ECE 6555 HW#2

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#1 $\vec{y} = H\vec{x} + \vec{v}$, R_x, R_v $R_{xv} = 0$ consider $\vec{z}$ where $R_{zv} = 0$, $R_{zx} \neq 0$

$\hat{z}_{|x}$ = LLMS estimate of $\vec{z}$ given $\vec{x}$

$\widehat{\hat{z}_{|x}}$ = LLMS of $\hat{z}_{|x}$ given $\vec{y}$

$\hat{z}_{|y}$ = LLMS of $\vec{z}$ given $\vec{y}$

Show that $\hat{z}_y = \widehat{\hat{z}_{|x}}$

$\cancel{R_y \to 0}$
$\cancel{R \to 0}$
$\cancel{R \to 0}$
$\cancel{R_y \to 0}$

let $\hat{z}_{|x} = K_x\vec{x} = E[z|x] = R_{zx}R_x^{-1}\vec{x}$

let $\widehat{\hat{z}_{|x}} = K_x\vec{x}_{|y} = E_y[\cancel{E[z|x]|y}] = E[\cancel{R_{zx}R_x^{-1}\vec{x}}|y] = E[\cancel{R_{zx}R_x^{-1}\vec{x}\vec{y}^T}] = R_{\hat{z}_{|x}y}R_y^{-1}y$

let $\hat{z}_{|y} = K_y\vec{y} = E[z|y] = R_{zy}R_y^{-1}y$

$R_{\hat{z}_{|x}y} \cancel{\hat{z}_{|x}} = E[R_{zx}R_x^{-1}\vec{x}\vec{y}^T] = R_{zx}R_x^{-1}E(\vec{x}\vec{y}^T)$ since R_{zx} and R_x are constant matrices. i.e.:

$= R_{zx}R_x^{-1}R_{xy}$

$R_{zx} = E[\vec{z}\vec{x}^T]$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{z}\vec{x}^T p(z,x)\, d\vec{z}\, d\vec{x}$

$R_x = \int_{-\infty}^{\infty} \vec{x}\vec{x}^T p(x)\, d\vec{x}$

and $\hat{z}_{|y} = E[z|y] = \cancel{E[\vec{z}\vec{y}^T]} = R_{zy}R_y^{-1}y$

$R_{zy} = E[\vec{z}\vec{y}^T] = E[\vec{z}(Hx+v)^T] = E[zx^TH^T + zv^T]$

$= R_{zx}H^T + \cancel{R_{zv}} \to 0$

$R_y = E[yy^T] = E[(Hx+v)(Hx+v)^T] = E[Hxx^TH^T + Hxv^T + vx^TH^T + vv^T]$

$= HR_xH^T + H\cancel{R_{xv}}^{0} + \cancel{R_{vx}}^{0}H^T + R_v$ $\boxed{ok}$

$+R_v$

$= HR_xH^T + R_v$

$\Rightarrow R_{zy}R_y^{-1}y = [R_{zx}H^T](HR_xH^T + R_v)^{-1}(Hx+v)$ ⟶

#1 cont'd

$$\hat{z}_{|y} = R_{zy} R_y^{-1} y = [R_{zx} H^T](H R_x H^T + R_v)^{-1}(Hx + v)$$
$$= R_{zx} H^T ((H R_x H^T)^{-1} + R_v^{-1})(Hx + v)$$
$$= [R_{zx} H^T (H^T)^{-1} R_x^{-1} H^{-1} + R_{zx} H^T R_v^{-1}](Hx + v)$$
$$= R_{zx} R_x^{-1} H^{-1} H x + R_{zx} R_x^{-1} H^{-1} v + R_{zx} H^T R_v^{-1} H x + R_{zx} H^T R_v^{-1} v$$
$$= R_{zx}(R_x^{-1} x + R_x^{-1} H^{-1} v + H^T R_v^{-1} H x + H^T R_v^{-1} v)$$

$$\hat{\hat{z}}_{|x} = R_{zx} R_x^{-1} \mathbb{E}(\vec{x}\vec{y}^T) = R_{zx} R_x^{-1} \mathbb{E}[\vec{x}(Hx+v)^T] = R_{zx} R_x^{-1}(R_x H^T + \cancelto{0}{R_{xv}})$$
$$= R_{zx} R_x^{-1} R_x H^T = R_{zx} H^T$$

$\hat{z}_{|x} = R_{zx} R_x^{-1} \vec{x}$ need $R_{zy} = R_{\hat{z}_{|x} y}$

$\hat{z}_{|y} = R_{zy} R_y^{-1} \vec{y}$ so: $R_{zy} = \mathbb{E}[z y^T] = \mathbb{E}[z(Hx+v)^T] = R_{zx} H^T + \cancelto{0}{R_{zv}} = R_{zx} H^T$

$\hat{\hat{z}}_{|x} = R_{\hat{z}_{|x} y} R_y^{-1} \vec{y}$

$$R_{\hat{z}_{|x} y} = \mathbb{E}[\hat{z}_{|x} y^T] = \mathbb{E}[R_{zx} R_x^{-1} \vec{x} y^T]$$, R_{zx} constant, R_x^{-1} constant

$$= R_{zx} R_x^{-1} R_{xy}$$
$$= R_{zx} R_x^{-1} \mathbb{E}[x(Hx+v)^T]$$
$$= R_{zx} R_x^{-1}(R_x H^T + \cancelto{0}{R_{xv}})$$
$$= R_{zx} H^T$$

$$\boxed{\text{Then } R_{zy} = R_{zx} H^T = R_{\hat{z}_{|x} y} \Rightarrow \hat{\hat{z}}_{|x} = R_{zx} H^T R_y^{-1} y = \hat{z}_{|y}}$$

#2

noisy measurement: $\vec{y} = (\cancel{\ \ })\vec{x}$ $\quad$ x, v are 0 mean, independent

$= (1+v)\vec{x}$ $\quad$ variance of v is $\sigma^2 \Rightarrow E(v^2) = \sigma_v^2$

1) LLMS of $\vec{x}$ given $\vec{y}$ $\qquad$ 2) Show that MMSE $< E(\vec{x}\vec{x}^T)$

$$\hat{x} = E(\vec{x}|\vec{y}) = R_{xy} R_y^{-1} \vec{y}$$

$\Rightarrow$ LLSE form: $\hat{x} = K_o \vec{y}$

need to find R_y^{-1}: $R_y = \mathbb{E}(\vec{y}\vec{y}^T) = \mathbb{E}((1+v)\vec{x}\vec{x}^T(1+v)^T)$

$$R_{xy} = \mathbb{E}(x y^T) = \mathbb{E}(\vec{x}\vec{x}^T \underbrace{(1+v)^T}_{\text{scalar}}) = R_x + E(\vec{x}\vec{x}^T v)$$

Note: if two random variables are un-correlated, then

$$Cov(\vec{x}\vec{y}) = \cancel{\mathbb{E}(\vec{x}\vec{y})} = \cancel{\mathbb{E}((\vec{x}-E(\vec{x}))(\vec{y}-E(\vec{y}))^T) = 0}$$

$$= \cancel{\mathbb{E}[\vec{x}\vec{y}^T - E(\vec{x})\vec{y}^T - \vec{x}E(\vec{y})^T + E(\vec{x})E(\vec{y})^T]} = \mathbb{E}(\vec{x}\vec{y}^T) - \mathbb{E}(\underbrace{E(\vec{x})}_{\text{constant so pull out}}\vec{y}^T) - E(\vec{x}E(\vec{y})^T) + E(\vec{x})E(\vec{y})^T = 0$$

$- \cancel{E(\vec{x})E(\vec{y}^T)} - E(\vec{x})E(\vec{y})^T + \cancel{E(\ )} = 0$ ✓

(if uncorrelated $= 0$)

$\Rightarrow \mathbb{E}(\vec{x}\vec{y}^T) = E(\vec{x})E(\vec{y})^T$

from wikipedia:

$$R_{x} = \mathbb{E}[(\vec{x}-\vec{\mu}_x)(\vec{x}-\vec{\mu}_x)^T] = \mathbb{E}[\vec{x}\vec{x}^T] - \vec{\mu}_x \vec{\mu}_x^T$$

$$R_{xy} = \mathbb{E}[(\vec{x}-\vec{\mu}_x)(\vec{y}-\vec{\mu}_y)^T] = \mathbb{E}[\vec{x}\vec{y}^T] - \vec{\mu}_x \vec{\mu}_y^T$$

$$R_y^{-1} \Rightarrow R_y = \mathbb{E}(\vec{y}\vec{y}^T) = \mathbb{E}[(1+v)\vec{x}\vec{x}^T(1+v)^T] = \mathbb{E}[(\vec{x}\vec{x}^T + v\vec{x}\vec{x}^T)(1+v)^T]$$

$$= \mathbb{E}[\vec{x}\vec{x}^T + \vec{x}\vec{x}^T v^T + v\vec{x}\vec{x}^T + v\vec{x}\vec{x}^T v^T]$$

$$= \mathbb{E}(\vec{x}\vec{x}^T) + \mathbb{E}(\vec{x}\vec{x}^T v) + \mathbb{E}(v\vec{x}\vec{x}^T) + \mathbb{E}(v\vec{x}\vec{x}^T v)$$

$\vec{x}, v$ uncorrelated... $\Rightarrow$

$$= R_x + \mathbb{E}(\vec{x}\vec{x}^T)\cancelto{0}{\mathbb{E}(v)} + \cancelto{0}{\mathbb{E}(v)}\mathbb{E}(\vec{x}\vec{x}^T) + \mathbb{E}(v)\mathbb{E}(\vec{x}\vec{x}^T)\mathbb{E}(v) \; ?$$

$$= R_x + \mathbb{E}(\underbrace{v\vec{x}\vec{x}^T v}_{\text{scalars...}}) = R_x + \mathbb{E}(\vec{x}\vec{x}^T)\mathbb{E}(v^2)$$

$$= R_x + R_x\sigma_v^2 = R_x(1+\sigma_v^2)$$

$$\Rightarrow R_y^{-1} = (R_x + R_x\sigma_v^2)^{-1} = \cancel{R_x^{-1} + \frac{1}{\sigma_v^2}R_x^{-1} = R_x^{-1}\left(1+\frac{1}{\sigma_v^2}\right)}$$

$$= (R_x(1+\sigma_v^2))^{-1} = \frac{1}{1+\sigma_v^2} R_x^{-1}$$

#2 cont'd) $R_{xy} = \mathbb{E}(\vec{x}\vec{y}^T) = \mathbb{E}(\vec{x}((1+v)\vec{x})^T) = \mathbb{E}(\vec{x}\vec{x}^T(1+v))$

$$= \mathbb{E}(\vec{x}\vec{x}^T + \vec{x}\vec{x}^T v)$$

$$= R_x + R_x \cancel{\mathbb{E}(v)}^{0} = R_x$$

then $\hat{x} = \mathbb{E}(\vec{x}|\vec{y}) = R_{xy} R_y^{-1} \vec{y} = R_x \left(\frac{1}{1+\sigma_v^2}\right) R_x^{-1} \vec{y} = \boxed{\frac{\vec{y}}{1+\sigma_v^2}}$

Now, show that $\mathbb{E}((\vec{x}-\hat{x})(\vec{x}-\hat{x})^T) < \mathbb{E}(\vec{x}\vec{x}^T)$

$$\mathbb{E}((\vec{x}-\hat{x})(\vec{x}-\hat{x})^T) = \mathbb{E}(\vec{x}\vec{x}^T - \hat{x}\vec{x}^T - \vec{x}\hat{x}^T + \hat{x}\hat{x}^T)$$

$$= R_x - \mathbb{E}\left(\frac{\vec{y}}{1+\sigma_v^2}\vec{x}^T\right) - \frac{\mathbb{E}(\vec{x}\vec{y}^T)}{1+\sigma_v^2} + \mathbb{E}(\vec{y}\vec{y}^T)\frac{1}{(1+\sigma_v^2)^2}$$

$$= R_x - \frac{R_{yx}}{1+\sigma_v^2} - \frac{R_{xy}}{1+\sigma_v^2} + \frac{R_y}{(1+\sigma_v^2)^2}$$

$R_{xy} = R_{yx}^T$

$R_{yx} = R_{xy}^T$

$$= R_x - \frac{R_{yx}}{1+\sigma_v^2} - \cancel{\frac{R_{xy}}{1+\sigma_v^2}}^{R_x} + \cancel{\frac{(R_x(1+\sigma_v^2))}{(1+\sigma_v^2)^2}} = R_x - \frac{R_{yx}}{1+\sigma_v^2}$$

$$= \cancel{\frac{R_x(1+\sigma_v^2) + R_x - R_{yx} - R_{xy}}{1+\sigma_v^2}}$$

$$R_{yx} = \mathbb{E}(\vec{y}\vec{x}^T) = \mathbb{E}((1+v)\vec{x}\vec{x}^T) = \mathbb{E}(\vec{x}\vec{x}^T + v\vec{x}\vec{x}^T) = R_x + \cancel{\mathbb{E}(v)}^{0}\mathbb{E}(\vec{x}\vec{x}^T) = R_x$$

$$\boxed{\Rightarrow \mathbb{E}((\vec{x}-\hat{x})(\vec{x}-\hat{x})^T) = R_x - \frac{R_x}{1+\sigma_v^2} = R_x\left(\frac{1+\sigma_v^2-1}{1+\sigma_v^2}\right) = \boxed{R_x\left(\underbrace{\frac{\sigma_v^2}{1+\sigma_v^2}}_{<1}\right) < R_x}}$$

implies $\square$

#3 Defective Measurement Noise

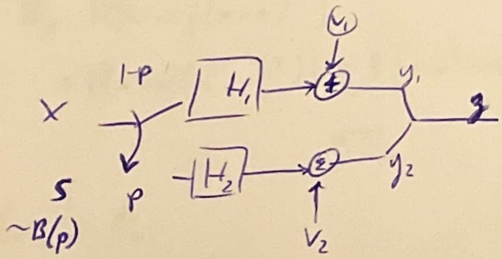
zero-mean random variable $\vec{x}$ w/ $var(\vec{x}) = \mathbb{E}((\vec{x}-\cancel{\vec{\mu}_x})(\vec{x}-\cancel{\vec{\mu}_x})^T) = \Pi_0$

two possible measurements for $\vec{x}$: $\vec{y}_1 = \underline{H}_1\vec{x}+\vec{v}_1$ +, $\vec{y}_2 = \underline{H}_2\vec{x}+\vec{v}_2$

$(\vec{v}_1, \vec{v}_2)$ are 0-mean, uncorrelated sensor noise w/ variance $\underline{R}_1$ and $\underline{R}_2$ respectively
is also uncorrelated w/ $\vec{x}$

One of the measurements is defective... either sensor 1 w/ probability 1-p or
sensor 2 w/ prob. p → this is measurement $\vec{z}$

3-1) LLMS estimator of $\vec{x}$ given $\vec{z}$ ⇒ $\hat{x}_z = \mathbb{E}(x|z) = R_{xz}R_z^{-1}\vec{z}$



joint distribution of data ...

