| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **12-13 Individual** | **1** | $\sqrt{61}-\sqrt{33}$ | **2** | 180 | **3** | 30° | **4** | 25 | **5** | 2012 |
| | **6** | 14 | **7** | 1 | **8** | 5 | **9** | 9 | **10** | 2 |
| **12-13 Group** | **1** | 13 | **2** | 192 | **3** | 6039 | **4** | 8028 | **5** | 10 |
| | **6** | 6 | **7** | 6 | **8** | 130 | **9** | 671 | **10** | $\frac{2}{2013}$ |

**Individual Events**

**I1** Simplify $\sqrt{94-2\sqrt{2013}}$.

$$\sqrt{94-2\sqrt{2013}}=\sqrt{61-2\sqrt{61\times 33}+33}$$
$$=\sqrt{\left(\sqrt{61}-\sqrt{33}\right)^2}$$
$$=\sqrt{61}-\sqrt{33}$$

**I2** A parallelogram is cut into 178 pieces of equilateral triangles with sides 1 unit. If the perimeter of the parallelogram is $P$ units, find the maximum value of $P$.

2 equilateral triangles joint to form a small parallelogram.

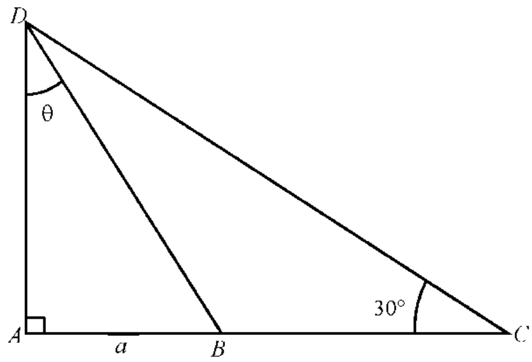
$178 = 2\times 89$

$\therefore$ The given parallelogram is cut into 89 small parallelograms, and 89 is a prime number.

The dimension of the given parallelogram is 1 unit × 89 units.

$P = 2(1 + 89) = 180$ units

**I3** Figure 1 shows a right-angled triangle $ACD$ where $B$ is a point on $AC$ and $BC = 2AB$. Given that $AB = a$ and $\angle ACD = 30°$, find the value of $\theta$.




In $\Delta ABD$, $AD=\dfrac{a}{\tan\theta}$

In $\Delta ACD$, $AC=\dfrac{AD}{\tan 30^\circ}=\dfrac{\sqrt{3}a}{\tan\theta}$

However, $AC = AB + BC = a + 2a = 3a$

$\therefore \dfrac{\sqrt{3}a}{\tan\theta}=3a$

$\tan\theta=\dfrac{\sqrt{3}}{3}$

$\Rightarrow \theta = 30°$

**I4** Given that $x^2 + 399 = 2^y$, where $x$, $y$ are positive integers. Find the value of $x$.

**Reference: 2018 HG6**

$2^9 = 512$, $512 - 399 = 113 \neq x^2$

$2^{10} = 1024$, $1024 - 399 = 625 = 25^2$, $x = 25$

**I5** Given that $y = (x + 1)(x + 2)(x + 3)(x + 4) + 2013$, find the minimum value of $y$.

**Reference 1993HG5, 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3**

$y = (x + 1)(x + 4)(x + 2)(x + 3) + 2013 = (x^2 + 5x + 4)(x^2 + 5x + 6) + 2013$

$= (x^2 + 5x)^2 + 10(x^2 + 5x) + 24 + 2013 = (x^2 + 5x)^2 + 10(x^2 + 5x) + 25 + 2012$

$= (x^2 + 5x + 5)^2 + 2012 \geq 2012$

The minimum value of $y$ is 2012.

**I6** In a convex polygon with $n$ sides, one interior angle is selected. If the sum of the remaining $n - 1$ interior angle is 2013°, find the value of $n$.

**Reference: 1989 HG2, 1990 FG10.3-4, 1992 HG3, 2002 FI3.4**

$1980° = 180°\times(13 - 2) < 2013° < 180°\times(14 - 2) = 2160°$

$n = 14$

**I7** Figure 2 shows a circle passes through two points $B$ and $C$, and a point $A$ is lying outside the circle. Given that $BC$ is a diameter of the circle, $AB$ and $AC$ intersect the circle at $D$ and $E$ respectively and $\angle BAC = 45°$, find $\dfrac{\text{area of } \Delta ADE}{\text{area of } BCED}$.

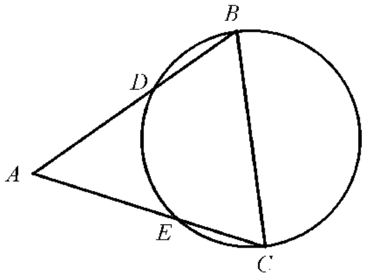
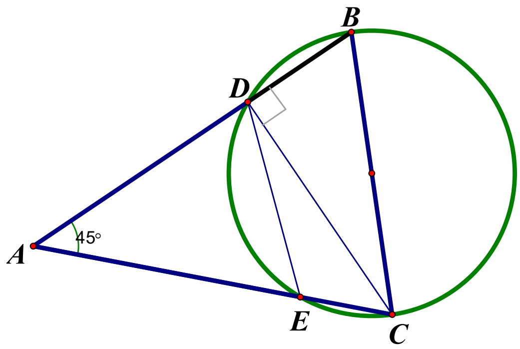



Figure 2

In $\Delta ACD$, $\angle BAC = 45°$ (given)

$\angle ADC = 90°$ (adj. $\angle$ on st. line, $\angle$ in semi-circle)

$\therefore \dfrac{AD}{AC} = \sin 45° = \dfrac{1}{\sqrt{2}}$ …… (1)




$\angle ADE = \angle ACB$ (ext. $\angle$ cyclic quad.)

