Georgia Institute of Technology

ECE 6555 - Assignment 2

due Thursday September 22, 2022 - v1.0

- There are 8 problems over 4 pages (including the cover page).
- The problems are not necessarily in order of difficulty.
- Every question in a problem is worth 2 points, so problems with many questions are worth more than problems with few questions.
- Each question is graded as follows: no credit without meaningful work, half credit for partial work, full credit if essentially correct.
- Unless otherwise specified, you should concisely indicate your reasoning and show all relevant work.
- The grade on each question is based on our judgment of your level of understanding as reflected by what you have written. If we cannot read it, we cannot grade it.
- Please use a pen and not a pencil if you handwrite your solution.
- **You must submit your assignment on Gradescope**.

Problem 1: A separation principle

All variables in this problem are zero-mean. Consider the linear model $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$ where $\mathbf{x}$ and $\mathbf{v}$ are uncorrelated with variances $\mathbf{R}_x$ and $\mathbf{R}_v$, respectively. Consider a random variable $\mathbf{z}$ uncorrelated to $\mathbf{v}$ but otherwise correlated to $\mathbf{x}$. Let $\hat{\mathbf{z}}_{|\mathrm{x}}$ denote the Linear Least Mean Square (LLMS) estimator of $\mathbf{z}$ given $\mathbf{x}$. Let $\widehat{\hat{\mathbf{z}}_{|\mathrm{x}}}$ denote the LLMS estimator of $\hat{\mathbf{z}}_{|\mathrm{x}}$ given $\mathbf{y}$. Let $\hat{\mathbf{z}}_{|\mathrm{y}}$ denote the LLMS estimator of $\mathbf{z}$ given $\mathbf{y}$.

Show that $\hat{\mathbf{z}}_{\mathrm{y}} = \widehat{\hat{\mathbf{z}}_{|\mathrm{x}}}$.

Problem 2: Multiplicative noise

Consider the noisy measurement $\mathbf{y} = (1+v)\mathbf{x}$ where $\mathbf{x}$ and v are zero mean independent random variables. The variance of v is σ_v^2. Determine the LLMS estimator of $\mathbf{x}$ given $\mathbf{y}$. Show that the Minimum Mean-Square Error (MMSE) is smaller than the variance of $\mathbf{x}$.

Problem 3: Defective measurement sensors

Consider a zero-mean random variable $\mathbf{x}$ with variance Π_0 and two possible measurements for $\mathbf{x}$:

$$\mathbf{y}_1 = \mathbf{H}_1\mathbf{x} + \mathbf{v}_1 \qquad \mathbf{y}_2 = \mathbf{H}_2\mathbf{x} + \mathbf{v}_2.$$

where $(\mathbf{v}_1, \mathbf{v}_2)$ are zero-mean uncorrelated sensor noise with variance $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively, also uncorrelated with $\mathbf{x}$. One of the measurements is defective and is either sensor 1 with probability $1-p$ or sensor 2 with probability p. Denote this measurement by $\mathbf{z}$.

[Q1] Find the LLMS estimator of $\mathbf{x}$ given $\mathbf{z}$

[Q2] Find the corresponding MMSE

[Q3] How does your answer change if $\mathbf{v}_1$ and $\mathbf{v}_2$ were correlated?

[Q4] What can you say about the special case $\mathbf{H}_1 = \mathbf{H}_2$.

Problem 4: Linear estimator of x^2

Consider $y = x + v$ where v and x are independent real-valued independent zero-mean Gaussian scalar random variables with variances σ_v^2 and σ_x^2, respectively. Find the LLMS estimator of the random variable x^2 using y and y^2. (*Hint:* recall that for a real-valued zero-mean Gaussian random variable z with variance σ^2, $\mathbb{E}\left[z^3\right] = 0$ ad $\mathbb{E}\left[z^4\right] = 3\sigma^4$.)

