05-06 Individual	**1**	250	**2**	$\frac{147\sqrt{3}}{242}$	**3**	–6	**4**	2006	**5**	60
	6	165	**7**	0	**8**	3	**9**	9	**10**	$\sqrt{5}$

05-06 Group	**1**	2	**2**	10^{10}	**3**	$\frac{\sqrt{2}}{4}$	**4**	1	**5**	59
	6	0	**7**	4	**8**	328	**9**	$7\sqrt{2}$	**10**	$\frac{1}{3}$

Individual Events

I1 Let $\sqrt{20+\sqrt{300}}=\sqrt{x}+\sqrt{y}$, where x and y are rational numbers and $w = x^2 + y^2$, find the value of w. **Reference 1993 FI1.4, 1999 HG3, 2001 FG2.1, 2011 HI7, 2015 FI4.2, 2015 FG3.1**

$$\sqrt{20+\sqrt{300}}=\sqrt{5+15+2\sqrt{5\times 15}}$$
$$=\sqrt{\left(\sqrt{5}+\sqrt{15}\right)^2}$$
$$=\sqrt{5}+\sqrt{15}$$

$x = 5$, $y = 15$.

$w = 5^2 + 15^2 = 250$

I2 In Figure 1, a regular hexagon is inscribed in a circle with circumference 4 m. If the area of the regular is A m^2, find the value of A. (Take $\pi = \frac{22}{7}$)

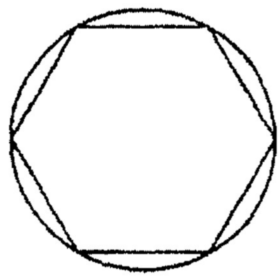

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Figure 1

Let the radius be r m.

$2\pi r = 4 \Rightarrow r = \frac{2}{\pi} = \frac{7}{11}$

A = area of 6 equilateral triangles each with side $= \frac{7}{11}$ m

$= 6\times\frac{1}{2}\times\frac{7}{11}\times\frac{7}{11}\times\sin 60°$

$= \frac{147}{121}\cdot\frac{\sqrt{3}}{2}$

$= \frac{147\sqrt{3}}{242}$

I3 Given that $\dfrac{1}{2+\dfrac{3}{1+\dfrac{1}{x}}}=\dfrac{5}{28}$, find the value of x.

$$2+\frac{3}{1+\frac{1}{x}}=\frac{28}{5}$$
$$\Rightarrow \frac{3}{1+\frac{1}{x}}=\frac{18}{5}$$
$$\Rightarrow 1+\frac{1}{x}=\frac{5}{6}$$
$$\Rightarrow \frac{1}{x}=-\frac{1}{6}$$
$$\Rightarrow x=-6$$

I4 Let $A=\dfrac{2006}{20052005^2-20052004\times 20052006}$, find the value of A.

Let x = 2005, then 20052005 = 2005×10001=10001x

20052004 = 20052005 – 1 = 10001x – 1

20052006 = 20052005 + 1 = 10001x + 1,

$$A=\frac{2006}{(10001x)^2-(10001x-1)\times(10001x+1)}$$

$$=\frac{2006}{(10001x)^2-(10001x)^2+1}$$

= 2006

I5 Given that 4 $\sec^2\theta°-\tan^2\theta°-7\sec\theta°+1=0$ and $0°\le\theta°\le 180°$, find the value of θ.

$4\sec^2\theta°-(\sec^2\theta°-1)-7\sec\theta°+1=0$

$3\sec^2\theta°-7\sec\theta°+2=0$

$(3\sec\theta°-1)(\sec\theta°-2)=0$

$\sec\theta°=\dfrac{1}{3}$ or 2

$\Rightarrow \cos\theta°=3$ (rejected) or $\dfrac{1}{2}$

$\Rightarrow \theta=60$

I6 Given that w, x, y and z are positive integers which satisfy the equation $w+x+y+z=12$. If there are W sets of different positive integral solutions of the equation, find the value of W.

