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$$\textcircled{0}) \quad L[t^2 \sin at]$$

$$= (-1)^2 \frac{d^2}{ds^2} L(\sin at)$$

$$= \frac{d^2}{ds^2} \left(\frac{a}{s^2 + a^2} \right)$$

$$= \frac{d}{ds} \left[\frac{-2as}{(s^2 + a^2)^2} \right]$$

$$= - \frac{(s^2 + a^2) \cdot 2a - 2as \cdot 2(s^2 + a^2) \cdot 2s}{(s^2 + a^2)^4}$$

$$= \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}$$

$$\textcircled{0}) \quad L[t e^{3t} \sin 4t]$$

$$L(\sin 4t) = \frac{4}{s^2 + 4^2}$$

$$L[e^{3t} \sin 4t] = \frac{4}{(s-3)^2 + 4^2}$$

$$\begin{aligned} L[t e^{3t} \sin 4t] &= \frac{d}{ds} \left(\frac{4}{(s-3)^2 + 4^2} \right) \\ &= \frac{4(2s-6)}{(s^2 - 6s + 25)^2} = \frac{8(s-3)}{(s^2 - 6s + 25)^2} \end{aligned}$$

Q) $L[t^n e^{at}]$

$$\Rightarrow L[e^{at}] = \frac{1}{s-a}$$

$$L[t^n e^{at}] = (-1)^n \frac{d^n}{ds^n} \left(\frac{1}{s-a} \right) = \frac{(-1)^n (-1)^n n!}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}}$$

Q) Find $L\left[t + \left(\frac{\sin t}{e^t} \right)^2 \right]$

$$\Rightarrow L\left[t \cdot e^{-2t} \sin^2 t \right] = t e^{-2t} \left(\frac{1 - \cos 2t}{2} \right)$$

$$= L\left[\frac{1}{2} \cdot t \cdot e^{-2t} [1 - \cos 2t] \right]$$

$$L(1 - \cos 2t) = L(1) - L(\cos 2t)$$

$$= \frac{1}{s} - \frac{s}{s^2 + 2^2}$$

$$L\left(\frac{1 - \cos 2t}{2} \right) = \frac{1}{s+2} - \frac{s+2}{(s+2)^2 + 2^2}$$

on multiplying + diff whole thing

$$\begin{aligned}
 L[t + e^{-2t} \left(\frac{1 - \cos 2t}{2} \right)] &= \frac{1}{2} \frac{d}{ds} \left(\frac{1}{s+2} - \frac{s+2}{s^2+4s+8} \right) \\
 &= \frac{1}{2} \left[-\frac{1}{(s+2)^2} - \frac{(s^2+4s+8) - (s+2)(2s+4)}{(s^2+4s+8)^2} \right] \\
 &= \frac{1}{2} \left[-\frac{1}{(s+2)^2} + \frac{s^2+4s}{s^2+4s+8} \right]
 \end{aligned}$$

Q) $\int_0^\infty \frac{t^2}{e^{st}} \sin 3t dt$

$$\int_0^\infty e^{-st} t^2 \sin 3t dt = L(t^2 \sin 3t)$$

$$= \frac{d^2}{ds^2} \left(\frac{3}{s^2+9} \right) = \frac{d}{ds} \left(\frac{-3 \cdot 2s}{(s^2+9)^2} \right)$$

$$= \frac{18(s^2-3)}{(s^2+9)^3}$$

putting $s=2$ in the above equality,

$$\int_0^\infty e^{-2t} t^2 \sin 3t dt = \frac{18(4-3)}{(4+9)^3} = \underline{\underline{\frac{18}{2197}}}$$

$$\Rightarrow \text{If } L(f(t)) = \phi(s) \text{ then } L\left[\frac{1}{t} f(t)\right] = \int_s^\infty \phi(s) ds$$

