Georgia Institute of Technology

ECE 6555 - Assignment 1

due Thursday September 8, 2022 - v1.2

- There are 8 problems over 3 pages (including the cover page).
- The problems are not necessarily in order of difficulty.
- Every question in a problem is worth 2 points, so problems with many questions are worth more than problems with few questions.
- Each question is graded as follows: no credit without meaningful work, half credit for partial work, full credit if essentially correct.
- Unless otherwise specified, you should concisely indicate your reasoning and show all relevant work.
- The grade on each question is based on our judgment of your level of understanding as reflected by what you have written. If we cannot read it, we cannot grade it.
- Please use a pen and not a pencil if you handwrite your solution.
- **You must submit your assignment on Gradescope**.

Problem 1: Space of solutions

Let $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{H} \in \mathbb{R}^{m \times n}$. Consider the equation $\mathbf{Hx} = \mathbf{y}$ and assume that $\mathbf{y} \in \operatorname{Im}(\mathbf{H})$ so that there exists at least one solution $\mathbf{x}_0$. Show that

$$\{\mathbf{x} : \mathbf{Hx} = \mathbf{y}\} = \mathbf{x}_0 + \operatorname{Ker}(\mathbf{H})$$

Problem 2: Inner product

[Q1] Consider a vector space V over the field $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle$. Show that $\|\mathbf{x}\| \triangleq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is indeed a norm.

[Q2] In this question, we will operate with the field of complex numbers $\mathbb{C}$ instead of $\mathbb{R}$. The operation $\dagger$ denotes the transpose conjugate, e.g., $\mathbf{x}^\dagger \triangleq (\mathbf{x}^*)^T$. An inner product must now be a symmetric sesquilinear form instead of symmetric linear form, i.e.,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$$

Show that the function $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} : (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}^T \mathbf{y}^* = \mathbf{y}^\dagger \mathbf{x}$ is a valid inner product. Conclude that $\|\mathbf{x}\| \triangleq \sqrt{\mathbf{x}^\dagger \mathbf{x}}$ is a norm.

Problem 3: Orthogonal complement

For a subvector space W of a vector space V (of finite dimension and over $\mathbb{R}$), the orthogonal complement is

$$W^\perp \triangleq \{v \in V : \forall w \in W \; \langle v, w \rangle = 0\}.$$

[Q1] Show that $W^\perp$ is a vector subspace.

[Q2] Show that $W \oplus W^\perp = V$. → looking at bases. can always write a vector as a bases. then orthogonalize it using Gram-Schmidt. Then normalize it
↳ in that case it really becomes easy to show 3-Q2

[Q3] Show that $(W^\perp)^\perp = W$.

Problem 4: Row space

Let $\mathbf{H} \in \mathbb{R}^{m \times n}$. Show that $\operatorname{Im}(\mathbf{H}^T\mathbf{H}) = \operatorname{Im}(\mathbf{H}^T)$. → decompose H^T into orthonormal bases

$H^T \in \mathbb{R}^{n \times m}$ if $H \in \mathbb{R}^{m \times n}$ → $H = \left(\sum \lambda_{i,1} b_i \;\; \sum \lambda_{i,2} b_i \;\; \cdots \;\; \sum \lambda_{i,n} b_i \right)$ $i = 1 \dots d$, $j = 1 \dots n$

$H^T = \begin{pmatrix} \sum \lambda_{i,1} b_i^T \\ \sum \lambda_{i,2} b_i^T \\ \vdots \\ \sum \lambda_{i,n} b_i^T \end{pmatrix}$

$d = \#$ lin. ind. vectors $= \operatorname{rank}(H)$

$H^T H = \begin{pmatrix} \sum \lambda_{i,1}^2 \|b_i\|^2 \\ \sum \lambda_{i,1} \lambda_{i,2} \|b_i\|^2 \end{pmatrix}$

$\|b_i\|^2$ → b/c $b_1^T b_2 = 0$ (orthogonal)

also $\|b_i\|^2 = 1$ b/c normalized

Problem 5: Weighted least-squares

Consider a modified cost function

$$J(\mathbf{x}) \triangleq (\mathbf{y} - \mathbf{Hx})^T \mathbf{W} (\mathbf{y} - \mathbf{Hx})$$

where $\mathbf{W}$ is symmetric positive definite.

[Q1] Show that the cost is always positive.

[Q2] Show that the cost is zero if an only if $\mathbf{y} = \mathbf{Hx}$.

[Q3] Show that the normal equations corresponding to that modified cost are $\mathbf{H}^T\mathbf{WHx} = \mathbf{H}^T\mathbf{Wy}$.

Problem 6: Projectors

In class, we defined $P_H \triangleq \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T$ as the projector on $\mathrm{Im}(\mathbf{H})$.

[Q1] Verify that P_H is symmetric and idempotent (i.e., $P_H = P_H^T$).

[Q2] Show that $\mathbf{I} - P_H$ is a projector on $\mathrm{Im}(\mathbf{H})^\perp$.

[Q3] Show that if $\mathbf{y} \in \mathrm{Im}(\mathbf{H})$, then $P_H\mathbf{y} = \mathbf{y}$.

Problem 7: Projection with Mahalanobis distance

Show that the projection matrix $P_H = \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T$ defined in class should be replaced by

$$P_H = \mathbf{H}(\mathbf{H}^T\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{W}$$

when dealing with weighted least square problem characterized by the symmetric positive definite matrix $\mathbf{W}$.

Problem 8: Constrained least-square

We are now interested in solving the least-square problem $\min_x \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_2^2$ subject to the linear constraint $\mathbf{A}\mathbf{x} = \mathbf{b}$. Note that this is different from a regularized least square. Using Lagrange multipliers, one can show that the constrained problem is equivalent to the minimization of the following unconstrained problem

$$\min_x \left(\|\mathbf{y} - \mathbf{H}\mathbf{x}\|_2^2 + \boldsymbol{\lambda}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) \right).$$

using the vector $\boldsymbol{\lambda}$ of Lagrange multipliers.

[Q1] Show that the stationarizing $\boldsymbol{\lambda}$ is

$$\boldsymbol{\lambda} = 2[\mathbf{A}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T]^{-1}(\mathbf{A}\mathbf{x}_{LS} - \mathbf{b})$$

where $\mathbf{x}_{LS}$ is the unconstrained least-square solution.

[Q2] Conclude that the constrained least-square solution is

$$\mathbf{x}_c = \mathbf{x}_{LS} - (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T[\mathbf{A}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T]^{-1}(\mathbf{A}\mathbf{x}_{LS} - \mathbf{b})$$