

12-13 Individual	1	$\sqrt{61} - \sqrt{33}$	2	180	3	30°	4	25	5	2012
	6	14	7	1	8	5	9	9	10	2
12-13 Group	1	13	2	192	3	6039	4	8028	5	10
	6	6	7	6	8	130	9	671	10	$\frac{2}{2013}$

Individual Events

- I1** Simplify $\sqrt{94 - 2\sqrt{2013}}$.

$$\begin{aligned}\sqrt{94 - 2\sqrt{2013}} &= \sqrt{61 - 2\sqrt{61 \times 33} + 33} \\ &= \sqrt{(\sqrt{61} - \sqrt{33})^2} \\ &= \sqrt{61} - \sqrt{33}\end{aligned}$$

- I2** A parallelogram is cut into 178 pieces of equilateral triangles with sides 1 unit. If the perimeter of the parallelogram is P units, find the maximum value of P .

2 equilateral triangles joint to form a small parallelogram.

$$178 = 2 \times 89$$

\therefore The given parallelogram is cut into 89 small parallelograms, and 89 is a prime number.

The dimension of the given parallelogram is 1 unit \times 89 units.

$$P = 2(1 + 89) = 180 \text{ units}$$

- I3** Figure 1 shows a right-angled triangle ACD where B is a point on AC and $BC = 2AB$. Given that $AB = a$ and $\angle ACD = 30^\circ$, find the value of θ .

$$\text{In } \triangle ABD, AD = \frac{a}{\tan \theta}$$

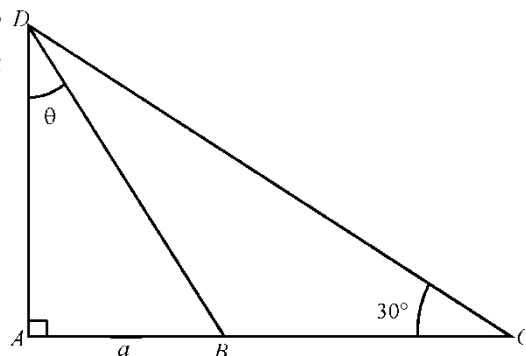
$$\text{In } \triangle ACD, AC = \frac{AD}{\tan 30^\circ} = \frac{\sqrt{3}a}{\tan \theta}$$

$$\text{However, } AC = AB + BC = a + 2a = 3a$$

$$\therefore \frac{\sqrt{3}a}{\tan \theta} = 3a$$

$$\tan \theta = \frac{\sqrt{3}}{3}$$

$$\Rightarrow \theta = 30^\circ$$



- I4** Given that $x^2 + 399 = 2^y$, where x, y are positive integers. Find the value of x .

Reference: 2018 HG6

$$2^9 = 512, 512 - 399 = 113 \neq x^2$$

$$2^{10} = 1024, 1024 - 399 = 625 = 25^2, x = 25$$

- I5** Given that $y = (x + 1)(x + 2)(x + 3)(x + 4) + 2013$, find the minimum value of y .

Reference 1993HG5, 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3

$$\begin{aligned}y &= (x + 1)(x + 4)(x + 2)(x + 3) + 2013 = (x^2 + 5x + 4)(x^2 + 5x + 6) + 2013 \\ &= (x^2 + 5x)^2 + 10(x^2 + 5x) + 24 + 2013 = (x^2 + 5x)^2 + 10(x^2 + 5x) + 25 + 2012 \\ &= (x^2 + 5x + 5)^2 + 2012 \geq 2012\end{aligned}$$

The minimum value of y is 2012.

- I6** In a convex polygon with n sides, one interior angle is selected. If the sum of the remaining $n - 1$ interior angle is 2013° , find the value of n .

Reference: 1989 HG2, 1990 FG10.3-4, 1992 HG3, 2002 FI3.4

$$1980^\circ = 180^\circ \times (13 - 2) < 2013^\circ < 180^\circ \times (14 - 2) = 2160^\circ$$

$$n = 14$$

- I7** Figure 2 shows a circle passes through two points B and C , and a point A is lying outside the circle. Given that BC is a diameter of the circle, AB and AC intersect the circle at D and E respectively and

$$\angle BAC = 45^\circ, \text{ find } \frac{\text{area of } \triangle ADE}{\text{area of } BCED}.$$

In $\triangle ACD$, $\angle BAC = 45^\circ$ (given)

$\angle ADC = 90^\circ$ (adj. \angle on st. line, \angle in semi-circle)

$$\therefore \frac{AD}{AC} = \sin 45^\circ = \frac{1}{\sqrt{2}} \quad \dots\dots (1)$$

$\angle ADE = \angle ACB$ (ext. \angle cyclic quad.)

$\angle AED = \angle ABC$ (ext. \angle cyclic quad.)

