$$\hat{x} = \cancel{\tfrac{1}{2}}\mathbb{E}[x] + \frac{(\mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y])}{\mathbb{E}[y^2] - \mathbb{E}[y]^2}(y - \mathbb{E}[y])$$

$\Rightarrow$ MATLAB

$\Rightarrow \hat{x} = \dfrac{y\mu^2 - \mu + \lambda}{\lambda^2 + \mu^2}$ <u>does not match</u> → this <u>is</u> the LLSE for part $\boxed{Q4}$

what went wrong?? How to get $e^{-\lambda y}$ and $e^{-\mu y}$ into $\hat{x}$??

↑ ~~comment:~~ ~~Similar form..~~ ~~$\hat{x} = -$~~

$$\hat{x} = \mu_x + R_{xy}R_y^{-1}(y - \mu_y)$$

$$\mu_x = \frac{1}{\lambda}$$

$$\mu_y = \mathbb{E}[y] = \mathbb{E}[x+v] = \frac{1}{\lambda} + \frac{1}{\mu}$$

$$\begin{aligned} R_{xy} = \mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y] &= \mathbb{E}[x(x+v)] - \mathbb{E}[x]\mathbb{E}[x+v] \\ &= \mathbb{E}[x^2] + \mathbb{E}[xv] - \mathbb{E}[x]\mathbb{E}[x] - \mathbb{E}[x]\mathbb{E}[v] \\ &= \cancel{\frac{1}{\lambda^2}} + \cancel{\frac{1}{\lambda}\left(\frac{1}{\mu}\right)} - \cancel{\frac{1}{\lambda^2}} - \cancel{\frac{1}{\lambda}\frac{1}{\mu}} = 0 \end{aligned}$$

<u>Non-linear, optimal estimate:</u>

let $h^* = \underset{h}{\operatorname{argmin}}\ \mathbb{E}[(x - h(y))(x - h(y))^T]$

since $\mathbb{E}[(x - h(y))(x - h(y))^T] \geq 0$ by definition,

then if we ~~can~~ find $h(y)$ s.t. $\mathbb{E}_{xh} = 0$, $h(y) = h^*$ since

$\mathbb{E}_{xh}$ cannot be any lower value.

~~calcula~~ calculation →

Let $h(y) = \hat{x} = \frac{1}{\lambda-\mu} - \left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right) y = \frac{1}{\lambda-\mu} - \left(\frac{e^{-\lambda(x+v)}}{e^{-\mu(x+v)} - e^{-\lambda(x+v)}}\right)(x+v)$

$\Rightarrow \mathbb{E}[(x-\hat{x})(x-\hat{x})^T] = \mathbb{E}[(x-\hat{x})^2] = \mathbb{E}[x^2] - \mathbb{E}[x\hat{x}] - \mathbb{E}[\hat{x}x] + \mathbb{E}[\hat{x}^2]$

$\mathbb{E}[x^2] = \frac{1}{\lambda^2}$

$\mathbb{E}[x\hat{x}] = \mathbb{E}\left[\frac{x}{\lambda-\mu} - \left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right) xy\right]$ $\Rightarrow$ MATLAB

$= \frac{1}{\lambda(\lambda-\mu)} -$

$\mathbb{E}[\hat{x}x] = \mathbb{E}[x\hat{x}]$ (scalar) $= \mathbb{E}\left[\frac{x}{\lambda-\mu} - x(x+v)\left(\frac{e^{-\lambda x}e^{-\lambda v}}{e^{-\mu x}e^{-\mu v} - e^{-\lambda x}e^{-\lambda v}}\right)\right]$

$\mathbb{E}[\hat{x}^2] =$

$= \frac{1}{\lambda^2}\left(\frac{1}{\lambda-\mu}\right) - \mathbb{E}\left[(x^2+xv)\left(\frac{e^{-\lambda x}e^{-\lambda v}}{e^{-\mu v}e^{-\mu v} - e^{-\lambda x}e^{-\lambda v}}\right)\right]$

too long...

#8) Optimal Nonlinear estimator for binary signals

#8-Q1) obs. $y_i = x + v_i$, x & v_i independent real-valued random variables

$\{v_i\}_{i\geq 0}$ is a white noise Gaussian process.

$E[v] = 0$, $\mathbb{E}[v^2] = R_v = 1$

$x = \pm 1$ w/ equal probability...

$\hat{x} = h(y) = \tanh\left(\sum_{i=0}^{n-1} y_i\right)$

$P(h) = \mathbb{E}[(x - h(y))(x - h(y))^T]$ ~~$\hat{x} = Ky$...~~ ?

$= \mathbb{E}[(x - \hat{x}_n)(x - \hat{x}_n)^T]$ $x, \hat{x}$ scalar...

$= \mathbb{E}[xx] - 2\mathbb{E}[x\hat{x}_n] + \mathbb{E}[\hat{x}_n^2]$ $\hat{x} = \tanh\left(\sum_{i=0}^{n-1} y_i\right)$

$\mathbb{P}(x=1) = \frac{1}{2}$, $\mathbb{E}[x=-1] = \frac{1}{2}$

$\mathbb{E}[(x-\hat{x}_n)^2] = \mathbb{E}_x[\mathbb{E}((x-\hat{x}_n)^2 | x)]$

$= \mathbb{E}((1-\hat{x}_n)^2)(0.5) + \mathbb{E}((-1-\hat{x}_n)^2)(0.5)$

$\mathbb{E}((1-\hat{x}_n)^2) = \mathbb{E}\left(\left(1 - \tanh\left(\sum_{i=0}^{n-1} y_i\right)\right)^2\right)$

$1 - 2\hat{x}_n + \hat{x}_n^2$

$= 1 - 2\mathbb{E}(\hat{x}_n) + \mathbb{E}[\hat{x}_n^2]$

$1 - 2\mathbb{E}\left(\tanh\left(\sum_{i=0}^{n-1}(x+v_i)\right)\right) + \mathbb{E}\left(\tanh\left(\sum_{i=0}^{n-1}(1+v_i)\right)^2\right)$ ($x \to 1$)

$\sum_{i=0}^{n-1}(1+v_i) = n$ b/c $\{v_i\}_{i\geq 0}$ is 0-mean, Gaussian

$n >> 1$

$= 1 - 2\mathbb{E}(\tanh(n)) + \mathbb{E}(\tanh(n)^2)$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\mathbb{E}[(1-\hat{x}_n)^2] = 1 - 2 + 1 = 0 = 0$

$\mathbb{E}((-1-\hat{x}_n)^2) = \mathbb{E}(1 + 2\hat{x}_n + \hat{x}_n^2)$ ~~$= 4$~~

$\mathbb{E}(\cancel{\hat{x}_n}) = 1$, $\mathbb{E}[\cancel{\hat{x}_n^2}] = 1$

$\mathbb{E}[\hat{x}_n] = \mathbb{E}\left[\tanh\left(\sum_{i=0}^{n-1}(-1+v_i)\right)\right]$

$= \mathbb{E}(\tanh(-n)) = -1$

$\mathbb{E}[\hat{x}_n^2] = \mathbb{E}[\tanh(-n)^2] = 1$

$\Rightarrow \mathbb{E}[(-1-\hat{x}_n)^2] = 1 - 2 + 1 = 0$

$\Rightarrow \mathbb{E}[(x-\hat{x}_n)^2] =$ ~~$0 + 4(0.5) = 2 \neq 0$~~ ... $\boxed{?}$

$= 0 + 0 = \boxed{0}$ ✓