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Due: 11/10/22

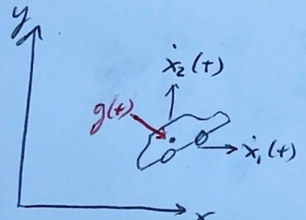
ECE6555, HW#4

#1 Kalman Filter in Action

→ use Jupyter notebooks + python (numpy + scipy)

State Space model of a vehicle subject to unknown forces

↑ single point w/ known mass, m, for simplicity



$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2$ (vehicle moving in a plane)

$\vec{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \in \mathbb{R}^2$ unknown Force as a time-varying vector

From Newtonian physics: $\vec{g}(t) = m \frac{\partial^2 \vec{x}(t)}{\partial t^2}$

"We choose to model $\vec{g}(t)/m$ as a 2D Gaussian white noise process with auto-correlation function $\delta(t)$" s.t. $R_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\frac{\partial^2 \vec{x}(t)}{\partial t^2} \triangleq \vec{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

Set $x_3(t) \triangleq \frac{\partial x_1(t)}{\partial t}$ and $x_4(t) \triangleq \frac{\partial x_2(t)}{\partial t}$ to consider augmented state of car:

$$\vec{z}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

Q1 Show the evolution of $\vec{z}(t)$ is governed by the matrix differential eqtn:

$$\frac{\partial \vec{z}(t)}{\partial t} = \underline{F}\,\vec{z}(t) + \underline{G}\,\vec{w}(t) \qquad (1)$$

$$\Rightarrow \frac{\partial \vec{z}(t)}{\partial t} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \partial^2 x_1/\partial t^2 \\ \partial^2 x_2/\partial t^2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ w_1(t) \\ w_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\triangleq \underline{F}} \vec{z} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\triangleq \underline{G}} \vec{w}$$

Q2] position of vehicle: measured through a noisy sensor $\vec{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \mathbb{R}^2$

which is a corrupted version of the true position by a 2D Gaussian white noise process $\vec{v}(t)$ w/ auto-correlation $\sigma^2 \delta(t) \Rightarrow R_v = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$

Show: measurement equation given by $\vec{y} = \underline{H}\vec{z} + \vec{v}(t)$ & specify $\underline{H}$:

$$\vec{y}(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\triangleq \underline{H}} \vec{z}(t) + \vec{v}(t) \quad \left(= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \right) \qquad (2)$$

Simulations & filters run in ~~real~~ discrete-time, need to sample at a period Δ.

Therefore only consider state of vehicle at times $t_k \in \{k\Delta : k \in \mathbb{N}\}$

Discretization: takes some care to get a meaningful time model.

One can show the solution of (1) satisfies for $k \geq 0$:

$$\vec{z}(t_{k+1}) = e^{\underline{F}\Delta}\vec{z}(t_k) + \underbrace{\int_0^{\Delta} \exp(\underline{F}(\Delta - \tau))\, \underline{G}\, \vec{w}(\tau + t_k)\, d\tau}_{\triangleq \vec{u}(t_k)} \qquad (3)$$

One can also show that the discrete-time process $\{\vec{u}(t_k)\}_{k\geq 0}$ is Gaussian and white with a covariance matrix $\underline{Q}$ that can be computed from $\underline{F}$, $\underline{G}$, Δ and the auto-correlation function of $\vec{w}(t)$. For $\Delta \ll 1$, one can show

$$\underline{Q} = \Delta \begin{bmatrix} \Delta^2/3 & 0 & \Delta/2 & 0 \\ 0 & \Delta^2/3 & 0 & \Delta/2 \\ \Delta/2 & 0 & 1 & 0 \\ 0 & \Delta/2 & 0 & 1 \end{bmatrix} \qquad (4)$$

Q3] writing $\vec{x}_k \triangleq \vec{z}(t_k)$ and $\vec{u}_k \triangleq \vec{u}(t_k)$ and using (3), show that the discretized state-space model when $\Delta \ll 1$ is:

$$\vec{x}_{k+1} = \begin{bmatrix} 1 & 0 & \Delta & 0 \\ 0 & 1 & 0 & \Delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{x}_k + \vec{u}_k$$, where $\{\vec{u}_k\}$ is white with covariance Q

