Theodore Wilkening

ECE_6555 Final

1) Assume that $U \sim N(0,1)$ and Set $V = |U|$

note: based on the assumptions of U, I assume that U is a scalar variable

1-a) optimal estimator of V from U of the form αU, linear estimator, to minimize the MSE. Provide corresponding MSE (numerical estimate)

~~let $\hat{V} = \alpha U$~~ reference lecture 6, pg 6

note: V is not a centered variable. So set $\tilde{V} = V - \mu_V$, $\mu_V = \mathbb{E}[V] (= \mathbb{E}[|U|])$

then, $\hat{\tilde{V}} = \hat{V} - \mu_V \Rightarrow$ that is $\mathbb{E}[\tilde{V}|U] = \mathbb{E}[V|U] - \mu_V$, also now $\hat{\tilde{V}} = \alpha U$

$$\text{let } P(\alpha) = \mathbb{E}[(\tilde{V} - \hat{\tilde{V}})(\tilde{V} - \hat{\tilde{V}})] = \mathbb{E}[(\tilde{V} - \alpha U)(\tilde{V} - \alpha U)]$$
$$= \mathbb{E}[\tilde{V}^2] - 2\alpha R_{\tilde{V}U} + \mathbb{E}[\alpha^2 U^2]$$
$$= R_{\tilde{V}} - 2\alpha R_{\tilde{V}U} + \alpha^2$$

$$\frac{\partial P(\alpha)}{\partial \alpha} = -2R_{\tilde{V}U} + 2\alpha = 0 \Rightarrow \alpha = R_{\tilde{V}U} = \mathbb{E}[\tilde{V}U] = \mathbb{E}[(V - \mu_V)U] = R_{VU} - \mu_V \cancel{\mathbb{E}[U]}^{0}$$

set equal to 0 to minimize w.r.t. α

$$\Rightarrow \boxed{\alpha = R_{VU}}$$

now, $\hat{\tilde{V}} = R_{VU} U$, $\hat{V} = R_{VU} U + \mu_V$

MSE

$$\mathbb{E}[(V - \hat{V})(V - \hat{V})] = \mathbb{E}[(V - R_{VU}U - \mu_V)(V - R_{VU}U - \mu_V)]$$
$$= \mathbb{E}[V^2 - 2VUR_{VU} + 2U\mu_V R_{VU} - 2V\mu_V + R_{VU}^2 U^2 + \mu_V^2]$$
$$= R_V - \cancel{2}R_{VU}^2 + 2\mu_V R_{VU} \cancel{\mathbb{E}[U]}^{0} - \cancel{2}\mu_V^2 + \cancel{R_{VU}^2} + \cancel{\mu_V^2}$$

$$\Rightarrow \boxed{MSE = R_V - R_{VU}^2 - \mu_V^2}$$

where

$R_V = \mathbb{E}[V^2] = \mathbb{E}[(\sqrt{U^2})^2]$

$R_{VU} = \mathbb{E}[VU] = \mathbb{E}[U\sqrt{U^2}]$

$\mu_V = \mathbb{E}[V] = \mathbb{E}[\sqrt{U^2}]$

See python code for estimate numerically

$\boxed{MSE \approx 0.336}$

$\boxed{\text{if we don't care about centering, then let } \hat{V} = \alpha U}$

$$\Rightarrow P(\alpha) = \mathbb{E}[(v-\alpha u)(v-\alpha u)]$$
$$= \mathbb{E}[v^2] - 2\alpha R_{uv} + \alpha^2 \cancelto{1}{\mathbb{E}[u^2]}$$
$$= R_v - 2\alpha R_{uv} + \alpha^2$$

$$\frac{\partial P(\alpha)}{\partial \alpha} = -2R_{uv} + 2\alpha = 0 \Rightarrow \boxed{\alpha = R_{uv}}$$

$$\Rightarrow MSE = \mathbb{E}[(v-\alpha u)(v-\alpha u)] = R_v - 2\alpha R_{uv} + \alpha^2$$
$$= R_v - 2R_{uv}^2 + R_{uv}^2$$

$$\Rightarrow \boxed{MSE = R_v - R_{uv}^2}$$

(un-centered)

$$R_v = \mathbb{E}[v^2] = \mathbb{E}[(\sqrt{u^2})^2]$$
$$R_{uv} = \mathbb{E}[uv] = \mathbb{E}[u\sqrt{u^2}]$$

$$\boxed{MSE \approx 0.929}$$

Q2] optimal estimator of V from U of form $\alpha + \beta U$ (affine estimator)