$\angle DAE = \angle CAB$ (common \angle)

$\therefore \triangle ADE \sim \triangle ACB$ (equiangular)

$$\frac{AD}{AC} = \frac{AE}{AB} \quad (\text{corr. of sides, } \sim \Delta\text{'s}) \quad \dots\dots (2)$$

$$\frac{\text{area of } \triangle ADE}{\text{area of } \triangle ABC} = \frac{\frac{1}{2} AD \cdot AE \sin 45^\circ}{\frac{1}{2} AC \cdot AB \sin 45^\circ} = \left(\frac{AD}{AC} \right)^2 \quad \text{by (2)}$$

$$= \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2} \quad \text{by (1)}$$

$$\Rightarrow \frac{\text{area of } \triangle ADE}{\text{area of } BCED} = 1$$

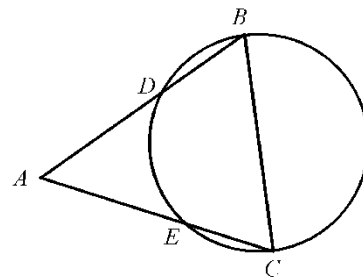
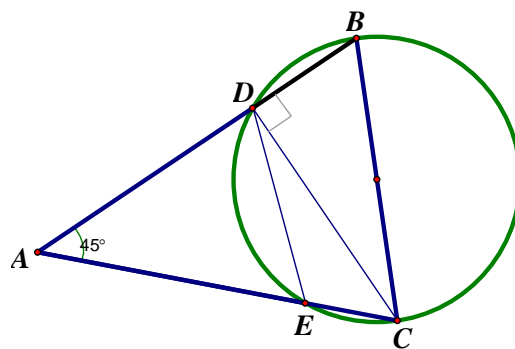


Figure 2



18 Solve $\sqrt{31-\sqrt{31+x}} = x$.

$$0 < \sqrt{31-\sqrt{31+x}} < \sqrt{36} = 6 \Rightarrow 0 < x < 6$$

$$\text{Then } 31 - \sqrt{31+x} = x^2$$

$$\Rightarrow (31 - x^2)^2 = 31 + x$$

$$x^4 - 62x^2 - x + 31^2 - 31 = 0$$

$$(x^4 - x) - 62x^2 + 930 = 0$$

$$(x^2 - x)(x^2 + x + 1) - 62x^2 + 930 = 0$$

$$\text{We want to factorise the above equation as } (x^2 - x + a)(x^2 + x + 1 + b) = 0$$

$$a(x^2 + x) + b(x^2 - x) = -62x^2 \dots\dots (1) \text{ and } a(1 + b) = 930 \dots\dots (2)$$

$$\text{From (1), } a + b = -62 \dots\dots (3), a - b = 0 \dots\dots (4)$$

$$\therefore a = b = -31 \dots\dots (5), \text{ sub. (5) into (2): L.H.S.} = -31(-30) = 930 = \text{R.H.S.}$$

$$(x^2 - x - 31)(x^2 + x + 1 - 31) = 0$$

$$x^2 - x - 31 = 0 \text{ or } x^2 + x - 30 = 0$$

$$x = \frac{1-5\sqrt{5}}{2}, \frac{1+5\sqrt{5}}{2}, 5, -6.$$

$$\frac{1-5\sqrt{5}}{2} < 0 \text{ and } \frac{1+5\sqrt{5}}{2} = \frac{1+\sqrt{125}}{2} > \frac{1+\sqrt{121}}{2} = \frac{1+11}{2} = 6$$

$$\therefore 0 < x < 6 \therefore x = 5 \text{ only.}$$

Method 2

$$\text{Let } \sqrt{31+\sqrt{31-y}} = y$$

$$\Rightarrow (y^2 - 31)^2 = 31 - y, \text{ then clearly } y > x$$

$$\text{and } y = \sqrt{31+\sqrt{31-y}} > \sqrt{30.25} = 5.5$$

$$y^4 - 62y^2 + y + 930 = 0 \dots\dots (2)$$

$$(2) - (1): y^4 - x^4 - 62(y^2 - x^2) + y + x = 0$$

$$(y+x)(y-x)(y^2+x^2) - 62(y+x)(y-x) + (y+x) = 0$$

$$(y+x)[(y-x)(y^2+x^2) - 62(y-x) + 1] = 0$$

$$\therefore y+x \neq 0 \therefore (y-x)(y^2+x^2-62) + 1 = 0 \dots\dots (3)$$

Assume x and y are positive integers. Then (3) becomes

$$(y-x)(y^2+x^2-62) = -1 \Rightarrow y-x = 1 \dots\dots (4) \text{ and } y^2+x^2-62 = -1 \dots\dots (5)$$

$$\text{From (4), } y = x + 1 \dots\dots (6)$$

$$\text{Sub. (6) into (5): } (x+1)^2 + x^2 - 62 = -1$$

$$x^2 + 2x + 1 + x^2 - 61 = 0$$

$$2x^2 + 2x - 60 = 0$$

$$x^2 + x - 30 = 0$$

$$\Rightarrow x = 5 \text{ or } -6 \text{ (rejected)}$$

- I9** Figure 3 shows a pentagon $ABCDE$. $AB = BC = DE = AE + CD = 3$ and $\angle A = \angle C = 90^\circ$, find the area of the pentagon.