Problem 5: Separation of signal and structured noise

Consider the model

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{S}\boldsymbol{\theta} + \mathbf{v}$$

where $\mathbf{v}$ is a zero mean additive noise random vector with unit variance and $\mathbf{x}, \boldsymbol{\theta}$ are *unknown* constant vectors. The matrices $\mathbf{H} \in \mathbb{C}^{m\times n}$ and $\mathbf{S} \in \mathbb{C}^{m\times p}$ are known and such that $\begin{bmatrix} \mathbf{H} & \mathbf{S} \end{bmatrix}$ is full rank and $m \geq n + p$. The term $\mathbf{S}\boldsymbol{\theta}$ is interpreted as a *perturbation* while the term $\mathbf{H}\mathbf{x}$ is the useful signal we wish to separate.

[Q1] Define the vector $\mathbf{z} = \begin{bmatrix} \mathbf{x} & \boldsymbol{\theta} \end{bmatrix}^{\mathsf{T}}$. Determine the optimal *unbiased* estimator $\hat{\mathbf{z}}$ of $\mathbf{z}$ given $\mathbf{y}$.

[Q2] Write $\hat{\mathbf{z}} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\boldsymbol{\theta}} \end{bmatrix}$ to separate the estimation of $\mathbf{x}$ and $\boldsymbol{\theta}$. Let $\hat{\mathbf{s}} \triangleq \mathbf{H}\hat{\mathbf{x}}$ denote the estimator of $\mathbf{s} \triangleq \mathbf{Hx}$. Show that

$$\hat{\mathbf{s}} = \mathbf{Ey} \text{ with } \mathbf{E} = P_{\mathsf{H}} \left[\mathbf{I} - \mathbf{S}(\mathbf{S}^\dagger P_{\mathsf{H}}^\perp \mathbf{S})^{-1}\mathbf{S}^\dagger P_{\mathsf{H}}^\perp\right] = \mathbf{H}(\mathbf{H}^\dagger P_{\mathsf{S}}^\perp \mathbf{H})^{-1}\mathbf{H}^\dagger P_{\mathsf{S}}^\perp.$$

with $P_{\mathsf{H}}^\perp = \mathbf{I} - P_{\mathsf{H}}$, $P_{\mathsf{S}}^\perp = \mathbf{I} - P_{\mathsf{S}}$, and P_{H} and P_{S} are the orthogonal projection matrices on the space spanned by the rows of their respective matrices.

[Q3] Conclude that $\mathbf{ES} = 0$ and provide a geometric interpretation.

[Q4] Let $\tilde{\mathbf{s}} = \mathbf{s} - \hat{\mathbf{s}}$. Show that the mean square error $\mathbb{E}\left[\tilde{\mathbf{s}}\tilde{\mathbf{s}}^\dagger\right]$ is $\mathbf{E}\mathbf{E}^\dagger$.

[Q5] Assume now that $\mathbf{x}$ is a zero mean random variable with known variance $\Pi_0 > 0$. Show that the LLMS of $\mathbf{s} = \mathbf{Hx}$ is now

$$\hat{\mathbf{s}} = \mathbf{Fy} \text{ with } \mathbf{F} = P_{\mathsf{H}} \left[\mathbf{I} - \mathbf{S}(\mathbf{S}^\dagger P_{\mathsf{H}}^\perp \mathbf{S})^{-1}\mathbf{S}^\dagger P_{\mathsf{H}}^\perp\right]$$

with $P_{\mathsf{H}}^\perp = \mathbf{I} - P_{\mathsf{H}}$ and $P_{\mathsf{H}} = \mathbf{H}(\mathbf{H}^\dagger\mathbf{H} + \Pi_0)^{-1}\mathbf{H}$. Make sure you understand the difference with the previous situation where $\mathbf{x}$ was modeled as a constant.

[Q6] Verify that $\mathbf{FS} = 0$.

[Q7] Compute the new MMSE and compare with the previous result.

