

Assignment 1

● Graded

Student

Theodore Johann Wilkening

Total Points

22.1 / 26 pts

Question 1

P1

0 / 0 pts

+ 2 pts Correct

+ 1 pt Click here to replace this description.

+ 0 pts I am not sure I can follow your solution

✓ + 0 pts Problem not graded

Question 2

P2

4 / 4 pts

2.1 (no title)

Resolved 2 / 2 pts

✓ + 2 pts Correct

+ 1.5 pts The problem asks you to consider an *arbitrary* norm. Your reasoning is correct though.

+ 1.5 pts You need to explicitly refer to Cauchy-Schwartz to justify the sub-additivity

+ 1 pt I am not following your discussion when you compare. I think you're making a slight mistake, you could have the inner product equal to zero and the norms pretty large.

+ 1 pt You're making a mistake in (5), squaring doesn't always give you an upper bound

+ 0 pts You are not really answering the question.

+ 1 pt Missing proof for subadditivity

C Regrade Request

Submitted on: Nov 06

Hi Professor,

Question 2-1 comment on why I missed 0.5 points mentions that I did not use the Cauchy-Schwarze Inequality explicitly, however at the bottom right of the page it is there.

I realize it may have been missed because it is a little cramped there... I will create more space in future homework submissions!

Got it!

Reviewed on: Dec 09

2.2 (no title)

2 / 2 pts

✓ + 2 pts Correct

+ 1.5 pts you need to check all properties of an inner product

+ 1 pt Be careful, you need to prove conjugate symmetry, not symmetry

+ 1 pt You have to check that the inner product satisfies three properties

+ 0 pts no work shown

Question 3

(no title)	3.5 / 6 pts
3.1 (no title)	1.5 / 2 pts
+ 2 pts Correct	
✓ + 1.5 pts Don't forget to check that 0 belongs to the space	
+ 0 pts No work shown	
3.2 (no title)	1 / 2 pts
+ 2 pts Correct	
✓ + 1 pt You're not fully proving it	
+ 0 pts No work shown	
💬 I don't understand why you introduce H	
3.3 (no title)	1 / 2 pts
+ 2 pts Correct	
✓ + 1 pt You don't prove everything	
+ 0 pts No work shown	
💬 There is no H in the problem	

Question 4

(no title)	0 / 0 pts
✓ + 0 pts Problem not graded	

Question 5

(no title)	5 / 6 pts
5.1 (no title)	2 / 2 pts
<div style="border: 1px solid #ccc; padding: 5px;"><p>✓ + 2 pts Correct</p></div>	
<p>+ 1 pt Click here to replace this description.</p>	
5.2 (no title)	2 / 2 pts
<div style="border: 1px solid #ccc; padding: 5px;"><p>✓ + 2 pts Correct</p></div>	
<p>+ 1 pt Click here to replace this description.</p>	
5.3 (no title)	1 / 2 pts
<p>+ 2 pts Correct</p>	
<div style="border: 1px solid #ccc; padding: 5px;"><p>✓ + 1 pt Click here to replace this description.</p></div>	
<p>+ 0 pts Click here to replace this description.</p>	
<div style="border: 1px solid #ccc; padding: 5px;"><p>Not sure what's going on</p></div>	

Question 6

(no title)	5 / 6 pts
6.1 (no title)	1 / 2 pts
<p>+ 2 pts Correct</p>	
<div style="border: 1px solid #ccc; padding: 5px;"><p>✓ + 1 pt Click here to replace this description.</p></div>	
<div style="border: 1px solid #ccc; padding: 5px;"><p>You only answer part of the question</p></div>	
6.2 (no title)	2 / 2 pts
<div style="border: 1px solid #ccc; padding: 5px;"><p>✓ + 2 pts Correct</p></div>	
<p>+ 1 pt Strictly speaking, the fact that that this is in $\text{Im}(H)^\perp$ is not enough to conclude. You also need to show that the error is orthogonal.</p>	
<p>+ 1 pt I can't follow what you're doing</p>	
6.3 (no title)	2 / 2 pts
<div style="border: 1px solid #ccc; padding: 5px;"><p>✓ + 2 pts Correct</p></div>	
<p>+ 1 pt I can't follow what you're doing</p>	
<p>+ 0 pts No work shown</p>	
<p>+ 1 pt Be careful, it's not true that you can write y as such</p>	
<p>+ 1.5 pts Click here to replace this description.</p>	

Question 7

(no title)

0 / 0 pts

✓ + 0 pts Problem not graded

Question 8

(no title)

4 / 4 pts

8.1 (no title)

2 / 2 pts

✓ + 2 pts Correct

+ 1 pt Incomplete

+ 0 pts No work shown

8.2 (no title)

