2/9/2020

Fourier Series

Dirichlet's conditions:-

→ Consider a function $f(x)$ in the interval $(a, a+2l)$ satisfying:-

1) $f(x)$ is single valued in $(a, a+2l)$
2) $f(x)$ is continuous or finite discontinuities
3) $f(x)$ has either no or finite maxima and minima
4) $f(x)$ is periodic with period $2l$

Then $f(x)$ in the interval $(a, a+2l)$ can be written in terms of an infinite series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\cos \frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

where, $$a_0 = \frac{1}{2l} \int_a^{a+2l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_a^{a+2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

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$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\frac{n\pi x}{l}$$

→ $\sin^{-1}x$ can not be expressed as a fourier series as it is not single valued function

→ $\tan x$ can not be expressed as a fourier series in $(0, 2\pi)$

★ **Some imp formulae :-**

1) $$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{if } f(x) \text{ even}$$
$$= 0 \quad \text{if } f(x) \text{ odd}$$

2) $$\int e^{ax}\sin bx\,dx = \int \mathrm{I}.\mathrm{II}.dx$$
$$= \frac{1}{a^2+b^2}\left[\left(\frac{d\mathrm{I}}{dx}\right)\mathrm{II} - \left(\frac{d\mathrm{II}}{dx}\right)\mathrm{I}\right]$$

3) $$\int e^{ax}\cos bx\,dx = \int \mathrm{I}.\mathrm{II}\,dx$$
$$= \frac{1}{a^2+b^2}\left[\left(\frac{d\mathrm{I}}{dx}\right)\mathrm{II} - \left(\frac{d\mathrm{II}}{dx}\right)\mathrm{I}\right]$$

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4) $\int u.v\,dx = uv_1 - u'v_2 + u''v_3 - \dots$

where $u \rightarrow$ polynomial
$v \rightarrow$ exp, sin, cos

5) If $f(x)$ is discontinuous at $x = c$, then the value of $f(x)\,x = c$ is given by

$$\frac{1}{2}\left[\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x)\right]$$

6) Parseval's ~~interval~~ identity for $f(x)$:-

$$\frac{1}{2l}\int_a^{a+2l} (f(x))^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

$\rightarrow$ $\int_c^{c+2\pi} \sin(mx)\cos(nx)\,dx = 0$ for all m, n

$\rightarrow$ $\int_c^{c+2\pi} \sin(mx)\sin(nx)\,dx = 0$ if $m \neq n$

$\rightarrow$ $\int_c^{c+2\pi} \cos(mx)\cos(nx) = 0$ if $m \neq n$
$= \pi$ if $m = n$

$\rightarrow$ $\sin(n\pi) = 0$, $\cos(n\pi) = (-1)^n$ if n is an integer

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03/09) Lec-14 03/09/2020

1) Obtain the fourier exp of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ with $f(x+2\pi) = f(x)$. Also deduce

i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots$

ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \dots$

$\rightarrow$ $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ — ①

$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4} dx$

$= \frac{1}{8\pi} \int_0^{2\pi} (\pi - x)^2$

$= \frac{1}{8\pi} \left[\cancel{\pi^3} \frac{(\pi - x)^3}{-3} \right]_0^{2\pi}$

$a_0 = \frac{\pi^2}{12}$

$$a_n = \frac{1}{\gamma}\int_0^{2\gamma} f(x)\cos nx\, dx$$

$$= \frac{1}{\gamma}\int_0^{2\gamma} \frac{(\gamma-x)^2}{4}\cos nx\, dx$$

$$= \frac{1}{4\gamma}\left[(\gamma-x)^2 \sin\left(\frac{nx}{n}\right) - 2(\gamma-x)\cdot(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\gamma}$$