→

Q3 cont'd

$$\underline{F}\Delta = \begin{bmatrix} 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \exp(\underline{F}\Delta) = e^{\underline{F}t}\Big|_{t=\Delta} = \mathcal{L}^{-1}\{(sI-\underline{F})^{-1}\}\Big|_{t=\Delta}$$

$$(sI-\underline{F})^{-1} = \begin{pmatrix} s & 0 & -1 & 0 \\ & s & 0 & -1 \\ & & s & 0 \\ 0 & & & s \end{pmatrix}^{-1} = \begin{pmatrix} 1/s & 0 & 1/s^2 & 0 \\ 0 & 1/s & 0 & 1/s^2 \\ 0 & 0 & 1/s & 0 \\ 0 & 0 & 0 & 1/s \end{pmatrix}$$

$$\mathcal{L}\{(sI-\underline{F})^{-1}\} = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathcal{L}\{(sI-\underline{F})^{-1}\}\Big|_{t=\Delta} = \begin{pmatrix} 1 & 0 & \Delta & 0 \\ & 1 & 0 & \Delta \\ & & 1 & 0 \\ 0 & & & 1 \end{pmatrix} = \exp(\underline{F}\Delta)$$

$\Rightarrow$ Thus the discretized state-space model from ③ when $\Delta \ll 1$ is

$$\vec{x}_{k+1} = \begin{bmatrix} 1 & 0 & \Delta & 0 \\ 0 & 1 & 0 & \Delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{x}_k + \vec{u}_k$$

, where $\{u_k\}$ is white w/ covariance Q given above.

Q4

See jupyter notebook for plot of data.

Q5 Kalman Filter equations

From class: Time Update:

1) (prediction)

$$\begin{cases} \hat{x}_{i|i-1} = F_{i-1}\hat{x}_{i-1|i-1} \\ P_{i|i-1} = F_{i-1}P_{i-1|i-1}F_{i-1}^T + G_{i-1}Q_{i-1}G_{i-1}^T \end{cases}$$

2) Measurement Update

(correction)

$$\begin{cases} \hat{x}_{i|i} = \hat{x}_{i|i-1} + K_{f,i}(y_i - H_i\hat{x}_{i|i-1}) \\ K_{f,i} = P_{i|i-1}H_i^T(H_iP_{i|i-1}H_i^T + R_i)^{-1} \\ P_{i|i} = P_{i|i-1} - K_{f,i}H_iP_{i|i-1} = (I - K_{f,i}H_i)P_{i|i-1} \end{cases}$$

To write out the equations further:

1) ~~Time Update:~~

$$\hat{x}_{i|i-1} = F \hat{x}_{i-1|i-1}$$

no dependence on time

$$P_{i|i-1} = F P_{i-1|i-1} F^T + G Q_{i-1} G^T$$

~~Note:~~

2) ~~Measurement Update:~~

$$\hat{x}_{i|i} = \hat{x}_{i|i-1} + K_{f,i} \ldots$$ same as previous actually

Q6) Implement Kalman Filter

(try both update first + prediction first)

State Space Model:

$$\vec{x}_{k+1} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{F} \vec{x}_k + G\vec{u}_k$$

$G = \text{Identity}(4)$

$$\vec{y}_k = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{H} \vec{x}_k + \vec{v}_k$$

$$K_{f,i} = P_{i|i-1} H_i^T \left(H_i P_{i|i-1} H_i^T + R_i \right)^{-1}$$

$$(4\times4)(4\times2)(2\times\cancel{4}\cancel{(4\times4)}(4\times2) + 2\times2)^{-1}$$

$$(4\times \cancel{4\times4\times2}(2\times2)^{-1}$$

$$= (4\times2)$$ ✓

$\boxed{\text{See Jupyter Notebook for code}}$

Markov chain ✓ (uses cond. indep.)