Draw the altitude $BN \perp DE$.

Let $AE = y$, $CD = 3 - y$

Cut $\triangle ABE$ out and then stick the triangle to BC as shown in the figure.

$\triangle ABE \cong \triangle CBF$ (by construction)

$CF = AE = y$ (corr. sides, $\cong \Delta$'s)

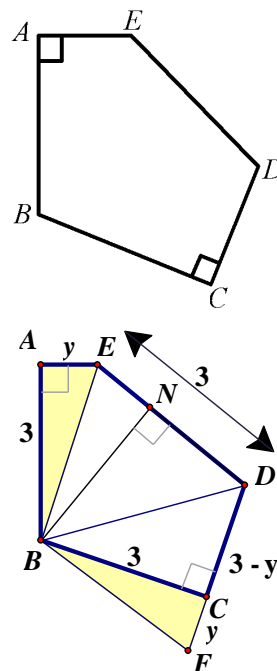
$\therefore DE = DF = 3$

$BD = BD$ (common side)

$BE = BF$ (corr. sides, $\cong \Delta$'s)

$\therefore \triangle BDE \cong \triangle BDF$ (S.S.S.)

\therefore The area of the pentagon = area of $\triangle BDE$ + area of $\triangle BDF$
 $= 2 \times \text{area of } \triangle BDE$
 $= 2 \times \frac{1}{2} \times 3 \times 3$
 $= 9 \text{ sq. units}$



- I10** If a and b are real numbers, and $a^2 + b^2 = a + b$. Find the maximum value of $a + b$.

$$a^2 + b^2 = a + b \Rightarrow \left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 = \frac{1}{2} \dots\dots (1)$$

$$\left[\left(a - \frac{1}{2}\right) - \left(b - \frac{1}{2}\right)\right]^2 \geq 0$$

$$\Rightarrow \left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 - 2\left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right) \geq 0$$

$$\text{By (1), } \frac{1}{2} - 2\left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right) \geq 0$$

$$\Rightarrow \frac{1}{2} \geq 2\left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right) \dots\dots (2)$$

$$\begin{aligned} (a + b - 1)^2 &= \left[\left(a - \frac{1}{2}\right) + \left(b - \frac{1}{2}\right)\right]^2 \\ &= \left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 + 2\left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right) \\ &\leq \frac{1}{2} + \frac{1}{2} = 1 \text{ (by (1) and (2))} \end{aligned}$$

$$\therefore (a + b - 1)^2 \leq 1$$

$$\Rightarrow -1 \leq a + b - 1 \leq 1$$

$$\Rightarrow 0 \leq a + b \leq 2$$

The maximum value of $a + b = 2$.

Group Events

- G1** Given that the length of the sides of a right-angled triangle are integers, and two of them are the roots of the equation $x^2 - (m + 2)x + 4m = 0$. Find the maximum length of the third side of the triangle. **Reference: 2000 FI5.2, 2001 FI2.1, 2010 FI2.2, 2011 FI3.1**

Let the 3 sides of the right-angled triangle be a , b and c .

If a , b are the roots of the quadratic equation, then $a + b = m + 2$ and $ab = 4m$

$$4a + 4b = 4m + 8 = ab + 8$$

$$4a - ab + 4b = 8$$

$$a(4 - b) - 16 + 4b = -8$$

$$a(4 - b) - 4(4 - b) = -8$$

$$(a - 4)(4 - b) = -8$$

$$(a - 4)(b - 4) = 8$$

$a - 4$	$b - 4$	a	b	c
1	8	5	12	13
2	4	6	8	10

\therefore The maximum value of the third side is 13.

- G2** Figure 1 shows a trapezium $ABCD$, where $AB = 3$, $CD = 5$ and the diagonals AC and BD meet at O . If the area of $\triangle AOB$ is 27, find the area of the trapezium $ABCD$.

Reference 1993 HI2, 1997 HG3, 2000 FI2.2, 2002 FI1.3, 2004 HG7, 2010 HG4

$AB \parallel DC$

$\therefore \triangle AOB \sim \triangle COD$ (equiangular)

$$\frac{\text{Area of } \triangle COD}{\text{Area of } \triangle AOB} = \left(\frac{5}{3}\right)^2 \Rightarrow \frac{\text{Area of } \triangle COD}{27} = \frac{25}{9}$$

$$\Rightarrow \text{Area of } \triangle COD = 75$$

$$\frac{\text{Area of } \triangle AOD}{\text{Area of } \triangle AOB} = \frac{DO}{OB} \Rightarrow \frac{\text{Area of } \triangle AOD}{27} = \frac{5}{3}$$

$$\Rightarrow \text{Area of } \triangle AOD = 45$$

$$\frac{\text{Area of } \triangle BOC}{\text{Area of } \triangle AOB} = \frac{CO}{OA} \Rightarrow \frac{\text{Area of } \triangle BOC}{27} = \frac{5}{3}$$

$$\Rightarrow \text{Area of } \triangle BOC = 45$$

$$\text{The area of the trapezium } ABCD = 27 + 75 + 45 + 45 = 192$$

- G3** Let x and y be real numbers such that $x^2 + xy + y^2 = 2013$. Find the maximum value of $x^2 - xy + y^2$.