$\angle AED = \angle ABC$ (ext. $\angle$ cyclic quad.)

$\angle DAE = \angle CAB$ (common $\angle$)

$\therefore \Delta ADE \sim \Delta ACB$ (equiangular)

$\dfrac{AD}{AC} = \dfrac{AE}{AB}$ (corr. of sides, ~Δ's) …… (2)

$$\frac{\text{area of } \Delta ADE}{\text{area of } \Delta ABC} = \frac{\frac{1}{2} AD \cdot AE \sin 45^\circ}{\frac{1}{2} AC \cdot AB \sin 45^\circ} = \left(\frac{AD}{AC}\right)^2 \text{ by (2)}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} \text{ by (1)}$$

$$\Rightarrow \frac{\text{area of } \Delta ADE}{\text{area of } BCED} = 1$$

**I8** Solve $\sqrt{31-\sqrt{31+x}}=x$.

$0<\sqrt{31-\sqrt{31+x}}<\sqrt{36}=6 \Rightarrow 0<x<6$

Then $31-\sqrt{31+x}=x^2$

$\Rightarrow(31-x^2)^2=31+x$

$x^4-62x^2-x+31^2-31=0$

$(x^4-x)-62x^2+930=0$

$(x^2-x)(x^2+x+1)-62x^2+930=0$

We want to factorise the above equation as $(x^2-x+a)(x^2+x+1+b)=0$

$a(x^2+x)+b(x^2-x)=-62x^2$ …… (1) and $a(1+b)=930$ …… (2)

From (1), $a+b=-62$ …… (3), $a-b=0$ …… (4)

$\therefore a=b=-31$…… (5), sub. (5) into (2): L.H.S. = –31(–30) = 930 = R.H.S.

$(x^2-x-31)(x^2+x+1-31)=0$

$x^2-x-31=0$ or $x^2+x-30=0$

$x=\frac{1-5\sqrt{5}}{2},\frac{1+5\sqrt{5}}{2},5,-6$.

$\frac{1-5\sqrt{5}}{2}<0$ and $\frac{1+5\sqrt{5}}{2}=\frac{1+\sqrt{125}}{2}>\frac{1+\sqrt{121}}{2}=\frac{1+11}{2}=6$

$\because 0<x<6 \therefore x=5$ only.

**Method 2**

Let $\sqrt{31+\sqrt{31-y}}=y$

$\Rightarrow (y^2-31)^2=31-y$, then clearly $y>x$

and $y=\sqrt{31+\sqrt{31-y}}>\sqrt{30.25}=5.5$

$y^4-62y^2+y+930=0$ …… (2)

(2) – (1): $y^4-x^4-62(y^2-x^2)+y+x=0$

$(y+x)(y-x)(y^2+x^2)-62(y+x)(y-x)+(y+x)=0$

$(y+x)[(y-x)(y^2+x^2)-62(y-x)+1]=0$

$\because y+x\neq 0 \therefore (y-x)(y^2+x^2-62)+1=0$ …… (3)

**Assume *x* and *y* are positive integers**. Then (3) becomes

$(y-x)(y^2+x^2-62)=-1 \Rightarrow y-x=1$ …… (4) and $y^2+x^2-62=-1$ …… (5)

From (4), $y=x+1$ …… (6)

Sub. (6) into (5): $(x+1)^2+x^2-62=-1$

$x^2+2x+1+x^2-61=0$

$2x^2+2x-60=0$

$x^2+x-30=0$

$\Rightarrow x=5$ or –6 (rejected)

**Method 3** Let $m=31$, then the equation becomes $\sqrt{m-\sqrt{m+x}}=x$

$m-\sqrt{m+x}=x^2 \Rightarrow (m-x^2)^2=m+x$

$m^2-2x^2m+x^4=m+x \Rightarrow m^2-(2x^2+1)m+x^4-x=0$

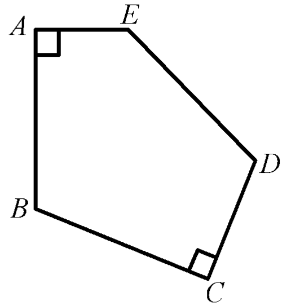
$$m=\frac{2x^2+1\pm\sqrt{(2x^2+1)^2-4(x^4-x)}}{2}=\frac{2x^2+1\pm\sqrt{4x^2+4x+1}}{2}=\frac{2x^2+1\pm\sqrt{(2x+1)^2}}{2}$$

$31=\frac{2x^2+1+(2x+1)}{2}$ or $\frac{2x^2+1-(2x+1)}{2} \Rightarrow 31=x^2+x+1$ or $31=x^2-x$

$x^2+x-30=0$ or $x^2-x-31=0$

$x=5$ or –6 (rejected) or $x=\frac{1\pm\sqrt{125}}{2}$ (both rejected, see **method 1**)

**I9** Figure 3 shows a pentagon $ABCDE$. $AB = BC = DE = AE + CD = 3$ and $\angle A = \angle C = 90°$, find the area of the pentagon.




