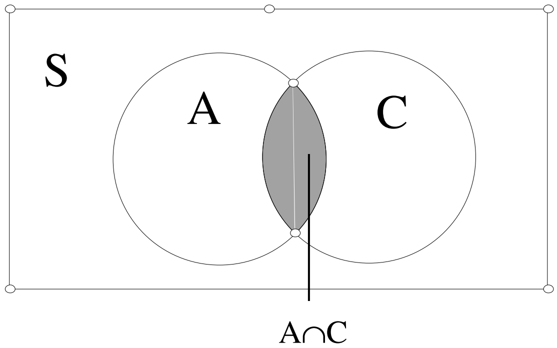
Conditional Probability

6.1 Suppose RE = a random experiment
S = sample space
C = subset of S $\neq \phi$ (i.e. $n(C) > 0$)
A = any event
Given that C must occur, then the probability that A happen is the conditional probability of A given C. It is denoted by:

$$P(A \mid C) = \frac{n(A \cap C)}{n(C)} = \frac{\text{number of elements in A and C}}{\text{number of elements in C}}$$



C is called the **Reduced Sample Space**.

Example 1 Two fair dice are thrown. Given that the sum of the dice is 6, what is the probability that both dice are even?

The sample space is not relevant to us.
Reduced Sample Space = RS = {sum = 6} = {(1,5), (2, 4), (3, 3), (4, 2), (5, 1)}, $n(\text{RS}) = 5$
A = both dice are even = {(2,2), (2, 4), (2, 6), (4, 2), (4, 4), (4,6), (6,2), (6,4), (6,6)}
A ∩ C = {(2, 4), (4, 2)}, $n(A \cap C) = 2$

$$P(A \mid C) = \frac{n(A \cap C)}{n(C)} = \frac{2}{5}$$

6.2 Theorem (a) $P(A \mid C) = \dfrac{P(A \cap C)}{P(C)}$

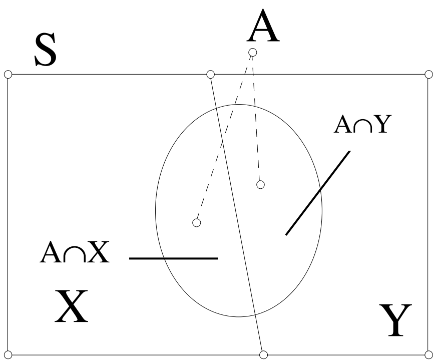
(b) If S = X ∪ Y and X ∩ Y = ϕ, then P(A) = P(A|X)P(X) + P(A|Y)P(Y)
(this is a typical case of theorem of total probability)

Proof:(a) $P(A \mid C) = \dfrac{n(A \cap C)}{n(C)} = \dfrac{\frac{n(A\cap C)}{n(S)}}{\frac{n(C)}{n(S)}} = \dfrac{P(A \cap C)}{P(C)}$, where S is the sample space.

(b) In the figure, $A = (A \cap X) \cup (A \cap Y)$
and $(A \cap X) \cap (A \cap Y) = \phi$
By axiom 3 of chapter 3, $E = A \cap X$, $F = A \cap Y$
$P(A) = P(A \cap X) + P(A \cap Y)$

$$P(A) = \frac{P(A \cap X)}{P(X)} \cdot P(X) + \frac{P(A \cap Y)}{P(Y)} \cdot P(Y)$$

P(A) = P(A|X)P(X) + P(A|Y)P(Y)



Example 2 Suppose that 10% of the women and 5% of the men at a certain college are members of a badminton club. Moreover, 70% of the students are male. If a randomly chosen student is a member of the badminton club, what is the probability that the student is male?

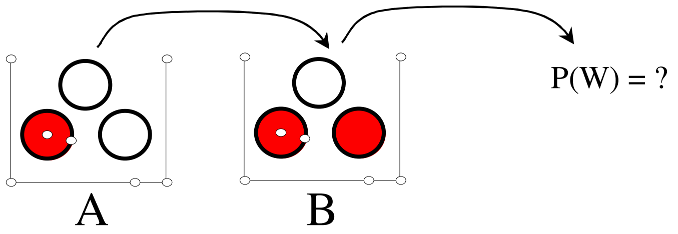
Solution S = {college member}
X = event of male students, P(X) = 0.7
Y = event of female students, P(Y) = 0.3
X ∩ Y = ϕ, X ∪ Y = S
A = {badminton club member}
P(A|X) = 0.05, P(A|Y) = 0.1
P(A) = P(A|X)P(X) + P(A|Y)P(Y) (theorem *b*)
= 0.05×0.7 + 0.1×0.3 = 0.065