Reference: 2001 HG2, 2010 HI1, 2012 HI2

$1\le w\le 9$, keep w fixed, we shall find the number of solutions to $x+y+z=12-w$ ......(1)

$1\le x\le 10-w$, keep x fixed, we shall find the number of solutions to $y+z=12-w-x$ ......(2)

$1\le y\le 11-w-x$, keep y fixed, the number of solution to $z=12-w-x-y$ is 1

$$\text{Total number of solutions}=\sum_{w=1}^{9}\sum_{x=1}^{10-w}\sum_{y=1}^{11-w-x}1=\sum_{w=1}^{9}\sum_{x=1}^{10-w}(11-w-x)$$

$$=\sum_{w=1}^{9}\left[(11-w)(10-w)-(1+2+\cdots 10-w)\right]$$

$$=\sum_{w=1}^{9}\left[(11-w)(10-w)-\frac{1}{2}(11-w)(10-w)\right]$$

$$=\sum_{w=1}^{9}\left[\frac{1}{2}(11-w)(10-w)\right]=\frac{1}{2}\sum_{w=1}^{9}\left(110-21w+w^2\right)$$

$$=\frac{1}{2}\left(990-21\times 45+\frac{9}{6}\cdot 10\cdot 19\right)$$

$$=\frac{45}{2}\left(22-21+\frac{19}{3}\right)=\frac{45}{2}\cdot\frac{22}{3}=165$$

Method 2

The problem is equivalent to: put 12 identical balls into 4 different boxes w, x, y and z. Each box should have at least one ball to ensure positive integral solutions.

Align the 12 balls in a row. There are 11 gaps between the 12 balls. Put 3 sticks into three of these gaps, so as to divide the balls into 4 groups.

The following diagrams show one possible division.

○ ○ | ○ ○ ○ ○ ○ | ○ ○ ○ ○ | ○

The three boxes contain 2 balls, 5 balls, 4 balls and 1 ball. $w=2$, $x=5$, $y=4$, $z=1$.

The number of ways is equivalent to the number of choosing 3 gaps as sticks from 11 gaps.

The number of ways is $C_3^{11}=\dfrac{11}{1}\cdot\dfrac{10}{2}\cdot\dfrac{9}{3}=165$

I7 Given that the number of prime numbers in the sequence 1001, 1001001, 1001001001, ..., $1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots1\underbrace{00}_{2}1$, ... is R, find the value of R.

$1001 = 7\times11\times13$ which is not a prime

Suppose there are n '1's in $1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots1\underbrace{00}_{2}1$.

If n is divisible by 3, then the number itself is divisible by 3.

Otherwise, $1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots1\underbrace{00}_{2}1 = 1 + 10^3 + 10^6 + \cdots + 10^{3(n-1)}$

$$= \frac{10^{3n}-1}{10^3-1} = \frac{10^{3n}-1}{999}$$

$$= \frac{(10^n-1)(10^{2n}+10^n+1)}{999}$$

$$= \frac{\underbrace{99\cdots9}_{n}(10^{2n}+10^n+1)}{999}$$

$$= \frac{\underbrace{11\cdots1}_{n}(10^{2n}+10^n+1)}{111}$$

$\because$ n is not divisible by 3, $\underbrace{11\cdots1}_{n}$ and 111 are relatively prime.

LHS = $1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots1\underbrace{00}_{2}1$ is an integer

$\Rightarrow$ RHS is an integer

$\Rightarrow \dfrac{10^{2n}+10^n+1}{111}$ is an integer $\neq 1$

$\therefore 1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots1\underbrace{00}_{2}1$ = product of two integers

$\therefore 1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots1\underbrace{00}_{2}1$ is not a prime number

$\Rightarrow$ there are no primes, $R = 0$

I8 Let $\lfloor x \rfloor$ be the largest integer not greater than x, for example, $\lfloor 2.5 \rfloor = 2$. If $B = \left\lfloor \log_7\left(462 + \log_2\lfloor \tan 60^\circ \rfloor + \sqrt{9872}\right)\right\rfloor$, find the value of B.

$$B = \left\lfloor \log_7\left(462 + \log_2\lfloor \sqrt{3} \rfloor + \sqrt{9872}\right)\right\rfloor$$
$$= \left\lfloor \log_7\left(462 + \log_2 1 + \sqrt{9872}\right)\right\rfloor$$
$$= \left\lfloor \log_7\left(462 + \sqrt{9872}\right)\right\rfloor$$

$\sqrt{99^2} = \sqrt{9801} < \sqrt{9872} < \sqrt{10000}$

$\sqrt{9872} = 99 + a$, $0 < a < 1$

$$\begin{aligned}462 + \sqrt{9872} &= 462 + 99 + a \\ &= 561 + a \\ &= 7\times80 + 1 + a \\ &= 7\times(7\times11 + 3) + 1 + a \\ &= 7^2\times11 + 7\times3 + 1 + a \\ &= 7^3 + 7^2\times4 + 7\times3 + 1 + a\end{aligned}$$