Draw the altitude $BN \perp DE$.

Let $AE = y$, $CD = 3 - y$

Cut $\Delta ABE$ out and then stick the triangle to $BC$ as shown in the figure.

$\Delta ABE \cong \Delta CBF$ (by construction)

$CF = AE = y$ (corr. sides, $\cong\Delta$'s)

$\therefore DE = DF = 3$

$BD = BD$ (common side)

$BE = BF$ (corr. sides, $\cong\Delta$'s)

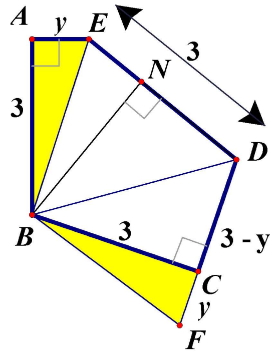
$\therefore \Delta BDE \cong \Delta BDF$ (S.S.S.)

$\therefore$ The area of the pentagon = area of $\Delta BDE$ + area of $\Delta BDF$

$= 2\times$ area of $\Delta BDF$

$= 2 \times \frac{1}{2} \times 3 \times 3$

$= 9$ sq. units




**I10** If $a$ and $b$ are real numbers, and $a^2 + b^2 = a + b$. Find the maximum value of $a + b$.

$a^2 + b^2 = a + b \Rightarrow \left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 = \frac{1}{2}$ …… (1)

$\left[\left(a - \frac{1}{2}\right) - \left(b - \frac{1}{2}\right)\right]^2 \geq 0$

$\Rightarrow \left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 - 2\left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right) \geq 0$

By (1), $\frac{1}{2} - 2\left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right) \geq 0$

$\Rightarrow \frac{1}{2} \geq 2\left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right)$ …… (2)

$(a + b - 1)^2 = \left[\left(a - \frac{1}{2}\right) + \left(b - \frac{1}{2}\right)\right]^2$

$= \left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 + 2\left(a - \frac{1}{2}\right)\left(b - \frac{1}{2}\right)$

$\leq \frac{1}{2} + \frac{1}{2} = 1$ (by (1) and (2))

$\therefore (a + b - 1)^2 \leq 1$

$\Rightarrow -1 \leq a + b - 1 \leq 1$

$\Rightarrow 0 \leq a + b \leq 2$

The maximum value of $a + b = 2$.

**Group Events**

**G1** Given that the length of the sides of a right-angled triangle are integers, and two of them are the roots of the equation $x^2-(m+2)x+4m=0$. Find the maximum length of the third side of the triangle.**Reference: 2000 FI5.2, 2001 FI2.1, 2010 FI2.2, 2011 FI3.1**

Let the 3 sides of the right-angled triangle be $a$, $b$ and $c$.

If $a$, $b$ are the roots of the quadratic equation, then $a + b = m + 2$ and $ab = 4m$

$4a + 4b = 4m + 8 = ab + 8$

$4a - ab + 4b = 8$

$a(4-b) - 16 + 4b = -8$

$a(4-b) - 4(4-b) = -8$

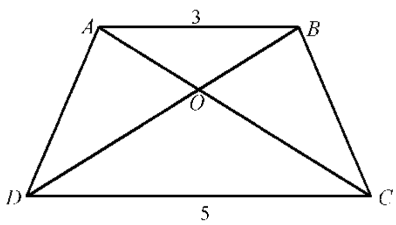
$(a-4)(4-b) = -8$

$(a-4)(b-4) = 8$

| $a-4$ | $b-4$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 1 | 8 | 5 | 12 | 13 |
| 2 | 4 | 6 | 8 | 10 |

$\therefore$ The maximum value of the third side is 13.

**G2** Figure 1 shows a trapezium $ABCD$, where $AB = 3$, $CD = 5$ and the diagonals $AC$ and $BD$ meet at $O$. If the area of $\Delta AOB$ is 27, find the area of the trapezium $ABCD$.




**Reference1993 HI2, 1997 HG3, 2000 FI2.2, 2002 FI1.3, 2004 HG7, 2010HG4**

$AB \mathbin{//} DC$

$\therefore \Delta AOB \sim \Delta COD$ (equiangular)

$$\frac{\text{Area of }\Delta COD}{\text{Area of }\Delta AOB}=\left(\frac{5}{3}\right)^2 \Rightarrow \frac{\text{Area of }\Delta COD}{27}=\frac{25}{9}$$

$\Rightarrow$ Area of $\Delta COD = 75$

$$\frac{\text{Area of }\Delta AOD}{\text{Area of }\Delta AOB}=\frac{DO}{OB} \Rightarrow \frac{\text{Area of }\Delta AOD}{27}=\frac{5}{3}$$

$\Rightarrow$ Area of $\Delta AOD = 45$

$$\frac{\text{Area of }\Delta BOC}{\text{Area of }\Delta AOB}=\frac{CO}{OA} \Rightarrow \frac{\text{Area of }\Delta BOC}{27}=\frac{5}{3}$$

$\Rightarrow$ Area of $\Delta BOC = 45$

The area of the trapezium $ABCD = 27 + 75 + 45 + 45 = 192$

**G3** Let $x$ and $y$ be real numbers such that $x^2 + xy + y^2 = 2013$.

Find the maximum value of $x^2 - xy + y^2$.

