ECE 6555 HW#2 Tev Willberg

#1] $\vec{y} = H\vec{x} + \vec{v}$, R_x, R_v $R_{xv} = 0$ consider $\vec{z}$ where $R_{zv} = 0$, $R_{zx} \neq 0$

$\hat{z}|_x$ = LLMS estimate of $\vec{z}$ given $\vec{x}$

$\widehat{\hat{z}|_x}$ = LLMS of $\hat{z}|_x$ given $\vec{y}$

$\hat{z}|_y$ = LLMS of $\vec{z}$ given $\vec{y}$

Show that $\hat{z}_y = \widehat{\hat{z}|_x}$

let $\hat{z}|_x = K_x \vec{x} = E[z|x] = R_{zx} R_x^{-1} \vec{x}$

let $\widehat{\hat{z}|_x} = K_x \vec{x}|_y = \cancel{E_y[E[z|x]|y]} = E[\cancel{R_{zx} R_x^{-1} \vec{x} | y}] = E[\cancel{R_{zx} R_x^{-1} \vec{x} \vec{y}^T}] = R_{\hat{z}|_x y} R_y^{-1} y$

let $\hat{z}|_y = K_y \vec{y} = E[z|y] = R_{zy} R_y^{-1} y$

$R_{\hat{z}|_x y} \cancel{\hat{z}|_x} = E[R_{zx} R_x^{-1} \vec{x} \vec{y}^T] = R_{zx} R_x^{-1} E(\vec{x} \vec{y}^T)$

$= R_{zx} R_x^{-1} R_{xy}$

since R_{zx} and R_x are constant matrices. i.e.:

$R_{zx} = E[\vec{z}\vec{x}^T]$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{z}\vec{x}^T p(z,x)\, d\vec{z}\, d\vec{x}$

$R_x = \int_{-\infty}^{\infty} \vec{x}\vec{x}^T p(x)\, d\vec{x}$

and $\hat{z}|_y = E[z|y] = \cancel{E[\vec{z}\vec{y}^T]} = R_{zy} R_y^{-1} y$

$R_{zy} = E[\vec{z}\vec{y}^T] = E[\vec{z}(Hx+v)^T] = E[z x^T H^T + z v^T]$

$= R_{zx} H^T + \cancel{R_{zv}}^{\,0}$

$R_y = E[yy^T] = E[(Hx+v)(Hx+v)^T] = E[Hxx^T H^T + Hxv^T + vx^T H^T + vv^T]$

$= HR_x H^T + H\cancel{R_{xv}}^{\,0} + \cancel{R_{vx}}^{\,0} H^T + R_v$ $\boxed{ok}$

$+ R_v$

$= HR_x H^T + R_v$

$\Rightarrow R_{zy} R_y^{-1} y = [R_{zx} H^T](HR_x H^T + R_v)^{-1}(Hx + v)$ ⟶

#1 con'd

$$\hat{z}_{|y} = R_{zy} R_y^{-1} y = [R_{zx} H^T](H R_x H^T + R_v)^{-1}(Hx + v)$$

$$= R_{zx} H^T((H R_x H^T)^{-1} + R_v^{-1})(Hx + v)$$

$$= [R_{zx} H^T (H^T)^{-1} R_x^{-1} H^{-1} + R_{zx} H^T R_v^{-1}](Hx + v)$$

$$= R_{zx} R_x^{-1} H^{-1} H x + R_{zx} R_x^{-1} H^{-1} v + R_{zx} H^T R_v^{-1} H x + R_{zx} H^T R_v^{-1} v$$

$$= R_{zx}(R_x^{-1} x + R_x^{-1} H^{-1} v + H^T R_v^{-1} H x + H^T R_v^{-1} v)$$

$$\hat{\hat{z}}_{|x} = R_{zx} R_x^{-1} \mathbb{E}(\vec{x}\vec{y}^T) = R_{zx} R_x^{-1} \mathbb{E}[\vec{x}(Hx + v)^T] = R_{zx} R_x^{-1}(R_x H^T + \cancelto{0}{R_{xv}})$$

$$= R_{zx} R_x^{-1} R_x H^T = R_{zx} H^T$$

$\hat{z}_{|x} = R_{zx} R_x^{-1} \vec{x}$ need $R_{zy} = R_{\hat{z}_{|x} y}$

$\hat{z}_{|y} = R_{zy} R_y^{-1} \vec{y}$

$\hat{\hat{z}}_{|x} = R_{\hat{z}_{|x} y} R_y^{-1} \vec{y}$

so: $R_{zy} = \mathbb{E}[z y^T] = \mathbb{E}[z(Hx + v)^T] = R_{zx} H^T + \cancelto{0}{R_{zv}} = R_{zx} H^T$

$R_{\hat{z}_{|x} y} = \mathbb{E}[\hat{z}_{|x} y^T] = \mathbb{E}[R_{zx} R_x^{-1} \vec{x} y^T]$, R_{zx} constant, R_x^{-1} constant

$$= R_{zx} R_x^{-1} R_{xy}$$

$$= R_{zx} R_x^{-1} \mathbb{E}[x(Hx + v)^T]$$

$$= R_{zx} R_x^{-1}(R_x H^T + \cancelto{0}{R_{xv}})$$

$$= R_{zx} H^T$$

$$\boxed{\text{Then } R_{zy} = R_{zx} H^T = R_{\hat{z}_{|x} y} \Rightarrow \hat{\hat{z}}_{|x} = R_{zx} H^T R_y^{-1} y = \hat{z}_{|y}}$$

#2

noisy measurement: $\vec{y} = (\cancel{\ \ })\vec{x}$ $\quad$ x, v are 0 mean, independent

$= (1+v)\vec{x}$ $\quad$ variance of v is $\sigma^2 \Rightarrow E(v^2) = \sigma_v^2$

1) LLMS of $\vec{x}$ given $\vec{y}$ $\quad$ 2) Show that MMSE < $E(\vec{x}\vec{x}^T)$

$$\hat{x} = E(\vec{x}|\vec{y}) = R_{xy} R_y^{-1} \vec{y}$$

$\Rightarrow$ LLSE form: $\hat{x} = K_0 \vec{y}$

need to find R_y^{-1}: $R_y = \mathbb{E}(\vec{y}\vec{y}^T) = \mathbb{E}((1+v)\vec{x}\vec{x}^T(1+v)^T)$

$$R_{xy} = \mathbb{E}(x y^T) = \mathbb{E}(\vec{x}\vec{x}^T \underset{\text{scalar}}{\underbrace{(1+v)^T}}) = R_x + \mathbb{E}(\vec{x}\vec{x}^T v)$$

Note: if two random variables ~~too~~ are un-correlated, then

$$Cov(\vec{x}\vec{y}) = \cancel{\mathbb{E}(\vec{x}\vec{y})} = \cancel{\mathbb{E}((\vec{x} - \mathbb{E}(\vec{x}))(\vec{y} - \mathbb{E}(\vec{y}))^T) = 0}$$

$$= \cancel{\mathbb{E}[\vec{x}\vec{y}^T - \mathbb{E}(\vec{x})\vec{y}^T - \vec{x}\mathbb{E}(\vec{y})^T + \mathbb{E}(\vec{x})\mathbb{E}(\vec{y})^T]} = \mathbb{E}(\vec{x}\vec{y}^T) - \mathbb{E}(\underset{\text{constant so pull out}}{\mathbb{E}(\vec{x})}\vec{y}^T) - \mathbb{E}(\vec{x}\mathbb{E}(\vec{y})^T) + \mathbb{E}(\vec{x})\mathbb{E}(\vec{y})^T = 0$$

$- \cancel{\mathbb{E}(\vec{x})\mathbb{E}(\vec{y}^T)} - \mathbb{E}(\vec{x})\mathbb{E}(\vec{y})^T + \cancel{\mathbb{E}(\ )} = 0$ ✓

(if uncorrelated $= 0$)

$\Rightarrow \mathbb{E}(\vec{x}\vec{y}^T) = \mathbb{E}(\vec{x})\mathbb{E}(\vec{y})^T$

from wikipedia:

$$R_{x} = \mathbb{E}[(\vec{x} - \vec{\mu}_x)(\vec{x} - \vec{\mu}_x)^T] = \mathbb{E}[\vec{x}\vec{x}^T] - \vec{\mu}_x\vec{\mu}_x^T$$

$$R_{xy} = \mathbb{E}[(\vec{x} - \vec{\mu}_x)(\vec{y} - \vec{\mu}_y)^T] = \mathbb{E}[\vec{x}\vec{y}^T] - \vec{\mu}_x \vec{\mu}_y^T$$

$$R_y^{-1} \Rightarrow R_y = \mathbb{E}(\vec{y}\vec{y}^T) = \mathbb{E}[(1+v)\vec{x}\vec{x}^T(1+v)^T] = \mathbb{E}[(\vec{x}\vec{x}^T + v\vec{x}\vec{x}^T)(1+v)^T]$$

$$= \mathbb{E}[\vec{x}\vec{x}^T + \vec{x}\vec{x}^T v^T + v\vec{x}\vec{x}^T + v\vec{x}\vec{x}^T v^T]$$

$$= \mathbb{E}(\vec{x}\vec{x}^T) + \mathbb{E}(\vec{x}\vec{x}^T v) + \mathbb{E}(v\vec{x}\vec{x}^T) + \mathbb{E}(v\vec{x}\vec{x}^T v)$$

$\vec{x}, v$ uncorrelated... $\Rightarrow$

$$= R_x + \mathbb{E}(\vec{x}\vec{x}^T)\overset{0}{\cancel{\mathbb{E}(v)}} + \overset{0}{\cancel{\mathbb{E}(v)}}\mathbb{E}(\vec{x}\vec{x}^T) + \mathbb{E}(v)\mathbb{E}(\vec{x}\vec{x}^T)\mathbb{E}(v) \ ?$$

$$= R_x + \mathbb{E}(\underset{\text{scalars...}}{v}\vec{x}\vec{x}^T v) = R_x + \mathbb{E}(\vec{x}\vec{x}^T)\mathbb{E}(v^2)$$

$$= R_x + R_x\sigma_v^2 = R_x(1+\sigma_v^2)$$

$$\Rightarrow R_y^{-1} = (R_x + R_x\sigma_v^2)^{-1} = \cancel{R_x^{-1} + \frac{1}{\sigma_v^2}R_x^{-1} = R_x^{-1}\left(1 + \frac{1}{\sigma_v^2}\right)}$$

$$= (R_x(1+\sigma_v^2))^{-1} = \frac{1}{1+\sigma_v^2} R_x^{-1}$$

#2 cont'd) $R_{xy} = \mathbb{E}(\vec{x}\vec{y}^T) = \mathbb{E}(\vec{x}((1+v)\vec{x})^T) = \mathbb{E}(\vec{x}\vec{x}^T(1+v))$

$$= \mathbb{E}(\vec{x}\vec{x}^T + \vec{x}\vec{x}^T v)$$

$$= R_x + R_x \cancel{\mathbb{E}(v)}^{0} = R_x$$

then $\hat{\vec{x}} = \mathbb{E}(\vec{x}|\vec{y}) = R_{xy} R_y^{-1} \vec{y} = R_x \left(\frac{1}{1+\sigma_v^2}\right) R_x^{-1} \vec{y} = \boxed{\frac{\vec{y}}{1+\sigma_v^2}}$