Conditional Independence. ✓

Q9

given our discrete-time probabilistic state-space model

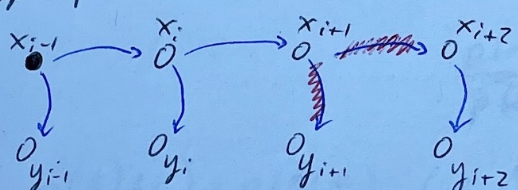
$$x_{i+1} \sim p(x_{i+1}|x_{0:i}\, y_{0:i}) \quad \text{and} \quad y_i \sim p(y_i|x_{0:i}\, y_{0:i-1})$$

that is Markovian & satisfies the conditional independent property of measurements:

$$p(x_{i+1}|x_{0:i}\, y_{0:i}) = p(x_{i+1}|x_i) \quad \text{and} \quad p(y_i|x_{0:i}\, y_{0:i-1}) = p(y_i|x_i)$$

show that $p(x_i|x_{i+1}, y_{0:n}) = p(x_i|x_{i+1}, y_{0:i})$

1) sub-graph of our probabilistic state-space model:



2) Condition on x_{i+1} shown by breaking the edges coming out of x_{i+1} (in red)

This shows us visually the conditional independent relationships.

As we can see, by breaking the edges out of x_{i+1}, we have that $y_{i+1:n}$ are no loger connected to x_i by any edge. ~~They~~ Thus, by the Functional Dependence Graph theorem and definition given in class,

$$\boxed{p(x_i|x_{i+1}, y_{0:n}) = p(x_i|x_{i+1}, y_{0:i})}$$

□

Q10 Using Bayes' rule and the properties of the state space model show that:

$$p(x_i | x_{i+1}, y_{0:n}) = \frac{p(x_{i+1} | x_i)\, p(x_i | y_{0:i})}{p(x_{i+1} | y_{0:i})}$$

Bayes' Rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$, $p(A,B) = P(A|B)P(B)$

let $A = x_i$, $B = x_{i+1}$, $C = y_{0:n}$

then we have: $p(A|B,C) = \frac{p(B|A)\,p(A|C)}{p(B|C)}$ ~~as~~ as desired relationship.

~~Using Bayes' Rule, $p(A|B,C) = \frac{p(BC|A)\,p(A)}{p(BC)}$~~

~~$= \frac{p(ABC)}{p(BC)} =$ (OK, but not where we want to go)~~

note: $p(B|AC) = p(B|A)$ (i.e. Markovian Property)

s.t. $p(A|BC) \overset{*}{=} \frac{p(ABC)}{p(BC)} \overset{*}{=} \frac{p(AC)\,p(B|AC)}{p(BC)} = \frac{p(A|C)\,\cancel{p(C)}\,p(B|AC)}{\cancel{p(C)}\,p(B|C)}$

$p(B|AC) = \frac{p(ABC)}{p(AC)}$

(Bayes' Rule)

$*$ = Bayes' Rule used

$= \frac{p(B|AC)\,p(A|C)}{p(B|C)}$

(using conditional independent property of Markovian) $\rightarrow = \frac{p(B|C)\,p(A|C)}{p(B|C)}$

$$\boxed{= \frac{p(x_{i+1}|x_i)\,p(x_i|y_{0:i})}{p(x_{i+1}|y_{0:i})}} \quad \square$$

Q11 Show that the joint distribution of x_i and x_{i+1} given $y_{0:n}$ is given by

$$p(x_i, x_{i+1} | y_{0:n}) = \frac{p(x_{i+1}|x_i)\, p(x_i|y_{0:i})\, p(x_{i+1}|y_{0:n})}{p(x_{i+1}|y_{0:i})}$$

So, $p(x_i, x_{i+1}|y_{0:n}) = \dfrac{p(x_i, x_{i+1}, y_{0:n})}{p(y_{0:n})} = \dfrac{p(x_{i+1}|x_i, y_{0:n})\, p(x_i, y_{0:n})}{p(x_{i+1}|y_{0:i})}$

did not get us where we wanted. Looking first at results from Q9 & Q10:

$$p(x_i | x_{i+1}\, y_{0:n}) = p(x_i | x_{i+1}, y_{0:i}) \qquad \text{11-1}$$

$$p(x_i | x_{i+1}, y_{0:n}) = \frac{p(x_{i+1}|x_i)\, p(x_i|y_{0:i})}{p(x_{i+1}|y_{0:i})} \qquad \text{11-2}$$

noting that 11-2 has the denominator we are looking for, we use Bayes' Theorem to get the term $p(x_i|x_{i+1}, y_{0:n})$ from our initial distribution. 11-3

$$p(x_i, x_{i+1}|y_{0:n}) \qquad \text{11-3}$$

$$\Rightarrow p(x_i, x_{i+1}|y_{0:n}) = \frac{p(x_i, x_{i+1}, y_{0:n})}{p(y_{0:n})} = \frac{p(x_i|x_{i+1}, y_{0:n})\, p(x_{i+1}, y_{0:n})}{p(y_{0:n})}$$