2 / 2 pts

✓ + 2 pts Correct

+ 0 pts No work shown

+ 1 pt Incomplete

Question 9

Bonus for early submission

0.6 / 0 pts

✓ + 0.6 pts Correct

+ 0 pts No bonus

Questions assigned to the following page: [2.1](#) and [1](#)

#1) Let $\vec{y} \in \mathbb{R}^m$ and $H \in \mathbb{R}^{m \times n}$. $\vec{y} \in \text{Im}(H)$ s.t. $H\vec{x}_0 = \vec{y}$. \vec{x}_0 is a solution.
show that $\{\vec{x} : H\vec{x} = \vec{y}\} = \vec{x}_0 + \ker(H)$

$$\textcircled{1} \quad \ker(H) = \{\vec{x} : H\vec{x} = 0\}, \vec{x} \in \mathbb{R}^n. \text{ let } \vec{x}_0 \in \ker(H)$$

$$\text{then take } H(\vec{x}_0 + \vec{x}_u) = H\vec{x}_0 + H\vec{x}_u^0 = H\vec{x}_0 = \vec{y} \Rightarrow (\vec{x}_0 + \ker(H)) \in \{\vec{x} : H\vec{x} = \vec{y}\} \quad (\text{part 1})$$

$$\textcircled{2} \quad \text{show that } \{\vec{x} : H\vec{x} = \vec{y}\} \subset \vec{x}_0 + \ker(H)$$

$$\text{let } \vec{x}_0 \text{ be another solution to } H\vec{x} = \vec{y}. \text{ then, } \vec{x}_0 = \vec{x}_0 + \vec{x}_0 - \vec{x}_0$$

$$= \vec{x}_0 + \vec{v}, \vec{v} = \vec{x}_0 - \vec{x}_0$$

if $\vec{v} \in \ker(H)$ then this implies

$$\text{that } \vec{x}_0 \in \vec{x}_0 + \ker(H), H\vec{x}_0 = \vec{y}$$

$$H\vec{v} = H\vec{x}_0 - H\vec{x}_0 = \vec{y} - \vec{y} = 0$$

$$\Rightarrow \vec{v} \in \ker(H)$$

Then by \textcircled{1} and \textcircled{2} we have shown that $\{\vec{x} : H\vec{x} = \vec{y}\} = \vec{x}_0 + \ker(H)$

#2) 2-1) $\vec{v} \in \mathbb{R}^n$ show that $\|\vec{x}\| \triangleq \sqrt{\langle \vec{x}, \vec{x} \rangle}$ is indeed a norm.

Norm properties: • positive definiteness: $\forall \vec{x} \in V, \forall \alpha \in \mathbb{R}, \|\alpha \vec{x}\| > 0$
and $\|\vec{x}\| = 0$ iff $\vec{x} = 0$

• homogeneity: $\forall \vec{x} \in V, \forall \alpha \in \mathbb{R}, \|\alpha \vec{x}\| = |\alpha| \|\vec{x}\|$

• subadditivity: $\forall \vec{x}, \vec{y} \in V, \|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

i) P.D.

$$\sqrt{\langle \vec{x}, \vec{x} \rangle} > 0 \text{ iff } \langle \vec{x}, \vec{x} \rangle > 0 \quad \checkmark$$

→ Inner product properties:

(1) Symmetry: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

(2) bilinearity: $\langle \vec{x}, \lambda \vec{y} + \mu \vec{z} \rangle = \lambda \langle \vec{x}, \vec{y} \rangle + \mu \langle \vec{x}, \vec{z} \rangle$

(3) P.D.: $\forall \vec{x} \neq 0, \langle \vec{x}, \vec{x} \rangle > 0$

ii) Homogeneity: $\alpha \vec{x}$

$$\|\alpha \vec{x}\| = \sqrt{\langle \alpha \vec{x}, \alpha \vec{x} \rangle} = \sqrt{\alpha^2 \langle \vec{x}, \vec{x} \rangle}$$

$$= |\alpha| \sqrt{\langle \vec{x}, \vec{x} \rangle} = |\alpha| \|\vec{x}\| \quad \checkmark$$

iii) Subadditivity:

$$\begin{aligned} \|\vec{x} + \vec{y}\| &\leq \|\vec{x}\| + \|\vec{y}\| \Rightarrow \sqrt{\langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle} \leq \sqrt{\langle \vec{x}, \vec{x} \rangle} + \sqrt{\langle \vec{y}, \vec{y} \rangle} ? \quad \checkmark \\ &\Rightarrow \sqrt{\langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle} \leq \sqrt{\langle \vec{x}, \vec{x} \rangle} + \sqrt{\langle \vec{y}, \vec{y} \rangle} \Rightarrow \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle \leq \langle \vec{x}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle + 2\sqrt{\langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle} \\ &\Rightarrow (\langle \vec{x}, \vec{y} \rangle + \sqrt{\langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle})^2 \leq \langle \vec{x}, \vec{x} \rangle \langle \vec{y}, \vec{y} \rangle \text{ by Cauchy-Schwarz inequality} \quad \rightarrow \square \end{aligned}$$

