To prove A.M. ≥ G.M. by differentiation

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Let $f(x) = \log x - x$ for $x > 0$

$f'(x) = \frac{1}{x} - 1$

Let $f'(x) = 0 \Rightarrow x = 1$

For $0 < x < 1$, $f'(x) > 0$; for $x > 1$, $f'(x) < 0$

$\therefore$ $f(1) = -1$ is the absolute maximum

$\log x - x \le -1$ $\forall x > 0$, equality holds when $x = 1$

$\log x \le x - 1$

Now suppose $x_1, x_2, \cdots, x_n > 0$ such that $x_1 + x_2 + \cdots + x_n = n$, where n is a positive integer.

$\log x_1 \le x_1 - 1$

$\log x_2 \le x_2 - 1$

………………

$\log x_n \le x_n - 1$

Add up these n equations:

$\log(x_1x_2 \cdots x_n) \le x_1 + x_2 + \cdots + x_n - n = 0$

$\Rightarrow x_1x_2 \cdots x_n \le 1$ for $x_1, x_2, \cdots, x_n > 0$ such that $x_1 + x_2 + \cdots + x_n = n$

Suppose $a_1, a_2, \cdots, a_n > 0$, let $x_i = \frac{na_i}{a_1 + a_2 + \cdots + a_n}$ for $i = 1, 2, \cdots, n$.

Then $x_i > 0$ and $x_1 + x_2 + \cdots + x_n = \frac{n(a_1 + a_2 + \cdots + a_n)}{a_1 + a_2 + \cdots + a_n} = n$

By the above result, $x_1x_2 \cdots x_n \le 1$

$$\Rightarrow \frac{na_1}{a_1 + a_2 + \cdots + a_n} \cdot \frac{na_2}{a_1 + a_2 + \cdots + a_n} \cdots \frac{na_n}{a_1 + a_2 + \cdots + a_n} \le 1$$

$$\Rightarrow \frac{n^n a_1 a_2 \cdots a_n}{(a_1 + a_2 + \cdots + a_n)^n} \le 1$$

$$\Rightarrow a_1 a_2 \cdots a_n \le \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n$$

$$\Rightarrow \sqrt[n]{a_1 a_2 \cdots a_n} \le \frac{a_1 + a_2 + \cdots + a_n}{n} \quad \text{for } a_1, a_2, \cdots, a_n > 0$$

If $a_m = 0$ for some m, then $0 = \sqrt[n]{a_1 a_2 \cdots a_n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}$ for $a_i \ge 0$

This proves A.M. ≥ G.M.

Let a , $b \ge 0$ and p, $q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then $\frac{a^p}{p}+\frac{b^q}{q}\ge ab$.

Proof: If $a = 0$ or $b = 0$, then the inequality holds obviously.

Assume $a > 0$ and $b > 0$.

If $a^p = b^q$, LHS $=\frac{a^p}{p}+\frac{b^q}{q}=a^p\left(\frac{1}{p}+\frac{1}{q}\right)=a^p$ RHS$=ab=a\left(b^q\right)^{1/q}=a\left(a^p\right)^{1/q}=a^{p/p}a^{p/q}=a^{p\left(\frac{1}{p}+\frac{1}{q}\right)}=a^p$

If $a^p \ne b^q$, $\ln(ab) = \ln a + \ln b = \frac{\ln a^p}{p}+\frac{\ln b^q}{q}$

Let $y = \ln x$, $y' = \frac{1}{x} > 0$ for all $x > 0 \Rightarrow y$ is strictly increasing for all $x > 0$

$\therefore a^p \ne b^q \Rightarrow \ln a^p \ne \ln b^q$

Let $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x > 0 \Rightarrow f(x)$ is a strictly convex function

$\therefore f(tx + (1-t)y) < tf(x) + (1-t)f(y)$ for all $0 < t < 1$.