$$= \frac{1}{4\gamma}\left[\frac{2\gamma}{n^2} + \frac{2\gamma}{n^2}\right]$$

$$a_n = \frac{1}{n^2} \qquad [\because \cos 2n\gamma = 1]$$

$$b_n = \frac{1}{\gamma}\int_0^{2\gamma} f(x)\sin nx\, dx$$

$$= \frac{1}{\gamma}\int_0^{2\gamma} \frac{(\gamma-x)^2}{4}\left(-\frac{\cos nx}{n}\right)$$

$$- 2(\gamma-x)(-1)\left(-\frac{\sin nx}{n^2}\right)$$

$$+ 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\Big]_0^{2\gamma}$$

$$= \frac{1}{4\gamma}\left[\left(\frac{-\gamma^2\cos 2n\gamma}{n} + 0 + \frac{2\cos 2n\gamma}{n^3}\right) - \left(-\frac{\gamma^2}{n} + 0 + \frac{2}{n^3}\right)\right]$$

$$b_n = \frac{1}{4\pi}\left(\frac{-\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right)$$

$$\underline{\underline{b_n = 0}}$$

Putting in (1)

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

$$= \frac{\pi^2}{12} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \dots$$

i) Now, let $x = 0$

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

ii) Now, let $x = \pi$

$$0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

iv) Using Parseval's identity,

$$\frac{1}{2\pi}\int_0^{2\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum (a_n^2 + b_n^2)$$

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Now, $\frac{1}{2r}\int_0^{2r}[f(x)]^2 dx = \frac{1}{2r}\int_0^{2r}\left(\frac{r-x}{2}\right)^4 dx$

$$= \frac{1}{32r}\left[r^4x - 2r^3x^3 - rx^4 + \frac{x^5}{5}\right]_0^{2r}$$

$$= \frac{1}{32r}\left[2r^5 - 8r^5 + 16r^5 - 16r^5 + \frac{32r^5}{5}\right]$$

$$= \frac{1}{32r}\left[\frac{2r^5}{5}\right]$$

$$= \frac{r^4}{80}$$

$$\therefore \frac{r^4}{80} = \frac{r^4}{144} + \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right)$$

$$\therefore r^4\left(\frac{1}{80} - \frac{1}{144}\right) = \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right)$$

$$\frac{r^4}{120} = \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right)$$

04/09 Lec-15 04/09/2020

Q2) Expand $f(x) = x\sin x$ in the interval $(0, 2\pi)$, Deduce that

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$$

$\rightarrow$ Let $f(x) = a_0 + \sum_{n=1} a_n \cos nx$

$$+ \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin x\, dx$$

$$= \frac{1}{2\pi}\Big[x(-\cos x) - (-\sin x)\Big]_0^{2\pi}$$

$$= -1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x \cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin x \cdot \cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$(-1)\left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

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$$= \frac{1}{2\pi}\left[(2\pi)\left[-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right] - 0\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1}$$

$$= \frac{2}{n^2-1} \quad \text{if } n \neq 1$$

If $n=1$, the above method fails,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin x \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos 4\pi}{2}\right) - 0\right]$$

$$= -\frac{1}{2}$$

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$$b_n = \frac{1}{\gamma}\int_0^{2\gamma} f(x)\sin nx\,dx$$

$$= \frac{1}{\gamma}\int_0^{2\gamma} x\sin x \;\sin nx\,dx \qquad -③$$

$$b_n = \frac{1}{2\gamma}\int_0^{2\gamma} x\,.\,2\sin x\;\sin nx\,dx$$

$$= \frac{1}{2\gamma}\int_0^{2\gamma} x\,.\,2\sin nx\,.\sin x\,dx$$

$$= -\frac{1}{2\gamma}\int_0^{2\gamma} x\left[\cos(n+1)x - \cos(n-1)x\right]dx$$

$$b_n = -\frac{1}{2\gamma}\left[x\left(\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right)\right.$$

$$\left.(-1)\left(-\frac{\cos(n+1)x}{(n+1)^2} + \frac{\cos(n-1)x}{(n-1)^2}\right)\right]$$

$$= -\frac{1}{2\gamma}\left[(-1)-\left(-\frac{\cos 2(n+1)\gamma}{(n+1)^2} + \frac{\cos(2)(n-1)\gamma}{(n-1)^2}\right)\right.$$

$$\left.+1\left[\frac{-1}{(n+1)^2} + \frac{1}{(n-1)^2}\right]\right]$$

$$= \frac{1}{2\gamma}\left[\frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right]$$

$$= 0 \quad \text{if } n \neq 1$$

if $n=1$, the above method fails

putting $n=1$ in (3)