Questions assigned to the following page: [2.1](#) and [2.2](#)

H2-1 cont'd)

Thus since $\|\vec{x}\| \triangleq \sqrt{\langle \vec{x}, \vec{x} \rangle}$ satisfies all of the properties of a norm, it is a norm. \square

- #2-2) Complex \mathbb{C}^n vs \mathbb{R}^n T = transpose conjugate. i.e. $\vec{x}^+ \triangleq (\vec{x}^*)^T$
- inner product now a sesquilinear form: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle^*$
- show that $\langle \cdot, \cdot \rangle: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}: (\vec{x}, \vec{y}) \mapsto \vec{x}^T \vec{y}^* = \vec{y}^+ \vec{x}$ is a valid inner product
 - conclude that $\|\vec{x}\| \triangleq \sqrt{\vec{x}^+ \vec{x}}$ is a norm

i-1) symmetry let $\vec{x}, \vec{y} \in \mathbb{C}^n$

$$\text{then } \vec{x}^T \vec{y}^* = \sum_{i=1}^n \vec{x}_i^* \vec{y}_i^* = \sum_{i=1}^n y_i^* x_i = \vec{y}^+ \vec{x}_i$$

i-2) bilinearity let $\vec{z} \in \mathbb{C}^n$ as well

$$\langle \vec{x}, \lambda \vec{y} + \mu \vec{z} \rangle = \vec{x}^T (\lambda \vec{y} + \mu \vec{z})^* = \sum_{i=1}^n x_i (\lambda y_i + \mu z_i)^*$$

$$= \sum_{i=1}^n \lambda x_i y_i^* + \mu x_i z_i^* = \lambda \vec{x}^T \vec{y}^* + \mu \vec{x}^T \vec{z}^* = \lambda \langle \vec{x}, \vec{y} \rangle + \mu \langle \vec{x}, \vec{z} \rangle$$

i-3) P.D. let $\vec{x} \neq 0$

$$\Rightarrow \langle \vec{x}, \vec{x} \rangle = \vec{x}^T \vec{x}^* = \sum_{i=1}^n x_i x_i^*, \quad x_i = a_i + i b_i, \quad a_i, b_i \in \mathbb{R}$$
$$x_i x_i^* = (a_i + i b_i)(a_i - i b_i)$$
$$= a_i^2 + b_i^2 > 0$$

$$\Rightarrow \sum_{i=1}^n x_i x_i^* > 0$$

Since all the properties of an inner product are satisfied, thus $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}^*$ is a valid inner product.
 $= \langle \vec{y}, \vec{x} \rangle^* = \vec{y}^+ \vec{x}$ \square

ii) Since $\langle \cdot, \cdot \rangle: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}: (\vec{x}, \vec{y}) \mapsto \vec{x}^T \vec{y}^* = \vec{y}^+ \vec{x}$

is an inner product of \mathbb{C}^n , it follows from the proof in H2-1 on this homework set that $\|\vec{x}\| \triangleq \sqrt{\vec{x}^+ \vec{x}}$ is a norm. \square

Questions assigned to the following page: [3.1](#) and [3.2](#)

H3-1] Orthogonal Complement

$$W^\perp = \left\{ \vec{v} \in V : \forall w \in W, \langle \vec{v}, \vec{w} \rangle = 0 \right\}$$

Show that W^\perp is a vector subspace

$$\text{let } \vec{x}, \vec{y} \in W^\perp, \text{ let } \lambda, \mu \in \mathbb{R}, \text{ then } \langle \lambda \vec{x} + \mu \vec{y}, \vec{w} \rangle = \langle \lambda \vec{x}, \vec{w} \rangle + \langle \mu \vec{y}, \vec{w} \rangle \\ = 0 + 0$$

$\Rightarrow \lambda \vec{x} + \mu \vec{y} \in W^\perp$ Thus, W^\perp is a vector subspace by the definition of a vector subspace \square

H3-2) Show that $W \oplus W^\perp = V$ (direct sum)

The definition of direct sum is: $V = W \oplus W^\perp$ iff $\forall \vec{v} \in V \exists \text{ a unique } (\vec{w}, \vec{w}^\perp) \in W \times W^\perp \text{ s.t. } \vec{v} = \vec{w} + \vec{w}^\perp$

Thus there are three conditions we must satisfy:

- $W \oplus W^\perp \subset V$
- $V \subset W \oplus W^\perp$
- $\vec{w} + \vec{w}^\perp$ is unique (i.e. $\vec{w}_1 + \vec{w}_1^\perp \neq \vec{w}_2 + \vec{w}_2^\perp$)

3-2a) $W \oplus W^\perp \subset V$

By the definition of an orthogonal complement,

$W \subset V, W^\perp \subset V \Rightarrow W$ and W^\perp are subspaces of V .