$$2013 = x^2 + xy + y^2 = (x + y)^2 - xy$$

$$\Rightarrow xy = (x + y)^2 - 2013 \geq 0 - 2013 = -2013 \dots (*)$$

$$\text{Let } T = x^2 - xy + y^2$$

$$2013 = x^2 + xy + y^2 = (x^2 - xy + y^2) + 2xy = T + 2xy$$

$$2xy = 2013 - T \geq -2013 \times 2 \quad \text{by } (*)$$

$$\therefore 2013 \times 3 \geq T$$

The maximum value of $x^2 - xy + y^2$ is 6039.

- G4** If α , β are roots of $x^2 + 2013x + 5 = 0$, find the value of $(\alpha^2 + 2011\alpha + 3)(\beta^2 + 2015\beta + 7)$. **Reference: 1993 HG2, 2010 HI2, 2019 HI9**

$$\alpha^2 + 2013\alpha + 5 = 0 \Rightarrow \alpha^2 + 2011\alpha + 3 = -2\alpha - 2$$

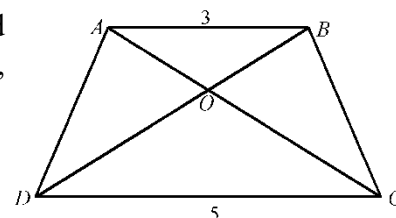
$$\beta^2 + 2013\beta + 5 = 0 \Rightarrow \beta^2 + 2015\beta + 7 = 2\beta + 2$$

$$(\alpha^2 + 2011\alpha + 3)(\beta^2 + 2015\beta + 7) = (-2\alpha - 2)(2\beta + 2)$$

$$= -4(\alpha + 1)(\beta + 1)$$

$$= -4(\alpha\beta + \alpha + \beta + 1)$$

$$= -4(5 - 2013 + 1) = 8028$$



- G5** As shown in Figure 2, $ABCD$ is a square of side 10 units, E and F are the mid-points of CD and AD respectively, BE and FC intersect at G . Find the length of AG .

Join BF and AG .

$$CE = DE = DF = FA = 5$$

Clearly $\triangle BCE \cong \triangle CDF \cong \triangle BAF$ (S.A.S.)

Let $\angle CBG = x = \angle ABF$ (corr. \angle s \cong Δ 's)

$$\angle BCE = 90^\circ$$

$$\angle BCG = 90^\circ - x$$

$$\angle BGC = 90^\circ \text{ (}\angle\text{s sum of } \Delta\text{)}$$

Consider $\triangle ABG$ and $\triangle FBC$.

$$\frac{AB}{BF} = \cos x$$

$$\angle ABG = x + \angle FBG = \angle FBC$$

$$\frac{GB}{BC} = \cos x$$

$\therefore \triangle ABG \sim \triangle FBC$ (ratio of 2 sides, included angle)

$$\therefore \frac{AB}{FB} = \frac{AG}{FC} \text{ (corr. sides, } \sim \Delta\text{'s)}$$

$\therefore FB = FC$ (corr. sides, $\triangle CDF \cong \triangle BAF$)

$$\therefore AG = AB = 10$$

Method 2 Define a rectangle coordinate system with A = origin, AB = positive x -axis, AD = positive y -axis.

$$B = (10, 0), C = (10, 10), E = (5, 10), D = (0, 10), F = (0, 5)$$

$$\text{Equation of } CF: y - 5 = \frac{10-5}{10-0}(x-0) \Rightarrow y = \frac{1}{2}x + 5 \dots\dots (1)$$

$$\text{Equation of } BE: y - 0 = \frac{10-0}{5-10}(x-10) \Rightarrow y = -2x + 20 \dots\dots (2)$$

$$(1) = (2): \frac{1}{2}x + 5 = -2x + 20 \Rightarrow x = 6$$

$$\text{Sub. } x = 6 \text{ into (1): } y = 8$$

$$\Rightarrow AG = \sqrt{6^2 + 8^2} = 10$$

- G6** Let a and b are positive real numbers, and the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ have real roots. Find the minimum value of $a + b$. (**Reference: 1999 FG5.2**)