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]$$

$$= \frac{1}{2\pi}\left[\left(2\pi(2\pi - 0) - \left(\frac{4\pi^2}{2} + \frac{1}{4}\right)\right) - \left(0 - \frac{1}{4}\right)\right]$$

$$= \frac{1}{2\pi}\left[2\pi^2\right] = \underline{\underline{\pi}}$$

Putting these in (1),

$$x\sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + \pi\sin x$$

putting $x=0$, we get

$$\frac{3}{4} = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

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09/09/20 Lec - 16 09/09/2020

Q3) Find the fourier series for $f(x) = e^{-x}$ in $0 < x < 2\pi$. Deduce the value of the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$

Also derive the series for cosec $h\pi$

→ let $f(x) = e^{-x} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1} b_n \sin nx$ — ①

$$\therefore a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-x}\, dx$$

$$= \frac{1}{2\pi}\left[-e^{-x}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(-e^{-2\pi}+1\right)$$

$$= \frac{1-e^{-2\pi}}{2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x} \cos nx\, dx$$

$$= \frac{1}{\pi} \cdot \frac{1}{n^2+1}\left[e^{-x}\left(-\cos nx + n \sin nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}\left(-\cos 2n\pi + n \sin n2\pi\right) - e^0\left(-\cos 0 + n \sin 0\right)\right]$$

$$= \frac{1}{\gamma(1+n^2)} \left[e^{-2\gamma}(-1) - (-1) \right]$$

$$= \frac{1 - e^{-2\gamma}}{\gamma(1+n^2)}$$

$$b_n = \frac{1}{\gamma} \int_0^{2\gamma} f(x) \sin nx \, dx$$

$$= \frac{1}{\gamma} \int_0^{2\gamma} e^{-x} \sin nx \, dx$$

$$= \frac{1}{\gamma} \cdot \frac{1}{1+n^2} \left[e^{-x} \left(-\sin nx \cdot n \cos nx \right) \right]_0^{2\gamma}$$

$$= \frac{n}{\gamma(1+n^2)} \left(1 - e^{-2\gamma} \right)$$

Putting these in (1)

$$e^{-\gamma} = \frac{1 - e^{-2\gamma}}{2\gamma} + \left(\frac{1 - e^{-2\gamma}}{\gamma} \right) \left[\sum_{n=1}^{\infty} \frac{1}{1+n^2} \cos nx \right]$$

$\because \sin n\gamma = 0$

$$= \frac{1 - e^{-2\gamma}}{2\gamma} + \frac{1 - e^{-2\gamma}}{\gamma} \left[-\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n^2+1} (-1)^n \right.$$

$$e^{-\gamma} = \frac{1 - e^{-2\gamma}}{\gamma} \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} \quad : \frac{\gamma}{e^{\gamma}(1 - e^{-2\gamma})}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{(1+n)^2}$$

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$$\therefore \frac{\pi}{e^{\pi}-e^{-\pi}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} \quad \therefore \frac{\pi}{2} \cdot \frac{2}{e^{\pi}-\pi^{-\pi}}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\therefore \frac{\pi}{2} \cdot \frac{1}{\sin h\pi} = \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$\rightarrow$ $$\operatorname{cosec} h\pi = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$= \frac{2}{\pi}\left[\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} \dots\right)$$

Q4) Find fourier exp of $\cos px$ in $(0, 2\pi)$ where p is not an integer then deduce that $\pi \operatorname{cosech} p\pi = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{p+n} + \frac{1}{p-n}\right]$

