Course EE5104: Adaptive Control System (Part I)

CA1 Project

Adaptive Controller Design for a Continuous-time System with only Input and Output Measurable

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Apr 17, 2025

Abstract

This project presents the design and simulation of a model reference adaptive controller (MRAC) for a continuous-time second-order plant with unknown parameters. The controller ensures boundedness, zero steady-state error for step inputs, and tracks reference models accurately. MATLAB simulation validates the design. Effects of observer polynomial $T(p)$ and adaptation gain matrix Γ are also analyzed. The performance of the system with sinusoid input reference signal is also analyzed.

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Chapter 1

Adaptive Controller Design

1.1 Problem Setup

The plant is defined by:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2}, \quad b_0 < 0 \tag{1.1}$$

And the plant has no zeros in the right-half of the s-plane.

The exact values of the transfer function coefficients are unknown. Simple tests conducted on the open loop plant have indicated that the plant is **stable**, **very lightly damped**, and has **natural frequency of approximately 2 rad/s**. For simulation, the plant is modeled as:

$$\frac{Y(s)}{U(s)} = \frac{-0.5s - 1}{s^2 + 0.22s + 6.1} \tag{1.2}$$

Only $y(t)$ and $u(t)$ are measurable. The control goal is to design an adaptive controller such that:

- The output tracks a reference model with fast dynamics and zero steady-state error.
- The closed-loop system remains bounded (ensure boundedness of $y(t)$ and $u(t)$).

The plant for simulation is given by Eq. 1.2. To understand the open-loop characteristics of the plant, we examine the denominator, which represents the characteristic polynomial:

$$D(s) = s^2 + 0.22s + 6.1$$

This corresponds to a standard second-order system:

$$D(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Comparing coefficients, we obtain: $\omega_n = \sqrt{6.2} \approx 2.47$ rad/s and $\zeta \approx 0.0445$. Thus, the plant is lightly damped ($\zeta \ll 1$) and has a relatively low natural frequency, confirming the information provided in the project description. This low damping ratio implies the plant exhibits oscillatory behavior, while the natural frequency of approximately 2.47 rad/s indicates a moderate response speed.

1.2 Reference Model Design

We rewrite the plant equation as:

$$R_p(p)y = k_p Z_p(p)u \tag{1.3}$$

where

$$\begin{aligned} p &\equiv \frac{d}{dt} \\ R_p(p) &= p^2 + a_1 p + a_2 \\ Z_p(p) &= p + \frac{b_1}{b_0} \\ k_p &= b_0 \end{aligned}$$

The relative degree $n^* = deg(R_p) - deg(Z_p)$ =1. We choose a first-order reference model:

$$\frac{Y_m(s)}{R(s)} = \frac{k_m}{s + a_m} \tag{1.4}$$

We choose $k_m = a_m$ = 5 rad/s, which is approximately twice the plant's natural frequency. This ensures that the reference model responds significantly faster than the plant while avoiding excessive aggressiveness that could destabilize the adaptive controller.

The choice of a_m = 5 yields a time constant of $\tau = \frac{1}{5}$ = 0.2 s, with an approximate rise time of 0.6 s. In contrast, the simulation plant's effective time constant is $\tau_p \approx \frac{1}{\zeta \omega_n} \approx$ 9.2 s, reflecting a much slower response.

The reference model is deliberately designed to be faster and more stable than the open-loop plant, ensuring improved transient performance and enabling the controller to correct tracking errors effectively with rise time $\approx$ 0.45 s.

1.3 Controller Structure

We rewrite the reference system as follows:

$$R_m(p)y = k_m r \tag{1.5}$$

Let $T(p) = p^2 + t_1 p + t_2$ be a stable, monic observer polynomial. We solve the equation:

$$T(p)R_m(p) = R_p(p)E(p) + F(p) \tag{1.6}$$

to obtain polynomials $E(p), F(p)$.

We define filtered signals:

$$\omega_y = \frac{1}{T(p)} y \tag{1.7}$$

$$\omega_u = \frac{1}{T(p)} u \tag{1.8}$$

Then the control low is:

$$u = \bar{\theta^*}^T \bar{\omega} \tag{1.9}$$

where

$$\bar{\omega} = [p\omega_y, \omega_y, p\omega_u, \omega_u, r]^T \tag{1.10}$$

$$\bar{\theta}^* = [-f_1, -f_2, -g_1, -g_2, k^*]^T \tag{1.11}$$

1.4 Adaptive Law

Since the exact parameters of the plant are unknown, the value of $\bar{\theta}^*$ is unknown. We make $\bar{\theta}$ a time-varying gain:

$$\boxed{u(t) = \bar{\theta}^T(t)\bar{\omega}(t)} \tag{1.12}$$

Let $e_1(t) = y(t) - y_m(t)$. The adaptive law is:

$$\dot{\theta}(t) = -\mathrm{sgn}(k^*)\Gamma\bar{\omega}e_1 \tag{1.13}$$

where Γ is a symmetric positive-definite adaptation gain matrix.

1.5 Boundedness Analysis

For the designed controller, we know that

1. the relative degree $n^* = 1$ is known;
2. the order of the plant $n = 2$ is known;
3. Z_p is a stable polynomial since the plant has no zeros in the right-half of the s-plane;
4. sgn(k_p) = sgn(b_0) = -1 is known

then $y(t), u(t), \bar{\theta}(t)$ are bounded $\forall t \geq 0$, and

$$\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty}(y(t) - y_m(t)) = 0$$

Chapter 2

Simulation

In this chapter, a simple simulation is carried out using MATLAB and Simulink to verify the performance of the designed adaptive controller when the reference signal $r(t)$ is a square wave of an appropriately chosen period.

2.1 Parameters

1. Plant: $\frac{-0.5s-1}{s^2+0.22s+6.1}$
2. Reference model: $\frac{5}{s+5}$
3. Reference signal $r(t)$:

   A square wave reference signal with amplitude 1 and period 10 seconds was selected to ensure sufficient excitation for parameter adaptation and to allow the system to reach steady state between transitions. The proper amplitude promotes observable tracking behavior, while the relatively long period avoids premature switching that would hinder convergence analysis.
4. Observer polynomial: the observer polynomial $T(p)$ is of second-order, which can be written as:

   $$T(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

   where ζ is the damping ratio and ω_n is the natural frequency of the filter $\frac{1}{T}$. As given by the experiment conditions, we know that the plant is very lightly damped, and has natural frequency of approximately 2 rad/s. I choose ω_n = 3 and ζ = 0.2 to ensure fast filtering dynamics and sufficient excitation in the regressor signals. A low damping ratio allows the filter to respond rapidly without overly suppressing signal variations, which promotes faster parameter adaptation while maintaining stability. The observer polynomial will then be

   $$T(s) = s^2 + 1.2s + 9 \tag{2.1}$$
5. Adaptation gain: For adaptation gain Γ, I have tried a few values and decided to choose

   $$\Gamma = \mathrm{diag}(20000, 20000, 2000, 1000, 1200) \tag{2.2}$$

   The adaptation gain matrix Γ was chosen as a positive definite diagonal matrix to ensure stability and allow independent tuning of each adaptive parameter. Higher values promote faster adaptation but may increase sensitivity to noise, while smaller values yield smoother but slower parameter convergence. A balanced setting was selected to achieve fast tracking without excessive control effort.

2.2 Results and Discussion

The simulation result is shown in Fig. 2.1 and Fig. 2.2.

The output of the plant as shown in Fig. 2.1 is able to track the reference model fairly well. Initially, there are some ripple at the rising and falling edge of the first three periods, however, the output seems to track the reference model well afterwards. The same response can be obtained from the error plot. As observed, in the first 30 seconds of the simulation, the system's output error is gradually reducing in general. After 30 seconds, the error is stable around the value 0, as desired. For the control effort of the plant, the plot shows that high control signal is needed to drive the plant to track the reference model. It is reasonable as the signal is bounded within certain value.

Fig. 2.2 shows the adaptive controller gains $\bar{\theta}$ and the exact controller gains $\bar{\theta}^*$. Since this is a simulated plant with exact plant transfer function is known, we can calculate the exact controller gains as follow:

$$T(p)R_m(p) = R_p(p)E(p) + F(p)$$

We have

$$\begin{aligned}
T(s) &= s^2 + 1.2s + 9 \\
R_m(s) &= s + 5 \\
R_p(s) &= s^2 + 0.22s + 6.1 \\
Z_p(s) &= -0.5s - 1
\end{aligned}$$

The solution to the Diophantine identity can then be obtained by a simple polynomial division,

$$E(s) = s + 5.98 \tag{2.3}$$

$$F(s) = 7.5844s + 8.522 \tag{2.4}$$

We then have

$$\bar{F}(s) = \frac{F(s)}{k_p} = -15.1688s - 17.044 \tag{2.5}$$

$$\bar{G}(s) = E(s)Z_p(s) = s^2 + 7.98s + 11.96 \tag{2.6}$$

$$G_1 = \bar{G}(s) - T(s) = 6.78s + 2.96 \tag{2.7}$$

Also,

$$k^* = \frac{k_m}{k_p} = -10 \tag{2.8}$$

Thus, the exact controller gains,

$$\bar{\theta}^* = \begin{bmatrix} 15.1688 & 17.0440 & -6.7800 & -2.9600 & -10.0000 \end{bmatrix}^T \tag{2.9}$$

From the plots shown in Fig. 2.2, the controller gains $\bar{f}_1, \bar{f}_2, \bar{g}_1, \bar{g}_1$ and $\bar{k}^*$ converge to the exact controller gains. It shows that the adaptive controller is correctly designed.

