ECE 6555 HW2

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Question 3

Q3-1

Q3-2

```
syms x K_1 K_2 p z Pi_0 H_1 H_2 R_1 R_2
xhat = K_1*z*(1-p) + K_2*z*p
```

xhat = $K_2\,p\,z - K_1\,z\,(p-1)$

```
MMSE = (x - xhat)*transpose(x - xhat)
```

MMSE = $(x - K_2\,p\,z + K_1\,z\,(p-1))^2$

```

syms z [3 1] matrix
syms K_1 K_2 [3 3] matrix

isequal(K_1*z*z.'*K_2.', K_2*z*z.'*K_1.')
```

```
ans = logical
   0
```

```
symmatrix2sym(K_2*z*z.'*K_1.')
```

ans =

$$\begin{pmatrix} \sigma_6 \sigma_3 & \sigma_5 \sigma_3 & \sigma_4 \sigma_3 \\ \sigma_6 \sigma_2 & \sigma_5 \sigma_2 & \sigma_4 \sigma_2 \\ \sigma_6 \sigma_1 & \sigma_5 \sigma_1 & \sigma_4 \sigma_1 \end{pmatrix}$$

where

$$\sigma_1 = K_{23,1} z_1 + K_{23,2} z_2 + K_{23,3} z_3$$

$$\sigma_2 = K_{22,1} z_1 + K_{22,2} z_2 + K_{22,3} z_3$$

$$\sigma_3 = K_{21,1} z_1 + K_{21,2} z_2 + K_{21,3} z_3$$

$$\sigma_4 = K_{13,1} z_1 + K_{13,2} z_2 + K_{13,3} z_3$$

$$\sigma_5 = K_{12,1} z_1 + K_{12,2} z_2 + K_{12,3} z_3$$

$$\sigma_6 = K_{11,1} z_1 + K_{11,2} z_2 + K_{11,3} z_3$$

```
symmatrix2sym(K_1*z*z.'*K_2.')
```

```
ans =
```

$$\begin{pmatrix} \sigma_6 \sigma_3 & \sigma_6 \sigma_2 & \sigma_6 \sigma_1 \\ \sigma_5 \sigma_3 & \sigma_5 \sigma_2 & \sigma_5 \sigma_1 \\ \sigma_4 \sigma_3 & \sigma_4 \sigma_2 & \sigma_4 \sigma_1 \end{pmatrix}$$

where

$$\sigma_1 = K_{23,1} z_1 + K_{23,2} z_2 + K_{23,3} z_3$$

$$\sigma_2 = K_{22,1} z_1 + K_{22,2} z_2 + K_{22,3} z_3$$

$$\sigma_3 = K_{21,1} z_1 + K_{21,2} z_2 + K_{21,3} z_3$$

$$\sigma_4 = K_{13,1} z_1 + K_{13,2} z_2 + K_{13,3} z_3$$

$$\sigma_5 = K_{12,1} z_1 + K_{12,2} z_2 + K_{12,3} z_3$$

$$\sigma_6 = K_{11,1} z_1 + K_{11,2} z_2 + K_{11,3} z_3$$

Question 5

Q5-2 Separate the Estimation

```
syms H [3 2] matrix
syms S [3 4] matrix
Hz = [H S]
```

Hz = $\begin{pmatrix} H & S \end{pmatrix}$

```
inv(Hz.'*Hz)
```

ans = $\left(\begin{pmatrix} H & S \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} H & S \end{pmatrix}\right)^{-1}$

Question 7

Q7-3

```
syms x y v mu lambda real
assume(lambda > 0); assume(mu > 0);
Ex = 1/lambda;
Ex2 = 1/lambda^2;
pdfxy = lambda*mu*exp(-(lambda - mu)*x)*exp(-mu*y)
```

pdfxy = $\lambda\,\mu\,\mathrm{e}^{-\mu\,y}\,\mathrm{e}^{-x\,(\lambda-\mu)}$

```
pdfy = (lambda*mu/(lambda - mu))*(exp(-mu*y) - exp(-lambda*y));
pdfx = lambda*exp(-lambda*x);
pdfv = mu*exp(-mu*v);
```

```
display(pdfxy)
```

pdfxy = $\lambda\,\mu\,\mathrm{e}^{-\mu\,y}\,\mathrm{e}^{-x\,(\lambda-\mu)}$

```
int(pdfxy,x,0,y)
```

