$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$."

$n = 1$, L.H.S. $= 1^2 = 1$, R.H.S. $= \frac{1}{6}(1)(1+1)(2+1) = 1$.

L.H.S. = R.H.S.

$P(1)$ is true

Suppose $P(k)$ is true for some positive integer k.

i.e. Assume $1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$ is true $\cdots\cdots$ (*)

When $n = k + 1$,

L.H.S. $= 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$

$= \frac{1}{6}k(k+1)(2k+1) + \frac{6}{6}(k+1)(k+1)$

$= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)]$

$= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$

$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)(k+1+1)[2(k+1)+1]$

= R.H.S.

$\therefore$ If it is true for $n = k$, then it is also true for $n = k + 1$.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

(b) (i) Length of side of one brick $= \frac{x}{n}$ m

Volume of one brick $= \left(\frac{x}{n}\right)^3$ m^3

Number of bricks in the rth layer $= r^2$

The volume of the rth layer counting from the top $= r^2\left(\frac{x}{n}\right)^3$ m^3

$$\text{Volume of the solid} = \left(\frac{x}{n}\right)^3 \left(1^2 + 2^2 + \cdots + n^2\right)$$
$$= \frac{1}{6}\left(\frac{x}{n}\right)^3 n(n+1)(2n+1)$$
$$= \frac{1}{6}x^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

(ii) Volume of the pyramid with the same base $= \frac{1}{3}x^2\left(n \times \frac{x}{n}\right) = \frac{1}{3}x^3$

$$\frac{1}{6}x^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) > \frac{1}{6}x^3(1)(2) = \frac{1}{3}x^3$$

$\therefore$ Volume of the solid > volume of the pyramid with the same base (and the same height)

$$\frac{1}{6}x^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{1}{3}x^3 = \frac{1}{6}x^3\left(2+\frac{3}{n}+\frac{1}{n^2}\right) - \frac{1}{3}x^3$$
$$= \frac{1}{6}x^3\left(\frac{3}{n}+\frac{1}{n^2}\right) \to 0 \text{ as } n \to \infty$$

If n is very large, the difference in volume be close to zero.

17. **1983 Paper 1 Q9**

(a) Prove, by mathematical induction, that for all positive integers n,

$1\times2 + 2\times3 + 3\times4 + \cdots + n(n + 1) = \frac{1}{3}n(n+1)(n+2)$.

(b) On a battle field, cannon-balls are stacked as shown in Figure 2. For a stack with n layers, the balls in the bottom layer are arranged as shown in Figure 3 with n balls on each side. For the second bottom layer, the arrangement is similar but each side consists of $(n - 1)$ balls; for the third bottom layer, each side has $(n - 2)$ balls, and so on. The top layer consists of only one ball.

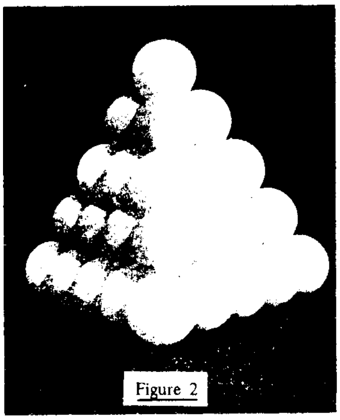


Figure 2

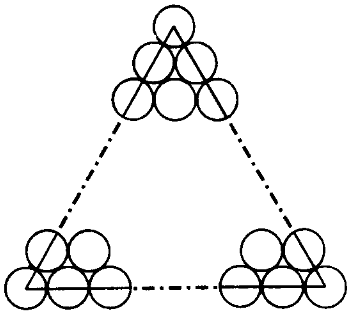


Figure 3

(i) Find the number of balls in the r-th layer counting from the top.

(ii) Using the result of (a), or otherwise, find the total number of cannon-balls in a stack consisting of n layers.

(iii) If the time required to deliver and fire a cannon-ball taken from the r-th layer is $\frac{2}{r}$ minutes, find the time required to deliver and fire all the cannon-balls in the r-th layer. Hence find the total time needed to use up all the cannon-balls in a stack of 10 layers.

(a) Let $P(n) \equiv$ "$1\times2 + 2\times3 + 3\times4 + \cdots + n(n + 1) = \frac{1}{3}n(n+1)(n+2)$ for $n = 1, 2, 3, \cdots$"

$n = 1$, L.H.S. $= 1\times2 = 2$, R.H.S. $= \frac{1}{3}\cdot1(1+1)(1+2) = 2$

L.H.S. = R.H.S. It is true for $n = 1$

Suppose it is true for $n = k$ for some positive integer k.

i.e. $1\times2 + 2\times3 + 3\times4 + \cdots + k(k + 1) = \frac{1}{3}(k+1)(k+2)(k+3)$

Add $(k + 1)(k + 2)$ to both sides.

