Lecture 17: Two-view Geometry

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Course Website:
Scan Me!

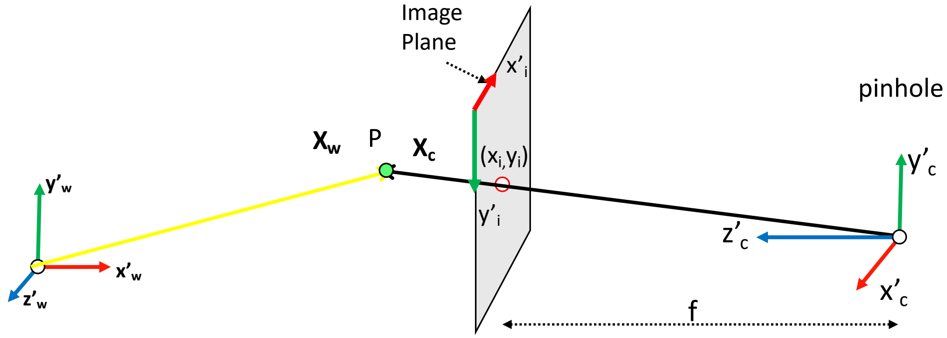


Image Coordinates

$$\mathbf{x}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

Perspective Projection

$$\begin{bmatrix} f_x & 0 & o_x & 0 \\ 0 & f_y & o_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Intrinsics

Camera Coordinates

$$\mathbf{X}_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

Coordinate Transformation

$$\begin{bmatrix} R_{3\times 3} & \mathbf{t} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

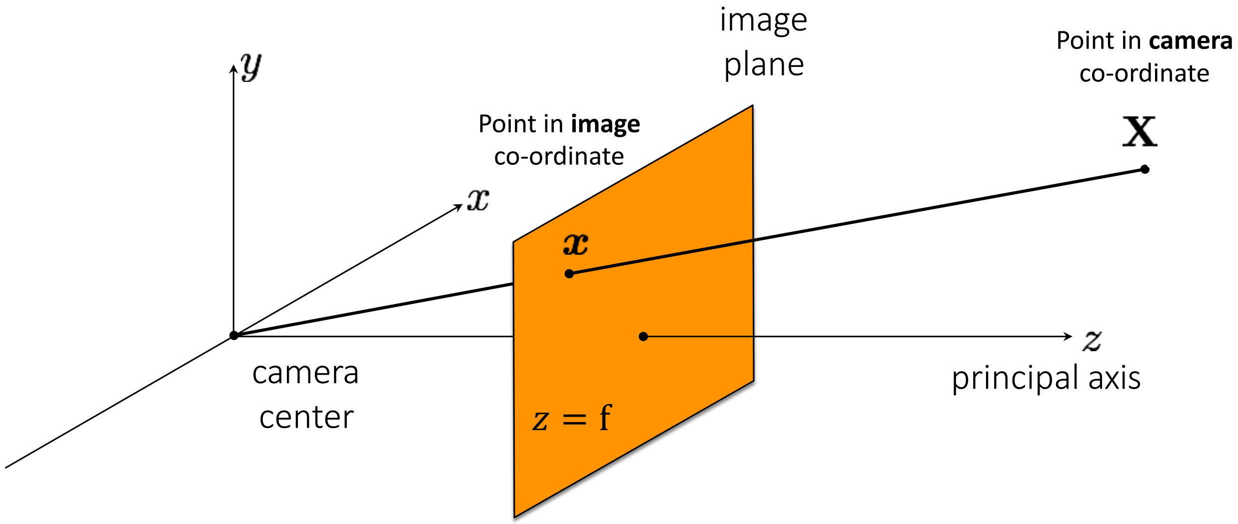
Extrinsics

World Coordinates

$$\mathbf{X}_w = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}$$

Slide inspired by Shree Nayar

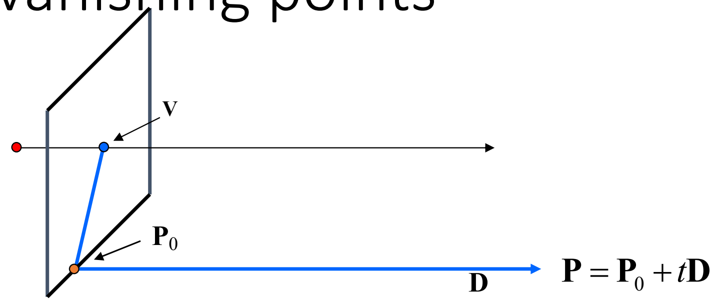
The (rearranged) pinhole camera



$$(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z}, f)$$

$$(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$$

Computing vanishing points



Any point on a line= P0 + t[Dx,Dy,Dz]

Projection of that point on image plane = $f*\left[\frac{(P0x + t*Dx)}{(P0z + t*Dz)}, \frac{(P0y + t*Dy)}{(P0z + t*Dz)}\right]$

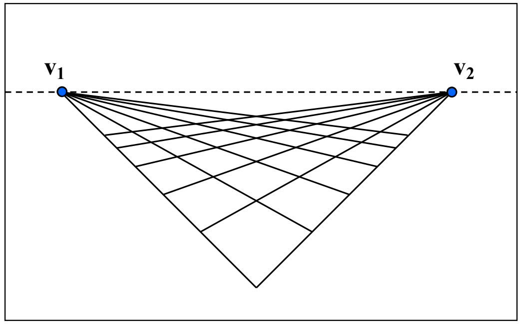
Vanishing Point = Limit t->infinity $f*\left[\frac{(P0x + t*Dx)}{(P0z + t*Dz)}, \frac{(P0y + t*Dy)}{(P0z + t*Dz)}\right]$

= (f*Dx/Dz, f*Dy/Dz)

Properties:

2 set of parallel lines project to the same vanishing point.

Vanishing lines (Horizon Lines)



Properties:

- Union of any 2 vanishing points create a vanishing line
- Horizon line is the projection of a plane at infinity
- A point on horizon line visible in the image is at the height of the camera.

Geometric camera calibration

Given a set of matched points

$$\{\mathbf{X}_i, \boldsymbol{x}_i\}$$

point in 3D space

point in the image

and camera model

$$\boldsymbol{x} = \boldsymbol{f}(\mathbf{X}; \boldsymbol{p}) = \mathbf{P}\mathbf{X}$$

projection model

parameters

Camera matrix

Find the (pose) estimate of

$$\mathbf{P}$$

Same setup as homography estimation

(slightly different derivation here)

Compute SVD of a measurement matrix to obtain P

$$\begin{aligned}\mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}| -\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M}| -\mathbf{M}\mathbf{c}]\end{aligned}$$

Find the camera center **C**

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of P!