$2013 = x^2 + xy + y^2 = (x + y)^2 - xy$

$\Rightarrow xy = (x + y)^2 - 2013 \ge 0 - 2013 = -2013$ …… (*)

Let $T = x^2 - xy + y^2$

$2013 = x^2 + xy + y^2 = (x^2 - xy + y^2) + 2xy = T + 2xy$

$2xy = 2013 - T \ge -2013\times 2$ by (*)

$\therefore 2013\times 3 \ge T$

The maximum value of $x^2 - xy + y^2$ is 6039.

**G4** If $\alpha$, $\beta$ are roots of $x^2 + 2013x + 5 = 0$, find the value of $(\alpha^2 + 2011\alpha + 3)(\beta^2 + 2015\beta + 7)$.

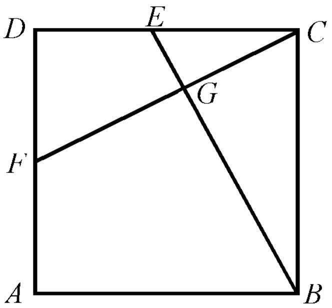
**Reference: 1993 HG2, 2010 HI2, 2019 HI9**

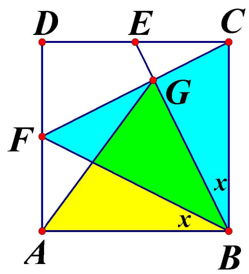
$\alpha^2 + 2013\alpha + 5 = 0 \Rightarrow \alpha^2 + 2011\alpha + 3 = -2\alpha - 2$

$\beta^2 + 2013\beta + 5 = 0 \Rightarrow \beta^2 + 2015\beta + 7 = 2\beta + 2$

$$\begin{aligned}(\alpha^2 + 2011\alpha + 3)(\beta^2 + 2015\beta + 7) &= (-2\alpha - 2)(2\beta + 2)\\ &= -4(\alpha + 1)(\beta + 1)\\ &= -4(\alpha\beta + \alpha + \beta + 1)\\ &= -4(5 - 2013 + 1) = 8028\end{aligned}$$

**G5** As shown in Figure 2, $ABCD$ is a square of side 10 units, $E$ and $F$ are the mid-points of $CD$ and $AD$ respectively, $BE$ and $FC$ intersect at $G$. Find the length of $AG$.




Join $BF$ and $AG$.

$CE = DE = DF = FA = 5$

Clearly $\Delta BCE \cong \Delta CDF \cong \Delta BAF$ (S.A.S.)

Let $\angle CBG = x = \angle ABF$ (corr. $\angle$s $\cong\Delta$'s)

$\angle BCE = 90°$

$\angle BCG = 90° - x$

$\angle BGC = 90°$ ($\angle$s sum of $\Delta$)

Consider $\Delta ABG$ and $\Delta FBC$.

$\dfrac{AB}{BF} = \cos x$

$\angle ABG = x + \angle FBG = \angle FBC$

$\dfrac{GB}{BC} = \cos x$

$\therefore \Delta ABG \sim \Delta FBC$ (ratio of 2 sides, included angle)

$\therefore \dfrac{AB}{FB} = \dfrac{AG}{FC}$ (corr. sides, $\sim\Delta$'s)

$\because FB = FC$ (corr. sides, $\Delta CDF \cong \Delta BAF$)

$\therefore AG = AB = 10$

**Method 2** Define a rectangle coordinate system with $A$ = origin, $AB$ = positive $x$-axis, $AD$ = positive $y$-axis.

$B = (10, 0)$, $C = (10, 10)$, $E = (5, 10)$, $D = (0, 10)$, $F = (0, 5)$

Equation of $CF$: $y - 5 = \dfrac{10-5}{10-0}(x-0) \Rightarrow y = \dfrac{1}{2}x + 5 \cdots\cdots (1)$

Equation of $BE$: $y - 0 = \dfrac{10-0}{5-10}(x-10) \Rightarrow y = -2x + 20 \cdots\cdots (2)$

(1) = (2): $\dfrac{1}{2}x + 5 = -2x + 20 \Rightarrow x = 6$

Sub. $x = 6$ into (1): $y = 8$

$\Rightarrow AG = \sqrt{6^2 + 8^2} = 10$

**G6** Let $a$ and $b$ are positive real numbers, and the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ have real roots. Find the minimum value of $a + b$. (**Reference: 1999 FG5.2**)