$$P(X|A) = \frac{P(A \cap X)}{P(A)} = \frac{P(A \mid X)P(X)}{P(A)} \text{ (theorem } a\text{)}$$

$$= \frac{0.05 \times 0.7}{0.065} \text{ (theorem } a\text{)}$$

$$= \frac{7}{13}$$

Example 3 Urn A contains one red and two white balls, urn B contains one white and two red balls. A ball is chosen at random from urn A and transferred to urn B. Now a ball is chosen at random from urn B, find the probability that the ball is white?



Let X = event that the first ball drawn from A is red, $P(X) = \frac{1}{3}$

Y = event that the first ball drawn from A is white, P(Y) = ____

C = event that the second ball drawn from B is white.

By theorem *b*, $P(C) = P(C|X)P(X) + P(C|Y)P(Y)$

$$= \rule{5cm}{0.4pt} = \frac{5}{12}$$

Example 4 In a family of 2 children, given that one of them is a girl, what is the probability having another girl? (Assuming equal probability of boys and girls.)

Solution: S = sample space = {BG, GB, GG}

$P(\text{another girl}) = P(GG) = \frac{1}{3}$

Example 5 Peter and Mary each chose a positive integer not exceeding 10000. It is known that Peter's number is divisible by 3 while Mary's number is divisible by 5. What is the probability that the two numbers are the same?

Solution: Let Peter chose the number x, Mary chose the number y.

Then the sample space $S = \{(x, y): x = 3k, y = 5m, k, m \in \mathbb{Z}^+ \text{ and } 1 \le k \le 3333, 1 \le m \le 2000\}$

Denote the number of elements in the set S by $n(S)$.

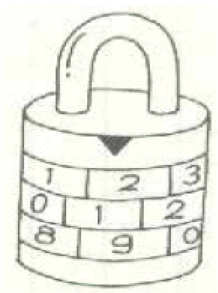
$n(S) = 3333\times2000 = 6666000$

If they chose the same number, the number may be 15, 30, ... , 9990.

$\because$ 9990 = 15×666, there are 666 favourable outcomes.

$$\text{Required probability} = \frac{666}{6666000} = \frac{111}{1111000}$$

Exercise 1 The lock can be opened if each row on the lock is turned to the correct number. There are three rows of numbers and each row has numbers from 0 to 9. It is known that at least one of the three numbers is 3. If you can guess only once, what is the probability of getting the correct number?



Answer $\frac{1}{271}$

Example 6 HKCEE 2006 Paper 2 Q53

There are two questions in a test. The probability that David answers the first question correctly is $\frac{1}{4}$ and the probability that David answers the second question correctly is $\frac{1}{3}$. Given that David answers at least one question correctly in the test, find the probability that he answers the second question correctly.

P(He answers the 2^{nd} correctly | he answers at least one correctly)

$$= \frac{\text{P(He answers the } 2^{\text{nd}} \text{ correctly and at least one correctly)}}{\text{P(at least one correctly)}}$$

$$= \frac{\frac{1}{3}}{1 - P(\text{all wrong})} = \frac{\frac{1}{3}}{1 - \frac{3}{4}\times\frac{2}{3}} = \frac{2}{3}$$

Example 7 Modified from HKCEE 2006 Paper 1 Q14

There are two classes A and B taking the same test, each class has 25 students. The result is as follows:

	Pass	Fail
Class A	18	7
Class B	10	15

From the 50 students, 3 students are randomly selected. Given that exactly 2 of the selected students pass the test, find the probability that both of them are in the same class.

