Laplace Inverse

$\Rightarrow L[f(t)] = \phi(s)$

$f(t) = L^{-1}\phi(s)$

① $L(1) = \frac{1}{s}$ $\qquad L^{-1}\left(\frac{1}{s}\right) = 1$

② $L(e^{-at}) = \frac{1}{s+a}$ $\qquad L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

③ $L(t^{n-1}) = \frac{(n-1)!}{s^n}$ $\qquad L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$

$L(\sin at) = \frac{a}{s^2+a^2}$ $\qquad L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

$L(\cos at) = \frac{s}{s^2+a}$ $\qquad L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

$L(\sinh(at)) = \frac{a}{s^2-a^2}$ $\qquad L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

$L(\cosh at) = \frac{s}{s^2-a^2}$ $\qquad L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

19/08

Q) $L^{-1}\left(\frac{1-\sqrt{s}}{s^2}\right)^2$

$= L^{-1}\left(\frac{1-2\sqrt{s}+s}{s^4}\right)$

$= L^{-1}\left(\frac{1}{s^4}\right) - 2L^{-1}\left(\frac{1}{s^{7/2}}\right) + L^{-1}\left(\frac{1}{s^3}\right)$

$= \cancel{\frac{1}{\lceil 4}}$

But $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\lceil n} = \frac{t^{n-1}}{(n-1)!}$

$= \cancel{L^{-1}\frac{1}{3!}} \cancel{L^{-1}} \left(\frac{t^3}{3!} - 2\cdot\frac{t^{5/2}}{\lceil 7/2} + \frac{t^2}{2!}\right.$

$= \frac{t^3}{6} - \frac{16}{15\sqrt{\pi}}\cdot t^{5/2} + \frac{t^2}{2}$

Q) Find $L^{-1}\left(\frac{3s+4}{s^2+16}\right)$

$= L^{-1}\left(\frac{3s}{s^2+16}\right) + L^{-1}\left(\frac{4}{s^2+4^2}\right)$

$= 3\cos 4t + \sin 4t$

$\Rightarrow$ If $L(f(t)) = \phi(s)$

then $L[e^{-at} f(t)] = \cancel{e^{-at}}\, \phi(s+a)$

$\therefore f(t) = L^{-1}(\phi(s))$

then $L^{-1}[\phi(s+a)] = e^{-at} f(t)$

eg) $L^{-1}\left(\frac{1}{(s+a)^2}\right) = e^{-2t} \cdot L^{-1} \frac{1}{s^2}$

$= e^{-2t} \cdot t$

eg) $L^{-1}\left(\frac{s+1}{(s+1)^2 + 3^2}\right)$

$= e^{-t} \cdot \cos 3t$

Formulae → Shifting theorem

1) $L^{-1}\left(\frac{1}{(s-b)^n}\right) = e^{bt} L^{-1}\left(\frac{1}{s^n}\right) = e^{bt} \cdot \frac{t^{n-1}}{\lceil n}$

2) $L^{-1}\left(\frac{1}{(s-b)^2+a^2}\right) = e^{bt} L^{-1}\left(\frac{1}{s^2+a^2}\right)$

$= \frac{e^{bt}}{a} \sin at$

3) $L^{-1}\left(\frac{s-b}{(s-b)^2+a^2}\right) = e^{bt} L^{-1} \frac{s}{s^2+a^2}$

$= e^{bt} \cos at$

4) $L^{-1}\left(\frac{1}{(s-b)^2-a^2}\right) = e^{bt} L^{-1}\left(\frac{1}{s^2-a^2}\right)$

$= \frac{e^{bt}}{a} \sinh at$

5) $L^{-1}\left(\frac{s-b}{(s-b)^2-a^2}\right) = e^{bt} L^{-1} \frac{s}{s^2-a^2}$

$= e^{bt} \cosh at$

Q) $L^{-1}\left(\frac{s+2}{s^2+4s+7}\right)$

$= L^{-1}\left(\frac{s+2}{(s+2)^2+(\sqrt{3})^2}\right)$

$= e^{-2t} L^{-1} \frac{s}{s^2+(\sqrt{3})^2}$

$= e^{-2t} \cos\sqrt{3}t$

Q) Find $L^{-1}\left(\frac{2s-1}{s^2+4s+29}\right)$

$= L^{-1}\left(\frac{2s+4-5}{(s+2)^2+5^2}\right)$

$= 2L^{-1}\left(\frac{s+2}{(s+2)^2+5^2}\right) - L^{-1}\frac{5}{(s+2)^2+5^2}$

$= 2e^{-2t}\cos 5t - e^{-2t}\sin 5t$

$= e^{-2t}(2\cos 5t - \sin 5t)$

Q) Find $L^{-1}\left(\frac{3s+7}{s^2-2s-3}\right)$

$= L^{-1}\left(\frac{3s+3-3+7}{s^2-2s+1-4}\right)$

$= 3L^{-1}\left(\frac{s-1}{(s-1)^2-2^2}\right) + 10\, L^{-1}\frac{1}{(s-1)^2-2^2}$