***c** is the singular vector corresponding to the smallest singular value*

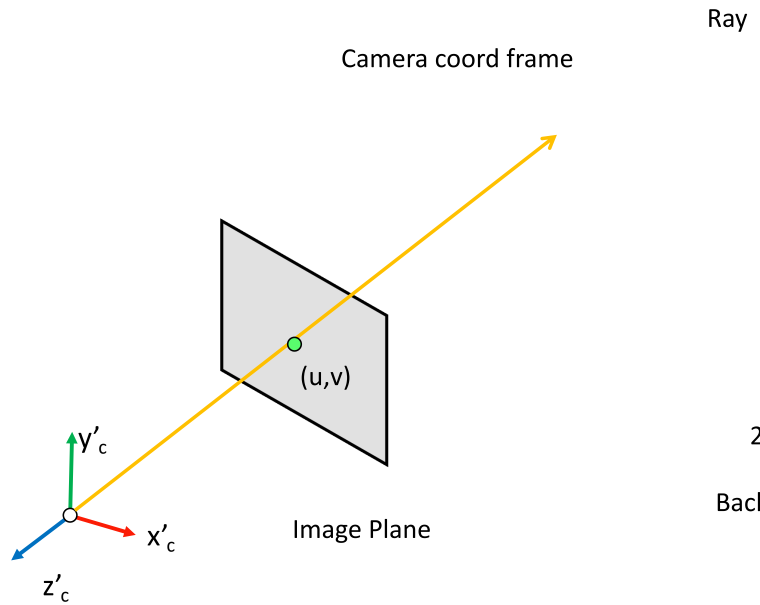
Find intrinsic **K** and rotation **R**

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

QR decomposition

Now that our cameras are calibrated, can we find the 3D scene point of a pixel?

You know we can't, but we know it'll be…
on the ray!



3D to 2D:
(point)

$$u = f_x \frac{x_c}{z_c} + o_x$$

$$v = f_y \frac{y_c}{z_c} + o_y$$

2D to 3D:
(ray)
Back projection

$$x = \frac{z}{f_x}(u - o_x)$$

$$y = \frac{z}{f_y}(v - o_y)$$

$$z > 0$$

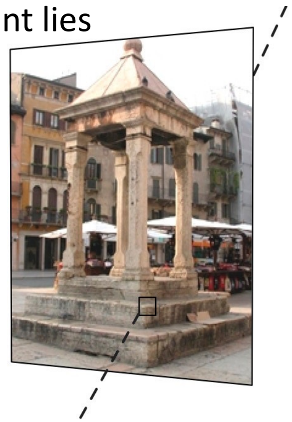
Slide credit: Shree Nayar

Our goal: Develop theories and study how a 3D point and its projection in 2 images are related to each other!

To find the actual location of the 3D point, you need:

an additional image captured from another viewpoint.

From a single image you can only back project a pixel to obtain a ray on which the actual 3D point lies



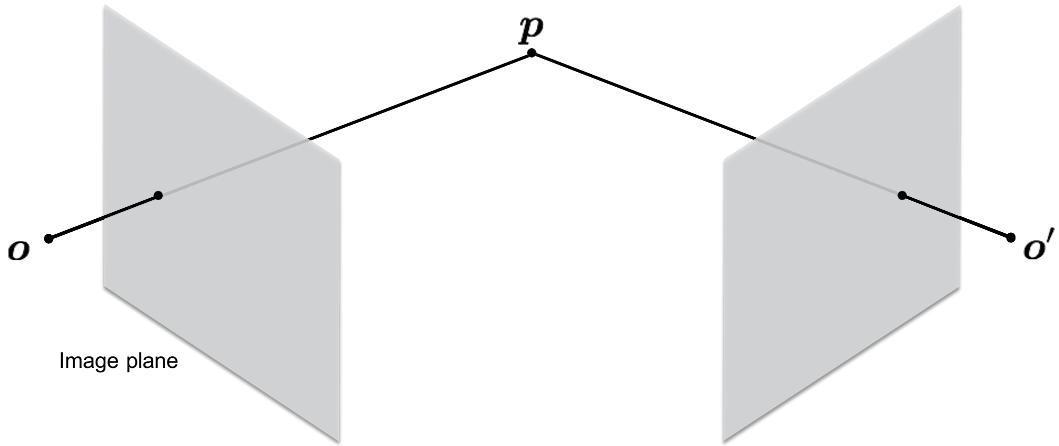
Today’s class

- Epipolar Geometry
- Essential Matrix
- Fundamental Matrix
- 8-point Algorithm
- Triangulation

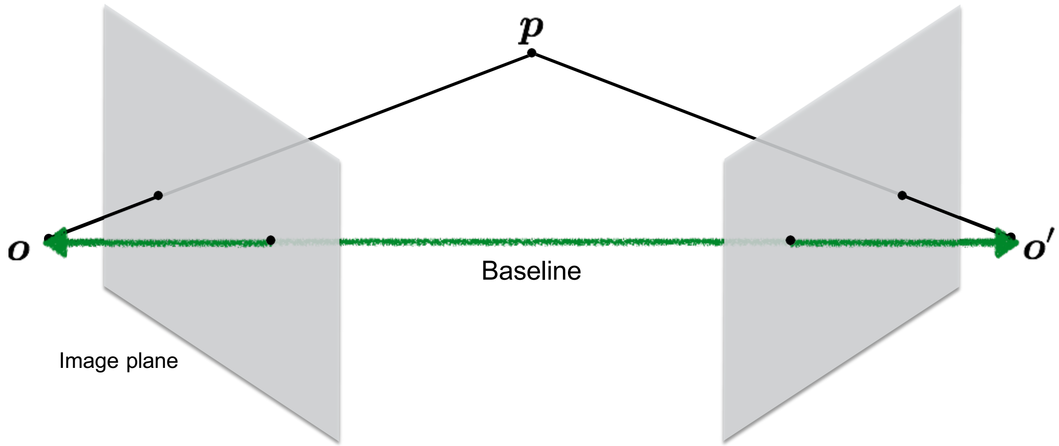
Today's class

- Epipolar Geometry (few definitions)
- Essential Matrix
- Fundamental Matrix
- 8-point Algorithm
- Triangulation

