(Q2-iii)

$$f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z-\mu_z}{\sigma}\right)^2\right)$$

$$= \frac{1}{\sqrt{2}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z^2}{2}\right)\right) \Rightarrow Z \sim N(0, \sqrt{2})$$

$Y \sim N(0,1)$

$X \sim N(0,1)$

$$f_{X|Z}(x|z) = \cancel{f_Y(z-x) f_X(x)} \quad \frac{f_{Z|X}(z|x)\, f_X(x)}{f_Z(z)} = \frac{f_Y(z-x)\, f_X(x)}{f_Z(z)}$$

$$= \frac{\cancel{\frac{1}{\sqrt{2\pi}}} \exp\left(-\frac{1}{2}(z-x)^2\right)\left(\frac{1}{\sqrt{2\pi}}\right) \exp\left(-\frac{1}{2}(x)^2\right)}{\frac{1}{\sqrt{2}\cancel{\sqrt{2\pi}}} \exp\left(-\frac{1}{2}\frac{z^2}{2}\right)}$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left[\left((z-x)^2\right) + x^2 - \frac{z^2}{2}\right]\right)$$

$$= \frac{1}{\frac{1}{\sqrt{2}}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left[z^2 - 2xz + x^2 + x^2 - \frac{z^2}{2}\right]\right)$$

$$\left[\frac{z^2}{2} - 2xz + 2x^2\right]$$

$$\left(2x - \frac{z}{2}\right)(x - z)$$

$$\left(x\sqrt{2} - \frac{z}{\sqrt{2}}\right)\left(x\sqrt{2} - \frac{z}{\sqrt{2}}\right) = 2x^2 + \frac{z^2}{2} - 2xz\frac{\sqrt{8}}{\sqrt{8}} \quad ✓$$

$$\frac{z^2}{\cancel{8}\,4} = \frac{z^2 a}{\cancel{8}} \Rightarrow a = \frac{1}{4}$$

$$= \frac{1}{\frac{1}{\sqrt{2}}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(x\sqrt{2} - \frac{z}{\sqrt{2}}\right)^2\right)$$

$$= \frac{x^2}{\frac{1}{2}} + \frac{z^2}{4\left(\frac{1}{2}\right)} - \frac{xz}{\frac{1}{2}}$$

$$= 2\cancel{8}\left(x^2 + \frac{z^2}{4} - xz\right)$$

$$= \frac{1}{\frac{1}{\sqrt{2}}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x - \frac{z}{2}}{\frac{1}{\sqrt{2}}}\right)^2\right)$$

$$= 2\cancel{8}\left(x - \frac{z}{2}\right)^2$$

$$= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}\left(x - \frac{z}{2}\right)^2 = \left(\frac{x - \frac{z}{2}}{\frac{1}{\sqrt{2}}}\right)^2$$

$\Rightarrow \cancel{f_{X|Z}}$

$X|Z \sim N\left(\frac{z}{2}, \frac{1}{\sqrt{2}}\right)$

Correlation? (between X & Z)

$$f_{X|Z}(x|z) = N(\mu_x + CB^{-1}(z-\mu_z), A - CB^{-1}C^T) \quad , \quad \begin{bmatrix} X \\ Z \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_x \\ \mu_z \end{bmatrix}, \begin{bmatrix} A & C \\ C^T & B \end{bmatrix}\right)$$

$$= N\left(\frac{z}{2}, \frac{1}{\sqrt{2}}\right)$$

$\rightarrow \mu_x = 0$, $A = 1$

$\mu_z = 0$, $B = \sqrt{2}$

$$\Rightarrow \mu_x + CB^{-1}(z - \mu_z) = z/2$$

$$0 + C\left(\frac{1}{\sqrt{2}}\right)(z) = \frac{z}{2} \Rightarrow C = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

check w/variance:

$$A - CB^{-1}C^T = \frac{1}{\sqrt{2}}$$

$$1 - \frac{C^2}{\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \sqrt{2} - C^2 = 1$$

$$\Rightarrow C^2 + 1 - \sqrt{2} = 0 \Rightarrow$$

$$C = \pm\sqrt{\sqrt{2} - 1}$$

$$\frac{2}{\sqrt{2}} - 1 = \sqrt{\frac{2 - \sqrt{2}}{\sqrt{2}}}$$

$$1 - \frac{1}{2\sqrt{2}} \stackrel{?}{=} \frac{1}{\sqrt{2}}$$

$$2\sqrt{2} - 1 = 2 \quad \times$$

$$CB^{-1}(z)$$

$$= \frac{\sqrt{\sqrt{2} - 1}}{\sqrt{2}} z$$

$$= \sqrt{\frac{\sqrt{2} - 1}{2}} z$$

$$= \sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}} z = \frac{z}{2} ?$$

$$= \sqrt{\frac{2 - 1}{2\sqrt{2}}}$$

$$= \sqrt{\frac{1}{2\sqrt{2}}} z \neq \frac{z}{2} \ldots ?$$

huh?