$= 3e^{t}\cosh 2t + \frac{10e^{t}\sinh 2t}{2}$

$= 3e^{t}\cosh 2t + 5.e^{t}\sinh 2t$

Q) $L^{-1}\left(\frac{5s+3}{(s-1)(s^2-2s+5)}\right) \to f$

Let $(f) = \frac{A}{s-1} + \frac{Bs+C}{s^2-2s+5}$

$5s+3 = A(s^2-2s+5) + (Bs+C)(s-1)$

coeff of $s^2 = 0$ $\therefore$ $A+B=0$

for $s=1$

$8 = 8A$
$A = 1$ $\qquad -A = B$
$\qquad B = -1$

for $s=0$,

$3 = 5A - C$ $\therefore$ $C = 2$

$$= \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}$$

$$= L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{-s+2}{(s+1)^2+2^2}\right)$$

$$= e^t \quad - \frac{(s+1)}{(\;)} - \frac{3}{(\;)}$$

$$= e^t - e^t \cos 2t - 3\,\frac{\sin 2t}{2} \cdot e^t$$

$$= e^t\left(1 - \cos 2t - \frac{3}{2}\sin 2t\right)$$

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20/08

d) $L^{-1} \dfrac{s+2}{(s+3)s^2}$

$$\frac{s+2}{(s+3)s^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+3}$$

$$s+2 = as(s+3) + b(s+3) + cs^2$$

putting $s=0$, $s=-3$

$2 = 3B$ $\qquad -1 = 9C$

$\therefore$ s^2 on lef $= 0$

$a + c = 0$

$a = \frac{1}{9}$

$$L^{-1}\left(\frac{s+2}{(s+3)s^2}\right) = \frac{1}{9} L^{-1}\left(\frac{1}{s}\right) + \frac{2}{3} L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{9} L^{-1}\left(\frac{1}{s+3}\right)$$

$L \quad 5 \quad L$

Q) $L^{-1} \dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

$$\frac{(s+1)^2+2}{((s+1)^2+2^2)((s+1)^2+1^2)}$$

$$e^{-1}\, L^{-1}\, \frac{s^2+2}{(s^2+4)(s^2+1)}$$

$s^2 = x \longrightarrow$ partial fraction

$\dfrac{a}{x+4} + \dfrac{b}{x+1}$ $\qquad a = \dfrac{2}{3} \qquad b = \dfrac{1}{3}$

On solving further,

$$= \frac{e^{-t}}{3}\left(\sin 2t + \sin t\right)$$

Convolation theorem

$\Rightarrow$ for $\int_0^t f_1(u) f_2(t-u)\,du$

$$f_1(t) \cdot f_2(t) = \int_0^t f_1(u) \cdot f_2(t-u)\,du$$

To find $L^{-1}\ \phi_1(s) \cdot \phi_2(s)$

1) $L^{-1}\ \phi_1(s) = f_1(u)$

2) $L^{-1}\ \phi_2(s) = f_2(u)$

3) $L^{-1}(\phi_1(s) \cdot \phi_2(s) = \int_0^t f_1(u) \cdot f_2(t-u)\,du$

① $L^{-1} \dfrac{1}{(s-2)^4 (s+3)}$

$$\phi_1(s) = \frac{1}{s+3} \qquad \phi_2 = \frac{1}{(s-2)^4}$$

$$L^{-1}\phi_1(s) = e^{-3t} \qquad L^{-1}\phi_2(s) = e^{2t} \frac{t^3}{\lfloor 3}$$

$$= \frac{e^{2t}\, t^3}{6}$$

$$L^{-1}\phi(s) = \int_0^t e^{-3u} \cdot e^{2(t-u)} \frac{(t-u)^3}{6}\, du$$

$$= \int_0^t e^{2t-5u} \frac{(t-u)^3}{6}$$

$$= e^{2t} \int_0^t e^{-5u} \left(\frac{(t-u)^3}{6} \right) du$$

$u \cdot v$

$$= \left[(t-u)(1)\left(\frac{-e^{-5u}}{125} \right) - (-1)\left(\frac{e^{-5u}}{625} \right) \right]_0^t$$

$$= \frac{e^{-3t}}{625} - e^{2t}\left[\frac{1}{625} - \frac{t}{125} + \frac{t^2}{50} - \frac{t^3}{30} \right]$$

