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Q) $L[t^2 \sin at]$

$$= (-1)^2 \frac{d^2}{ds^2} L(\sin at)$$

$$= \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right]$$

$$= -\frac{(s^2+a^2)^2 \cdot 2a - 2as \cdot 2(s^2+a^2)2s}{(s^2+a^2)^{\cancel{2}4}}$$

$$= \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

Q) $L[t e^{3t} \sin 4t]$

$$L(\sin 4t) = \frac{4}{s^2+4^2}$$

$$L[e^{3t} \sin 4t] = \frac{4}{(s-3)^2+4^2}$$

$$L[t e^{3t} \sin 4t] = \frac{d}{ds}\left(\frac{4}{(s-3)^2+4^2}\right)$$

$$= \frac{4(2s-6)}{(s^2-6s+25)^2} = \frac{8(s-3)}{(s^2-6s+25)^2}$$

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Q) $L[t^n e^{at}]$

$\Rightarrow L\, e^{at} = \frac{1}{s-a}$

$$L[t^n e^{at}] = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s-a}\right) = \frac{(-1)^n (-1)^n\, n!}{(s-a)^{n+1}}$$

$$= \frac{n!}{(s-a)^{n+1}}$$

Q) Find $L\left[t\left(\frac{\sin t}{e^t}\right)^2\right]$

$\Rightarrow L[t.e^{-2t}\sin^2 t]$

$= te^{-2t}\left(\frac{1-\cos 2t}{2}\right)$

$= L\left[\frac{1}{2}.t.e^{-2t}[1-\cos 2t]\right]$

$L(1-\cos 2t) = L(1) - L(\cos 2t)$

$= \frac{1}{s} - \frac{s}{s^2+2^2}$

$L e^{-2t}\frac{(1-\cos 2t)}{2} = \frac{1}{s+2} - \frac{s+2}{(s+2)^2+2^2}$

on multiplying t, diff whole thing

$$L\left[t\, e^{-2t}\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{s+2} - \frac{s+2}{s^2+4s+8}\right)\right.$$

$$= \frac{1}{2}\left[\frac{-1}{(s+2)^2} - \frac{(s^2+4s+8) - (s+2)(2s+4)}{(s^2+4s+8)^2}\right.$$

$$= \frac{1}{2}\left[\frac{-1}{(s+2)^2} + \frac{s^2+4s}{s^2+4s+8}\right]$$

Q) $\int_0^\infty \frac{t^2}{e^{2t}} \sin 3t$

$$\int_0^\infty e^{-st} t^2 \sin 3t \, dt = L(t^2 \sin 3t)$$

$$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) = \frac{d}{ds}\left(\frac{-3 \cdot 2s}{(s^2+9)^2}\right)$$

$\vdots$

$$= \frac{18(s^2-3)}{(s^2+9)^3}$$

putting $s=2$ in the above equality,

$$\int_0^\infty e^{-2t} t^2 \sin 3t = \frac{18(4-3)}{(4+9)^3} = \frac{18}{2197}$$

$\Rightarrow$ If $L(f(t)) = \phi(s)$ then $L\left[\frac{1}{t} f(t)\right] = \int_s^\infty \phi(s)\,ds$

Q) $L\left(\frac{1}{t}(1-\cos t)\right)$

$$L(1-\cos t) = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left(\frac{1}{t}(1-\cos t)\right) = \int_0^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$= -\frac{1}{2}\left[\log(s^2+1) - \log s^2\right]_0^\infty$$

$$= \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

Q) $L\left(\frac{\sin^2 t}{t^2}\right)$

$\Rightarrow L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right)$

$= \frac{1}{2}\left(L(1) - L(\cos 2t)\right)$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$

$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$

$= \frac{1}{4}\left[\log \frac{s^2}{s^2+4}\right]_s^\infty$

$= -\frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right)$

$= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{1}{4}\left[\log(\,)\cdot s - \int s \cdot \frac{s^2}{s^2+4}\left(\frac{s^2\cdot 2s - (s^2+4)2\cdot s}{s^4}\right) ds\right]_s^\infty$

$$= \frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right) + 8\int \frac{ds}{s^2+4} \right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right) + 4\tan^{-1}\left(\frac{s}{2}\right) \right]_0^\infty$$

$$= \frac{1}{4}\left[0 + 2\cdot\frac{\pi}{2} - s\log\left(\frac{s^2+4}{s^2}\right) - 2\tan^{-1}\frac{s}{2} \right]$$

$$= \frac{\pi}{4} - \frac{s}{4}\left(\frac{s^2+4}{s^2}\right) - \frac{1}{2}\tan^{-1}\frac{s}{2}$$

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Q) Evaluate $\int_0^\infty \frac{\cos at - \cos bt}{t} \, dt$

$f(t) = \cos at - \cos bt$

$$L\left[\frac{1}{t} f(t)\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= -\frac{1}{2}\left[\log\left(\frac{s^2+b^2}{s+a^2}\right)\right]_0^\infty$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$\therefore \int_0^\infty e^{-st}\left(\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

putting $s = 0$,

$$\int_0^\infty \frac{\cos at - \cos bt}{t} \, dt = \frac{1}{2}\log\left(\frac{b^2}{a^2}\right)$$

$$= \log \frac{b}{a}$$

Assumed e^{-st} to convert it into La Place and put $s = 0$.