Problem 6: General combined estimator

Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two separate observations of a zero-mean random variable $\mathbf{x}$ using the linear models

$$\mathbf{y}_1 = \mathbf{H}_1\mathbf{x} + \mathbf{v}_1 \qquad \mathbf{y}_2 = \mathbf{H}_2\mathbf{x} + \mathbf{v}_2$$

and

$$\left\langle \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{x} \end{bmatrix} \right\rangle = \begin{bmatrix} R_1 & 0 \\ 0 & M_1 \end{bmatrix} \qquad \left\langle \begin{bmatrix} \mathbf{v}_2 \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \mathbf{v}_2 \\ \mathbf{x} \end{bmatrix} \right\rangle = \begin{bmatrix} R_2 & 0 \\ 0 & M_2 \end{bmatrix}.$$

Note that this is a bit different from what we saw in class because the covariance matrix of $\mathbf{x}$ *is different in both experiments.*

Let $\hat{\mathbf{x}}_1$ be the LLMS estimate of $\mathbf{x}$ from $\mathbf{y}_1$ with corresponding error covariance matrix P_1. Let $\hat{\mathbf{x}}_2$ be the LLMS estimate of $\mathbf{x}$ from $\mathbf{y}_2$ with corresponding error covariance matrix P_2. Let $\hat{\mathbf{x}}$ be the LLMS estimate of $\mathbf{x}$ from $\mathbf{y}_1$ and $\mathbf{y}_2$ assuming $\langle \mathbf{x}, \mathbf{x} \rangle = \Pi$ with corresponding error covariance matrix P.

Show that $P^{-1}\hat{\mathbf{x}} = P_1^{-1}\hat{\mathbf{x}}_1 + P_2^{-1}\hat{\mathbf{x}}_2$ and

$$P^{-1} = P_1^{-1} + P_2^{-1} + \Pi^{-1} - M_1^{-1} - M_2^{-1}.$$

Make sure your provide enough details to justify your answer, but you can of course use any result seen in class. Just make sure that you clearly indicate which results you use!

Problem 7: Optimal estimation for exponential distribution

Suppose $y = x + v$ where x and v are independent real-valued random variables with exponential distribution of parameters $\lambda > 0$ and $\mu > 0$, respectively. Recall that that an exponential distribution is of the form $\lambda e^{-\lambda x}$ for $x \geq 0$, with mean λ^{-1} and variance λ^{-2}.

[Q1] Show that $p(x, y) = \lambda\mu e^{-(\lambda-\mu)x}e^{-\mu y}$ for $x \leq y$ and 0 else.

[Q2] Show that $p(y) = \frac{\lambda\mu}{\lambda-\mu}(e^{-\mu y} - e^{-\lambda y})$

[Q3] Show that the optimal (non linear) least mean square estimate of x given y is

$$\hat{x} = \frac{1}{\lambda - \mu} - \frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}} y.$$

[Q4] Calculate the LLMS (*Hint:* x and y are not centered!) and compare with the result above.

Problem 8: An optimal nonlinear estimator for binary signals

Consider the observations $y_i = x + v_i$ where x and v_i are independent real-valued random variables and $\{v_i\}_{i\geq 0}$ is a white noise Gaussian process with zero mean and unit variance.

[Q1] Assume that the variable x takes the values ± 1 with equal probability. Show that the optimal nonlinear Least Mean Square (LMS) estimator of x given n observations $\{y_i\}_{i=0}^{n-1}$ is

$$\hat{x}_n = \tanh\left(\sum_{i=0}^{n-1} y_i\right)$$

[Q2] Now assume that the variable x takes the value $+1$ with probability p and -1 with probability $1-p$. Show that the optimal nonlinear LMS estimator of x given n observations $\{y_i\}_{i=0}^{n-1}$ is now

$$\hat{x}_n = \tanh\left(\frac{1}{2}\ln\left(\frac{p}{1-p}\right) + \sum_{i=0}^{n-1} y_i\right).$$