(i) $L^{-1} \dfrac{s^2}{(s^2-a^2)^2}$

$$\phi_1 = \phi_2 = \frac{s}{s^2-a^2}$$

$$L^{-1}\phi_1 = L^{-1}\frac{s}{s^2-a^2} = \cosh at$$

$$\int_0^t \cosh au \cdot \cosh(t-u)\, du$$

$$\frac{1}{2}\int_0^t \cosh at + \cosh a(2u-t) \quad \text{wrt } u$$

$$= \frac{1}{2}\left[u \cosh at + \frac{1}{2a} \sinh a(2u-t) \right]_0^t$$

$$= \frac{1}{2}\left[\sinh at + at \cosh at \right]$$

21/08

$$\int u \cdot v \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4$$

where $u \rightarrow$ polynomial
$v \rightarrow$ exponential

eg) $\int$

eg) $$\phi(s) = \frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$$

$$\frac{x-4}{(x+5)(x+2)} \quad \text{where } x = s^2+2s$$

eg) $$\phi(s) = \frac{s}{(s^2+2as+2)(s^2-2as+2)}$$

consider $$\frac{1}{(s^2+2as+2)} - \frac{1}{(s^2-2as+2)}$$

$$= \frac{s^2-2s+2-s^2-2s-2}{(s^2+2as+2)(s^2-2as+2)}$$

$$= \frac{-4s}{(s^2+2as+2)(s^2-2as-2)}$$

$$\therefore = \frac{-1}{4a}\left(\frac{-4as}{(s^2+2as+2)(s^2-2as+2)}\right)$$

$$= -\frac{1}{4a}\left[\frac{1}{s^2+2a^2+2as} - \frac{1}{s^2+2a^2-2as}\right]$$

8) $\dfrac{1}{(s+3)(s^2+2s+2)} = f(x)$

$\Rightarrow L^{-1}(f(s)) = L^{-1}\left(\dfrac{1}{[(s+1)+2][(s+1)^2+1]}\right)$

$= e^{-t} L^{-1}\left(\dfrac{-1}{(s+2)(s^2+1)}\right)$

$\phi_1(s) = \dfrac{1}{s^2+1} \quad \phi_2(s) = \dfrac{-1}{s+2}$

$L^{-1}\phi_1(s) = \sin at$

$L^{-1}\phi_2(s) = e^{-2t}$

$L^{-1}\phi_1 . \phi_2 = \int_0^t \sin u . e^{-2(t-u)} du$

$= \int_0^t \sin u . e^{-2t} . e^{2u} du$

$= e^{-2t} \int_0^t e^{2u} \sin u \, du$

$\dfrac{1}{a^2+b^2}\left[I\left(\dfrac{dII}{dx}\right) - II\dfrac{dI}{dx}\right]$

$= e^{-2t}\left[\dfrac{1}{5}\left(e^{2u}(-\cos u) + \sin u . 2 . e^{2u}\right)\right]$

$= \dfrac{e^{-2t}}{5}\left[e^{2u}(2\sin u - \cos u)\right]_0^t$

$$= \frac{1}{5} e^{-2t} \left[e^{2t} (2\sin t - \cos t) + 1 \right]$$

$$\therefore L^{-1}\left(F(s)\right) = e^{-t} \cdot L^{-1}[\phi_1 \phi_2]$$

$$= \frac{e^{-3t}}{5} \left[e^{2t} (2\sin t - \cos t) + 1 \right]$$

$$= \frac{1}{5} \left[e^{-t} (2\sin t - \cos t) + e^{-3t} \right]$$

Q) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$

$\Rightarrow$ $\phi_1(s) = \dfrac{s+1}{s^2+2s+2}$ $\qquad \phi_2(s) = \dfrac{s}{s^2+1}$