Discriminants of the two equations ≥ 0

$$a^2 - 8b \geq 0 \dots\dots (1)$$

$$(2b)^2 - 4a \geq 0 \Rightarrow b^2 - a \geq 0 \dots\dots (2)$$

$$a^2 \geq 8b \Rightarrow a^4 \geq 64b^2 \geq 64a$$

$$\Rightarrow a^4 - 64a \geq 0$$

$$\Rightarrow a(a-4)(a^2+4a+16) \geq 0$$

$$\Rightarrow a(a-4)[(a+2)^2+12] \geq 0$$

$$\Rightarrow a(a-4) \geq 0$$

$$\Rightarrow a \leq 0 \text{ or } a \geq 4$$

$$\therefore a > 0 \therefore a \geq 4 \text{ only}$$

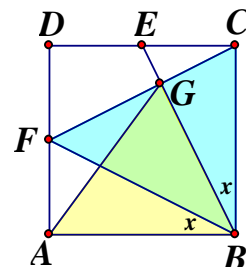
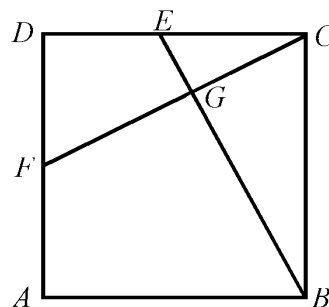
$$\text{When } a = 4, \text{ sub. into (2): } b^2 - 4 \geq 0$$

$$\Rightarrow (b+2)(b-2) \geq 0$$

$$\Rightarrow b \leq -2 \text{ or } b \geq 2$$

$$\therefore b > 0 \therefore b \geq 2 \text{ only}$$

$$\text{The minimum value of } a + b = 4 + 2 = 6$$



- G7** Given that the length of the three sides of $\triangle ABC$ form an arithmetic sequence, and are the roots of the equation $x^3 - 12x^2 + 47x - 60 = 0$, find the area of $\triangle ABC$.

Let the roots be $a - d, a$ and $a + d$.

$$a - d + a + a + d = 12 \Rightarrow a = 4 \dots\dots (1)$$

$$(a - d)a + a(a + d) + (a - d)(a + d) = 47$$

$$\Rightarrow 3a^2 - d^2 = 47 \Rightarrow d = \pm 1 \dots\dots (2)$$

$$(a - d)a(a + d) = 60 \Rightarrow a^3 - ad^2 = 60 \dots\dots (3)$$

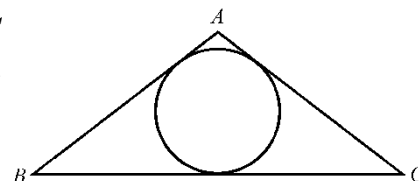
Sub. (1) and (2) into (3): L.H.S. = $64 - 4 = 60 =$ R.H.S.

\therefore The 3 sides of the triangle are 3, 4 and 5.

$$\text{The area of } \triangle ABC = \frac{1}{2} \cdot 3 \cdot 4 = 6 \text{ sq. units.}$$

- G8** In Figure 3, $\triangle ABC$ is an isosceles triangle with $AB = AC$, $BC = 240$. The radius of the inscribed circle of $\triangle ABC$ is 24. Find the length of AB . **Reference 2007 FG4.4, 2022 P1Q15**

Let I be the centre. The inscribed circle touches AB and CA at F and E respectively. Let $AB = AC = x$.



Let D be the mid-point of BC .

$$\triangle ABD \cong \triangle ACD \text{ (S.S.S.)}$$

$$\angle ADB = \angle ADC = 90^\circ$$

corr. \angle s, $\cong \Delta$'s, adj. \angle s on st. line

BC touches the circle at D

(converse, tangent \perp radius)

$$ID = IE = IF = \text{radii} = 24$$

$IE \perp AC$, $IF \perp AB$ (tangent \perp radius)

$$AD = \sqrt{x^2 - 120^2} \text{ (Pythagoras' theorem)}$$

$$S_{\triangle ABC} = S_{\triangle IBC} + S_{\triangle ICA} + S_{\triangle IAB} \text{ (where } S \text{ stands for areas)}$$

$$\frac{1}{2} \cdot 240 \cdot \sqrt{x^2 - 120^2} = \frac{1}{2} \cdot 240 \cdot 24 + \frac{1}{2} \cdot x \cdot 24 + \frac{1}{2} \cdot x \cdot 24$$

$$\frac{1}{2} \cdot 10 \cdot \sqrt{x^2 - 120^2} = \frac{1}{2} \cdot 240 + \frac{1}{2} \cdot x + \frac{1}{2} \cdot x$$

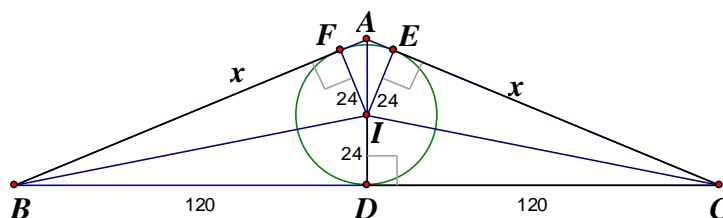
$$5\sqrt{(x-120)(x+120)} = \sqrt{(120+x)^2}$$

$$5\sqrt{x-120} = \sqrt{120+x}$$

$$25(x-120) = 120+x$$

$$24x = 25 \times 120 + 120 = 26 \times 120$$

$$AB = x = 26 \times 5 = 130$$



G9 At most how many numbers can be taken from the set of integers: $1, 2, 3, \dots, 2012, 2013$ such that the sum of any two numbers taken out from the set is not a multiple of the difference between the two numbers?