$$0 + \frac{0.6436}{\sqrt{2}}(z)$$

$$1 - \sqrt{2} - 1$$

$$1 - \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) = 1 - \frac{1}{4} = \frac{3}{4} \ldots$$

$$\Sigma = \rho(\sigma_x \sigma_z)^{1/2}$$

$$= \frac{1}{2}(1(\sqrt{2}))^{1/2} = \frac{1}{2}\sqrt{\sqrt{2}}$$

$$= \sqrt{\frac{\sqrt{2}}{4}} = \sqrt{\frac{1}{2\sqrt{2}}} \quad ??$$

$$\sqrt{\frac{1}{2\sqrt{2}}}\left(\frac{1}{\sqrt{2}}\right) z = \sqrt{\frac{1}{2(2)}} z = \frac{z}{2}$$

$$\left(x - \frac{z}{2}\right)^2 = x^2 + \frac{z^2}{4} - 2x\frac{z}{2} = x^2 + \frac{z^2}{4} - xz$$

$$\frac{\left(x - \frac{z}{2}\right)^2}{\frac{1}{2}} = 2\left(x^2 + \frac{z^2}{4} - xz\right) = 2x^2 + \frac{z^2}{2} - 2xz$$

$$= x^2 + x^2 + z^2 - \frac{z^2}{2} - 2xz$$

$$= \left[(z-x)^2 + x^2 - \frac{z^2}{2}\right] \checkmark \Rightarrow \sigma_{x|z} = \frac{1}{\sqrt{2}}$$

$$\mu_{x|z} = \frac{z}{2}$$

$$0 + \frac{c}{\sqrt{2}}(z) = \frac{z}{2} \Rightarrow c = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} = \rho\left(\sqrt{1}\left(\frac{1}{\sqrt{2}}\right)\right) \Rightarrow$$

$$\frac{1}{\sqrt{2}} = \rho\sqrt{\frac{1}{\sqrt{2}}} \Rightarrow \rho = \frac{1}{\sqrt{2}}\sqrt{\frac{\sqrt{2}}{1}} = \sqrt{\frac{\sqrt{2}}{2}} = \sqrt{\frac{1}{\sqrt{2}}}$$

$$1 - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = 1 - \frac{1}{2\sqrt{2}} = \frac{2\sqrt{2}-1}{2\sqrt{2}}$$

$$f_{x|z}(x|z) = \frac{f_{xz}(x,z)}{f_z(z)} \Rightarrow f_{xz}(x,z) = f_{x|z}(x|z)\, f_z(z)$$

$$= N_{x|z}\left(\frac{z}{2}, \frac{1}{\sqrt{2}}\right) N_z(0, \sqrt{2})$$

$$= \frac{1}{\frac{1}{\sqrt{2}}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{\left(x - \frac{z}{2}\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^2}\right) \frac{1}{\sqrt{2}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z^2}{2}\right)\right)$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left[\frac{\left(x - \frac{z}{2}\right)^2}{\frac{1}{2}} + \frac{z^2}{2}\right]\right)$$

$$f_{xz}(x,z) = \frac{1}{2\pi} \cancel{\exp}\left(-\frac{1}{2}\left[\frac{2(x-\frac{z}{2})^2 + \frac{1}{2}z^2}{\frac{1}{2}\cancel{(2)}}\right]\right)$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left[2\left(x^2 + \frac{z^2}{4} - xz\right) + \frac{z^2}{2}\right]\right)$$

$$= \left[2x^2 + \frac{z^2}{2} \cancel{2xz} + \frac{z^2}{2}\right]$$

$$= \left[2x^2 + z^2 \cancel{2xz}\right] = \left(\begin{pmatrix}x\\z\end{pmatrix} - \begin{pmatrix}\mu_x\\\mu_z\end{pmatrix}\right)^T \Sigma^{-1} \left(\begin{pmatrix}x\\z\end{pmatrix} - \begin{pmatrix}\mu_x\\\mu_z\end{pmatrix}\right)$$

$$= (\sqrt{2}x \quad)(\quad)$$

$$\Sigma = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix} = \begin{bmatrix} 1 & C \\ C^T & \sqrt{2} \end{bmatrix}$$