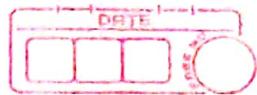
Q) $L\left(\frac{1}{t}(1-\cos t)\right)$

$$L(1-\cos t) = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left(\frac{1}{t}(1-\cos t)\right) = \int_0^\infty \left(\frac{1}{s} - \frac{s}{s^2+1} \right)$$

$$= -\frac{1}{2} \log(s^2+1) - \log s^2 \Big|_0^\infty$$

$$= \frac{1}{2} \log \left(\frac{s^2+1}{s^2} \right)$$



$$Q) L\left(\frac{\sin 2t}{t^2}\right)$$

$$\Rightarrow L(\sin^2 t) = L\left(\frac{1 - \cos 2t}{2}\right)$$

$$= \frac{1}{2} (L(1) - L(\cos 2t))$$

$$= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2 + 4} \right]$$

$$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2} \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2 + 4} \right] ds$$

$$= \frac{1}{2} \left[\log s - \frac{1}{2} \log(s^2 + 4) \right]_s^\infty$$

$$= \frac{1}{4} \left[\log \frac{s^2}{s^2 + 4} \right]_s^\infty$$

$$= -\frac{1}{4} \log \left(\frac{s^2}{s^2 + 4} \right)$$

$$= \frac{1}{4} \log \left(\frac{s^2 + 4}{s^2} \right)$$

$$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{1}{4} \left[\log() \cdot s - \int s \cdot \frac{s^2}{s^2 + 4} \left(\frac{s^2 \cdot 2s - (s^2 + 4) \cdot 2s}{s^4} \right) ds \right]_s^\infty$$

$$= \frac{1}{5} \left[s \log \left(\frac{s^2 + 5}{s^2} \right) + 8 \int \frac{ds}{s^2 + 5} \right]_s^\infty$$

$$= \frac{1}{5} \left[s \log \left(\frac{s^2 + 5}{s^2} \right) + 5 \tan^{-1} \left(\frac{s}{2} \right) \right]_s^\infty$$

$$= \frac{1}{5} \left[0 + 2 \cdot \frac{\pi}{2} - s \log \left(\frac{s^2 + 5}{s^2} \right) \right]$$

$$- 2 \tan^{-1} \frac{s}{2}$$

$$= \frac{\pi}{5} - \frac{3}{5} \left(\frac{s^2 + 5}{s^2} \right) - \frac{1}{2} \tan^{-1} \frac{s}{2}$$

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Q) Evaluate $\int_0^\infty \frac{\cos at - \cos bt}{t} dt$

$$f(t) = \cos at - \cos bt$$

$$\begin{aligned} L\left[\frac{1}{t} f(t)\right] &= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \right) ds \\ &= -\frac{1}{2} \left[\log \left(\frac{s^2+b^2}{s^2+a^2} \right) \right]_s^\infty \\ &= \frac{1}{2} \log \left(\frac{s^2+b^2}{s^2+a^2} \right) \end{aligned}$$

$$\therefore \int_0^\infty e^{-st} \left(\frac{\cos at - \cos bt}{t} \right) dt = \frac{1}{2} \log \left(\frac{s^2+b^2}{s^2+a^2} \right)$$

putting $s = 0$,

$$\int_0^\infty \frac{\cos at - \cos bt}{t} dt = \frac{1}{2} \log \left(\frac{b^2}{a^2} \right)$$
$$= \underline{\underline{\log \frac{b}{a}}}$$



Assumed e^{-st} to convert it into La Place and put $s = 0$.

Q) $\int e^{-t} \sin 2t + \sin 3t dt$

$$L(\sin 2t) = \frac{2}{s^2 + 4} \quad L(\sin 3t) = \frac{3}{s^2 + 9}$$

$$\begin{aligned} L(\sin 2t + \sin 3t) &= \int_s^\infty \frac{2}{s^2 + 4} + \frac{3}{s^2 + 9} ds \\ &= \left[\tan^{-1} \frac{s}{2} \right]_s^\infty + \left[\tan^{-1} \frac{s}{3} \right]_s^\infty \end{aligned}$$

$$= \frac{\pi}{2} - \tan^{-1} \frac{s}{2} + \frac{\pi}{2} - \tan^{-1} \frac{s}{3}$$

$$= \pi - \left(\tan^{-1} \frac{s}{2} + \tan^{-1} \frac{s}{3} \right)$$

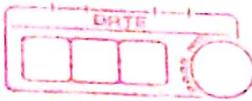
$$\tan^{-1} \frac{a+b}{1-ab}$$

$$L\left(\frac{s^2 + s}{t}\right) = \pi - \tan^{-1} \frac{5s}{6-s^2}$$

$$\int e^{-st} \left(\frac{\sin 2t + \sin 3t}{t} \right) dt = \pi - \tan^{-1} \frac{5s}{6-s^2}$$