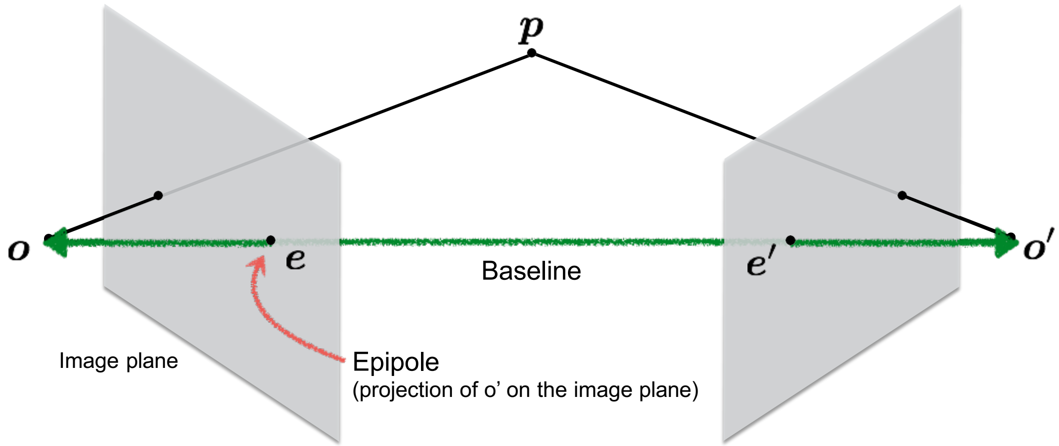
Epipolar geometry



Epipolar geometry



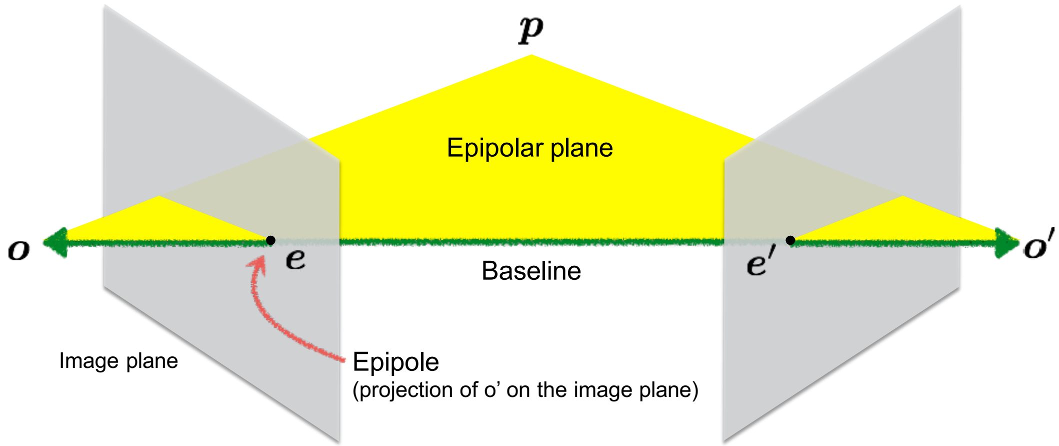
Epipolar geometry



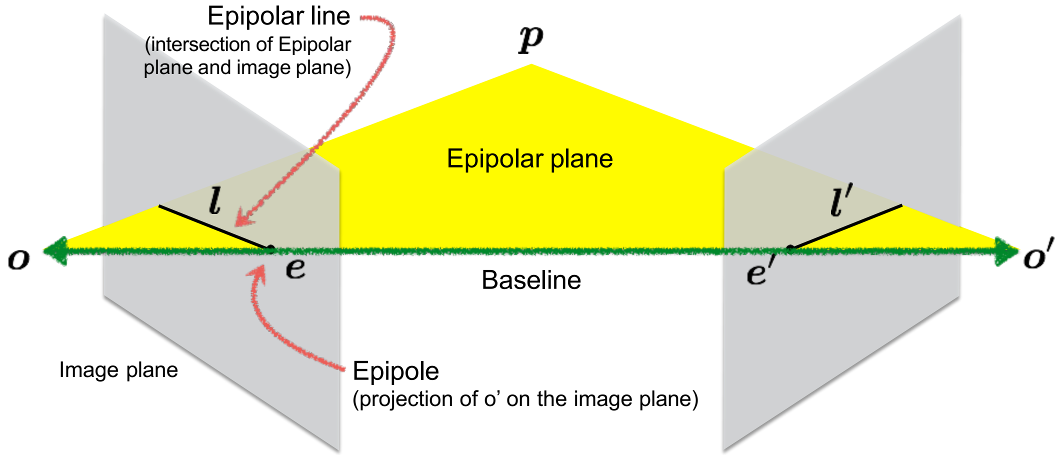
The Epipole

Photo by Frank Dellaert

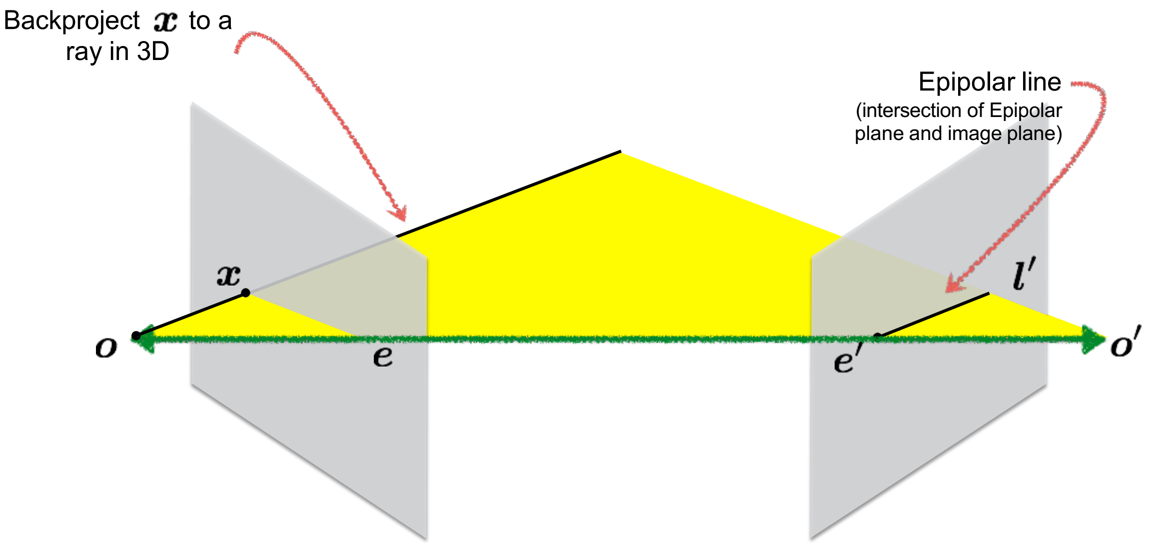
Epipolar geometry



Epipolar geometry

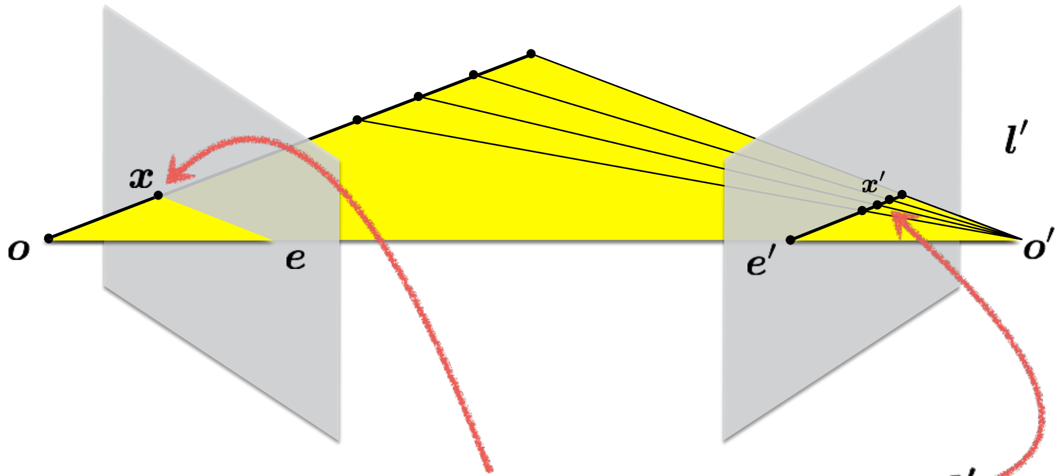


Epipolar constraint



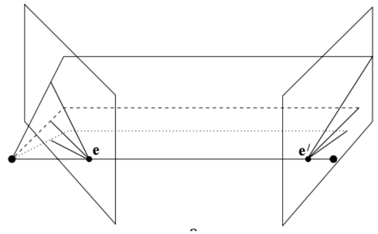
Another way to construct the epipolar plane, this time given $\boldsymbol{x}$