Now, show that $\mathbb{E}((\vec{x}-\hat{\vec{x}})(\vec{x}-\hat{\vec{x}})^T) < \mathbb{E}(\vec{x}\vec{x}^T)$

$$\mathbb{E}((\vec{x}-\hat{\vec{x}})(\vec{x}-\hat{\vec{x}})^T) = \mathbb{E}(\vec{x}\vec{x}^T - \hat{\vec{x}}\vec{x}^T - \vec{x}\hat{\vec{x}}^T + \hat{\vec{x}}\hat{\vec{x}}^T)$$

$$= R_x - \mathbb{E}\left(\frac{\vec{y}}{1+\sigma_v^2}\vec{x}^T\right) - \frac{\mathbb{E}(\vec{x}\vec{y}^T)}{1+\sigma_v^2} + \mathbb{E}(\vec{y}\vec{y}^T)\frac{1}{(1+\sigma_v^2)^2}$$

$$= R_x - \frac{R_{yx}}{1+\sigma_v^2} - \frac{R_{xy}}{1+\sigma_v^2} + \frac{R_y}{(1+\sigma_v^2)^2}$$

$R_{xy} = R_{yx}^T$

$R_{yx} = R_{xy}^T$

$$= R_x - \frac{R_{yx}}{1+\sigma_v^2} - \cancel{\frac{R_{xy}}{1+\sigma_v^2}}^{R_x} + \cancel{\frac{(R_x(1+\sigma_v^2))}{(1+\sigma_v^2)^2}} = R_x - \frac{R_{yx}}{1+\sigma_v^2}$$

$$= \cancel{\frac{R_x(1+\sigma_v^2) + R_x - R_{yx} - R_{xy}}{1+\sigma_v^2}}$$

$$R_{yx} = \mathbb{E}(\vec{y}\vec{x}^T) = \mathbb{E}((1+v)\vec{x}\vec{x}^T) = \mathbb{E}(\vec{x}\vec{x}^T + v\vec{x}\vec{x}^T) = R_x + \cancel{\mathbb{E}(v)}^{0}\mathbb{E}(\vec{x}\vec{x}^T) = R_x$$

$$\boxed{\Rightarrow \mathbb{E}((\vec{x}-\hat{\vec{x}})(\vec{x}-\hat{\vec{x}})^T) = R_x - \frac{R_x}{1+\sigma_v^2} = R_x\left(\frac{1+\sigma_v^2-1}{1+\sigma_v^2}\right) = \boxed{R_x\left(\underbrace{\frac{\sigma_v^2}{1+\sigma_v^2}}_{<1}\right) < R_x}}$$

implies $\square$

#3 Defective Measurement Noise

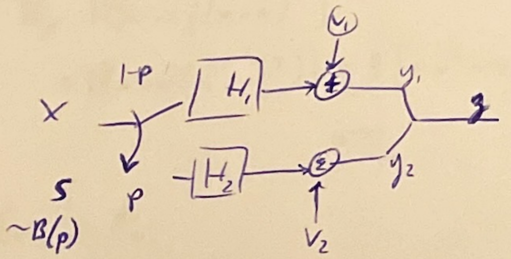
zero-mean random variable $\vec{x}$ w/ var($\vec{x}$) = $\mathbb{E}((\vec{x}-\cancel{\vec{\mu}_x})(\vec{x}-\cancel{\vec{\mu}_x})^T) = \Pi_0$

two possible measurements for $\vec{x}$: $\vec{y}_1 = \underline{H}_1\vec{x}+\vec{v}_1$ &, $\vec{y}_2 = \underline{H}_2\vec{x}+\vec{v}_2$

$(\vec{v}_1, \vec{v}_2)$ are 0-mean, uncorrelated sensor noise w/ variance $\underline{R}_1$ and $\underline{R}_2$ respectively.
is also uncorrelated w/ $\vec{x}$

One of the measurements is defective... either sensor 1 w/ probability 1-p or sensor 2 w/ prob. p → this is measurement $\vec{z}$

3-1) LLMS estimator of $\vec{x}$ given $\vec{z}$ ⇒ $\hat{x}_z = \mathbb{E}(x|z) = R_{xz}R_z^{-1}\vec{z}$



joint distribution of data...

Hint: introduce s explicitly...

$$P_{zx}(z,x) = \sum_s P_{zxs}(z,x,s)$$

law of total probability: (like a weighted probability)

$$\mathbb{E}_z(f(z)) = \mathbb{E}_s(\mathbb{E}_{z|s}(f(z)|s))$$

$$\Rightarrow P_{zx}(z,x) = p(z,x|s_1)\,p(s_1) + p(z,x|s_2)\,p(s_2)$$
$$= p(z,x|s_1)(1-p) + p(z,x|s_2)(p)$$

$$\Rightarrow \mathbb{E}(\vec{x}|\vec{z}) = \mathbb{E}_s(\mathbb{E}(\vec{x}|\vec{z})|s) = \mathbb{E}(\vec{x}|\vec{z}|s_1)\,\mathbb{E}(s_1) + \mathbb{E}(\vec{x}|\vec{z}|s_2)\,\mathbb{E}(s_2)$$

keep this expectation?? $= \mathbb{E}(\vec{x}|\vec{z}|s_1)(1-p) + \mathbb{E}(\vec{x}|\vec{z}|s_2)(p)$

(I don't know if the notation is correct but it feels right)

from class notes: $\mathbb{E}(\vec{x}|\vec{z}_1)|_{s_1} = R_{xy_1}R_{y_1}^{-1}\vec{z}_1 = K_{o1}\vec{z}_1$, $K_{o1} = (R_x^{-1} + \underline{H}_1^T R_{v_1}^{-1}\underline{H}_1)^{-1}\underline{H}_1^T R_{v_1}^{-1}$

$\mathbb{E}(\vec{x}|\vec{z}_2)|_{s_2} = R_{xy_2}R_{y_2}^{-1}\vec{z}_2 = K_{o2}\vec{z}_2$, $K_{o2} = (R_x^{-1} + H_2^T R_{v_2}^{-1}H_2)^{-1}H_2^T R_{v_2}^{-1}$

→

$$\cancel{\mathbb{E}((\vec{x}-\vec{z}W)(\vec{x}-\vec{z}W)^T)} = \mathbb{E}(\vec{x}\vec{x}^T - \cancel{\vec{x}W^T\vec{z}^T - \vec{z}W^T\vec{x}^T + \vec{z}WW^T\vec{z}^T})$$

$$\cancel{= R_x \ \ldots ?}$$

$\Rightarrow$ let $\hat{x} = \mathbb{E}(\vec{x}|z) = \mathbb{E}(\vec{x}|\vec{y}_1)E(y_1) + \mathbb{E}(\vec{x}|\vec{y}_2)E(y_2)$

$$= K_{01}\vec{z}(1-p) + K_{02}\vec{z}\,p$$

$$= K_{01}\vec{z} - K_{01}\vec{z}p + K_{02}\vec{z}\,p$$

$$= K_{01}\vec{z} + (K_{02}\cancel{\vec{z}} - K_{01})\vec{z}\,p$$

, p is a scalar

$$\boxed{\hat{x} = [K_{01} + (K_{02} - K_{01})p]\vec{z}}$$

where $K_{01} = (\pi_0^{-1} + H_1^T R_1^{-1} H_1)^{-1} H_1^T R_1^{-1}$

$K_{02} = (\pi_0^{-1} + H_2^T R_2^{-1} H_2)^{-1} H_2^T R_2^{-1}$

p = probability switch is set for model 2

H3-Q2

$\Rightarrow \cancel{R_x} = \mathbb{E}[(x-\hat{x})(x-\hat{x})^T]$

$$= \mathbb{E}[(x-\hat{x})(x^T-\hat{x}^T)] = \mathbb{E}[xx^T - x\hat{x}^T - \hat{x}x^T + \hat{x}\hat{x}^T]$$

$$\cancel{= R_x}$$

$\mathbb{E}[xx^T] = R_x$

$\mathbb{E}[x\hat{x}^T] = \mathbb{E}[x[(K_{01} + (K_{02}-K_{01})p)\vec{z}]^T]$

K_{01}, K_{02}, p are constants (not random)

$$= \mathbb{E}[xz^T(K_{01} + (K_{02}-K_{01})p)^T]$$

[let $W = (K_{01} + (K_{02}-K_{01})p)$]

$$= \cancel{R_{xz}(K_{01} + (K_{02}-K_{01})p)^T} = R_{xz}W^T$$

$\mathbb{E}[\hat{x}x^T] = \mathbb{E}[(K_{01} + (K_{02}-K_{01})p)zx^T]$

$$= \cancel{(K_{01} + (K_{02}-K_{01})p)R_{zx}} = WR_{zx}$$

$\mathbb{E}[\hat{x}\hat{x}^T] = \mathbb{E}[(K_{01} + (K_{02}-K_{01})p)zz^T(K_{01} + (K_{02}-K_{01})p)^T]$

$$= \cancel{(K_{01} + (K_{02}-K_{01})p)R_z(K_{01} + (K_{02}-K_{01})p)^T} \rightarrow$$

$$= \cancel{WR_zW^T}$$

#3 - Q2 cont'd)

$$\mathbb{E}[(x-\hat{x})(x-\hat{x})^T] = \mathbb{E}[xx^T - x\hat{x}^T - \hat{x}x^T - \hat{x}\hat{x}^T]$$

$$= R_x - R_{xz}W^T - WR_{zx} - WR_zW^T \quad , \quad W = (K_{01} + (K_{02} - K_{01})p)$$

$$= \Pi_0 - R_{xz}W^T - WR_{zx} - WR_zW^T \qquad K_{01} = (\Pi_0^{-1} + H_1^T R_1 H_1)^{-1} H_1^T R_1$$

$$K_{02} = (\Pi_0^{-1} + H_2^T R_2 H_2)^{-1} H_2^T R_2$$

$$\Pi_0 = R_x$$

Not complete yet.