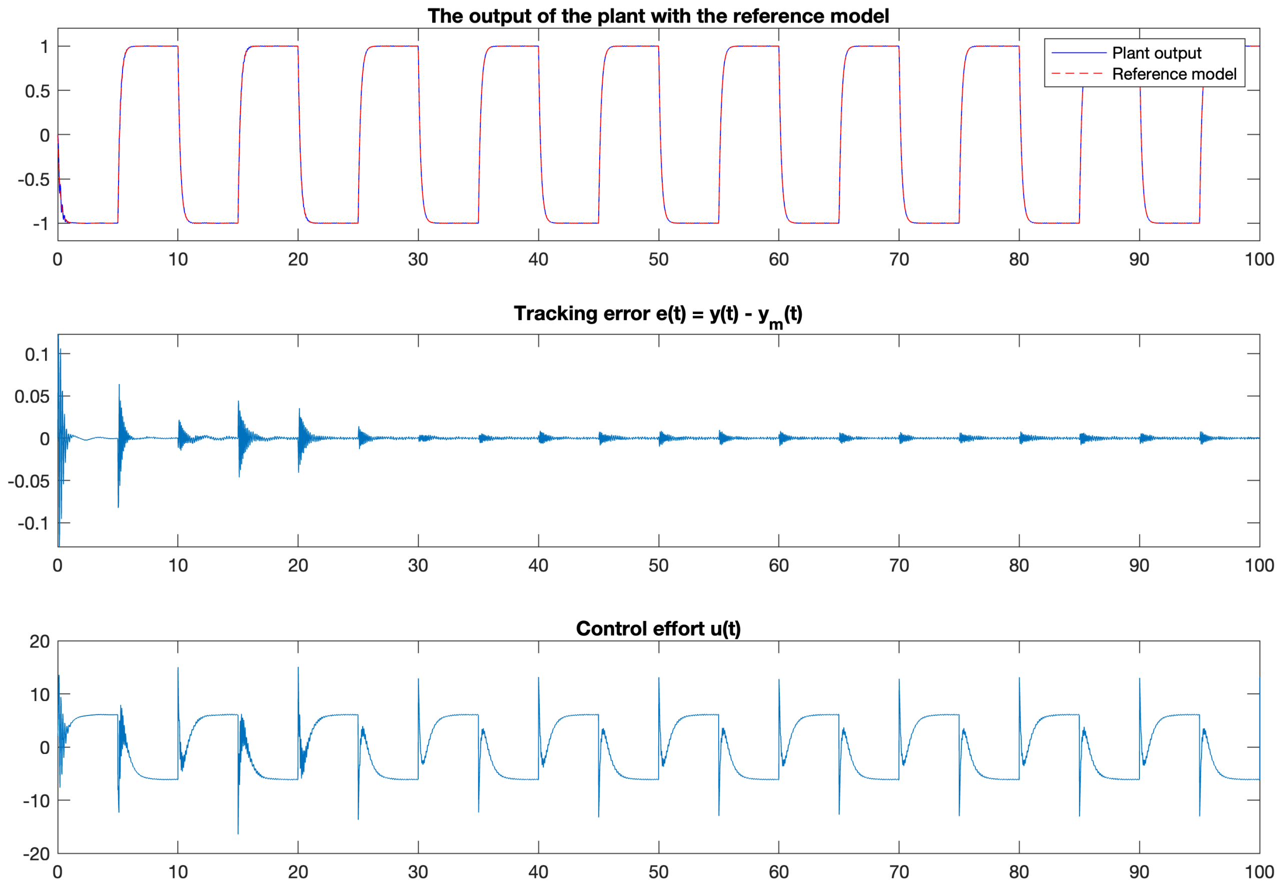

Figure 2.1: Reference model output y_m, plant output y, error e, and control signal u

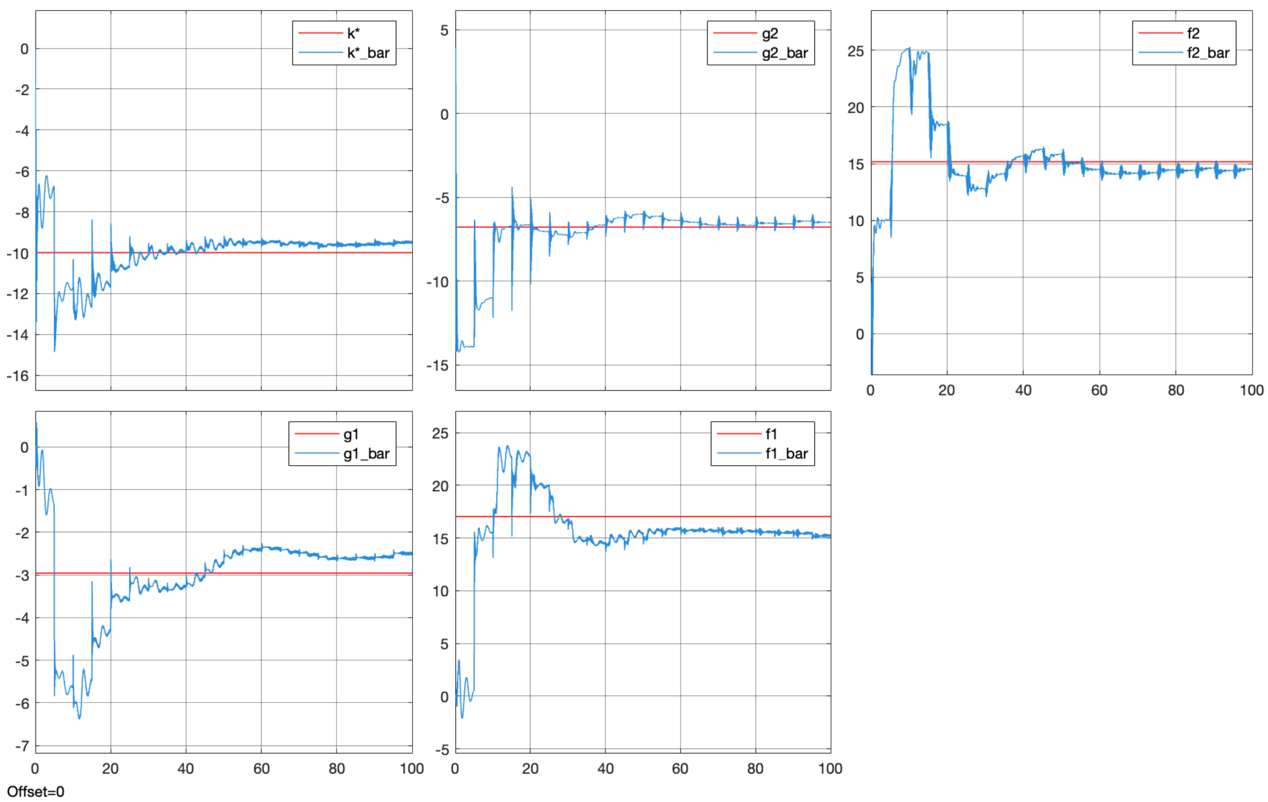

Figure 2.2: The adapted controller gains $\bar{\theta}$ with the exact gains $\bar{\theta}^*$

Chapter 3

Effect of Different T and Γ

In this chapter, we will investigate the performance of the system under different adaptation gains and observer polynomial.

3.1 Adaptation Gain, Γ

For simplicity, we will observe the performance of the system when Γ is both increased and decreased by 100 times, while the observer polynomial remains unchanged.

3.1.1 Increased $\Gamma_1 = \Gamma \times 100$

The simulation results are illustrated in Fig.3.1 and Fig.3.2. It can be observed that after the first period, the system output tracks the reference model almost perfectly. Only minimal ripples are present during the initial transient phase. Similarly, the tracking error $e(t)$ is significantly reduced—approximately one-tenth of its magnitude when using the original Γ matrix. However, the control input $u(t)$ exhibits increased oscillations. This is expected, as greater control effort is required to achieve faster and more accurate tracking performance.

As shown in Fig.3.2, the adapted gains converge to their ideal values, albeit at a slower rate compared to the case with the original Γ. Additionally, the gains exhibit more noticeable oscillations during the transient phase before settling to their steady-state values.