Q) $\int_0^\infty e^{-t} \dfrac{\sin 2t + \sin 3t}{t}\, dt$

$L(\sin 2t) = \dfrac{2}{s^2+4}$ $\qquad L(\sin 3t) = \dfrac{3}{s^2+9}$

$L\left(\dfrac{\sin 2t + \sin 3t}{t}\right) = \int_s^\infty \dfrac{2}{s^2+4} + \dfrac{3}{s^2+9}\, ds$

$= \left[\tan^{-1}\dfrac{s}{2}\right]_s^\infty + \left[\tan^{-1}\dfrac{s}{3}\right]_s^\infty$

$= \dfrac{\pi}{2} - \tan^{-1}\dfrac{s}{2} + \dfrac{\pi}{2} - \tan^{-1}\dfrac{s}{3}$

$= \pi - \left(\tan^{-1}\dfrac{s}{2} + \tan^{-1}\dfrac{s}{3}\right)$

$\tan^{-1}\dfrac{a+b}{1-a.b}$

$L\left(\dfrac{s \cdot t \cdot s ..}{t}\right) = \pi - \tan^{-1}\dfrac{5s}{6-s^2}$

$\int_0^\infty e^{-st}\left(\dfrac{\sin 2t + \sin 3t}{t}\right) = \pi - \tan^{-1}\dfrac{5s}{6-s^2}$

putting $s = 1$

$Q = \pi - \tan^{-1}\dfrac{5}{5} = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

$$\Rightarrow L[f'(t)] = -f(0) + s \cdot L[f(t)]$$

$$\Rightarrow L[f''(t)] = s^2 \cdot L[f(t)] - sf(0) - f'(0)$$

Laplace Transform of derivatives

$$\Rightarrow L[f^{(n)}(t)] = s^n \cdot L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \ldots$$

Q) L.T. of $\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)$

$$L(1-\cos 2t) = L(1) - L(\cos 2t)$$

$$= \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left(\frac{1-\cos 2t}{t}\right) = \int_s^\infty \frac{1}{s} - \frac{s}{s^2+4} \, ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \log\left(\frac{s}{\sqrt{s^2+4}}\right)\Bigg]_s^\infty$$

$$= \left[0 - \log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]$$

$$= \log \frac{\sqrt{s^2+4}}{s}$$

$$\therefore L\left(\frac{1-\cos 2t}{t}\right) = \log \frac{\sqrt{s^2+4}}{s}$$

By property,

$$L[f'(t)] = s \cdot L[f(t)] - f(0)$$

$$f(0) = \frac{1-\cos 2t}{t} = \frac{2\sin^2 t}{t}$$

$$= 2\sin t \cdot \cancel{\frac{\sin t}{t}} \rightarrow 0$$

$$= 2 \cdot 0$$

$$= \underline{\underline{0}}$$

$$= s \cdot \log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

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$\Rightarrow L\left[f(t)\right] = \phi(s)$ then

$$L \int_0^t f(u)\,du = \frac{1}{s}\phi(s)$$

:

$\Rightarrow L\left[\int_0^t \int_0^t \cdots \int_0^t f(u)\,du\right] : \frac{1}{s^n}\ \cancel{\phi(s)}\ L(f(t))$

Q) L.T. of $\int_0^t u \cdot \cosh u\ du$

$$L(\cosh t) = \frac{s}{s^2-a^2}$$

$$L(t\cdot\cosh t) : \frac{d}{ds}\left(\frac{s}{s^2-a^2}\right)$$

$$= -\frac{s^2+a^2}{(s^2-a^2)^2} - \phi(s)$$

$$L\int_0^t u\cosh u\ du = \frac{1}{s}\phi(s) = -\frac{s^2+a^2}{s(s^2-a^2)^2}$$

Q) $L\left[\int_0^t\int_0^t\int_0^t t\sin t\,(dt)^3\right] = \frac{1}{s^3}\phi(s)$

$L(\sin t) = \frac{1}{s^2+1}$

$L(t\sin t) = \frac{2s}{(s^2+1)^2}$

$L\left[\int_0^t\int_0^t\int_0^t t\sin t\,(dt)^3\right] = \frac{1}{s^3}\cdot\frac{2s}{(s^2+1)^2}$