Discriminants of the two equations $\geq 0$

$a^2 - 8b \geq 0 \ldots\ldots (1)$

$(2b)^2 - 4a \geq 0 \Rightarrow b^2 - a \geq 0 \ldots\ldots (2)$

$a^2 \geq 8b \Rightarrow a^4 \geq 64b^2 \geq 64a$

$\Rightarrow a^4 - 64a \geq 0$

$\Rightarrow a(a-4)(a^2 + 4a + 16) \geq 0$

$\Rightarrow a(a-4)[(a+2)^2 + 12] \geq 0$

$\Rightarrow a(a-4) \geq 0$

$\Rightarrow a \leq 0$ or $a \geq 4$

$\because a > 0$ $\therefore a \geq 4$ only

When $a = 4$, sub. into (2): $b^2 - 4 \geq 0$

$\Rightarrow (b+2)(b-2) \geq 0$

$\Rightarrow b \leq -2$ or $b \geq 2$

$\because b > 0$ $\therefore b \geq 2$ only

The minimum value of $a + b = 4 + 2 = 6$

**G7** Given that the length of the three sides of $\Delta ABC$ form an arithmetic sequence, and are the roots of the equation $x^3 - 12x^2 + 47x - 60 = 0$, find the area of $\Delta ABC$.

Let the roots be $a-d$, $a$ and $a+d$.

$a-d+a+a+d=12 \Rightarrow a=4$…… (1)

$(a-d)a+a(a+d)+(a-d)(a+d)=47$

$\Rightarrow 3a^2-d^2=47 \Rightarrow d=\pm 1$ …… (2)

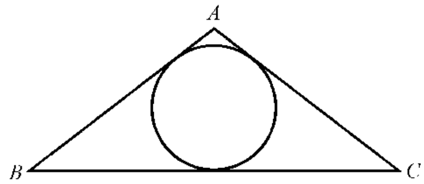
$(a-d)a(a+d)=60 \Rightarrow a^3-ad^2=60$ …… (3)

Sub. (1) and (2) into (3): L.H.S. = 64 – 4 = 60 = R.H.S.

$\therefore$ The 3 sides of the triangle are 3, 4 and 5.

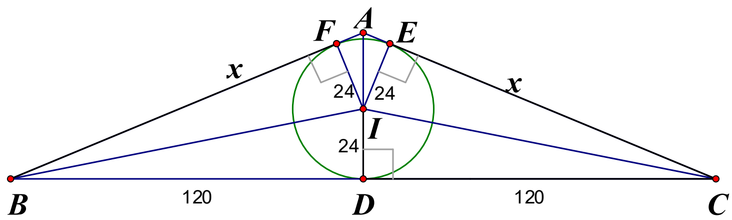
The area of $\Delta ABC = \frac{1}{2}\cdot 3\cdot 4 = 6$ sq. units.

**G8** In Figure 3, $\Delta ABC$ is an isosceles triangle with $AB = AC$, $BC = 240$. The radius of the inscribed circle of $\Delta ABC$ is 24. Find the length of $AB$. **Reference 2007 FG4.4, 2022 P1Q15**




Let $I$ be the centre. The inscribed circle touches $AB$ and $CA$ at $F$ and $E$ respectively. Let $AB = AC = x$.

Let $D$ be the mid-point of $BC$.




$\Delta ABD \cong \Delta ACD$ (S.S.S.)

$\angle ADB = \angle ADC = 90°$

corr. $\angle$s, $\cong\Delta$'s, adj. $\angle$s on st. line

$BC$ touches the circle at $D$

(converse, tangent $\perp$ radius)

$ID = IE = IF$ = radii = 24

$IE\perp AC$, $IF\perp AB$ (tangent$\perp$ radius)

$AD = \sqrt{x^2-120^2}$ (Pythagoras' theorem)

$S_{\Delta ABC} = S_{\Delta IBC} + S_{\Delta ICA} + S_{\Delta IAB}$ (where $S$ stands for areas)

$\frac{1}{2}\cdot 240\cdot\sqrt{x^2-120^2} = \frac{1}{2}\cdot 240\cdot 24 + \frac{1}{2}\cdot x\cdot 24 + \frac{1}{2}\cdot x\cdot 24$

$\frac{1}{2}\cdot 10\cdot\sqrt{x^2-120^2} = \frac{1}{2}\cdot 240 + \frac{1}{2}\cdot x + \frac{1}{2}\cdot x$

$5\sqrt{(x-120)(x+120)} = \sqrt{(120+x)^2}$

$5\sqrt{x-120} = \sqrt{120+x}$

$25(x-120) = 120 + x$

$24x = 25\times 120 + 120 = 26\times 120$

$AB = x = 26\times 5 = 130$

**G9** At most how many numbers can be taken from the set of integers: 1, 2, 3, $\cdots$ , 2012, 2013 such that the sum of any two numbers taken out from the set is not a multiple of the difference between the two numbers?

In order to understand the problem, let us take out a few numbers and investigate the property.

Take 1, 5, 9.

$1+5 = 6$, $5 - 1 = 4$, $6 \neq 4k$, for any integer $k$

$1 + 9 = 10$, $9 - 1 = 8$, $10 \neq 8k$, for any integer $k$

$5 + 9 = 14$, $9 - 5 = 4$, $14 \neq 4k$, for any integer $k$

Take 3, 6, 8.