Let $t=\frac{1}{p}$, $1-t=\frac{1}{q}$, $x = \ln a^p$, $y = \ln b^b$, $tx + (1-t)y=\frac{\ln a^p}{p}+\frac{\ln b^q}{q}$

$\therefore\ e^{\frac{\ln a^p}{p}+\frac{\ln b^q}{q}}<\frac{1}{p}e^{\ln a^p+}+\frac{1}{q}e^{\ln b^q}$

$a\,b<\frac{a^p}{p}+\frac{b^q}{q}$

HKAL Pure Mathematics 1968 Paper 2 Q6 (*b*)

Let $\varphi(x) = 1-\lambda+\lambda x - x^\lambda$ be defined on the interval $0 \le x < +\infty$, where λ is a constant and $0 < \lambda < 1$. Find the minimum value of $\varphi(x)$. Hence or otherwise, prove that for any non-negative real numbers α and β $(1-\lambda)\alpha + \lambda\beta \ge \alpha^{1-\lambda}\beta^\lambda$ with equality if and only if $\alpha = \beta$.

$\varphi'(x) = \lambda - \lambda x^{\lambda-1} = \lambda(1 - x^{\lambda-1})$

For $0 \le x < 1$, $\varphi'(x) < 0$; $x = 1$, $\varphi'(1) = 0$; for $x > 1$, $\varphi'(x) > 0$

$\therefore$ The absolute minimum $= \varphi(1) = 1-\lambda+\lambda-1 = 0$

$\varphi(x) = 1-\lambda+\lambda x - x^\lambda \ge \varphi(1) = 0$, with equality holds if and only if $x = 1$ …… (1)

For any non-negative real numbers α and β, if $\alpha = 0$, then $(1-\lambda)\alpha + \lambda\beta = \lambda\beta \ge 0 = \alpha^{1-\lambda}\beta^\lambda$

If $\alpha > 0$, let $x=\frac{\beta}{\alpha}\ge 0$, by the result of (1),

$1-\lambda+\lambda\cdot\frac{\beta}{\alpha}-\left(\frac{\beta}{\alpha}\right)^\lambda \ge 0$, with equality holds if and only if $\frac{\beta}{\alpha}=1$.

$(1-\lambda)\alpha + \lambda\beta - \alpha^{1-\lambda}\beta^\lambda \ge 0$

$\therefore (1-\lambda)\alpha + \lambda\beta \ge \alpha^{1-\lambda}\beta^\lambda$ with equality if and only if $\alpha = \beta$.

Let $a = \alpha^{1-\lambda}$, $b = \beta^\lambda$, $\frac{1}{p}=1-\lambda$, $\frac{1}{q}=\lambda$

Then $\frac{1}{p}+\frac{1}{q}=1$, $\alpha=a^{\frac{1}{1-\lambda}}=a^p$, $\beta=b^{\frac{1}{\lambda}}=b^q$

$\frac{a^p}{p}+\frac{b^q}{q}\ge ab$ with equality if and only if $a^p = b^q$.

Hölder's Inequality

HKAL Pure Mathematics 1982 Paper 1 Q1 Last updated: 2021-09-01

(a) Let $0 < \lambda < 1$. Show that $\lambda t + (1 - \lambda) \geq t^{\lambda}$ for all $t > 0$.
Deduce that $\lambda\alpha + (1 - \lambda)\beta \geq \alpha^{\lambda}\,\beta^{1-\lambda}$ for all $\alpha, \beta \geq 0$.

(b) Let $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$ be two sets of non-negative real numbers such that $\sum_{i=1}^{n} a_i^p = \sum_{i=1}^{n} b_i^q = 1$. Using the result in (a), show that $\sum_{i=1}^{n} a_i b_i \leq 1$. Hence show that, for any two sets of non-negative real numbers $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$, $\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}$.

(a) Let $f(t) = \lambda t + (1 - \lambda) - t^{\lambda}$, $t > 0$, $0 < \lambda < 1$.
$f'(t) = \lambda - \lambda t^{\lambda-1} = \lambda(1 - t^{\lambda-1})$
For $0 < t < 1$, $f'(t) < 0$; $t = 1$, $f'(1) = 0$; for $t > 1$, $f'(t) > 0$
$\therefore$ The absolute minimum $= f(1) = 1 - \lambda + \lambda - 1 = 0$
$\lambda t + (1 - \lambda) \geq t^{\lambda}$ for all $t > 0$, with equality holds if and only if $t = 1$
For any non-negative real numbers α and β, if $\alpha = 0$, then $(1 - \lambda)\alpha + \lambda\beta = \lambda\beta \geq 0 = \alpha^{1-\lambda}\,\beta^{\lambda}$
If $\alpha > 0$, let $x = \frac{\beta}{\alpha} \geq 0$, by the result of (1),

$1 - \lambda + \lambda\cdot\frac{\beta}{\alpha} - \left(\frac{\beta}{\alpha}\right)^{\lambda} \geq 0$, with equality holds if and only if $\frac{\beta}{\alpha} = 1$.