Thus any two vectors $\vec{w} \in W$ and $\vec{w}^\perp \in W^\perp$ must also be in the vector space V . ~~Therefore~~ $\vec{w} + \vec{w}^\perp = \vec{z}, \vec{z} \in V$

Therefore $W \oplus W^\perp \subset V$ \square

3-2b) $V \subset W \oplus W^\perp$

let $W = \text{Im}(H)$, let $\vec{v} \in V$. Then $\vec{v} = P_H \vec{v} + (I - P_H) \vec{v} = \vec{y} + \vec{y}^\perp$.

(H is full rank) Thus where $\vec{y} \in W$ and $\vec{y}^\perp \in W^\perp$ by the definition (H can always be a basis) of the orthogonal projector P_H .

For W .

Thus, $\forall \vec{v} \in V, \exists \vec{y} \in W$ and $\vec{y}^\perp \in W^\perp$ s.t. $\vec{v} = \vec{y} + \vec{y}^\perp$.

Therefore $V \subset W \oplus W^\perp$ \square

\boxed{B}

Questions assigned to the following page: [3.2](#) and [3.3](#)

#3-2c) $\vec{w} + \vec{w}^\perp$ unique

assume $\vec{w} + \vec{w}^\perp = \vec{v}$ is not unique, then the following would be true:

$$\vec{w}_1 + \vec{w}_1^\perp = \vec{v} = \vec{w}_2 + \vec{w}_2^\perp, \quad \vec{w}_1, \vec{w}_2 \in W; \vec{w}_1^\perp, \vec{w}_2^\perp \in W^\perp, \vec{w}_1 \neq \vec{w}_2, \vec{w}_1^\perp \neq \vec{w}_2^\perp$$

$$\Rightarrow \vec{w}_1 - \vec{w}_2 + \vec{w}_1^\perp - \vec{w}_2^\perp = 0$$

Now, represent the vectors using bases:

let $A = \{\vec{z}_1, \vec{z}_2, \dots, \vec{z}_i\}$ be an orthonormal basis for W

let $B = \{\vec{\beta}_1, \vec{\beta}_2, \dots, \vec{\beta}_j\}$ be an orthonormal basis for W^\perp

Note that $\vec{z}_i, \vec{\beta}_j$ are linearly independent b/c constituent vectors are orthogonal. i.e. $\langle \vec{z}_i, \vec{\beta}_j \rangle = z_{ii}\beta_{j1} + z_{i2}\beta_{j2} + \dots + z_{in}\beta_{jn} = 0$

$$\text{then, } \vec{w}_1 = \sum_{i=1}^i z_i \vec{z}_i, \quad z_i \in \mathbb{R}$$

$$\vec{w}^\perp = \sum_{j=1}^j y_j \vec{\beta}_j, \quad y_j \in \mathbb{R}$$

$$\begin{aligned} \Rightarrow \vec{w}_1 - \vec{w}_2 + \vec{w}_1^\perp - \vec{w}_2^\perp &= \sum_{i=1}^i (z_i - \bar{z}_i) \vec{z}_i + \sum_{j=1}^j (y_j - \bar{y}_j) \vec{\beta}_j - \sum_{j=1}^j y_j \vec{\beta}_j = 0 \\ &= \sum_{i=1}^i (z_{ii} - \bar{z}_{ii}) \vec{z}_i + \sum_{j=1}^j (y_{jj} - \bar{y}_{jj}) \vec{\beta}_j = 0 \end{aligned}$$

By linear independence of $\vec{z}_i, \vec{\beta}_j$, the above can be true iff

$$z_{ii} = \bar{z}_{ii} \text{ and } y_{jj} = \bar{y}_{jj}. \quad \text{Thus for } \vec{w}_1 + \vec{w}_1^\perp = \vec{w}_2 + \vec{w}_2^\perp \text{ to be}$$

But then ~~$\vec{w}_1 = \vec{w}_2$~~ $\vec{w}_1 = \vec{w}_2$ and $\vec{w}_1^\perp = \vec{w}_2^\perp$ and this contradicts our assumption.