Epipolar constraint



Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l'}$

Example : Converging Cameras



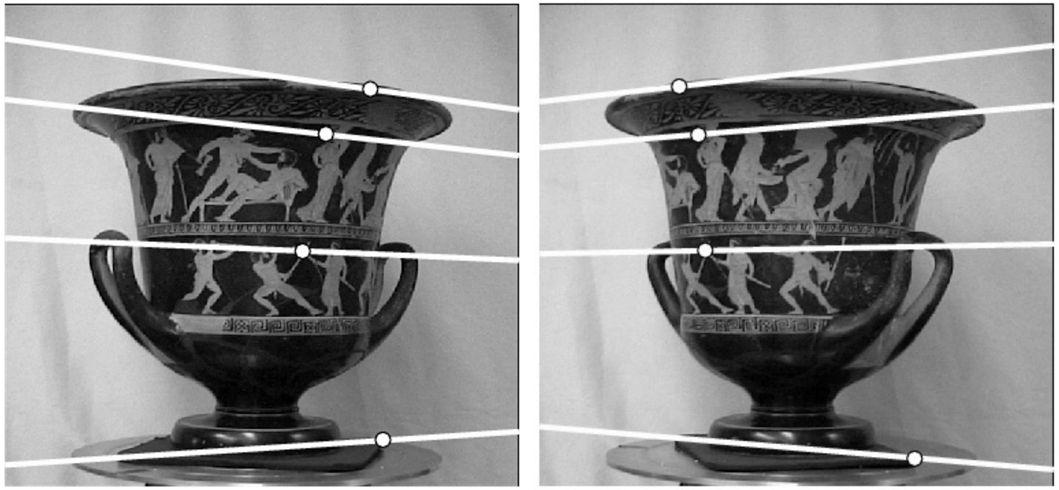
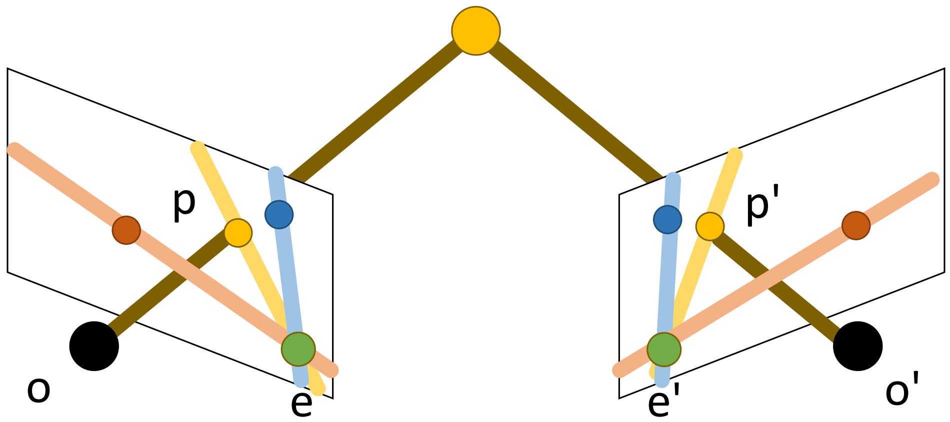


Image Credit: Hartley & Zisserman

Example: Converging Cameras



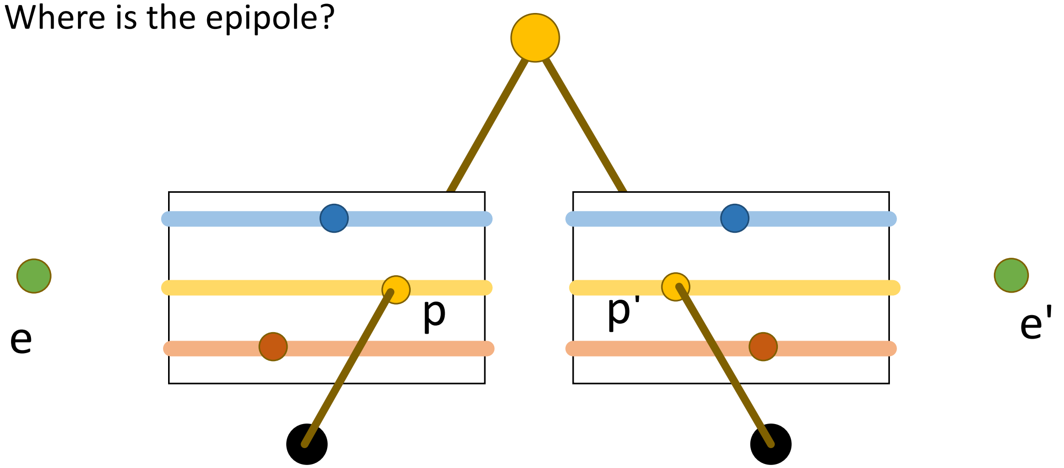
Epipoles finite, maybe in image; epipolar lines converge

Epipolar lines come in pairs:
given a point p, we can construct the epipolar line for p’.

Slide credit: David Fouhey

Example: Parallel to Image Plane

Where is the epipole?



Epipoles *infinitely* far away, epipolar lines parallel

Slide credit: David Fouhey

Example: Forward Motion

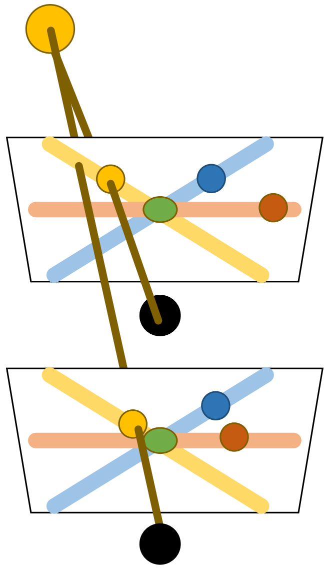
Image Credit: Hartley & Zisserman

Example: Forward Motion

Image Credit: Hartley & Zisserman

Example: Forward Motion

Epipole is focus of expansion / principal point of the camera.