~~#3 - Q3)~~

~~if v_1 and v_2 are correlated, it does not change the MMSE.~~

(Law of total Expectation)

$$\mathbb{E}[xx^T] = R_x$$

$$\mathbb{E}[x\hat{x}^T] = \mathbb{E}[xz^TW^T] = \mathbb{E}_s[xz^T|s]W^T = (\mathbb{E}[xy_1^T](1-p) + \mathbb{E}(xy_2^T)(p))W^T$$

$$\mathbb{E}[xy_1^T] = \mathbb{E}[x(H_1x+v_1)^T] = \mathbb{E}[xx^TH_1^T + xv_1^T] = R_xH_1 + \cancelto{0}{R_{xv_1}}$$

$$\mathbb{E}[xy_2^T] = \quad '' \quad {}_2 = \quad '' \quad {}_2 = R_xH_2 + \cancelto{0}{R_{xv_2}}$$

$$\Rightarrow \mathbb{E}[x\hat{x}^T] = (R_xH_1(1-p) + R_xH_2p)W^T$$

$$\mathbb{E}[\hat{x}x^T] = \mathbb{E}[Wzx^T] = \mathbb{E}_s[\mathbb{E}[Wzx^T|s]] = W(\mathbb{E}_{z|s}(y_1x^T)(1-p) + \mathbb{E}_{z|s}(y_2x^T)p)$$

$$\mathbb{E}(y_1x^T) = \mathbb{E}((H_1x+v)x^T) = \mathbb{E}(H_1xx^T + vx^T) = H_1R_x$$

$$\mathbb{E}(y_2x^T) = \quad '' \quad {}_2 = \quad '' \quad {}_2 = H_2R_x$$

$$\Rightarrow \mathbb{E}[\hat{x}x^T] = W(H_1R_x(1-p) + H_2R_xp)$$

$$\mathbb{E}[\hat{x}\hat{x}^T] = \mathbb{E}[Wzz^TW^T] = W\mathbb{E}_s[\mathbb{E}_{z|s}[zz^T]]W^T = W(\mathbb{E}_{z|s}(y_1y_1^T)(1-p) + \mathbb{E}(y_2y_2^T)p)W^T$$

$$\mathbb{E}(y_1y_1^T) = \mathbb{E}((H_1x+v)(H_1x+v)^T) = \mathbb{E}(H_1xx^TH_1^T + \cancelto{0}{H_1xv^T} + \cancelto{0}{vx^TH_1^T} + v_1v_1^T)$$

$$= H_1R_xH_1^T + R_{v_1}$$

$$\mathbb{E}(y_2y_2^T) = \quad '' \quad {}_2 = H_2R_xH_2^T + R_{v_2}$$

$$\Rightarrow \mathbb{E}[\hat{x}\hat{x}^T] = W((H_1R_xH_1^T + R_{v_1})(1-p) + (H_2R_xH_2^T + R_{v_2})p)W^T$$

3-3

#3-Q-4 | ~~Let $H_1 = H_2 = H$, then $y_1 = Hx + v_1$ and $K_{01} = (\Pi_0^{-1} + H^T R_1^{-1} H)^{-1} H^T R_1^{-1}$~~

~~$y_2 = Hx + v_2$~~ $K_{02} = (\Pi_0^{-1} + H^T R_2^{-1} H)^{-1} H^T R_2^{-1}$

#3-Q2 cont'd |

$$\boxed{\begin{aligned} \mathbb{E}[(x-\hat{x})(x-\hat{x})^T] = R_x &- (R_x H_1 (1-p) + R_x H_2 p) W^T \\ &- W(H_1 R_x (1-p) + H_2 R_x p) \\ &+ W((H_1 R_x H_1^T + R_{v_1})(1-p) + (H_2 R_x H_2^T + R_{v_2}) p) W^T \end{aligned}}$$

$\square$

#3-Q3 |

Answer does not change if v_1, v_2 are correlated.

#3-Q4 |

For $H_1 = H_2$ ... what say?

In this case, our ~~MMSE~~ ~~depends~~ MMSE will ~~change~~ improve or decline based ~~largely~~ on the variance of the random noise R_{v_1} and R_{v_2} and our knowledge of the probability p.

#4] Linear estimator of x^2

$y = x + v$, v, x are 0 mean, independent, $\in \mathbb{R}$, Gaussian random variables

$\mathbb{E}(v^2) = \sigma_v^2$, $\mathbb{E}(x^2) = \sigma_x^2$

Note: for a real-valued, zero-mean Gaussian random variable z, $\text{var}(z) = \sigma_z^2$, $\mathbb{E}(z^3) = 0$ and $\mathbb{E}(z^4) = 3\sigma^4$

4-1] Find LLMS estimator of x^2 using ~~$\bar{y}$~~ y :

let $\hat{x^2} = ky$ (linear estimator)

$$\hat{x}^2 = \mathbb{E}(x^2|y) = \mathbb{E}((x^2 - ky)(x^2 - ky)^T)$$

$$= \mathbb{E}(x^4 - kyx^2 - kyx^2 + k^2y^2)$$

$$= \mathbb{E}(x^4) - 2\mathbb{E}(\underset{\text{(scalar)}}{kyx^2}) + k^2\mathbb{E}(y^2)$$

$$= 3\sigma_x^4 - 2kR_{yx^2} + k^2 R_y$$

$\mathbb{E}(\cancel{yx^2}) = \mathbb{E}_y(\mathbb{E}(yx^2|y))$

$= \cancel{\mathbb{E}_y(\mathbb{E}(x^2|y)y)}$

$= \mathbb{E}_y$

~~$\hat{x}$~~ $P(k) = \mathbb{E}((x^2 - ky)(x^2 - ky))$

$\Rightarrow$ ~~$\hat{x}$~~ $\frac{\partial P(k)}{\partial k} = -2R_{yx^2} + 2kR_y = 0$

(minimization w.r.t. estimator k)

$\Rightarrow k_0 = R_{yx^2} R_y^{-1}$

$$P(k) = \mathbb{E}(x^2/y) = \mathbb{E}((x^2 - ky)(x^2 - ky)^T) = \mathbb{E}((x^2 - k(x+v))^2)$$

$$= \mathbb{E}(x^4 - 2x^2k(x+v) + k^2(x+v)^2)$$

$$= \mathbb{E}(x^4) - 2k(\mathbb{E}(x^3) + \mathbb{E}(x^2v)) + k^2\mathbb{E}(x^2 + 2xv + v^2)$$

$$= 3\sigma_x^4 - 2k(0 + \mathbb{E}(x^2)\cancelto{0}{\mathbb{E}(v)}) + k^2(\sigma_x^2 + 2\cancelto{0}{\mathbb{E}(x)\mathbb{E}(v)} + \mathbb{E}(v^2))$$

$$= 3\sigma_x^4 - 0 + k^2(\sigma_x^2 + \sigma_v^2)$$

$$\frac{\partial P(k)}{\partial k} = 2k(\sigma_x^2 + \sigma_v^2) = 0 \Rightarrow \boxed{k = 0} \Rightarrow \boxed{\hat{x^2} = ky = 0 = \mathbb{E}[x^2/y]}$$

$\rightarrow$

4-2) $\underline{\text{LLMS of } x^2 \text{ using } y^2}$.

let estimated $\hat{x}^2 = ky^2$

$$P(k) = \mathbb{E}(x^2|y^2) = \mathbb{E}((x^2-ky^2)(x^2-ky^2))$$

$y = x+v$

$$= \mathbb{E}(x^4 - 2kx^2(x+v)^2 + k^2(x+v)^4)$$

$$= 3\sigma_x^4 - 2k\,\mathbb{E}(x^2(x^2+2xv+v^2)) + k^2\mathbb{E}(x^4+4x^3v+6x^2v^2+4xv^3+v^4)$$

$$= 3\sigma_x^4 - 2k\,\mathbb{E}(x^4+2x^3v+x^2v^2) + k^2[\mathbb{E}(x^4)+4\mathbb{E}(x^3)\overset{0}{\mathbb{E}(v)}+6\mathbb{E}(x^2)\mathbb{E}(v^2)+4\mathbb{E}(x)\overset{0}{\mathbb{E}(v^3)}+\mathbb{E}(v^4)]$$

$$= 3\sigma_x^4 - 2k[3\sigma_x^4+0+\sigma_x^2\sigma_v^2] + k^2[3\sigma_x^4+6\sigma_x^2\sigma_v^2+3\sigma_v^4]$$

$$\frac{\partial P(k)}{\partial k} = -\cancel{6}_{3}\sigma_x^4 - \cancel{2}\sigma_x^2\sigma_v^2 + \cancel{2}k[3\sigma_x^4+6\sigma_x^2\sigma_v^2+3\sigma_v^4] = 0$$

(minimization w.r.t. k)

$$\Rightarrow \boxed{k = \frac{3\sigma_x^4+\sigma_x^2\sigma_v^2}{3\sigma_x^4+6\sigma_x^2\sigma_v^2+3\sigma_v^4}}$$

#5] Separation of Signal and Structured Noise

$\vec{y} = \underline{H}\vec{x} + \underline{S}\vec{\theta} + \vec{v}$

$\vec{v}$ is 0-mean additive noise random vector

$E(\vec{v}\vec{v}^*) = I = R_v$

$\vec{x}, \vec{\theta}$ are unknown vectors, constant

$\underline{H} \in \mathbb{C}^{m \times n}$, $\underline{S} \in \mathbb{C}^{m \times p}$ $\Rightarrow$ $\underline{H}, \underline{S}$ known, $[\underline{H}\ \underline{S}]$ full rank, $m \geq n+p$ $\Rightarrow \text{rank}([\underline{H}\ \underline{S}]) = n+p$

$\underline{S}\vec{\theta}$ = perturbation

$\underline{H}\vec{x}$ = useful signal to separate

5-Q1) let $\vec{z} = [\vec{x}^T\ \vec{\theta}^T]^T = \begin{bmatrix} \vec{x} \\ \vec{\theta} \end{bmatrix}$ Determine optimal unbiased estimator $\hat{z}$ of z given y

Un-biased $\Rightarrow$ $\mathbb{E}(\hat{z}) = z$ note $\vec{y} = [\underline{H}\ \underline{S}]\vec{z} + \vec{v}$, let $\boxed{\underline{H}_z = [\underline{H}\ \underline{S}]}$

$\hat{z} \triangleq Ky$ (general linear estimate K)

By the Gauss-Markov Thm, the optimal, unbiased LLSE of $\vec{z}$ given $\vec{y}$ is

$$\boxed{\hat{z}_{\infty} = (\underline{H}_z^{*}\underline{H}_z)^{-1}\underline{H}_z^{*}\vec{y}}$$

recall that if z is deterministic then $R_z = \alpha I$, $\alpha \to \infty$

5-Q2) let $\hat{z} = \begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix}$, $\hat{s} \triangleq \underline{H}\hat{x}$ is the estimate of $\vec{s} \triangleq \underline{H}\vec{x}$