↳ minimize mean-square error. provide MSE calc. (assume scalar variables)

~~let $P(\alpha,\beta) = \mathbb{E}((v-\hat{v})(v-\hat{v}))$~~ , $\hat{v} = \alpha + \beta U$ per problem statement

~~$= \mathbb{E}((v-\alpha-\beta U)(v-\alpha-\beta U))$~~

~~$= \mathbb{E}[v^2 - 2\alpha v - 2v\beta U + 2\alpha\beta U + \alpha^2 + \beta^2 U^2]$~~

V is not centered, so define $\tilde{v} \triangleq v - \mu_v$, $\mu_v = \mathbb{E}[V]$

now, $\hat{\tilde{v}} = \alpha + \beta U$

$$P(\alpha,\beta) = \mathbb{E}((\tilde{v}-\hat{\tilde{v}})(\tilde{v}-\hat{\tilde{v}})) = \mathbb{E}[(\tilde{v}-\alpha-\beta U)(\tilde{v}-\alpha-\beta U)]$$

$$= \mathbb{E}[\tilde{v}^2 - 2\alpha\tilde{v} - 2\tilde{v}\beta U + 2\alpha\beta U + \alpha^2 + \beta^2 U^2]$$

$$= R_{\tilde{v}} - 2\alpha\mu_{\tilde{v}} - 2\beta R_{\tilde{v}u} + 2\alpha\beta \overset{0}{\mu_u} + \alpha^2 + \beta^2 \overset{1}{R_u}$$

$$= R_{\tilde{v}} - 2\alpha\mu_{\tilde{v}} - 2\beta R_{\tilde{v}u} + \alpha^2 + \beta^2$$

$$\frac{\partial P(\alpha,\beta)}{\partial\alpha} = 0 - 2\mu_{\tilde{v}} - 0 + 2\alpha = 0 \Rightarrow \boxed{\alpha = \mu_{\tilde{v}} = \mathbb{E}(v-\mu_v) = \mu_v - \mu_v = 0}$$

$$\frac{\partial P(\alpha,\beta)}{\partial\beta} = 0 - 0 - 2R_{\tilde{v}u} + 0 + 2\beta = 0 \Rightarrow \boxed{\beta = R_{\tilde{v}u}}$$

$$R_{\tilde{v}u} = \mathbb{E}[\tilde{v}u] = \mathbb{E}[(v-\mu_v)u] = R_{vu}$$

$$\boxed{\Rightarrow \beta = R_{vu}}$$

$MSE = \mathbb{E}[(v-\hat{v})(v-\hat{v})]$, where $\hat{\tilde{v}} = \alpha + \beta u \Rightarrow \hat{v} - \mu_v = \alpha + \beta u$

$\Rightarrow \hat{v} = \alpha + \beta u + \mu_v$

$= R_{vu} U + \mu_v$

$$= \mathbb{E}[(v - R_{vu}U - \mu_v)(v - R_{vu}U - \mu_v)]$$

$\Rightarrow \boxed{MSE = R_v - R_{vu}^2 - \mu_v^2}$, per [Q1]

$R_{vu} = \mathbb{E}[vu] = \mathbb{E}[\sqrt{u^2}\, u]$

$R_v = \mathbb{E}[|u|^2] = ~~\mathbb{E}[u^2]~~$ → see code for estimate of MSE

$\mu_v = \mathbb{E}(v) = \mathbb{E}[\sqrt{u^2}]$ $\boxed{MSE \approx 0.336}$

Q3 opt. est. of $V|U$, form $\alpha + \beta U + \gamma U^2$ → quadratic estimator for min. MSE.

→ provide MSE estimate.