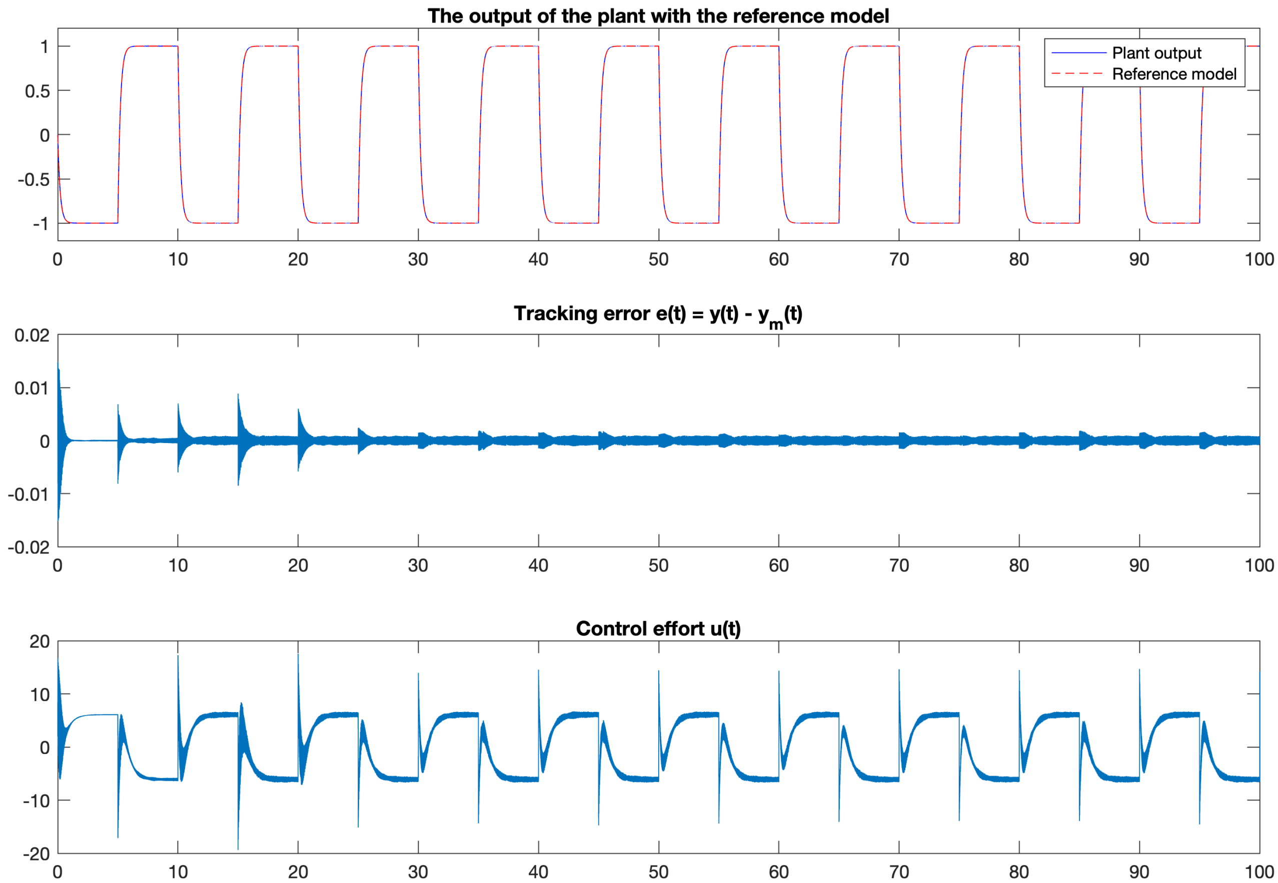

Figure 3.1: y_m, y, e, u with high Γ values

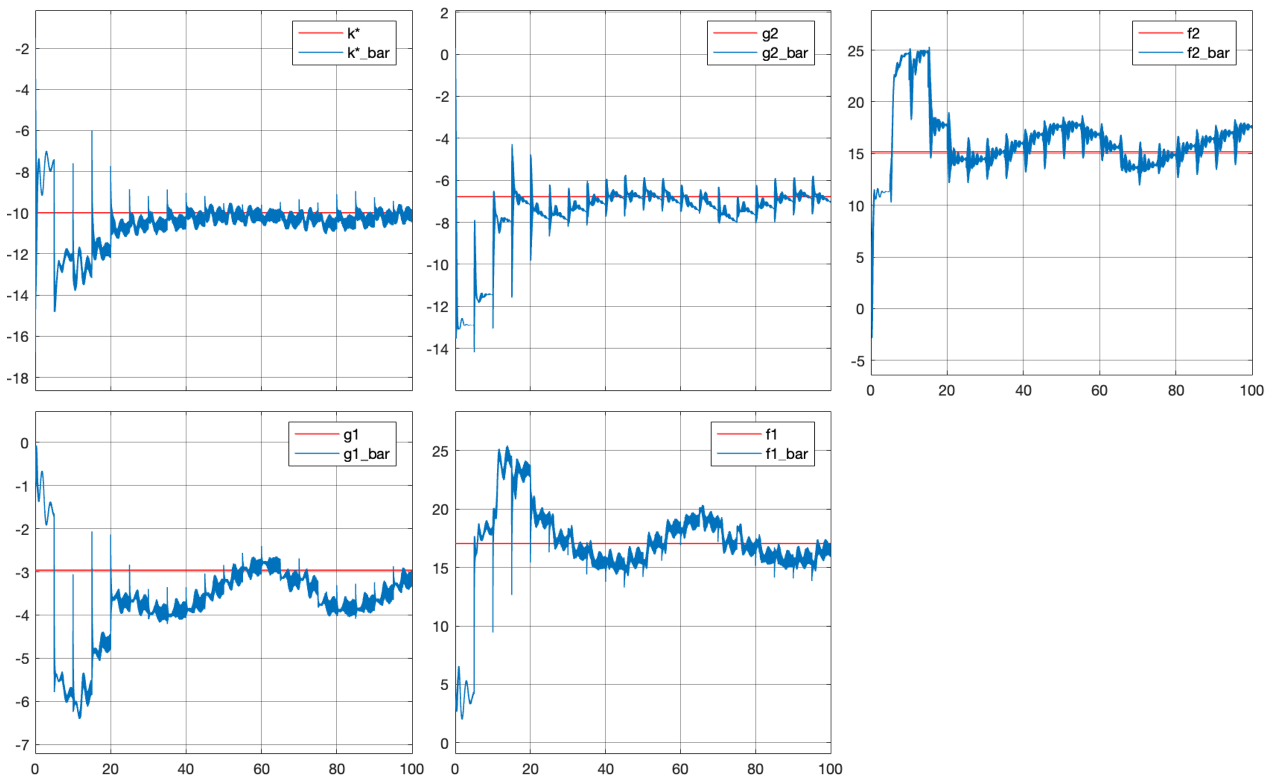

Figure 3.2: $\bar{\theta}$, $\bar{\theta}^*$ with high Γ values

3.1.2 Decreased $\Gamma_2 = \Gamma / 100$

The simulation results are shown in Fig.3.3 and Fig.3.4. It can be observed that throughout the entire 100-second simulation, the output signal fails to track the reference signal accurately. Although oscillations are present in every period, their amplitudes gradually decrease over time. This indicates that the system output could eventually track the reference model, albeit very slowly. From the error dynamics $e(t)$, it is evident that the tracking error does converge to zero, but the convergence is significantly slower compared to the previous cases.

Regarding the control input $u(t)$, the control effort is noticeably smaller than in earlier scenarios. This is likely due to the reduced aggressiveness required to drive the system towards the reference model under the current conditions.

As shown in Fig.3.4, the adapted gains appear to converge toward the exact values, but at a much slower rate.