$3 + 6 = 9$, $6 - 3 = 3$, $9 = 3\times3$

$3 + 8 = 11$, $8 - 3 = 5$, $11 \neq 5k$ for any integer $k$

$6 + 8 = 14$, $8 - 6 = 2$, $14 = 2\times7$

Take 12, 28, 40.

$12 + 28 = 40$, $28 - 12 = 16$, $40 \neq 16k$ for any integer $k$

$28 + 40 = 68$, $40 - 28 = 12$, $68 \neq 12k$ for any integer $k$

$12 + 40 = 52$, $40 - 12 = 28$, $52 \neq 28k$ for any integer $k$

$\therefore$ We can take three numbers 1, 5, 9 or 12, 28, 40 (but not 3, 6, 8).

Take the arithmetic sequence 1, 3, 5, … , 2013. (1007 numbers)

The general term = $T(n) = 2n - 1$ for $1 \leq n \leq 1007$

$T(n) + T(m) = 2n + 2m - 2 = 2(n + m - 1)$

$T(n) - T(m) = 2n - 2m = 2(n - m)$

$T(n) + T(m) = [T(n) - T(m)]k$ for some integer $k$. For example, $3 + 5 = 8 = (5 - 3)\times4$.

$\therefore$ The sequence 1, 3, 5, … , 2013 does not satisfy the condition.

Take the arithmetic sequence 1, 4, 7, … , 2011. (671 numbers)

The general term = $T(n) = 3n - 2$ for $1 \leq n \leq 671$

$T(n) + T(m) = 3n + 3m - 4 = 3(n + m - 1) - 1$

$T(n) - T(m) = 3n - 3m = 3(n - m) \Rightarrow T(n) + T(m) \neq [T(n) - T(m)]k$ for any non-zero integer $k$

We can take at most 671 numbers to satisfy the condition.

**G10** For all positive integers $n$, define a function f as

(i) $f(1) = 2012$,

(ii) $f(1) + f(2) + \cdots + f(n-1) + f(n) = n^2 f(n)$，$n > 1$ .

Find the value of f(2012) .

**Reference: 2014 FG1.4, 2022 P2Q8**

$f(1) + f(2) + \cdots + f(n-1) = (n^2-1)\, f(n) \Rightarrow f(n) = \dfrac{f(1)+f(2)+\cdots+f(n-1)}{n^2-1}$

$f(2) = \dfrac{f(1)}{3} = \dfrac{2012}{3}$

$f(3) = \dfrac{f(1)+f(2)}{8} = \dfrac{2012+\frac{2012}{3}}{8} = \dfrac{1+\frac{1}{3}}{8}\cdot 2012 = \dfrac{1}{6}\cdot 2012$

$f(4) = \dfrac{f(1)+f(2)+f(3)}{15} = \dfrac{2012+\frac{2012}{3}+\frac{2012}{6}}{15} = \dfrac{\frac{3}{2}}{15}\cdot 2012 = \dfrac{1}{10}\cdot 2012$

It is observed that the answer is 2012 divided by the $n^{\text{th}}$ triangle number.

Claim: $f(n) = \dfrac{2}{n(n+1)}\cdot 2012$ for $n \geq 1$

$n = 1, 2, 3, 4$, proved above.

Suppose $f(k) = \dfrac{2}{k(k+1)}\cdot 2012$ for $k = 1, 2, \cdots, m$ for some positive integer $m$ .

$$f(m+1) = \frac{f(1)+f(2)+\cdots+f(m)}{(m+1)^2-1} = \frac{\frac{2}{1\times 2}+\frac{2}{2\times 3}+\frac{2}{3\times 4}+\frac{2}{4\times 5}+\frac{2}{5\times 6}+\cdots+\frac{2}{m(m+1)}}{m(m+2)}\cdot 2012$$

$$= 2\cdot\frac{\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{m}-\frac{1}{m+1}\right)}{m(m+2)}\cdot 2012$$

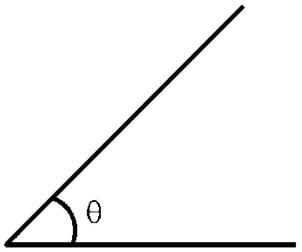
$$= 2\cdot\frac{1-\frac{1}{m+1}}{m(m+2)}\cdot 2012 = \frac{2}{(m+1)(m+2)}\cdot 2012$$

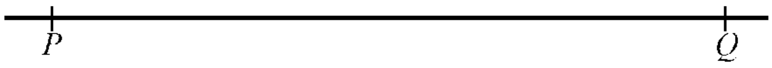
$\therefore$ It is also true for $m$. By the principle of mathematical induction, the formula is true.

