SEMESTER III.
MODULE 3

RELATIONS AND FUNCTIONS

UNIT NO :3.2

Relations

Cartesian product

Consider two arbitrary sets X and Y.

The set of all ordered pairs(x,y) where x∈X and y∈Y is called the **Cartesian product**, of X and Y.

it is denoted by $X \times Y$, which is read "X cross Y."

Definition

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$$

Definition : Relation

A **relation** from a set X to a set Y is any subset of the Cartesian product $\mathrm{X} \times \mathrm{Y}$

EXAMPLE

Let $\mathrm{X} = \{1, 2\}$ and $\mathrm{Y} = \{10, 15, 20\}$.

R ={(1,10) (2,20)}

EXAMPLE

Let $X = \{1, 2\}$ and $Y = \{10, 15, 20\}$. Then write

$X \times Y, Y \times X, X \times X$

$X \times Y$
$= \{(1, 10), (1, 15), (1, 20), (2, 10), (2,15), (2, 20)\}$
$Y \times X$
$= \{(10, 1), (15, 1), (20, 1), (10, 2), (15, 2), (20, 2)\}$

Also, $X \times X = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

NOTE

The first components in the ordered pairs is called the **domain** of the relation and the set of second ordered pairs is called the **range** of the relation.

Let $\mathrm{X} = \{1, 2\}$ and $\mathrm{Y} = \{10, 15, 20\}$.

R ={(1,10) (2,20)}

Domain of R ={1,2}

R**ange** of R ={10,20}

NOTE

Suppose R is a relation from X to Y. Then R is a set of ordered pairs where each first element comes from X and each second element comes from Y. That is, for each pair $x \in X$ and $y \in Y$, exactly one of the following is true:

i. $(x,y) \in R$; we then say "x is R – related to y", written xRy.

ii. $(x,y) \notin R$; we then say "x is not R – related to y", written $x \not{R} y$

Definition : Inverse of *R*

Let R be any relation from a set A to set B.
The **inverse** of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs, when reversed, belong to R.
That is: $R^{-1} = \{(b, a) : (a, b) \in R\}$

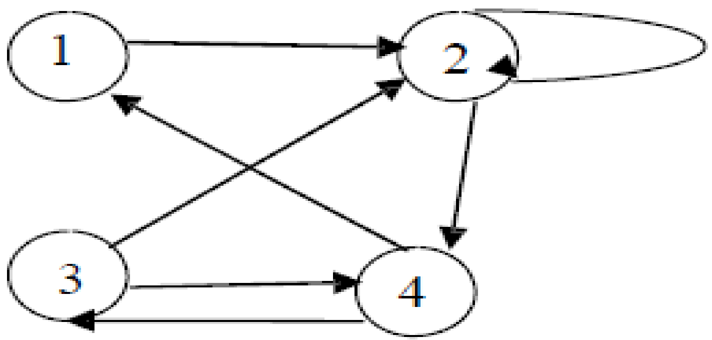
REPRESENTATION OF RELATIONS: Matrices can be easily used to represent relation

If R={(1,x)(2,x)(3,y)(3,z)} then matrix of R is

$$\begin{array}{c|ccc} & x & y & z \\ \hline 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 4 & 0 & 0 & 0 \end{array}$$

REPRESENTATION OF RELATIONS: Another way of pictorial representation, known as **diagraph**.

R = {(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)} Then, the diagraph of R is drawn as follows:



The directed graphs are very important data structures that have applications in Computer Science (in the area of networking).