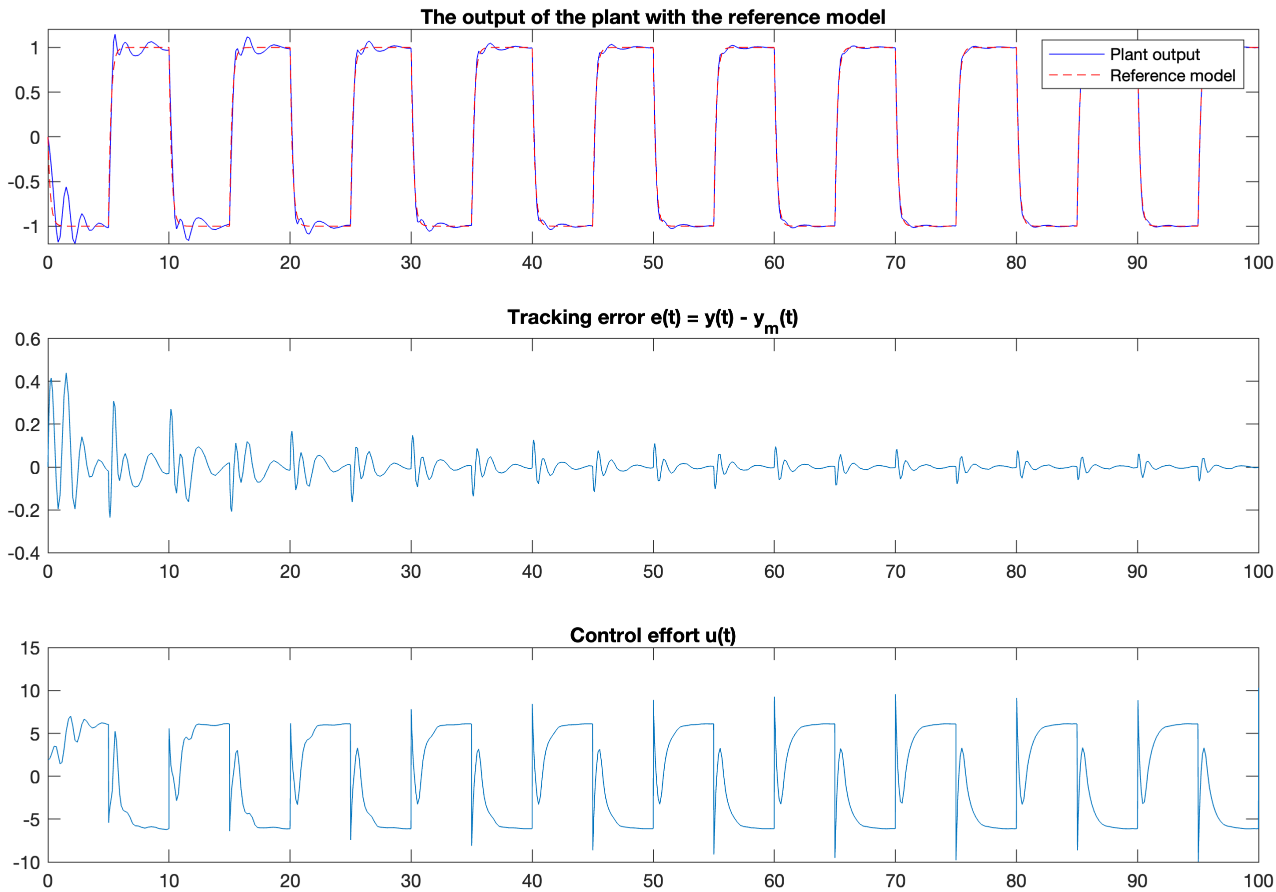

Figure 3.3: y_m, y, e, u with low Γ values

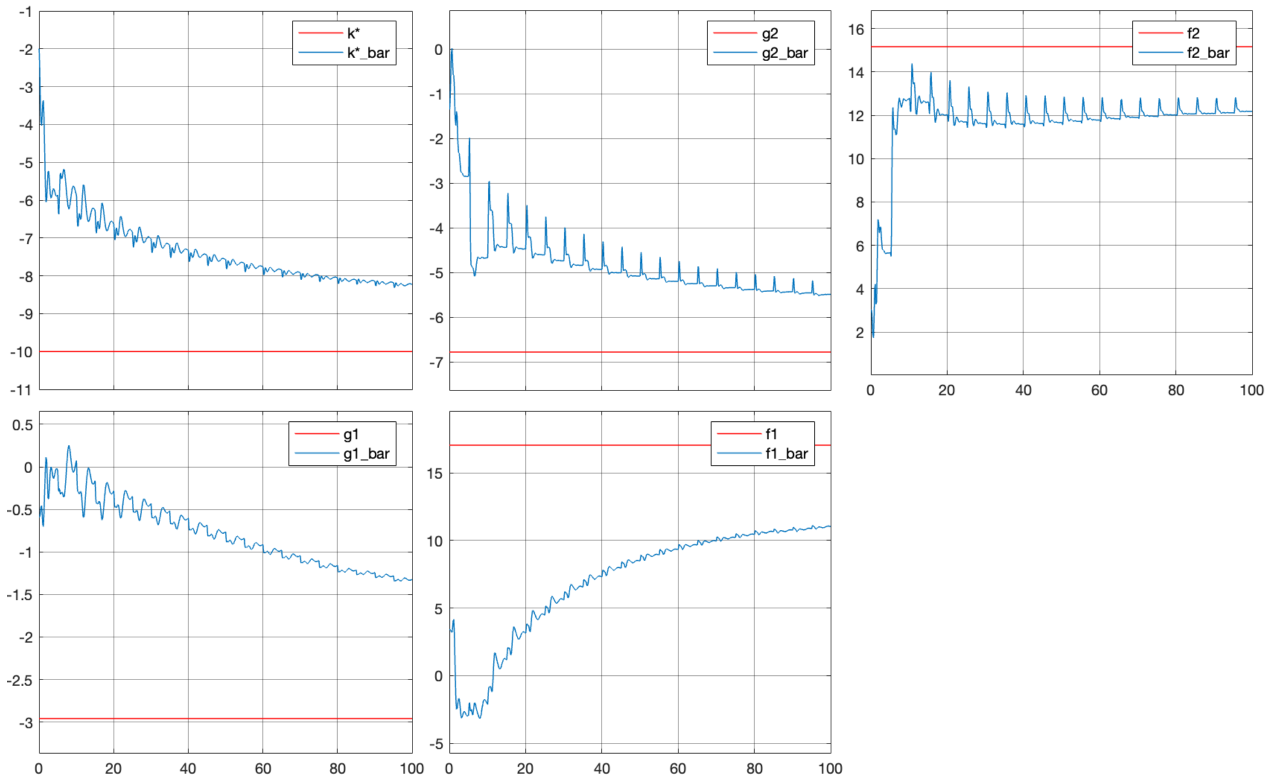

Figure 3.4: $\bar{\theta}$, $\bar{\theta}^*$ with low Γ values

3.1.3 Conclusion

1. Higher Γ: faster convergence, larger control effort
2. Lower Γ: slower convergence, less aggressive control

3.2 Observer Polynomial, T

For the initial simulation, the observer polynomial is as follows:

$$T(s) = s^2 + 1.2s + 9$$

which corresponds to $\zeta = 0.2$ and $\omega_n = 3$ rad/s. Here we will observe the performance of the system in the following two cases:

1. ω_n unchanged, ζ both increased and decreased by 5 times;
2. ζ unchanged, ω_n both increased and decreased by 5 times.

3.2.1 Increased $\zeta_1 = 5\zeta$

$$\zeta_1 = 1$$
$$\omega_n = 3$$

Thus the observer polynomial is

$$T(s) = s^2 + 6s + 9$$

The simulation results are shown in Fig. 3.5 and Fig. 3.6. Increasing ζ to 1.0 resulted in smoother waveforms and more stable behavior, albeit with slightly higher final error.

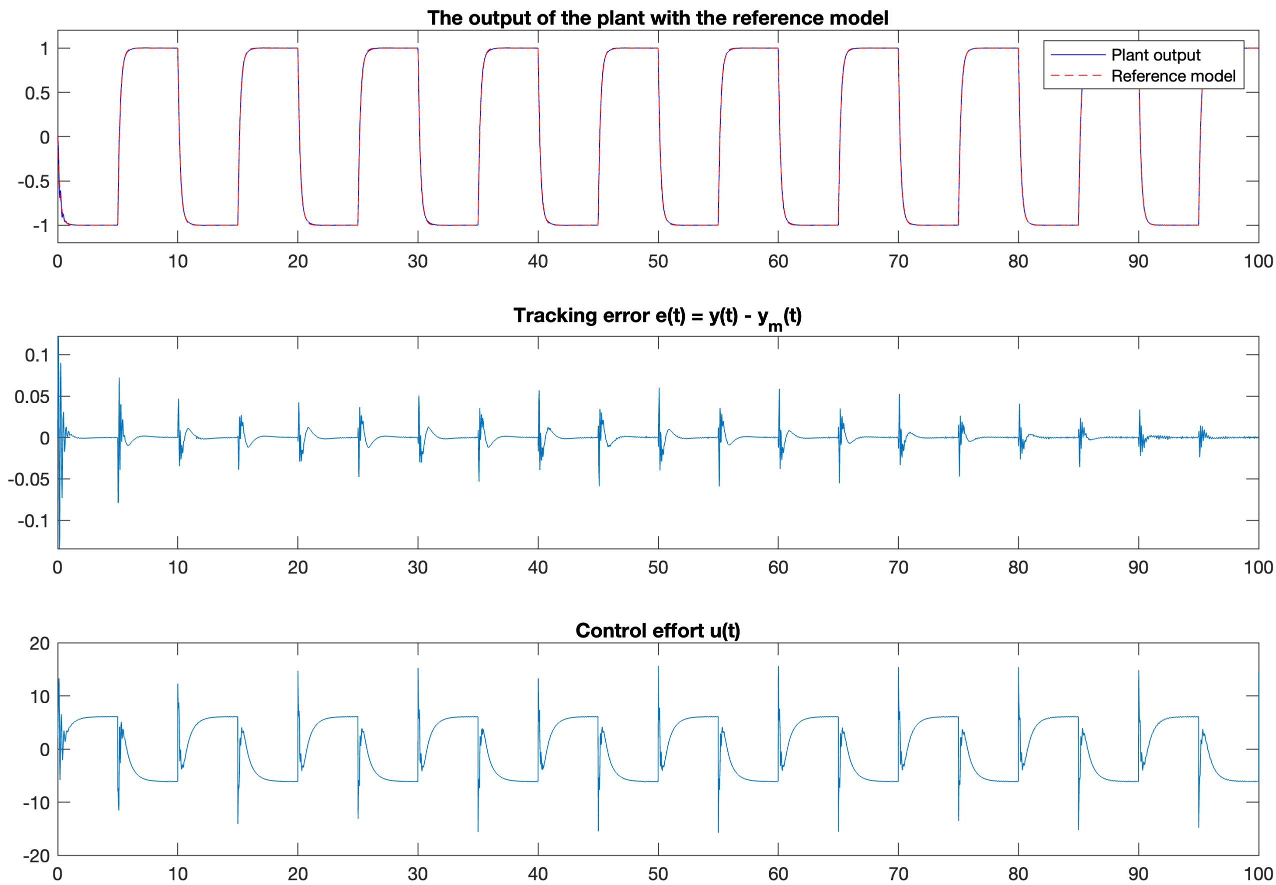

Figure 3.5: y_m, y, e, u with high ζ

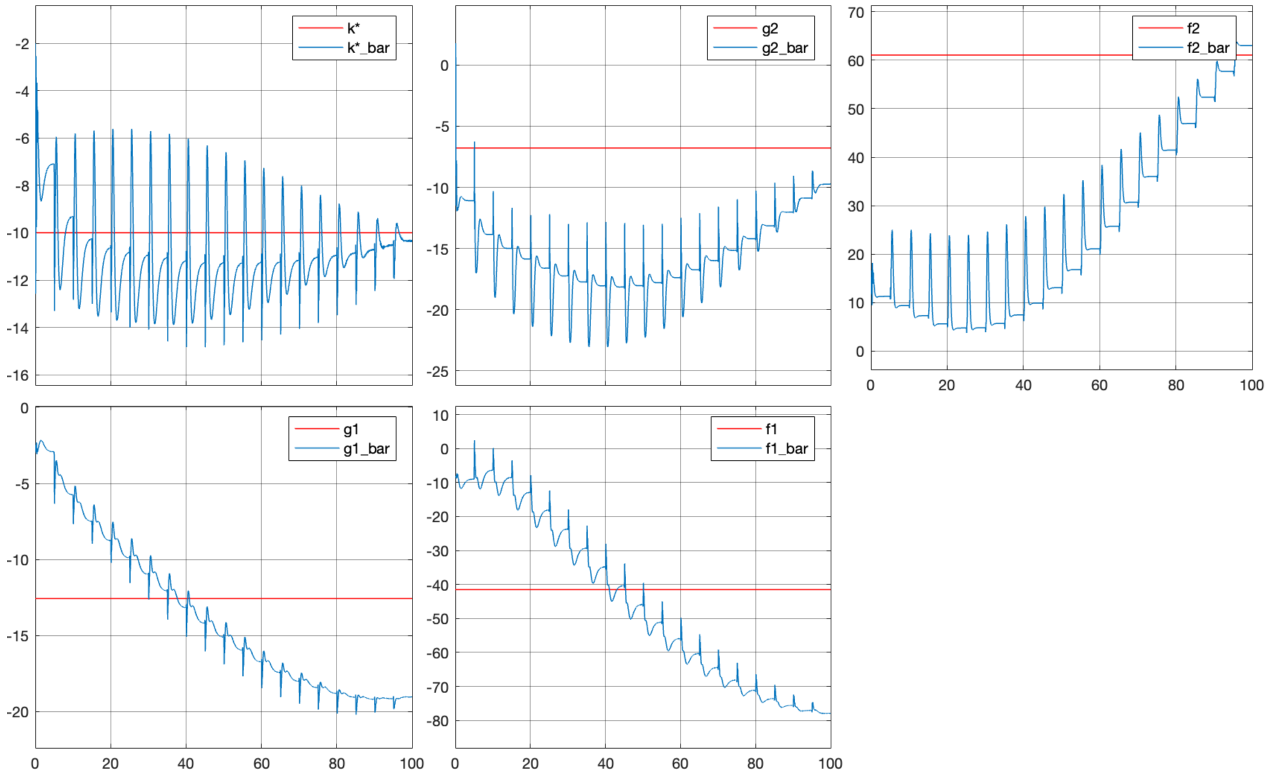

Figure 3.6: $\bar{\theta}$, $\bar{\theta}^*$ with high ζ

3.2.2 Decreased $\zeta_2 = \zeta/5$

$$\zeta_2 = 0.04$$
$$\omega_n = 3$$

Thus the observer polynomial is

$$T(s) = s^2 + 0.24s + 9$$

The simulation results are shown in Fig. 3.7 and Fig. 3.8. When ζ was reduced from 0.2 to 0.04, the adaptive controller demonstrated faster convergence and lower final tracking error. However, it also introduced significant oscillations in the early transient phase and produced larger control effort.