In order to understand the problem, let us take out a few numbers and investigate the property.

Take 1, 5, 9.

$$1 + 5 = 6, 5 - 1 = 4, 6 \neq 4k, \text{ for any integer } k$$

$$1 + 9 = 10, 9 - 1 = 8, 10 \neq 8k, \text{ for any integer } k$$

$$5 + 9 = 14, 9 - 5 = 4, 14 \neq 4k, \text{ for any integer } k$$

Take 3, 6, 8.

$$3 + 6 = 9, 6 - 3 = 3, 9 = 3 \times 3$$

$$3 + 8 = 11, 8 - 3 = 5, 11 \neq 5k \text{ for any integer } k$$

$$6 + 8 = 14, 8 - 6 = 2, 14 = 2 \times 7$$

Take 12, 28, 40.

$$12 + 28 = 40, 28 - 12 = 16, 40 \neq 16k \text{ for any integer } k$$

$$28 + 40 = 68, 40 - 28 = 12, 68 \neq 12k \text{ for any integer } k$$

$$12 + 40 = 52, 40 - 12 = 28, 52 \neq 28k \text{ for any integer } k$$

\therefore We can take three numbers 1, 5, 9 or 12, 28, 40 (but not 3, 6, 8).

Take the arithmetic sequence 1, 3, 5, \dots , 2013. (1007 numbers)

The general term = $T(n) = 2n - 1$ for $1 \leq n \leq 1007$

$$T(n) + T(m) = 2n + 2m - 2 = 2(n + m - 1)$$

$$T(n) - T(m) = 2n - 2m = 2(n - m)$$

$$T(n) + T(m) = [T(n) - T(m)]k \text{ for some integer } k. \text{ For example, } 3 + 5 = 8 = (5 - 3) \times 4.$$

\therefore The sequence 1, 3, 5, \dots , 2013 does not satisfy the condition.

Take the arithmetic sequence 1, 4, 7, \dots , 2011. (671 numbers)

The general term = $T(n) = 3n - 2$ for $1 \leq n \leq 671$

$$T(n) + T(m) = 3n + 3m - 4 = 3(n + m - 1) - 1$$

$$T(n) - T(m) = 3n - 3m = 3(n - m) \Rightarrow T(n) + T(m) \neq [T(n) - T(m)]k \text{ for any non-zero integer } k$$

We can take at most 671 numbers to satisfy the condition.

G10 For all positive integers n , define a function f as

(i) $f(1) = 2012$,

(ii) $f(1) + f(2) + \dots + f(n-1) + f(n) = n^2 f(n)$, $n > 1$.

Find the value of $f(2012)$.

Reference: 2014 FG1.4, 2022 P2Q8

$$f(1) + f(2) + \dots + f(n-1) = (n^2 - 1) f(n) \Rightarrow f(n) = \frac{f(1) + f(2) + \dots + f(n-1)}{n^2 - 1}$$

$$f(2) = \frac{f(1)}{3} = \frac{2012}{3}$$

$$f(3) = \frac{f(1) + f(2)}{8} = \frac{2012 + \frac{2012}{3}}{8} = \frac{1 + \frac{1}{3}}{8} \cdot 2012 = \frac{1}{6} \cdot 2012$$

$$f(4) = \frac{f(1) + f(2) + f(3)}{15} = \frac{2012 + \frac{2012}{3} + \frac{2012}{6}}{15} = \frac{\frac{2}{3}}{15} \cdot 2012 = \frac{1}{10} \cdot 2012$$

It is observed that the answer is 2012 divided by the n^{th} triangle number.

$$\text{Claim: } f(n) = \frac{2}{n(n+1)} \cdot 2012 \text{ for } n \geq 1$$

$n = 1, 2, 3, 4$, proved above.

$$\text{Suppose } f(k) = \frac{2}{k(k+1)} \cdot 2012 \text{ for } k = 1, 2, \dots, m \text{ for some positive integer } m.$$

$$f(m+1) = \frac{f(1) + f(2) + \dots + f(m)}{(m+1)^2 - 1} = \frac{\frac{2}{1 \times 2} + \frac{2}{2 \times 3} + \frac{2}{3 \times 4} + \frac{2}{4 \times 5} + \frac{2}{5 \times 6} + \dots + \frac{2}{m(m+1)}}{m(m+2)} \cdot 2012$$

$$= 2 \cdot \frac{\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{m} - \frac{1}{m+1}\right)}{m(m+2)} \cdot 2012$$