$\mu_x = 0$

$\mu_z = 0$

$$2\left(x^2 - 2x\frac{z}{2} + \frac{z^2}{4}\right)$$

$$2x^2 - \cancel{2xz} + \frac{z^2}{2}$$

$$(x \quad z)\begin{bmatrix} 1 & C \\ C^T & \sqrt{2} \end{bmatrix}^{-1} \begin{pmatrix}x\\z\end{pmatrix} = 2x^2 + z^2 - 2xz$$

$\det \Sigma = \sqrt{2} - C^2$

$$\Sigma^{-1} = \frac{1}{\sqrt{2} - C^2}\begin{bmatrix} \sqrt{2} & -C \\ -C & 1 \end{bmatrix}$$

$$\Rightarrow (x \quad z)\left(\frac{1}{\sqrt{2}-C^2}\begin{bmatrix} \sqrt{2} & -C \\ -C & 1 \end{bmatrix}\right)\begin{pmatrix}x\\z\end{pmatrix}$$

$$= \frac{(x \quad z)}{\sqrt{2}-C^2}\begin{pmatrix} x\sqrt{2} - Cz \\ xC + z \end{pmatrix} = \frac{x^2\sqrt{2} - Cxz - zxC + z^2}{\sqrt{2} - C^2}$$

$$\Rightarrow \frac{\sqrt{2}x^2 - 2Cxz + z^2}{\sqrt{2} - C^2} = 2x^2 + z^2 - 2xz$$

$$\text{if } C = \frac{1}{\sqrt{2}} \Rightarrow \sqrt{2}x^2 - 2x\frac{z}{\sqrt{2}} + z^2 = (2x^2 + z^2 - 2xz)\left(\sqrt{2} - \frac{1}{2}\right)$$

$$\frac{2\sqrt{2}-1}{2}$$

$$2\sqrt{2}x^2 - 2\sqrt{2}xz + 2z^2 = (2x^2 + z^2 - 2xz)(2\sqrt{2}-1)^2$$

$$= (4\sqrt{2} - 2)x^2$$

$$\begin{pmatrix} x|z - \mu_{x|z} \\ z - \mu_z \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} x - \mu_{x|z} \\ z - \mu_z \end{pmatrix}$$

$\mu_{x|z} = \frac{z}{2}$

$\Sigma_{x|z} = \frac{1}{\sqrt{2}} = A$

$B = \Sigma_z = \sqrt{2}$

$\mu_z = \cancel{\sqrt{2}}\ 0$

$$\Sigma^{-1} = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix}^{-1}$$

$$= \frac{1}{AB - C^2} \begin{bmatrix} B & -C \\ -C^T & A \end{bmatrix}$$

$$= \frac{1}{\frac{1}{\sqrt{2}}\sqrt{2} - C^2} \begin{bmatrix} \sqrt{2} & -C \\ -C^T & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{1 - C^2} \begin{bmatrix} \sqrt{2} & -C \\ -C^T & \frac{1}{\sqrt{2}} \end{bmatrix} =$$

$= \cancel{[\frac{z}{2}]}$

$$= \frac{\left[x - \frac{z}{2},\ z\right]}{1 - C^2} \begin{bmatrix} \sqrt{2} & -C \\ -C & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x - \frac{z}{2} \\ z \end{bmatrix}$$

$$= \frac{\left[x - \frac{z}{2},\ z\right]}{1 - C^2} \begin{bmatrix} \sqrt{2}\left(x - \frac{z}{2}\right) - Cz \\ -C\left(x - \frac{z}{2}\right) + \frac{z}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{\left[x - \frac{z}{2},\ z\right]}{1 - C^2} \begin{bmatrix} x\sqrt{2} + z\left(-C - \frac{\sqrt{2}}{2}\right) \\ -Cx + z\left(\frac{C}{2} + \frac{1}{\sqrt{2}}\right) \end{bmatrix} = \frac{1}{1 - C^2}\left(\left(x - \frac{z}{2}\right)\left(x\sqrt{2} + z\left(-C - \frac{\sqrt{2}}{2}\right)\right) + z\left(-Cx + z\left(\frac{C}{2} + \frac{1}{\sqrt{2}}\right)\right)\right)$$