Also deduce that $\pi \cot 2p\pi$
$= \frac{1}{2p} + p\sum_{n=1}^{\infty} \frac{1}{p^2-n^2}$

$\rightarrow$ $$\cos px = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$\therefore a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos px\,dx$$

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$$= \frac{1}{2\gamma}\left[\frac{\sin px}{p}\right]_0^{2\gamma}$$

$$= \frac{1}{2\gamma} \frac{\sin 2px}{p}$$

$$= \frac{\sin 2px}{2\gamma p}$$

$$a_n = \frac{1}{\gamma}\int_0^{2\gamma} f(x) \cos nx \, dx$$

$$= \frac{1}{\gamma}\int_0^{2\gamma} \cos px \cos nx \, dx$$

$$= \frac{1}{2\gamma}\int_0^{2\gamma} 2\cos px . \cos nx \, dx$$

$$= \frac{1}{2\gamma}\int_0^{2\gamma} \left[\cos (p+n)x + \cos(p-n)x\right] dx$$

$$= \frac{1}{2\gamma}\left[\frac{\sin (p+n)x}{p+n} + \frac{\sin (p-n)x}{p-n}\right]_0^{2\gamma}$$

$$= \frac{1}{2\gamma}\left[\frac{\sin 2\gamma(p+n)}{p+n} + \frac{\sin 2\gamma(p-n)}{p-n}\right]$$

$$a_n = \frac{1}{2\gamma}\left[\frac{\sin 2p\gamma}{p+n} + \frac{\sin 2p\gamma}{p-n}\right]$$

$$a_n = \frac{1}{2\gamma} \sin 2p\gamma \left[\frac{1}{p+n} + \frac{1}{p-n}\right]$$

$$= \frac{p}{\gamma} \frac{\sin 2p\gamma}{p^2-n^2}$$

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$$b_n = \frac{1}{\gamma}\int_0^{2\gamma} f(x)\,\sin nx\,dx$$

$$= \frac{1}{\gamma}\int_0^{2\gamma} \cos px\,\sin nx\,dx$$

$$= \frac{1}{2\gamma}\int_0^{2\gamma} 2\sin nx\,\cos px\,dx$$

$$= \frac{1}{2\gamma}\int_0^{2\gamma} \left[\sin (n+p)x + \sin (n-p)x\right] dx$$

$$= \frac{1}{2\gamma}\left[\frac{-\cos (n+p)x}{n+p} - \frac{\cos (n-p)x}{n-p}\right]_0^{2\gamma}$$

$$= \frac{1}{2\gamma}\left[\frac{-\cos 2\gamma(n+p)}{n+p} - \frac{\cos 2\gamma(n-p)}{n-p} + \frac{1}{n-p} + \frac{1}{n+p}\right]$$

$$= \frac{1-\cos 2p\gamma}{2\gamma}\left[\frac{1}{n+p} + \frac{1}{n-p}\right]$$

$$= \frac{1-\cos 2p\gamma}{2\gamma} \cdot \frac{2n}{n^2-p^2}$$

$$b_n = \frac{-n}{\gamma}\left(\frac{1-\cos 2p\gamma}{\gamma^2-n^2}\right)$$

putting these values in (1)

$$\cos px = \frac{\sin 2p\pi}{2p\pi} + \frac{p \sin 2p\pi}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{p^2 - n^2}$$

$$= \left(\frac{1-\cos 2p\pi}{\pi}\right) \sum_{n=1}^{\infty} \frac{n \sin nx}{\pi^2 - n^2}$$

Lec 17 10/09/2020

Q) Fourier series in the interval $(-\pi, \pi)$

① $$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ \pi, & 0 < x < \pi \end{cases}$$

State the value of $f(x)$ at $x = 0$ and hence deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \quad \frac{-\pi^2}{8}$$