V not centered, so define $\tilde{v} = v - \mu_v$, $\mu_v = \mathbb{E}(v) = \sum_{i=1}^{n} v_i / n$

now, $\hat{\tilde{v}} = \alpha + \beta u + \gamma u^2$ (quadratic estimator for v)

then, $P(\alpha, \beta, \gamma) = \mathbb{E}[(\tilde{v} - \hat{\tilde{v}})(\tilde{v} - \hat{\tilde{v}})]$

$= \mathbb{E}[(\tilde{v} - \alpha - \beta u - \gamma u^2)(\tilde{v} - \alpha - \beta u - \gamma u^2)]$ → MATLAB

$= \mathbb{E}[\alpha^2 + 2\alpha\beta u + 2\alpha\gamma u^2 - 2\alpha\tilde{v} + \beta^2 u^2 + 2\beta\gamma u^3 - 2\beta u \tilde{v} + \gamma^2 u^4 - 2\gamma u^2 \tilde{v} + \tilde{v}^2]$

$= \alpha^2 + 2\alpha\beta(0) + 2\alpha\gamma(1) - 2\alpha\mu_{\tilde{v}} + \beta^2 + 2\beta\gamma(0) - 2\beta R_{\tilde{v}u} + \gamma^2 3 - 2\gamma\mathbb{E}[u^2\tilde{v} + \tilde{v}^2]$

$= \alpha^2 + 2\alpha\gamma - 2\alpha\mu_{\tilde{v}} + \beta^2 - 2\beta R_{\tilde{v}u} + 3\gamma^2 - 2\gamma\mathbb{E}(u^2\tilde{v}) + \mathbb{E}(\tilde{v}^2)$

$\mathbb{E}(u^2\tilde{v}) = \mathbb{E}[u^2(v - \mu_v)] = \mathbb{E}[u^2 v] - \mu_v$

$\frac{\partial P(\alpha, \beta, \gamma)}{\partial \alpha} = 2\alpha + 2\gamma - 2\mu_{\tilde{v}}^{\,0} = 0 \Rightarrow \alpha = \mu_{\tilde{v}}^{\,0} - \gamma$

$\frac{\partial P}{\partial \beta} = 2\beta - 2R_{\tilde{v}u} = 0 \Rightarrow \boxed{\beta = R_{\tilde{v}u} = R_{vu}}$

$\frac{\partial P}{\partial \gamma} = 2\alpha + 6\gamma - 2\mathbb{E}[u^2\tilde{v}] = 0 \Rightarrow \alpha + 3\gamma - \mathbb{E}[u^2\tilde{v}] = 0$, let $\mathbb{E}[u^2\tilde{v}] = \emptyset$

$\Rightarrow \mu_{\tilde{v}} - \gamma + 3\gamma - \emptyset = 0$ $= \mathbb{E}[u^2\sqrt{u^2}] - \mu_v$

$\mathbb{E}[\tilde{v}] = \mathbb{E}[v - \mu_v] = 0$

$\Rightarrow 2\gamma = \emptyset - \mu_{\tilde{v}}^{\,0}$

$\gamma = (\mathbb{E}[u^2 v] - \mu_v)/2$ ✓

$\Rightarrow \boxed{\gamma = (\emptyset - \mu_{\tilde{v}}^{\,0})/2}$

$\emptyset = \mathbb{E}[u^2\tilde{v}] = \mathbb{E}[u^2(v - \mu_v)]$

$= \mathbb{E}[u^2 v] - \mu_v$

$\Rightarrow \alpha = \mu_{\tilde{v}}^{\,0} - \frac{(\emptyset - \mu_{\tilde{v}})}{2}$

$\boxed{\alpha = (3\mu_{\tilde{v}}^{\,0} - \emptyset)/2}$

$\alpha = -\frac{\emptyset}{2}$

$\gamma = \frac{\emptyset}{2}$

MSE ⟶

$\underline{MSE}$ $= \mathbb{E}[(v-\hat{v})(v-\hat{v})]$, $\quad \hat{\tilde{v}} = \hat{v} - \mu_v = \alpha + \beta u + \gamma u^2$