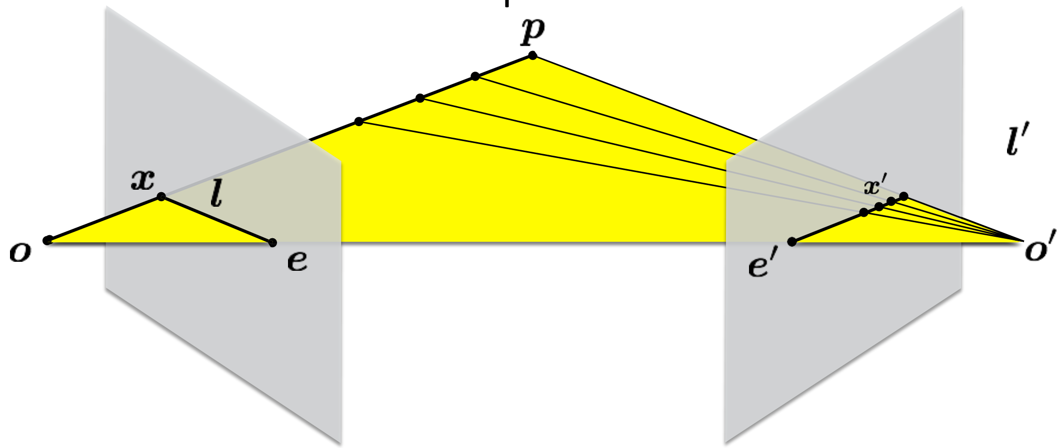
Epipolar lines go out from principal point

Motion perpendicular to image plane

http://vimeo.com/48425421

Slide credit: David Fouhey

Recap Time!



The point **x** (left image) maps to a ___________ in the right image

The baseline connects the ___________ and ____________

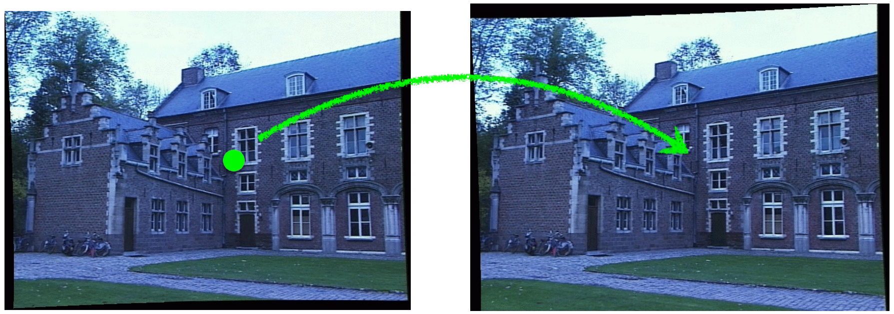
An epipolar line (left image) maps to a __________ in the right image

An epipole **e** is a projection of the______________ on the image plane

All epipolar lines in an image intersect at the ______________

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image

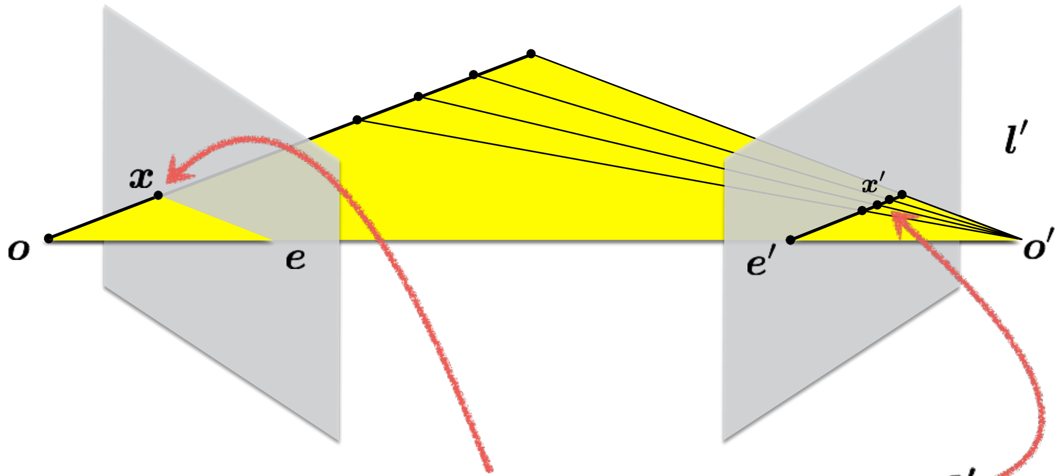


Left image

Right image

How would you do it?