→ let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx +$

$$\sum_{n=1}^{\infty} b_n \sin nx \, \cancel{dx}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} -\pi \, dx + \int_{0}^{\pi} \pi \, dx \right]$$

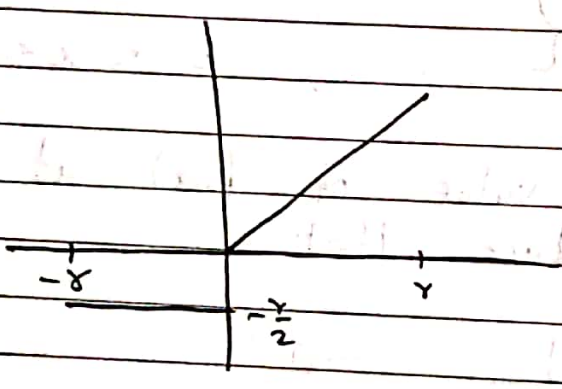
$$= \frac{1}{2\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{-\pi^2}{4}$$

Now $f(x)$ is discontinuous at $x=0$
At a point of discontinuity $x = c$

$$f(x) = \frac{1}{2}\left[\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x)\right]$$

$$f(0) = \frac{1}{2}\left[-\pi + 0\right] = \frac{-\pi}{2}$$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx \, dx + \int_{0}^{\pi} x \cos nx \, dx\right.$$

$$= \left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}\right]$$

$$= \cancel{\frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)\right.}$$

$$= \frac{1}{\pi}\left[-\pi(0-0) + \left(\pi(0) + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

Similarly,

$$b_n = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Putting the values in (1)

$$f(x) = \frac{-\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$$

$$+ \sum_{n=1}^{\infty} 1 - \frac{2}{n}(-1)^n \sin nx$$

$$= \frac{-\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2}\cdots\right]$$

$$+ \left[\frac{3\sin x}{1} - \frac{\sin 2x}{2} + \sin 3x - \cdots\right]$$

Now $f(x)$ is discontinuous at $x = 0$

~~At~~ a point of discontinuity $x = c$

$$f(x) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right]$$

$$f(0) = \frac{1}{2}\left[-\pi + 0\right]$$

$$= \frac{-\pi}{2}$$

$$\therefore \frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Lec 18 11/09/2020

Q) obtain fourier expansion of $f(x) = |\cos x|$ in $(-\pi, \pi)$

→ Here $f(x)$ is even
$f(-x) = |\cos(-x)| = f(x)$

let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ ①

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} |\cos x|\,dx$$

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$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \cos x \, dx - \int_{\pi/2}^{\pi} \cos x \, dx\right]$$

$\because |\cos x| = \cos x \qquad 0 < x < \frac{\pi}{2}$

$|\cos x| = -\cos x \qquad \frac{\pi}{2} < x < \pi$

$$= \frac{1}{\pi}\left([\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}\right)$$

$$a_n = \begin{cases} \frac{-4}{\pi(n^2-1)} \cos\left(\frac{n\pi}{2}\right) & n = \text{even} \\ 0 & n = \text{odd} \end{cases}$$

To find a_1,

$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx\right]$$

$= 0$

Q) Fourier series in interval $(-l, l)$

1) Find the fourier exp of

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

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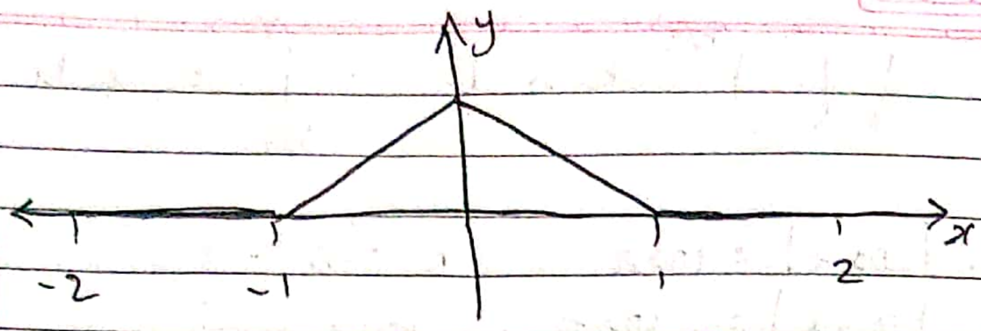