Hint: introduce s explicitly...

$$P_{zx}(z,x) = \sum_s P_{zxs}(z,x|s)$$

Given

law of total probability: (like a weighted probability)

$$\mathbb{E}_z(f(z)) = \mathbb{E}_s(\mathbb{E}_{z|s}(f(z)|s))$$

$$\Rightarrow P_{zx}(z,x) = p(z,x|s_1)\,p(s_1) + p(z,x|s_2)\,p(s_2)$$
$$= p(z,x|s_1)(1-p) + p(z,x|s_2)(p)$$

$$\Rightarrow \mathbb{E}(\vec{x}|\vec{z}) = \mathbb{E}_s(\mathbb{E}(\vec{x}|\vec{z})|s) = \mathbb{E}(\vec{x}|\vec{z}|s_1)\,\mathbb{E}(s_1) + \mathbb{E}(\vec{x}|\vec{z}|s_2)\,\mathbb{E}(s_2)$$

keep this expectation?!

$$= \mathbb{E}(\vec{x}|\vec{z}s_1)(1-p) + \mathbb{E}(\vec{x}|\vec{z}s_2)(p)$$

(I don't know if the notation is correct but it feels right)

from class notes: $\mathbb{E}(\vec{x}|\vec{y}_1)_{s_1} = R_{xy_1}R_{y_1}^{-1}\vec{y}_1 = K_{o1}\vec{y}_1$, $K_{o1} = (R_x^{-1} + \underline{H}_1^T R_{v_1}^{-1}\underline{H}_1)^{-1}\underline{H}_1^T R_{v_1}^{-1}$

$\mathbb{E}(\vec{x}|\vec{y}_2)_{s_2} = R_{xy_2}R_{y_2}^{-1}\vec{y}_2 = K_{o2}\vec{y}_2$, $K_{o2} = (R_x^{-1} + H_2^T R_{v_2}^{-1}H_2)^{-1}H_2^T R_{v_2}^{-1}$

→

$$\cancel{\mathbb{E}\left((\vec{x}-\vec{z}W)(\vec{x}-\vec{z}W)^T\right)} = \mathbb{E}\left(\vec{x}\vec{x}^T - \vec{x}W^T\vec{z}^T - \vec{z}W^T\vec{x}^T + \vec{z}WW^T\vec{z}^T\right)$$

$$\cancel{= R_x \ldots ?}$$

$\Rightarrow$ let $\hat{x} = \mathbb{E}(\vec{x}|z) = \mathbb{E}(\vec{x}|\vec{y}_1)\,\mathbb{E}(y_1) + \mathbb{E}(\vec{x}|\vec{y}_2)\,\mathbb{E}(y_2)$

$$= K_{01}\vec{z}(1-p) + K_{02}\vec{z}\,p$$

$$= K_{01}\vec{z} - K_{01}\vec{z}p + K_{02}\vec{z}\,p$$

$$= K_{01}\vec{z} + (K_{02}\cancel{\vec{z}} - K_{01})\vec{z}\,p$$

, p is a scalar

$$\boxed{\hat{x} = \left[K_{01} + (K_{02} - K_{01})p\right]\vec{z}}$$

where $K_{01} = (\Pi_0^{-1} + H_1^T R_1^{-1} H_1)^{-1} H_1^T R_1^{-1}$

$K_{02} = (\Pi_0^{-1} + H_2^T R_2^{-1} H_2)^{-1} H_2^T R_2^{-1}$

p = probability switch is set for model 2

H3-Q2

$\Rightarrow \cancel{R_x} = \mathbb{E}[(x-\hat{x})(x-\hat{x})^T]$

$$= \mathbb{E}[(x-\hat{x})(x^T-\hat{x}^T)] = \mathbb{E}[xx^T - x\hat{x}^T - \hat{x}x^T + \hat{x}\hat{x}^T]$$

$$\cancel{= R_x}$$

$\mathbb{E}[xx^T] = R_x$

$\mathbb{E}[x\hat{x}^T] = \mathbb{E}[x[(K_{01} + (K_{02}-K_{01})p)\vec{z}]^T]$

K_{01}, K_{02}, p are constants (not random)

$$= \mathbb{E}[xz^T(K_{01} + (K_{02}-K_{01})p)^T]$$

$$= \cancel{R_{xz}(K_{01} + (K_{02}-K_{01})p)^T} = R_{xz}W^T$$

[let $W = (K_{01} + (K_{02}-K_{01})p)$]

$\mathbb{E}[\hat{x}x^T] = \mathbb{E}[(K_{01} + (K_{02}-K_{01})p)z\,x^T]$

$$= \cancel{(K_{01} + (K_{02}-K_{01})p)R_{zx}} = WR_{zx}$$

$\mathbb{E}[\hat{x}\hat{x}^T] = \mathbb{E}[(K_{01} + (K_{02}-K_{01})p)zz^T(K_{01} + (K_{02}-K_{01})p)^T]$

$$= \cancel{(K_{01} + (K_{02}-K_{01})p)R_z(K_{01} + (K_{02}-K_{01})p)^T} \rightarrow$$

$$= \cancel{WR_zW^T}$$

#3-Q2 cont'd

$$\mathbb{E}[(x-\hat{x})(x-\hat{x})^T] = \mathbb{E}[xx^T - x\hat{x}^T - \hat{x}x^T - \hat{x}\hat{x}^T]$$

$$= R_x - R_{xz}W^T - WR_{zx} - WR_zW^T \quad , \quad W = (K_{01} + (K_{02} - K_{01})p)$$

$$K_{01} = (\Pi_0^{-1} + H_1^T R_v H_1)^{-1} H_1^T R_1$$

$$= \Pi_0 - R_{xz}W^T - WR_{zx} - WR_zW^T$$

$$K_{02} = (\Pi_0^{-1} + H_2^T R_2 H_2)^{-1} H_2^T R_2$$

Not complete yet.

$$\Pi_0 = R_x$$

~~#3-Q3~~

~~if v_1 and v_2 are correlated, it does not change the MMSE.~~

(Law of total Expectation)

$$\mathbb{E}[xx^T] = R_x$$

$$\mathbb{E}[x\hat{x}^T] = \mathbb{E}[xz^TW^T] = \mathbb{E}_s[xz^T|s]W^T = \left(\mathbb{E}[xy_1^T](1-p) + \mathbb{E}(xy_2^T)(p)\right)W^T$$

$$\mathbb{E}[xy_1^T] = \mathbb{E}[x(H_1x+v_1)^T] = \mathbb{E}[xx^TH_1 + xv_1^T] = R_xH_1 + \cancelto{0}{R_{xv_1}}$$

$$\mathbb{E}[xy_2^T] = \text{''}_2 = \text{''}_2 = R_xH_2 + \cancelto{0}{R_{xv_2}}$$

$$\Rightarrow \mathbb{E}[x\hat{x}^T] = (R_xH_1(1-p) + R_xH_2p)W^T$$

$$\mathbb{E}[\hat{x}x^T] = \mathbb{E}[Wzx^T] = \mathbb{E}_s[\mathbb{E}[Wzx^T|s]] = W\left(\mathbb{E}_{z|s}(y_1x^T)(1-p) + \mathbb{E}_{z|s}(y_2x^T)p\right)$$

$$\mathbb{E}(y_1x^T) = \mathbb{E}((H_1x+v)x^T) = \mathbb{E}(H_1xx^T + vx^T) = H_1R_x$$

$$\mathbb{E}(y_2x^T) = \text{''}_2 = \text{''}_2 = H_2R_x$$

$$\Rightarrow \mathbb{E}[\hat{x}x^T] = W(H_1R_x(1-p) + H_2R_xp)$$

$$\mathbb{E}[\hat{x}\hat{x}^T] = \mathbb{E}[Wzz^TW^T] = W\mathbb{E}_s[\mathbb{E}_{z|s}[zz^T]]W^T = W\left(\mathbb{E}_{z|s}(y_1y_1^T)(1-p) + \mathbb{E}(y_2y_2^T)p\right)W^T$$

$$\mathbb{E}(y_1y_1^T) = \mathbb{E}((H_1x+v)(H_1x+v)^T) = \mathbb{E}(H_1xx^TH_1^T + \cancelto{0}{H_1xv^T} + \cancelto{0}{vx^TH_1^T} + v_1v_1^T)$$

$$= H_1R_xH_1^T + R_{v_1}$$

$$\mathbb{E}(y_2y_2^T) = \text{''}_2 = H_2R_xH_2^T + R_{v_2}$$

$$\Rightarrow \mathbb{E}[\hat{x}\hat{x}^T] = W\left((H_1R_xH_1^T + R_{v_1})(1-p) + (H_2R_xH_2^T + R_{v_2})p\right)W^T$$

→

#3-Q-4 Let $H_1 = H_2 = H$, then $y_1 = Hx + v_1$ and $K_{01} = (\Pi_0^{-1} + H^T R_1^{-1} H)^{-1} H^T R_1^{-1}$

$y_2 = Hx + v_2$ $\quad K_{02} = (\Pi_0^{-1} + H^T R_2^{-1} H)^{-1} H^T R_2^{-1}$

#3-Q2 cont'd

$$\boxed{\begin{aligned} \mathbb{E}[(x-\hat{x})(x-\hat{x})^T] = R_x &- (R_x H_1 (1-p) + R_x H_2 p) W^T \\ &- W (H_1 R_x (1-p) + H_2 R_x p) \\ &+ W ((H_1 R_x H_1^T + R_{v_1})(1-p) + (H_2 R_x H_2^T + R_{v_2}) p) W^T \end{aligned}}$$

$\square$

#3-Q3

Answer does not change if v_1, v_2 are correlated.

#3-Q4

For $H_1 = H_2$ ... what say?

In this case, our ~~MMSE~~ ~~depends~~ MMSE will ~~change~~ improve or decline based ~~largely~~ on the variance of the random noise R_{v_1} and R_{v_2} and our knowledge of the probability p.

#4] Linear estimator of x^2

$y = x + v$, v, x are 0 mean, independent, $\in \mathbb{R}$, Gaussian random variables

$\mathbb{E}(v^2) = \sigma_v^2$, $\mathbb{E}(x^2) = \sigma_x^2$

Note: for a real-valued, zero-mean Gaussian random variable z, $\mathrm{var}(z) = \sigma_z^2$, $\mathbb{E}(z^3) = 0$ and $\mathbb{E}(z^4) = 3\sigma^4$

4-1] Find LLMS estimator of x^2 using y : let $\hat{x^2} = ky$ (linear estimator)

$$\hat{x}^2 = \mathbb{E}(x^2|y) = \mathbb{E}\left((x^2 - ky)(x^2 - ky)^T\right)$$

$$= \mathbb{E}\left(x^4 - kyx^2 - kyx^2 + k^2y^2\right)$$

(scalar)

$$= \mathbb{E}(x^4) - 2\mathbb{E}(kyx^2) + k^2\mathbb{E}(y^2)$$

$$= 3\sigma_x^4 - 2kR_{yx^2} + k^2 R_y$$

$$\mathbb{E}(\cancel{yx^2}) = \mathbb{E}_y\left(\mathbb{E}(yx^2|y)\right)$$

$$\cancel{= \mathbb{E}_y\left(\mathbb{E}(x^2|y)\,y\right)}$$

$$= \mathbb{E}_y$$

$$P(k) = \mathbb{E}\left((x^2 - ky)(x^2 - ky)\right)$$

$$\Rightarrow \frac{\partial P(k)}{\partial k} = -2R_{yx^2} + 2kR_y = 0$$

(minimization w.r.t. estimator k)

$$\Rightarrow k_0 = R_{yx^2} R_y^{-1}$$

$$P(k) = \mathbb{E}(x^2/y) = \mathbb{E}\left((x^2 - ky)(x^2 - ky)^T\right) = \mathbb{E}\left(\left(x^2 - k(x+v)\right)^2\right)$$

$$= \mathbb{E}\left(x^4 - 2x^2k(x+v) + k^2(x+v)^2\right)$$

$$= \mathbb{E}(x^4) - 2k\left(\mathbb{E}(x^3) + \mathbb{E}(x^2v)\right) + k^2\mathbb{E}\left(x^2 + 2xv + v^2\right)$$

$$= 3\sigma_x^4 - 2k\left(0 + \mathbb{E}(x^2)\cancelto{0}{\mathbb{E}(v)}\right) + k^2\left(\sigma_x^2 + 2\cancelto{0}{\mathbb{E}(x)\mathbb{E}(v)} + \mathbb{E}(v^2)\right)$$

$$= 3\sigma_x^4 - 0 + k^2(\sigma_x^2 + \sigma_v^2)$$

$$\frac{\partial P(k)}{\partial k} = 2k(\sigma_x^2 + \sigma_v^2) = 0 \Rightarrow \boxed{k = 0} \Rightarrow \boxed{\hat{x^2} = ky = 0 = \mathbb{E}[x^2/y]}$$

→

4-2] <u>LLMS of x^2 using y^2.</u>

let estimated $\hat{x}^2 = ky^2$

$$P(k) = \mathbb{E}(x^2|y^2) = \mathbb{E}\left((x^2-ky^2)(x^2-ky^2)\right) \qquad y = x+v$$

$$= \mathbb{E}\left(x^4 - 2kx^2(x+v)^2 + k^2(x+v)^4\right)$$

$$= 3\sigma_x^4 - 2k\,\mathbb{E}\left(x^2(x^2+2xv+v^2)\right) + k^2\,\mathbb{E}\left(x^4+4x^3v+6x^2v^2+4xv^3+v^4\right)$$

$$= 3\sigma_x^4 - 2k\,\mathbb{E}\left(x^4+2x^3v+x^2v^2\right) + k^2\left[\mathbb{E}(x^4) + 4\,\mathbb{E}(x^3)\overset{0}{\mathbb{E}(v)} + 6\,\mathbb{E}(x^2)\mathbb{E}(v^2) + 4\,\mathbb{E}(x)\overset{0}{\mathbb{E}(v^3)} + \mathbb{E}(v^4)\right]$$

$$= 3\sigma_x^4 - 2k\left[3\sigma_x^4 + 0 + \sigma_x^2\sigma_v^2\right] + k^2\left[3\sigma_x^4 + 6\sigma_x^2\sigma_v^2 + 3\sigma_v^4\right]$$

$$\frac{\partial P(k)}{\partial k} = -\underset{3}{\cancel{6}}\sigma_x^4 - \cancel{2}\sigma_x^2\sigma_v^2 + \cancel{2}k\left[3\sigma_x^4 + 6\sigma_x^2\sigma_v^2 + 3\sigma_v^4\right] = 0 \qquad \text{(minimization w.r.t. } k\text{)}$$

$$\Rightarrow \boxed{k = \frac{3\sigma_x^4 + \sigma_x^2\sigma_v^2}{3\sigma_x^4 + 6\sigma_x^2\sigma_v^2 + 3\sigma_v^4}}$$

#5] Separation of Signal and Structured Noise

$\vec{y} = \underline{H}\vec{x} + \underline{S}\vec{\theta} + \vec{v}$ $\qquad$ $\vec{v}$ is 0-mean additive noise random vector

$E(\vec{v}\vec{v}^+) = I = R_v$

$\vec{x}, \vec{\theta}$ are unknown vectors, constant

$\underline{H} \in \mathbb{C}^{m \times n}$, $\underline{S} \in \mathbb{C}^{m \times p}$ $\Rightarrow$ $\underline{H}, \underline{S}$ known, $[\underline{H}\ \underline{S}]$ full rank, $m \geq n+p$ $\Rightarrow \text{rank}([\underline{H}\ \underline{S}]) = n+p$

$\underline{S}\vec{\theta}$ = perturbation

$\underline{H}\vec{x}$ = useful signal to separate

5-Q1) let $\vec{z} = [\vec{x}^T\ \vec{\theta}^T]^T = \begin{bmatrix} \vec{x} \\ \vec{\theta} \end{bmatrix}$ Determine optimal unbiased estimator $\hat{z}$ of z given y

Un-biased $\Rightarrow \mathbb{E}(\hat{z}) = z$ $\qquad$ note $\vec{y} = [\underline{H}\ \underline{S}]\vec{z} + \vec{v}$, let $\boxed{\underline{H}_z = [\underline{H}\ \underline{S}]}$

$\hat{z} \triangleq Ky$ (general linear estimate K)

By the Gauss-Markov Thm, the optimal, unbiased LLSE of $\vec{z}$ given $\vec{y}$ is

$$\boxed{\hat{z}_\infty = (\underline{H}_z^+ \underline{H}_z)^{-1} \underline{H}_z^+ \vec{y}}$$

recall that if z is deterministic then $R_z = \alpha I$, $\alpha \to \infty$

5-Q2) let $\hat{z} = \begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix}$, $\hat{s} \triangleq \underline{H}\hat{x}$ is the estimate of $\vec{s} \triangleq \underline{H}\vec{x}$

Show that $\hat{s} = \underline{E}y$ with $\underline{E} = P_H[I - S(S^+ P_H^\perp S)^{-1} S^+ P_H^\perp] = \underline{H}(\underline{H}^+ P_S^\perp \underline{H})^{-1} \underline{H}^+ P_S^\perp$

P_H, P_S are orthogonal projection matrices on the space spanned by the rows of H, S respectively

#5-Q1

generic linear estimate: $\hat{z} \triangleq ky$ ~~note~~ let $H_z = [H \ \ S]$

$\Rightarrow \vec{y} = H_z \vec{z} + \vec{v}$

$\Rightarrow \mathbb{E}(\hat{z}) = k\,\mathbb{E}(\vec{y}) = k\,\mathbb{E}(H_z \vec{z} + \vec{v})$

$= kH_z z + \cancel{\mathbb{E}(\vec{v})}^{0} = kH_z z$ $\rightarrow$ unbiased so require $\underline{kH_z = I}$

<u>Normal equation</u> for complex numbers?: (assume 0-mean random vectors)

must have $\forall a \in \mathbb{C}^n$, $a^+ P(k) a > a^+ P(k_0) a \Rightarrow P(k) \geq P(k_0)$

$a^+ P(k) a = a^+ \mathbb{E}((z - k\vec{y})(z - ky)^+) a = \mathbb{E}(a^+(z-ky)(z-ky)^+ a)$

$= \mathbb{E}(a^+(zz^+ - kyz^+ - zy^+k^+ + kyy^+k^+)a)$

$= a^+ R_z a \underbrace{- a^+ k R_{yz} a}_{\text{scalar}} - a^+ R_{zy} k^+ a + a^+ k R_y k^+ a$