3.2.3 Conclusion

In conclusion, the system performs better when the ζ is lower. This reflects a typical tradeoff between tracking accuracy and robustness, suggesting that moderately damped filters provide the best balance for practical implementations.

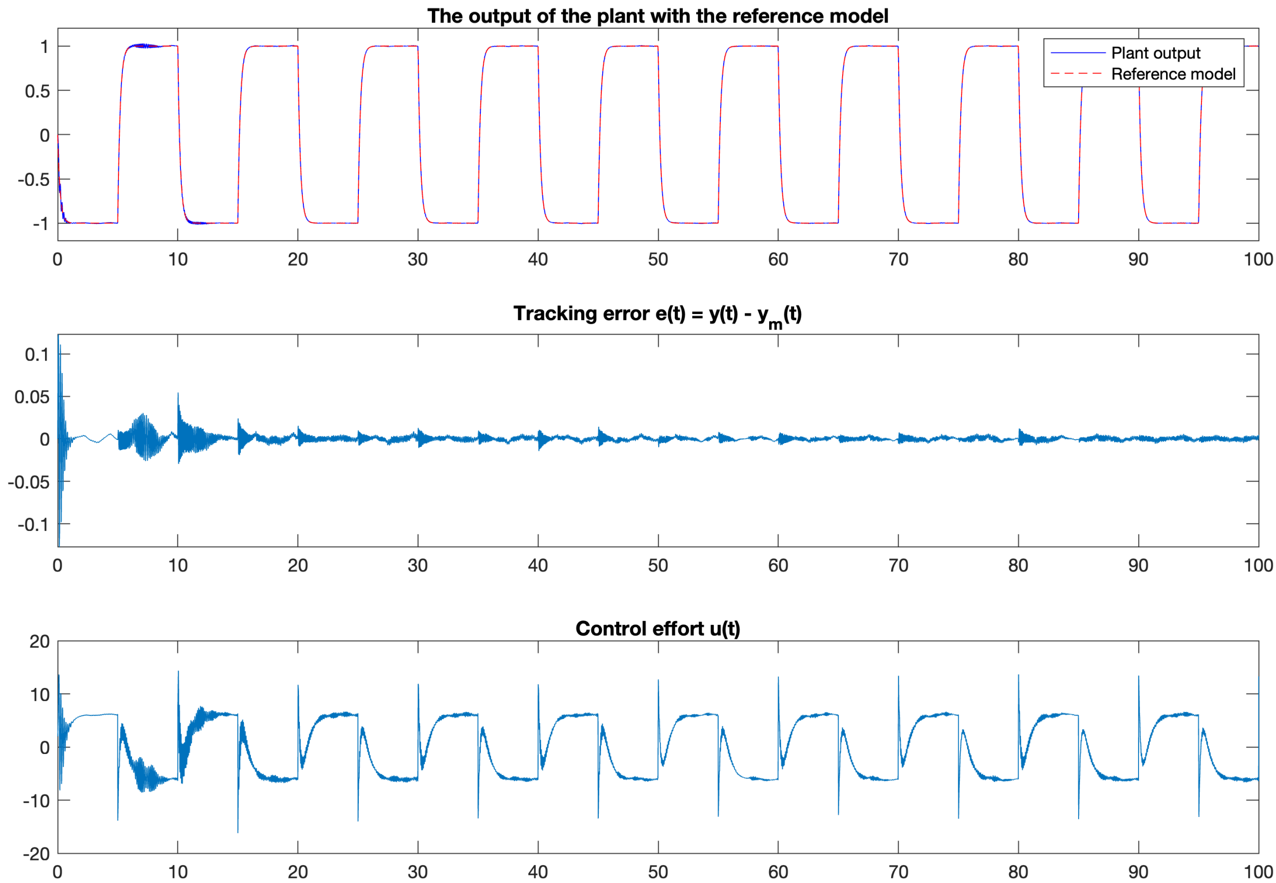

Figure 3.7: y_m, y, e, u with low ζ

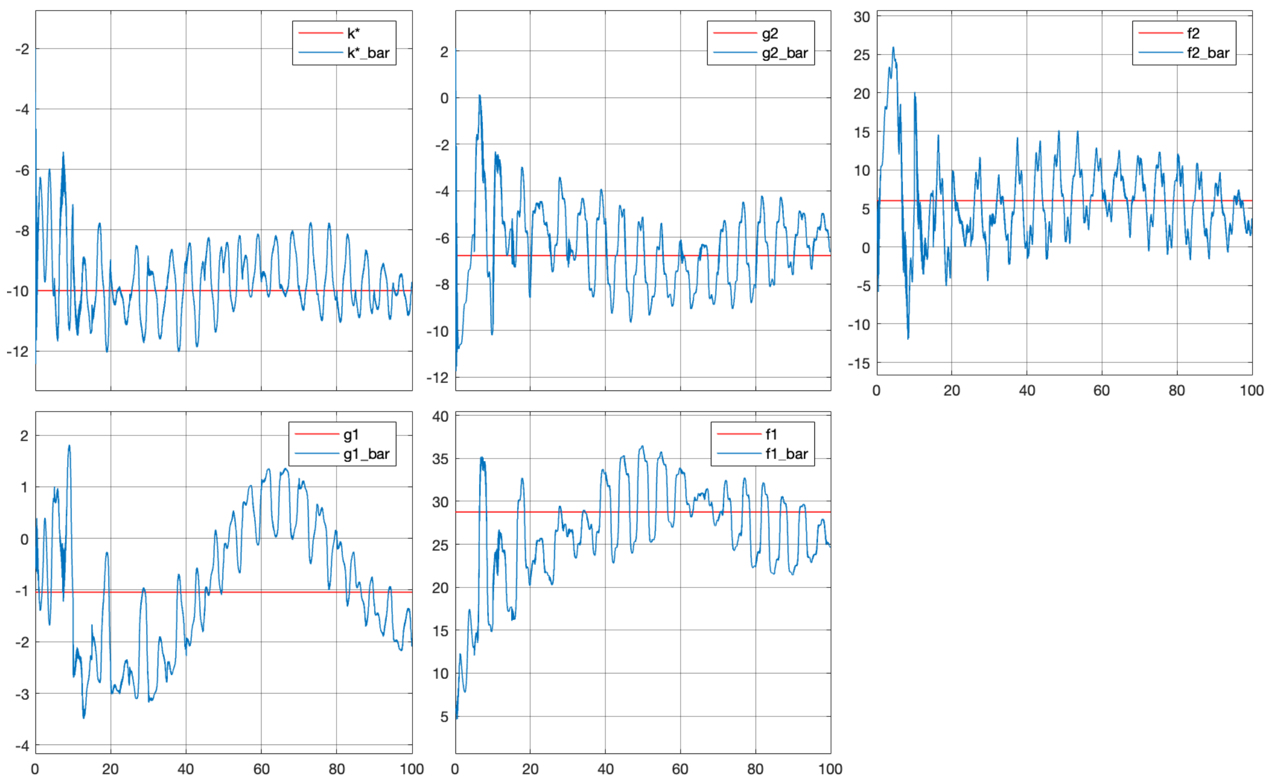

Figure 3.8: $\bar{\theta}$, $\bar{\theta}^*$ with low ζ

3.2.4 Increased $\omega_{n_1} = 5\omega_n$

$$\zeta = 0.2$$
$$\omega_{n_1} = 15$$

Thus the observer polynomial is

$$T(s) = s^2 + 6s + 9$$

The simulation results are shown in Fig. 3.9 and Fig. 3.10. When ω_n was increased fivefold, the observer filter became significantly faster, allowing it to track changes in the plant output and reference signals with minimal lag. This resulted in a smoother and more responsive regressor signal, which in turn facilitated faster parameter adaptation. The tracking error converged relatively faster, stabilizing around 0.01 with minimal oscillation. Additionally, the control effort $u(t)$ remained bounded and relatively smooth, indicating that the adaptive controller was able to act decisively without overcompensating.