ans =

$$-\frac{\lambda\,\mu\,(\mathrm{e}^{-\lambda\,y}-\mathrm{e}^{-\mu\,y})}{\lambda-\mu}$$

```
% Exxhat
```

pdfxy = $\lambda\,\mu\,\mathrm{e}^{-\mu\,y}\,\mathrm{e}^{-x\,(\lambda-\mu)}$

```
integrandxxhat = x*y*exp(-lambda*y)/(exp(-mu*y)-exp(-lambda*y))*pdfxy
```

integrandxxhat =

$$-\frac{\lambda\,\mu\,x\,y\,\mathrm{e}^{-\lambda\,y}\,\mathrm{e}^{-\mu\,y}\,\mathrm{e}^{-x\,(\lambda-\mu)}}{\mathrm{e}^{-\lambda\,y}-\mathrm{e}^{-\mu\,y}}$$

```
xhat = 1/(lambda - mu) - y*exp(-lambda*y)/(exp(-mu*y) - exp(-lambda*y))
```

xhat =

$$\frac{1}{\lambda-\mu}+\frac{y\,e^{-\lambda\,y}}{e^{-\lambda\,y}-e^{-\mu\,y}}$$

```
xhat_v = subs(xhat,y,v+x)
```

xhat_v =

$$\frac{1}{\lambda-\mu}+\frac{e^{-\lambda\,(v+x)}\,(v+x)}{e^{-\lambda\,(v+x)}-e^{-\mu\,(v+x)}}$$

```
% Exxhat = int(int(x*xhat*pdfxy,x,0,y),y,0,inf)
% Exhatxhat = int(simplify(expand(xhat*xhat*pdfy)),y,0,inf)

% Exxhat_v = int(int(x*xhat_v*pdfx*pdfv,x,0,inf),v,0,inf)
```

Q7-4

```
syms x y mu lambda real
pdfxy = lambda*mu*exp(-(lambda - mu)*x)*exp(-mu*y)
```

pdfxy = $\lambda\,\mu\,e^{-\mu\,y}\,e^{-x\,(\lambda-\mu)}$

```
Exy = int(int(x*y*pdfxy,x,0,y),y,0,inf)
```

Exy =

$$\frac{\lim_{y\to\infty} -\frac{\lambda\,e^{-y\,\mu}}{\mu}+y\,e^{-y\,\lambda}\left(3\,\mu-\frac{2\,\mu^2}{\lambda}\right)+\frac{e^{-y\,\lambda}\,\sigma_2}{\lambda^2}-y\,\lambda\,e^{-y\,\mu}+y^2\,\mu\,e^{-y\,\lambda}\,(\lambda-\mu)}{\sigma_1}+\frac{\frac{\lambda}{\mu}-\frac{\sigma_2}{\lambda^2}}{\sigma_1}$$

where

$$\sigma_1=\lambda^2-2\,\lambda\,\mu+\mu^2$$

$$\sigma_2=3\,\lambda\,\mu-2\,\mu^2$$

```
expand(Exy)
```

ans =

$$\frac{2\,\mu^2}{\lambda^4-2\,\lambda^3\,\mu+\lambda^2\,\mu^2}+\frac{\lambda}{\lambda^2\,\mu-2\,\lambda\,\mu^2+\mu^3}-\frac{3\,\mu}{\lambda^3-2\,\lambda^2\,\mu+\lambda\,\mu^2}+\frac{\lim_{y\to\infty}\frac{3\,\mu\,e^{-\lambda\,y}}{\lambda}-\frac{\lambda\,e^{-\mu\,y}}{\mu}-\frac{2\,\mu^2\,e^{-\lambda\,y}}{\lambda^2}-\mu^2\,y^2\,e^{-}}{\lambda^2-}$$

```
% check my calculation of E[xy]
```

```
expand((-lambda*mu/(lambda - mu)^2)*(2*(lambda - mu)/lambda^3 + 1/lambda^2 - 1/mu^2))
```

ans =

$$\frac{2\,\mu^2}{\lambda^4 - 2\,\lambda^3\,\mu + \lambda^2\,\mu^2} + \frac{\lambda}{\lambda^2\,\mu - 2\,\lambda\,\mu^2 + \mu^3} - \frac{3\,\mu}{\lambda^3 - 2\,\lambda^2\,\mu + \lambda\,\mu^2}$$

```
pdfy = (lambda*mu/(lambda - mu))*(exp(-mu*y) - exp(-lambda*y))
```