$= a^+ R_z a - a^+ R_{yz}^{+} k^+ a - a^+ R_{zy} k^+ a + a^+ k R_y k^+ a$

R_{zy}

$R_{zy} = \mathbb{E}(zy^+)$

$R_{zy}^{+} = \mathbb{E}(zy^+)^+ = \mathbb{E}((zy^+)^+) = \mathbb{E}(yz^+) = R_{yz}$ ✓

transpose commutes

$\dfrac{d x^T B x}{dx} = (B + B^T)x$

$\dfrac{d\, a^+ k R_y k^+ a}{d k^+ a} = (R_y + R_y^+) k^+ a$

$R_y = R_y^+$

$= 2 R_y k^+ a$

$\dfrac{\partial a^T x}{\partial x} = a$

$\Rightarrow \dfrac{\partial\, a^+ R_{zy} k^+ a}{\partial k^+ a} = (a^+ R_{zy})^+ = R_{yz} a$

$\dfrac{\partial\, a^+ P(k) a}{\partial \underset{\text{vector}}{k^+ a}} = 0 \Leftrightarrow -\cancel{2}R_{zy} + 2 R_y k^+ a = 0$

$-\cancel{2} R_{yz} \cancel{a} + \cancel{2} R_y k^+ \cancel{a} = 0$

$(R_y k^+ = R_{yz})^+ \Rightarrow k R_y = R_{zy}$

$\Rightarrow$ if R_y^{-1} non-singular

$\Rightarrow k_0 = R_{zy} R_y^{-1}$ ✓

$\longrightarrow$

Now that we have confirmed the normal equation for a general linear estimate, we restrict ourselves to the linear model $y = Hz + v$ $v \to$ zero mean, uncorrelated w/ z

$$K_o = R_{zy} R_y^{-1} \Rightarrow R_{zy} = \mathbb{E}((z - \cancel{m_z}^0)(y - \cancel{m_y}^0)^+) = \mathbb{E}(z y^+) = \mathbb{E}(z(Hz+v)^+) = \mathbb{E}(zz^+H^+ + zv^+)$$

$\hat{z} = K_o y$

$$= R_z H^+ + \cancel{R_{zv}}^0 = R_z H^+$$

$$R_y = \mathbb{E}(yy^+) = \mathbb{E}((Hz+v)(Hz+v)^+) = \mathbb{E}(Hzz^+H^+ + vz^+H^+ + Hzv^+ + vv^+)$$

$$= HR_zH^+ + 0 + 0 + R_v = HR_zH^+ + R_v$$

$$\Rightarrow \hat{z} = \underbrace{R_z H^+ (HR_zH^+ + R_v)^{-1}}_{K_o} y$$

Matrix inversion lemma: $(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$

let $A = R_v$, $B = H$, $C = R_z$, $D = H^+$

$\Rightarrow$ then $(R_v + HR_zH^+)^{-1} = R_v^{-1} - R_v^{-1}H(R_z^{-1} + H^+R_v^{-1}H)^{-1}H^+R_v^{-1}$

$$\Rightarrow K_o = R_z H^+ R_v^{-1} - R_z H^+ R_v^{-1} H (R_z^{-1} + H^+ R_v^{-1} H)^{-1} H^+ R_v^{-1}$$

similar algebra as in class $\to$ no transpose so keeps form

$$\Rightarrow K_o = (R_z^{-1} + H^+R_v^{-1}H)^{-1} H^+ R_v^{-1}$$

Now, if z is deterministic, then $R_z = \alpha I$, $\alpha \to \infty$.

$$\Rightarrow K_o = (0 + H^+R_v^{-1}H)^{-1}H^+R_v^{-1} \quad \text{if } R_v = I,\ R_v^{-1} = I$$

$$\Rightarrow K_o = (H^+H)^{-1}H^+$$

thus, by the Gauss-Markov Thm: $\boxed{\hat{z}_\infty = K_o y = (H^+H)^{-1}H^+ y}$

And in our specific case, $H_z = [H\ S]$, $y = H_z z + v$, $z = \begin{pmatrix} x \\ \theta \end{pmatrix}$

$$\boxed{\hat{z}_\infty = (H_z^+ H_z)^{-1} H_z^+ y}$$

#5-Q21

$\hat{z} = \begin{bmatrix}\hat{x}\\ \hat{\theta}\end{bmatrix}$, $\hat{y} = H_z \hat{z} + v$

$$\begin{bmatrix}\hat{x}\\ \hat{\theta}\end{bmatrix} = (H_z^+ H_z)^{-1} H_z^+ \vec{y} = \left(\begin{bmatrix}H^+\\ S^+\end{bmatrix}[H\ S]\right)^{-1}\begin{bmatrix}H^+\\ S^+\end{bmatrix}\vec{y}$$

want $H\hat{x}$

$$= \begin{pmatrix}H^+H & H^+S\\ S^+H & S^+S\end{pmatrix}^{-1}\begin{bmatrix}H^+\\ S^+\end{bmatrix}\vec{y}$$

Schur Complement!

For $\left[\begin{array}{c|c}A_{11} & A_{12}\\ \hline A_{21} & A_{22}\end{array}\right]^{-1} = \left[\begin{array}{c|c} I & 0\\ \hline -A_{22}^{-1}A_{21} & I\end{array}\right]\left[\begin{array}{c|c}(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1} & 0\\ \hline 0 & A_{22}^{-1}\end{array}\right]\begin{bmatrix} I & -A_{12}A_{22}^{-1}\\ 0 & I\end{bmatrix}$

let $A_{11} = H^+H$

$A_{12} = H^+S$

$A_{21} = S^+H$

$A_{22} = S^+S$

$$= \left[\begin{array}{c|c} I & 0\\ \hline -A_{22}^{-1}A_{21} & I\end{array}\right]\begin{bmatrix} '' & -''(A_{12}A_{22}^{-1})\\ 0 & A_{22}^{-1}\end{bmatrix} = \left[\begin{array}{c|c} '' & -''(A_{12}A_{22}^{-1})\\ \hline -A_{22}^{-1}A_{21}('') & A_{22}^{-1}A_{21}('')(A_{12}A_{22}^{-1}) + A_{22}^{-1}\end{array}\right]$$

$$\Rightarrow \begin{pmatrix}\hat{x}\\ \hat{\theta}\end{pmatrix} = \left(\begin{array}{c|c} I & 0\\ \hline -(S^+S)^{-1}S^+H & I\end{array}\right)\left(\begin{array}{c|c}(H^+H - H^+S(S^+S)^{-1}S^+H)^{-1} & 0\\ \hline 0 & (S^+S)^{-1}\end{array}\right)\begin{pmatrix} I & -H^+S(S^+S)^{-1}\\ 0 & I\end{pmatrix}$$

= SEE MATLAB

$P_H = H(H^+H)^{-1}H^+$

$P_S = S(S^+S)^{-1}S^+$

Schur Complement:

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}^{-1} = \begin{bmatrix}A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1}\\ -S^{-1}CA^{-1} & S^{-1}\end{bmatrix}, \quad S \triangleq D - CA^{-1}B$$

→ arises naturally from $\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}u\\ v\end{bmatrix}$

→ can also get Schur complement of D if solve for y first

let $A = H^+H$

$B = H^+S \Rightarrow$

$C = S^+H$

$D = S^+S$

$$S_2 = (S^+S) - (S^+H)\underbrace{(H^+H)^{-1}H^+}_{P_H}S \longrightarrow$$

#5 Q2 cont'd

$$\begin{pmatrix} H^+H & H^+S \\ S^+H & S^+S \end{pmatrix}^{-1} = \begin{bmatrix} (H^+H)^{-1} + (H^+H)^{-1}(H^+S)\left(S^+S - S^+H(H^+H)^{-1}H^+S\right)^{-1}S^+H(H^+H)^{-1} & -(H^+H)^{-1}H^+S\, S_2^{-1} \\ -\left(S^+S - S^+H(H^+H)^{-1}H^+S\right)^{-1}S^+H(H^+H)^{-1} & \left(S^+S - S^+H(H^+H)^{-1}H^+S\right)^{-1} \end{bmatrix}$$

$$\Rightarrow \hat{x} = \left[(H^+H)^{-1}H^+ + (H^+H)^{-1}H^+S(S^+S - S^+P_H S)^{-1}S^+H(H^+H)^{-1}H^+ - (H^+H)^{-1}H^+S(S^+S - S^+P_H S)^{-1}S^+\right]\vec{y}$$

$$H\hat{x} = \left[H(H^+H)^{-1}H^+ + H(H^+H)^{-1}H^+S(S^+S - S^+P_H S)^{-1}S^+P_H - H(H^+H)^{-1}H^+S(S^+S - S^+P_H S)^{-1}S^+\right]\vec{y}$$

$$= \left[P_H + P_H S(S^+S - S^+P_H S)^{-1}S^+P_H - P_H S(S^+S - S^+P_H S)^{-1}S^+\right]\vec{y}$$

$$= P_H\left[I + S\left((S^+S)^{-1} - (S^+P_H S)^{-1}\right)S^+P_H - S\left((S^+S)^{-1} - (S^+P_H S)^{-1}\right)S^+\right]\vec{y}$$

$$= P_H\left[I - S\left(\underbrace{(S^+S)^{-1}}_{=(S^+IS)^{-1}} - (S^+P_H S)^{-1}\right)S^+\underbrace{(I - P_H)}_{P_H^\perp}\right]\vec{y}$$

$$= P_H\left[I - S(S^+IS - S^+P_H S)^{-1}S^+P_H^\perp\right]\vec{y}$$

$$= P_H\left[I - S(S^+(I - P_H)S)^{-1}S^+P_H^\perp\right]\vec{y} =$$

$$\boxed{= \underbrace{P_H\left[I - S(S^+P_H^\perp S)^{-1}S^+P_H^\perp\right]}_{E}\vec{y} = Ey = \hat{s} \quad \square}$$

Now show $E = H(H^+P_S^\perp H)^{-1}H^+P_S^\perp$

$$= H\left(H^+(I - S(S^+S)^{-1}S^+)H\right)^{-1}H^+\left(I - S(S^+S)^{-1}S^+\right)$$

$$= H\left(H^+H - H^+S(S^+S)^{-1}S^+H\right)^{-1}H^+\left(I - S(S^+S)^{-1}S^+\right)$$

$$= \left(H(H^+H)^{-1} - H(H^+S(S^+S)^{-1}S^+H)^{-1}H^+\right)\left(I - S(S^+S)^{-1}S^+\right)$$

$$= \left(H(H^+H)^{-1}H^+ - H(H^+S(S^+S)^{-1}S^+H)^{-1}H^+\right)\left(I - S(S^+S^{-1})S^+\right)$$

$$= H(H^+H)^{-1}H^+ - H(H^+S(S^+S)^{-1}S^+H)^{-1}H^+ - P_H S(S^+S)^{-1}S^+ + H(H^+P_S H)^{-1}H^+P_S$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \to \begin{array}{l} Ax + By = u \quad (1) \\ Cx + Dy = v \quad (2) \end{array}$$

$$\xrightarrow{(2)} y = D^{-1}(-Cx + v)$$

Sub into (1) $\to Ax + BD^{-1}(-Cx + v) = u$

$$\Rightarrow (A - BD^{-1}C)x = u - BD^{-1}v$$

let $S_2 = A - BD^{-1}C$

$$\Rightarrow x = S_2^{-1}(u - BD^{-1}v)$$

$$\Rightarrow y = D^{-1}(v - C S_2^{-1}(u - BD^{-1}v))$$

$$= D^{-1}v - D^{-1}CS_2^{-1}(u - BD^{-1}v)$$

$$= D^{-1}v - D^{-1}CS_2^{-1}u + D^{-1}CS_2^{-1}BD^{-1}v$$

$$= \begin{bmatrix} -D^{-1}CS_2^{-1} & D^{-1} + D^{-1}CS_2^{-1}BD^{-1} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$x = \begin{bmatrix} S_2^{-1} & -S_2^{-1}BD^{-1} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} S_2^{-1} & -S_2^{-1}BD^{-1} \\ -D^{-1}CS_2^{-1} & D^{-1} + D^{-1}CS_2^{-1}BD^{-1} \end{bmatrix}$$

$$\Rightarrow \hat{x} = \left[S_2^{-1}H^+ - S_2^{-1}BD^{-1}S^+\right]\vec{y}$$

$A = H^+H$ \
$B = H^+S$ \
$C = S^+H$ \
$D = S^+S$

$$S_2 = A - BD^{-1}C$$
$$= H^+H - H^+S(S^+S)^{-1}S^+H$$
$$= H^+H - H^+P_S H$$

$$\Rightarrow H\hat{x} = \left[H(H^+H - H^+P_SH)^{-1}H^+ - H(H^+H - H^+P_SH)^{-1}BD^{-1}S^+\right]\vec{y}$$

$$\Rightarrow \hat{s} = \left[H(H^+(I-P_S)H)^{-1}H^+ - H(H^+(I-P_S)H)^{-1}H^+S(S^+S)^{-1}S^+\right]\vec{y}$$

$$= \left[H(H^+P_S^\perp H)^{-1}H^+ - H(H^+P_S^\perp H)^{-1}H^+P_S\right]\vec{y}$$

$$= \left[H(H^+P_S^\perp H)^{-1}H^+(I - P_S)\right]\vec{y}$$

$$\boxed{\hat{s} = \left[\underbrace{H(H^+P_S^\perp H)^{-1}H^+P_S^\perp}_{=E}\right]\vec{y} \qquad \square}$$

HS-Q3) Since $ES = H(H^+P_S^\perp H)^{-1}H^+\underbrace{P_S^\perp S}_{=0}$ and $P_S^\perp S = 0$, $ES = 0$

Geometric Interpretation: Since the ~~components~~ measurements of $y \in \text{span}([H\ S])$ and $y = Hx + \theta S + v$, when performing the LLSE estimation for $\hat{x}$, we are looking for the influence of the measurements that exist in the span(H), which are necessarily due to x.

Continue →

Thus, ~~we~~ since $y \in \underbrace{\mathrm{Span}([H\ S])}_{=Y}$, $\quad Im(s) \subset Im(H^\perp)$

$Im(S) \subset Y$ (b/c $[H\ S]$ full rank)

$Im(H) \subset Y$

$Im(H) \subset Im(S^\perp)$

and we have that any measurement ~~due to~~ in the $Im(S)$ is discarded when estimating $\hat{x}$. That is, $ES = 0$.

#5-Q4)

Let $\tilde{s} = s - \hat{s}$ show $\mathbb{E}[\tilde{s}\tilde{s}^+] = EE^+$

$s = Hx, \quad \hat{s} = H\hat{x} = E\vec{y}$

$$\mathbb{E}[\tilde{s}\tilde{s}^+] = \mathbb{E}[(s-\hat{s})(s-\hat{s})^+] = \mathbb{E}[ss^+ - s\hat{s}^+ - \hat{s}s^+ + \hat{s}\hat{s}^+] =$$

$y = Hx + S\theta + v$

$$\mathbb{E}[ss^+] = \mathbb{E}[Hxx^+H^+] = HR_xH^+ =$$

$$\cancel{\mathbb{E}[s\hat{s}^+] = \mathbb{E}[Hx(Ey)^+] = \mathbb{E}[Hxy^+E] = HR_{xy}E}$$

$$\cancel{\mathbb{E}[\hat{s}s^+] = \mathbb{E}[Ey(Hx)^+] = \mathbb{E}[Eyx^+H] = ER_{yx}H}$$

$$\cancel{\mathbb{E}[\hat{s}\hat{s}^+] = \mathbb{E}[Eyy^+E^+] = ER_yE^+}$$

$R_v = I$

$R_{xv} = 0$ (uncorrelated sensor noise)

$R_{vx} = 0$

$$\mathbb{E}[s\hat{s}^+] = \mathbb{E}[Hx(Ey)^+] = \mathbb{E}[Hx(E(Hx + S\theta + v))^+] = \mathbb{E}[Hx(EHx + \overset{0}{\cancel{ES\theta}} + Ev)^+]$$

$$= \mathbb{E}[Hxx^+H^+E^+ + Hxv^+E^+] = HR_xH^+E^+ + H\overset{0}{\cancel{R_{xv}}}E^+$$

$$\mathbb{E}[\hat{s}s^+] = \mathbb{E}[Ey(Hx)^+] = \mathbb{E}[E(Hx + S\theta + v)(Hx)^+] = \mathbb{E}[(EHx + Ev)(Hx)^+]$$

$$= \mathbb{E}[EHxx^+H^+ + Evx^+H^+] = EHR_xH^+ + E\overset{0}{\cancel{R_{vx}}}H^+$$

$$\mathbb{E}[\hat{s}\hat{s}^+] = \mathbb{E}[Ey(Ey)^+] = \mathbb{E}[E(Hx + S\theta + v)(E(Hx + S\theta + v))^+] = \mathbb{E}[(EHx + Ev)(EHx + Ev)^+]$$

$$= \mathbb{E}[EHxx^+H^+E^+ + EHxv^+E^+ + Evx^+H^+E^+ + Evv^+E^+]$$

$$= EHR_xH^+E^+ + EH\overset{0}{\cancel{R_{xv}}}E^+ + E\overset{0}{\cancel{R_{vx}}}H^+E^+ + ER_vE^+$$

$$\Rightarrow \mathbb{E}[\tilde{s}\tilde{s}^+] = HR_xH^+ - HR_xH^+E^+ - EHR_xH^+ + EHR_xH^+E^+ + ER_vE^+$$