L.H.S. $= 1\times2 + 2\times3 + 3\times4 + \cdots + k(k + 1) + (k + 1)(k + 2)$

$= \frac{1}{3}k(k+1)(k+2) + (k + 1)(k + 2)$ by induction assumption

$= \frac{1}{3}(k+1)(k+2)(k+3) =$ R.H.S.

If it is true for $n = k$, then it is also true for $n = k + 1$.

By the principle of mathematical induction, the formula is true for all positive integer n.

(b) (i) The number of balls in the r-th layer counting from the top

$= 1 + 2 + \cdots + r$

$= \frac{1}{2} r(r+1)$

(ii) The total number of balls in a stack of n layers

$= \frac{1}{2} \cdot 1 \times 2 + \frac{1}{2} \cdot 2 \times 3 + \cdots + \frac{1}{2} r(r+1)$

$= \frac{1}{2}\left[1 \times 2 + 2 \times 3 + \cdots + r(r+1)\right]$

$= \frac{1}{2} \cdot \frac{1}{3} r(r+1)(r+2)$ by the result of (a), $n = r$

$= \frac{1}{6} r(r+1)(r+2)$

(iii) Time required to deliver and fire all the cannon-balls in the r-th layers

$= \frac{1}{2} r(r+1) \times \frac{2}{r}$ minutes

$= (r + 1)$ min.

Total time needed to used up the cannon-balls in a stack of 10 layers

$= (2 + 3 + \cdots + 11)$ minutes

$= 65$ minutes

18. **HKAL 1970 Paper 1 Q8**

Define $X^+ = X \cup \{X\}$. Sequence of sets $A_0, A_1, A_2, \cdots$ are defined as follows:

$A_0 = \phi$, $A_1 = \phi^+$, $A_2 = \phi^{++}$, $\cdots$, $A_n = A_{n-1}^+$ for $n \ge 1$.

Prove that (a) $A_j = \{A_i : i < j\}$ for $j \ge 0$

(b) For $0 \le i < j$, A_i is a proper subset of A_j.

(Need to assume that it is true for $n = 1, 2, \cdots , k - 1$.)

(a) When $j = 0$, $A_0 = \phi = \{A_i : i < 0\}$ $\because$ there is no non-negative integer < 0

When $j = 1$, $A_1 = \phi^+ = \{\phi\} = \{A_0\} = \{A_i : i < 1\}$

Assume $A_k = \{A_i : i < k\}$ for some non-negative integer k.

Then for $j = k + 1$.

$A_{k+1} = \phi^{+\cdots+}$ ($k + 1$ times),

$= A_k \cup \{A_k\} = \{A_i : i < k\} \cup \{A_k\} = \{A_i : i < k + 1\}$

It is also true for $j = k + 1$

By the principle of mathematical induction, $A_j = \{A_i : i < j\}$ for every non-negative integer j;

(b) $A_0 = \phi \subsetneq \{\phi\} = A_1$

$A_1 = \{\phi\} \subsetneq \{\phi, \{\phi\}\} = A_2$

Suppose $A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_k$ for some non-negative integer k.

$A_{k+1} = \{A_i : i < k + 1\} = A_k \cup \{A_k\}$

If $i < k + 1$, then $A_i \subsetneq A_k \subsetneq A_{k+1}$

($\because$ $A_{k+1} = A_k \cup \{A_k\}$ and $A_k \in A_{k+1} \setminus A_k$ $\therefore$ $A_k \subsetneq A_{k+1}$)

It is also true for $j = k + 1$

By M.I., if $i < j$, then $A_i \subsetneq A_j$. for every non-negative integer j.

19. Let a, b are any non-zero integers and $n = a + b$. Prove by MI on n that there exist integers m_0, n_0 such that $m_0 a + n_0 b = \gcd(a, b)$.

20. **M2 SP Q10**

Let $0° < \theta < 180°$ and define $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(a) Prove, by induction, that $A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$ for all positive integers n.

(a) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, it is true for $n = 1$

Suppose $M^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$ for some positive integer k.

$$M^{k+1} = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & -\cos k\theta\sin\theta - \sin k\theta\cos\theta \\ \sin k\theta\cos\theta + \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{pmatrix}$$, $\therefore$ it is also true for $n = k + 1$

By mathematical induction, $M^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$ for all positive integers n.

21. **Mathematics 1973 Syllabus B Paper 2 Q13(a)**

Prove by Mathematical Induction that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for $n \in \mathbf{N}$.

(Note: If A is a square matrix, $A^n = \underbrace{AA \cdots A}_{n \text{ times}}$.)

Let $P(n) \equiv$ '$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for $n \in \mathbf{N}$'

$n = 1$, L.H.S. $= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, R.H.S. $= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

L.H.S = R.H.S.

$P(1)$ is true

Suppose $P(k)$ is true for some natural number k .

i.e. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ for some $k \in \mathbf{N}$

Whewn $n = k + 1$,

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

L.H.S. = R.H.S.

If $P(k)$ is true then $P(k + 1)$ is also true.

By the principle of mathematical induction, $P(n)$ is true for all natural number n .