Further $f(-x) = \begin{cases} 0 & -2 < -x < -1 \\ 1-x & -1 < -x < 0 \\ 1+x & 0 < -x < 1 \\ 0 & 1 < -x < 2 \end{cases}$

$$= \begin{cases} 0 & 2 > x > 1 \\ 1-x & 1 > x > 0 \\ 1+x & 0 > x > -1 \\ 0 & -1 > x > -2 \end{cases}$$

$$= f(x)$$

$\therefore$ $f(x)$ is even

$\therefore$ $b_n = 0$ and $l = 2$

$$f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= a_0 + \sum a_n \cos \frac{n\pi x}{2}$$

$$\therefore a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{2}\int_0^2 f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1 (1-x)\,dx + \int_1^2 0\,dx\right]$$

$$= \frac{1}{2}\left[x - \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 (1-x)\cos\frac{n\pi x}{2}\,dx + \int_1^2 0\,dx$$

$$= \left[(1-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \left[0 - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right] - \left[0 - \frac{4}{n^2\pi^2}\right]$$

$$= \frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)$$

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Half Range Series

1) Half range cosine series :-

if $f(x)$ is defined on $(0, l)$ then take it's reflection about y axis so $f(x)$ becomes even on $(-l, l)$ hence $b_n = 0$

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} \right]$$

where $a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx$

$$= \cancel{2} \frac{1}{l} \int_{0}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

2) Half range sine series

If $f(x)$ is defined on $(0, l)$ then take it's reflection in opposite quadrant so $f(x)$ becomes odd on $(-l, l)$ hence $a_0 = 0$

$\therefore a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

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$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx$$

<u>Lec 19</u> 16/09/2020

Q1) Find half range cosine series for $f(x) = x(\pi - x)$ in $(0, \pi)$. Hence show that

i) $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ ii) $\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

iii) $\sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

→ Let $f(x) = a_0 + \sum a_n \cos nx$ $\quad \because l = \pi$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) \, dx = \frac{1}{\pi} \int_0^{\pi} \pi x - x^2 \, dx$$

$$= \frac{1}{\pi} \left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{\pi^2}{6}$$

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$$a_n = \frac{2}{\gamma}\int_0^{\gamma} f(x)\cos nx \, dx$$

$$= \frac{2}{\gamma}\int_0^{\gamma} (\gamma x - x^2)\cos nx \, dx$$

$$= \frac{2}{\gamma}\left[(\gamma x - x^2)\frac{\sin nx}{n} - (\gamma - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\gamma}$$

$$= -2\left(\frac{1+\cos n\gamma}{n^2}\right)$$

$$= \begin{cases} 0 & n = \text{odd} \\ -\frac{4}{n^2} & n = \text{even} \end{cases}$$

$$x(\gamma - x) = \frac{\gamma^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \cdots\right]$$

$$= \frac{\gamma^2}{6} - \left[\frac{1}{1^2}\cos 2x + \frac{1}{2^2}\cos 4x + \frac{1}{3^2}\cos 6x + \cdots\right]$$

i) Now put $x = 0$

$$0 = \frac{\gamma^2}{6} - \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

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$$\therefore \frac{\pi^2}{6} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right].$$

ii) put $x = \frac{\pi}{2}$

$$\frac{\pi^2}{4} = \frac{\pi^2}{6} - \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$$