Definition :Composite Relation

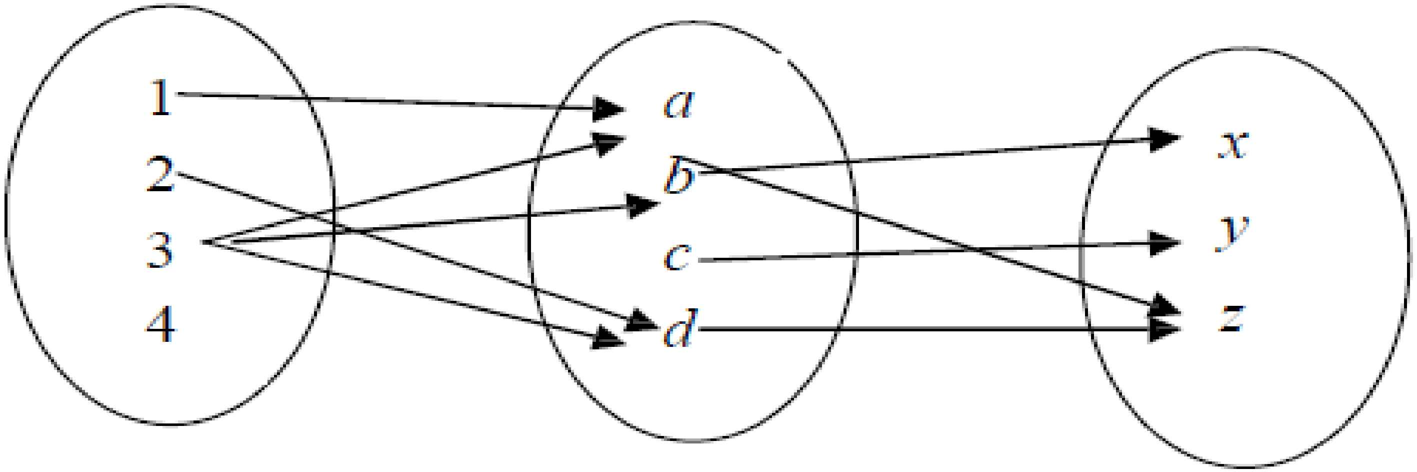
Let A, B and C be three sets.

Let R be a relation from A to B and S be a relation from B to C.

Then, composite relation ROS is a relation from A to C defined by,

$a(\mathrm{ROS})c$, if there is some $b \in B$, such that $a\,R\,b$ and $b\,S\,c$.

Example : Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$,$C = \{x, y, z\}$ and let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ and $S = \{(b, x), (b, z), (c, y), (d, z)\}$.Write ROS



$R0S$ will be given as below.

$$R0S = \{(2, z), (3, x), (3, z)\}.$$

Definition :Reflexive Relation

Let A be a nonempty set, a relation R on A is said to be reflexive if for each a ∈ A, (a; a) ∈ R.

Example

Let $A = \{a, b, c, d\}$ and R be defined as follows:

$R = \{(a, a), (a, c), (b, a), (b, b), (c, c), (d, c), (d, d)\}$.

Is R a reflexive relation ?

YES

Definition : Symmetric Relation

Let A be a nonempty set, a relation R on A is said to be symmetric if for each pair of elements a; b ∈ A,

(a; b) ∈ R implies (b; a) ∈ R.

Example

Let $A = \{1, 2, 3, 4\}$ and R be defined as:

$R = \{(1, 2), (2, 3), (2, 1), (3, 2), (3, 3)\}$,

Is R a symmetric relation ?

YES

NOTE

If we draw a diagraph of a reflexive relation, then all the vertices will have a loop.

Also if we represent reflexive relation using a matrix, then all its diagonal entries will be 1.

Also if we represent symmetric relation using a matrix then the matrix will be symmetric matrix

Definition : Antisymmetric Relation

Let A be a nonempty set,

A relation R on A is said to be antisymmetric

if if for $a, b \in A$, if $a\ R\ b$ and $b\ R\ a$, then $a = b$.

Thus, R is not anti-symmetric if there exists $a, b \in A$ such that $a\ R\ b$ and $b\ R\ a$ but $a \neq b$.

Example

Example1: Let $A = \{a, b, c, d\}$

R be defined as: $R = \{(a, b), (b, a), (a, c), (c, d), (d, b)\}$.

Check whether R is symmetric , anti-symmetric ?

R is not symmetric, as $a\ R\ c$ but c R a .

R is not anti-symmetric, because $a\ R\ b$ and $b\ R\ c$, but $a \neq b$

Example2: The relation "less than or equal to (≤)", is an anti- symmetric relation

Definition : Transitive Relation

Let A be a nonempty set, a relation R on A is said to be transitive if for each triple of elements a, b, c $\in$ A, (a, b), (b, c) $\in$ R imply (a,c) $\in$ R.