P(both are in the same class | exactly 2 students pass)

$$= \frac{\text{P(both are in the same class and exactly 2 students pass)}}{\text{P(exactly 2 students pass)}}$$

$$= \frac{3\times\text{P(pass in A, pass in A, fail)} + 3\times\text{P(pass in B, pass in B, fail)}}{\text{P(pass, pass, fail or pass, fail, pass or fail, pass, pass)}}$$

$$= \frac{3\times\frac{18}{50}\times\frac{17}{49}\times\frac{22}{48} + 3\times\frac{10}{50}\times\frac{9}{49}\times\frac{22}{48}}{3\times\frac{28}{50}\times\frac{27}{49}\times\frac{22}{48}}$$

$$= \frac{\frac{1089}{4900}}{\frac{297}{700}} = \frac{11}{21} \quad (= 0.523)$$

Example 8 Modified from HKCEE 2007 Paper 1 Q15

The following table shows the results of a survey about the sizes of shirts dressed by 80 students on a certain school day.

Student \ Size	Small	Medium	Large	Total
Boy	8	28	12	48
Girl	20	8	4	32

On that school day, a student is randomly selected from the 80 students. Given that the selected student is a boy, find the probability that he dresses a shirt of large size.

$$\text{P(large shirt | boy)} = \frac{P(\text{boy and large shirt})}{P(\text{boy})} = \frac{\frac{12}{80}}{\frac{48}{80}} = \frac{1}{4}$$

Exercise 2 Two cards are drawn randomly from nine cards numbered 1, 2, 3, 4, 5, 6, 7, 8 and 9 respectively one by one without replacement. Given that the sum of the two numbers drawn is even, find the probability that the second card drawn is odd.

Answer $\frac{5}{8}$

Example 9 The weights (in kg) of a sample of twenty F.5 students are shown below.

45	47	48	50	54
55	56	58	60	60
60	71	75	75	76
85	85	85	85	90

(a) Show that the mean weight is 66 kg .

(b) If two of the above data is deleted at random, find the probability that the mean weight of the remaining 18 students is 65 kg .

(c) If two of the above data is deleted at random and it is known that one of them is greater than 85 kg, find the probability that the mean weight of the remaining 18 students is 65 kg.

(b) Let the weights of the deleted data be x kg and y kg.

$18\times65 + x + y = 20\times66$

$x + y = 150$

The favourable outcomes are (60,90), (60, 90), (60, 90), (90, 60), (90, 60), (90, 60), (75, 75), (75, 75)

Probability $= \frac{4\times2}{20\times19} = \frac{2}{95}$

(c) P(mean of 18 students = 65 kg | deleted one > 85 kg)

$= \frac{\text{P(mean} = \text{65kg and deleted one} > \text{85 kg)}}{\text{P(Deleted one} > \text{85kg)}}$

$= \frac{\text{P(The deleted two weights are 60kg and 90 kg)}}{\text{P(Deleted one} = \text{90 kg)}}$

$= \frac{\frac{3\times2}{20\times19}}{\frac{1}{20}+\frac{19}{20}\cdot\frac{1}{19}} = \frac{3}{19}$

6.3 Theorem of total probability

Let $A_1, A_2, \cdots, A_n$ be mutually exclusive events (n is a positive integer),

then $P(A) = P(A_1)P(A|A_1) + P(A_2)P(A|A_2) + \cdots + P(A_n)P(A|A_n)$

Proof: $A = (A \cap A_1) \cup (A \cap A_2) \cup \cdots \cup (A \cap A_n)$ and

the subsets $A \cap A_1, A \cap A_2, \cdots, A \cap A_n$ are mutually exclusive.

By an extension of axiom 3 in Chapter 3,

$P(A) = P(A \cap A_1) + P(A \cap A_2) + \cdots + P(A \cap A_n)$

$= P(A_1)\cdot\frac{P(A\cap A_1)}{P(A_1)} + P(A_2)\cdot\frac{P(A\cap A_2)}{P(A_2)} + \cdots + P(A_n)\cdot\frac{P(A\cap A_n)}{P(A_n)}$

$= P(A_1)P(A|A_1) + P(A_2)P(A|A_2) + \cdots + P(A_n)P(A|A_n)$

Bayes Theorem

Let $A_1, A_2, \cdots, A_n$ be mutually exclusive events and the sample space $S = A_1 \cup A_2 \cup \cdots \cup A_n$,

then $P(A_k \mid A) = \frac{P(A_k)P(A\mid A_k)}{\sum_{k=1}^{n} P(A_k)P(A\mid A_k)}$

Proof: $P(A_k \mid A) = \frac{P(A_k \cap A)}{P(A)} = \frac{P(A_k)\frac{P(A_k\cap A)}{P(A_k)}}{P(A)} = \frac{P(A_k)P(A\mid A_k)}{P(A)}$

$= \frac{P(A_k)P(A\mid A_k)}{P(A_1)P(A\mid A_1)+P(A_2)P(A\mid A_2)+\cdots+P(A_n)P(A\mid A_n)}$ by the total probability.