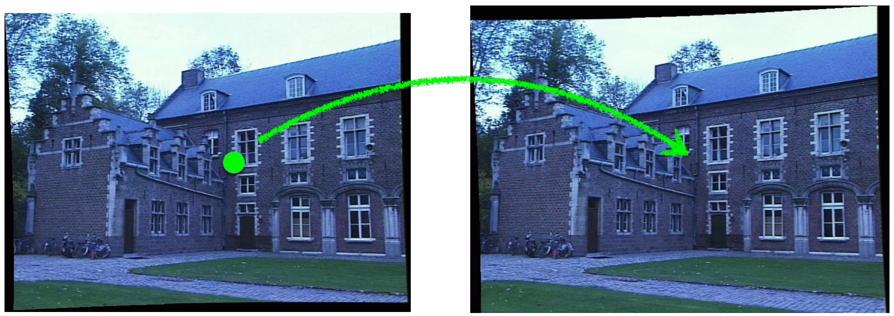
Epipolar constraint



Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l'}$

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image



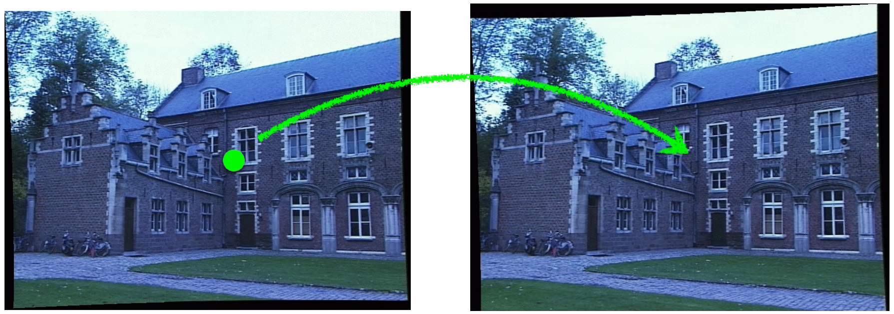
Left image Right image

Want to avoid search over entire image

Epipolar constraint reduces search to a single line

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image



Left image

Right image

Want to avoid search over entire image

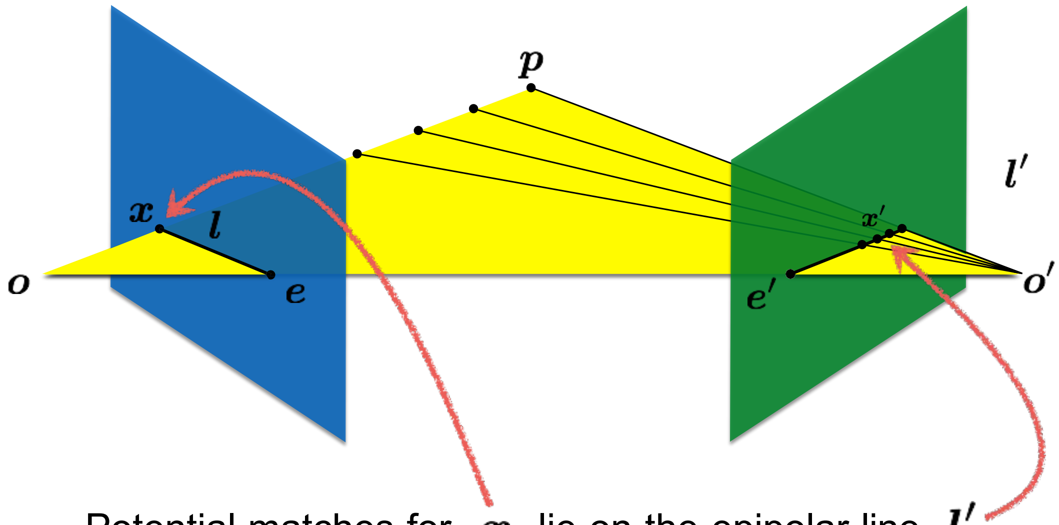
Epipolar constraint reduces search to a single line

How do you compute the epipolar line?

Today’s class

- Epipolar Geometry
- Essential Matrix
- Fundamental Matrix
- 8-point Algorithm
- Triangulation

Recall:Epipolar constraint

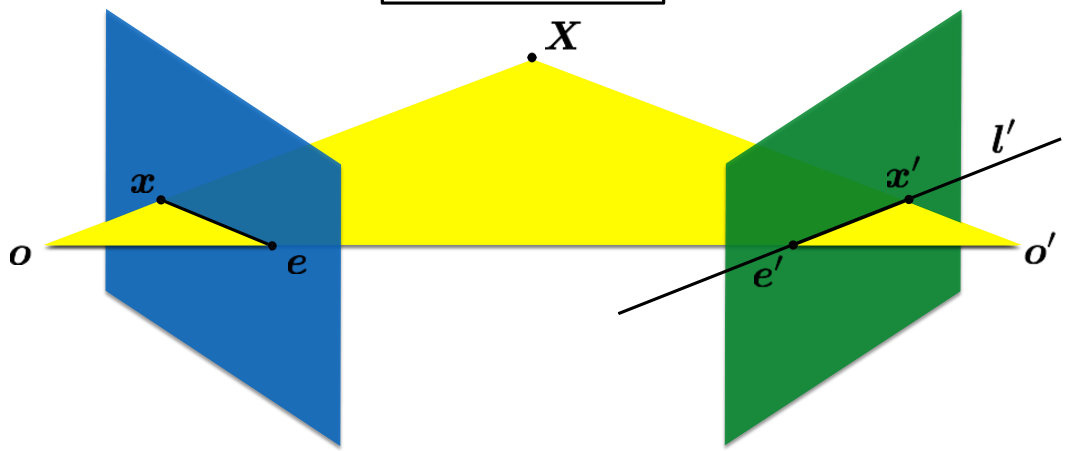


Potential matches for x lie on the epipolar line l'

Given a point in one image,
multiplying by the **essential matrix** will tell us
the **epipolar line** in the second view.

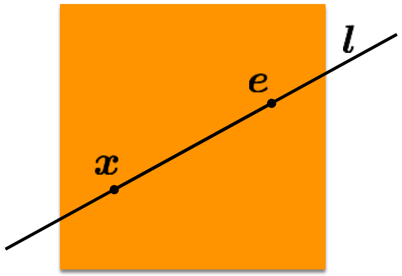
$$\boxed{\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'}$$

Essential matrix is 3x3 and encodes epipolar geometry.



Epipolar Line

$$ax + by + c = 0$$ in vector form $$\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

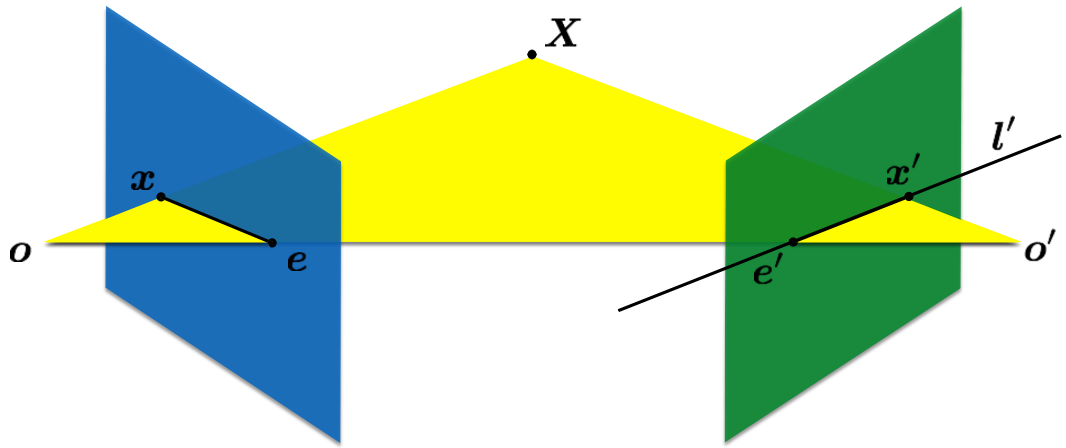


If the point $\boldsymbol{x}$ is on the epipolar line $\boldsymbol{l}$ then

$$\boldsymbol{x}^\top \boldsymbol{l} = 0$$

So if $\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$ and $\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'$ then

$$\boxed{\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0}$$



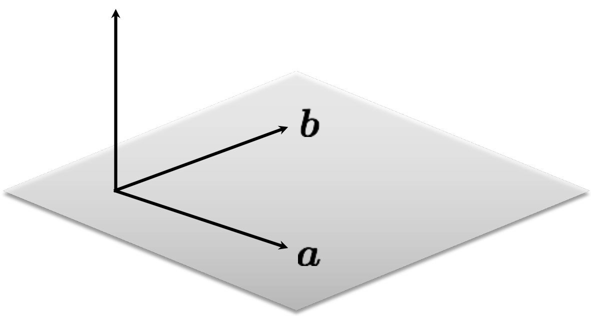
Where does the essential matrix come from?

Can we express essential matrix as function of camera parameters?

Linear algebra reminder: cross product

Vector (cross) product
takes two vectors and returns a vector perpendicular to both

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$



$$\boldsymbol{c} \cdot \boldsymbol{a} = 0 \qquad \boldsymbol{c} \cdot \boldsymbol{b} = 0$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

cross product of two vectors in the same direction is zero vector

$$\boldsymbol{a} \times \boldsymbol{a} = 0$$

remember this!!!