Sub in 11-2:

$$= \frac{p(x_{i+1}|x_i)\, p(x_i|y_{0:i})\, p(x_{i+1}, y_{0:n})}{p(x_{i+1}|y_{0:i})\, p(y_{0:n})}$$

Use Bayes' Rule:

$$= \frac{p(x_{i+1}|x_i)\, p(x_i|y_{0:i})\, p(x_{i+1}|y_{0:n})\, \cancel{p(y_{0:n})}}{p(x_{i+1}|y_{0:i})\, \cancel{p(y_{0:n})}} = \boxed{\frac{p(x_{i+1}|x_i)\, p(x_i|y_{0:i})\, p(x_{i+1}|y_{0:n})}{p(x_{i+1}|y_{0:i})}}$$

$\square$

Q-12)

Assume $x_{i-1}|y_{0:i-1}$ is distributed according to Gaussian distribution

$$\mathcal{N}(\hat{x}_{i-1|0:i-1}, P_{i-1|0:i-1})$$

~~R~~ Show that joint distribution of x_{i-1} and x_i given $y_{0:i-1}$ is:

$$\mathcal{N}\left(\begin{bmatrix}\hat{x}_{i-1|0:i-1} \\ \hat{x}_{i|0:i-1}\end{bmatrix}, \begin{bmatrix}P_{i-1|0:i-1} & P_{i-1|0:i-1}F^T \\ FP_{i-1|0:i-1} & P_{i|0:i-1}\end{bmatrix}\right)$$

where $\hat{x}_{i|0:i-1} = F\hat{x}_{i-1|0:i-1}$

$P_{i|0:i-1} = FP_{i-1|0:i-1}F^T + Q$

Gauss-Markov model: $x_{i+1} = Fx_i + u_i$, $y_i = Hx_i + v_i$

$R_u = Q\delta_{ij}$, $R_v = R\delta_{ij}$, $\{u_i\}, \{v_i\}$ are white & Gaussian

from page 8 of Lecture 18 we have the following Lemma:

Lemma: let $x \sim \mathcal{N}(\mu_x, \Sigma_x)$ and $y|x \sim \mathcal{N}(Hx, R)$ (eg. $y = Hx + v$)

Then $\begin{bmatrix}x \\ y\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}\mu_x \\ H\mu_x\end{bmatrix}, \begin{bmatrix}\Sigma_x & \Sigma_x H^T \\ H\Sigma_x & H\Sigma_x H^T + R\end{bmatrix}\right)$

we are given the distribution of $x_{i-1}|y_{0:i-1}$ and the state-space mode $x_{i+1} = Fx_i + u_i$. Thus if we set $x = x_{i-1}|y_{0:i-1}$ and $y = x_i|y_{0:i-1}$ then we have $y = Fx + u$, which matches the form given in the example of the Lemma.

To prove the result to ourselves, we will actually calculate the joint distribution:

let $\mathcal{N}(\hat{x}_{i-1|0:i-1}, P_{i-1|0:i-1}) = \mathcal{N}(\mu_x, \Sigma_x)$ i.e. $\mu_x \triangleq \hat{x}_{i-1|0:i-1}$

$\Sigma_x \triangleq P_{i-1|0:i-1}$

First, we calculate the mean of $x_i|y_{0:i-1}$:

$$\hat{x}_{i|0:i-1} = \mathbb{E}(x_i|y_{0:i-1}) = \mathbb{E}(Fx_{i-1} + u_i|y_{0:i-1}) = F\hat{x}_{i-1|0:i-1} + \underbrace{\mathbb{E}(u_i|y_{0:i-1})}_{\to 0} = F\hat{x}_{i-1|0:i-1} = F\mu_x$$

process noise is ~~independent~~ uncorrelated of past measurements & $\mathbb{E}(u_i) = 0$