Putting $s = 1$

$$Q = \pi - \tan^{-1} \frac{5}{5} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$



$$\Rightarrow L[f'(t)] = -f(0) + s \cdot L[f(t)]$$

$$\Rightarrow L[f''(t)] = s^2 \cdot L[f(t)] - sf(0) - f'(0)$$

Laplace Transform of derivatives

$$\Rightarrow L[f^{(n)}(t)] = s^n \cdot L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0)$$

(Q) L.T. of $\frac{d}{dt} \left(\frac{1 - \cos 2t}{t} \right)$

$$L(1 - \cos 2t) = L(1) - L(\cos 2t)$$

$$= \frac{1}{s} - \frac{s}{s^2 + 4}$$

$$L\left(\frac{1 - \cos 2t}{t}\right) = \int_s^\infty \frac{1}{s} - \frac{s}{s^2 + 4} ds$$

$$= \left[\log s - \frac{1}{2} \log(s^2 + 4) \right]_s^\infty$$

$$= \left. \log \left(\frac{s}{\sqrt{s^2 + 4}} \right) \right]_s^\infty$$

$$= \left[0 - \log \left(\frac{s}{\sqrt{s^2+4}} \right) \right]$$

$$= \log \frac{\sqrt{s^2+4}}{s}$$

$$\therefore L \left(\frac{1-\cos 2t}{t} \right) = \log \frac{\sqrt{s^2+4}}{s}$$

By property

$$L[f'(t)] = s \cdot L[f(t)] - f(0)$$

$$f(0) = \frac{1-\cos 2t}{t} = \frac{1-\sin^2 t}{t} = \frac{\cos^2 t}{t}$$

$$= 2 \sin t \cdot \frac{\sin t}{t} \rightarrow 0$$

$$= 0$$

$$= s \cdot \log \left(\frac{\sqrt{s^2+4}}{s} \right)$$

$$\Rightarrow L[f(t)] = \phi(s) \text{ then}$$

$$L^{-1} \int_0^t f(u) du = \frac{1}{s} \phi(s)$$

$$\Rightarrow L \left[\int_0^t \dots \int_0^t f(u) du \right] = \frac{1}{s^n} \cancel{\phi(s)} L(f(t))$$

(i) L.T. of $\int_0^t u \cosh u du$

$$L(\cosh t) = \frac{s}{s^2 - a^2}$$

$$L(t \cdot \cosh t) = \frac{d}{ds} \left(\frac{s}{s^2 - a^2} \right)$$

$$= -\frac{s^2 + a^2}{(s^2 - a^2)^2} \cancel{\phi(s)}$$

$$L \int_0^t u \cosh u du = \frac{1}{s} \phi(s) = -\frac{s^2 + a^2}{s(s^2 - a^2)^2}$$

$$①) L \left[\int_0^t \int_0^s \int_0^u \sin t \, dt \right] + \sin t \cdot (dt)^3 = \frac{1}{s^3} \phi(s)$$

$$L(\sin t) = \frac{1}{s^2 + 1}$$

$$L(t \sin t) = \frac{2s}{(s^2 + 1)^2}$$

$$L \left[\int_0^t \int_0^s \int_0^u \sin t \, dt \right] + \sin t \cdot (dt)^3 = \frac{1}{s^3} \cdot \frac{2s}{(s^2 + 1)^2}$$

$$= \frac{2}{s^2(s^2 + 1)^2}$$

$$②) I.T. of \int_0^t u^{-1} e^{-u} \sin u \, du$$

$$L(\sin u) = \frac{1}{s^2 + 1}$$

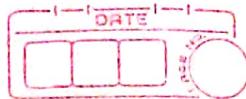
$$L(e^{-u} \sin u) = \frac{-1}{(s+1)^2 + 1} \quad \text{by shifting}$$

$$L(u^{-1} e^{-u} \sin u) = \int_s^\infty \frac{1}{(s+1)^2 + 1} \, ds$$

$$= \left. \tan^{-1}(s+1) \right|_s^\infty$$

$$= \cot^{-1}(s+1) = \phi s$$

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$$\rightarrow \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