$(1 - \lambda)\alpha + \lambda\beta - \alpha^{1-\lambda}\beta^{\lambda} \geq 0$
$\therefore (1 - \lambda)\alpha + \lambda\beta \geq \alpha^{1-\lambda}\,\beta^{\lambda}$ with equality if and only if $\alpha = \beta$.

(b) Let $a_i = \alpha^{1-\lambda}$, $b_i = \beta^{\lambda}$, $\frac{1}{p} = 1 - \lambda$, $\frac{1}{q} = \lambda$

Then $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha = a_i^{\frac{1}{1-\lambda}} = a_i^p$, $\beta = b_i^{\frac{1}{\lambda}} = b_i^q$

$$\frac{a_i^p}{p} + \frac{b_i^q}{q} \geq a_i b_j$$

Take summation from $i = 1, 2, \cdots, n$

$$\frac{\sum_{i=1}^{n} a_i^p}{p} + \frac{\sum_{i=1}^{n} b_i^q}{q} \geq \sum_{i=1}^{n} a_i b_j \Rightarrow \frac{1}{p} + \frac{1}{q} \geq \sum_{i=1}^{n} a_i b_j \Rightarrow \sum_{i=1}^{n} a_i b_i \leq 1$$

For any two sets of non-negative real numbers $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$,

if all items are zeros, then the inequality is obviously true. Otherwise, $\left(\sum_{j=1}^{n} x_j^p\right)^{\frac{1}{p}}, \left(\sum_{j=1}^{n} y_j^q\right)^{\frac{1}{q}} > 0$

Let $a_i = \frac{x_i}{\left(\sum_{j=1}^{n} x_j^p\right)^{\frac{1}{p}}}$, $b_i = \frac{y_i}{\left(\sum_{j=1}^{n} y_j^q\right)^{\frac{1}{q}}}$, then $\sum_{i=1}^{n} a_i^p = \sum_{i=1}^{n} b_i^q = 1$; by the result of above,

$$\sum_{i=1}^{n} \frac{x_i}{\left(\sum_{j=1}^{n} x_j^p\right)^{\frac{1}{p}}} \cdot \frac{y_i}{\left(\sum_{j=1}^{n} y_j^q\right)^{\frac{1}{q}}} \leq 1 \Rightarrow \sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}$$

Minkowski's Inequality

(a) Let $p > 1$ and $f(x) = x^p - px$ for all $x > 0$.

(i) Find the absolute minimum of $f(x)$ on the interval $(0, \infty)$.

(ii) Deduce that $x^p - 1 \ge p(x - 1)$ for all $x > 0$.

(b) (i) Let α, β, γ and δ be positive numbers such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $\gamma + \delta = 1$.

By taking $x = \alpha\gamma$ and $\beta\delta$ respectively, prove that, for $p > 1$, $\alpha^{p-1}\gamma^p + \beta^{p-1}\delta^p \ge 1$, where the equality holds if and only if $\alpha\gamma = \beta\delta = 1$.

(ii) Deduce that, if a, b, c and d are positive and $p > 1$, then

$$\left(\frac{a+b}{a}\right)^{p-1} c^p + \left(\frac{a+b}{b}\right)^{p-1} d^p \ge (c+d)^p .$$

(c) Suppose $\{a_i\}_{i=1,2,\ldots}$ and $\{b_i\}_{i=1,2,\ldots}$ are two sequences of positive numbers and $p > 1$. By considering $a = \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}}$ and $b = \left(\sum_{j=1}^{n} b_j^p\right)^{\frac{1}{p}}$,

prove that $\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \ge \left(\sum_{i=1}^{n} (a_i+b_i)^p\right)^{\frac{1}{p}}$,

where the equality holds if and only if $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}=\frac{a}{b}$.

