Georgia Institute of Technology

ECE 6555 - Bonus assignment

Monday November 7, 2022 - v1.0

- There are 3 problems over 2 pages (including the cover page).
- The problems are not necessarily in order of difficulty.
- Every question in a problem is worth 2 points, so problems with many questions are worth more than problems with few questions.
- Each question is graded as follows: no credit without meaningful work, half credit for partial work, full credit if essentially correct.
- Unless otherwise specified, you should concisely indicate your reasoning and show all relevant work.
- The grade on each question is based on our judgment of your level of understanding as reflected by what you have written. If we cannot read it, we cannot grade it.
- Please use a pen and not a pencil if you handwrite your solution.
- **You must submit your assignment on Gradescope**.

Problem 1: A non-standard state-space model

Consider the state-space model

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{F}_i \mathbf{x}_i + \mathbf{G}_i \mathbf{u}_{i+1} \\ \mathbf{y}_i = \mathbf{H}_i \mathbf{x}_i + \mathbf{v}_i \end{cases} \quad \text{for } i \geq 0.$$

with random variables $\mathbf{x}_0, \mathbf{u}_i, \mathbf{v}_i$ that satisfy

$$\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & S_i\delta_{ij} & 0 \\ 0 & S_i^\dagger\delta_{ij} & R_i\delta_{ij} & 0 \end{bmatrix}$$

[Q1] Show that $\langle \mathbf{u}_i, \mathbf{x}_i \rangle = Q_i \mathbf{G}_{i-1}^\dagger$ and that for $i > j$ $\langle \mathbf{u}_i, \mathbf{x}_j \rangle = 0$ and $\langle \mathbf{v}_i, \mathbf{x}_j \rangle = 0$.

[Q2] Show that $\langle \mathbf{u}_i, \mathbf{y}_i \rangle = Q_i \mathbf{G}_{i-1}^\dagger \mathbf{H}_i^\dagger + S_i$ and that for $i > j$ $\langle \mathbf{u}_i, \mathbf{y}_j \rangle = 0$ and $\langle \mathbf{v}_i, \mathbf{y}_j \rangle = 0$.

Problem 2: A different state-space model

In some application, the state-space equations are

$$\mathbf{x}_{i+1} = \mathbf{F}\mathbf{x}_i + \mathbf{G}\mathbf{u}_i \qquad \mathbf{y}_i = \mathbf{H}_0\mathbf{x}_i + \mathbf{H}_1\mathbf{x}_{i-1} + \mathbf{v}_i \quad i \geq 0$$

where $\mathbf{u}_i, \mathbf{v}_i$ satisfy the conditions of our standard state space model

$$\left\langle \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} Q_i\delta_{ij} & S_i\delta_{ij} \\ S_i^\dagger\delta_{ij} & R_i\delta_{ij} \end{bmatrix}.$$

Define $\mathbf{z}_0 \triangleq \begin{bmatrix} \mathbf{x}_{-1}, \mathbf{x}_0 \end{bmatrix}^\mathsf{T}$ denote the initial state vector with covariance matrix Π_0, and assumed uncorrelated with all $\{\mathbf{u}_i, \mathbf{v}_i\}_{i\geq 0}$.

Find recursions for computing the innovations.

Problem 3: A modified state-space model

Consider the modified state-space model

$$\mathbf{x}_{i+1} = \mathbf{F}\mathbf{x}_i + \mathbf{G}_1\mathbf{u}_i + \mathbf{G}_2\mathbf{u}_{i+1} \qquad \mathbf{y}_i = \mathbf{H}\mathbf{x}_i + \mathbf{v}_i \quad i \geq 0$$

with the conditions

$$\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & S_i\delta_{ij} & 0 \\ 0 & S_i^\dagger\delta_{ij} & R_i\delta_{ij} & 0 \end{bmatrix}.$$

Find recursive equations for $\hat{\mathbf{x}}_{i|i-1}$ and $P_{i|i-1}$.