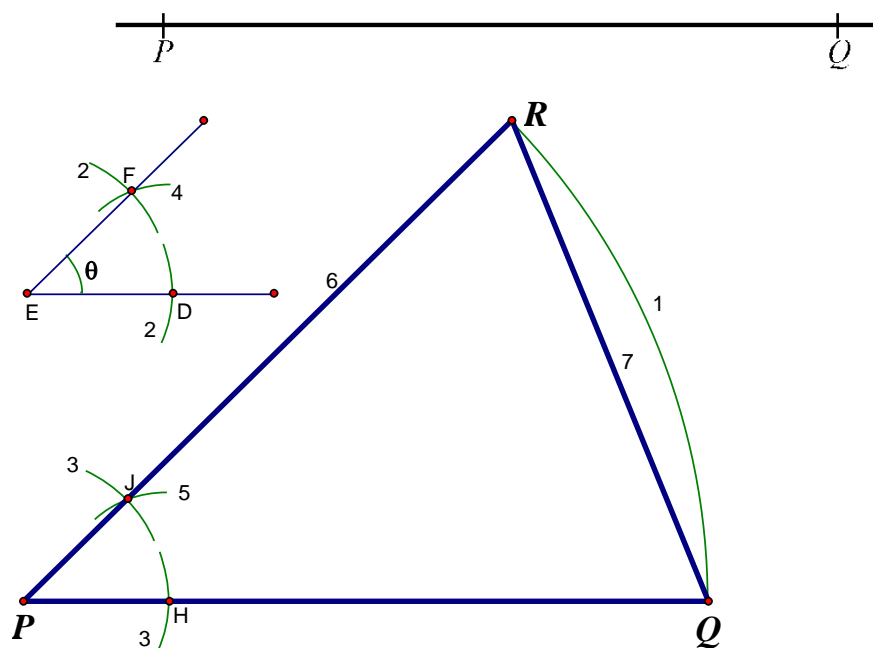
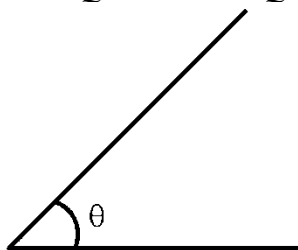
$$= 2 \cdot \frac{1 - \frac{1}{m+1}}{m(m+2)} \cdot 2012 = \frac{2}{(m+1)(m+2)} \cdot 2012$$

\therefore It is also true for m . By the principle of mathematical induction, the formula is true.

$$f(2012) = \frac{2}{2012 \times 2013} \cdot 2012 = \frac{2}{2013}$$

Geometrical Construction

- Line segment PQ and an angle of size θ are given below. Construct the isosceles triangle PQR with $PQ = PR$ and $\angle QPR = \theta$.

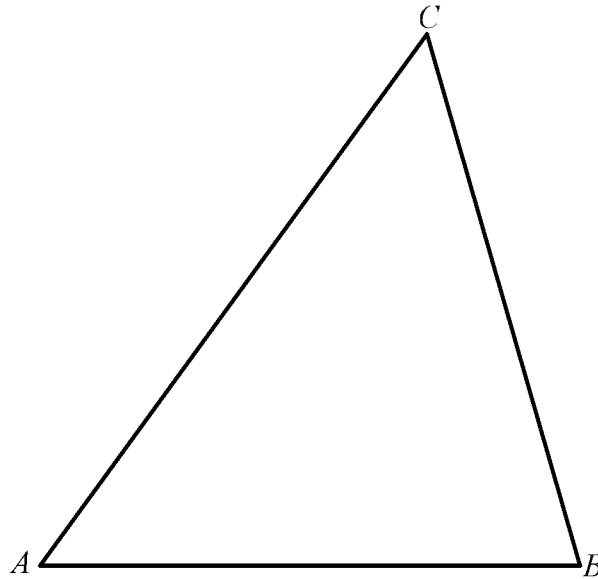


Steps. Let the vertex of the given angle be E .

- Use P as centre, PQ as radius to draw a circular arc QR .
- Use E as centre, a certain radius to draw an arc, cutting the given angle at D and F respectively.
- Use P as centre, the same radius in step 2 to draw an arc, cutting PQ at H .
- Use D as centre, DF as radius to draw an arc.
- Use H as centre, DF as radius to draw an arc, cutting the arc in step 3 at J .
- Join PJ , and extend PJ to cut the arc in step 1 at R .
- Join QR .

$\triangle PQR$ is the required triangle.

2. Construct a rectangle with AB as one of its sides and with area equal to that of $\triangle ABC$ below.



Theory

Let the height of the rectangle be h .