pdfy =

$$-\frac{\lambda\,\mu\,(e^{-\lambda\,y} - e^{-\mu\,y})}{\lambda - \mu}$$

```
% E[y]
expand(int(y*pdfy,y,0,inf))
```

ans =

$$\frac{\lambda}{\lambda\,\mu - \mu^2} + \frac{\mu}{\lambda\,\mu - \lambda^2} - \frac{\lambda\,\mu\,\left(\lim_{y\to\infty} \frac{e^{-\mu\,y}}{\mu^2} - \frac{e^{-\lambda\,y}}{\lambda^2} - \frac{y\,e^{-\lambda\,y}}{\lambda} + \frac{y\,e^{-\mu\,y}}{\mu}\right)}{\lambda - \mu}$$

```
% E[y^2]
expand(int(y^2*pdfy,y,0,inf))
```

ans =

$$\frac{2\,\lambda}{\lambda\,\mu^2 - \mu^3} + \frac{2\,\mu}{\lambda^2\,\mu - \lambda^3} - \frac{\lambda\,\mu\,\left(\lim_{y\to\infty} \frac{2\,e^{-\mu\,y}}{\mu^3} - \frac{2\,e^{-\lambda\,y}}{\lambda^3} - \frac{2\,y\,e^{-\lambda\,y}}{\lambda^2} + \frac{2\,y\,e^{-\mu\,y}}{\mu^2} - \frac{y^2\,e^{-\lambda\,y}}{\lambda} + \frac{y^2\,e^{-\mu\,y}}{\mu}\right)}{\lambda - \mu}$$

```
% part of the integral for E[y^2]:
int(y^2*exp(-mu*y),y,0,inf)
```

ans =

$$\frac{2}{\mu^3} - \frac{\lim_{y\to\infty} e^{-\mu\,y}\,(\mu^2\,y^2 + 2\,\mu\,y + 2)}{\mu^3}$$

```
% Calculating estimation of x given y:
Ex = 1/lambda;
Ey = simplify(expand((lambda^2 - mu^2)/(lambda - mu)/(lambda*mu)))
```

Ey =

$$\frac{\lambda + \mu}{\lambda\,\mu}$$

```
Exy = (- lambda*mu/(lambda - mu)^2)*(2*(lambda - mu)/lambda^3 + 1/lambda^2 - 1/mu^2)
```

Exy =

$$-\frac{\lambda\,\mu\,\left(\frac{2\,\lambda-2\,\mu}{\lambda^3}+\frac{1}{\lambda^2}-\frac{1}{\mu^2}\right)}{(\lambda-\mu)^2}$$

```
Ey2 = (lambda*mu/(lambda - mu))*(2/mu^3 - 2/lambda^3)
```

Ey2 =

$$-\frac{\lambda\,\mu\,\left(\frac{2}{\lambda^3}-\frac{2}{\mu^3}\right)}{\lambda-\mu}$$

```
xhat = Ex + (Exy - Ex*Ey)*(y - Ey)/(Ey2 - Ey^2);
xhat = simplify(expand(xhat))
```

xhat =

$$\frac{y\,\mu^2-\mu+\lambda}{\lambda^2+\mu^2}$$

```
% Calculating K_0
syms x v K lambda mu y
% P(K) = 1/lambda^2 - 2*K/lambda - 2*K/mu + K^2/lambda^2 +K^2/mu^2
% K0 = simplify(expand(solve(diff(P(K),K)==0,K)))
% simplify(P(K0))
% simplify(P(mu^2/(lambda^2 + mu^2)))
% just use the solution for a linear estimate given y = x + v ; (H = 1)

% K0 = RxyRy^-1
% see written work

% xhat affine solution
xhat_affine = simplify(expand(1/lambda + (mu^2/(lambda^2+mu^2))*(y - (mu + lambda)/(mu*lambda)
```

xhat_affine =

$$\frac{y\,\mu^2-\mu+\lambda}{\lambda^2+\mu^2}$$

```
xhat_affine_y = collect(xhat_affine,y)
```

xhat_affine_y =

$$\frac{\mu^2}{\lambda^2+\mu^2}\,y+\frac{\lambda-\mu}{\lambda^2+\mu^2}$$

Question 8

Q8-1

Q8-2

```
syms p
assume(p > 0)
assumeAlso(p,'real')
simplify(expand((log(1-p) - 1/2*log(p/(1-p)))))
```

```
ans =
```

$$\log(1-p) - \frac{\log(p)}{2} - \frac{\log\left(-\frac{1}{p-1}\right)}{2}$$