$\longrightarrow$

$\mathbb{E}[\tilde{s}s^+] = HR_xH^+$

$\mathbb{E}[s\hat{s}^+] = HR_xH^+E^+ = \cancel{HR_xH^+}$

<u>note</u>

$$\boxed{EH} = \left(P_H - P_H S(S^+P_H^\perp S)^{-1}S^+P_H^\perp\right)H = \underbrace{P_H H}_{H(H^+H)^{-1}H^+H = H} - P_H S(S^+P_H^\perp S)^{-1}S^+\cancel{P_H^\perp H}^{\,0} = H$$

$H(\underbrace{H^+H)^{-1}H^+H}_{I} = H$

$\mathbb{E}[s\hat{s}^+] = HR_xH^+E^+ = HR_x(EH)^+ = HR_xH^+$

$\mathbb{E}[\hat{s}s^+] = EHR_xH^+ = HR_xH^+$

$\mathbb{E}[\hat{s}\hat{s}^+] = EHR_x(EH)^+ + E\overset{I}{\cancel{R_v}}E^+ = HR_xH^+ + EE^+$

$\Rightarrow \mathbb{E}[\tilde{s}\tilde{s}^+] = \mathbb{E}[ss^+ - s\hat{s}^+ - \hat{s}s^+ + \hat{s}\hat{s}^+]$

$= \cancel{HR_xH^+} - \cancel{HR_xH^+} - \cancel{HR_xH^+} + \cancel{HR_xH^+} + EE^+ = \boxed{EE^+}$ □

#5-Q5

Now assume x is a zero-mean random variable w/ $\mathbb{E}(xx^+) = R_x = \Pi_0 > 0$

Show the LLMS of $s = Hx$ is now $\hat{s} = Fy$ w/ $F = P_H[I - S(S^+P_H^\perp S)^{-1}S^+P_H^\perp]$

$\hat{s} = H\hat{x}$

$P_H = H(H^+H + \Pi_0^{-1})^{-1}H^+$

from #5-Q1, if z is not a deterministic ~~var~~ variable, then for $y = H_z z + v$,

$K_0 = (R_z^{-1} + H_z^+R_v^{-1}H_z)^{-1}H_z^+R_v^{-1} = (R_z^{-1} + H^+H)^{-1}H^+$, $(R_v = I)$

$\hat{z} = K_0 y$

$= (R_z^{-1} + H_z^+H_z)^{-1}H_z^+\vec{y}$

$\Rightarrow \hat{y} = H_z\hat{z} + v = H_z(R_z^{-1} + H_z^+H_z)^{-1}H_z^+\vec{y} + v$

$\hat{z} = \begin{bmatrix}\hat{x}\\ \hat{\theta}\end{bmatrix} = (R_z^{-1} + H_z^+H_z)^{-1}H_z^+$, $H_z = [H \;\; S]$, $R_z = \mathbb{E}[zz^+] = \mathbb{E}\left[\begin{pmatrix}x\\ \theta\end{pmatrix}(x^+ \;\; \theta^+)\right]$

$= \mathbb{E}\begin{bmatrix}xx^+ & x\theta^+\\ \theta x^+ & \theta\theta^+\end{bmatrix}$

→

$$R_z = \mathbb{E}\begin{bmatrix} xx^+ & x\theta^+ \\ \theta x^+ & \theta\theta^+ \end{bmatrix} = \begin{bmatrix} R_x & R_{x\theta} \\ R_{\theta x} & R_\theta \end{bmatrix}$$

θ is deterministic

$\Rightarrow R_\theta = \alpha I,\ \alpha \to \infty$

$\Rightarrow R_\theta^{-1} = \frac{1}{\alpha} I = 0$

$$R_z^{-1} = \begin{bmatrix} R_x & R_{x\theta} \\ R_{\theta x} & R_\theta \end{bmatrix}^{-1}$$

Schur Complement of D: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} S_2^{-1} & -S_2^{-1}BD^{-1} \\ -D^{-1}CS_2^{-1} & D^{-1}+D^{-1}CS_2^{-1}BD^{-1} \end{bmatrix}$

$$S_2 = A - BD^{-1}C$$

let $A = R_x$

$B = R_{x\theta}$

$C = R_{\theta x} =$

$D = R_\theta$

$$\Rightarrow R_z^{-1} = \begin{bmatrix} R_x^{-1} & -R_x^{-1}R_{x\theta}\cancel{R_\theta^{-1}}^{\to 0} \\ -\cancel{R_\theta^{-1}}^{\to 0} R_{\theta x}R_x^{-1} & \cancel{R_\theta^{-1}}^{\to 0} + \cancel{R_\theta^{-1}}^{\to 0} R_{\theta x}R_x^{-1}R_{x\theta}\cancel{R_\theta^{-1}}^{\to 0} \end{bmatrix} = \begin{bmatrix} R_x^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$S_2 = R_x - R_{x\theta}\cancel{R_\theta^{-1}}^{\to 0}R_{\theta x}$

$= R_x$

$$\hat{z} = K_0 y = (R_z^{-1} + H_z^+H_z)^{-1}H_z^+\vec{y}$$

So, $\begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = \left[\begin{bmatrix} R_x^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} H^+ \\ S^+ \end{bmatrix}\begin{bmatrix} H & S \end{bmatrix}\right]^{-1}\begin{bmatrix} H^+ \\ S^+ \end{bmatrix}\vec{y}$

$$= \begin{bmatrix} H^+H + R_x^{-1} & H^+S \\ S^+H & S^+S \end{bmatrix}^{-1}\begin{bmatrix} H^+ \\ S^+ \end{bmatrix}\vec{y}$$

Schur Complement of A: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}+A^{-1}BS_2^{-1}CA^{-1} & -A^{-1}BS_2^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$

let $A = H^+H + R_x^{-1}$, $S_2 = D - CA^{-1}B$

$B = H^+S$

$C = S^+H$

$D = S^+S$

$\longrightarrow$

#5-Q5 cont'd

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} (H^+H+R_x^{-1})^{-1}(I + H^+S(S^+S - S^+H(H^+H+R_x^{-1})^{-1}H^+S)^{-1}S^+H(H^+H+R_x^{-1})^{-1}, & -(H^+H+R_x^{-1})^{-1}H^+S(S^+S - S^+H(H^+H+R_x^{-1})^{-1}H^+S)^{-1} \\ -(S^+S - S^+H(H^+H+R_x^{-1})^{-1}H^+S)^{-1}S^+H(H^+H+R_x^{-1})^{-1}, & (S^+S - S^+H(H^+H+R_x^{-1})^{-1}H^+S)^{-1} \end{bmatrix} \begin{bmatrix} H^+ \\ S^+ \end{bmatrix} \vec{y}$$

let $P_H = H(H^+H+R_x^{-1})^{-1}H^+$

$$\Rightarrow H\hat{x} = \Big[H(H^+H+R_x^{-1})^{-1}H^+ + H(H^+H+R_x^{-1})^{-1}H^+S(S^+S - S^+P_HS)^{-1}S^+H(H^+H+R_x^{-1})^{-1}H^+$$

$$- H(H^+H+R_x^{-1})^{-1}H^+S(S^+S - S^+P_HS)^{-1}S^+\Big]\vec{y}$$

$$= \left[P_H + P_HS(S^+S - S^+P_HS)^{-1}S^+P_H - P_HS(S^+S - S^+P_HS)^{-1}S^+\right]\vec{y}$$

$$= P_H\left[I + S(S^+S - S^+P_HS)^{-1}S^+P_H - S(S^+S - S^+P_HS)^{-1}S^+\right]\vec{y}$$

$$= P_H\Big[I - (S(\underbrace{S^+S - S^+P_HS}_{S^+(I-P_H)S})^{-1}S^+)(I - P_H)\Big]\vec{y}$$

$$= P_H\left[I - (S(S^+P_H^\perp S)^{-1})P_H^\perp\right]\vec{y}$$

$$\boxed{\begin{aligned} &\hat{s} = H\hat{x} = P_H[I - S(S^+P_H^\perp S)^{-1}P_H^\perp]\vec{y}, \quad P_H = H(H^+H+R_x^{-1})^{-1}H^+ = H(H^+H+\Pi_0^{-1})^{-1}H^+ \\ &\Rightarrow \hat{s} = H\hat{x} = Fy, \quad F = P_H[I - S(S^+P_H^\perp S)^{-1}P_H^\perp] \end{aligned}}$$

□

PS-Q6

Schur Complement of D: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} S_2^{-1} & -S_2^{-1}BD^{-1} \\ -D^{-1}CS_2^{-1} & D^{-1}+D^{-1}CS_2^{-1}BD^{-1} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} H^+H + R_x^{-1} & H^+S \\ S^+H & S^+S \end{bmatrix}^{-1} \begin{bmatrix} H^+ \\ S^+ \end{bmatrix} \vec{y}$$

let $A = H^+H + R_x^{-1}$, $B = H^+S$, $C = S^+H$, $D = S^+S$, $S_2 = A - BD^{-1}C$ $= \cancel{D} - \cancel{CA^{-1}B}$

let $P_s = S(S^+S)^{-1}S^+$

$$\Rightarrow \hat{x} = [S_2^{-1}H^+ - S_2^{-1}BD^{-1}S^+]\vec{y}$$

$$H\hat{x} = [HS_2^{-1}H^+ - HS_2^{-1}BD^{-1}S^+]\vec{y}$$

$$= [H(\underbrace{\cancel{D-CA^{-1}B}}_{S_2})^{-1}H^+ - H(\underbrace{\cancel{D-CA^{-1}B}}_{S_2})^{-1}H^+S(S^+S)^{-1}S^+]\vec{y}$$

$$S_2^{-1} = (\cancel{S^+S - S^+H(H^+H+R_x^{-1})^{-1}H^+S})^{-1}$$

$$= H^+H + R_x^{-1} - H^+S(S^+S)^{-1}S^+H$$

$$= ((H^+H+R_x^{-1}) - H^+P_sH)^{-1}$$

$$= [H((H^+H+R_x^{-1}) - H^+P_sH)^{-1}H^+$$
$$- H((H^+H+R_x^{-1}) - H^+P_sH)^{-1}H^+P_s]\vec{y}$$

$$= [H(H^+(I-P_s)H + R_x^{-1})^{-1}H^+ - H(H^+(I-P_s)H + R_x^{-1})^{-1}H^+P_s]\vec{y}$$

$$= [H(H^+P_s^\perp H + R_x^{-1})^{-1}H^+ - H(H^+P_s^\perp H + R_x^{-1})^{-1}H^+P_s]\vec{y}$$

$$\hat{s} = H\hat{x} = \underbrace{H(H^+P_s^\perp H + R_x^{-1})^{-1}H^+P_s^\perp}_{F}\vec{y}$$

$$\boxed{\Rightarrow FS = H(H^+P_s^\perp H + R_x^{-1})^{-1}H^+P_s^\perp S, \quad P_s^\perp S = 0 \Rightarrow FS = 0 \quad \square}$$

#5-Q7 $\tilde{s} = s - \hat{s}$, $\hat{s} = Fy$

MMSE: $\mathbb{E}[\tilde{s}\tilde{s}^+] = \mathbb{E}[(s-\hat{s})(s-\hat{s})^+] = \mathbb{E}[ss^+ - s\hat{s}^+ - \hat{s}s^+ + \hat{s}\hat{s}^+]$

$\mathbb{E}[ss^+] = \mathbb{E}[Hxx^+H^+] = HR_xH^+$

$\mathbb{E}[s\hat{s}^+] = \mathbb{E}[Hx(Fy)^+] = \cancel{HR_{xy}F^+} = \mathbb{E}[Hx(F(Hx+S\theta+v))^+]$

$= \mathbb{E}[Hx(FHx + \cancel{FS\theta}^{0} + Fv)^+] = \mathbb{E}[Hx(FHx)^+] + \mathbb{E}[Hxv^+F]$

note: $FH = P_H[I - S(S^+P_H^\perp S)^{-1}P_H^\perp]H$

$= [P_H - P_HS(S^+P_H^\perp S)^{-1}P_H^\perp]H$

$= P_HH - P_HS(S^+P_H^\perp S)^{-1}\cancel{P_H^\perp H}$

=

$P_HH = H(H^+H + \pi_0^{-1})^{-1}H^+H$

$= H(H^+H)^{-1}H^+H + H\pi_0H^+H$

$= H + H\pi_0H^+H = H(I + \pi_0H^+H)$

$= (I + H\pi_0H^+)H$

$P_H^\perp H = (I - P_H)H$

$= H - (I + H\pi_0H^+)H$

$= (I - (I + H\pi_0H^+))H$

$= (I - I - H\pi_0H^+)H = -H\pi_0H^+H$

$\Rightarrow FH = P_HH - P_HS(S^+P_H^\perp S)^{-1}P_H^\perp H$

$= (I + H\pi_0H^+)H - P_HS(S^+P_H^\perp S)^{-1}(-H\pi_0H^+H)$

or... $FH = H(H^+P_S^\perp H + R_x^{-1})^{-1}H^+(I - P_S)H$

$= H(H^+P_S^\perp H + R_x^{-1})^{-1}H^+H$

$= H(H^+P_S^\perp H)^{-1}H^+H + H(R_x)H^+H$

$P_SH = 0$ b/c $Im(S) \subset Im(H^\perp)$

~~$Im(H^\perp) \subset Im(S)$~~

~~$Im(S^\perp)$~~

$Im(H) \subset Im(S^\perp)$

$\mathbb{E}[s\hat{s}^+] = \mathbb{E}[Hxx^+(FH)^+] + H\cancel{R_{xv}}^{0}F = HR_x(FH)^+$

$\mathbb{E}[\hat{s}s^+] = \mathbb{E}[Fy(Hx)^+] = \mathbb{E}[F(Hx + S\theta + v)(Hx)^+] = \mathbb{E}[(FHx + \cancel{FS\theta}^{0} + Fv)(Hx)^+]$

$= \mathbb{E}[FHxx^+H^+] + \mathbb{E}[Fvx^+H^+]$

$= FHR_xH^+ + F\cancel{R_{vx}}^{0}H^+$

$\rightarrow$

#5-Q7 cont'd

$$\mathbb{E}[\hat{s}\hat{s}^{+}] = \mathbb{E}[F_y(F_y)^{+}] = \mathbb{E}[(FHx + \cancel{FS\theta}^{0} + Fv)(FHx + \cancel{FS\theta}^{0} + Fv)^{+}]$$

$$= \mathbb{E}[FHxx^{+}(\cancel{FH})^{+} + FHxv^{+}F^{+} + Fvx^{+}(FH)^{+} + Fvv^{+}F^{+}]$$

$$= FHR_x(FH)^{+} + FH\cancel{R_{xv}}^{0}F^{+} + F\cancel{R_{vx}}^{0}(FH)^{+} + FR_v \overset{I}{\cancel{F}}^{+} =$$

$$\boxed{\begin{aligned} \Rightarrow \mathbb{E}[\tilde{s}\tilde{s}^{+}] &= \mathbb{E}[ss^{+}] - \mathbb{E}[s\hat{s}^{+}] - \mathbb{E}[\hat{s}s^{+}] + \mathbb{E}[\hat{s}\hat{s}^{+}] \\ &= HR_xH^{+} - HR_x(FH)^{+} - FHR_xH^{+} + FHR_x(FH)^{+} + \underbrace{FR_vF^{+}}_{FF^{+}} \end{aligned}}$$

Now, the expectation $\mathbb{E}[\tilde{s}\tilde{s}^{+}]$ depends on the co-variance of the random variable x. (i.e. $R_x = \mathbb{E}(xx^{+})$)

#6 General Combined estimator

$y_1 = H_1 x + v_1 \qquad , \; y_2 = H_2 x + v_2$

$\left\langle \begin{bmatrix} v_1 \\ x \end{bmatrix}, \begin{bmatrix} v_1 \\ x \end{bmatrix} \right\rangle = \left\| \begin{bmatrix} v_1 \\ x \end{bmatrix} \right\|^2 = \begin{bmatrix} R_1 & 0 \\ 0 & M_1 \end{bmatrix} \quad , \left\langle \begin{bmatrix} v_2 \\ x \end{bmatrix}, \begin{bmatrix} v_2 \\ x \end{bmatrix} \right\rangle = \begin{bmatrix} R_2 & 0 \\ 0 & M_2 \end{bmatrix}$

i.e. covariance matrices →

$\hat{x}_1 = R_{xy_1} R_{y_1}^{-1} y_1 = K_1 y_1 = R_{x_1} H_1^T (H_1 R_{x_1} H_1^T + R_{v_1})^{-1} y$, per slide 3 of Lecture 7 for Linear Models of form $y = Hx + v$

$= \underbrace{M_1 H_1^T (H_1 M_1 H_1^T + R_1)^{-1}}_{K_1} y$

$P_1 = M_1 - M_1 H_1^T (H_1 M_1 H_1^T + R_1)^{-1} H_1 R_1 = (M_1^{-1} + H_1^T R_1^{-1} H_1)^{-1}$ (same slide)

$K_1 = M_1 H_1^T (H_1 M_1 H_1^T + R_1)^{-1} = (M_1^{-1} + H_1^T R_1^{-1} H_1)^{-1} H_1^T R_1^{-1}$

$\hat{x}_1 = K_1 y_1$

Similarly, $P_2 =$ " with sub-scripts of 2

$K_2 =$ "

$\hat{x}_2 = K_2 y_2$

Joint estimate:

$y = Hx + v \Rightarrow \cancel{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} x + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ w/ $R_x = \Pi$, $R_v = \begin{pmatrix} R_{v_1} & \cancel{R_{v_1 v_2}} \\ \cancel{R_{v_2 v_1}} & R_{v_2} \end{pmatrix} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$

per Lect. 8.

per Lec. 7.