Q2-iv

$Z = X + Y$, $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\lambda)$

$$P_Z(z) = \sum_{x \in X} P_X(x) P(z-x)$$

$$= \text{Poisson}(2\lambda) = \frac{(2\lambda)^z e^{-2\lambda}}{z!}$$

$W = \frac{1}{Z+1}$

$$\Rightarrow F_W(w) = P(W \le w) = P\left(\frac{1}{Z+1} \le w\right) = P\left(\frac{1}{w} \le Z+1\right)$$

$$= P\left(\frac{1}{w} - 1 \le Z\right) = 1 - P\left(Z \le \frac{1}{w} - 1\right)$$

$\frac{dw}{dz} = -1(z+1)^{-2}$

$= -\left(\frac{1}{w} - 1 + 1\right) = -w^2$ ✓

$$\Rightarrow \frac{d}{dw}\left(F_w(w)\right) = \frac{d}{dw}\left(1 - P\left(Z \le \frac{1}{w} - 1\right)\right)$$

$Z + 1 = \frac{1}{w} \Rightarrow Z = \frac{1}{w} - 1$

$$P_W(w) = -P_Z\left(\frac{1}{w} - 1\right)\underbrace{\frac{dz}{dw}}_{(-1/w^2)}, \quad \frac{dz}{dw} = \frac{1}{\frac{dw}{dz}}$$

$\frac{dz}{dw} = -\frac{1}{w^2}$

$$P_W(w) = P_Z\left(\frac{1}{w} - 1\right)\frac{1}{w^2}$$

$$= \frac{(2\lambda)^{\left(\frac{1-w}{w}\right)} e^{-2\lambda}}{\left(\frac{1-w}{w}\right)!\, w^2}$$

* need a scaling factor (inspired by Lec07, pg 14)

correct!!

$$\Rightarrow \mathbb{E}_W[W] = \sum_{w \in W} \frac{(2\lambda)^{\left(\frac{1}{w}-1\right)} e^{-2\lambda}}{\left(\frac{1}{w}-1\right)!\, w^{\cancel{2}}}, \quad w = 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots 0$$

$$= \sum_{n=1}^{\infty} \frac{(2\lambda)^{n-1} e^{-2\lambda}}{(n-1)!\left(\frac{1}{n}\right)^{\cancel{2}}}$$

Let $w = \frac{1}{n} \to n = 1, 2, 3, 4, \ldots \infty$

OR

$$\cancel{\mathbb{E}[f(z)] = \sum_{z \in Z} f(z) P_Z(z)}$$

$$\cancel{= \sum_{z=0}^{\infty} \frac{1}{z+1} \frac{e^{-2\lambda}(2\lambda)^z}{z!} = e^{-2\lambda}\left(1\left(\frac{1}{0!}\right) + \frac{1}{2}\left(\frac{2\lambda}{1!}\right) + \frac{1}{3}\left(\frac{(2\lambda)^2}{2!}\right) + \cdots\right)}$$

$$\cancel{\sum_{n=1}^{\infty} \frac{(2\lambda)^{n-1} e^{-2\lambda}}{(n-1)!} n^{2} = e^{-2\lambda}\left(\frac{1}{0!} + \frac{(2\lambda) 2^2}{1!} + \frac{(2\lambda)^2 3^2}{2!} + \cdots\right)}$$

if sum of all $P(w) = 1$... scale factor! $\rightarrow$ sum(P_w)

Start: $\cancel{Z = X + Y}$

1) $w = \frac{1}{Z+1} \Rightarrow P_w(w) = P_z\left(\frac{1}{w} - 1\right)\frac{1}{w^2}$

2) $P_z = ? \rightarrow Z = X + Y \Rightarrow P(z) = \sum_{x \in X} P_x(x) P(z - x)$

$= \text{Poisson}(2\lambda)$

3) plug in to $P_w(w) = P_z\left(\frac{1}{w} - 1\right)\frac{1}{w^2}$

4) but sum $w \in W \neq 1 \Rightarrow$ "proportionality factor": $\sum_{w \in W} P_w(w)$

$\Rightarrow$ <u>new</u> $\hat{P}_w(w) = \frac{P_w(w)}{\sum_{w \in W} P_w(w)}$

5) Expectation: $\mathbb{E}[w] = \sum_{w \in W} w \hat{P}_w(w) \rightarrow \mathbb{E}[w] = \sum_{n=1}^{\infty} \frac{1}{n} \hat{P}_w\left(\frac{1}{n}\right)$

6) Plots!