Linear algebra reminder: cross product

Cross product

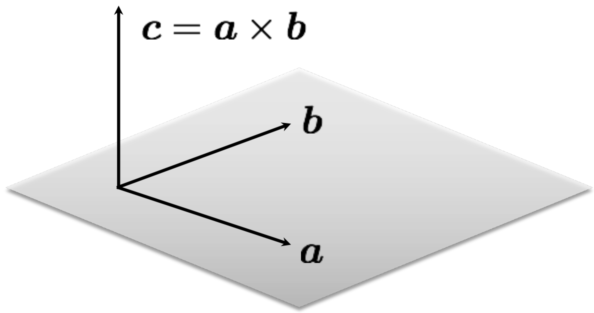
$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Can also be written as a matrix multiplication

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

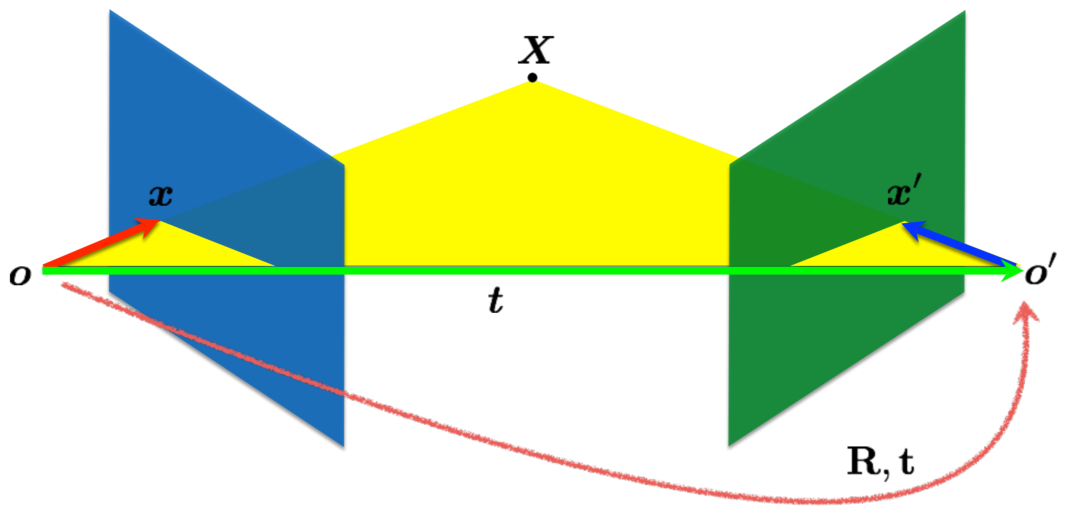
Skew symmetric

Compare with: dot product



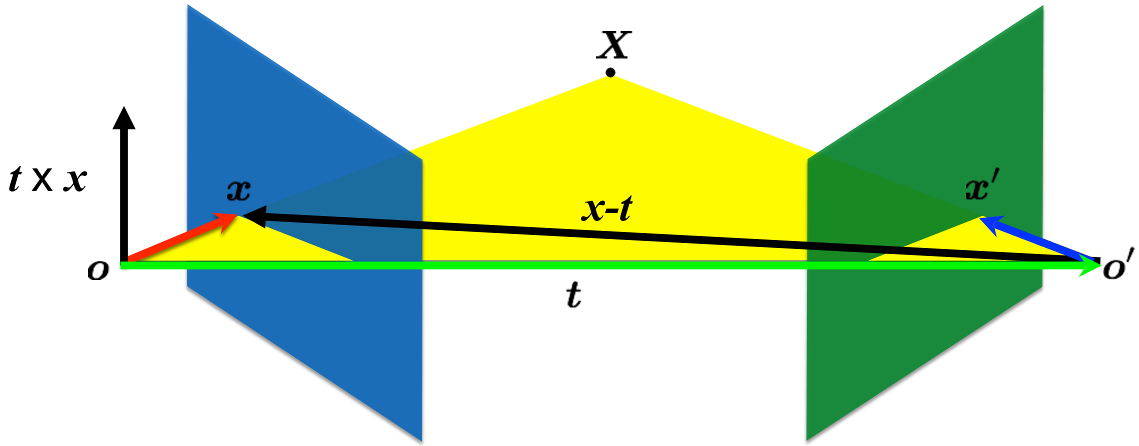
$\boldsymbol{c} \cdot \boldsymbol{a} = 0$ $\boldsymbol{c} \cdot \boldsymbol{b} = 0$

dot product of two orthogonal vectors is (scalar) zero



$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

Camera-camera transform just like **world-camera** transform



$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

dot product of orthogonal vectors

cross-product: vector orthogonal to plane

Putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

use skew-symmetric matrix to represent cross product

$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

$$\boxed{\boldsymbol{x}'^\top \mathbf{E} \boldsymbol{x} = 0}$$

Essential Matrix
[Longuet-Higgins 1981]

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b} = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{\textbf{Skew symmetric}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_\times}$$

Properties of the E matrix

$$\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_\times}$$

Longuet-Higgins equation	$\boldsymbol{x}'^\top \mathbf{E}\boldsymbol{x} = 0$	
Epipolar lines	$\boldsymbol{x}^\top \boldsymbol{l} = 0$ $\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$	$\boldsymbol{x}'^\top \boldsymbol{l}' = 0$ $\boldsymbol{l} = \mathbf{E}^T\boldsymbol{x}'$
Epipoles	$\boldsymbol{e}'^\top \mathbf{E} = \mathbf{0}$	$\mathbf{E}\boldsymbol{e} = \mathbf{0}$

(2D points expressed in camera coordinate system)

Properties of the E matrix

$$\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_\times}$$

- E has 5 degrees of freedom, why?
  - R has 3 degree of freedom
  - T has 3 degree of freedom
  - However since this is a projective transformation one can apply an arbitrary scale to E. Thus 1 degree of freedom less.

- E is rank 2, why?
  - $[t_x]$ is skew symmetric, hence rank 2.
  - Thus Det(E) = 0.

- E has 2 singular value both of which are equal.
  - $[t_x]$ a skew symmetric matrix has 2 equal singular values

2 possible notation

x' = R (x – t) $\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_\times}$

x' = Rx – Rt
= Rx + t' $\mathbf{E} = [\tilde{\mathbf{t}}]_\times \mathbf{R}$

Today’s class

- Epipolar Geometry
- Essential Matrix
- Fundamental Matrix
- 8-point Algorithm
- Triangulation

$$\hat{\boldsymbol{x}}'^{\top}\mathbf{E}\hat{\boldsymbol{x}} = 0$$

In practice we have points in image coordinate, i.e. pixel values.