$7^3 < 462 + \sqrt{9872} < 7^4$

$\log_7 7^3 < \log_7\left(462 + \sqrt{9872}\right) < \log_7 7^4$

$3 < \log_7\left(462 + \sqrt{9872}\right) < 4$

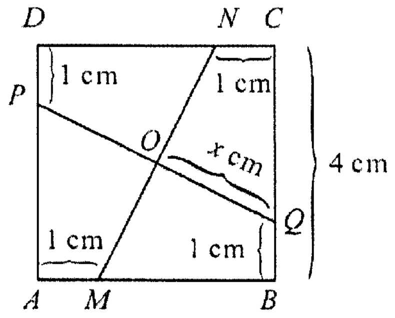
$\left\lfloor \log_7\left(462 + \sqrt{9872}\right)\right\rfloor = 3$

I9 Given that the units digit of 7^{2006} is C, find the value of C.

$7^1 = 7$, $7^2 \equiv 9 \pmod{10}$, $7^3 \equiv 3 \pmod{10}$, $7^4 \equiv 1$ (mode 10)

$7^{2006} = (7^4)^{501} \times 7^2 \equiv 9 \bmod 10$

I10 In Figure 2, $ABCD$ is a square with side length equal to 4 cm. The line segments PQ and MN intersect at the point O. If the lengths of PD, NC, BQ and AM are 1 cm and the length of OQ is x cm, find the value of x.



$AP = BM = CQ = DN = 3$ cm

$PMQN$ is a rhombus (4 sides equal)

$PQ \perp MN$, $PO = OQ$, $MO = NO$ (property of rhombus)

$MN = PQ = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$ cm

$\Rightarrow x = \sqrt{5}$

Group Events

G1 Let a, b and c are three prime numbers. If $a < b < c$ and $c = a^2 + b^2$, find the value of a.

Note that 2 is the only prime number which is even.

If $a \neq 2$ and $b \neq 2$, then a and b must be odd prime number,

Then c = odd + odd = even prime number > 2, which is a contradiction.

$\therefore a = 2$

By trail and error, $b = 3$, $c = 13 = 2^2 + 3^2$

G2 If $\log\left(\log\left(\log\left(1\overbrace{00\cdots0}^{n \text{ zeros}}\right)\right)\right) = 1$, find the value of n.

$\log\left(\log\left(1\overbrace{00\cdots0}^{n \text{ zeros}}\right)\right) = 10$

$\Rightarrow \log(10^n) = 10^{10}$

$\Rightarrow 10^n = 10^{\left(10^{10}\right)}$

$\Rightarrow n = 10^{10}$

G3 Given that $0° < \theta < 90°$ and $1 + \sin\theta + \sin^2\theta + \cdots = \frac{3}{2}$. If $y = \tan\theta$, find the value of y.

Similar question: 2014 HG3

$\frac{1}{1-\sin\theta} = \frac{3}{2}$

$\Rightarrow 2 = 3 - 3\sin\theta$

$\Rightarrow \sin\theta = \frac{1}{3}$

$\Rightarrow y = \tan\theta = \frac{1}{\sqrt{8}} = \frac{\sqrt{2}}{4}$

G4 Consider the quadratic equation $x^2 - (a-2)x - a - 1 = 0$, where a is a real number. Let α and β be the roots of the equation. Find the value of a such that the value of $\alpha^2 + \beta^2$ will be the least.

$\alpha + \beta = a - 2$, $\alpha\beta = -a - 1$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (a-2)^2 - 2(-a-1)$

$= a^2 - 4a + 4 + 2a + 2$

$= a^2 - 2a + 6$

$= (a-1)^2 + 5$

$\alpha^2 + \beta^2$ will be the least when $(a-1)^2 = 0$

$\Rightarrow a = 1$

G5 Given that the sum of k consecutive positive integers is 2006, find the maximum possible value of k. (**Reference: 2004 HG5**)

Let the smallest positive integer be x: $x + (x + 1) + \cdots + (x + k - 1) = 2006$, $x > 0$

$\frac{k}{2}\left[2x+(k-1)\right] = 2006 \Rightarrow k(2x + k - 1) = 4\times17\times59$

$\Rightarrow 2x + k - 1 = \frac{4\times17\times59}{k}$, an integer

$\therefore$ k is an factor of $4\times17\times59$.