Example 10 Suppose that if a person with SARS disease is given a diagnose test the probability that his condition will be detected is 0.95 and that if a person without SARS is given a diagnose test the probability that he will be diagnosed incorrectly as having SARS is 0.002. Suppose, furthermore, that 0.1 percent of the residents of a certain city have SARS. If one of these persons (selected at random, that is, equal probabilities) is diagnosed as having SARS on the basis of the diagnose test, what is the probability that he actually has SARS?

Solution: P(diagnose positive | SARS) = 0.95

P(diagnose positive | no SARS) = 0.002

P(SARS) = 0.001

$$\begin{aligned}
&\mathrm{P}(\text{SARS} \mid \text{diagnose positive})\\
&= \frac{\mathrm{P}(\text{SARS} \cap \text{diagnose positive})}{\mathrm{P}(\text{diagnose positve})}\\
&= \frac{\mathrm{P}(\text{SARS})\mathrm{P}(\text{diagnose positive} \mid \text{SARS})}{\mathrm{P}(\text{SARS})\mathrm{P}(\text{diagnose positive} \mid \text{SARS}) + \mathrm{P}(\text{no SARS})\mathrm{P}(\text{diagnose positive} \mid \text{no SARS})}\\
&= \frac{0.001 \times 0.95}{0.001 \times 0.95 + 0.999 \times 0.002}\\
&= 0.32225
\end{aligned}$$

Example 11 (A Paradox) Three students A, B, C are nominated for a scholarship. The three students have equally sound academic records, so let us assume that each has the same chance of getting the scholarship.

Student A, knowing the situation, is eager to have more information. So he asks a professor who already knows the result of the competition, "Professor, of the other two nominees, please name one who will not get the scholarship?". The professor ponders a moment and replies, "student B".

Student A then goes happily to see his girl friend, saying, "My chance has now increased from $\frac{1}{3}$ to $\frac{1}{2}$, because student B is out". "But what", replied the girlfriend, "if the professor's answer is student C?" "Well, it's a similar situation and my chance would also be increased to $\frac{1}{2}$."

"Come on", laughed the girlfriend, "if your chance will be $\frac{1}{2}$ whatever the professor's answer, why did you have to ask? Isn't your chance should always be $\frac{1}{2}$ then?"

Question: Is this a paradox?

Solution: Assume the professor answered student A honestly.

If A gets the scholarship, then the professor may reply "B" or "C" with equal chance $\frac{1}{2}$.

If B gets the scholarship, then the professor replies "B" with no chance.

If C gets the scholarship, then the professor must reply "B" with chance 1.

$$\begin{aligned}
&\mathrm{P}(\text{A gets the scholarship} \mid \text{professor replied "student B"})\\
&= \frac{\mathrm{P}(\text{A and professor replied "B"})}{\mathrm{P}(\text{professor replied "B"})}\\
&= \frac{P(A)\mathrm{P}(\text{professor "B"} \mid \mathrm{A})}{P(A)\mathrm{P}(\text{professor "B"} \mid \mathrm{A}) + P(B)\mathrm{P}(\text{professor "B"} \mid \mathrm{B}) + P(C)\mathrm{P}(\text{professor "B"} \mid \mathrm{C})}\\
&= \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1} = \frac{1}{3}
\end{aligned}$$

Example 12 Eric, Fred and Gary are playing a shooting game together. The probabilities that they hit a flying target are $\frac{3}{10}$, $\frac{2}{5}$ and $\frac{1}{2}$ respectively.

(a) If each of them fires once, find the probability that
   (i) all of them hit the target.
   (ii) at least one of them hits the target.

(b) If each of them fires twice, find the probability that
   (i) each of them hits the target at least once,
   (ii) at least one of them hits the target once.

(c) If each of them fires twice and it is given that at least one of them hits the target once, find the conditional probability that Fred hits the target at least once.