The essential matrix operates on image points expressed in **2D coordinates** in the camera coordinate system.

$$\hat{\boldsymbol{x}}' = \mathbf{K}'^{-1}\boldsymbol{x}'$$

$$\hat{\boldsymbol{x}} = \mathbf{K}^{-1}\boldsymbol{x}$$

camera point

image point

Writing out the epipolar constraint in terms of image coordinates

$$\boldsymbol{x}'^{\top}(\mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1})\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$$

Fundamental Matrix

Properties of the E matrix

$$\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_{\times}} \qquad \mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1} \qquad \mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_{\times}]\mathbf{R}\mathbf{K}^{-1}$$

Longuet-Higgins equation $\quad \boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$

Epipolar lines

$$\boldsymbol{x}^{\top}\boldsymbol{l} = 0 \qquad \boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$$

$$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x} \qquad \boldsymbol{l} = \mathbf{E}^{T}\boldsymbol{x}'$$

Epipoles $\quad \boldsymbol{e}'^{\top}\mathbf{E} = \mathbf{0} \qquad \mathbf{E}\boldsymbol{e} = \mathbf{0}$

(2D points expressed in image coordinate system)

Properties of the ~~E~~ F matrix

$$\boxed{\mathbf{E} = \mathbf{R}\,[\mathbf{t}]_{\times}} \qquad \mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1} \qquad \mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_{\times}]\mathbf{R}\mathbf{K}^{-1}$$

- ~~E~~ F has ~~5~~ 7 degrees of freedom, why?
  - F is 3x3, has 8 degrees of freedom, since it is a projective transformation.
  - F is rank 2. So 1 less degree of freedom.

- ~~E~~ F is rank 2, why?
  - Same reason as E
  - $[t_x]$ is skew symmetric, hence rank 2.

- ~~E~~ F has 2 singular value both of which are ~~equal.~~

Essential Matrix vs Homography

What's the difference between the essential matrix and a homography?

They are both 3 x 3 matrices but …

$$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$$

Essential matrix maps a **point** to a **line**

- Rank 2
- 5 DoF

$$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$$

Fundamental matrix maps a **point** to a **line**

- Rank 2
- 7 DoF

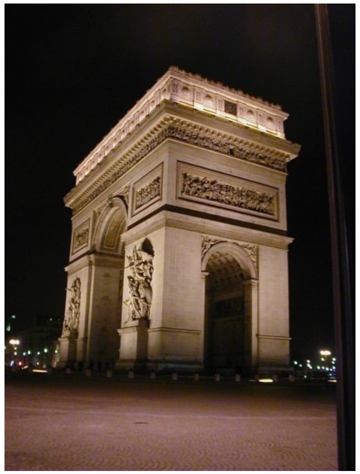
$$\boldsymbol{x}' = \mathbf{H}\boldsymbol{x}$$

Homography maps a **point** to a **point**

- Rank 3
- 8 DoF

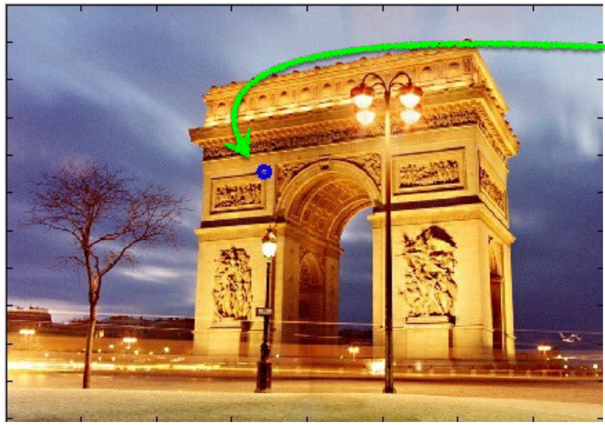
Homography is a special case of the Essential/Fundamental matrix, for planar scenes

Example



epipolar lines

$$\mathbf{F} = \begin{bmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{bmatrix}$$

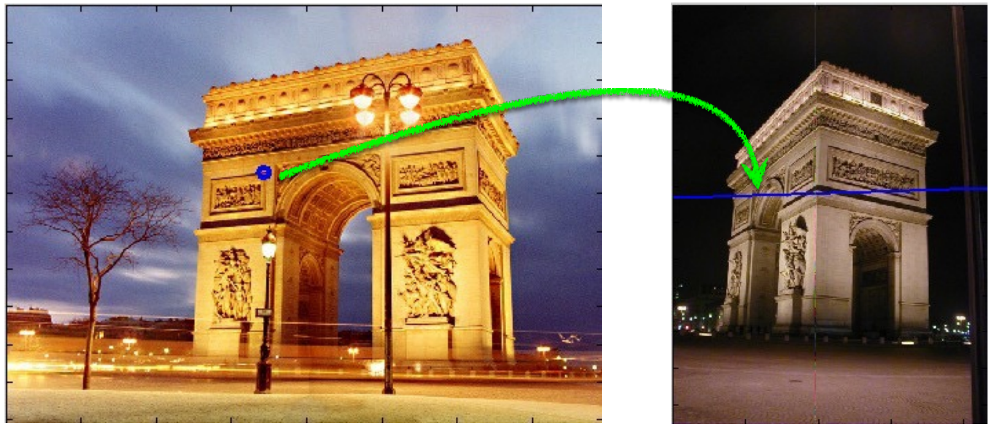


$$\boldsymbol{x} = \begin{bmatrix} 343.53 \\ 221.70 \\ 1.0 \end{bmatrix}$$

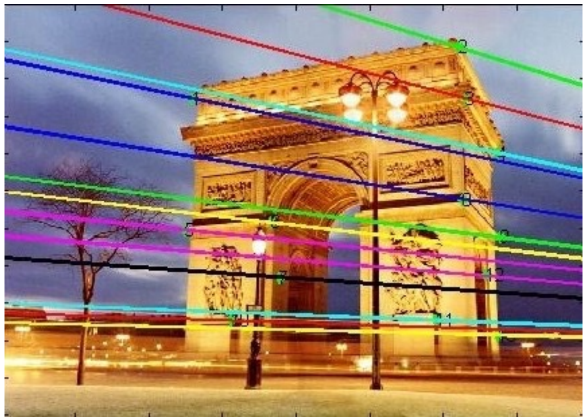
$$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$$

$$= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}$$

$$
\begin{aligned}
\boldsymbol{l}' &= \mathbf{F}\boldsymbol{x} \\
&= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}
\end{aligned}
$$



Where is the epipole?



How would you compute it?

$$\mathbf{F}\boldsymbol{e} = \mathbf{0}$$

The epipole is in the right null space of **F**

How would you solve for the epipole?

$$\mathbf{F}\boldsymbol{e} = \mathbf{0}$$

The epipole is in the right null space of **F**

How would you solve for the epipole?

SVD!

SVDs are pretty useful, huh?

Continue to next Lecture 18

Slide Credits

- CS5670, Introduction to Computer Vision, **Cornell Tech, by Noah Snavely.**
- CS 194-26/294-26: Intro to Computer Vision and Computational Photography, **UC Berkeley, by Angjoo Kanazawa.**
- CS 16-385: Computer Vision, **CMU, by Matthew O'Toole**

Additional Reading

- Multiview Geometry, Hartley & Zisserman,
  - Chapter 9 (focus on topics discussed or mentioned in the slides).
  - Chapter 10.1, 10.2 (not discussed in class, no midterm ques, but imp to understand, practical importance.)
  - Chapter 11.1, 11.2
  - Chapter 12.1, 12.2, 12.3, 12.4 (no midterm ques, but imp to understand)