Q+2 cont'd

Covariance of $x_i | y_{0:i-1}$;

$$\mathbb{E}\left((x_i - \mathbb{E}(x_i|y_{0:i-1}))(x_i - \mathbb{E}(x_i|y_{0:i-1}))^T | y_{0:i-1}\right)$$

$$= \mathbb{E}\left((x_i - F\mu_x)(x_i - F\mu_x)^T | y_{0:i-1}\right)$$

$$= \mathbb{E}\left[x_i x_i^T - x_i \mu_x^T F^T - F\mu_x x_i^T + F\mu_x \mu_x^T F^T | y_{0:i-1}\right)$$

$$= \mathbb{E}\left((Fx_{i-1} + u_{i-1})(Fx_{i-1}+u_{i-1})^T | y_{0:i-1}\right) - F\mu_x\mu_x^T F^T - \cancel{F\mu_x\mu_x^T F^T} + \cancel{F\mu_x\mu_x^T F^T}$$

$$= F R_x F^T + F\,\mathbb{E}[x_{i-1}u_{i-1}^T | y_{0:i-1}] + \mathbb{E}[u_{i-1}x_{i-1}^T | y_{0:i-1}]F^T + \mathbb{E}[u_{i-1}u_{i-1}^T | \underbrace{\cancel{y_{0:i-1}}}_{\text{uncorrelated}}] - F\mu_x\mu_x^T F^T$$

$$= F\,\mathbb{E}(x_{i-1}x_{i-1}^T | y_{0:i-1}) + F\,\mathbb{E}[x_{i-1}|y_{0:i-1}]\,\mathbb{E}[\cancel{u_{i-1}}|y_{0:i-1}]^T + \mathbb{E}[\cancel{u_{i-1}}|y_{0:i-1}]\,\mathbb{E}[x_{i-1}^T|y_{0:i-1}]F^T + Q - F\mu_x\mu_x F^T$$

x_{i-1} & u_{i-1} independent $\{u_i\}$ white noise, (0-mean)

$$= F\left(\underbrace{\mathbb{E}(x_{i-1}x_{i-1}^T | y_{0:i-1}) - \mu_x\mu_x^T}\right)F^T + Q$$

$$= \Sigma_x = \mathbb{E}\left((x_{i-1}-\mu_x)(x_{i-1}-\mu_x)^T | y_{0:i-1}\right) = P_{i-1|0:i-1}$$

$$= F P_{i-1|0:i-1} F^T + Q$$

$\triangleq P_{i|0:i-1}$ as given in problem statement. □

Similarly, cross-covariance of $x_{i-1}|y_{0:i-1}$ and $x_i | y_{0:i-1}$;

$$\mathbb{E}\left((x_{i-1}-\mu_x)(x_i - F\mu_x)^T | y_{0:i-1}\right) = \mathbb{E}\left(x_{i-1}x_i^T + \mu_x\mu_x^T F^T - \mu_x x_i^T - x_{i-1}\mu_x^T F^T | y_{0:i-1}\right)$$

$$= \mathbb{E}\left(x_{i-1}(Fx_{i-1}+u_{i-1})^T | y_{0:i-1}\right) - \mu_x \mathbb{E}(Fx_{i-1}+u_{i-1}|y_{0:i-1})^T - \cancel{\mu_x\mu_x^T F^T} + \cancel{\mu_x\mu_x^T F^T}$$

$$= \mathbb{E}\left(x_{i-1}x_{i-1}^T F^T | y_{0:i-1}\right) + \mu_x\,\mathbb{E}[\cancel{u_{i-1}^T | y_{0:i-1}}] - \mu_x\left((F\mu_x)^T + \mathbb{E}[\cancel{u_{i-1}}|y_{0:i-1}]^T\right)$$

~~uncorrelated~~ d
u_{i-1} independent of
$y_{0:i-1}$ ($\{u_i\}$, $\{v_i\}$ white, Gaussian)
0-mean

$$= \mathbb{E}(x_{i-1}x_{i-1}^T | y_{0:i-1})F^T + 0 - \mu_x\mu_x^T F^T = \left(\mathbb{E}(x_{i-1}x_{i-1}^T|y_{0:i-1}) - \mu_x\mu_x\right)F^T$$

$$= \Sigma_x F^T \quad \square$$

Since $R_{xy} = R_{yx}^T$, $\mathbb{E}\left((x_i - F\mu_x)(x_{i-1}-\mu_x)^T | y_{0:i-1}\right) = (\Sigma_x F^T)^T = F\Sigma_x$ □