Show that $\hat{s} = \underline{E}y$ with $\underline{E} = P_H[I - S(S^* P_H^{\perp} S)^{-1} S^* P_H^{\perp}] = \underline{H}(\underline{H}^* P_S^{\perp} \underline{H})^{-1}\underline{H}^* P_S^{\perp}$

P_H, P_S are orthogonal projection matrices on the space spanned by the rows of H, S respectively

#5-Q1

generic linear estimate: $\hat{z} \triangleq ky$ ~~note~~ let $H_z = [\underline{H} \ \underline{S}]$

$\Rightarrow \vec{y} = H_z \vec{z} + \vec{v}$

$\Rightarrow \mathbb{E}(\hat{z}) = k\,\mathbb{E}(\vec{y}) = k\,\mathbb{E}(H_z \vec{z} + \vec{v})$

$= kH_z(z + \cancel{\mathbb{E}(\vec{v})}^{0}) = kH_z z$ → unbiased so require $\underline{kH_z = I}$

Normal equation for complex numbers?: (assume 0-mean random vectors)

must have $\forall a \in \mathbb{C}^n$, $a^+ P(k) a > a^+ P(k_0) a \Rightarrow P(k) \geq P(k_0)$

$a^+ P(k) a = a^+ \mathbb{E}((z - k\vec{y})(z - ky)^+) a = \mathbb{E}(a^+(z - ky)(z - ky)^+ a)$

$= \mathbb{E}(a^+(zz^+ - kyz^+ - zy^+k^+ + kyy^+k^+) a)$

$= a^+ R_z a \underbrace{- a^+ k R_{yz} a}_{\text{scalar}} - a^+ R_{zy} k^+ a + a^+ k R_y k^+ a$

$= a^+ R_z a - a^+ R^+_{yz} k^+ a - a^+ R_{zy} k^+ a + a^+ k R_y k^+ a$

R_{zy}

$R_{zy} = \mathbb{E}(zy^+)$

$R_{zy}^+ = \mathbb{E}(zy^+)^+ = \mathbb{E}((zy^+)^+) = \mathbb{E}(yz^+) = R_{yz}$ ✓

transpose commutes

$\frac{dx^T B x}{dx} = (B + B^T)x$

$\frac{d\, a^+ k R_y k^+ a}{d k^+ a} = (R_y + R_y^+) k^+ a$

$R_y = R_y^+$

$= 2 R_y k^+ a$

$\frac{\partial a^T x}{\partial x} = a$

$\Rightarrow \frac{\partial\, a^+ R_{zy} k^+ a}{\partial k^+ a} = (a^+ R_{zy})^+ = R_{yz} a$

$\frac{\partial\, a^+ P(k) a}{\partial \underset{\text{vector}}{k^+ a}} = 0 \iff -\cancel{2} R_{zy} + 2 R_y k^+ a = 0$

$-\cancel{2} R_{yz} \cancel{a} + \cancel{2} R_y k^+ \cancel{a} = 0$

$(R_y k^+ = R_{yz})^+ \Rightarrow k R_y = R_{zy}$

$\Rightarrow$ if R_y^{-1} non-singular

$\Rightarrow k_0 = R_{zy} R_y^{-1}$ ✓

→

Now that we have confirmed the normal equation for a general linear estimate, we restrict ourselves to the linear model $y = Hz + v$ $v \rightarrow$ zero mean, uncorrelated w/ z

$K_o = R_{zy} R_y^{-1} \Rightarrow R_{zy} = \mathbb{E}((z - \cancel{m_z}^{0})(y - \cancel{m_y}^{0})^+) = \mathbb{E}(z y^+) = \mathbb{E}(z(Hz+v)^+) = \mathbb{E}(zz^+H^+ + zv^+)$

$\hat{z} = K_o y$ $= R_z H^+ + \cancel{R_{zv}}^{0} = R_z H^+$

$R_y = \mathbb{E}(yy^+) = \mathbb{E}((Hz+v)(Hz+v)^+) = \mathbb{E}(Hzz^+H^+ + vz^+H^+ + Hzv^+ + vv^+)$

$= HR_z H^+ + 0 + 0 + R_v = HR_z H^+ + R_v$

$$\Rightarrow \hat{z} = \underbrace{R_z H^+ (HR_z H^+ + R_v)^{-1}}_{K_o} y$$

Matrix inversion lemma: $(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$

let $A = R_v$, $B = H$, $C = R_z$, $D = H^+$

$\Rightarrow$ then $(R_v + HR_z H^+)^{-1} = R_v^{-1} - R_v^{-1} H (R_z^{-1} + H^+ R_v^{-1} H)^{-1} H^+ R_v^{-1}$

$\Rightarrow K_o = R_z H^+ R_v^{-1} - R_z H^+ R_v^{-1} H (R_z^{-1} + H^+ R_v^{-1} H)^{-1} H^+ R_v^{-1}$

similar algebra as in class $\rightarrow$ no transpose so keeps form

$\Rightarrow K_o = (R_z^{-1} + H^+ R_v^{-1} H)^{-1} H^+ R_v^{-1}$

Now, if z is deterministic, then $R_z = \alpha I$, $\alpha \rightarrow \infty$.

$\Rightarrow K_o = (0 + H^+ R_v^{-1} H)^{-1} H^+ R_v^{-1}$ if $R_v = I$, $R_v^{-1} = I$

$\Rightarrow K_o = (H^+ H)^{-1} H^+$

thus, by the Gauss-Markov Thm: $\boxed{\hat{z}_\infty = K_o y = (H^+ H)^{-1} H^+ y}$

And in our specific case, $H_z = [H \; S]$, $y = H_z z + v$, $z = \binom{x}{\theta}$

$$\boxed{\hat{z}_\infty = (H_z^+ H_z)^{-1} H_z^+ y}$$

#5-Q21

$\hat{z} = \begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix}$, $\hat{y} = H_z \hat{z} + v$

$$\begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = (H_z^+ H_z)^{-1} H_z^+ \vec{y} = \left(\begin{bmatrix} H^+ \\ S^+ \end{bmatrix} [H \; S] \right)^{-1} \begin{bmatrix} H^+ \\ S^+ \end{bmatrix} \vec{y}$$

want $H\hat{x}$

$$= \begin{pmatrix} H^+H & H^+S \\ S^+H & S^+S \end{pmatrix}^{-1} \begin{bmatrix} H^+ \\ S^+ \end{bmatrix} \vec{y}$$

Schur Complement!

$$\text{For } \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]^{-1} = \left[\begin{array}{c|c} I & 0 \\ \hline -A_{22}^{-1}A_{21} & I \end{array}\right] \left[\begin{array}{c|c} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & 0 \\ \hline 0 & A_{22}^{-1} \end{array}\right] \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}$$

let $A_{11} = H^+H$

$A_{12} = H^+S$

$A_{21} = S^+H$

$A_{22} = S^+S$

$$= \left[\begin{array}{c|c} I & 0 \\ \hline -A_{22}^{-1}A_{21} & I \end{array}\right] \begin{bmatrix} '' & -''(A_{12}A_{22}^{-1}) \\ 0 & A_{22}^{-1} \end{bmatrix} = \left[\begin{array}{c|c} '' & -''(A_{12}A_{22}^{-1}) \\ \hline -A_{22}^{-1}A_{21}('') & A_{22}^{-1}A_{21}('')(A_{12}A_{22}^{-1}) + A_{22}^{-1} \end{array}\right]$$

$$\Rightarrow \begin{pmatrix} \hat{x} \\ \hat{\theta} \end{pmatrix} = \left(\begin{array}{c|c} I & 0 \\ \hline -(S^+S)^{-1}S^+H & I \end{array}\right) \left(\begin{array}{c|c} (H^+H - H^+S(S^+S)^{-1}S^+H)^{-1} & 0 \\ \hline 0 & (S^+S)^{-1} \end{array}\right) \begin{pmatrix} I & -H^+S(S^+S)^{-1} \\ 0 & I \end{pmatrix}$$

= SEE MATLAB

$P_H = H(H^+H)^{-1}H^+$

$P_S = S(S^+S)^{-1}S^+$

Schur Complement:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}, \quad S \triangleq D - CA^{-1}B$$

→ arises naturally from

$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

→ can also get Schur complement of D if solve for y first

let $A = H^+H$

$B = H^+S$ ⇒

$C = S^+H$

$D = S^+S$

$$S_2 = (S^+S) - (S^+H)\underbrace{(H^+H)^{-1}H^+S}_{P_H} \longrightarrow$$

#5-Q2 cont'd

$$\begin{pmatrix} H^+H & H^+S \\ S^+H & S^+S \end{pmatrix}^{-1} = \begin{bmatrix} (H^+H)^{-1} + (H^+H)^{-1}(H^+S)\left(S^+S - S^+H(H^+H)^{-1}H^+S\right)^{-1}S^+H(H^+H)^{-1} & -(H^+H)^{-1}H^+S\,S_2^{-1} \\ -\left(S^+S - S^+H(H^+H)^{-1}H^+S\right)^{-1}S^+H(H^+H)^{-1} & \left(S^+S - S^+H(H^+H)^{-1}H^+S\right)^{-1} \end{bmatrix}$$

$$\Rightarrow \hat{x} = \left[(H^+H)^{-1}H^+ + (H^+H)^{-1}H^+S\left(S^+S - S^+P_H S\right)^{-1}S^+H(H^+H)^{-1}H^+ - (H^+H)^{-1}H^+S\left(S^+S - S^+P_H S\right)^{-1}S^+\right]\vec{y}$$

$$H\hat{x} = \left[H(H^+H)^{-1}H^+ + H(H^+H)^{-1}H^+S\left(S^+S - S^+P_H S\right)^{-1}S^+P_H - H(H^+H)^{-1}H^+S\left(S^+S - S^+P_H S\right)^{-1}S^+\right]\vec{y}$$

$$= \left[P_H + P_H S\left(S^+S - S^+P_H S\right)^{-1}S^+P_H - P_H S\left(S^+S - S^+P_H S\right)^{-1}S^+\right]\vec{y}$$

$$= P_H\left[I + S\left((S^+S)^{-1} - (S^+P_H S)^{-1}\right)S^+P_H - S\left((S^+S)^{-1} - (S^+P_H S)^{-1}\right)S^+\right]\vec{y}$$

$$= P_H\left[I - S\big(\underbrace{(S^+S)^{-1}}_{=(S^+IS)^{-1}} - (S^+P_H S)^{-1}\big)S^+\underbrace{(I - P_H)}_{P_H^\perp}\right]\vec{y}$$

$$= P_H\left[I - S\left(S^+IS - S^+P_H S\right)^{-1}S^+P_H^\perp\right]\vec{y}$$

$$= P_H\left[I - S\left(S^+(I - P_H)S\right)^{-1}S^+P_H^\perp\right]\vec{y} =$$

$$\boxed{= \underbrace{P_H\left[I - S\left(S^+P_H^\perp S\right)^{-1}S^+P_H^\perp\right]}_{E}\vec{y} = Ey = \hat{s} \qquad \square}$$