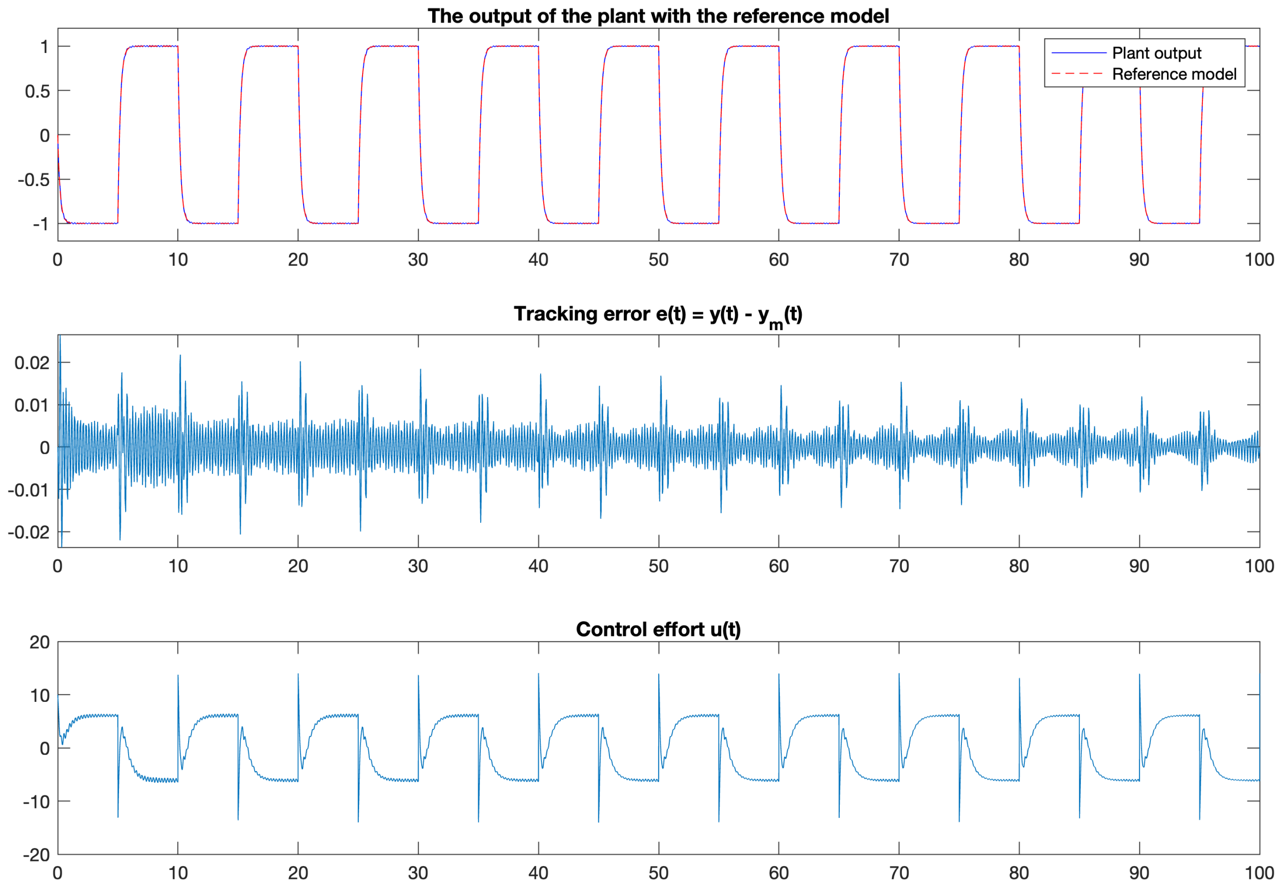

Figure 3.9: y_m, y, e, u with high ω_n

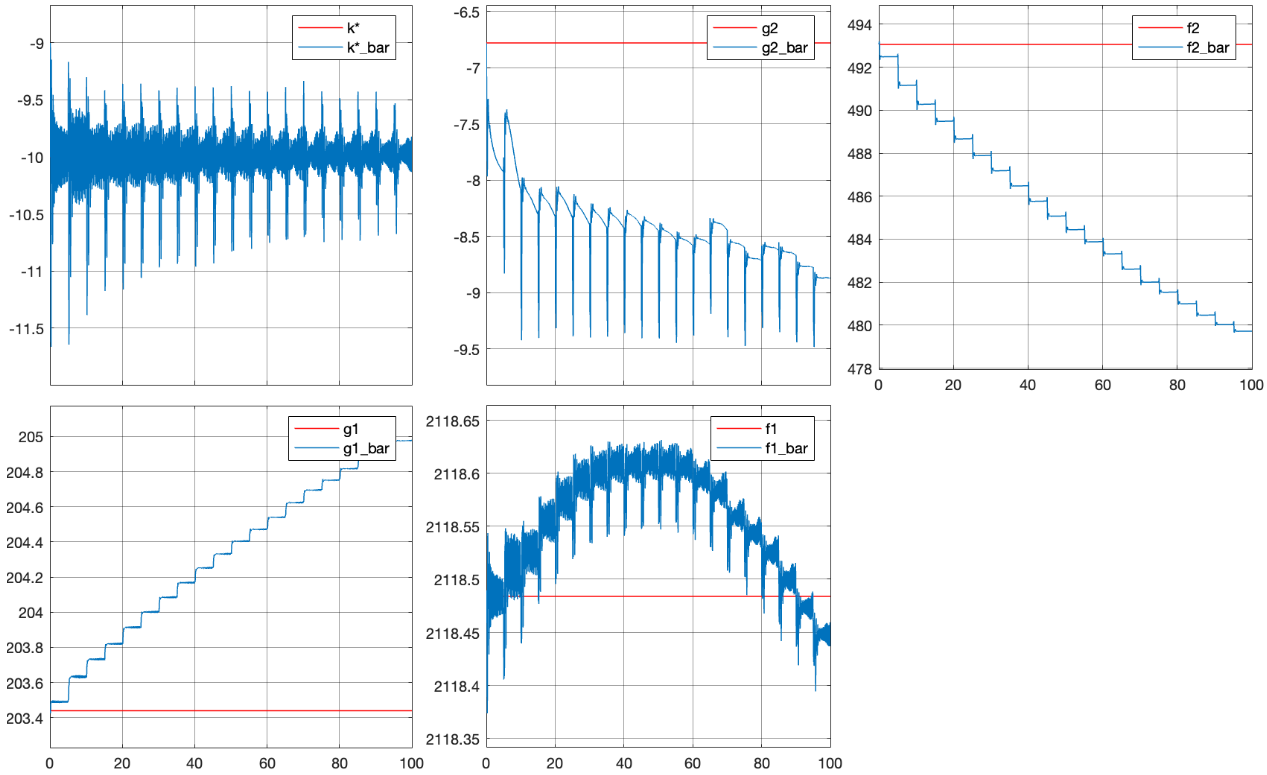

Figure 3.10: $\bar{\theta}$, $\bar{\theta}^*$ with high ω_n

3.2.5 Decreased $\omega_{n_2} = \omega_n/5$

$$\zeta = 0.2$$
$$\omega_{n_2} = 0.6$$

Thus the observer polynomial is

$$T(s) = s^2 + 6s + 9$$

The simulation results are shown in Fig. 3.11 and Fig. 3.12. In contrast, when ω_n was reduced to one-fifth of the nominal value, the filter became overly sluggish. It failed to respond to fast-changing dynamics, resulting in delayed and distorted regressor signals. As a consequence, the tracking error converged more slowly and fluctuated persistently around 0.02. Furthermore, the control input $u(t)$ exhibited high-amplitude oscillations and jittery behavior, particularly in the transient phase. This suggests that the controller was overreacting to inaccurate or stale filtered signals, leading to aggressive and unstable control actions.