$= \frac{2}{s^2(s^2+1)^2}$

Q) L.T. of $\int_0^t u^{-1}e^{-u}\sin u\,du$

$L(\sin u) = \frac{1}{s^2+1}$

$L(e^{-u}\sin u) = \frac{1}{(s+1)^2+1}$ by shifting

$L(u^{-1}e^{-u}\sin u) = \int_s^\infty \frac{1}{(s+1)^2+1}\,ds$

$= \tan^{-1}(s+1)\Big]_s^\infty$

$= \cot^{-1}(s+1) = \phi s$

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$$\rightarrow \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}$$

$x = \sqrt{t}$

$$\operatorname{erf} \sqrt{t} = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2}$$

$u = 0$ then $v = 0$ $\quad$ $u = \sqrt{t}$, $V = t$

$u = \sqrt{V}$, $V = t$ $\quad$ $du = \frac{1}{2\sqrt{V}} dv$

$$\therefore \operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-v} \cdot \frac{1}{2\sqrt{v}} dv$$

$$= \frac{1}{\sqrt{\pi}} \int_0^t e^{-v} v^{-1/2} dv$$

$$L(v^{1/2}) = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$L(e^{-v} v^{-1/2}) = \frac{\sqrt{\pi}}{\sqrt{s+1}}$$

$$L \int_0 e^{-v} v^{1/2} dv = \frac{1}{s} \frac{\sqrt{\pi}}{\sqrt{s+1}}$$

$$L \operatorname{erf} \sqrt{t} = \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{s\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}$$

$\therefore \; \text{erf}\, x + \text{erf}_c\, x = 1$

$\therefore \; \text{erf}_c \sqrt{t} = 1 - \text{erf}\sqrt{t}$

$$L(\text{erf}_c\sqrt{t}) = L(1) - L(\text{erf}\sqrt{t})$$

$$= \frac{1}{s} - \frac{1}{s\sqrt{s+1}}$$

$$= \frac{s\sqrt{s+1} - 1}{s\sqrt{s+1}}$$

Q) $L \displaystyle\int_t^\infty \frac{\cos u}{u}\, du$

Let $f(t) = \displaystyle\int_t^\infty \frac{\cos u}{u}\, du$

$u = vt$

$du = t\, dv$

$u = 1 \qquad u \to \infty$

$\to v = 1 \qquad v \to \infty$

$$f(t) = \int_t^\infty \frac{\cos vt}{vt} \cdot t\, dv$$

$$= \int_1^\infty \frac{\cos vt}{v}\, dv$$

$$L[f(t)] = \int_0^\infty e^{-st} \left[\int_1^\infty \frac{\cos vt}{v}\, dv\right] dt$$

$$= \int_1^\infty \frac{dv}{v} \int_0^\infty e^{-st} \cos vt \, dt$$

$$= \int_1^\infty \frac{dv}{v} \, L[\cos vt] \, dt$$

$$= \int_1^\infty \frac{s \, dv}{v(s^2+v^2)}$$

$$= \frac{1}{s} \int_1^\infty \left[\frac{1}{v} - \frac{v}{s^2+v^2} \right] dv$$

$$= \frac{1}{s} \left[\log v - \frac{1}{2} (s^2+v^2) \right]$$

$$= \frac{1}{2s} \left[\log \left(\frac{v^2}{s^2+v^2} \right) \right]_1^\infty$$

$$= \frac{1}{2s} \left[\log 1 - \log \left(\frac{1}{s^2+1} \right) \right]$$

$$= \frac{1}{2s} \log (s^2+1)$$

Q) $\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u . \cosh u . du\right) dt$

$$L[\sinh u . \cosh u] = L\left[\frac{1}{2}\sinh 2u\right]$$

$$= \frac{1}{2} . \frac{2}{s^2-4}$$

$$= \frac{1}{s^2-4}$$

$$L\left[u^2 . \left(\sinh u . \cosh u\right)\right] = (-1)^2 \frac{d}{ds^2}\left(\frac{1}{s^2-4}\right)$$

$$\vdots$$

$$= \frac{2(3s^2-4)}{(s^2-4)^3} = \phi(s)$$

$$L\left[\int_0^t u^2 \sinh u \cosh u \, du\right] = \frac{1}{s}\phi(s)$$

$$= \frac{2}{s}\left(\frac{3s^2-4}{(s^2-4)^3}\right)$$

$$L\left(\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u\right)\right) = \frac{2}{s}\left(\frac{3s^2-4}{(s^2-4)^3}\right) \quad @ \, s=1$$

$$= \frac{2}{27}$$