Now show $E = H(H^+P_S^\perp H)^{-1}H^+P_S^\perp$

$$= H\left(H^+\left(I - S(S^+S)^{-1}S^+\right)H\right)^{-1}H^+\left(I - S(S^+S)^{-1}S^+\right)$$

$$= H\left(H^+H - H^+S(S^+S)^{-1}S^+H\right)^{-1}H^+\left(I - S(S^+S)^{-1}S^+\right)$$

$$= \left(H(H^+H)^{-1} - H\left(H^+S(S^+S)^{-1}S^+H\right)^{-1}H^+\right)\left(I - S(S^+S)^{-1}S^+\right)$$

$$= \left(H(H^+H)^{-1}H^+ - H\left(H^+S(S^+S)^{-1}S^+H\right)^{-1}H^+\right)\left(I - S(S^+S^{-1})S^+\right)$$

$$= H(H^+H)^{-1}H^+ - H\left(H^+S(S^+S)^{-1}S^+H\right)^{-1}H^+ - P_H S(S^+S)^{-1}S^+ + H(H^+P_S H)^{-1}H^+P_S$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \rightarrow \begin{matrix} Ax + By = u \quad (1) \\ Cx + Dy = v \quad (2) \end{matrix} \overset{(2)}{\Rightarrow} y = D^{-1}(-Cx + v)$$

Sub into (1) $\rightarrow Ax + BD^{-1}(-Cx+v) = u$

$\Rightarrow (A - BD^{-1}C)x = u - BD^{-1}v$

let $S_2 = A - BD^{-1}C$

$\Rightarrow x = S_2^{-1}(u - BD^{-1}v)$

$\Rightarrow y = D^{-1}(v - CS_2^{-1}(u - BD^{-1}v))$

$= D^{-1}v - D^{-1}CS_2^{-1}(u - BD^{-1}v)$

$= D^{-1}v - D^{-1}CS_2^{-1}u + D^{-1}CS_2^{-1}BD^{-1}v$

$= \begin{bmatrix} -D^{-1}CS_2^{-1} & D^{-1} + D^{-1}CS_2^{-1}BD^{-1} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

$x = \begin{bmatrix} S_2^{-1} & -S_2^{-1}BD^{-1} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} S_2^{-1} & -S_2^{-1}BD^{-1} \\ -D^{-1}CS_2^{-1} & D^{-1} + D^{-1}CS_2^{-1}BD^{-1} \end{bmatrix}$$

$\Rightarrow \hat{x} = \left[S_2^{-1}H^+ - S_2^{-1}BD^{-1}S^+ \right] \vec{y}$

$A = H^+H$, $B = H^+S$, $C = S^+H$, $D = S^+S$

$S_2 = A - BD^{-1}C$

$= H^+H - H^+S(S^+S)^{-1}S^+H$

$= H^+H - H^+P_SH$

$\Rightarrow H\hat{x} = \left[H(H^+H - H^+P_SH)^{-1}H^+ - H(H^+H - H^+P_SH)^{-1}BD^{-1}S^+ \right] \vec{y}$

$\Rightarrow \hat{s} = \left[H(H^+(I-P_S)H)^{-1}H^+ - H(H^+(I-P_S)H)^{-1}H^+S(S^+S)^{-1}S^+ \right] \vec{y}$

$= \left[H(H^+P_S^\perp H)^{-1}H^+ - H(H^+P_S^\perp H)^{-1}H^+P_S \right] \vec{y}$

$= \left[H(H^+P_S^\perp H)^{-1}H^+(I - P_S) \right] \vec{y}$

$$\boxed{\hat{s} = \left[\underbrace{H(H^+P_S^\perp H)^{-1}H^+P_S^\perp}_{=E} \right] \vec{y} \qquad \square}$$

HS-Q3] Since $ES = H(H^+P_S^\perp H)^{-1}H^+\underbrace{P_S^\perp S}_{=0}$ and $P_S^\perp S = 0$, $ES = 0$

Geometric Interpretation: Since the ~~components~~ measurements of $y \in \text{span}([H \ S])$ and $y = Hx + \theta S + v$, when performing the LLSE estimation for $\hat{x}$, we are looking for the influence of the measurements that exist in the span(H), which are necessarily due to x.

Continue →

Thus, ~~we~~ since $y \in \underbrace{\mathrm{Span}([H\ S])}_{=Y}$, $Im(S) \subset Im(H^\perp)$

$\cancel{Im(H^\perp) \subset} Im(S) \subset Y$ (b/c $[H\ S]$ full rank)

$\cancel{Im(S^\perp) \subset} Im(H) \subset Y$

$Im(H) \subset Im(S^\perp)$

and we have that any measurement ~~due to~~ in the $Im(S)$ is discarded when estimating $\hat{x}$. That is, $ES = 0$ ~~and ...~~.

#5-Q4

Let $\tilde{s} = s - \hat{s}$ show $\mathbb{E}[\tilde{s}\tilde{s}^+] = EE^+$

$s = Hx$, $\hat{s} = H\hat{x} = E\vec{y}$

$\mathbb{E}[\tilde{s}\tilde{s}^+] = \mathbb{E}[(s-\hat{s})(s-\hat{s})^+] = \mathbb{E}[ss^+ - s\hat{s}^+ - \hat{s}s^+ + \hat{s}\hat{s}^+] =$

$y = Hx + S\theta + v$

$\mathbb{E}[ss^+] = \mathbb{E}[Hxx^+H^+] = HR_xH^+ =$

$\cancel{\mathbb{E}[s\hat{s}^+] = \mathbb{E}[Hx(Ey)^+] = \mathbb{E}[Hxy^+E] = HR_{xy}E}$

$\cancel{\mathbb{E}[\hat{s}s^+] = \mathbb{E}[Ey(Hx)^+] = \mathbb{E}[Eyx^+H] = ER_{yx}H}$

$\cancel{\mathbb{E}[\hat{s}\hat{s}^+] = \mathbb{E}[Eyy^+E^+] = ER_yE^+}$

$R_v = I$

$R_{xv} = 0$ (uncorrelated sensor noise)

$R_{vx} = 0$

$\mathbb{E}[s\hat{s}^+] = \mathbb{E}[Hx(Ey)^+] = \mathbb{E}[Hx(E(Hx + S\theta + v))^+] = \mathbb{E}[Hx(EHx + \cancelto{0}{ES\theta} + Ev)^+]$

$= \mathbb{E}[Hxx^+H^+E^+ + Hxv^+E^+] = HR_xH^+E^+ + H\cancelto{0}{R_{xv}}E^+$

$\mathbb{E}[\hat{s}s^+] = \mathbb{E}[Ey(Hx)^+] = \mathbb{E}[E(Hx + S\theta + v)(Hx)^+] = \mathbb{E}[(EHx + Ev)(Hx)^+]$

$= \mathbb{E}[EHxx^+H^+ + Evx^+H^+] = EHR_xH^+ + E\cancelto{0}{R_{vx}}H^+$

$\mathbb{E}[\hat{s}\hat{s}^+] = \mathbb{E}[Ey(Ey)^+] = \mathbb{E}[E(Hx + S\theta + v)(E(Hx + S\theta + v))^+] = \mathbb{E}[(EHx + Ev)(EHx + Ev)^+]$

$= \mathbb{E}[EHxx^+H^+E^+ + EHxv^+E^+ + Evx^+H^+E^+ + Evv^+E^+]$

$= EHR_xH^+E^+ + EH\cancelto{0}{R_{xv}}E^+ + E\cancelto{0}{R_{vx}}H^+E^+ + ER_vE^+$

$\Rightarrow \mathbb{E}[\tilde{s}\tilde{s}^+] = HR_xH^+ - HR_xH^+E^+ - EHR_xH^+ + EHR_xH^+E^+ + ER_vE^+$

$\rightarrow$

$\mathbb{E}[s s^+] = H R_x H^+$

$\mathbb{E}[s \hat{s}^+] = H R_x H^+ E^+ = \cancel{H R_x H^+}$

<u>note</u>

$$\boxed{E H} = \left(P_H - P_H S (S^+ P_H^\perp S)^{-1} S^+ P_H^\perp\right) H = \underbrace{P_H H}_{H(H^+H)^{-1}H^+H = H} - P_H S (S^+ P_H^\perp S)^{-1} S^+ \cancel{P_H^\perp H}^{\,0} = H$$

$\mathbb{E}[s \hat{s}^+] = H R_x H^+ E^+ = H R_x (E H)^+ = H R_x H^+$

$\mathbb{E}[\hat{s} s^+] = E H R_x H^+ = H R_x H^+$

$\mathbb{E}[\hat{s}\hat{s}^+] = E H R_x (E H)^+ + E \overset{I}{R_v} E^+ = H R_x H^+ + E E^+$

$\Rightarrow \mathbb{E}[\tilde{s}\tilde{s}^+] = \mathbb{E}[s s^+ - s\hat{s}^+ - \hat{s} s^+ + \hat{s}\hat{s}^+]$

$= \cancel{H R_x H^+} - \cancel{H R_x H^+} - \cancel{H R_x H^+} + \cancel{H R_x H^+} + E E^+ = \boxed{E E^+ \quad \square}$

#5-Q5

Now assume x is a zero-mean random variable w/ $\mathbb{E}(x x^+) = R_x = \Pi_0 > 0$

Show the LLMS of $s = H x$ is now $\hat{s} = F y$ w/ $F = P_H [I - S(S^+ P_H^\perp S)^{-1} S^+ P_H^\perp]$

$\hat{s} = H \hat{x}$

$P_H = H(H^+ H + \Pi_0^{-1})^{-1} H^+$

from #5-Q1, if z is <u>not</u> a deterministic ~~...~~ variable, then for $y = H_z z + v$,

$K_o = (R_z^{-1} + H_z^+ R_v^{-1} H_z)^{-1} H_z^+ R_v^{-1} = (R_z^{-1} + H^+ H)^{-1} H^+$, $(R_v = I)$