$L^{-1} \phi_1 = \dfrac{s+1}{(s+1)^2+1}$ $\qquad L^{-1} \phi_2 = \cos u$

$$= e^{-u} \frac{s}{s^2+1}$$

$$= e^{-u} \cdot \cos u$$

$$L^{-1}[\phi_1 \cdot \phi_2] = \int_0^t e^{-u} \cos u \cdot \cos(u-t)\, du$$

$$= \frac{1}{2} \int_0^t e^{-u} \left[\cos(2u-t) + \cos t \right] du$$

$$= \frac{1}{2}\left[\frac{1}{5}e^{-u}\left(-\cos(2u-t)+2\sin(2u-t)\right) - e^{-u}\cos t\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{5}e^{-t}\left(-\cos t+2\sin t\right) - e^{-t}\cos t - \frac{1}{5}\left(-\cos t - 2\sin t\right) + \cos t\right]$$

0) $L^{-1}\left(\frac{1}{s\sqrt{s+4}}\right)$

$L^{-1}\phi_1 = \frac{1}{(s+4)^{1/2}}$ $\qquad$ $L^{-1}\phi_2 = \frac{1}{s}$

$= e^{-4u}\frac{u^{-1/2}}{\Gamma 1/2}$ $\qquad$ $= 1$

$$L^{-1}\phi_1 \cdot \phi_2 = \left[\frac{1}{s}\cdot\frac{1}{\sqrt{s+4}}\right]$$

$$= \int_0^t \frac{e^{-4u}\cdot u^{-1/2}}{\sqrt{\pi}} \cdot 1 \cdot du$$

put $4u = x^2$ $\quad du = \frac{x\,dx}{2}$, $\sqrt{u} = x/2$

$$\therefore \int_0^{2\sqrt{t}} \frac{e^{-x^2}}{\sqrt{\pi}}\,dx = \frac{1}{2}\operatorname{erf}(2\sqrt{t})$$

26/08/2020

Q) Find $L^{-1}\left[\log\left(1+\frac{a^2}{s^2}\right)\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(1+\frac{a^2}{s^2}\right)\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log(a^2+s^2) - \log s^2\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{2s}{a^2+s^2} - \frac{2\cancel{s}}{s^{\cancel{2}}}\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{2s}{a^2+s^2} - \frac{2}{s}\right]$

$= -\frac{1}{t} [2\cos at - 2.1]$

$= \frac{2}{t}[1-\cos at]$

Q) Find $L^{-1}\left[s\log\left(\frac{s+1}{s-1}\right)\right]$

$= \frac{-1}{t} . L^{-1}\left[\frac{d}{ds}(s\log(s+1) - s\log(s-1))\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{s}{s+1} + \log(s+1) - \frac{s}{s-1} - \log(s-1)\right]$

$$= -\frac{1}{t} L^{-1}\left(\frac{-2s}{s^2-1}\right) - \frac{1}{t}\left(\frac{-1}{t}\right) L^{-1}\left[\frac{d}{ds}\log - \log\right]$$

$$= \frac{2}{t} L^{-1}\left(\frac{s}{s^2-1}\right) + \frac{1}{t^2} L^{-1}\left(\frac{1}{s+1} - \frac{1}{s-1}\right)$$

Ans $$= \frac{2}{t}\cosh t + \frac{1}{t^2}\left(e^{-t} - e^{t}\right)$$

Q) Find $L^{-1}\left(\frac{1}{s(s^2+4)}\right)$

$$= \int_0^t L^{-1}\left[\frac{1}{s^2+4}\right] du$$

$$= \int_0^t \frac{1}{2} \sin 2u \, du$$

$$= \cancel{\tfrac{1}{4}}\ \frac{1}{2}\left[-\frac{\cos 2u}{4}\right]_0^t$$

$$= \frac{1}{4}\left[1 - \cos 2t\right]$$

$$\Rightarrow L^{-1}\left(\frac{1}{s} \cdot f(u)\right) = \int_0^t L^{-1} f(u)$$

Q) $L^{-1}\left(\frac{1}{s^3(s^2+a^2)}\right)$

$$= \int_0^t L^{-1} \frac{1}{s^2} \cdot \frac{1}{(s^2+a^2)} du$$

$$= \int_0^t L^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+a^2}\right] du$$

$$= \frac{1}{a^2} \int_0^t L^{-1}\left[\frac{1}{s^2 \cancel{a}} - \frac{1}{s^2+a^2}\right] du$$

$$= \frac{1}{a^2} \int_0^t \left[u - \frac{1}{a} \sin au\right] du = \frac{1}{a}\left[\left[\frac{u^2}{2}\right]_0^t + \frac{1}{a^2}\left[\cos au\right]_0^t\right.$$

$$= \frac{1}{a^2}\left[\frac{t^2}{2}\right] + \frac{1}{a^4}\left[\cos at - 1\right]$$