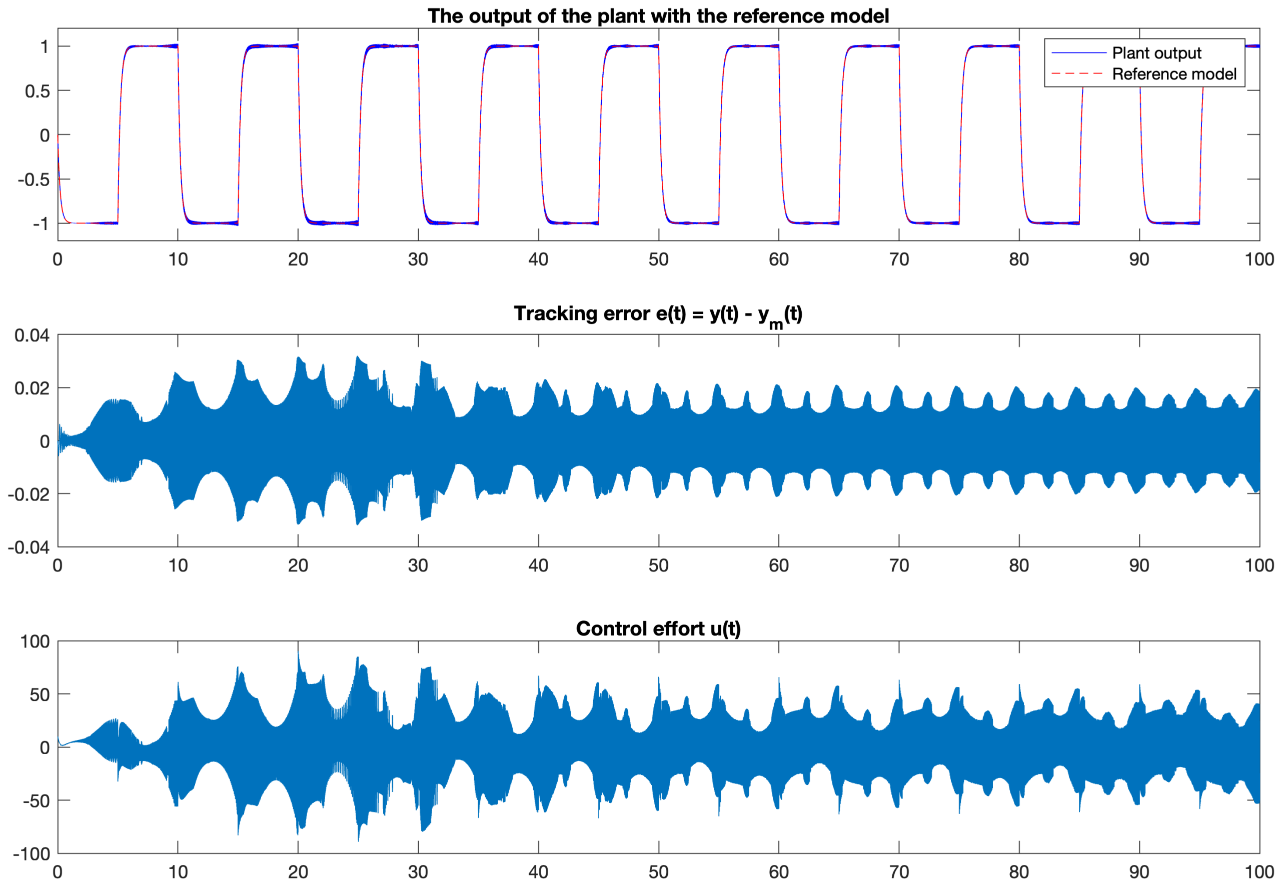

Figure 3.11: y_m, y, e, u with low ω_n

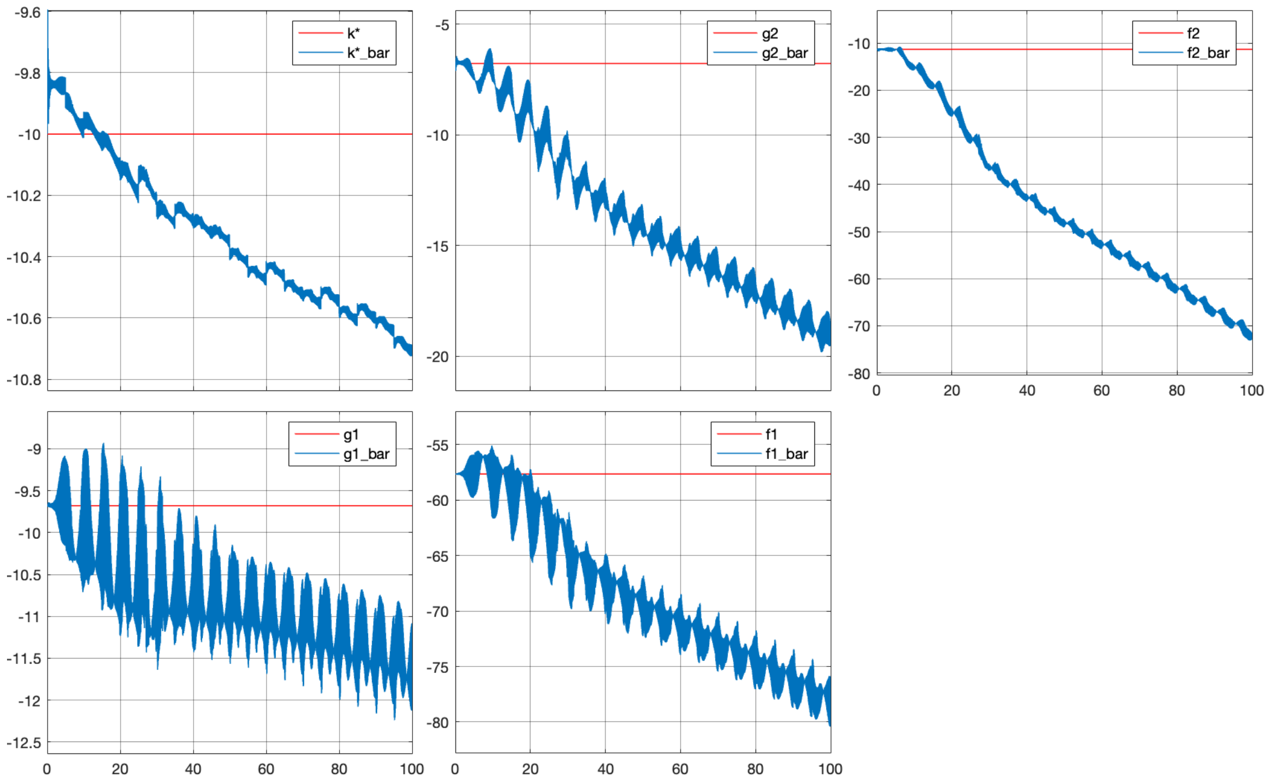

Figure 3.12: $\bar{\theta}$, $\bar{\theta}^*$ with low ω_n

3.2.6 Conclusion

These results highlight the importance of choosing an appropriate observer speed. A faster observer (larger ω_n) can provide sharper and more informative regressor signals, enhancing adaptation quality and ensuring more accurate tracking. However, if the observer is too slow (smaller ω_n), it fails to capture the relevant system dynamics in a timely manner, impairing both learning speed and stability. Therefore, while extremely high ω_n may lead to noise amplification in practice, moderately increasing ω_n relative to the plant dynamics is often beneficial for adaptive performance.

Chapter 4

Sinusoidal Reference Tracking

In this chapter, the performance of the adaptive controller will be evaluated by using a single sinusoid input of

$$r(t) = 10\sin(0.5t)$$

I will use the observer polynomial $T(s)$ and adaptation controller gains Γ, which perform the best in the previous chapters.

$$T(s) = s^2 + 1.2s + 9$$

$$\Gamma = \mathrm{diag}(20000, 20000, 2000, 1000, 1200)$$

4.1 Simulation Result and Discussion

The simulation results are shown in Fig.4.1 and Fig.4.2. According to Fig.4.1, we observe that the plant output tracks the reference model well, with no significant overshoot or unwanted oscillation beyond the expected sinusoidal behavior. The tracking error dynamics also show that the error becomes negligible after the initial few seconds. In terms of the control input, due to the oscillatory nature of the reference signal, it is reasonable to observe high-magnitude fluctuations in the control effort. Overall, we can conclude that the controller performs well for a sinusoidal input.

As shown in Fig.4.2, most of the adaptation gains do not converge to their exact values. All the controller gains exhibit persistent oscillations and remain distant from their true values, suggesting that they are not converging.

The convergence of the tracking error in adaptive control does not necessarily imply the convergence of the parameter estimates. This issue is well known and is typically addressed by ensuring that the regressor signal satisfies the condition known as ”persistence of excitation”.

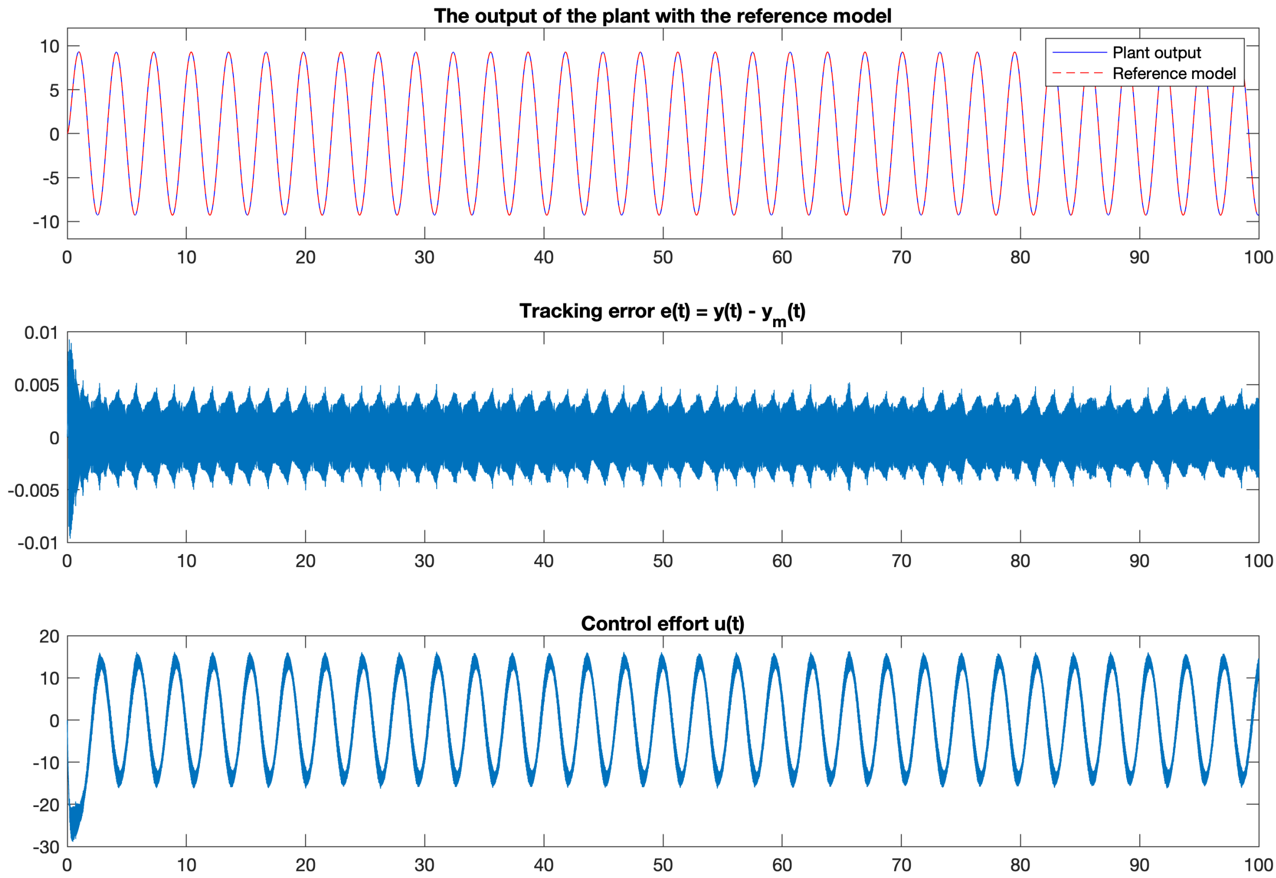

Figure 4.1: Reference model output y_m, plant output y, error e, and control signal u (sinusoid $r(t)$)