Now by parsevals identity,

$$\frac{1}{\pi}\int_0^{\pi} [f(x)^2 dx] = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + \cdots\right]$$

$$\frac{1}{\pi}\int_0^{\pi} x^2(\pi - x)^2 dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + \cdots\right]$$

$$\frac{1}{\pi}\int_0^{\pi} x^2(\pi - x^2) dx = \frac{1}{\pi}\int_0^{\pi} \pi^2x^2 - 2\pi x^3 + x^4 dx$$

$$= \frac{1}{\pi}\left[\frac{\pi^2x^3}{3} - \frac{2\pi x^4}{4} + \frac{x^5}{5}\right]_0^{\pi}$$

$$= \frac{1}{\pi} \cdot \frac{\pi^5}{30}$$

$$= \frac{\pi^4}{30}$$

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$$\frac{\pi^4}{30} = \frac{1}{2}\left[\frac{2\pi^4}{36} + \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right]$$

$$\frac{\pi^4}{15} - \frac{\pi^4}{18} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

$$\rightarrow \quad \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

Lec 20 17/09/2020

Q2) Expand $f(x) = lx - x^2$, $0 < x < l$ in half range sine series. Hence deduce,

i) $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots$

ii) $\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots$

iii) $\frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \dots$

$$\rightarrow \quad f(x) = \Sigma b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

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$$= \frac{2}{l}\Big[(lx - x^2)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\Big]_0^l$$

$$= \frac{2}{l}\Big[\left\{0 - 0 - \frac{2l^3}{n^3\pi^3}\cos n\pi\right\} - \left(0 - 0 - \frac{2l^3}{n^3\pi^3}\right)\Big]$$

$$= \frac{2}{l}\cdot\frac{2l^3}{n^3\pi^3}\left(-\cos n\pi + 1\right)$$

$$= \begin{cases} \dfrac{8l^2}{n^3\pi^3} & \text{if } n = \text{odd} \\ 0 & \text{if } n = \text{even} \end{cases}$$

$$f(x) = lx - x^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \cdots\right]$$

i) putting $x = \frac{l}{2}$

$$l\left(\frac{l}{2}\right) - \left(\frac{l}{2}\right)^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right]$$

$$\Rightarrow \frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$$

ii) By parseval's identity,

$$\frac{1}{l}\int_0^l [f(x)]^2 dx = \frac{1}{2}[b_1^2 + b_2^2 + b_3^2 + \dots]$$

$$\frac{1}{l}\int_0^l lx - x^2 \, dx = \frac{1}{2}\left[\frac{8l^2}{\pi^3}\right]^2\left[\left(\frac{1}{1^2}\right)^2 + \left(\frac{1}{3^3}\right)^2 + \dots\right]$$

$$\therefore \frac{1}{l}\left[\frac{l^2x^3}{3} - 2\frac{lx^4}{4} + \frac{x^5}{5}\right]_0^l$$

$$= \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots\right]$$

$$\frac{1}{30}l^4 = \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots\right]$$

$$\rightarrow \cancel{l^4}\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots$$

iii) $$S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} \dots$$

$$= \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots\right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} \dots\right)$$

$$= \frac{\pi^6}{960} + \frac{1}{2^6}\left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \dots\right)$$

$$S = \frac{\pi^6}{960} + \frac{S}{64}$$

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$$\frac{63S}{64} = \frac{\pi^6}{960}$$

$$S = \frac{\pi^6}{945}$$

$$\therefore \frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots$$

Q3) Show that if $0 < x < \pi$, $\cos x = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2 - 1} \sin 2mx$

→ $\because$ only sine terms, $\therefore$ half range sine series.