Factors of 4012 are 1, 2, 4, 17, 34, 59, 68, 118, 236, 1003, 2006, 4012

When $k = 4m + 2$, where m is an integer,

$(4m + 2)(2x + 4m + 2 - 1) = 4\times17\times59$

$\Rightarrow (2m + 1)(2x + 4m + 1) = 2\times17\times59$

L.H.S. is odd and R.H.S. is even

$\therefore$ reject 2, 34, 118, 2006

$2x + k - 1 = \frac{4\times17\times59}{k}$

$\Rightarrow \frac{4\times17\times59}{k} > k - 1$

$\Rightarrow 4012 > k(k - 1)$

$\Rightarrow \sqrt{4012} > k - 1$

$\Rightarrow 64 > k$

$\therefore$ Possible values of k = 1, 4, 17, 59 only

When $k = 1$, $1(2x) = 4012 \Rightarrow x = 2006$

When $k = 4$, $4(2x + 3) = 4012 \Rightarrow x = 500$

When $k = 17$, $17(2x + 16) = 4012 \Rightarrow x + 8 = 118 \Rightarrow x = 110$

When $k = 59$, $59(2x + 58) = 4\times17\times59 \Rightarrow x + 29 = 34 \Rightarrow x = 5$

$\Rightarrow$ The maximum possible $k = 59$

G6 Let a, b, c and d be real numbers such that $a^2 + b^2 = c^2 + d^2 = 1$ and $ac + bd = 0$. If $R = ab + cd$, find the value of R. (**Reference: 2002 HI7, 2009 FI3.3, 2014 HG7**)

Let $a = \sin A$, $b = \cos A$, $c = \cos B$, $d = \sin B$

$ac + bd = 0$

$\Rightarrow \sin A \cos B + \cos A \sin B = 0$

$\Rightarrow \sin(A + B) = 0$

$R = ab + cd$

$= \sin A \cos A + \cos B \sin B$

$= \frac{1}{2}(\sin 2A + \sin 2B)$

$= \frac{1}{2}[2 \sin(A + B)\cos(A - B)] = 0$

Method 2

There is no need to let the sum = 1.

$R^2 = R^2 - 0^2 = (ab + cd)^2 - (ac + bd)^2$

$= a^2b^2 + c^2d^2 - a^2c^2 - b^2d^2$

$= a^2(b^2 - c^2) - d^2(b^2 - c^2)$

$R^2 = (b^2 - c^2)(a^2 - d^2) \cdots\cdots\cdots\cdots (1)$

$a^2 + b^2 = c^2 + d^2 \Rightarrow b^2 - c^2 = d^2 - a^2 \cdots\cdots\cdots\cdots (2)$

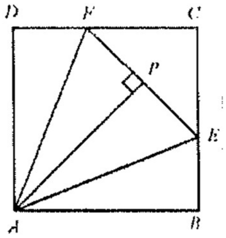
Sub. (2) into (1): $R^2 = (d^2 - a^2)(a^2 - d^2) = -(a^2 - d^2)^2$

LHS ≥ 0, whereas RHS ≤ 0

$\Rightarrow$ LHS = RHS = 0

$\therefore R = ab + cd = 0$

G7 In Figure 1, $ABCD$ is a square with perimeter equal to 16 cm, $\angle EAF = 45°$ and $AP \perp EF$. If the length of AP is equal to R cm, find the value of R.



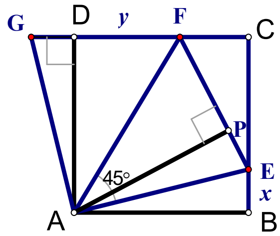
Reference: http://www.hkedcity.net/ihouse/fh7878/Geometry/transform/Q5.pdf, 2017 HG3

$AB = BC = CD = DA = 4$ cm

Let $BE = x$ cm, $DF = y$ cm.

Rotate ΔABE about A in anti-clockwise direction by 90°

Then $\Delta ABE \cong \Delta ADG$; $GD = x$ cm, $AG = AE$ (corr. sides $\cong$ Δ's)



$AE = AE$ (common side)

$\angle EAG = 90°$ ($\angle$ of rotation)

$\angle GAF = 90° - 45° = 45° = \angle EAF$

$\therefore \Delta AEF \cong \Delta AGF$ (SAS)

$\angle AGD = \angle AEP$ (corr. $\angle$s. $\cong$ Δ's)

$\angle ADG = 90° = \angle APE$ (by rotation)

$AD = AB$ (sides of a square)

$\therefore \Delta ADG \cong \Delta APE$ (AAS)