$\hat{z} = K_o y$

$= (R_z^{-1} + H_z^+ H_z)^{-1} H_z^+ \vec{y}$

$\Rightarrow \hat{y} = H_z \hat{z} + v = H_z (R_z^{-1} + H_z^+ H_z)^{-1} H_z^+ \vec{y} + v$

$\hat{z} = \begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = (R_z^{-1} + H_z^+ H_z)^{-1} H_z^+$, $H_z = [H \;\; S]$, $R_z = \mathbb{E}[z z^+] = \mathbb{E}\left[\begin{pmatrix} x \\ \theta \end{pmatrix} (x^+ \; \theta^+)\right]$

$= \mathbb{E}\begin{bmatrix} x x^+ & x\theta^+ \\ \theta x^+ & \theta\theta^+ \end{bmatrix}$

$\longrightarrow$

#5-Q5 cont'd)

$$R_z = \mathbb{E}\begin{bmatrix} xx^+ & x\theta^+ \\ \theta x^+ & \theta\theta^+ \end{bmatrix} = \begin{bmatrix} R_x & R_{x\theta} \\ R_{\theta x} & R_\theta \end{bmatrix}$$

θ is deterministic

$\Rightarrow R_\theta = \alpha I,\ \alpha \to \infty$

$\Rightarrow R_\theta^{-1} = \frac{1}{\alpha} I = 0$

$$R_z^{-1} = \begin{bmatrix} R_x & R_{x\theta} \\ R_{\theta x} & R_\theta \end{bmatrix}^{-1}$$

Schur Complement of D: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} S_2^{-1} & -S_2^{-1}BD^{-1} \\ -D^{-1}CS_2^{-1} & D^{-1}+D^{-1}CS_2^{-1}BD^{-1} \end{bmatrix}$

$$S_2 = A - BD^{-1}C$$

let $A = R_x$

$B = R_{x\theta}$

$C = R_{\theta x} =$

$D = R_\theta$

$$\Rightarrow R_z^{-1} = \begin{bmatrix} R_x^{-1} & -R_x^{-1}R_{x\theta}\cancelto{0}{R_\theta^{-1}} \\ -\cancelto{0}{R_\theta^{-1}}R_{\theta x}R_x^{-1} & \cancelto{0}{R_\theta^{-1}} + \cancelto{0}{R_\theta^{-1}}R_{\theta x}R_x^{-1}R_{x\theta}\cancelto{0}{R_\theta^{-1}} \end{bmatrix} = \begin{bmatrix} R_x^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$S_2 = R_x - R_{x\theta}\cancelto{0}{R_\theta^{-1}}R_{\theta x}$$

$$= R_x$$

$$\hat{z} = K_0 y = (R_z^{-1} + H_z^+ H_z)^{-1} H_z^+ \vec{y}$$

So, $$\begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = \left[\begin{bmatrix} R_x^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} H^+ \\ S^+ \end{bmatrix}\begin{bmatrix} H & S \end{bmatrix}\right]^{-1}\begin{bmatrix} H^+ \\ S^+ \end{bmatrix}\vec{y}$$

$$= \begin{bmatrix} H^+H + R_x^{-1} & H^+S \\ S^+H & S^+S \end{bmatrix}^{-1}\begin{bmatrix} H^+ \\ S^+ \end{bmatrix}\vec{y}$$

Schur Complement of A: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}+A^{-1}BS_2^{-1}CA^{-1} & -A^{-1}BS_2^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$

let $A = H^+H + R_x^{-1}$

$B = H^+S$

$C = S^+H$

$D = S^+S$

$$S_2 = D - CA^{-1}B$$

$\longrightarrow$

#5-Q5 cont'd

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} (H^+H + R_x^{-1})^{-1}(I + H^+S(S^+S - S^+H(H^+H+R_x^{-1})^{-1}H^+S)^{-1}S^+H(H^+H+R_x^{-1})^{-1}, & -(H^+H + R_x^{-1})^{-1}H^+S(S^+S - S^+H(H^+H+R_x^{-1})^{-1}H^+S)^{-1} \\ -(S^+S - S^+H(H^+H+R_x^{-1})^{-1}H^+S)^{-1}S^+H(H^+H+R_x^{-1})^{-1}, & (S^+S - S^+H(H^+H+R_x^{-1})^{-1}H^+S)^{-1} \end{bmatrix} \begin{bmatrix} H^+ \\ S^+ \end{bmatrix} \vec{y}$$

Let $P_H = H(H^+H + R_x^{-1})^{-1}H^+$

$$\Rightarrow H\hat{x} = \left[H(H^+H+R_x^{-1})^{-1}H^+ + H(H^+H+R_x^{-1})^{-1}H^+S(S^+S - S^+P_H S)^{-1}S^+H(H^+H+R_x^{-1})^{-1}H^+ - H(H^+H+R_x^{-1})^{-1}H^+S(S^+S-S^+P_H S)^{-1}S^+\right]\vec{y}$$

$$= \left[P_H + P_H S(S^+S - S^+P_H S)^{-1}S^+P_H - P_H S(S^+S - S^+P_H S)^{-1}S^+\right]\vec{y}$$

$$= P_H\left[I + S(S^+S - S^+P_H S)^{-1}S^+P_H - S(S^+S-S^+P_H S)^{-1}S^+\right]\vec{y}$$

$$= P_H\left[I - (S(\underbrace{S^+S - S^+P_H S}_{S^+(I-P_H)S})^{-1}S^+)(I - P_H)\right]\vec{y}$$

$$= P_H\left[I - (S(S^+P_H^{\perp}S)^{-1})P_H^{\perp}\right]\vec{y}$$

$$\boxed{\hat{s} = H\hat{x} = P_H\left[I - S(S^+P_H^{\perp}S)^{-1}P_H^{\perp}\right]\vec{y}}, \quad P_H = H(H^+H + R_x^{-1})^{-1}H^+ = H(H^+H + \Pi_0^{-1})^{-1}H^+$$

$$\Rightarrow \hat{s} = H\hat{x} = Fy, \quad F = P_H\left[I - S(S^+P_H^{\perp}S)^{-1}P_H^{\perp}\right]$$

$\square$

#5-Q6

Schur Complement of D: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} S_2^{-1} & -S_2^{-1} B D^{-1} \\ -D^{-1} C S_2^{-1} & D^{-1} + D^{-1} C S_2^{-1} B D^{-1} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} H^+H + R_x^{-1} & H^+S \\ S^+H & S^+S \end{bmatrix}^{-1} \begin{bmatrix} H^+ \\ S^+ \end{bmatrix} \vec{y}$$

let $A = H^+H + R_x^{-1}$, $B = H^+S$, $C = S^+H$, $D = S^+S$, $S_2 = A - BD^{-1}C$ ~~$= D - CA^{-1}B$~~

let $P_s = S(S^+S)^{-1}S^+$

$$\Rightarrow \hat{x} = [S_2^{-1} H^+ - S_2^{-1} B D^{-1} S^+] \vec{y}$$

$$H\hat{x} = [H S_2^{-1} H^+ - H S_2^{-1} B D^{-1} S^+] \vec{y}$$

$$= [H(\underbrace{\cancel{D - CA^{-1}B}}_{S_2})^{-1} H^+ - H(\underbrace{\cancel{D - CA^{-1}B}}_{S_2})^{-1} H^+S(S^+S)^{-1}S^+] \vec{y}$$

$S_2^{-1} = (\cancel{S^+S - S^+H(H^+H + R_x^{-1})^{-1} H^+S})^{-1}$

$= H^+H + R_x^{-1} - H^+S(S^+S)^{-1}S^+H$

$= ((H^+H + R_x^{-1}) - H^+P_s H)^{-1}$

$$= [H((H^+H + R_x^{-1}) - H^+P_sH)^{-1}H^+ - H((H^+H + R_x^{-1}) - H^+P_s H)^{-1} H^+ P_s] \vec{y}$$

$$= [H(H^+(I - P_s)H + R_x^{-1})^{-1} H^+ - H(H^+(I - P_s)H + R_x^{-1})^{-1} H^+ P_s] \vec{y}$$

$$= [H(H^+P_s^\perp H + R_x^{-1})^{-1} H^+ - H(H^+P_s^\perp H + R_x^{-1})^{-1} H^+ P_s] \vec{y}$$

$$\hat{s} = H\hat{x} = \underbrace{H(H^+P_s^\perp H + R_x^{-1})^{-1} H^+ P_s^\perp}_{F} \vec{y}$$

$$\boxed{\Rightarrow FS = H(H^+P_s^\perp H + R_x^{-1})^{-1} H^+ P_s^\perp S, \quad P_s^\perp S = 0 \Rightarrow FS = 0 \quad \square}$$

#5-Q7 $\quad \tilde{s} = s - \hat{s}$, $\hat{s} = Fy$

MMSE: $\mathbb{E}[\tilde{s}\tilde{s}^+] = \mathbb{E}[(s-\hat{s})(s-\hat{s})^+] = \mathbb{E}[ss^+ - s\hat{s}^+ - \hat{s}s^+ + \hat{s}\hat{s}^+]$

$\mathbb{E}[ss^+] = \mathbb{E}[Hxx^+H^+] = HR_xH^+$

$\mathbb{E}[s\hat{s}^+] = \mathbb{E}[Hx(Fy)^+] = \cancel{HR_{xy}F^+} = \mathbb{E}[Hx(F(Hx+S\theta+v))^+]$

$= \mathbb{E}[Hx(FHx + \overset{0}{\cancel{FS\theta}} + Fv)^+] = \mathbb{E}[Hx(FHx)^+] + \mathbb{E}[Hxv^+F]$

Note: $FH = P_H[I - S(S^+P_H^\perp S)^{-1}P_H^\perp]H$

$= [P_H - P_H S(S^+P_H^\perp S)^{-1}P_H^\perp]H$

$= P_H H - P_H S(S^+P_H^\perp S)^{-1}\cancel{P_H^\perp H}$

$=$

$P_H H = H(H^+H + \pi_0^{-1})^{-1}H^+H$

$= H(H^+H)^{-1}H^+H + H\pi_0 H^+H$

$= H + H\pi_0 H^+H = H(I + \pi_0 H^+H)$

$= (I + H\pi_0 H^+)H$

$P_H^\perp H = (I - P_H)H$

$= H - (I + H\pi_0 H^+)H$

$= (I - (I + H\pi_0 H^+))H$

$= (I - I - H\pi_0 H^+)H = -H\pi_0 H^+H$

$\Rightarrow FH = P_H H - P_H S(S^+P_H^\perp S)^{-1}P_H^\perp H$

$= (I + H\pi_0 H^+)H - P_H S(S^+P_H^\perp S)^{-1}(-H\pi_0 H^+H)$

or... $FH = H(H^+P_S^+H + R_x^{-1})^{-1}H^+(I - P_S)H$

$= H(H^+P_S^+H + R_x^{-1})^{-1}H^+H$

$= H(H^+P_S^+H)^{-1}H^+H + H(R_x)H^+H$

$P_S H = 0$ b/c $Im(S) \subset Im(H^\perp)$ ~~$Im(H^\perp) \subset Im(S)$~~

~~$Im(S^\perp)$~~

$Im(H) \subset Im(S^\perp)$

$\mathbb{E}[s\hat{s}^+] = \mathbb{E}[Hxx^+(FH)^+] + H\overset{0}{\cancel{R_{xv}}}F = HR_x(FH)^+$