Thus, $\vec{w}_1 + \vec{w}_1^\perp \neq \vec{w}_2 + \vec{w}_2^\perp$ and hence $\vec{w}_1 + \vec{w}_1^\perp = \vec{v}$ is unique. \square

#3-3] Show that $(W^\perp)^\perp = W$, $W = \text{Im}(E) = \{\vec{w}\}, \vec{w} \in W$

by definition: $W^\perp = \{\vec{w}^\perp \in V : \forall \vec{w} \in W, \langle \vec{w}^\perp, \vec{w} \rangle = 0\} \stackrel{\text{def}}{=} \{\vec{w}^\perp\}$

+ Len, $(W^\perp)^\perp = \{\vec{y} \in V : \forall \vec{w}^\perp \in W^\perp, \langle \vec{y}, \vec{w}^\perp \rangle = 0\} \stackrel{\text{def}}{=} \{\vec{y}\}$

$$\Rightarrow \langle \vec{w}^\perp, \vec{w} \rangle = 0 = \langle \vec{y}, \vec{w}^\perp \rangle = \langle \vec{w}^\perp, \vec{y} \rangle \Rightarrow \langle \vec{w}^\perp, \vec{w} \rangle = \langle \vec{w}^\perp, \vec{y} \rangle$$

$$\Rightarrow \vec{w} = \vec{y}. \quad \text{Thus, } \{\vec{w}\} = \{\vec{y}\}, \text{ that is, } W = (W^\perp)^\perp$$

\square

Question assigned to the following page: [4](#)

H4) Let $\underline{H} \in \mathbb{R}^{m \times n}$, Show that $\text{Im}(\underline{H}^T) = \overline{\text{Im}}(\underline{H}^T \underline{H})$

a) $\text{Im}(\underline{H}^T \underline{H}) \subset \text{Im}(\underline{H}^T)$:

$$\text{Im}(\underline{H}^T \underline{H}) \Rightarrow \vec{y} = \underline{H}^T \underline{H} \vec{x} \quad \text{for some } \vec{x} \in \mathbb{R}^n.$$

$$\text{Im}(\underline{H}^T) \Rightarrow \vec{y} = \underline{H}^T \vec{u}, \quad \vec{u} \in \mathbb{R}^m$$

$$\text{Let } \vec{w} = \underline{H} \vec{x} \Rightarrow \vec{w} \in \mathbb{R}^m$$

$$\text{Then } \vec{y} = \underline{H}^T \underline{H} \vec{x} = \underline{H}^T \vec{w} \quad \text{and} \quad \underline{H}^T \vec{w} \in \text{Im}(\underline{H}^T)$$

Therefore $\text{Im}(\underline{H}^T \underline{H}) \subset \text{Im}(\underline{H}^T)$

□

b) $\text{Im}(\underline{H}^T) \subset \text{Im}(\underline{H}^T \underline{H})$

$$\text{Im}(\underline{H}^T) \subset \text{Im}(\underline{H}^T \underline{H}) \Leftrightarrow \text{Im}(\underline{H}^T)^\perp \subset \text{Im}(\underline{H}^T \underline{H})^\perp$$
$$\Leftrightarrow \text{ker}(\underline{H}^T) \subset \text{ker}(\underline{H}^T \underline{H})$$

$$\underline{H} \vec{x} \in \text{ker}(\underline{H}^T) \text{ s.t. } \underline{H}^T \vec{x} = 0$$

$$\text{Let } \vec{y} \in \text{Im}(\underline{H}^T), \text{ then } \exists \vec{u} \text{ s.t. } \vec{y} = \underline{H}^T \vec{u}, \quad \vec{u} \in \mathbb{R}^m$$

$$\text{note: } \mathbb{R}^n = \text{Im}(\underline{H}) \oplus \text{ker}(\underline{H})^\perp$$

$$\text{note also: } \text{Im}(\underline{H})^\perp = \text{ker}(\underline{H}^T)$$

$$\text{then, } \vec{u} = \tilde{\vec{u}} + \hat{\vec{u}}, \quad \tilde{\vec{u}} \in \text{Im}(\underline{H}), \quad \hat{\vec{u}} \in \text{ker}(\underline{H}^T), \quad \tilde{\vec{u}} = \underline{H} \vec{v}, \quad \vec{v} \in \mathbb{R}^n$$

$$\text{and } \vec{y} = \underline{H}^T (\hat{\vec{u}} + \tilde{\vec{u}}) = \underline{H}^T (\underline{H} \vec{v} + \hat{\vec{u}}) = \underline{H}^T \underline{H} \vec{v}$$

$$\Rightarrow \exists \vec{v} \text{ s.t. } \vec{y} \in \text{Im}(\underline{H}^T \underline{H}) \Rightarrow \text{Im}(\underline{H}^T) \subset \text{Im}(\underline{H}^T \underline{H})$$

□

Therefore, $\text{Im}(\underline{H}^T \underline{H}) = \text{Im}(\underline{H}^T)$

□

Questions assigned to the following page: [5.1](#), [5.2](#), and [5.3](#)

H3-2 contd)

$$V = \text{span}(\{\underline{v}_i\}_{i=1}^n) \subset \mathbb{R}^n \in \mathbb{R}^n \quad \text{vector space}$$

$$= \text{span}(\{\underline{w}_i\}_{i=1}^n, \{\underline{w}_i^\perp\}_{i=1}^{n-h}) \subset \mathbb{R}^n \in \mathbb{R}^n \quad \text{vector space}$$

n orthonormal basis vectors $\rightarrow \in \mathbb{R}^n \Rightarrow \text{span}(\{\underline{v}_i\}) = \text{span}(\{\underline{w}_i, \underline{w}_i^\perp\})$ ~~vector space~~

numbers? or something else? \uparrow identically.
i.e. dimension.