Solution: (a) (i) P(all of them hit the target) $= \frac{3}{10}\times\frac{2}{5}\times\frac{1}{2}=\frac{3}{50}$

(ii) P(at least one of them hits the target) = 1 – P(all of them don't hit the target)
$= 1-\frac{7}{10}\times\frac{3}{5}\times\frac{1}{2}=\frac{79}{100}$

(b) (i) P(Eric hits the target at least once) $= 1-\frac{7}{10}\times\frac{7}{10}=\frac{51}{100}$

P(Fred hits the target at least once) $= 1-\frac{3}{5}\times\frac{3}{5}=\frac{16}{25}$

P(Gary hits the target at least once) $= 1-\frac{1}{2}\times\frac{1}{2}=\frac{3}{4}$

P(each of them hits the target at least once) $= \frac{51}{100}\times\frac{16}{25}\times\frac{3}{4}=\frac{153}{625}$

(ii) P(Eric hits the target once) $= \frac{3}{10}\times\frac{7}{10}\times 2=\frac{21}{50}$

P(Fred hits the target once) $= \frac{3}{5}\times\frac{2}{5}\times 2=\frac{12}{25}$

P(Gray hits the target once) $= \frac{1}{2}\times\frac{1}{2}\times 2=\frac{1}{2}$

P(at least one of them hits the target once)
= 1 – P(all of them don't hits the target once)
$= 1-\left(1-\frac{21}{50}\right)\left(1-\frac{12}{25}\right)\left(1-\frac{1}{2}\right)=\frac{2123}{2500}$

(c) P(Fred hits the target at least once| at least one of them hits the target once)

$= \frac{\text{P(Fred hits the target at least once and at least one of them hits the target once)}}{\text{P(at least one of them hits the target once)}}$

$= \frac{16}{25}\div\frac{2123}{2500}$

$= \frac{1600}{2123}$

Example 13 Monty Hall Problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Let C = the number of the door hiding the car,
H = the number of the door opened by the Host.

As the car behind each door is equal likely, $P(C) = \frac{1}{3}$

The probability of winning by switching the door, given the player choose door no. 1

$= P((C = 2 \text{ and } H = 3) \text{ or } (C = 3 \text{ and } H = 2))$

$= P(C = 2 \text{ and } H = 3) + P(C = 3 \text{ and } H = 2)$

$= \frac{P(C = 2 \text{ and } H = 3)}{P(C = 2)} \times P(C = 2) + \frac{P(C = 3 \text{ and } H = 2)}{P(C = 3)} \times P(C = 3)$

$= P(H = 3 \mid C = 2) \times \frac{1}{3} + P(H = 2 \mid C = 3) \times \frac{1}{3}$

$= 1 \times \frac{1}{3} + 1 \times \frac{1}{3}$

$= \frac{2}{3}$

$\therefore$ If the player switches, the probability of getting a car is higher.

Example 14 **(Modified from CLSMSS 2022 S.6 Mock Examination Paper 1 Q16)**

There are 2 female social workers, n female students, 5 male students and 4 male volunteers in a classroom.

(a) A person is randomly selected from the classroom. If the probability of selecting a student is $\frac{4}{7}$, find the value of n .

(b) 4 people are randomly selected from the classroom . Given that **exactly** 2 of the selected people are students, find the probability that the 4 selected people are of the same gender.

(c) 4 people are randomly selected from the classroom . Given that **at least** 2 of the selected people are students, find the probability that the 4 selected people are of the same gender.

(a) $\frac{n+5}{2+n+5+4}=\frac{4}{7}$

$7n + 35 = 4n + 44$

$3n = 9$

$n = 3$

(b) Let X be the event that exactly 2 of the selected people are students,

Y be the event that the 4 selected people are of the same gender.

$$P(X)=\frac{C_2^8 C_2^6}{C_4^{14}}=\frac{\frac{8\times 7}{2}\cdot\frac{6\times 5}{2}}{\frac{14\times 13\times 12\times 11}{1\times 2\times 3\times 4}}=\frac{60}{143}$$

$X\cap Y$ = Event that the 4 selected people of **male** and exactly 2 of them are male students or the 4 selected people of **female** and exactly 2 of them are female students

$$P(X\cap Y)=\frac{C_2^5 C_2^4+C_2^2 C_2^3}{C_4^{14}}=\frac{10\times 6+1\times 3}{\frac{14\times 13\times 12\times 11}{1\times 2\times 3\times 4}}=\frac{9}{143}$$