$\Rightarrow P = (R_x^{-1} + H^T R_v^{-1} H)^{-1}$

$K = (R_x^{-1} + H^T R_v^{-1} H)^{-1} H^T R_v^{-1}$

$x = K y$

→

#6 cont'd

$$P^{-1}\hat{x} = (\Pi^{-1} + H^T R_v^{-1} H)(Ky) = H^T R_v^{-1} y \quad \text{(per Lec. 7 slide 3)}$$

$$= \left(\Pi^{-1} + [H_1^T \;\; H_2^T]\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}\right)\left(\Pi^{-1} + [H_1^T \;\; H_2^T]\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}\right)^{-1}[H_1^T \;\; H_2^T]\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}^{-1} y$$

$$= \left(\Pi^{-1} + [H_1^T \;\; H_2^T]\begin{bmatrix} R_1 H_1 \\ R_2 H_2 \end{bmatrix}\right)\left(\Pi^{-1} + [H_1^T \;\; H_2^T]\begin{bmatrix} R_1 H_1 \\ R_2 H_2 \end{bmatrix}\right)^{-1}[H_1^T \;\; H_2^T]\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}^{-1} \vec{y}$$

$$= (\Pi^{-1} + H_1^T R_1 H_1 + H_2^T R_2 H_2)(\Pi^{-1} + H_1^T R_1 H_1 + H_2^T R_2 H_2)^{-1}[H_1^T \;\; H_2^T]\begin{pmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{pmatrix} y$$

$$= [H_1^T R_1^{-1} \;\; H_2^T R_2^{-1}]\begin{pmatrix} \vec{y}_1 \\ y_2 \end{pmatrix} = H^T R_1^{-1} \vec{y}_1 + H_2^T R_2^{-1} \vec{y}_2$$

$$P_1^{-1}\hat{x}_1 = H_1^T R_1^{-1} \vec{y}_1$$

$$P_2^{-1}\hat{x}_2 = H_2^T R_2^{-1} \vec{y}_2$$

$$\Rightarrow \boxed{P^{-1}\hat{x} = H_1^T R_1^{-1} y_1 + H_2^T R_2^{-1} \vec{y}_2 = P_1^{-1}\hat{x}_1 + P_2^{-1}\hat{x}_2} \quad \square$$

$$P^{-1} = \Pi^{-1} + H^T R_v^{-1} H = \Pi^{-1} + H_1^T R_1 H_1 + H_2^T R_2 H_2$$

$$P_1^{-1} = M_1^{-1} + H_1^T R_1^{-1} H_1$$

$$P_2^{-1} = M_2^{-1} + H_2^T R_2^{-1} H_2$$

$$\boxed{P_1^{-1} + P_2^{-1} + \Pi^{-1} - M_1^{-1} - M_2^{-1} = H_1^T R_1^{-1} H_1 + H_2^T R_2^{-1} H_2 + \Pi^{-1} = P^{-1}} \quad \square$$

#7 Optimal Estimation for exponential distribution

$y = x + v$, x, v are independent real-valued random variables w/ exponential distribution of parameters $\lambda > 0$ and $\mu > 0$ respectively

exponential distribution: of form $\begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & x < 0 \end{cases}$, mean $= \lambda^{-1}$, variance $= \lambda^{-2}$

$p(x) = \lambda e^{-\lambda x}$, $x \geq 0$

$p(v) = \mu e^{-\mu v}$, $v \geq 0$

#7-Q1 → next page (7-2)

#7-Q2 The ~~sum~~ pdf of a sum of two indep. random var. is the convolution of their individual pdf's :

$\because$ $y = x + v$

$$P_y(y) = \int_{-\infty}^{\infty} p_x(\tau)\, p_v(y-\tau)\, d\tau$$

$$= \int_0^y \lambda e^{-\lambda\tau} \mu e^{-\mu(y-\tau)}\, d\tau$$

$$= \lambda\mu e^{-\mu y} \int_0^y e^{-\lambda\tau} e^{\mu\tau}\, d\tau = \lambda\mu e^{-\mu y} \int_0^y e^{(\mu-\lambda)\tau}\, d\tau$$

$$= \lambda\mu e^{-\mu y} \left[\frac{e^{(\mu-\lambda)y}}{(\mu-\lambda)} - \frac{e^{(\mu-\lambda)0}}{\mu-\lambda} \right]$$

$$= \left[\frac{\lambda\mu e^{\mu y}}{e^{\mu y} e^{\lambda y}} - \frac{\lambda\mu e^{\lambda y}}{e^{\mu y}} \right] \Big/ (\mu-\lambda) = \lambda\mu \left[\frac{e^{\mu y} - e^{\lambda y}}{e^{\lambda y} e^{\mu y}} \right] \left(\frac{1}{\mu-\lambda} \right)$$

$$= \frac{\lambda\mu}{\mu-\lambda} \left[\frac{e^{\mu y} - e^{\lambda y}}{e^{\lambda y} e^{\mu y}} \right] = \frac{\lambda\mu}{\mu-\lambda} \left(e^{-\lambda y} - e^{-\mu y} \right)$$

$$\boxed{= \frac{\lambda\mu}{\lambda-\mu} \left(e^{-\mu y} - e^{-\lambda y} \right)} \quad , y \geq 0$$

$\boxed{\#7 - Q1}$ ~~assume Gaussian distributions.~~

from https://faculty.math.illinois.edu/~r-ash/Stat/StatLec1-5.pdf , pg 7

if X, Y are independent random variables w/ pdf's f_X, f_Y

let $Z = XY$, $W = Y$

note $x>0, y>0 \Leftrightarrow z>0, w>0$

(Jacobian)

then $f_{ZW}(z,w) = \dfrac{f_{XY}(x,y)}{\left|\dfrac{\partial(z,w)}{\partial(x,y)}\right|}$, $\left|\dfrac{\partial(z,w)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y = w$

$\Rightarrow f_{ZW}(z,w) = \dfrac{f_{XY}(x,y)}{w} = \dfrac{f_X(x) f_X(y)}{w} = \dfrac{f_X(z/w) f_Y(w)}{w}$

also, $f_Z(z) = \displaystyle\int_{-\infty}^{\infty} f_{ZW}(z,w)\, dw = \int_0^{\infty} \frac{1}{w} f_X(z/w) f_Y(w)\, dw$

marginal density from joint density

---

for our case, we want $p(x,y)$, the joint density of x and y. $y = x+v$

~~so, let $Z = XV$, $W = V$~~

given $Y = X + V$, let $W = X$ $\rightarrow V = Y - X \Rightarrow \boxed{y \geq x \text{ b/c } v \geq 0}$

then $\cancel{f_{XY}(x,y) = f_{XV}(x,v)}$

$\cancel{f_{YW}(y,w) = f_{YX}(y,x) = f_{XY}(x,y) = \dfrac{f_{XV}(x,v)}{\left|\frac{\partial(y,w)}{\partial(x,v)}\right|}}$, $\cancel{\left|\dfrac{\partial(y,w)}{\partial(x,v)}\right| = \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial v} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial v} \end{vmatrix}}$

$f_{XY}(x,y) = \dfrac{f_{XV}(x,v)}{\left|\dfrac{\partial(x,y)}{\partial(x,v)}\right|}$, $\left|\dfrac{\partial(x,y)}{\partial(x,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

$f_{X,Y}(x,y) = f_{XV}(x,v) = f_X(x) f_V(y-x) = \lambda e^{-\lambda x} \mu e^{-\mu(y-x)} = \lambda\mu e^{-(\lambda-\mu)x} e^{-\mu y}$

$\boxed{f_{XY}(x,y) = \begin{cases} \lambda\mu e^{-(\lambda-\mu)x} e^{-\mu y} & x \leq y \\ 0 & \text{else} \end{cases}}$

#7 - ~~Q3~~

(non-linear)

Q4 Show ~~optimal~~ ~~least mean square estimate of~~ x ~~given~~ y ~~is~~

$$\hat{x} = \cancel{\frac{1}{\lambda-\mu}} - \left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right) y$$

→ instead, performed calc for LLSE (#7-Q4)

From class, $\hat{x} = \mathbb{E}(x|y) = R_{xy} R_y^{-1} y$ (for centered variables) | we have affine estimation: $\hat{x} = \mu_x + R_{xy} R_y^{-1} (y - \mu_y)$

$R_{xy} = \cancel{\mathbb{E}(xy^T)} = \cancel{\mathbb{E}(x(x+v)^T) = \mathbb{E}(xx^T) + \mathbb{E}(xv^T)}$ (not 0-mean!)

$= \mathbb{E}[(x - E(x))(y - E(y))^T] = E[xy^T - xE(y)^T - E(x)y^T + E(x)E(y)^T]$

note: $\mathbb{E}(y) = \int_0^\infty y p_y(y)\, dy = \frac{\lambda\mu}{\lambda-\mu} \int_0^\infty y(e^{-\mu y} - e^{-\lambda y}) dy$

$= \frac{\lambda\mu}{\lambda-\mu}\left[\int_0^\infty y e^{-\mu y} dy - \int_0^\infty y e^{-\lambda y} dy\right]$

similar to $\mathbb{E}[x]$ or $\mathbb{E}[v]$

$= \frac{\lambda\mu}{\lambda-\mu}\left[\frac{1}{\mu^2} - \frac{1}{\lambda^2}\right]$

$= \frac{1}{\lambda-\mu}\left[\frac{\lambda}{\mu} - \frac{\mu}{\lambda}\right] = \frac{1}{\lambda-\mu}\left[\frac{\lambda^2 - \mu^2}{\lambda\mu}\right] = \boxed{\frac{\lambda^2 - \mu^2}{(\lambda-\mu)\lambda\mu}}$

note 2: $\mathbb{E}[x] = \int_0^\infty x p_x(x) dx$

$= \int_0^\infty x\lambda e^{-\lambda x} dx$

$= \frac{1}{\lambda}$

$\Rightarrow \cancel{\mathbb{E}[xy] = \int_0^\infty \int_0^\infty}$

$\Rightarrow R_{xy} = \mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y] - \cancel{\mathbb{E}[x]\mathbb{E}[y]} + \cancel{\mathbb{E}[x]\mathbb{E}[y]} = \mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y]$

$R_y = \mathbb{E}[(y - \mathbb{E}(y))^2] = \mathbb{E}[y^2] - \mathbb{E}(y)^2 \cancel{- \mathbb{E}(y)^2} + \cancel{\mathbb{E}(y)^2}$

$R_y^{-1} = \frac{1}{R_y}$ (scalar) $\Rightarrow \frac{1}{\mathbb{E}(y^2) - \mathbb{E}(y)^2}$

$$\Rightarrow R_{xy} R_y^{-1} = \frac{\mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y]}{\mathbb{E}(y^2) - \mathbb{E}(y)^2} = \frac{\mathbb{E}(xy)}{\mathbb{E}(y^2) - \mathbb{E}(y)^2} - \frac{\mathbb{E}(x)\mathbb{E}(y)}{E(y^2) - E(y)^2}$$

→

$\mathbb{E}[xy] = \int_0^\infty \int_0^\infty xy\, p_{xy}(x,y)\,dx\,dy$ note that $p_{XY}(x,y) = 0$ for $x \geq y$

$= \int_0^\infty \int_0^\infty xy(\lambda\mu e^{-(\lambda-\mu)x} e^{-\mu y})\,dx\,dy$

$\cancel{= \mathbb{E}(xg(y)) = \mathbb{E}_y[\mathbb{E}[x|y]] = \mathbb{E}_y(\mathbb{E}_{xy}(xg(y)|y)) = \mathbb{E}_y(g(y)\,\mathbb{E}_{x|y}(x|y))}$

(let $g(y) = y$)

$= \int_0^\infty \int_0^y xy(\lambda\mu e^{-(\lambda-\mu)x} e^{-\mu y})\,dx\,dy$

$= \int_0^\infty \lambda\mu y e^{-\mu y}\left(\int_0^y x e^{-(\lambda-\mu)x}\,dx\right)dy$

$= \int_0^\infty \lambda\mu y e^{-\mu y}\Big[$

Integrate by parts:

$\int f\,dg = fg - \int g\,df$

let $f = x$, $dg = e^{-(\lambda-\mu)x}\,dx$

$df = dx$, $g = \frac{e^{-(\lambda-\mu)x}}{-(\lambda-\mu)}$

$\Rightarrow \frac{-xe^{-(\lambda-\mu)x}}{(\lambda-\mu)} + \int_0^y \frac{e^{-(\lambda-\mu)x}}{(\lambda-\mu)}\,dx$

$= \left(\frac{1}{\lambda-\mu}\right)\left[-xe^{-(\lambda-\mu)x} + \left[\frac{e^{-(\lambda-\mu)x}}{-(\lambda-\mu)}\right]\right]_0^y$

$= \left(\frac{-1}{\lambda-\mu}\right)\left[ye^{-(\lambda-\mu)y} + \left[\frac{e^{-(\lambda-\mu)y} - 1}{(\lambda-\mu)}\right]\right]$

$= \left(\frac{-1}{(\lambda-\mu)^2}\right)\left[ye^{-(\lambda-\mu)y}(\lambda-\mu) + e^{-(\lambda-\mu)y} - 1\right]$

$= \frac{-1}{(\lambda-\mu)^2}\left[e^{-(\lambda-\mu)y}(y(\lambda-\mu) + 1) - 1\right]$

$=$

$\longrightarrow$

$$\Rightarrow \mathbb{E}[xy] = \int_0^\infty \lambda\mu y e^{-\mu y}\left[\left(\frac{-1}{(\lambda-\mu)^2}\right)\left[y e^{-(\lambda-\mu)y}(\lambda-\mu) + e^{-(\lambda-\mu)y} - 1\right]\right] dy$$

$$= \frac{\lambda\mu}{(\lambda-\mu)^2}\int_0^\infty y e^{-\mu y}\left[y e^{-(\lambda-\mu)y}(\lambda-\mu) + e^{-(\lambda-\mu)y} - 1\right] dy$$

$$= \left(\frac{-\lambda\mu}{(\lambda-\mu)^2}\right)\underbrace{\left[\int_0^\infty y^2 e^{-\mu y} e^{-(\lambda-\mu)y}(\lambda-\mu)\,dy + \int_0^\infty y e^{-\mu y} e^{-(\lambda-\mu)y}\,dy - \int_0^\infty y e^{-\mu y}\,dy\right]}_{\Downarrow}$$

$$(\lambda-\mu)\int_0^\infty y^2 \underbrace{e^{-\mu y} e^{-(\lambda-\mu)y}}_{e^{-\mu y-(\lambda-\mu)y}}\,dy + \int_0^\infty y e^{-\mu y-(\lambda-\mu)y}\,dy - \int_0^\infty y e^{-\mu y}\,dy$$

$$-\mu y - (\lambda-\mu)y = \cancel{-\mu y} - \lambda y + \cancel{\mu y} = -\lambda y$$

$$\Rightarrow (\lambda-\mu)\int_0^\infty y^2 e^{-\lambda y}\,dy + \int_0^\infty y e^{-\lambda y}\,dy - \int_0^\infty y e^{-\mu y}\,dy$$

$$\Rightarrow (\lambda-\mu)\left(\frac{2}{\lambda^3}\right) + \frac{1}{\lambda^2} - \frac{1}{\mu^2}$$

$$\Rightarrow \boxed{\mathbb{E}[xy] = \left(\frac{-\lambda\mu}{(\lambda-\mu)^2}\right)\left(\frac{2(\lambda-\mu)}{\lambda^3} + \frac{1}{\lambda^2} - \frac{1}{\mu^2}\right)}$$