(a) (i) Let $p > 1$ and $f(x) = x^p - px$ for all $x > 0$.

$f'(x) = px^{p-1} - p = p(x^{p-1} - 1)$

For $0 < x < 1$, $f'(x) < 0$; $x = 1$, $f'(1) = 0$; for $x > 1$, $f'(x) > 0$

$\therefore$ The absolute minimum $= f(1) = 1 - p$

(ii) $f(x) \ge f(1) \Rightarrow x^p - px \ge 1 - p$ for all $x > 0$.

(b) (i) Let α, β, γ and δ be positive numbers such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $\gamma + \delta = 1$.

Take $x = \alpha\gamma$, $(\alpha\gamma)^p - p(\alpha\gamma) \ge 1 - p$, equality holds iff $\alpha\gamma = 1$

Take $x = \beta\delta$, $(\beta\delta)^p - p(\beta\delta) \ge 1 - p$, equality holds iff $\beta\delta = 1$

$\therefore$ $\alpha^{p-1}\gamma^p - p\gamma \ge \frac{1-p}{\alpha}$ and $\beta^{p-1}\delta^p - p\delta \ge \frac{1-p}{\beta}$

Add up these equations: $\alpha^{p-1}\gamma^p + \beta^{p-1}\delta^p - p(\gamma + \delta) \ge (1-p)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$

$\alpha^{p-1}\gamma^p + \beta^{p-1}\delta^p - p \ge 1 - p \Rightarrow \alpha^{p-1}\gamma^p + \beta^{p-1}\delta^p \ge 1$,

where the equality holds if and only if $\alpha\gamma = \beta\delta = 1$.

(ii) Let $\alpha = \frac{a+b}{a}$, $\beta = \frac{a+b}{b}$, $\gamma = \frac{c}{c+d}$, $\delta = \frac{d}{c+d}$.

$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{a}{a+b}+\frac{b}{a+b}=1$, $\gamma + \delta = \frac{c}{c+d}+\frac{d}{c+d}=1$

By the result of (b)(i), $\left(\frac{a+b}{a}\right)^{p-1}\left(\frac{c}{c+d}\right)^p + \left(\frac{a+b}{b}\right)^{p-1}\left(\frac{d}{c+d}\right)^p \ge 1$

$\therefore$ $\left(\frac{a+b}{a}\right)^{p-1} c^p + \left(\frac{a+b}{b}\right)^{p-1} d^p \ge (c+d)^p$.

(c) Let $a=\left(\sum_{j=1}^{n}a_j^p\right)^{\frac{1}{p}}$, $b=\left(\sum_{j=1}^{n}b_j^p\right)^{\frac{1}{p}}$ $c = a_i$, $d = b_i$, by the result of (b)(ii),

$$\left[\frac{\left(\sum_{j=1}^{n}a_j^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}b_j^p\right)^{\frac{1}{p}}}{\left(\sum_{j=1}^{n}a_j^p\right)^{\frac{1}{p}}}\right]^{p-1}a_i^p+\left[\frac{\left(\sum_{j=1}^{n}a_j^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}b_j^p\right)^{\frac{1}{p}}}{\left(\sum_{i=1}^{n}b_j^p\right)^{\frac{1}{p}}}\right]^{p-1}b_i^p\ge(a_i+b_i)^p$$