$\therefore$ The required probability = $P(Y\mid X)=\frac{P(X\cap Y)}{P(X)}$

$$=\frac{9}{143}\div\frac{60}{143}=\frac{3}{20}$$

(c) Let Z be the event that **at least** 2 of the selected people are students

Z = 2 of the selected people are students or 3 of the selected people are students or 4 of the selected people are students.

$$P(Z)=\frac{60}{143}+\frac{48}{143}+\frac{10}{143}=\frac{118}{143}$$

$Y\cap Z$ = Event that the 4 selected people of **male** and exactly 2 of them are male students or the 4 selected people of **female** and exactly 2 of them are female students or the 4 selected people of **male** and exactly 3 of them are male students or the 4 selected people of **female** and exactly 3 of them are female students or the 4 selected people of **male** and exactly 4 of them are male students or the 4 selected people of **female** and exactly 4 of them are female students (impossible)

$$P(Y\cap Z)=\frac{9}{143}+\frac{C_3^5 C_1^4+C_1^2 C_3^3}{C_4^{14}}+\frac{C_0^4 C_4^5}{C_4^{14}}$$

$$=\frac{9}{143}+\frac{47}{7\times 143}$$

$$=\frac{110}{7\times 143}$$

$\therefore$ The required probability = $P(Y\mid Z)=\frac{P(Y\cap Z)}{P(Z)}=\frac{110}{7\times 143}\div\frac{118}{143}=\frac{55}{413}$

Example 15 (Mathematics in Action 5B (Pearson Longman) Chapter 10 Exercise 9E Q 19)

A bag contains 50 fair coins. One of the coins has heads on both sides. 6 coins are drawn from the bag at random without replacement. Each of the first 5 coins drawn is then tossed. What are the probabilities that the sixth coin drawn will be the coin with both heads if

(a) 5 tails turns up in the 5 tosses?

(b) 5 heads turn up in the 5 tosses?

(a) Let A be the event that the sixth coin drawn will be the coin with both heads.
Let B be the event that 5 tails turns up in the 5 tosses.
Let C be the event that 5 heads turns up in the 5 tosses.
B = 5 tails turns up in the 5 tosses
= The first 5 coins are true coins and 5 tails turns up

$$P(B)=\frac{49}{50}\times\frac{48}{49}\times\frac{47}{48}\times\frac{46}{47}\times\frac{45}{46}\times\left(\frac{1}{2}\right)^5=\frac{9}{320}$$

$A\cap B$ = the event that 5 tails turns up in the 5 tosses and the sixth coin is a fake coin
= The first 5 coins are true coins and 5 tails turns up and the sixth coin is a fake coin

$$P(A\cap B)=\frac{49}{50}\times\frac{48}{49}\times\frac{47}{48}\times\frac{46}{47}\times\frac{45}{46}\times\left(\frac{1}{2}\right)^5\times\frac{1}{45}=\frac{1}{1600}$$

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{1600}}{\frac{9}{320}}=\frac{1}{45}$$

(b) C = 5 heads turns up in the 5 tosses
= Either the first 5 coins are true coins and 5 heads turns up
OR the fake coin in one of the first 5 coins and 5 heads turns up

$$P(C)=\frac{49}{50}\times\frac{48}{49}\times\frac{47}{48}\times\frac{46}{47}\times\frac{45}{46}\times\left(\frac{1}{2}\right)^5+C_1^5\cdot\frac{49}{50}\times\frac{48}{49}\times\frac{47}{48}\times\frac{46}{47}\times\frac{1}{46}\times\left(\frac{1}{2}\right)^4\times 1$$

$$=\frac{9}{320}+\frac{1}{160}=\frac{11}{320}$$

$A\cap C$ = the event that 5 heads turns up in the 5 tosses and the sixth coin is a fake coin
= The first 5 coins are true coins and 5 heads turns up and the sixth coin is a fake coin

$$P(A\cap C)=\frac{49}{50}\times\frac{48}{49}\times\frac{47}{48}\times\frac{46}{47}\times\frac{45}{46}\times\left(\frac{1}{2}\right)^5\times\frac{1}{45}=\frac{1}{1600}$$

$$P(A\mid C)=\frac{P(A\cap C)}{P(C)}=\frac{\frac{1}{1600}}{\frac{11}{320}}=\frac{1}{55}$$