✓ Checked w/ MATLAB

$$\mathbb{E}[y^2] = \int_0^\infty y^2 p_y(y)\,dy = \int_0^\infty y^2\left(\frac{\lambda\mu}{\lambda-\mu}\right)\left(e^{-\mu y} - e^{-\lambda y}\right) dy$$

$$\boxed{\mathbb{E}[y^2] = \left(\frac{\lambda\mu}{\lambda-\mu}\right)\left(\frac{2}{\mu^3} - \frac{2}{\lambda^3}\right)}$$

$$\mathbb{E}[x] = \frac{1}{\lambda} = \mu_x\,,$$

$$\Rightarrow \hat{x} = \mu_x + R_{xy} R_y^{-1}(y - \mu_y)$$

$$= \frac{1}{\lambda} + \frac{\left(\mathbb{E}[xy] - \mathbb{E}[x]\,\mathbb{E}[y]\right)}{\left(\mathbb{E}[y^2] - \mathbb{E}[y]^2\right)}\left(y - \mathbb{E}[y]\right) \longrightarrow$$

$$\hat{x} = \mathbb{E}[x] + \frac{(\mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y])}{\mathbb{E}[y^2] - \mathbb{E}[y]^2}(y - \mathbb{E}[y])$$

$\Rightarrow$ MATLAB

See page 7-12 for further comments

$\Rightarrow \hat{x} = \dfrac{y\mu^2 - \mu + \lambda}{\lambda^2 + \mu^2}$ no exponentials

does not match → this is the LLSE for part $\boxed{Q4}$

↑ comment:

~~similar form..~~

~~$\hat{x} = -$~~

What went wrong?? How to get $e^{-\lambda y}$ and $e^{-\mu y}$ into $\hat{x}$??

$$\hat{x} = \mu_x + R_{xy} R_y^{-1}(y - \mu_y)$$

$$\mu_x = \frac{1}{\lambda}$$

$$\mu_y = \mathbb{E}[y] = \mathbb{E}[x+v] = \frac{1}{\lambda} + \frac{1}{\mu}$$

$$R_{xy} = \mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y] = \mathbb{E}[x(x+v)] - \mathbb{E}[x]\mathbb{E}[x+v]$$

$$= \mathbb{E}[x^2] + \mathbb{E}[xv] - \mathbb{E}[x]\mathbb{E}[x] - \mathbb{E}[x]\mathbb{E}[v]$$

$$= \cancel{\frac{1}{\lambda^2}} + \cancel{\frac{1}{\lambda}\left(\frac{1}{\mu}\right)} - \cancel{\frac{1}{\lambda^2}} - \cancel{\frac{1}{\lambda}\frac{1}{\mu}} = 0$$

$\boxed{\#7\text{-}Q3}$

Non-linear, optimal estimate:

let $h^* = \underset{h}{\operatorname{argmin}}\ \mathbb{E}[(x-h(y))(x-h(y))^T]$

since $\mathbb{E}[(x-h(y))(x-h(y))^T] \geq 0$ by definition,

then if we can find $h(y)$ s.t. $\mathbb{E}_{xh} = 0$, $h(y) = h^*$ since $\mathbb{E}_{xh}$ cannot be any lower value.

~~Calculat~~ calculation →

#7-Q3 Start on page 7-8

$\mathbb{E}$ Let $h(y) = \hat{x} = \frac{1}{\lambda-\mu} - \left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right) y = \frac{1}{\lambda-\mu} - \left(\frac{e^{-\lambda(x+v)}}{e^{-\mu(x+v)} - e^{-\lambda(x+v)}}\right)(x+v)$

$\Rightarrow \mathbb{E}\left[(x-\hat{x})(x-\hat{x})^T\right] = \mathbb{E}\left[(x-\hat{x})^2\right] = \mathbb{E}[x^2] - \mathbb{E}[x\hat{x}] - \mathbb{E}[\hat{x}x] + \mathbb{E}[\hat{x}^2]$

$\mathbb{E}[x^2] = \frac{1}{\lambda^2}$

$\mathbb{E}[x\hat{x}] = \mathbb{E}\left[\frac{x}{\lambda-\mu} - \left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right) xy\right]$ $\Rightarrow$ MATLAB

$= \frac{1}{\lambda(\lambda-\mu)} -$

$\mathbb{E}[\hat{x}x] = \mathbb{E}[x\hat{x}]$ (scalar) $= \mathbb{E}\left[\frac{x}{\lambda-\mu} - x(x+v)\left(\frac{e^{-\lambda x}e^{-\lambda v}}{e^{-\mu x}e^{-\mu v} - e^{-\lambda x}e^{-\lambda v}}\right)\right]$

$\mathbb{E}[\hat{x}^2] =$ $= \frac{1}{\lambda^2}\left(\frac{1}{\lambda-\mu}\right) - \mathbb{E}\left[(x^2+xv)\left(\frac{e^{-\lambda x}e^{-\lambda v}}{e^{-\mu x}e^{-\mu v} - e^{-\lambda x}e^{-\lambda v}}\right)\right]$

too long...

#7-Q3 on page 7-8

#7 - Q3 Show that the optimal (non-linear) LMS estimate of x given y is:

$$\hat{x} = \frac{1}{\lambda-\mu} - \left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right) y$$

$p(x) = \lambda e^{-\lambda x}$ $\quad \lambda > 0$

$p(v) = \mu e^{-\mu v}$ $\quad \mu > 0$

$y = x + v$ $\quad x, v \in \mathbb{R}$, random, independent

$\Rightarrow R_{xv} = R_{vx} = 0$

$P_{xv}(x,v) = p(x)p(v)$

note: $p(x,y) = \lambda\mu e^{-(\lambda-\mu)x} e^{-\mu y}$

and; $p(y) = \frac{\lambda\mu}{\lambda-\mu}(e^{-\mu y} - e^{-\lambda y})$

~~$\mathbb{E}(x|y) = ? = \int_{-\infty}^{\infty} x\, p(x,y)\, dy$ (?)~~

note: marginal density from a joint density:

$$p(y) = \int_{-\infty}^{\infty} p(x,y)\, dx$$

in our case: $x \geq 0$, $x \leq y$

$$\Rightarrow p(y) = \int_0^y p(x,y)\, dx = \frac{\lambda\mu}{\lambda-\mu}(e^{-\mu y} - e^{-\lambda y}) \quad ✓$$

note: expectation of a variable $\mathbb{E}(x) = \int_{-\infty}^{\infty} x\, p_x(x)\, dx$

note: the conditional pdf is: $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

note: conditional expectation:

$$= \mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

$$\Rightarrow \mathbb{E}[X|Y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, d\cancel{y}x$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\cancel{\lambda\mu}(e^{-(\lambda-\mu)x} e^{-\mu y})}{\frac{\cancel{\lambda\mu}}{(\lambda-\mu)}(e^{-\mu y} - e^{-\lambda y})} \rightarrow$$

#7 - Q3 cont'd

$$f_{X|Y}(x|y) = (\lambda-\mu)\,\frac{\left(e^{-(\lambda-\mu)x}\,e^{-\mu y}\right)}{e^{-\mu y}-e^{-\lambda y}}$$

$\mathbb{E}[X|Y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dy$ where $x \geq 0$, $x \leq y$ due to the constituent pdf's

$$= \int_0^y x\,(\lambda-\mu)\,\frac{\left(e^{-(\lambda-\mu)x}\,e^{-\mu y}\right)}{e^{-\mu y}-e^{-\lambda y}}\,dx$$

$$= \frac{(\lambda-\mu)\,e^{-\mu y}}{e^{-\mu y}-e^{-\lambda y}}\int_0^y x\,e^{-(\lambda-\mu)x}\,dx$$

Integration by parts:

$\int f\,dg = fg - \int g\,df$

$f = x \quad dg = e^{-(\lambda-\mu)x}\,dx$

$df = dx \quad g = \frac{-e^{-(\lambda-\mu)x}}{\lambda-\mu}$

$$\Rightarrow -\frac{x\,e^{-(\lambda-\mu)x}}{(\lambda-\mu)} + \int \frac{e^{-(\lambda-\mu)x}}{\lambda-\mu}\,dx$$

$$+\frac{(-1)\,e^{-(\lambda-\mu)x}}{(\lambda-\mu)^2}$$

$$= \left.\frac{-x(\lambda-\mu)e^{-(\lambda-\mu)x} - e^{-(\lambda-\mu)x}}{(\lambda-\mu)^2}\right|_0^y$$

$$\Rightarrow \int_0^y x\,e^{-(\lambda-\mu)x}\,dx = \frac{-y(\lambda-\mu)e^{-(\lambda-\mu)y} - e^{-(\lambda-\mu)y}}{(\lambda-\mu)^2} + \frac{1}{(\lambda-\mu)^2}$$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{\cancel{(\lambda-\mu)}\,e^{-\mu y}}{e^{-\mu y}-e^{-\lambda y}}\left[\left(\frac{1}{(\lambda-\mu)^{\cancel{2}}}\right)\left(-y(\lambda-\mu)e^{-(\lambda-\mu)y} - e^{-(\lambda-\mu)y} + 1\right)\right]$$

$$= \left(\frac{e^{-\mu y}}{e^{-\mu y}-e^{-\lambda y}}\right)\left[-y\,e^{-(\lambda-\mu)y} - \frac{e^{-(\lambda-\mu)y}}{\lambda-\mu} + \frac{1}{\lambda-\mu}\right]$$

$\longrightarrow$

#7 - Q3 cont'd

$$= \left[\left(\frac{1}{\lambda-\mu}\right)\left(1-e^{-(\lambda-\mu)y}\right) - y e^{-(\lambda-\mu)y}\right]\left(\frac{e^{-\mu y}}{e^{-\mu y}-e^{-\lambda y}}\right)$$

$$= \left(\frac{1}{\lambda-\mu}\right)\left(\frac{\left(1-e^{-(\lambda-\mu)y}\right)e^{-\mu y}}{e^{-\mu y}-e^{-\lambda y}}\right) - y\,\frac{e^{(-\lambda+\mu-\mu)y}}{e^{-\mu y}-e^{-\lambda y}}$$

$$= \left(\frac{1}{\lambda-\mu}\right)\left(\frac{\cancel{e^{-\mu y}-e^{-\lambda y}}}{\cancel{e^{-\mu y}-e^{-\lambda y}}}\right) - y\left(\frac{e^{-\lambda y}}{e^{-\mu y}-e^{-\lambda y}}\right)$$

$$\boxed{\mathbb{E}[X|Y] = \frac{1}{\lambda-\mu} - y\left(\frac{e^{-\lambda y}}{e^{-\mu y}-e^{-\lambda y}}\right)} \quad \square$$

#7-Q4)

let $\hat{x} = Ky$, $P(K) = \cancel{E(x-y)} \, E\left((x-\hat{x})(x-\hat{x})^T\right)$,

$= E\left((x-Ky)(x-Ky)\right)$, $y = x+v$

$= E\left((x-K(x+v))(x-K(x+v))\right)$

$= E\left(x^2 - 2K(x+v) + K^2(x+v)^2\right)$

$= E\left(x^2 - 2K(x+v) + K^2(x^2+2xv+v^2)\right)$

$= E\left(x^2 - 2Kx - 2Kv + K^2x^2 + 2K^2xv + K^2v^2\right)$

$= R_x - 2KE(x) - 2KE(v) + K^2R_x + 2K^2R_{xv}^{\,0} + K^2R_v$

$= \frac{1}{\lambda^2} - 2K\frac{1}{\lambda} - 2K\frac{1}{\mu} + K^2\frac{1}{\lambda^2} + 2K^2(0) + K^2\frac{1}{\mu^2} \rightarrow$ (P(K) is P.S.D by definition)

$p(x) = \lambda e^{-\lambda x}$

$E(x) = \frac{1}{\lambda}$

$R_x = \frac{1}{\lambda^2}$

$p(v) = \mu e^{-\mu v}$

$E_v = \frac{1}{\mu}$

$\sigma_v^2 = \frac{1}{\mu^2}$

independent ⇒ uncorrelated ⇓ $R_{xv} = 0$

$\frac{\partial P(K)}{\partial K} = 0 - \frac{2}{\lambda} - \frac{2}{\mu} + \frac{2K}{\lambda^2} + \frac{2K}{\mu^2} = 0$ ← set equal to 0 to find minimum since P(K) is P.S.D. and is 2nd order function of K

$\Rightarrow 2K\left(\frac{1}{\lambda^2} + \frac{1}{\mu^2}\right) = 2\left(\frac{1}{\lambda} + \frac{1}{\mu}\right)$

$K\left(\frac{\mu^2+\lambda^2}{\mu^2\lambda^2}\right) = \left(\frac{\mu+\lambda}{\lambda\mu}\right)$

$\Rightarrow \boxed{K_0 = \left(\frac{\mu+\lambda}{\mu\lambda}\right)\left(\frac{\mu^2\lambda^2}{\mu^2+\lambda^2}\right) = \frac{(\mu+\lambda)\mu\lambda}{\mu^2+\lambda^2}}$

for a linear equation of $y = x+v$, a LMS estimate (per class notes Lec. 6) is $\hat{x} = K_0 y$,

$K_0 = R_{xy} R_y^{-1}$

⇒ plug K_0 back in to $P(K)$:

$P(K_0) = \frac{1}{\lambda^2} - \frac{2K_0}{\lambda} - \frac{2K_0}{\mu} + \frac{K_0^2}{\lambda^2} + \frac{K_0^2}{\mu^2}$

⇒ MATLAB

⇒ $P(K_0) =$

→

#7-Q4 cont'd

per Lec. 6 notes, the affine solution to a linear estimator (where the random variables do not have a 0-mean), we have the following scenario:

$$\hat{x} = \mu_x + R_{xy} R_y^{-1} (y - \mu_y)$$

So, compute it:

$$R_{xy} = \mathbb{E}((x-\mu_x)(y-\mu_y)) = \mathbb{E}((x-\mu_x)((x+v) - \mathbb{E}(x+v)))$$
$$= \mathbb{E}((x-\mu_x)((x+v) - \mu_x - \mu_v))$$
$$= \mathbb{E}((x-\mu_x)((x-\mu_x) + (v-\mu_v)))$$
$$= \mathbb{E}((x-\mu_x)(x-\mu_x) + (x-\mu_x)(v-\mu_v))$$
$$= R_x + \cancelto{0}{R_{xv}}$$

(b/c independent, $\mathbb{E}[xv] = \mathbb{E}[x]\mathbb{E}[v]$)

$R_{xv} = \mathbb{E}(xv) - \mathbb{E}(x)\mathbb{E}(v)$ by definition of covariance.

$$= \frac{1}{\lambda^2} + 0$$
$$= \frac{1}{\lambda^2}$$

$$R_y^{-1} = (\mathbb{E}((y-\mu_y)^2))^{-1} = [\mathbb{E}((x-\mu_x + v-\mu_v)(x-\mu_x + v-\mu_v))]^{-1}$$
$$= [\mathbb{E}((x-\mu_x)^2 + 2(x-\mu_x)(v-\mu_v) + (v-\mu_v)^2)]^{-1}$$
$$= [R_x + 2\cancelto{0}{R_{xv}} + R_v]^{-1}$$
$$= \left[\frac{1}{\lambda^2} + \frac{1}{\mu^2}\right]^{-1} = \left(\frac{\mu^2+\lambda^2}{\mu^2\lambda^2}\right)^{-1} = \frac{\lambda^2\mu^2}{\lambda^2+\mu^2}$$

$$\Rightarrow K = R_{xy} R_y^{-1} = \frac{1}{\lambda^2}\left(\frac{\lambda^2\mu^2}{\lambda^2+\mu^2}\right) = \frac{\mu^2}{\lambda^2+\mu^2}$$

$$\Rightarrow \hat{x} = \frac{1}{\lambda} + \frac{\mu^2}{(\lambda^2+\mu^2)}(y-\mu_y) = \frac{1}{\lambda} + \frac{\mu^2}{\lambda^2+\mu^2}\left(y - \frac{\lambda+\mu}{\lambda\mu}\right)$$

Also, $\hat{x}_{LLMS}$ has no exponential terms with y

$\Rightarrow$ MATLAB $\Rightarrow$ $\boxed{= \frac{\lambda-\mu}{\mu^2+\lambda^2} + \left(\frac{\mu^2}{\mu^2+\lambda^2}\right) y = \hat{x}_{LLMS}}$

comment:
both $\hat{x}_{non\text{-}LLMS}$ and $\hat{x}_{LLMS}$ have an offset term due to non-0 mean distributions. But the coefficient term for y in $\hat{x}_{LLMS}$ is constant, whereas it changes as a function of y in $\hat{x}_{non\text{-}LLMS}$.