Example

Relation "*a* divides *b*", on the set of integers, is a transitive relation.

The relation "less than or equal to (≤)", is a transitive relation.

Definition : Antisymmetric Relation

Let A be a nonempty set,

A relation R on A is said to be antisymmetric

if if for a, $b \in A$, if a R b and b R a, then $a = b$.

Thus, R is not anti-symmetric if there exists a, $b \in A$ such that a R b and b R a but $a \neq b$.

Definition : Equivalence relation

Let A be a nonempty set.

A relation R on set A is said to be equivalence relation if R is reflexive , symmetric and transitive

Example : Consider the set *L* of lines in the Euclidean plane. Two lines in the plane are said to be related, if they are parallel to each other.

Is this relation an equivalence relation?

Yes .

R = {(L_1, L_2): L_1 is parallel to L_2}

R is reflexive as any line L_1 is parallel to itself i.e., (L_1, L_1) ∈ R.

Now,

Let (L_1, L_2) ∈ R.

⇒ L_1 is parallel to L_2.

⇒ L_2 is parallel to L_1.

⇒ (L_2, L_1) ∈ R

∴ R is symmetric.

Now,

Let (L_1, L_2), (L_2, L_3) ∈R.

⇒ L_1 is parallel to L_2. Also, L_2 is parallel to L_3.

⇒ L_1 is parallel to L_3.

∴R is transitive.

Hence, R is an equivalence relation.

Example Determine whether the relation R on a set A is reflexive, symmetric, antisymmetric or transitive.
A = set of all positive integers, a R b iff $|a - b| \leq 2$

R is reflexive because $|a-a| = 0 < 2, \forall\, a \in A$

R is symmetric because $|a-b| \leq 2 \Rightarrow |b-a| \leq 2 \;\therefore a\,R\,b \Rightarrow b\,R\,a$

R is not antisymmetric because $1\,R\,2$ & $2\,R\,1$ $1\,R\,2 \Rightarrow |1-2| \leq 2$ & $2\,R\,1 \Rightarrow |2-1| \leq 2$. But $1 \neq 2$

R is not transitive because 5 R 4, 4 R 2 but 5 $\not{R}$ 2

Definition : Partition

A partition of a set A is a collection of non-empty subsets $A_1, A_2, A_3, \ldots$ of A which are pairwise disjoint and whose union equals A

$$1.\ A_i \cap A_j = \Phi \qquad \text{for } i \neq j$$

$$2.\ \cup_n A_n = A$$

Example: Is P= {{1,2}{3,5}{4,5,6}} partition of A={1,2,34,5,6}?

Let $A = \{1, 2, 3, 4, 5, 6\}$.

$A_1 = \{1, 2\}; A_2 = \{3, 5\}; A_3 = \{4, 5, 6\}$.

$A = A_1 \cup A_2 \cup A_3$ but $A_2 \cap A_3 \neq \phi$..

P is not partition of A

Example:

Is P= {{1,2}{3,5}{4}} partition of A={1,2,3,4,5} ?

$A_1 = \{1, 2\}; A_2 = \{3, 5\}; A_3 = \{4\}.$

$A_1 \cap A_2 = \phi, A_1 \cap A_3 = \phi, \text{ and } A_2 \cap A_3 = \phi.$

$A = A_1 \cup A_2 \cup A_3$

P is partition of A

Equivalence Class

Let R be an equivalence relation on a set A

Let $\mathrm{x} \in \mathrm{A}$

the set of elements of A related to x is called the equivalence class of x, represented $[\mathrm{x}]$

$[\mathrm{x}] = \{\mathrm{y} \in \mathrm{A} \mid \mathrm{y\,Rx}\}$.

The collection of equivalence classes, represented $\mathrm{A/R}$

$\mathrm{A/R} = \{[\mathrm{x}] \mid \mathrm{x} \in \mathrm{A}\}$, is called quotient set of A by R

Note: If *R* is an equivalence relation on *A*, then sets [a] or *R*(*a*) are called as equivalence classes of *R*.

Theorem

Let R be an equivalence relation on a set A.