#5

$$J(x) \triangleq (\vec{y} - \underline{H}\vec{x})^T \underline{W} (\vec{y} - \underline{H}\vec{x}) \quad , \quad \underline{W} > 0 \text{ symmetric}$$

$$\text{i.e. } \vec{x}^T \underline{W} \vec{x} > 0 \text{ for } \vec{x} \neq \vec{0}, \vec{x} \in \mathbb{R}^n$$

Q1) Show $J(x) > 0$

Since \underline{W} is positive definite, then by definition for a real vector $\vec{z} \in \mathbb{R}^n$, $\vec{z} \neq \vec{0}$

$\vec{z}^T \underline{W} \vec{z} > 0$. If we let $\vec{z} = (\vec{y} - \underline{H}\vec{x})$ then we have

$$\vec{z}^T \underline{W} \vec{z} = (\vec{y} - \underline{H}\vec{x})^T \underline{W} (\vec{y} - \underline{H}\vec{x}) > 0$$

□

Q2) Show that $\text{cost is zero iff } \vec{y} = \underline{H}\vec{x}$:

$$\text{if } \vec{y} = \underline{H}\vec{x} \text{ then } J(x) = (\underline{H}\vec{x} - \underline{H}\vec{x})^T \underline{W} (\underline{H}\vec{x} - \underline{H}\vec{x}) = \vec{0}^T \underline{W} \vec{0} = 0$$

if $J(x) = 0$ then by the definition of a positive definite matrix, this can only be true if $\vec{z} = (\vec{y} - \underline{H}\vec{x}) = \vec{0} \Rightarrow \vec{y} = \underline{H}\vec{x}$

Thus, $J(x) = 0$ iff $\vec{y} = \underline{H}\vec{x}$ □

Q3) Normal equation for modified cost: $\underline{H}^T \underline{W} \underline{H} \vec{x} = \underline{H}^T \underline{W} \vec{y}$

$$\begin{aligned} \text{So, } J(x) &\triangleq (\vec{y} - \underline{H}\vec{x})^T \underline{W} (\vec{y} - \underline{H}\vec{x}) = (\vec{y} - \underline{H}\vec{x})^T (\underline{W}\vec{y} - \underline{W}\underline{H}\vec{x}) \\ &= (\vec{y}^T - \vec{x}^T \underline{H}^T)(\underline{W}\vec{y} - \underline{W}\underline{H}\vec{x}) \\ &= \vec{y}^T \underline{W} \vec{y} - \vec{x}^T \underline{H}^T \underline{W} \vec{y} - \vec{y}^T \underline{W} \underline{H} \vec{x} + \vec{x}^T \underline{H}^T \underline{W} \underline{H} \vec{x} \end{aligned}$$

$$(\vec{x}^T \underline{H}^T \underline{W} \vec{y})^T = \vec{y}^T \underline{W}^T \underline{H} \vec{x} = \vec{y}^T \underline{W} \underline{H} \vec{x} \rightarrow$$

$$\text{b/c } \vec{x}^T \underline{H}^T \underline{W} \vec{y} \in \mathbb{R}, \underline{W} = \underline{W}^T$$

(6)
qed

Questions assigned to the following page: [6.1](#) and [6.2](#)

#6] Projectors $P_H \triangleq H(H^T H)^{-1} H^T$ is the projector onto $\text{Im}(H)$

6-1) Show $P_{H^T} = P_H^T$:

$$\begin{aligned} P_{H^T}^T &= (H(H^T H)^{-1} H^T)^T = H(H^T H)^{-T} H^T ; \quad (H^T H)^{-T} = ((H^T H)^T)^{-1} = (H^T H)^{-1} \\ &= H(H^T H)^{-1} H^T \\ &= P_H \quad \square \end{aligned}$$

6-2) Show $I - P_H$ is a projector onto $\text{Im}(H)^\perp$:

Let ~~W~~ W be a linear subspace of $V \in \mathbb{R}^n$ and let \tilde{y} be the orthogonal projection of $\vec{y} \in V$ onto W , where $W \neq \text{Im}(H)$

then $\hat{y} = P_H \vec{y} = H(H^T H)^{-1} H^T \vec{y}$

Now, $(I - P_H) \vec{y} = \vec{y} - P_H \vec{y} = \vec{y} - \hat{y}$

Since $\mathbb{R}^n = \text{Im}(H) \oplus \text{Im}(H)^\perp$, $\vec{y} = \vec{y}_0 + \vec{y}_1$, where $\vec{y}_0 \in W$, $\vec{y}_1 \in W^\perp$

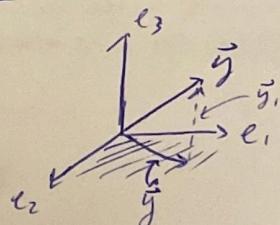
Since $\hat{y} = P_H \vec{y}$, it contains all of the components of \vec{y} contained in W . Thus $\hat{y} = \vec{y}_0$ and $\vec{y}_1 = \vec{y} - \hat{y} \in W^\perp$

Graphically in \mathbb{R}^3 :

Therefore, $\vec{y}_1 = (I - P_H) \vec{y}$ and $(I - P_H)$

is the projector of \vec{y} onto $\text{Im}(H)^\perp$

□



Questions assigned to the following page: [8.1](#), [6.3](#), and [7](#)

H6-Q3 show that if $\vec{y} \in \text{Im}(H)$ then $P_H \vec{y} = \vec{y}$

$$\Rightarrow \vec{y} = H\vec{x} \Rightarrow P_H \vec{y} = H(H^T H)^{-1} H^T H \vec{y}$$

$$= H I = H \vec{y}$$

□

P_H presumes H full rank.

$\Rightarrow H^T H = \text{Gramian Matrix } \in \mathbb{R}^{n \times n}$

\Rightarrow by properties of Gramian Matrix

$(H^T H)^{-1}$ is nonsingular for H full rank

$$\text{and } (H^T H)^{-1} H^T H = I$$

H7] Projection w/ Mahalanobis distance

Show $P_H = H(H^T H)^{-1} H^T$ should be replaced by $P_H = H(H^T W H)^{-1} H^T W$

$$\text{where } J(\vec{x}) = (\vec{y} - H\vec{x})^T W (\vec{y} - H\vec{x})$$

Result from Q5-3 that the normal equation is:

$$H^T W \vec{y} = H^T W H \vec{x} \Rightarrow \vec{x}_0 = (H^T W H)^{-1} H^T W \vec{y} \quad (\text{best estimate of } \vec{x})$$

$$\Rightarrow \hat{\vec{y}} = H \vec{x}_0 = \underbrace{H (H^T W H)^{-1} H^T W \vec{y}}_{P_H} \quad (\text{best estimate or "projection" of } \vec{y} \text{ onto solution space } W = \text{span } \text{Im}(H))$$

$$\Rightarrow P_H = H(H^T W H)^{-1} H^T W$$

□

H8] Constrained Least Square

$\min_{\vec{x}} \| \vec{y} - H\vec{x} \|_2^2$ subject to ~~$A\vec{x} = \vec{b}$~~ . Equivalent to minimization of

$$\text{unconstrained } \min_{\vec{x}} \left(\| \vec{y} - H\vec{x} \|_2^2 + \vec{\lambda}^T (A\vec{x} - \vec{b}) \right), \quad \vec{\lambda} = \text{Lagrange multipliers}$$

~~allow us to perform~~
~~minimization/max.~~ for a
constrained problem analytically
 (converts constraint into a
cost)

→
 (behind page 1)
 (physical paper) 8
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Question assigned to the following page: [8.1](#)

#8 cont'd)

$$\text{note: } \vec{x}_0 = \vec{x}_{LS} = (\underline{H}^T \underline{H})^{-1} \underline{H}^T \vec{y} \quad , \quad \vec{y} = \underline{H} \vec{x}$$

$$\begin{aligned} \text{let } J(\vec{x}) &= \| \vec{y} - \underline{H} \vec{x} \|^2 + \vec{\lambda}^T (\underline{A} \vec{x} + \vec{b}) \\ &= (\vec{y} - \underline{H} \vec{x})^T (\vec{y} - \underline{H} \vec{x}) + \vec{\lambda}^T (\underline{A} \vec{x} + \vec{b}) \end{aligned}$$

$$\min J(\vec{x}) \text{ is where } \nabla_{\vec{x}, \vec{\lambda}} J(\vec{x}, \vec{\lambda}) = \vec{0}$$

$$\Rightarrow \frac{\partial J}{\partial \vec{x}} = 2 \underline{H}^T \underline{H} \vec{x} - 2 \underline{H}^T \vec{y} + \underline{A}^T \vec{\lambda} = 0$$

$$\text{note: } \frac{\partial \vec{x}^T \underline{A} \vec{x}}{\partial \vec{x}} = (\underline{A} + \underline{A}^T) \vec{x}$$

$$\frac{\partial J}{\partial \vec{\lambda}} = \underline{A}^T \vec{x} - \vec{b} = 0 \Rightarrow \vec{x} = \underline{A}^{-1} \vec{b}$$