$\Rightarrow f(t) \rightarrow$ periodic f^n with period $= a$

$$L(f(t)) = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$$

Q) $f(t) = |\sin pt|$

$$L(f(t)) = \frac{1}{1-e^{-\frac{\pi}{p}s}}\int_0^{\pi/p} e^{-st}\sin pt\,dt$$

$$= \left(\frac{1}{1-e^{-\pi/ps}}\right)\left[\frac{e^{-st}}{s^2+p^2}\left(-s\cdot\sin pt - p\cos pt\right)\right]_0^{\pi/p}$$

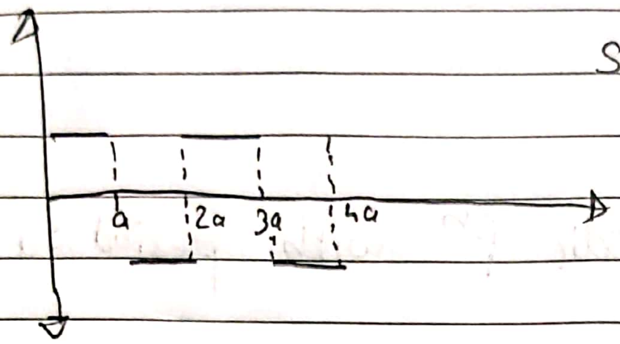
$$= \left(\frac{1}{1-e^{-\frac{\pi s}{p}}}\right)\cdot\frac{1}{s^2+p^2}\left[e^{-s\pi/p}(0+p) - (0-p)\right]$$

$$= \frac{1}{s^2+p^2}\cdot\frac{1}{(1-e^{-\pi/ps})}\cdot p\left(1+e^{-\frac{s\pi}{p}}\right)$$

$$= \frac{p}{s^2+p^2}\left(\frac{e^{\pi s/2p}+e^{-\pi s/2p}}{e^{\pi s/2p}-e^{-\pi s/2p}}\right)$$

$$= \frac{p}{s^2+p^2}\,\cancel{\cot}\,\coth\left(\frac{\pi s}{2p}\right)$$

Q) Find $L(f(t) = 1, \quad 0 < t < a$
$f(t) = -1, \quad a < t < 2a$



square wave f^n

period = 2a

$$Lf(t) = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\, 1\, dt + \int_a^{2a} e^{-st} \cdot -1\, dt \right]$$

$$= \left(\frac{1}{1-e^{-2as}}\right) \left[\left(-\frac{e^{-st}}{s} \right)_0^a + \left(\frac{e^{-st}}{s} \right)_a^{2a} \cdot (\;) \right]$$

$$= \frac{1}{s} \left(\frac{1}{1-e^{-2as}}\right) (1 - e^{-as})^2$$

$$= \frac{1}{s} \frac{(1-e^{-as})}{(1+e^{-as})}$$

$$= \frac{1}{s} \left(\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \right)$$

$$= \frac{1}{s} \tanh\left(\frac{as}{2}\right)$$

27/08

Q)
$$f(t) = a \sin pt \quad 0 \leq t \leq \frac{\pi}{p}$$
$$0 \quad \frac{\pi}{p} < t \leq \frac{2\pi}{p}$$

$$L(f(t)) = \frac{1}{1 - e^{-2\pi/p\, s}} \int_0^{\pi/p} e^{-st} a \sin pt \, dt$$

$$= \frac{a}{(1 - e^{-2\pi s/p})} \left[\frac{1}{s^2+p^2} \cdot e^{-st} (-s \sin pt - p \cos pt) \right]_0^{\pi/p}$$

$$= \frac{a}{(1-e^{-2\pi/ps})} \cdot \frac{1}{s^2+p^2} \left[p \cdot e^{-s\pi/p} + p \right]$$