$\#8-$ ~~Q1~~ Optimal Nonlinear estimator for binary signals

observations: $y_i = x + v_i$, x & v_i are independent, real-valued, random

$\{v_i\}_{i\geq 0} \neq V$ is a white-noise Gaussian process

$R_v = 1$, $\mathbb{E}(v) = \mu_v = 0$

$\#8$-Q1 let $x = \pm 1$ w/ equal probability

Show the optimal nonlinear LMS estimator of x given n obs $\{y_i\}_{i=0}^{n-1}$ is

$$\hat{x}_n = \tanh\left(\sum_{i=0}^{n-1} y_i\right)$$

note: expectation of a discrete variable: $\mathbb{E}(X) = \sum_{i=0}^{n-1} x_i p(x_i)$ for n possible states

note: joint probability mass function (pmf):

$$p_{X,Y}(x,y) = P(X=x \text{ and } Y=y) = P(X \cap Y)$$

in terms of conditional distributions:

$$p_{X,Y}(x,y) = P(Y=y \mid X=x) \cdot P(X=x) = P(X=x \mid Y=y) \cdot P(Y=y)$$ (chain rule of probability)

note: conditional pmf:

$$f(x|X) = P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$$

example: if we want to know the prob. that two die = 3, and we know one die = 3, then $p(x|y) = \frac{1}{6}$ (and not $\frac{1}{36}$)

note: conditional expectation of a pmf:

$$\mathbb{E}[X|Y] = \sum_x x \, f_{X|Y}(x|Y)$$

$$= \sum_x x \, \frac{P(X,Y)}{P(Y)}$$

$\longrightarrow$

#8-Q1 cont'd)

optimal estimator as shown in class is $\mathbb{E}[X|Y]$, thus:

$$\mathbb{E}[X|Y] = \sum_i x_i \frac{P(X_i, Y)}{P(Y)}$$

$$= \cancel{(\sum)} \, x_1 \frac{P(X_1, Y)}{P(Y)} + x_2 \frac{P(X_2, Y)}{P(Y)}$$

$P(y) = ?$

n observations: $\{y_i\}_{i=0}^{n-1}$, $P(Y) = \frac{1}{n}$? $x = y_i - v_i$

~~Similar to #7:~~

$y = x + v$, x, v independent. $\Rightarrow$ $f_{XY}(x,y) = \dfrac{f_{XV}(x,v)}{\left|\frac{\partial(x,y)}{\partial(x,v)}\right|} = \dfrac{f_X(x) f_V(v)}{\left|\frac{\partial(x,y)}{\partial(x,v)}\right|}$

$v = y - x$

$$\left|\frac{\partial(x,y)}{\partial(x,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \Rightarrow f_{XY}(x,y) = f_X(x) f_V(y-x)$$

OK...

$$\boxed{f_X(x) = \begin{cases} 0.5 & \text{if } x = \pm 1 \\ 0 & \text{else} \end{cases}}$$

$f_V(v) = \frac{1}{\sigma_v \sqrt{2\pi}} \exp\left(\frac{-(v-\mu)^2}{2\sigma_v}\right)$, $\mu_v = 0$, $\sigma_v = 1$, $\sigma_v^2 = 1$ $\Rightarrow f_V(v) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-v^2}{2}\right)$

Gaussian pdf

$$\Rightarrow f_V(y-x) = \frac{\exp\left(\frac{-(y-x)^2}{2}\right)}{\sqrt{2\pi}}$$

$$\Rightarrow f_{X,Y}(x,y) = f_X(x) \left(\frac{1}{\sqrt{2\pi}}\right) \exp\left(\frac{-(y-x)^2}{2}\right) \longrightarrow$$

#8-Q1 cont'd

$$f_Y = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \sum_{x_i} f_X(x_i) \frac{\exp\left(\frac{-(y-x_i)^2}{2}\right)}{\sqrt{2\pi}} \qquad \{x\} = \{1, -1\}$$

$$f_Y = 0.5 \frac{\exp(-(y-1)^2/2)}{\sqrt{2\pi}} + 0.5 \frac{\exp\left(\frac{-(y+1)^2}{2}\right)}{\sqrt{2\pi}}$$

$$\Rightarrow \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)\left(\cancel{\frac{1}{\sqrt{2\pi}}}\right)\exp\left(\frac{-(y-x)^2}{2}\right)}{\frac{0.5}{\cancel{\sqrt{2\pi}}}\left[\exp\left(\frac{-(y-1)^2}{2}\right) + \exp\left(\frac{-(y+1)^2}{2}\right)\right]}$$

$$= \frac{2 f_X(x) \exp\left(\frac{-(y-x)^2}{2}\right)}{\left(\exp\left(\frac{-(y-1)^2}{2}\right) + \exp\left(\frac{-(y+1)^2}{2}\right)\right)}$$

$$\Rightarrow E[X|Y] = \sum_{x_i} x_i \frac{f_{X,Y}(x_i, y)}{f(y)}$$

$$= 1\left(\frac{\cancel{2(0.5)}\exp\left(\frac{-(y-1)^2}{2}\right)}{\exp\left(\frac{-(y-1)^2}{2}\right) + \exp\left(\frac{-(y+1)^2}{2}\right)}\right) + (-1)\left(\frac{\cancel{2(0.5)}\exp\left(\frac{-(y+1)^2}{2}\right)}{\exp\left(\frac{-(y-1)^2}{2}\right) + \exp\left(\frac{-(y+1)^2}{2}\right)}\right)$$

$$= \frac{\exp\left(\frac{-(y-1)^2}{2}\right) - \exp\left(\frac{-(y+1)^2}{2}\right)}{\exp\left(\frac{-(y-1)^2}{2}\right) + \exp\left(\frac{-(y+1)^2}{2}\right)} \cdot \frac{e^2}{e^2}$$

$\boxed{??}$

$$e^{(-(y-1)^2)(\frac{1}{2})} = \sqrt{e^{(-(y+1)^2)}}$$

$$\rightarrow \sum_{i=0}^{n-1} y_i \qquad \frac{(y-1)^2}{2} + x = y$$

$$\frac{y^2 - 2y + 1}{2} \qquad x = 2y - \frac{(y)^2+1}{2}$$

treat the sum as one variable?

So I'm almost there... just need to figure out how to get $\underline{f_{X,Y}(x,y) = f_X(x)\exp\left(\sum_{i=0}^{n-1} y_i\right)}$

treat the sum as a single variable?...

hopefully other terms cancel...?

$$x^2 x^{-3} = x^{-1}?$$

??...

$$\frac{x^2}{x^3} = x^{-1}$$

#8-Q1 cont'd)

let $Y = \sum_{i=0}^{n-1} y_i = \sum_{i=0}^{n-1} x + v_i = xn + \sum_{i=0}^{n-1} v_i$, let $V = \sum_{i=0}^{n-1} v_i$

note: b/c v_i is a white noise Gaussian process w/ 0 mean & unit variance,

$\Rightarrow \mathbb{E}(V) = 0$, $\cancel{\mathbb{E}(VV^T) = R_v = I}$

$\Rightarrow \mathbb{E}[V^2] = n\sigma_v^2$ $\quad$ $\mathbb{E}(v_i v_j) = 0$ b/c white noise i,j are uncorrelated

(b/c $\mathbb{E}\{v_i v_j\} = 0$ b/c white noise

$\Rightarrow \sigma_V = \sqrt{n}\,\sigma_v = \sqrt{n}$

$(\sigma_V^2 = n\sigma_v^2) = n$

$\cancel{f_V(V) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(V)^2}{2R_v}\right)}$

pdf of V: still Gaussian:

$$f_V(V) = \frac{1}{\sigma_V\sqrt{2\pi}} \exp\left(\frac{-(V-\cancel{\mu_V})^2}{2\sigma_V^2}\right)$$

$$= \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{(V-\cancel{\mu_v}\sqrt{n})^2}{2\sigma_v^2\sqrt{n}^2}\right)$$

$$= \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{(V-\cancel{0})^2}{2\cancel{\sqrt{n}}\,n}\right) = \frac{1}{\sqrt{2\pi n}} \exp\left(\frac{-V^2}{2\cancel{\sqrt{n}}\,n}\right)$$

$$f_{x,Y}(x,Y) = \frac{f_{xV}(x,\cancel{Y}V)}{\left|\frac{\partial(x,Y)}{\partial(x,\cancel{Y}V)}\right|} = \frac{f_x(x)\,f_V(\cancel{Y}V)}{\left|\frac{\partial(x,Y)}{\partial(x,\cancel{Y}V)}\right|} \quad , \quad \left|\frac{\partial(x,Y)}{\partial(x,V)}\right| = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial V} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ n & \cancel{1}\,n \end{vmatrix} = \cancel{1}\,n$$

$$= f_x(x)\,f_V(\cancel{Y}V)\cancel{/n}$$

$\cancel{f_V = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-V^2}{2}\right)}$

$\boxed{\text{blue is OK, green is best}}$

$\cancel{V = y - x}$?

$$f_Y(Y) = \int_{-\infty}^{\infty} f_{x,Y}(x,Y)\,dx = \sum_{x_i} f_x(x) \frac{\exp\left(-\cancel{Y}V^2/2\sqrt{n}\right)}{\cancel{n}\sqrt{2\pi\sqrt{n}\,\cancel{n}}} = \sum_{x_i} f_x(x) \frac{\exp\left(-\frac{(Y-xn)^2}{2\sqrt{n}}\right)}{\cancel{n}\sqrt{2\pi\sqrt{n}\,\cancel{n}}}$$

$$= 0.5\,\frac{\exp\left(-\frac{(Y-n)^2}{2\sqrt{n}}\right)}{\cancel{n}\sqrt{2\pi\sqrt{n}}} + 0.5\,\frac{\exp\left(\frac{-(Y+n)^2}{2\sqrt{n}}\right)}{\cancel{n}\sqrt{2\pi\sqrt{n}}}$$

$$\frac{f_{x,Y}(x,Y)}{f_Y(Y)} = \frac{f_x(x)\exp\left(-\frac{(Y-xn)^2}{n2\cancel{\sqrt{n}}}\right)\cancel{/\sqrt{2\pi\sqrt{n}}}}{\frac{0.5}{\cancel{\sqrt{2\pi\sqrt{n}}}}\left[\exp\left(-\frac{(Y-n)^2}{n2\sqrt{n}}\right) + \exp\left(-\frac{(Y+n)^2}{n2\sqrt{n}}\right)\right]}$$

$\longrightarrow$

$$\mathbb{E}[X|Y] = \sum_{x_i} x_i \frac{f_{X,Y}(x_i, Y)}{f_Y(Y)}$$

$$= (1)\frac{\overset{0.5}{\cancel{f_x(1)}} \exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right)}{0.5\left(\exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right) + \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)\right)} + \frac{(-1)\cancel{(0.5)} \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)}{\cancel{0.5}(\quad " \quad)}$$

$$= \frac{\exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right) - \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)}{\exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right) + \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)}$$

$$\frac{-(Y-n)^2}{n2\sqrt{n}} + \alpha = Y \qquad \frac{-(Y+n)^2}{n2\sqrt{n}} + \alpha_2 = Y$$

$$\frac{-(Y^2 - 2Yn + n^2)}{n2\sqrt{n}} + \alpha = Y \qquad \frac{-(Y^2 + 2Yn + n^2)}{n2\sqrt{n}} + \alpha_2 = Y$$

$$\Rightarrow \alpha = Y + \frac{Y^2 - 2Yn + n^2}{n2\sqrt{n}} \qquad \alpha_2 = Y + \frac{(Y^2 + 2Yn + n^2)}{n2\sqrt{n}}$$

$$= Y + \frac{Y^2 + n^2}{n2\sqrt{n}} - \frac{2Yn}{n2\sqrt{n}}$$

---

multiply by $\exp\left(\frac{Y^2+n^2}{n2\sqrt{n}}\right) / \exp\left(\frac{Y^2+n^2}{n2\sqrt{n}}\right)$:

$$\mathbb{E}[X|Y] = \frac{\exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right) - \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)}{\exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right) + \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)} \left(\frac{\exp\left(\frac{Y^2+n^2}{n2\sqrt{n}}\right)}{\exp\left(\frac{Y^2+n^2}{n2\sqrt{n}}\right)}\right)$$

$$\frac{-(Y-n)^2}{2\sqrt{n}} \to \frac{-(\cancel{Y^2+n^2} - 2Yn)}{n2\sqrt{n}} + \frac{\cancel{Y^2+n^2}}{n2\sqrt{n}} = \frac{2Yn}{n2\sqrt{n}} = Y\sqrt{n}$$

$$\frac{-(Y+n)^2}{n2\sqrt{n}} + \frac{(Y^2+n^2)}{n2\sqrt{n}} = \frac{-(\cancel{Y^2+n^2} + 2Yn)}{n2\sqrt{n}} + \frac{\cancel{Y^2+n^2}}{n2\sqrt{n}} = -Y\sqrt{n} \quad \Rightarrow$$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{\exp(Y\sqrt{n}) - \exp(-Y\sqrt{n})}{\exp(Y\sqrt{n}) + \exp(-Y\sqrt{n})} = \tanh(\sqrt{n}\,Y)$$

$$Y = \sum_{i=0}^{n-1} y_i$$

$Y\sqrt{n} + \beta = Y$

$\beta = Y - Y\sqrt{n}$

$$\mathbb{E}[V^2] = \mathbb{E}[(\sum v_i)^2] = \mathbb{E}[(v_0 + v_1 + \dots + v_n)(v_0 + v_1 + \dots + v_n)] \quad , \quad \mathbb{E}[v_i v_j] = 0$$

$$= \mathbb{E}[\sum v_i^2] \quad , \quad i = 0, 1, \dots n \qquad \mathbb{E}[v_i^2] = \sigma_v^2$$

$$= n\sigma_v^2$$

$$\Rightarrow \sigma_V = \sqrt{n}\,\sigma_v = \sqrt{n} \quad \checkmark, \ \sigma_V^2 = n$$

$$\hat{x}_n = \boxed{\mathbb{E}[X|Y] = \frac{\exp(Y) - \exp(-Y)}{\exp(Y) + \exp(-Y)} = \tanh(Y) = \tanh\left(\sum_{i=0}^{n-1} y_i\right)}$$

$\square$

#8-Q2) Now assume x takes values of 1 w/ probability p, and -1 w/ probability $(1-p)$

from #8-Q1) we have that: $y_i = x + v_i \Rightarrow \sum_{i=0}^{n-1} y_i = xn + \sum_{i=0}^{n-1} v_i$

$$\Rightarrow Y = xn + V$$

$$f_V(V) = \cancel{f_X(x) f_V(v)} = \frac{1}{\sqrt{2\pi n}} \exp\left(\frac{-V^2}{2n}\right)$$

$$f_{XY}(x,Y) = f_X(x) f_V(V) = f_X(x) \exp\left(\frac{-(Y-xn)^2}{2n}\right) \Big/ \sqrt{2\pi n}$$

$$f_Y(Y) = \sum_{x_i} \frac{f_X(x_i) \exp\left(\frac{-(Y-x_i n)^2}{2n}\right)}{\sqrt{2\pi n}}$$

and $\mathbb{E}[X|Y] = \sum_i x_i \frac{P(x_i, Y)}{P(Y)} = \sum_i x_i \frac{f_{XY}(x_i, Y)}{f_Y(Y)}$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{(+1)(p)\left(\cancel{\frac{1}{\sqrt{2\pi n}}}\right)\left(\exp\left(\frac{-(Y-1n)^2}{2n}\right)\right)}{\frac{(p)\exp\left(\frac{-(Y-n)^2}{2n}\right)}{\cancel{\sqrt{2\pi n}}} + \frac{(1-p)\exp\left(\frac{-(Y+n)^2}{2n}\right)}{\cancel{\sqrt{2\pi n}}}}$$

$$+ \frac{(-1)(1-p)\left(\cancel{\frac{1}{\sqrt{2\pi n}}}\right)\exp\left(\frac{-(Y+n)^2}{2n}\right)}{\frac{(p)\exp\left(\frac{-(Y-n)^2}{2n}\right) + (1-p)\exp\left(\frac{-(Y+n)^2}{2n}\right)}{\cancel{\sqrt{2\pi n}}}}$$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{p\exp\left(\frac{-(Y-n)^2}{2n}\right) - (1-p)\exp\left(\frac{-(Y+n)^2}{2n}\right)}{p\exp\left(\frac{-(Y-n)^2}{2n}\right) + (1-p)\exp\left(\frac{-(Y+n)^2}{2n}\right)} \left(\frac{\exp\left(\frac{Y^2+n^2}{2n}\right)}{\exp\left(\frac{Y^2+n^2}{2n}\right)}\right)$$

$\longrightarrow$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{p\exp(Y) - (1-p)\exp(-Y)}{p(\exp(Y)) + (1-p)\exp(-Y)}$$