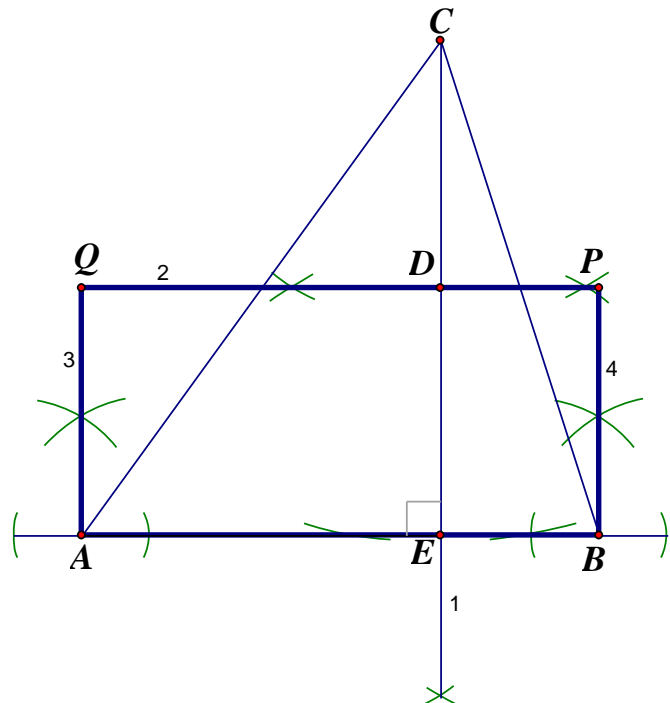
Let the height of the triangle be k .

\therefore Area of rectangle = area of triangle

$$AB \times h = \frac{1}{2} AB \times k$$

$$h = \frac{1}{2} k$$

\therefore The height of rectangle is half of the height of the triangle.

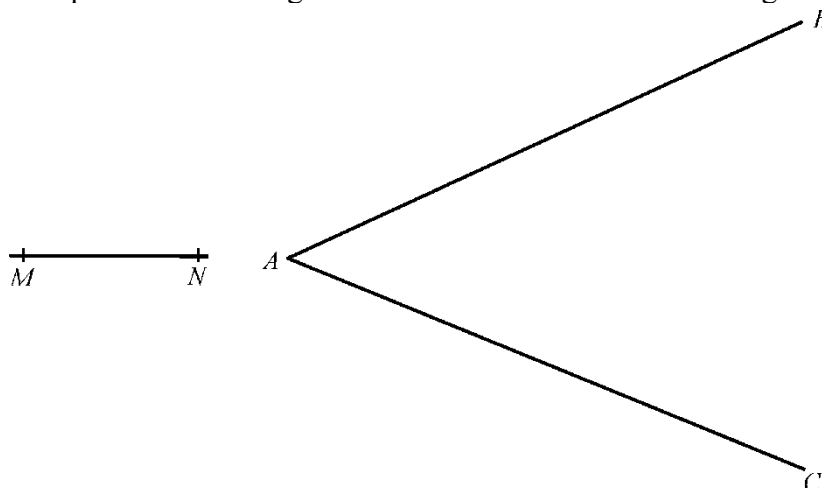


Steps.

1. Draw a line segment $CE \perp AB$. (E lies on AB , CE is the altitude of $\triangle ABC$)
2. Draw the perpendicular bisector PQ of CE , D is the mid-point of CE .
3. Draw a line segment $AQ \perp AB$, cutting PQ and Q .
4. Draw a line segment $BP \perp AB$, cutting PQ and P .

$ABPQ$ is the required rectangle.

3. The figure below shows two straight lines AB and AC intersecting at the point A . Construct a circle with radius equal to the line segment MN so that AB and AC are tangents to the circle.



Lemma:

如圖，已給一線段 AB ，過 B 作一線段垂直於 AB 。

作圖方法如下：

(1) 取任意點 C (C 在 AB 之間的上方) 為圓心， CB 為半徑作一圓，交 AB 於 P 。

(2) 連接 PC ，其延長線交圓於 Q ；連接 BQ 。

BQ 為所求的垂直線。

作圖完畢。

證明如下：

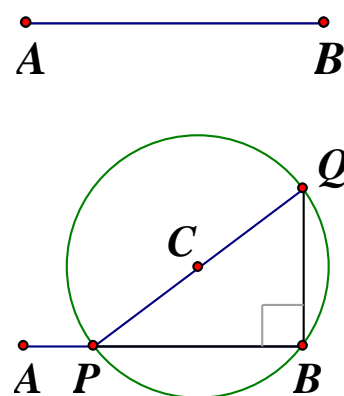
PCQ 為圓之直徑

$\angle PBQ = 90^\circ$

證明完畢。

(由作圖所得)

(半圓上的圓周角)



Steps.

1. Draw the angle bisector AQ
2. Use A as centre, MN as radius to draw an arc.
3. Use the lemma to draw $AP \perp AC$, AP cuts the arc in step 2 at P .
4. Draw $PQ \perp AP$, PQ cuts the angle bisector at Q .
5. Draw $QR \perp PQ$, QR cuts AC at R .
6. Use Q as centre, QR as radius to draw a circle.

This is the required circle.

Proof:

$\angle ARQ = 90^\circ$ (\angle s sum of polygon)

$APQR$ is a rectangle.

AC is a tangent touching the circle at R (converse, tangent \perp radius)

Let S be the foot of perpendicular drawn from Q onto AB , $QS \perp AB$.

$\triangle AQR \cong \triangle AQS$ (A.A.S.)

$\therefore SQ = SR$ (corr. sides, $\cong \Delta$'s)

S lies on the circle and $QS \perp AB$

$\therefore AB$ is a tangent touching the circle at S (converse, tangent \perp radius)