$\mathbb{E}[\hat{s}s^+] = \mathbb{E}[Fy(Hx)^+] = \mathbb{E}[F(Hx + S\theta + v)(Hx)^+] = \mathbb{E}[(FHx + \overset{0}{\cancel{FS\theta}} + Fv)(Hx)^+]$

$= \mathbb{E}[FHxx^+H^+] + \mathbb{E}[Fvx^+H^+]$

$= FHR_xH^+ + F\overset{0}{\cancel{R_{vx}}}H^+$ $\rightarrow$

#5-Q7 cont'd

$$\mathbb{E}[\hat{s}\hat{s}^+] = \mathbb{E}[F_y(F_y)^+] = \mathbb{E}[(FHx + FS\cancel{\theta}^{0} + Fv)(FHx + FS\cancel{\theta}^{0} + Fv)^+]$$

$$= \mathbb{E}[FHxx^+(FH)^+ + FHxv^+F^+ + Fvx^+(FH)^+ + Fvv^+F^+]$$

$$= FHR_x(FH)^+ + FH\cancel{R_{xv}}^{0}F^+ + F\cancel{R_{vx}}^{0}(FH)^+ + FR_vF^+ =$$

$$\Rightarrow \mathbb{E}[\tilde{s}\tilde{s}^+] = \mathbb{E}[ss^+] - \mathbb{E}[s\hat{s}^+] - \mathbb{E}[\hat{s}s^+] + \mathbb{E}[\hat{s}\hat{s}^+]$$

$$= HR_xH^+ - HR_x(FH)^+ - FHR_xH^+ + FHR_x(FH)^+ + \underbrace{FR_vF^+}_{FF^+}$$

Now, the expectation $\mathbb{E}[\tilde{s}\tilde{s}^+]$ depends on the covariance of the random variable x. (i.e. $R_x = \mathbb{E}(xx^+)$)

#6 General Combined estimator

$y_1 = H_1 x + v_1$, $y_2 = H_2 x + v_2$

$$\left\langle \begin{bmatrix} v_1 \\ x \end{bmatrix}, \begin{bmatrix} v_1 \\ x \end{bmatrix} \right\rangle = \left\| \begin{bmatrix} v_1 \\ x \end{bmatrix} \right\|^2 = \begin{bmatrix} R_1 & 0 \\ 0 & M_1 \end{bmatrix} , \left\langle \begin{bmatrix} v_2 \\ x \end{bmatrix}, \begin{bmatrix} v_2 \\ x \end{bmatrix} \right\rangle = \begin{bmatrix} R_2 & 0 \\ 0 & M_2 \end{bmatrix}$$

i.e. covariance matrices →

$$\hat{x}_1 = R_{xy_1} R_{y_1}^{-1} y_1 = K_1 y_1 = R_{x_1} H_1^T (H_1 R_{x_1} H_1^T + R_{v_1})^{-1} y$$

$$= \underbrace{M_1 H_1^T (H_1 M_1 H_1^T + R_1)^{-1}}_{K_1} y$$

, per slide 3 of Lecture 7 for Linear Models of form $y = Hx + v$

$$P_1 = M_1 - M_1 H_1^T (H_1 M_1 H_1^T + R_1)^{-1} H_1 R_1 = (M_1^{-1} + H_1^T R_1^{-1} H_1)^{-1} \quad \text{(same slide)}$$

$$K_1 = M_1 H_1^T (H_1 M_1 H_1^T + R_1)^{-1} = (M_1^{-1} + H_1^T R_1^{-1} H_1)^{-1} H_1^T R_1^{-1}$$

$$\hat{x}_1 = K_1 y_1$$

Similarly, $P_2 =$ " with sub-scripts of 2

$K_2 =$ "

$\hat{x}_2 = K_2 y_2$

Joint estimate:

$$y = Hx + v \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} x + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

w/ $R_x = \Pi$, $R_v = \begin{pmatrix} R_{v_1} & \cancel{R_{v_1 v_2}}^0 \\ \cancel{R_{v_2 v_1}}^0 & R_{v_2} \end{pmatrix} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$

per Lect. 8.

per Lec. 7.

$$\Rightarrow P = (R_x^{-1} + H^T R_v^{-1} H)^{-1}$$

$$K = (R_x^{-1} + H^T R_v^{-1} H)^{-1} H^T R_v^{-1}$$

$$x = Ky$$

→

#6 cont'd

$$P^{-1}\hat{x} = (\Pi^{-1} + H^T R_v^{-1} H)(Ky) = H^T R_v^{-1} y \qquad \text{(per Lec. 7 slide 3)}$$

$$= \left(\Pi^{-1} + [H_1^T \;\; H_2^T]\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}\right)\left(\Pi^{-1} + [H_1^T \;\; H_2^T]\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}\right)^{-1}[H_1^T \;\; H_2^T]\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}^{-1}$$

$$= \left(\Pi^{-1} + [H_1^T \;\; H_2^T]\begin{bmatrix} R_1 H_1 \\ R_2 H_2 \end{bmatrix}\right)\left(\Pi^{-1} + [H_1^T \;\; H_2^T]\begin{bmatrix} R_1 H_1 \\ R_2 H_2 \end{bmatrix}\right)^{-1}[H_1^T \;\; H_2^T]\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}^{-1}\vec{y}$$

$$= (\Pi^{-1} + H_1^T R_1 H_1 + H_2^T R_2 H_2)(\Pi^{-1} + H_1^T R_1 H_1 + H_2^T R_2 H_2)^{-1}[H_1^T \;\; H_2^T]\begin{pmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{pmatrix} y$$

$$= [H_1^T R_1^{-1} \;\; H_2^T R_2^{-1}]\begin{pmatrix} \vec{y}_1 \\ y_2 \end{pmatrix} = H^T R_1^{-1}\vec{y}_1 + H_2^T R_2^{-1}\vec{y}_2$$

$$P_1^{-1}\hat{x}_1 = H_1^T R_1^{-1}\vec{y}_1$$

$$P_2^{-1}\hat{x}_2 = H_2^T R_2^{-1}\vec{y}_2$$

$$\Rightarrow \boxed{P^{-1}\hat{x} = H_1^T R_1^{-1} y_1 + H_2^T R_2^{-1}\vec{y}_2 = P_1^{-1}\hat{x}_1 + P_2^{-1}\hat{x}_2} \quad \square$$

$$P^{-1} = \Pi^{-1} + H^T R_v^{-1} H = \Pi^{-1} + H_1^T R_1 H_1 + H_2^T R_2 H_2$$

$$P_1^{-1} = M_1^{-1} + H_1^T R_1^{-1} H_1$$

$$P_2^{-1} = M_2^{-1} + H_2^T R_2^{-1} H_2$$

$$\boxed{P_1^{-1} + P_2^{-1} + \Pi^{-1} - M_1^{-1} - M_2^{-1} = H_1^T R_1^{-1} H_1 + H_2^T R_2^{-1} H_2 + \Pi^{-1} = P^{-1}} \quad \square$$

#7 Optimal Estimation for exponential distribution

$y = x + v$, x, v are independent real-valued random variables w/ exponential distribution of parameters $\lambda > 0$ and $\mu > 0$ respectively

exponential distribution: of form $\begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & x < 0 \end{cases}$, mean $= \lambda^{-1}$, variance $= \lambda^{-2}$

$p(x) = \lambda e^{-\lambda x}$, $x \geq 0$

$p(v) = \mu e^{-\mu v}$, $v \geq 0$

#7-Q1 → next page (7-2)

#7-Q2 The ~~sum~~ pdf of a sum of two indep. random var. is the convolution of their individual pdf's :

$\because$ $y = x + v$

$$p_y(y) = \int_{-\infty}^{\infty} p_x(\tau)\, p_v(y-\tau)\, d\tau$$

$$= \int_0^y \lambda e^{-\lambda\tau} \mu e^{-\mu(y-\tau)}\, d\tau$$

$$= \lambda\mu e^{-\mu y} \int_0^y e^{-\lambda\tau} e^{\mu\tau}\, d\tau = \lambda\mu e^{-\mu y} \int_0^y e^{(\mu-\lambda)\tau}\, d\tau$$

$$= \lambda\mu e^{-\mu y}\left[\frac{e^{(\mu-\lambda)y}}{(\mu-\lambda)} - \frac{e^{(\mu-\lambda)0}}{\mu-\lambda}\right]$$

$$= \left[\frac{\lambda\mu e^{\mu y}}{e^{\mu y} e^{\lambda y}} - \frac{\lambda\mu e^{\mu y}}{e^{\mu y}}\right] \Big/ (\mu-\lambda) = \lambda\mu\left[\frac{e^{\mu y} - e^{\lambda y}}{e^{\lambda y} e^{\mu y}}\right]\left(\frac{1}{\mu-\lambda}\right)$$

$$= \frac{\lambda\mu}{\mu-\lambda}\left[\frac{e^{\mu y} - e^{\lambda y}}{e^{\lambda y} e^{\mu y}}\right] = \frac{\lambda\mu}{\mu-\lambda}\left(e^{-\lambda y} - e^{-\mu y}\right)$$

$$\boxed{= \frac{\lambda\mu}{\lambda-\mu}\left(e^{-\mu y} - e^{-\lambda y}\right)} \quad , y > 0$$

#7 - Q1

~~assume Gaussian distributions.~~

from https://faculty.math.illinois.edu/~r-ash/Stat/StatLec1-5.pdf , pg 7

if X, Y are independent random variables w/ pdf's f_X, f_Y

let $Z = XY$, $W = Y$

note $x>0, y>0 \Leftrightarrow z>0, w>0$

(Jacobian)

then $f_{ZW}(z,w) = \dfrac{f_{XY}(x,y)}{\left|\frac{\partial(z,w)}{\partial(x,y)}\right|}$, $\left|\dfrac{\partial(z,w)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y = w$