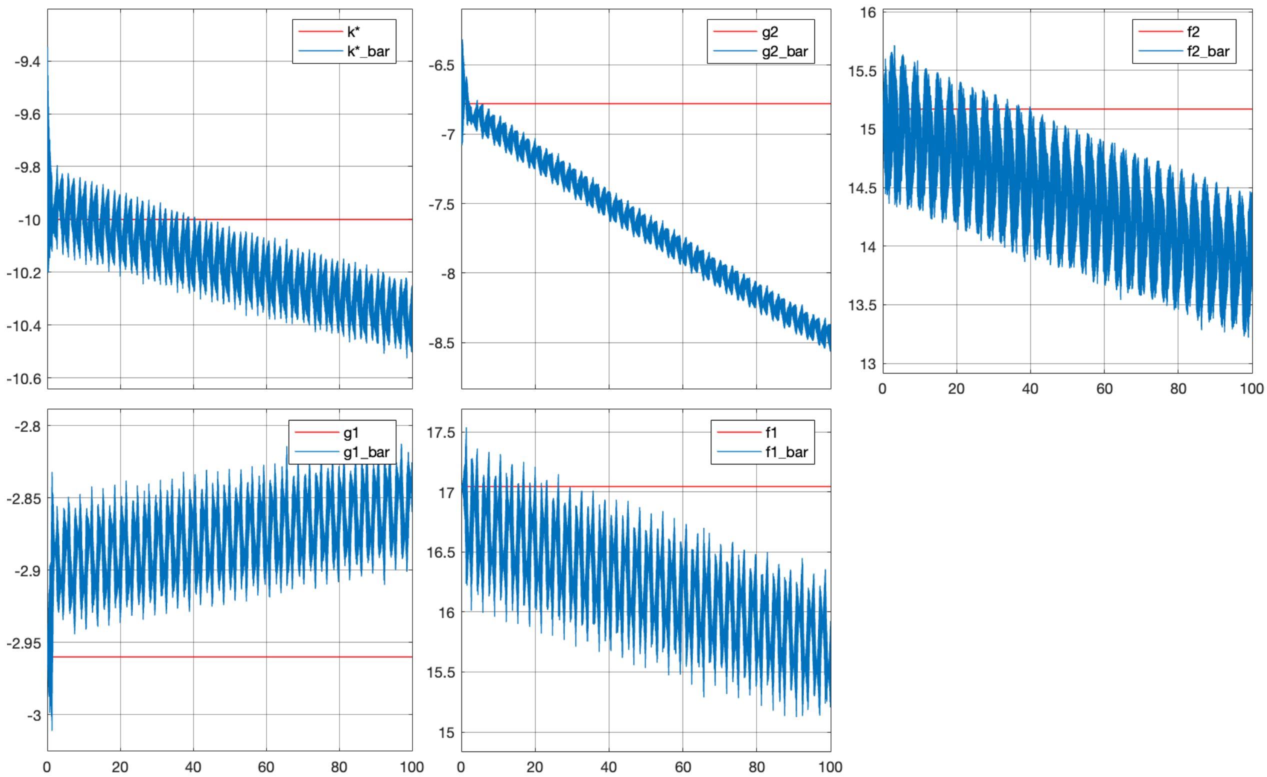

Figure 4.2: The adapted controller gains $\bar{\theta}$ with the exact gains $\bar{\theta}^*$ (sinusoid $r(t)$)

Chapter 5

Conclusion

In this project, an adaptive controller was designed and simulated for a second-order plant with partially unknown dynamics. The goal was to achieve accurate tracking of a given reference model while ensuring system stability and boundedness. The design process followed a systematic approach as outlined in the CA1 assignment, with each part addressed as follows:

1. **Design of the reference model:**
   The plant was identified as a stable, lightly damped second-order system with a natural frequency of approximately 2 rad/s. A first-order reference model was designed with $a_m = 5$, ensuring faster dynamics compared to the plant and guaranteeing zero steady-state error. The reference model transfer function $Y_m(s)/R(s) = 5/(s+5)$ was chosen to balance speed and stability.

2. **Derivation of the adaptive controller structure:**
   The controller was derived based on filtered signals using an observer polynomial $T(s)$. The control law $u = \bar{\theta}^T \bar{\omega}$ was constructed with a regressor vector $\bar{\omega}$ and adaptive gains $\bar{\theta}(t)$.

3. **Implementation of adaptive law:**
   A gradient-based adaptive law was implemented:

   $$\dot{\bar{\theta}}(t) = -\text{sgn}(k^*)\Gamma\bar{\omega}(t)e_1(t)$$

   where $e_1(t) = y(t) - y_m(t)$ is the tracking error. The adaptation gain matrix Γ was selected as a positive-definite diagonal matrix and tuned to achieve a tradeoff between adaptation speed and robustness.

4. **Simulation and analysis:**
   Simulations were carried out using MATLAB and Simulink under various reference inputs and parameter configurations. The results confirmed that:

   - With properly tuned Γ and observer polynomial $T(s)$, the plant output could accurately track the reference model.
   - Increasing Γ accelerated adaptation but induced higher oscillations in $u(t)$.
   - Decreasing Γ led to slow convergence but smoother control.
   - Lower ζ or higher ω_n in the observer polynomial enhanced tracking speed and gain convergence, but at the cost of transient overshoot or sensitivity.

- Under sinusoidal reference, although tracking was good, parameter convergence was not guaranteed due to lack of persistent excitation.

5. **Discussion of parameter convergence:**
   It was observed that perfect tracking (i.e., $e(t) \to 0$) does not imply convergence of the estimated parameters $\bar{\theta}(t)$ to the exact values $\bar{\theta}^*$. This is consistent with adaptive control theory, which states that parameter convergence requires the regressor $\bar{\omega}(t)$ to be persistently exciting (PE). In the case of sinusoidal input, PE is not satisfied, leading to oscillatory behavior in parameter estimates even when tracking error is minimized.

Overall, the designed controller successfully achieved the required tracking performance, and the simulation results matched theoretical expectations under various test conditions. The study also highlighted the importance of filter design and adaptation gain selection in practical adaptive control systems.

Chapter 6

Code

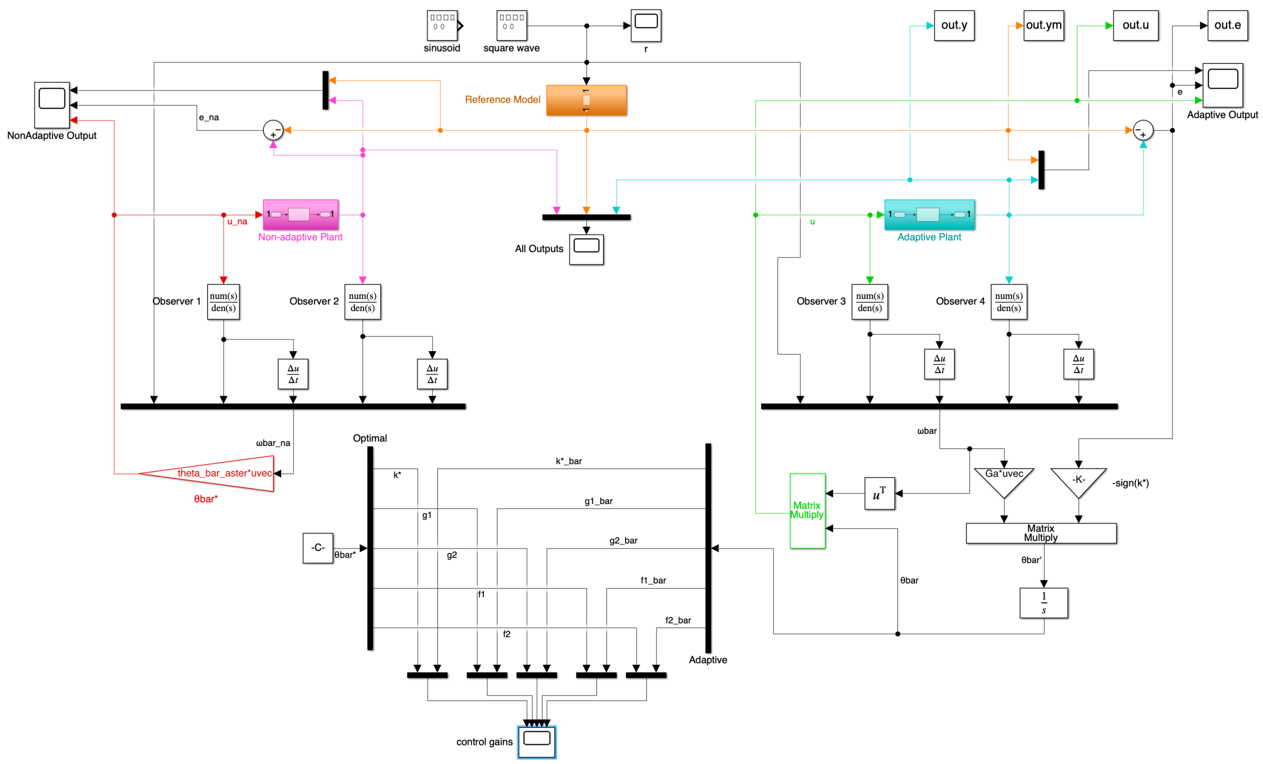

Figure 6.1: Simulink Block Diagram

Simulation Parameters (Original Settings)

```
%parameters
zeta=0.2;                   % damping coefficient
wn=3;                       % natural frequency
Tp=[1 2*zeta*wn wn^2];      % observer
disp("Observer poles: ")    % Observer poles
disp(roots(Tp))

ga=100;                     % weight
Ga=diag([12*ga,10*ga,20*ga,200*ga,200*ga]);
disp("Gamma: ")
disp(Ga)

%% Plant model
a1=0.22;
a2=6.1;
```

```
b0=-0.5;
b1=-1;

Kp=b0;
Zp=[1 b1/b0];              % Plant zero
Rp=[1 a1 a2];              % Plant pole
disp("Plant poles: ")      % Observer poles
disp(roots(Rp))

%% Reference model
am=5;
Rm=[1 am];
Km=am;

%% Optimal control gains
[E,F]=deconv(conv(Tp,Rm), Rp); % T*Rm=Rp*E+F
Fbar = F/Kp;
Gbar = conv(E, Zp);
G1 = Gbar - Tp;

Kaster = Km/Kp;
f1 = Fbar(3);
f2 = Fbar(4);
g1 = G1(2);
g2 = G1(3);
theta_bar_aster = [Kaster, -g2, -g1 , -f2, -f1];
disp("Exact control gains (theta_bar_aster): ")
disp(theta_bar_aster')
```

Simulation Plot

```
% Get simulation data
t = out.tout;                           % time
y = out.y.Data;                         % plant output
ym = out.ym.Data;                       % reference model output
e = out.e.Data;                         % tracking error
u = out.u.Data;                         % control input

% Plot
f = figure('WindowStyle','docked');
subplot(3,1,1); plot(t, y, 'b', t, ym, 'r--');
ylim([-12,12]);
title('The output of the plant with the reference model');
legend('Plant output', 'Reference model');

subplot(3,1,2); plot(t, e);
title('Tracking error e(t) = y(t) - y_m(t)');

subplot(3,1,3); plot(t, u);
title('Control effort u(t)');
```