$$x = \sqrt{t}$$

$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du$$

$$u=0$$

$$u = \sqrt{t}$$

$$\text{then } v=0$$

$$v=t$$

$$u = \sqrt{v} \quad du = \frac{1}{2\sqrt{v}} dv$$

$$\therefore \operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-v} \frac{1}{2\sqrt{v}} dv$$

$$= \frac{1}{\sqrt{\pi}} \int_0^t e^{-v} v^{-1/2} dv$$

$$L(v^{1/2}) = \frac{1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$L(e^{-v} v^{-1/2}) = \frac{\sqrt{\pi}}{\sqrt{s+1}}$$

$$L \int_0^t e^{-v} v^{-1/2} dv = \frac{1}{s} \frac{\sqrt{\pi}}{\sqrt{s+1}}$$

$$L \operatorname{erf}(\sqrt{t}) = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{s \sqrt{s+1}} = \frac{1}{s \sqrt{s+1}}$$

$$\therefore \operatorname{erf} x + \operatorname{erfc} x = 1$$

$$\therefore \operatorname{erfc} \sqrt{t} = 1 - \operatorname{erf} \sqrt{t}$$

$$L(\operatorname{erfc} \sqrt{t}) = L(1) - L(\operatorname{erf} \sqrt{t})$$

$$= \frac{1}{s} - \frac{1}{s\sqrt{s+1}}$$

$$= \frac{\sqrt{s+1} - 1}{s\sqrt{s+1}}$$

$$(i) L \int_t^{\infty} \frac{\cos u}{u} du$$

$$\text{Let } f(t) = \int_t^{\infty} \frac{\cos u}{u} du$$

$$u = vt$$

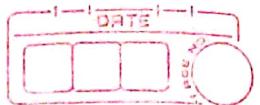
$$du = t dv$$

$$\begin{aligned} u &= 1 & u &\rightarrow \infty \\ \rightarrow v &= 1 & v &\rightarrow \infty \end{aligned}$$

$$f(t) = \int_t^{\infty} \frac{\cos vt \cdot t}{vt} dt$$

$$= \int_1^{\infty} \frac{\cos vt}{v} dv$$

$$L[f(t)] = \int_0^{\infty} e^{-st} \left[\int_1^{\infty} \frac{\cos vt}{v} dv \right] dt$$



$$= \int_1^\infty \frac{dv}{v} \int_0^\infty e^{-st} \cos vt dt$$

$$= \int_1^\infty \frac{dv}{v} \left[\cos vt \right] dt$$

$$= \int_1^\infty \frac{s dv}{v(s^2 + v^2)}$$

$$= \frac{1}{s} \int_1^\infty \left[\frac{1}{v} - \frac{v}{s^2 + v^2} \right] dv$$

$$= \frac{1}{s} \left[\log v - \frac{1}{2} \log(s^2 + v^2) \right]$$

$$= \frac{1}{2s} \left[\log \left(\frac{v^2}{s^2 + v^2} \right) \right]_1^\infty$$

$$= \frac{1}{2s} \left[\log 1 - \log \left(\frac{1}{s^2 + 1} \right) \right]$$

$$= \underline{\underline{\frac{1}{2s} \log(s^2 + 1)}}$$

$$0) \int_0^{\infty} e^{-t} \left(\int_0^t u^2 \sinhu \cosh u \, du \right) dt$$

$$\begin{aligned} L[\sinhu \cosh u] &= L\left[\frac{1}{2} \sinh 2u\right] \\ &= \frac{1}{2} \cdot \frac{2}{s^2 - 4} \end{aligned}$$

$$L\left[\frac{u^2 (\sinhu)}{\cosh u}\right] = (-1)^2 \frac{d}{ds^2} \left(\frac{1}{s^2 - 4}\right)$$

$$= 2 \left(\frac{3s^2 - 4}{(s^2 - 4)^3} \right) = \phi(s)$$

$$L\left[\int_0^t u^2 \sinhu \cosh u \, du\right] = \frac{1}{s} \phi(s)$$

$$= \frac{2}{s} \left(\frac{3s^2 - 4}{(s^2 - 4)^3} \right)$$

$$L\left(\int_0^{\infty} e^{-t} \left(\int_0^t u^2 \sinhu \cosh u \, du \right) dt\right) = \frac{2}{s} \left(\frac{3s^2 - 4}{(s^2 - 4)^3} \right) @ s=1$$

$$= \frac{2}{27}$$