(assume \underline{A} non-singular)

$$\frac{\partial \vec{y}^T \underline{A} \vec{x}}{\partial \vec{x}} = \underline{A}^T \vec{y}$$

\Rightarrow solve for $\vec{\lambda}$:

$$\frac{\partial J}{\partial \vec{x}} = \underline{A}^T \underline{H} \vec{x} - \underline{H}^T \vec{y} + \underline{A}^T \vec{x} = 0$$

$$\Rightarrow \cancel{\underline{A}^T \underline{H}^T \underline{H} \vec{x}} /$$

$$\begin{aligned} \underline{A}^T \vec{\lambda} &= -2 \underline{H}^T \underline{H} \underline{A}^{-1} \vec{b} + 2 \underline{H}^T \vec{y} \\ &= -2 \underline{H}^T \underline{H} \underline{A}^{-1} \vec{b} + 2 \underline{H}^T \underline{H} \underbrace{\underline{A}^{-1} \underline{A} (\underline{H}^T \underline{H})^{-1} \underline{H}^T \vec{y}}_{\vec{x}_{LS}} \end{aligned}$$

$$\underline{A}^T \vec{\lambda} = 2 [\underline{H}^T \underline{H} \underline{A}^{-1}] (\underline{A} \vec{x}_{LS} - \vec{b})$$

$$\text{note: } A B^{-1} A = B A^{-1} B$$

$$\Rightarrow \vec{\lambda} = 2 [\underline{A}^{-T} \underline{H}^T \underline{H} \underline{A}^{-1}] (\underline{A} \vec{x}_{LS} - \vec{b})$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$= 2 [\underline{A} (\underline{H}^T \underline{H})^{-1} \underline{A}^T]^{-1} (\underline{A} \vec{x}_{LS} - \vec{b})$$

$$\Rightarrow [A (\underline{H}^T \underline{H})^{-1} A^T]^{-1}$$

□

$$= \underline{A}^{-T} (\underline{H}^T \underline{H})^{-1} \underline{A}^{-1}$$

(#8-2 see last page)

Question assigned to the following page: [8.2](#)

H8 cont'd

8-2] Conclude that the constrained LS solution is:

$$\vec{x}_c = \vec{x}_{ls} - (\underline{H}^T \underline{H})^{-1} \underline{A}^T [\underline{A} (\underline{H}^T \underline{H})^{-1} \underline{A}^T]^{-1} (\underline{A} \vec{x}_{ls} - \vec{b})$$

so, $\frac{\partial J(\vec{x})}{\partial \vec{x}} = 2 \underline{H}^T \underline{H} \vec{x} - 2 \underline{H}^T \vec{y} + \underline{A}^T \vec{A} = 0$

$$= 2 \underline{H}^T \underline{H} \vec{x} - 2 \underline{H}^T \vec{y} + \underline{A}^T \left(2 [\underline{A} (\underline{H}^T \underline{H})^{-1} \underline{A}^T]^{-1} (\underline{A} \vec{x}_{ls} - \vec{b}) \right) = 0$$

$$= \underline{H}^T \underline{H} \vec{x} - \cancel{2 \underline{H}^T \vec{y}} + \underline{A}^T [\underline{A} (\underline{H}^T \underline{H})^{-1} \underline{A}^T]^{-1} (\underline{A} \vec{x}_{ls} - \vec{b})$$

\Rightarrow normal equation:

$$\underline{H}^T \underline{H} \vec{x} = \underline{H}^T \vec{y} + \underline{A}^T [\underline{A} (\underline{H}^T \underline{H})^{-1} \underline{A}^T]^{-1} (\underline{A} \vec{x}_{ls} - \vec{b})$$

\Rightarrow solution to normal equations:

$$\vec{x}_c = \underbrace{(\underline{H}^T \underline{H})^{-1} \underline{H}^T \vec{y}}_{\vec{x}_{ls}} + \underbrace{\underline{A}^T [\underline{A} (\underline{H}^T \underline{H})^{-1} \underline{A}^T]^{-1} (\underline{A} \vec{x}_{ls} - \vec{b})}_{\vec{x}_2}$$

$$\boxed{\vec{x}_c = \vec{x}_{ls} + (\underline{H}^T \underline{H})^{-1} \underline{A}^T [\underline{A} (\underline{H}^T \underline{H})^{-1} \underline{A}^T]^{-1} (\underline{A} \vec{x}_{ls} - \vec{b})}$$