$y = \ln x$ $\iff x = e^y$

note:
$$p\exp(Y) = \exp(Y + \ln(p))$$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{\exp(Y+\ln p) - \exp(-Y+\ln(1-p))}{\exp(Y+\ln(p)) + \exp(-Y + \underbrace{\ln(1-p)}_{-(-\ln(1-p))})}$$

note:
$$\ln(p) - \frac{1}{2}\ln\left(\frac{p}{1-p}\right) = \ln(p) - \frac{1}{2}(\ln(p) - \ln(1-p)) = \frac{\ln(p)}{2} - \frac{\ln(1-p)}{2} = \frac{1}{2}\ln\frac{(p)}{(1-p)}$$

$$+\ln(1-p) - \frac{1}{2}\ln\left(\frac{p}{1-p}\right) = +\ln(1-p) - \frac{1}{2}(\ln(p) - \ln(1-p))$$

$$= \cancel{? - \frac{1}{2}(\ln(p)}$$

$$\ln(1-p) + \alpha = \frac{1}{2}\ln\left(\frac{p}{1-p}\right) = -\frac{1}{2}(\ln(p) - \ln(1-p)) = -\frac{1}{2}\ln(p) + \frac{1}{2}\ln(1-p)$$

$$\left[\Rightarrow \alpha = -\frac{1}{2}\ln(p) - \frac{1}{2}\ln(1-p) = \underbrace{-\frac{1}{2}\ln\left(\frac{p}{1-p}\right)}\right]$$

$(-\frac{1}{2})(-\frac{3}{2})$

$$\cancel{\ln(1-p) - \frac{1}{2}(\ln(p) - \ln(1-p))} = \ln(1-p) - \frac{1}{2}\ln(p) + \frac{1}{2}\ln(1-p) = -\cancel{\frac{1}{2}\ln(p) + \frac{3}{2}}\ln(1-p)$$

$$= \ln(\cancel{1-p}) - \frac{1}{2}\ln(p) + \ln(1-p)^{1/2} = \cancel{-\frac{1}{2}(\ln(p) + \ln(1-p)^{-3/2})}$$

$$= -(-\ln(1-p) + \frac{1}{2}\ln(p)^{1/2} + \ln(1-p)^{1/2})$$

$$= -\cancel{\left(\ln\left(\frac{p^{1/2}}{(1-p)(1-p)^{1/2}}\right)\right)} \; ?$$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{\exp(Y+\ln(p)) - \exp(-Y+\ln(1-p))}{\exp(Y+\ln(p)) + \exp(-Y+\ln(1-p))}\left(\frac{\exp\left(-\frac{1}{2}\ln\left(\frac{p}{1-p}\right)\right)}{\exp\left(-\frac{1}{2}\ln\left(\frac{p}{1-p}\right)\right)}\right)$$

$$\Rightarrow \boxed{\mathbb{E}[X|Y] = \frac{\exp\left(\frac{1}{2}\ln\left(\frac{p}{1-p}\right) + Y\right) - \exp\left(-\frac{1}{2}\ln\left(\frac{p}{1-p}\right) - Y\right)}{\exp\left(\frac{1}{2}\ln\left(\frac{p}{1-p}\right) + Y\right) + \exp\left(-\frac{1}{2}\ln\left(\frac{p}{1-p}\right) - Y\right)} = \tanh\left(\frac{1}{2}\ln\left(\frac{p}{1-p}\right) + \sum_{i=0}^{n-1} y_i\right)}$$

$= \hat{x}_n$

$\square$

8-8

ECE 6555 HW2

Teo Wilkening

Due: 11:59pm 9/22/2022

Table of Contents

Question 3

Q3-1

Q3-2

```
syms x K_1 K_2 p z Pi_0 H_1 H_2 R_1 R_2
xhat = K_1*z*(1-p) + K_2*z*p
```

xhat = $K_2\, p\, z - K_1\, z\, (p-1)$

```
MMSE = (x - xhat)*transpose(x - xhat)
```

MMSE = $(x - K_2\, p\, z + K_1\, z\, (p-1))^2$

```
syms z [3 1] matrix
syms K_1 K_2 [3 3] matrix

isequal(K_1*z*z.'*K_2.', K_2*z*z.'*K_1.')
```

```
ans = logical
   0
```

```
symmatrix2sym(K_2*z*z.'*K_1.')
```

ans =

$$\begin{pmatrix} \sigma_6\,\sigma_3 & \sigma_5\,\sigma_3 & \sigma_4\,\sigma_3 \\ \sigma_6\,\sigma_2 & \sigma_5\,\sigma_2 & \sigma_4\,\sigma_2 \\ \sigma_6\,\sigma_1 & \sigma_5\,\sigma_1 & \sigma_4\,\sigma_1 \end{pmatrix}$$

where

$$\sigma_1 = K_{23,1}\,z_1 + K_{23,2}\,z_2 + K_{23,3}\,z_3$$

$$\sigma_2 = K_{22,1}\,z_1 + K_{22,2}\,z_2 + K_{22,3}\,z_3$$

$$\sigma_3 = K_{21,1}\,z_1 + K_{21,2}\,z_2 + K_{21,3}\,z_3$$

$$\sigma_4 = K_{13,1}\,z_1 + K_{13,2}\,z_2 + K_{13,3}\,z_3$$

$$\sigma_5 = K_{12,1}\,z_1 + K_{12,2}\,z_2 + K_{12,3}\,z_3$$

$$\sigma_6 = K_{11,1}\,z_1 + K_{11,2}\,z_2 + K_{11,3}\,z_3$$

```
symmatrix2sym(K_1*z*z.'*K_2.')
```

```
ans =
```

$$\begin{pmatrix} \sigma_6\,\sigma_3 & \sigma_6\,\sigma_2 & \sigma_6\,\sigma_1 \\ \sigma_5\,\sigma_3 & \sigma_5\,\sigma_2 & \sigma_5\,\sigma_1 \\ \sigma_4\,\sigma_3 & \sigma_4\,\sigma_2 & \sigma_4\,\sigma_1 \end{pmatrix}$$

where

$$\sigma_1 = K_{23,1}\,z_1 + K_{23,2}\,z_2 + K_{23,3}\,z_3$$

$$\sigma_2 = K_{22,1}\,z_1 + K_{22,2}\,z_2 + K_{22,3}\,z_3$$

$$\sigma_3 = K_{21,1}\,z_1 + K_{21,2}\,z_2 + K_{21,3}\,z_3$$

$$\sigma_4 = K_{13,1}\,z_1 + K_{13,2}\,z_2 + K_{13,3}\,z_3$$

$$\sigma_5 = K_{12,1}\,z_1 + K_{12,2}\,z_2 + K_{12,3}\,z_3$$

$$\sigma_6 = K_{11,1}\,z_1 + K_{11,2}\,z_2 + K_{11,3}\,z_3$$

Question 5

Q5-2 Separate the Estimation

```
syms H [3 2] matrix
syms S [3 4] matrix
Hz = [H S]
```

Hz = $\begin{pmatrix} \boldsymbol{H} & \boldsymbol{S} \end{pmatrix}$

```
inv(Hz.'*Hz)
```

ans = $\left(\begin{pmatrix} \boldsymbol{H} & \boldsymbol{S} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \boldsymbol{H} & \boldsymbol{S} \end{pmatrix}\right)^{-1}$

Question 7

Q7-3

```
syms x y v mu lambda real
assume(lambda > 0); assume(mu > 0);
Ex = 1/lambda;
Ex2 = 1/lambda^2;
pdfxy = lambda*mu*exp(-(lambda - mu)*x)*exp(-mu*y)
```

pdfxy = $\lambda\,\mu\,\mathrm{e}^{-\mu\,y}\,\mathrm{e}^{-x\,(\lambda-\mu)}$

```
pdfy = (lambda*mu/(lambda - mu))*(exp(-mu*y) - exp(-lambda*y));
pdfx = lambda*exp(-lambda*x);
pdfv = mu*exp(-mu*v);
```

```
display(pdfxy)
```

pdfxy = $\lambda\,\mu\,\mathrm{e}^{-\mu\,y}\,\mathrm{e}^{-x\,(\lambda-\mu)}$

```
int(pdfxy,x,0,y)
```

ans =

$$-\frac{\lambda\,\mu\,\left(\mathrm{e}^{-\lambda\,y}-\mathrm{e}^{-\mu\,y}\right)}{\lambda-\mu}$$

```
% Exxhat
```

pdfxy = $\lambda\,\mu\,\mathrm{e}^{-\mu\,y}\,\mathrm{e}^{-x\,(\lambda-\mu)}$

```
integrandxxhat = x*y*exp(-lambda*y)/(exp(-mu*y)-exp(-lambda*y))*pdfxy
```

integrandxxhat =

$$-\frac{\lambda\,\mu\,x\,y\,\mathrm{e}^{-\lambda\,y}\,\mathrm{e}^{-\mu\,y}\,\mathrm{e}^{-x\,(\lambda-\mu)}}{\mathrm{e}^{-\lambda\,y}-\mathrm{e}^{-\mu\,y}}$$

```
xhat = 1/(lambda - mu) - y*exp(-lambda*y)/(exp(-mu*y) - exp(-lambda*y))
```

xhat =

$$\frac{1}{\lambda-\mu}+\frac{y\,e^{-\lambda\,y}}{e^{-\lambda\,y}-e^{-\mu\,y}}$$

```
xhat_v = subs(xhat,y,v+x)
```

xhat_v =

$$\frac{1}{\lambda-\mu}+\frac{e^{-\lambda\,(v+x)}\,(v+x)}{e^{-\lambda\,(v+x)}-e^{-\mu\,(v+x)}}$$

```
% Exxhat = int(int(x*xhat*pdfxy,x,0,y),y,0,inf)
% Exhatxhat = int(simplify(expand(xhat*xhat*pdfy)),y,0,inf)

% Exxhat_v = int(int(x*xhat_v*pdfx*pdfv,x,0,inf),v,0,inf)
```

Q7-4

```
syms x y mu lambda real
pdfxy = lambda*mu*exp(-(lambda - mu)*x)*exp(-mu*y)
```

pdfxy = $\lambda\,\mu\,e^{-\mu\,y}\,e^{-x\,(\lambda-\mu)}$

```
Exy = int(int(x*y*pdfxy,x,0,y),y,0,inf)
```

Exy =

$$\lim_{y\to\infty}\frac{-\frac{\lambda\,e^{-y\,\mu}}{\mu}+y\,e^{-y\,\lambda}\left(3\,\mu-\frac{2\,\mu^2}{\lambda}\right)+\frac{e^{-y\,\lambda}\,\sigma_2}{\lambda^2}-y\,\lambda\,e^{-y\,\mu}+y^2\,\mu\,e^{-y\,\lambda}\,(\lambda-\mu)}{\sigma_1}+\frac{\frac{\lambda}{\mu}-\frac{\sigma_2}{\lambda^2}}{\sigma_1}$$

where

$$\sigma_1=\lambda^2-2\,\lambda\,\mu+\mu^2$$

$$\sigma_2=3\,\lambda\,\mu-2\,\mu^2$$

```
expand(Exy)
```

ans =

$$\frac{2\,\mu^2}{\lambda^4-2\,\lambda^3\,\mu+\lambda^2\,\mu^2}+\frac{\lambda}{\lambda^2\,\mu-2\,\lambda\,\mu^2+\mu^3}-\frac{3\,\mu}{\lambda^3-2\,\lambda^2\,\mu+\lambda\,\mu^2}+\frac{\lim_{y\to\infty}\frac{3\,\mu\,e^{-\lambda\,y}}{\lambda}-\frac{\lambda\,e^{-\mu\,y}}{\mu}-\frac{2\,\mu^2\,e^{-\lambda\,y}}{\lambda^2}-\mu^2\,y^2\,e^{-}}{\lambda^2-}$$

```
% check my calculation of E[xy]
```

```
expand((-lambda*mu/(lambda - mu)^2)*(2*(lambda - mu)/lambda^3 + 1/lambda^2 - 1/mu^2))
```

ans =

$$\frac{2\,\mu^2}{\lambda^4 - 2\,\lambda^3\,\mu + \lambda^2\,\mu^2} + \frac{\lambda}{\lambda^2\,\mu - 2\,\lambda\,\mu^2 + \mu^3} - \frac{3\,\mu}{\lambda^3 - 2\,\lambda^2\,\mu + \lambda\,\mu^2}$$

```
pdfy = (lambda*mu/(lambda - mu))*(exp(-mu*y) - exp(-lambda*y))
```

pdfy =

$$-\frac{\lambda\,\mu\,(e^{-\lambda\,y} - e^{-\mu\,y})}{\lambda - \mu}$$

```
% E[y]
expand(int(y*pdfy,y,0,inf))
```

ans =

$$\frac{\lambda}{\lambda\,\mu - \mu^2} + \frac{\mu}{\lambda\,\mu - \lambda^2} - \frac{\lambda\,\mu\left(\lim_{y\to\infty} \frac{e^{-\mu\,y}}{\mu^2} - \frac{e^{-\lambda\,y}}{\lambda^2} - \frac{y\,e^{-\lambda\,y}}{\lambda} + \frac{y\,e^{-\mu\,y}}{\mu}\right)}{\lambda - \mu}$$

```
% E[y^2]
expand(int(y^2*pdfy,y,0,inf))
```

ans =

$$\frac{2\,\lambda}{\lambda\,\mu^2 - \mu^3} + \frac{2\,\mu}{\lambda^2\,\mu - \lambda^3} - \frac{\lambda\,\mu\left(\lim_{y\to\infty} \frac{2\,e^{-\mu\,y}}{\mu^3} - \frac{2\,e^{-\lambda\,y}}{\lambda^3} - \frac{2\,y\,e^{-\lambda\,y}}{\lambda^2} + \frac{2\,y\,e^{-\mu\,y}}{\mu^2} - \frac{y^2\,e^{-\lambda\,y}}{\lambda} + \frac{y^2\,e^{-\mu\,y}}{\mu}\right)}{\lambda - \mu}$$

```
% part of the integral for E[y^2]:
int(y^2*exp(-mu*y),y,0,inf)
```

ans =

$$\frac{2}{\mu^3} - \frac{\lim_{y\to\infty} e^{-\mu\,y}\,(\mu^2\,y^2 + 2\,\mu\,y + 2)}{\mu^3}$$

```
% Calculating estimation of x given y:
Ex = 1/lambda;
Ey = simplify(expand((lambda^2 - mu^2)/(lambda - mu)/(lambda*mu)))
```

Ey =

$$\frac{\lambda + \mu}{\lambda\,\mu}$$

```
Exy = (- lambda*mu/(lambda - mu)^2)*(2*(lambda - mu)/lambda^3 + 1/lambda^2 - 1/mu^2)
```

Exy =

$$-\frac{\lambda\,\mu\,\left(\frac{2\,\lambda-2\,\mu}{\lambda^3}+\frac{1}{\lambda^2}-\frac{1}{\mu^2}\right)}{(\lambda-\mu)^2}$$

```
Ey2 = (lambda*mu/(lambda - mu))*(2/mu^3 - 2/lambda^3)
```

Ey2 =

$$-\frac{\lambda\,\mu\,\left(\frac{2}{\lambda^3}-\frac{2}{\mu^3}\right)}{\lambda-\mu}$$

```
xhat = Ex + (Exy - Ex*Ey)*(y - Ey)/(Ey2 - Ey^2);
xhat = simplify(expand(xhat))
```

xhat =

$$\frac{y\,\mu^2-\mu+\lambda}{\lambda^2+\mu^2}$$

```
% Calculating K_0
syms x v K lambda mu y
% P(K) = 1/lambda^2 - 2*K/lambda - 2*K/mu + K^2/lambda^2 +K^2/mu^2
% K0 = simplify(expand(solve(diff(P(K),K)==0,K)))
% simplify(P(K0))
% simplify(P(mu^2/(lambda^2 + mu^2)))
% just use the solution for a linear estimate given y = x + v ; (H = 1)

% K0 = RxyRy^-1
% see written work

% xhat affine solution
xhat_affine = simplify(expand(1/lambda + (mu^2/(lambda^2+mu^2))*(y - (mu + lambda)/(mu*lambda)
```

xhat_affine =

$$\frac{y\,\mu^2-\mu+\lambda}{\lambda^2+\mu^2}$$

```
xhat_affine_y = collect(xhat_affine,y)
```

xhat_affine_y =

$$\frac{\mu^2}{\lambda^2+\mu^2}\,y+\frac{\lambda-\mu}{\lambda^2+\mu^2}$$

Question 8

Q8-1

Q8-2

```
syms p
assume(p > 0)
assumeAlso(p,'real')
simplify(expand((log(1-p) - 1/2*log(p/(1-p)))))
```

```
ans =
```

$$\log(1-p) - \frac{\log(p)}{2} - \frac{\log\left(-\frac{1}{p-1}\right)}{2}$$