$$= \frac{ap}{1 - e^{-(s\pi/p)}} \cdot \frac{1}{s^2+p^2}$$

Heavy side

$$H(t-a) = \begin{cases} 1 : t > a \\ 0 : t < a \end{cases}$$

① $L(f(t) H(t-a)) = e^{-as} L(f(t+a))$

② $L[f(t-a) H(t-a)] = e^{-as} L(f(t)) = e^{-as} \phi(s)$

③ $L^{-1}(e^{-as} \phi(s)) = f(t-a) H(t-a)$

Q) Find $L\ \sin(t)\, u\left(t-\frac{\pi}{2}\right)$

$$= e^{-\pi/2\, s}\, L\left(\sin\left(t+\frac{\pi}{2}\right)\right)$$

$$= e^{-\pi/2\, s}\, L(\cos t)$$

$$= e^{-\pi/2\, s}\, \frac{s}{s^2+\cancel{a^2}1}$$

Q) Find L of $f(t) = (1+2t-t^2+t^3)\, u(t-1)$

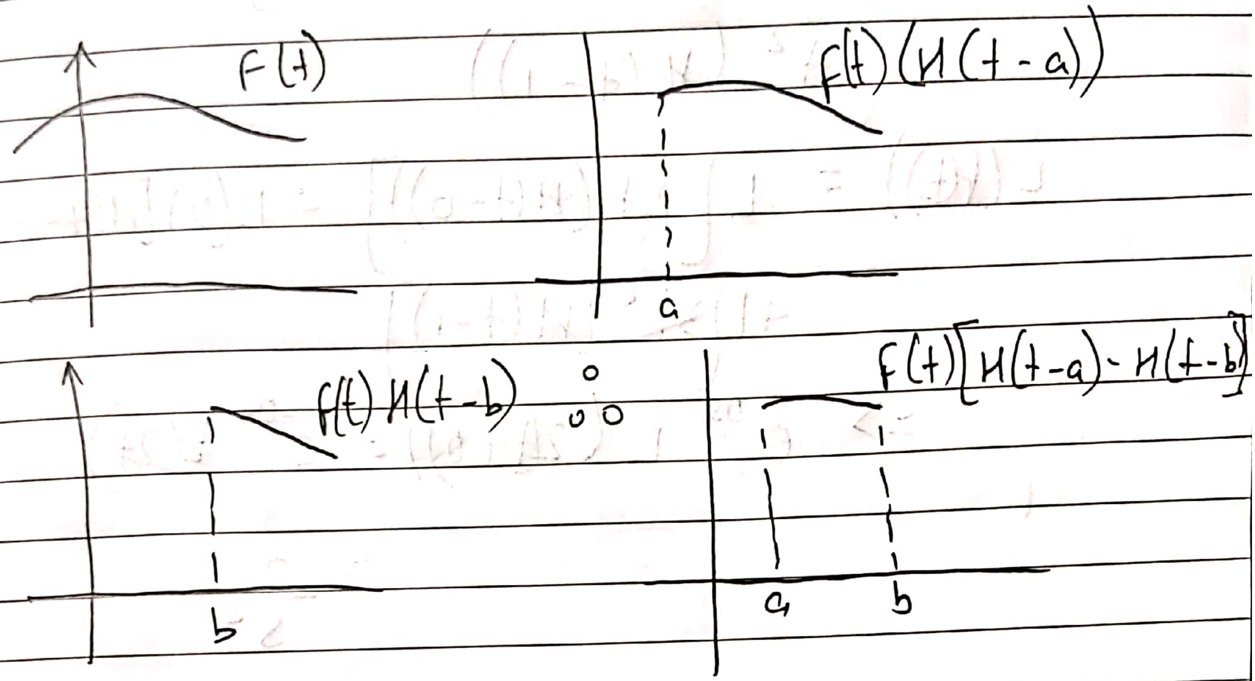
$$= e^{-s}\, L\left(1+2(t+1)-(t+1)^2+(t+1)^3\right)$$

$$= e^{-s}\, L\left(3+3t+2t^2+t^3\right)$$

$$= e^{-s}\left[3\cdot\frac{1}{s}+3\cdot\frac{1}{s^2}+2\cdot\frac{2}{s^3}+\frac{6}{s^4}\right]$$

$$= e^{-s}\left[\frac{3}{s}+\frac{3}{s^2}+\frac{4}{s^3}+\frac{6}{s^4}\right]$$

Q) Graph of $f(t)\left[H(t-a) - H(t-b)\right]$ is graph of f^n $f(t)$ on the interval (a,b)



$$f(t) = \begin{cases} f_1(t) & 0 < t < a \\ f_2(t) & a < t < b \\ f_3(t) & b < t \end{cases}$$

$$f(t) = f_1(t)\left[H(t-0) - H(t-a)\right] + f_2(t)\left[H(t-a) - H(t-b)\right] + f_3(t)\left[H(t-b)\right]$$

