Cauchy Schwarz's inequality

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$(a_i x - b_i)^2 \geq 0$ for $i = 1, 2, \cdots, n$, where n is a positive integer.

$$\sum_{i=1}^{n} (a_i x - b_i)^2 \geq 0$$

$$\sum_{i=1}^{n} (a_i^2 x^2 - 2a_i b_i x + b_i^2) \geq 0$$

$$\sum_{i=1}^{n} a_i^2 x^2 - 2\sum_{i=1}^{n} a_i b_i x + \sum_{i=1}^{n} b_i^2 \geq 0$$

$$\left(\sum_{i=1}^{n} a_i^2\right) x^2 - 2\left(\sum_{i=1}^{n} a_i b_i\right) x + \left(\sum_{i=1}^{n} b_i^2\right) \geq 0$$

For real values of x, $\Delta = 4\left(\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leq 0$

$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$ (square product ≥ product square S.P. ≥ P.S.)

i.e. $(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$

Equality holds when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Example 1

Let $b_1 = b_2 = \cdots = b_n = 1$, then $(a_1^2 + a_2^2 + \cdots + a_n^2)(1 + 1 + \cdots + 1) \geq (a_1 + a_2 + \cdots + a_n)^2$

$n(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 + a_2 + \cdots + a_n)^2$

Equality holds when $a_1 = a_2 = \cdots = a_n$.

Example 2

Real numbers $a_1, a_2, \cdots, a_n$, not all zero, are given, and $x_1, x_2, \cdots, x_n$ are real variables satisfying the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 1$. Prove that the least value of $x_1^2 + x_2^2 + \cdots + x_n^2$ is $(a_1^2 + a_2^2 + \cdots + a_n^2)^{-1}$.

By Cauchy Schwarz's inequality,

$(x_1^2 + x_2^2 + \cdots + x_n^2)(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^2$

But the right-hand side of this inequality is unity, so that

$(x_1^2 + x_2^2 + \cdots + x_n^2) \geq \frac{1}{(a_1^2 + a_2^2 + \cdots + a_n^2)}$, which is the required result.

Example 3

If $a_1, a_2, \cdots, a_n$ are all positive, prove that $(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$.

By Cauchy Schwarz's inequality,

$$\left(\sqrt{a_1}^2 + \sqrt{a_2}^2 + \cdots + \sqrt{a_n}^2\right)\left(\frac{1}{\sqrt{a_1}^2} + \frac{1}{\sqrt{a_2}^2} + \cdots + \frac{1}{\sqrt{a_n}^2}\right) \geq \left(\sqrt{a_1} \cdot \frac{1}{\sqrt{a_1}} + \sqrt{a_2} \cdot \frac{1}{\sqrt{a_2}} + \cdots + \sqrt{a_n} \cdot \frac{1}{\sqrt{a_n}}\right)^2$$

Hence, $(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$.

Example 4

If $a_1, a_2, \cdots, a_n$ are n positive numbers whose sum is unity, prove that $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\ge n^2$.

By Cauchy Schwarz's inequality,

$$\left(\sqrt{a_1}^2+\sqrt{a_2}^2+\cdots+\sqrt{a_n}^2\right)\left(\frac{1}{\sqrt{a_1}^2}+\frac{1}{\sqrt{a_2}^2}+\cdots+\frac{1}{\sqrt{a_n}^2}\right)\ge\left(\sqrt{a_1}\cdot\frac{1}{\sqrt{a_1}}+\sqrt{a_2}\cdot\frac{1}{\sqrt{a_2}}+\cdots+\sqrt{a_n}\cdot\frac{1}{\sqrt{a_n}}\right)^2$$

Hence, $(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\ge n^2 \Rightarrow \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\ge n^2$.

Example 5 (HKMO 2017 FG2.2)

If real numbers x, y and z satisfy $(x + y + z) = 30$ and $C = x^2 + y^2 + z^2$, determine the least value of C.

Consider $t^2 - 2xt + x^2 = (t - x)^2 \cdots (1)$

$t^2 - 2yt + y^2 = (t - y)^2 \cdots (2)$

$t^2 - 2zt + z^2 = (t - z)^2 \cdots (3)$

(1) + (2) + (3):

L.H.S. $= 3t^2 - 2(x + y + z)t + (x^2 + y^2 + z^2)$

The function is always non-negative

$\Delta = 4(x + y + z)^2 - 4(3)(x^2 + y^2 + z^2) \le 0$

$(1^2 + 1^2 + 1^2)(x^2 + y^2 + z^2) \ge (x + y + z)^2$

$3C \ge 30^2$

$C \ge 300$

The minimum value of $C = 300$

Example 6 (SEVENTH USA MO 1978 Q1)

Given that a, b, c, d, e are real numbers such that

$a + b + c + d + e = 8$

$a^2 + b^2 + c^2 + d^2 + e^2 = 16$

Determine the maximum value of e .