Then A|R is a partition of A. Specifically:

(i) For each a in A, we have $a \in [a]$.

(ii) [a] = [b] if and only if (a, b) ∈ R.

(iii) If $[a] \neq [b]$, then [a] and [b] are disjoint.

Example let A = {1, 2, …., 8}. Let R be the equivalence relation defined by $x \equiv y \bmod(4)$ Write R as a set of ordered pairs Find the partition of A induced by R .

$$R = \{(1,1), 1,5), (2,2), (2,6), (3,3), (3,7), (4,4), (4,8), (5,1), (5,5), (6,2), (6,6), (7,3), (7,7), (8,4), (8,8)\}$$

ii) [1] = {1, 5} ,[2] = {2, 6} ,$[3] = \{3,7\}$,[4] = {4,8}

So, the following is the partition of A induced by R:

$$A|R = \{[1], [2], [3], [4]\}$$

Note: If R is an equivalence relation on A, then sets [a] or $R(a)$ are called as equivalence classes of R.

Example

Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3,4), (4, 3), (3, 3), (4, 4)\}$.

Show that R is an equivalence relation on A hence find partition of A induced by R

We observe that $R(1) = R(2)$ and $R(3) = R(4)$ and

hence $P = \{ \{1, 2\}, \{3, 4\} \}$.

Construction of Z_n

Let A = Z (set of integers) and define R as
$R = \{(a, b) \in A \times A\!: a \equiv b \pmod 5\}$. Then, we have,
$R(1) = \{......, -14, -9, -4, 1, 6, 11, ... \}$
$R(2) = \{......, -13, -8, -3, 2, 7, 12, ... \}$
$R(3) = \{......, -12, -7, -2, 3, 8, 13, ... \}$
$R(4) = \{......, -11, -6, -1, 4, 9, 14, ... \}$
$R(5) = \{......, -10, -5, 0, 5, 10, 15, ... \}$.
$R(1)$, $R(2)$, $R(3)$, $R(4)$ and $R(5)$ form partition on Z with respect to given equivalence relation.
Z|R= {R(1),R(2),R(3),R(4),R(5)}
$Z_{5=} \{\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$

Definition: PARTIAL ORDER RELATION

A relation R on the set A is said to be ***partial order relation***, if it is reflexive, anti-symmetric and transitive.

Example : Let $A = \{a, b, c, d, e\}$.

Relation R, represented using following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Is R partial order relation ?

ANS :Yes

Example : Let A be a set of natural numbers and relation R be "less than or equal to relation ($\leq$)". Then R is a partial order relation on A.

For any m, n, $k \in N$,
$n \leq n$ (reflexive);
if $m \leq n$ and $n \leq$ m, then $m = n$ (anti-symmetric);
lastly, if $m \leq n$ and $n \leq k$, then $m \leq k$ (transitive)

Example 7.47 : *Let R and S are equivalence relation on $A = \{1, 2, 3, 4\}$ given by*

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$

$S = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 1)\}$

Determine partition of A induced by

(i) R^{-1} *(ii)* $R \cap S$

Solution : (i)

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$$

$$R^{-1} = \{(1, 1), (2, 1), (1, 2), (2, 2), (4, 3), (3, 4), (3, 3), (4, 4)\}$$

$\therefore$

$$[1]_{R^{-1}} = \{1, 2\}$$

$$[2]_{R^{-1}} = \{1, 2\}$$

$$[3]_{R^{-1}} = \{3, 4\}$$

$$[4]_{R^{-1}} = \{3, 4\}$$

Here, $[1]_{R^{-1}} = [2]_{R^{-1}}$ and $[3]_{R^{-1}} = [4]_{R^{-1}}$

$\therefore$ Partition of A induced by $R^{-1} = [\{1, 2\}, \{3, 4\}]$

(ii)

$$R \cap S = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$[1]_{R \cap S} = \{1\}$$

$$[2]_{R \cap S} = \{2\}$$

$$[3]_{R \cap S} = \{3\}$$

$$[4]_{R \cap S} = \{4\}$$

$\therefore$ Partition of A induced by $R \cap S = [\{1\}, \{2\}, \{3\}, \{4\}]$