$\Rightarrow f_{ZW}(z,w) = \dfrac{f_{XY}(x,y)}{w} = \dfrac{f_X(x) f_X(y)}{w} = \dfrac{f_X(z/w)\, f_Y(w)}{w}$

also, $f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\, dw = \int_0^\infty \frac{1}{w} f_X(z/w)\, f_Y(w)\, dw$

marginal density from joint density

---

for our case, we want $p(x,y)$, the joint density of x and y. $y = x + v$

~~so, let $Z = XV$, $W = V$~~

given $Y = X + V$, let $W = X$ $\rightarrow$ $V = Y - X$ $\Rightarrow$ $\boxed{y \geq x}$ b/c $v \geq 0$

then ~~$f_{XY}(x,y) = f_{XV}(x,v)$~~

~~$f_{YW}(y,w) = f_{YX}(y,x) = f_{XY}(x,y) = \dfrac{f_{XV}(x,v)}{\left|\frac{\partial(y,w)}{\partial(x,v)}\right|}$, $\left|\dfrac{\partial(y,w)}{\partial(x,v)}\right| = \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial v} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial v} \end{vmatrix}$~~

$f_{XY}(x,y) = \dfrac{f_{XV}(x,v)}{\left|\frac{\partial(x,y)}{\partial(x,v)}\right|}$, $\left|\dfrac{\partial(x,y)}{\partial(x,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

$f_{X,Y}(x,y) = f_{XV}(x,v) = f_X(x) f_V(y-x) = \lambda e^{-\lambda x} \mu e^{-\mu(y-x)} = \lambda\mu e^{-(\lambda-\mu)x} e^{-\mu y}$

$\boxed{f_{XY}(x,y) = \lambda\mu e^{-(\lambda-\mu)x} e^{-\mu y}}$ $\begin{cases} x \leq y \\ 0 \text{ else} \end{cases}$

#7-~~Q3~~

(non-linear)

Q4 Show optimal, least mean square estimate of x given y is

$$\cancel{\hat{x} = \frac{1}{\lambda-\mu} - \left(\frac{e^{-\lambda y}}{e^{-\mu y}-e^{-\lambda y}}\right) y}$$

→ instead, performed calc for LLSE (#7-Q4)

From class, $\hat{x} = \mathbb{E}(x|y) = R_{xy} R_y^{-1} y$ (for centered variables) | we have affine estimation: $\hat{x} = \mu_x + R_{xy} R_y^{-1}(y-\mu_y)$

$$R_{xy} = \cancel{\mathbb{E}(xy^T)} = \cancel{\mathbb{E}(x(x+v)^T) = \mathbb{E}(xx^T) + \mathbb{E}(xv^T)}$$ (not 0-mean!)

$$= \mathbb{E}\left[(x-E(x))(y-E(y))^T\right] = E\left[xy^T - xE(y)^T - E(x)y^T + E(x)E(y)^T\right]$$

note: $\mathbb{E}(y) = \int_0^\infty y p_y(y)\, dy = \frac{\lambda\mu}{\lambda-\mu}\int_0^\infty y(e^{-\mu y} - e^{-\lambda y})dy$

$$= \frac{\lambda\mu}{\lambda-\mu}\left[\int_0^\infty y e^{-\mu y} dy - \int_0^\infty y e^{-\lambda y} dy\right]$$

similar to $\mathbb{E}[x]$ or $\mathbb{E}[v]$

$$= \frac{\lambda\mu}{\lambda-\mu}\left[\frac{1}{\mu^2} - \frac{1}{\lambda^2}\right]$$

$$= \frac{1}{\lambda-\mu}\left[\frac{\lambda}{\mu} - \frac{\mu}{\lambda}\right] = \frac{1}{\lambda-\mu}\left[\frac{\lambda^2-\mu^2}{\lambda\mu}\right] = \boxed{\frac{\lambda^2-\mu^2}{(\lambda-\mu)\lambda\mu}}$$

note 2: $\mathbb{E}[x] = \int_0^\infty x p_x(x) dx$

$$= \int_0^\infty x\lambda e^{-\lambda x} dx$$

$$= \frac{1}{\lambda}$$

$\Rightarrow$ ~~$\mathbb{E}[xy] = \int_0^\infty \int_0^\infty$~~

$$\Rightarrow R_{xy} = \mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y] - \cancel{\mathbb{E}[x]\mathbb{E}[y]} + \cancel{\mathbb{E}[x]\mathbb{E}[y]} = \mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y]$$

$$R_y = \mathbb{E}\left[(y-\mathbb{E}(y))^2\right] = \mathbb{E}[y^2] - \mathbb{E}(y)^2 \cancel{- \mathbb{E}(y)^2 + \mathbb{E}(y)^2}$$

$$R_y^{-1} = \frac{1}{R_y} \text{ (scalar)} \Rightarrow \frac{1}{\mathbb{E}(y^2) - \mathbb{E}(y)^2}$$

$$\Rightarrow R_{xy}R_y^{-1} = \frac{\mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y]}{\mathbb{E}(y^2) - \mathbb{E}(y)^2} = \frac{\mathbb{E}(xy)}{\mathbb{E}(y^2) - \mathbb{E}(y)^2} - \frac{\mathbb{E}(x)\mathbb{E}(y)}{\mathbb{E}(y^2) - \mathbb{E}(y)^2}$$

→

$\mathbb{E}[xy] = \int_0^\infty \int_0^\infty xy \, p_{xy}(x,y)\,dx\,dy$ note that $p_{XY}(x,y) = 0$ for $x \geq y$

$= \int_0^\infty \int_0^\infty xy(\lambda\mu e^{-(\lambda-\mu)x} e^{-\mu y})\,dx\,dy$

~~$= \mathbb{E}(xg(y)) = \mathbb{E}_y[\mathbb{E}[x|y]] = \mathbb{E}_y(\mathbb{E}_{xy}(xg(y)|y)) = \mathbb{E}_y(g(y)\mathbb{E}_{x|y}(x|y))$~~

(let $g(y) = y$)

$= \int_0^\infty \int_0^y xy(\lambda\mu e^{-(\lambda-\mu)x} e^{-\mu y})\,dx\,dy$

$= \int_0^\infty \lambda\mu y e^{-\mu y}\left(\int_0^y x e^{-(\lambda-\mu)x}\,dx\right)dy$

$= \int_0^\infty \lambda\mu y e^{-\mu y}[$

Integrate by parts:

$\int F\,dg = Fg - \int g\,dF$

let $F = x$, $dg = e^{-(\lambda-\mu)x}\,dx$

$dF = dx$, $g = \frac{e^{-(\lambda-\mu)x}}{-(\lambda-\mu)}$

$\Rightarrow \frac{-xe^{-(\lambda-\mu)x}}{(\lambda-\mu)} + \int_0^y \frac{e^{-(\lambda-\mu)x}}{(\lambda-\mu)}\,dx$

$= \left(\frac{1}{\lambda-\mu}\right)\left[-xe^{-(\lambda-\mu)x} + \left[\frac{e^{-(\lambda-\mu)x}}{-(\lambda-\mu)}\right]\right]_0^y$

$= \left(\frac{-1}{\lambda-\mu}\right)\left[ye^{-(\lambda-\mu)y} + \left[\frac{e^{-(\lambda-\mu)y} - 1}{(\lambda-\mu)}\right]\right]$

$= \left(\frac{-1}{(\lambda-\mu)^2}\right)\left[ye^{-(\lambda-\mu)y}(\lambda-\mu) + e^{-(\lambda-\mu)y} - 1\right]$

$= \frac{\mp 1}{(\lambda-\mu)^2}\left[e^{-(\lambda-\mu)y}(y(\lambda-\mu) \pm 1) \mp 1\right]$

$=$

$\longrightarrow$

7-4

$$\Rightarrow \mathbb{E}[xy] = \int_0^\infty \lambda\mu y e^{-\mu y}\left[\left(\frac{-1}{(\lambda-\mu)^2}\right)\left[y e^{-(\lambda-\mu)y}(\lambda-\mu) + e^{-(\lambda-\mu)y} - 1\right]\right] dy$$

$$= \frac{-\lambda\mu}{(\lambda-\mu)^2}\int_0^\infty y e^{-\mu y}\left[y e^{-(\lambda-\mu)y}(\lambda-\mu) + e^{-(\lambda-\mu)y} - 1\right] dy$$

$$= \left(\frac{-\lambda\mu}{(\lambda-\mu)^2}\right)\Big[\underbrace{\int_0^\infty y^2 e^{-\mu y} e^{-(\lambda-\mu)y}(\lambda-\mu)\,dy + \int_0^\infty y e^{-\mu y} e^{-(\lambda-\mu)y}\,dy - \int_0^\infty y e^{-\mu y}\,dy}\Big]$$

$$\Downarrow$$

$$(\lambda-\mu)\int_0^\infty y^2 \underbrace{e^{-\mu y} e^{-(\lambda-\mu)y}}_{e^{-\mu y-(\lambda-\mu)y}}\,dy + \int_0^\infty y e^{-\mu y-(\lambda-\mu)y}\,dy - \int_0^\infty y e^{-\mu y}\,dy$$

$$-\mu y - (\lambda-\mu)y = \cancel{-\mu y} - \lambda y + \cancel{\mu y} = -\lambda y$$

$$\Rightarrow (\lambda-\mu)\int_0^\infty y^2 e^{-\lambda y}\,dy + \int_0^\infty y e^{-\lambda y}\,dy - \int_0^\infty y e^{-\mu y}\,dy$$

$$\Rightarrow (\lambda-\mu)\left(\frac{2}{\lambda^3}\right) + \frac{1}{\lambda^2} - \frac{1}{\mu^2}$$

$$\Rightarrow \boxed{\mathbb{E}[xy] = \left(\frac{-\lambda\mu}{(\lambda-\mu)^2}\right)\left(\frac{2(\lambda-\mu)}{\lambda^3} + \frac{1}{\lambda^2} - \frac{1}{\mu^2}\right)}$$

✓ Checked w/ MATLAB

$$\mathbb{E}[y^2] = \int_0^\infty y^2 p_y(y)\,dy = \int_0^\infty y^2\left(\frac{\lambda\mu}{\lambda-\mu}\right)\left(e^{-\mu y} - e^{-\lambda y}\right)dy$$

$$\boxed{\mathbb{E}[y^2] = \left(\frac{\lambda\mu}{\lambda-\mu}\right)\left(\frac{2}{\mu^3} - \frac{2}{\lambda^3}\right)}$$

$$\mathbb{E}[x] = \frac{1}{\lambda} = \mu_x,$$

$$\Rightarrow \hat{x} = \mu_x + R_{xy}R_y^{-1}(y - \mu_y)$$

$$= \frac{1}{\lambda} + \frac{\left(\mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y]\right)}{\left(\mathbb{E}[y^2] - \mathbb{E}[y]^2\right)}\left(y - \mathbb{E}[y]\right) \longrightarrow$$