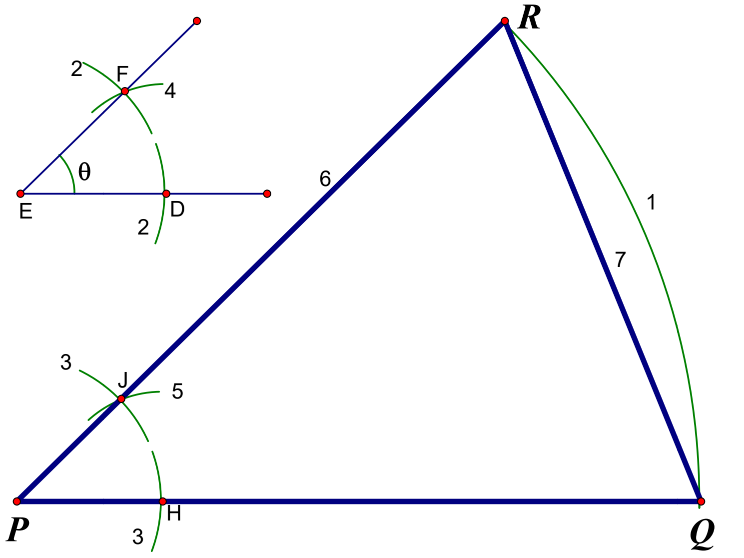
$f(2012) = \dfrac{2}{2012\times 2013}\cdot 2012 = \dfrac{2}{2013}$

**Geometrical Construction**

**1.** Line segment *PQ* and an angle of size θ are given below. Construct the isosceles triangle *PQR* with *PQ* = *PR* and ∠*QPR* = θ.



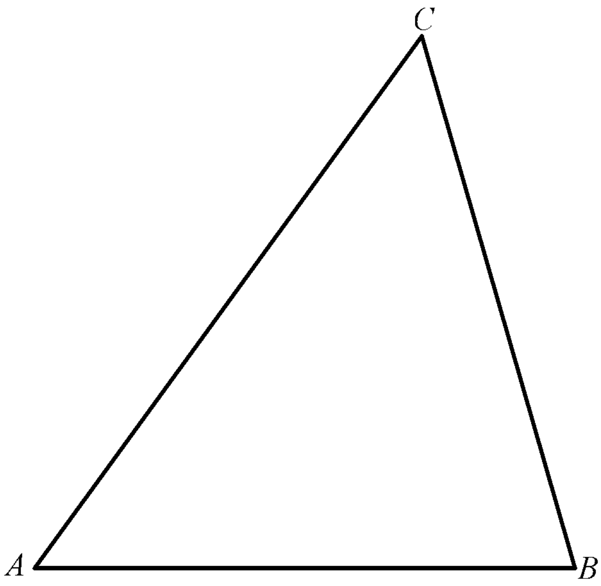




Steps. Let the vertex of the given angle be *E*.

1. Use *P* as centre, *PQ* as radius to draw a circular arc *QR*.
2. Use *E* as centre, a certain radius to draw an arc, cutting the given angle at *D* and *F* respectively.
3. Use *P* as centre, the same radius in step 2 to draw an arc, cutting *PQ* a *H*.
4. Use *D* as centre, *DF* as radius to draw an arc.
5. Use *H* as centre, *DF* as radius to draw an arc, cutting the arc in step 3 at *J*.
6. Join *PJ*, and extend *PJ* to cut the arc in step 1 at *R*.
7. Join *QR*.

Δ*PQR* is the required triangle.

**2.** Construct a rectangle with $AB$ as one of its sides and with area equal to that of $\Delta ABC$ below.




Theory

Let the height of the rectangle be $h$.

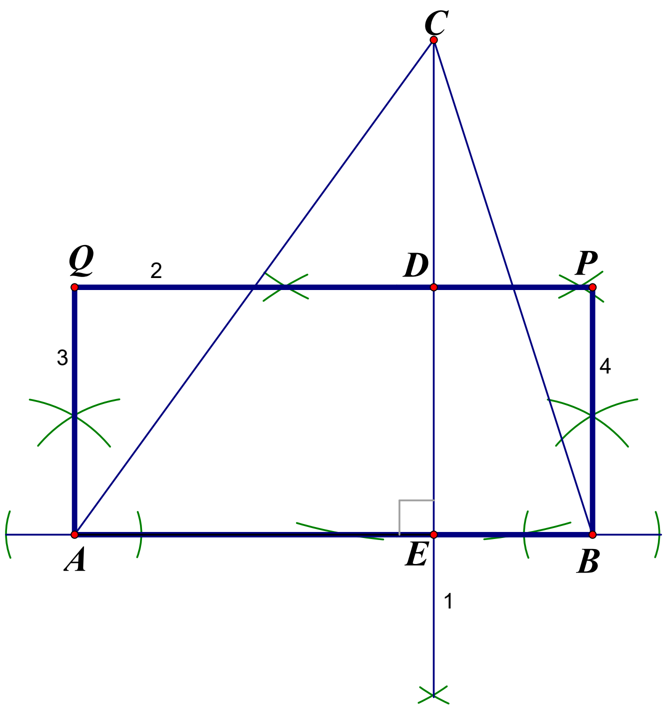
Let the height of the triangle be $k$.

$\because$ Area of rectangle = area of triangle

$AB\times h = \frac{1}{2} AB \times k$

$h = \frac{1}{2} k$

$\therefore$ The height of rectangle is half of the height of the triangle.