Q) $f(t) = 2t \quad 0 < t < 1$

$3t^2 \quad t > 1$

$f(t) = 2t\,(H(t-0) - H(t-1)] +$

$3t^2\,(H(t-1))$

$L(f(t)) = L\left[2t\,(H(t-0))\right] - L(2t[H(t-1)]$

$+ L\left[3t^2\,[H(t-1)]\right]$

$\Rightarrow e^{-0s}\,L(2(t+0)) = e^{-0s}\,L(2t)$

$= \dfrac{2}{s^2}$

$\Rightarrow e^{-1s}\,L(2(t+1))$

$= e^{-s}\,2 \cdot L(t+1)$

$= e^{-s} \cdot 2 \cdot \left[\dfrac{1}{s^2} + \dfrac{1}{s}\right]$

$\Rightarrow e^{-1s}\,L(3(t+1)^2)$

$= e^{-1s}\,3 \cdot L(t^2 + 2t + 1)$

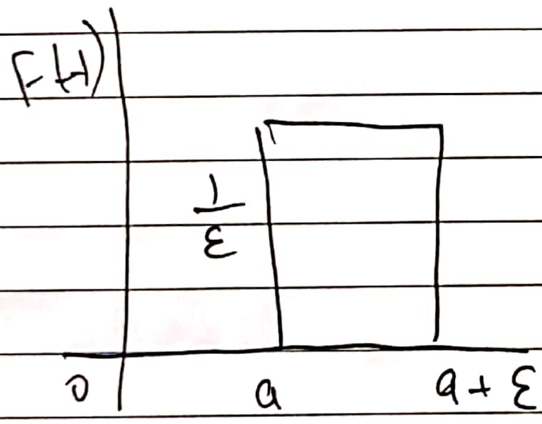
$= e^{-s}\left(3\left[\dfrac{2}{s^3} + \dfrac{2}{s^2} + \dfrac{1}{s}\right]\right.$

$= e^{-s}\left[\dfrac{6}{s^3} + \dfrac{6}{s^2} + \dfrac{3}{s}\right]$

$$\therefore L[f(t)] = \frac{2}{s^2} - e^{-s}\left[\frac{2}{s^2} + \frac{2}{s}\right] + e^{-s}\left[\frac{6}{s^3} + \frac{6}{s^2} + \frac{3}{s}\right]$$

Q) Dirac Delta f^n (Unit impulse f^n)

$$f(t) = \begin{cases} 0, & t < a \\ \frac{1}{\varepsilon}, & a \leq t \leq a+\varepsilon \\ 0, & t > a+\varepsilon \end{cases}$$



$$\int_0^{\infty} f(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = 1 \text{ for all } \varepsilon$$

28/8

① $(D^3 + 2D^2 - D - 2)y = 0$ where $D = \frac{d}{dt}$

$y(0) = y'(0) = 0$, $y''(0) = 6$

→ Taking Laplace on both side

$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] +$$

$$2[s^2(\bar{y}) - sy(0) - y'(0)]$$

$$- [s\bar{y} - y(0)] - 2\bar{y} = 0$$

Using the given cond^n's,

$$(s^3 + 2s^2 - s - 2)\bar{y} = 6$$

$$\bar{y} = \frac{6}{(s-1)(s+1)(s+2)}$$

$$= \frac{6}{(s-1)(6)} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}$$

On inversion, we get

$$y = L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\left(\frac{1}{s+1}\right) + 2L^{-1}\left(\frac{1}{s+2}\right)$$

$$= e^{+s} - 3e^{-s} + 2e^{-2s}$$

② Solve using laplace theorem

$$\frac{d^2x}{dt^2} + 9x = \cos 2t \quad , \quad x(0) = 1 \, , \, x\left(\frac{\pi}{2}\right) = -$$