By Cauchy Schwarz's inequality S.P. ≥ P.S.

$(a^2 + b^2 + c^2 + d^2)(1^2 + 1^2 + 1^2 + 1^2) \ge (a + b + c + d)^2$

$4(16 - e^2) \ge (8 - e)^2$

$64 - 4e^2 \ge 64 - 16e + e^2$

$0 \ge 5e^2 - 16e$

$0 \le e \le 3.2$

The maximum value of e is 3.2 .

(The maximum value of $e = 3.2$ when $a = b = c = d = (8 - 3.2) \div 4 = 1.2$.)

Example 7 Show that $\left(\sum_{i=1}^n a_i^4\right)\left(\sum_{i=1}^n b_i^4\right)\left(\sum_{i=1}^n c_i^4\right)\left(\sum_{i=1}^n d_i^4\right)\ge\left(\sum_{i=1}^n a_ib_ic_id_i\right)^4$.

By giving a suitable value to each d_i, deduce that $\left(\sum_{i=1}^n \alpha_i^3\right)\left(\sum_{i=1}^n \beta_i^3\right)\left(\sum_{i=1}^n \gamma_i^3\right)\ge\left(\sum_{i=1}^n \alpha_i\beta_i\gamma_i\right)^3$.

$$\left(\sum_{i=1}^n a_i^4\right)\left(\sum_{i=1}^n b_i^4\right)\left(\sum_{i=1}^n c_i^4\right)\left(\sum_{i=1}^n d_i^4\right)=\left[\sum_{i=1}^n (a_i^2)^2\sum_{i=1}^n (b_i^2)^2\right]\left[\sum_{i=1}^n (c_i^2)^2\sum_{i=1}^n (d_i^2)^2\right]\ge\left[\sum_{i=1}^n (a_i^2)(b_i^2)\right]^2\left[\sum_{i=1}^n (c_i^2)(d_i^2)\right]^2$$

$$=\left\{\left[\sum_{i=1}^n (a_ib_i)^2\right]\left[\sum_{i=1}^n (c_id_i)^2\right]\right\}^2\ge\left\{\left[\sum_{i=1}^n (a_ib_ic_id_i)\right]^2\right\}^2=\left(\sum_{i=1}^n a_ib_ic_id_i\right)^4.$$

Put $d_i=(a_ib_ic_i)^{\frac{1}{3}}$, then $\left(\sum_{i=1}^n a_i^4\right)\left(\sum_{i=1}^n b_i^4\right)\left(\sum_{i=1}^n c_i^4\right)\left[\sum_{i=1}^n (a_ib_ic_i)^{\frac{4}{3}}\right]\ge\left[\sum_{i=1}^n a_ib_ic_i(a_ib_ic_i)^{\frac{1}{3}}\right]^4=\left[\sum_{i=1}^n (a_ib_ic_i)^{\frac{4}{3}}\right]^4$

$$\Rightarrow \left[\sum_{i=1}^n \left(a_i^{\frac{4}{3}}\right)^3\right]\left[\sum_{i=1}^n \left(b_i^{\frac{4}{3}}\right)^3\right]\left[\sum_{i=1}^n \left(c_i^{\frac{4}{3}}\right)^3\right]\ge\frac{\left[\sum_{i=1}^n (a_ib_ic_i)^{\frac{4}{3}}\right]^4}{\left[\sum_{i=1}^n (a_ib_ic_i)^{\frac{4}{3}}\right]}$$

$$\left[\sum_{i=1}^n \left(a_i^{\frac{4}{3}}\right)^3\right]\left[\sum_{i=1}^n \left(b_i^{\frac{4}{3}}\right)^3\right]\left[\sum_{i=1}^n \left(c_i^{\frac{4}{3}}\right)^3\right]\ge\left[\sum_{i=1}^n (a_ib_ic_i)^{\frac{4}{3}}\right]^3$$

$$\left[\sum_{i=1}^n \left(a_i^{\frac{4}{3}}\right)^3\right]\left[\sum_{i=1}^n \left(b_i^{\frac{4}{3}}\right)^3\right]\left[\sum_{i=1}^n \left(c_i^{\frac{4}{3}}\right)^3\right]\ge\left[\sum_{i=1}^n (a_i)^{\frac{4}{3}}(b_i)^{\frac{4}{3}}(c_i)^{\frac{4}{3}}\right]^3$$

Let $\alpha=a_i^{\frac{4}{3}}$, $\beta=b_i^{\frac{4}{3}}$, $\gamma=c_i^{\frac{4}{3}}$, then $\left(\sum_{i=1}^n \alpha_i^3\right)\left(\sum_{i=1}^n \beta_i^3\right)\left(\sum_{i=1}^n \gamma_i^3\right)\ge\left(\sum_{i=1}^n \alpha_i\beta_i\gamma_i\right)^3$.