$AP = AD = 4$ cm (corr. sides $\cong$ Δ's)

G8 Given that x and y are real numbers and satisfy the system of the equations

$$\begin{cases} \dfrac{100}{x+y} + \dfrac{64}{x-y} = 9 \\ \dfrac{80}{x+y} + \dfrac{80}{x-y} = 9 \end{cases}$$. If $V = x^2 + y^2$, find the value of V.

$$\frac{100}{x+y} + \frac{64}{x-y} = 9 = \frac{80}{x+y} + \frac{80}{x-y}$$

$$\Rightarrow \frac{20}{x+y} = \frac{16}{x-y}$$

$$\Rightarrow \frac{x+y}{5} = \frac{x-y}{4} = k$$

Let $x + y = 5k$, $x - y = 4k$.

$$\Rightarrow \frac{100}{5k} + \frac{64}{4k} = 9$$

$\Rightarrow 20 + 16 = 9k$

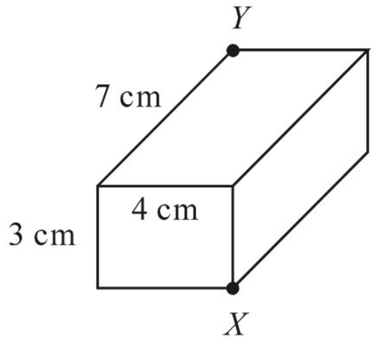
$\Rightarrow k = 4$

$x + y = 20$, $x - y = 16$

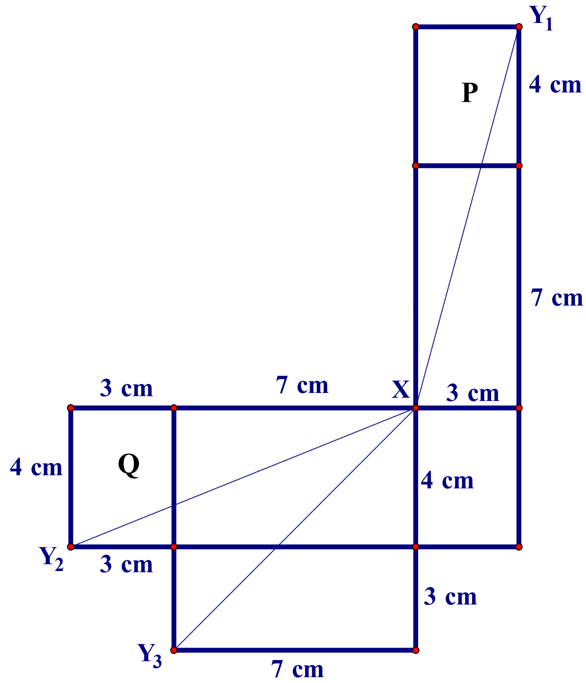
$\Rightarrow x = 18$, $y = 2$

$V = 18^2 + 2^2 = 328$

G9 In Figure 2, given a rectangular box with dimensions 3 cm, 4 cm, and 7 cm respectively. If the length of the shortest path on the surface of the box from point X to point Y is K cm, find the value of K.



Unfold the rectangular box as follow. Note that the face P is identical to the face Q. There are 3 possible routes to go from X to Y:



$XY_1 = \sqrt{3^2 + (4+7)^2}\text{ cm} = \sqrt{130}\text{ cm}$

$XY_2 = \sqrt{4^2 + (3+7)^2}\text{ cm} = \sqrt{116}\text{ cm}$

$XY_3 = \sqrt{7^2 + (3+4)^2}\text{ cm} = 7\sqrt{2}\text{ cm}$

The length of the shortest path is $7\sqrt{2}$ cm.

G10 Given that x is a positive real number which satisfy the inequality $|x-5| - |2x+3| \le 1$, find the least value of x.

Reference: 2001 HG9 ...... $|x-3| + |x-5| = 2$.......

Case 1: $x \le -1.5$

$5 - x + 2x + 3 \le 1$

$x \le -7$

Case 2: $-1.5 < x \le 5$

$5 - x - 2x - 3 \le 1$

$\frac{1}{3} \le x$

$\Rightarrow \frac{1}{3} \le x \le 5$

Case 3: $5 < x$

$x - 5 - 2x - 3 \le 1$

$-9 \le x$

$\Rightarrow 5 < x$

Combined solution: $x \le -7$ or $\frac{1}{3} \le x$

The least positive value of $x = \frac{1}{3}$.