Therefore,

$$\begin{bmatrix} x_{i-1} | y_{0:i-1} \\ x_i | y_{0:i-1} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_x \\ F\mu_x \end{bmatrix}, \begin{bmatrix} \Sigma_x & \Sigma_x F^T \\ F\Sigma_x & F\Sigma_x F^T + Q \end{bmatrix} \right)$$

where $\Sigma_x = P_{i-1|0:i-1}$, $\mu_x = \hat{x}_{i-1|0:i-1}$

$\square$

Q-13] Using the result of Q-12, show that the distribution of x_{i-1} given x_i and $y_{0:n}$, which is equal to distribution of x_{i-1} given x_i and $y_{0:i-1}$ by Q-9, is

$$x_{i-1} | x_i, y_{0:n} \sim \mathcal{N}(\tilde{x}_{i-1|i-1}, \tilde{P}_{i-1|i-1})$$

where
$$\tilde{x}_{i-1|i-1} = \hat{x}_{i-1|0:i-1} + k_i (x_i - F\hat{x}_{i-1|0:i-1})$$
$$\tilde{P}_{i-1|i-1} = P_{i-1|0:i-1} - k_i (F P_{i-1|0:i-1} F^T + Q) k_i^T$$
$$k_i = P_{i-1|0:i-1} F^T (F P_{i-1|0:i-1} F^T + Q)^{-1}$$

---

mean: $\mathbb{E}(x_{i-1} | x_i, y_{0:n}) = \mathbb{E}(x_{i-1} | x_i, y_{0:i-1})$ by Q-9 (independence).

$$\mathbb{E}(x_{i-1} | x_i, y_{0:i-1}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_{i-1}\, p(x_{i-1} | x_i, y_{0:i-1})\, d\vec{x}_{i-1}$$

note from Q-10: $p(x_{i-1} | x_i, y_{0:n}) = \dfrac{p(x_{i-1} | x_{i-1})\, p(x_{i-1} | y_{0:i-1})}{p(x_{i-1} | y_{0:i-1})}$

$$\hat{x}_{i-1|0:i-1} = \mathbb{E}(x_{i-1} | y_{0:i-1}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_{i-1}\, p(x_{i-1} | y_{0:i-1})\, d\vec{x}_{i-1}$$

From Lecture; #18, pg 7: if $\begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} R_x & R_{xy} \\ R_{yx} & R_y \end{bmatrix} \right)$

then $x \sim \mathcal{N}(\mu_x, R_x)$, $y \sim \mathcal{N}(\mu_y, R_y)$ and

$$x | y \sim \mathcal{N}(\mu_x + R_{xy} R_y^{-1} (y - \mu_y),\ R_x - R_{xy} R_y^{-1} R_{yx})$$ use this

$$y | x \sim \mathcal{N}(\mu_y + R_{yx} R_x^{-1} (x - \mu_x),\ R_y - R_{yx} R_x^{-1} R_{xy})$$

---

let $x_{i-1} | y_{0:i-1} = x$ and $x_i | y_{0:i-1} = y$ s.t.

$$\left.\begin{array}{lll} \hat{x}_{i-1|0:i-1} = \mu_x & P_{i-1|0:i-1} = R_x & P_{i-1|0:i-1} F^T = R_{xy} \\ F\hat{x}_{i-1|0:i-1} = \hat{x}_{i|0:i-1} = \mu_y & F P_{i-1|0:i-1} F^T + Q = R_y & F P_{i-1|0:i-1} = R_{yx} \end{array}\right\}$$ from Q-12 →

and $x_{i-1}|x_i\, y_{0:i-1} = x_{i-1}|x_i\, y_{0:i-1} = x|y$

then:

$$\cancel{\;} = \hat{x} = \mu_x + R_{xy}R_y^{-1}(y-\mu_y)$$

$$= \mathbb{E}(x_{i-1}|x_i\, y_{0:i-1}) = \hat{x}_{i-1|0:i-1} + \underbrace{P_{i-1|0:i-1}F^T(FP_{i-1|0:i-1}F^T+Q)^{-1}}_{K_i}(\underset{\substack{\downarrow \\ \text{Markovian} \\ = x_i}}{x_i|y_{0:i-1}} - \underset{\substack{\downarrow \\ = F\hat{x}_{i-1|0:i-1}}}{\hat{x}_{i|0:i-1}})$$