Take summation from $i = 1, 2, \dots , n$

$$\left[\frac{\left(\sum_{j=1}^{n}a_j^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}b_j^p\right)^{\frac{1}{p}}}{\left(\sum_{j=1}^{n}a_j^p\right)^{\frac{1}{p}}}\right]^{p-1}\sum_{i=1}^{n}a_i^p+\left[\frac{\left(\sum_{j=1}^{n}a_j^p\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}b_j^p\right)^{\frac{1}{p}}}{\left(\sum_{i=1}^{n}b_j^p\right)^{\frac{1}{p}}}\right]^{p-1}\sum_{i=1}^{n}b_i^p\ge\sum_{i=1}^{n}(a_i+b_i)^p$$

$$\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}\right]^{p-1}\left(\sum_{i=1}^{n}a_i^p\right)^{1-\frac{p-1}{p}}+\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}\right]^{p-1}\left(\sum_{i=1}^{n}b_i^p\right)^{1-\frac{p-1}{p}}\ge\sum_{i=1}^{n}(a_i+b_i)^p$$

$$\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}\right]^{p-1}\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}\right]^{p-1}\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}\ge\sum_{i=1}^{n}(a_i+b_i)^p$$

$$\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}\right]^{p-1}\cdot\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}\right]\ge\sum_{i=1}^{n}(a_i+b_i)^p$$

$$\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}b_i^p\right)^{\frac{1}{p}}\right]^{p}\ge\sum_{i=1}^{n}(a_i+b_i)^p$$

where the equality holds if and only if $\alpha\gamma = \beta\delta = 1$.

i.e. $\frac{a+b}{a}\cdot\frac{c}{c+d}=\frac{a+b}{b}\cdot\frac{d}{c+d}=1$

$\Leftrightarrow\ \frac{a+b}{a}\cdot\frac{a_i}{a_i+b_i}=1$ and $\frac{a+b}{b}\cdot\frac{b_i}{a_i+b_i}=1$

$\Leftrightarrow\ \frac{a_i+b_i}{a_i}=\frac{a+b}{a}$ and $\frac{a_i+b_i}{b_i}=\frac{a+b}{b}$

$\Leftrightarrow\ 1+\frac{b_i}{a_i}=1+\frac{b}{a}$ and $\frac{a_i}{b_i}+1=\frac{a}{b}+1$

$\Leftrightarrow\ \frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}=\frac{a}{b}$.

Other Inequalities

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1989 Paper 2 Q1

4. Let $f(x) = \dfrac{e^x}{x^e}$ for $x > 0$,

   (a) Find the least value of $f(x)$.

   (b) Hence show that $e^\pi > \pi^e$.

4. Let $f(x) = \dfrac{e^x}{x^e}$ for $x > 0$,

   (a) Find the least value of $f(x)$.

$$f'(x) = \frac{x^e e^x - e^x x^{e-1}\cdot e}{\left(x^e\right)^2}$$

$$= \frac{x^e e^x\left(1-\frac{e}{x}\right)}{\left(x^e\right)^2} = \frac{e^x\left(1-\frac{e}{x}\right)}{x^e}$$

Let $f'(x) = 0 \Rightarrow x = e$

When $x > e$, $f'(x) > 0\ \forall\ x > 0$

When $0 < x < e$, $f'(x) < 0$

$\therefore$ $f(e)$ = absolute minimum

$f(x) \ge f(e) = \dfrac{e^e}{e^e} = 1$

$\therefore$ Least value of $f(x) = 1$

(b) Hence show that $e^\pi > \pi^e$.

$\pi \approx 3.14 > 2.718 \approx e$

$\therefore$ $f(\pi) > f(e)$

$\dfrac{e^\pi}{\pi^e} > 1$

$e^\pi > \pi^e$

1992 Paper 2 Q12 (a)

5. If $x > 0$, prove that $\dfrac{x}{1+x} < \ln(1+x) < x$.

Let $f(x) = x - \ln(1+x)$

$f'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$ for all $x > 0$

$\therefore$ $f(x)$ is an strictly increasing function

$f(x) > f(0) = 0 - \ln 1 = 0$

$x - \ln(1+x) > 0$

$x > \ln(1+x)$ …… (1)

Let $g(x) = (1+x)\ln(1+x) - x$, where $x > 0$

$g'(x) = 1 + \ln(1+x) - 1$

$g'(x) = \ln(1+x) > 0$ for all $x > 0$

$g'(x) > g(0)$

$(1+x)\ln(1+x) - x > 0$

$(1+x)\ln(1+x) > x$

$\therefore$ $\dfrac{x}{1+x} < \ln(1+x)$ …… (2)

Combine (1) and (2), $\dfrac{x}{1+x} < \ln(1+x) < x$.