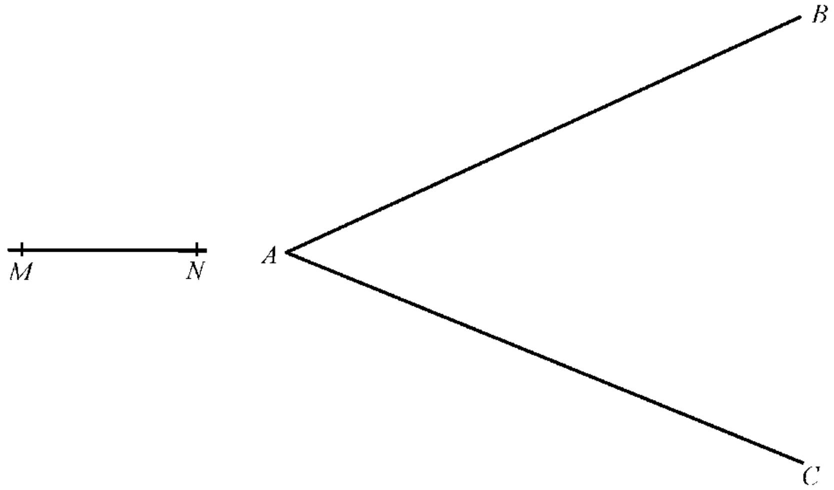



Steps.

1. Draw a line segment $CE\perp AB$. ($E$ lies on $AB$, $CE$ is the altitude of $\Delta ABC$)
2. Draw the perpendicular bisector $PQ$ of $CE$, $D$ is the mid-point of $CE$.
3. Draw a line segment $AQ\perp AB$, cutting $PQ$ and $Q$.
4. Draw a line segment $BP\perp AB$, cutting $PQ$ and $P$.

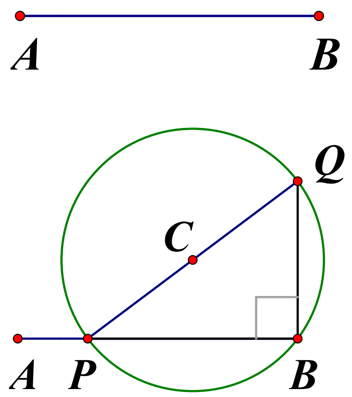
$ABPQ$ is the required rectangle.

**3.** The figure below shows two straight lines $AB$ and $AC$ intersecting at the point $A$. Construct a circle with radius equal to the line segment $MN$ so that $AB$ and $AC$ are tangents to the circle.




Lemma:

如圖，已給一綫段 $AB$，過 **$B$** 作一綫段垂直於 **$AB$**。




作圖方法如下：

(1) 取任意點 $C$ ($C$ 在 $AB$ 之間的上方)為圓心，$CB$ 為半徑作一圓，交 $AB$ 於 $P$。

(2) 連接 $PC$，其延長綫交圓於 $Q$；連接 $BQ$。

$BQ$ 為所求的垂直綫。

作圖完畢。

證明如下：

$PCQ$ 為圓之直徑 (由作圖所得)

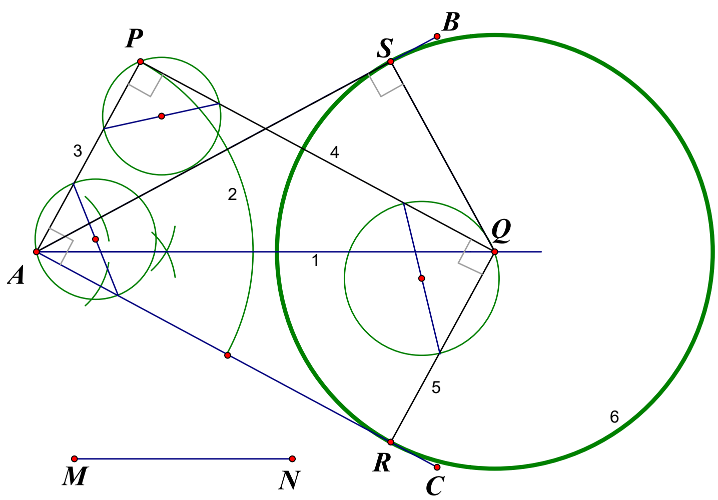
$\angle PBQ = 90°$ (半圓上的圓周角)

證明完畢。

Steps.

1. Draw the angle bisector $AQ$
2. Use $A$ as centre, $MN$ as radius to draw an arc.
3. Use the lemma to draw $AP \perp AC$, $AP$ cuts the arc in step 2 at $P$.
4. Draw $PQ \perp AP$, $PQ$ cuts the angle bisector at $Q$.
5. Draw $QR \perp PQ$, $QR$ cuts $AC$ at $R$.
6. Use $Q$ as centre, $QR$ as radius to draw a circle.

This is the required circle.




Proof:

$\angle ARQ = 90°$ ($\angle$s sum of polygon)

$APQR$ is a rectangle.

$AC$ is a tangent touching the circle at $R$ (converse, tangent $\perp$ radius)

Let $S$ be the foot of perpendicular drawn from $Q$ onto $AB$, $QS \perp AB$.

$\Delta AQR \cong \Delta AQS$ (A.A.S.)

$\therefore SQ = SR$ (corr. sides, $\cong\Delta$'s)

$S$ lies on the circle and $QS \perp AB$

$\therefore AB$ is a tangent touching the circle at $S$ (converse, tangent $\perp$ radius)