$$= \mathbb{E}(x|y)$$

$$= \hat{x}_{i-1|0:i-1} + K_i(x_i - F\hat{x}_{i-1|0:i-1})$$

$$= \tilde{x}_{i-1|i-1} \qquad \square$$

~~Now~~ Now for the covariance:

$$\mathrm{Cov}(x|y) = R_x - R_{xy}R_y^{-1}R_{yx}$$

$$\Rightarrow \tilde{P}_{i-1|i-1} = P_{i-1|0:i-1} - \underbrace{P_{i-1|0:i-1}F^T(FP_{i-1|0:i-1}F^T+Q)^{-1}}_{K_i}FP_{i-1|0:i-1}$$

$$= P_{i-1|0:i-1} - K_i FP_{i-1|0:i-1}$$

$$= P_{i-1|0:i-1} - K_i(FP_{i-1|0:i-1}F^T+Q)(FP_{i-1|0:i-1}F^T+Q)^{-1}FP_{i-1|0:i-1}$$

Note: $K_i^T = (FP_{i-1|0:i-1}F^T+Q)^{-T}F\,P_{i-1|0:i-1}$ where $P_{i-1|0:i-1} = P_{i-1|0:i-1}^T$

self-covariance matrices are symmetric

$= (FP_{i-1|0:i-1}^T F^T + Q^T)^{-1}FP_{i-1|0:i-1}$ and $Q^T = Q = R_n$ (symmetric

$= (FP_{i-1|0:i-1}F^T+Q)^{-1}FP_{i-1|0:i-1}$

$$\boxed{\Rightarrow \tilde{P}_{i-1|i-1} = P_{i-1|0:i-1} - K_i(FP_{i-1|0:i-1}F^T+Q)K_i^T \qquad \square}$$

Q-14) Implement the Kalman Filter for smoothing

Summary:

(predict)

$$\hat{x}_{i|0:i-1} = F\hat{x}_{i-1|0:i-1}$$

$$P_{i|0:i-1} = FP_{i-1|0:i-1}F^T + Q$$

(update)

$$\left.\begin{aligned} \hat{x}_{i|0:i} &= \hat{x}_{i|0:i-1} + K_{f,i}(y_i - H_i\hat{x}_{i|0:i-1}) \\ K_{f,i} &= P_{i|0:i-1}H_i^T(H_iP_{i|0:i-1}H_i^T + R_i)^{-1} \\ P_{i|0:i} &= P_{i|0:i-1} - K_{f,i}H_iP_{i|0:i-1} \end{aligned}\right\} \text{Kalman Filter}$$

$$\left.\begin{aligned} \tilde{x}_{i-1|i-1} = \hat{x}_{i-1|x_i,0:i-1} &= \hat{x}_{i-1|0:i-1} + K_i(x_i - F\hat{x}_{i-1|0:i-1}) \\ K_i &= P_{i-1|0:i-1}F^T(FP_{i-1|0:i-1}F^T + Q)^{-1} \\ \tilde{P}_{i-1|i-1} &= P_{i-1|i-1} - K_i(FP_{i-1|0:i-1}F^T + Q)K_i^T \\ \hat{x}_{i-1|0:n} &= \hat{x}_{i-1|0:i-1} + K_i(\hat{x}_{i|0:n} - F\hat{x}_{i-1|0:i-1}) \\ P_{i-1|0:n} &= K_iP_{i|0:n}K_i^T + \tilde{P}_{i-1|i-1} \end{aligned}\right\} \text{Kalman Smoother}$$

So, in Q-6 we produced $\hat{x}_{i|0:i}$ and $P_{i|0:i}$, now we will use them to go backwards and produce:

K_i, $\hat{x}_{i-1|0:n}$

$\boxed{\text{See code in Jupyter}}$

Q-15) $\boxed{\text{Plot the smoothed trajectory} \rightarrow \text{see Jupyter}}$

Q-16) compute RMS for smoothed vs. true trajectory: $\sqrt{\sum_{i=1}^{n} \|\hat{x}_{i|0:n} - x_i\|_2^2}$

& How does this compare? $\boxed{\text{See Jupyter code}}$