→ $\cos x = \sum b_n \sin nx \quad \because l = \pi$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(1+n)x - \sin(1-n)x \,] \, dx$$

$$= \frac{1}{\pi} \left[-\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} \right]$$

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$$= \frac{1}{\gamma}\left[\frac{\cos n\gamma}{n+1} + \frac{\cos n\gamma}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$\because \cos(n \pm 1)\gamma = -\cos n\gamma$

$$= \frac{1}{\gamma}\left[\frac{1}{n+1} + \frac{1}{n-1}\right](1+\cos n\gamma)$$

$$= \frac{1}{\gamma} \cdot \frac{2n}{n^2-1}\left[1 + (-1)^n\right]$$

$$= \begin{cases} 0 & n = \text{odd} \ \& \ n \neq 1 \\ \frac{1}{\gamma} \cdot \frac{4n}{n^2-1} & n = \text{even} \end{cases}$$

when $n=1$, we get

$$b_1 = \frac{2}{\gamma}\int_0^{\gamma} \cos x \sin x \, dx = \frac{1}{\gamma}\int_0^{\gamma} \sin 2x \, dx$$

$$b_1 = \frac{1}{\gamma}\left[-\frac{\cos 2x}{2}\right]_0^{\gamma} = -\frac{1}{2\gamma}\left[1-1\right] = \underline{\underline{0}}$$

$$\cos x = \frac{4}{\gamma}\left[\frac{2}{2^2-1}\sin 2x + \frac{4}{4^2-1}\sin 4x + \frac{6}{6^2-1}\sin 6x\right.$$

$$= \frac{8}{\gamma}\left[\frac{1}{2^2-1}\sin 2x + \frac{2}{4^2-1}\sin 4x + \frac{3}{6^2-1}\sin 6x\right]$$

$$= \frac{8}{\gamma}\sum \frac{m}{4m^2-1}\sin 2m\gamma$$

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Lec 21 18/09/2020

Q) obtain fourier series for $f(x) = \begin{cases} 1 + \frac{2x}{\gamma} & -\gamma < x < 0 \\ 1 - \frac{2x}{\gamma} & 0 < x < \gamma \end{cases}$

$\therefore f(x) = \begin{cases} 1 + \frac{2x}{\gamma} & -\gamma < x < 0 \\ 1 - \frac{2x}{\gamma} & 0 \leq x < \gamma \end{cases}$

$f(-x) = \begin{cases} 1 - \frac{2x}{\gamma} & -\gamma \leq -x \leq 0 \\ 1 + \frac{2x}{\gamma} & 0 \leq -x \leq \gamma \end{cases}$

$= \begin{cases} 1 - \frac{2x}{\gamma} & \gamma \geq x \geq 0 \\ 1 + \frac{2x}{\gamma} & 0 \geq x \geq -\gamma \end{cases}$

$= f(x)$

$f(x)$ = even on $(-\gamma, \gamma)$ then $b_n = 0$

$a_0 = \frac{1}{2\gamma} \int_{-\gamma}^{\gamma} f(x)\, dx = \frac{1}{\gamma} \int_0^{\gamma} f(x)\, dx$

$= \frac{1}{\gamma} \int_0^{\gamma} f(x)\, dx = \frac{1}{\gamma} \int_0^{\gamma} \left(1 - \frac{2x}{\gamma}\right) dx$

$= \frac{1}{\gamma} \left[x - \frac{x^2}{\gamma} \right]_0^{\gamma} = \frac{1}{\gamma} (\gamma - \gamma) = 0$

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$$a_n = \frac{2}{\gamma}\int_0^{\gamma} f(x)\cos nx\,dx$$

$$= \frac{2}{\gamma}\int_0^{\gamma}\left(1-\frac{2x}{\gamma}\right)\cos nx\,dx$$

$$= \frac{2}{\gamma}\left[\left(1-\frac{2x}{\gamma}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{-2}{\gamma}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\gamma}$$

$$= \frac{2}{\gamma}\left[\left(0-\frac{2\cos n\gamma}{\gamma n^2}\right)-\left(-\frac{2}{\gamma n^2}\right)\right]$$

$$= \frac{4}{\gamma^2 n^2}\left[1-\cos n\gamma\right]$$

$$= \begin{cases} 0 & n = \text{even} \\ \dfrac{8}{\gamma^2 n^2} & n = \text{odd} \end{cases}$$

$$f(x) = \frac{8}{\gamma^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

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