→ Since $x'(0)$ is not given, we assume $x'(0) = a$

Taking laplace on both sides,

$$L(x'') + 9L(x) = L(\cos 2t)$$

$$\left[s^2\bar{x} - sx(0) - x'(0)\right] + 9\bar{x} = \frac{s}{s^2+4}$$

$$(s^2+9)\bar{x} = s + a + \frac{s}{s^2+4}$$

$$\bar{x} = \frac{s+a}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

On inversion, we get

$$x = \frac{a}{3}\sin(3t) + \frac{1}{5}\cos(2t) + \frac{4}{5}\cos 3t$$

when, $t = \frac{\pi}{2}$, $-1 = -\frac{a}{3} - \frac{1}{5}$

$$\frac{a}{3} = \frac{4}{5}$$

$$\therefore x = \frac{1}{5}\cos 2t + 4\sin 3t + 4\cos 3t$$

02/09/2020

Q) $\int_0^{\infty} \sin(tx)^2 dx$ and hence find $\int_0^{\infty} \sin x^2 dx$

$$L(f(t)) = \int_0^{\infty} \sin(tx)^2 dx$$

$$= \int_0^{\infty} e^{-st}(f(t))\, dt$$

$$= \int_0^{\infty} e^{-st}\left[\int_0^{\infty} \sin(tx^2)\, dx\right] dt$$

$$= \int_0^{\infty}\left[\int_0^{\infty} e^{-st} \sin tx^2\, dt\right] dx$$

$$= \int_0^{\infty} L(\sin t x^2)\, dx$$

$$= \int_0^{\infty} \frac{x^2}{s^2+x^4}\, dx$$

$$x^4 = s^2 \tan^2\theta$$

$$= \int_0^{\pi/2} \frac{s\tan\theta}{s^2+s^2\tan^2\theta} \cdot \frac{s.\sec\theta}{2\sqrt{s\tan\theta}}\, d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{1/2}\theta . \cos^{1/2}\theta\, d\theta$$

$\hookrightarrow B(m,n)$

$$= \frac{1}{2\sqrt{s}} \frac{\Gamma 3/4 . \Gamma 1/4}{2.1}$$

let $p = \frac{1}{4}$

$1-p = \frac{3}{4}$

$$= \frac{1}{2\sqrt{s}} \frac{\sqrt{2} . \pi}{2}$$

$$= \frac{\pi}{2\sqrt{2}} \cdot \frac{1}{\sqrt{s}}$$

$$f(t) = \frac{\pi}{2\sqrt{2}} L^{-1}\left(\frac{1}{\sqrt{s}}\right)$$

$$= \frac{\pi}{2\sqrt{2}} \frac{t^{-1/2}}{\Gamma 1/2}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{1}{\sqrt{t}}$$

put $t = 1$ $\therefore \int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Q) $\int_0^\infty e^{-tx^2} dx$ and hence find $\int_0^\infty e^{-x^2} dx$

Let $f(t) = \int_0^\infty e^{-tx^2} dx$

$$Lf(t) = \int_0^\infty e^{-st} \left[\int_0^\infty e^{-tx^2} dx\right] dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \cdot e^{-tx^2} dt\right] dx$$

$$= \int_0^\infty L\left(e^{-tx^2}\right) dx$$

$$= \int_0^\infty \frac{dx}{s+x^2}$$

$$= \frac{1}{\sqrt{s}} \tan^{-1} \frac{x}{\sqrt{s}} \Bigg]_0^\infty$$

$$= \frac{\pi}{2\sqrt{s}}$$

$$f(t) = \frac{\pi}{2} L^{-1}\left(\frac{1}{\sqrt{s}}\right)$$

$$= \frac{\pi}{2} \frac{t^{-1/2}}{\Gamma(1/2)}$$

$\therefore$ $t = 1$ $\quad f(t) = \cancel{\frac{\sqrt{\pi}}{2}} \; \frac{1}{2}\sqrt{\pi}$

Q) $J_0(t) = \dfrac{1}{\pi} \displaystyle\int_0^\pi \cos(t\cos\theta)\, d\theta$

P.T. $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$

$\because \pi/2$ is easier

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$$J_0(t) = \frac{2}{\pi}\int_0^{\pi/2} \cos(t\cos\theta)\, d\theta$$

$$L(J_0(t)) = \frac{2}{\pi}\int_0^{\infty} e^{-st}\left[\int_0^{\pi/2} \cos(t\cos\theta)\, d\theta\right] dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left[\int_0^{\infty} e^{-st}\cos(t\cos\theta)\, dt\right] d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2} L(\cos(t\cos\theta))\, d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{s^2}{s^2+\cos^2\theta}\, d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{s^2\sec^2\theta}{s^2\sec^2\theta+1}\, d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{s^2\sec^2\theta}{s^2(1+\tan^2\theta)+1}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{s\sec^2\theta}{(s^2+1)+s^2\tan^2\theta}\, d\theta$$

put $s\tan\theta = t$

$s\sec^2\theta\, d\theta = dt$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{dt}{t^2+(s^2+1)}$$

$$= \frac{2}{\pi}\left[\tan^{-1}\frac{1}{s^2+1}\right]_0^{\infty}$$

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$$= \frac{2}{\pi} \cdot \frac{1}{\sqrt{s^2+1}} \cdot \frac{\pi}{2}$$

$$L(J_0(t)) = \frac{1}{\sqrt{s^2+1}}$$