$\Rightarrow \hat{v} = \alpha + \beta u + \gamma u^2 + \mu_v$

$\Rightarrow MSE = \mathbb{E}[(v-(\alpha+\beta u+\gamma u^2+\mu_v))^2] = \mathbb{E}[v^2 - 2v\hat{v} + \hat{v}^2]$

$= \mathbb{E}[v^2] - 2\mathbb{E}[v(\alpha+\beta u+\gamma u^2+\mu_v)] + \mathbb{E}[(\alpha+\beta u+\gamma u^2+\mu_v)^2]$

$= R_v - 2(\alpha\mu_v + \cancel{0+\gamma\, \mathbb{E}\mu_v^2} + \beta R_{uv} + \gamma\mathbb{E}(u^2 v) + \mu_v^2) + \mathbb{E}[(\alpha+\beta u+\gamma u^2+\mu_v)^2]$

$= R_v - 2(\alpha\mu_v + \beta R_{uv} + \gamma\mathbb{E}(u^2 v) + \mu_v^2)$
$\quad + \mathbb{E}(\alpha^2 + 2\alpha\beta u + 2\alpha\gamma u^2 + 2\alpha\mu_v + \beta^2 u^2 + 2\beta\gamma u^3$
$\quad + 2\beta\mu_v u + \gamma^2 u^4 + 2\gamma\mu_v u^2 + \mu_v^2)$

principle of orthogonality?

$\mathbb{E}[(\tilde{v} - \hat{\tilde{v}})(\tilde{v} - \hat{\tilde{v}})]$

$= \mathbb{E}[(\tilde{v} - \hat{\tilde{v}})\tilde{v} - (\tilde{v} - \hat{\tilde{v}})\hat{\tilde{v}}]$

$= \mathbb{E}[(\tilde{v} - \hat{\tilde{v}})\tilde{v}] - 0$ due to orthogonality

$= \mathbb{E}[\tilde{v}^2 - (\alpha + \beta u + \gamma u^2)\tilde{v}]$

$= R_{\tilde{v}} - \alpha\mu_{\tilde{v}} - \beta R_{\tilde{v}u} - \gamma\mathbb{E}[u^2\tilde{v}]$

$\mathbb{E}(u^3) = 0$

$\mathbb{E}(u^4) = 3\sigma^2 = 3$

$= \cancel{\mathbb{E}[\alpha}$

$= \alpha^2 + 0 + 2\alpha\gamma + 2\alpha\mu_v + \beta^2 + 0$
$\quad + 0 + 3\gamma^2 + 2\gamma\mu_v + \mu_v^2$

$\Rightarrow$

$MSE = R_v - 2(\cancel{\alpha}\mu_v + \beta R_{uv} + \gamma\mathbb{E}[u^2 v] + \mu_v^2) + \alpha^2 + 2\alpha\gamma + \cancel{2\alpha\mu_v} + \beta^2 + 3\gamma^2 + 2\gamma\mu_v + \mu_v^2$

$= R_v - 2\beta R_{uv} - 2\gamma\mathbb{E}[u^2 v] - 2\mu_v^2 + \alpha^2 + 2\alpha\gamma + \beta^2 + 3\gamma^2 + 2\gamma\mu_v + \mu_v^2$

$\alpha =$

What if we let $\alpha = \mu_v$ from beginning?

Simpler method $\longrightarrow$

let $\hat{v} = \alpha + \beta u + \gamma u^2$ (form of estimate)

since v is not centered, then $\tilde{v} \triangleq v - \mu_v$ s.t. $\hat{v} = \hat{\tilde{v}} + \mu_v$

$$\Rightarrow \hat{v} = \alpha + \beta u + \gamma u^2 = \hat{\tilde{v}} + \mu_v \quad \boxed{\text{let } \alpha = \mu_v} \text{ s.t. } \boxed{\hat{\tilde{v}} = \beta u + \gamma u^2}$$

then, $\rho(\alpha,\beta,\gamma) = \mathbb{E}[(\tilde{v} - \hat{\tilde{v}})(\tilde{v} - \hat{\tilde{v}})] = \mathbb{E}[(\tilde{v} - \beta u - \gamma u^2)(\tilde{v} - \beta u - \gamma u^2)]$

$$= \mathbb{E}[\tilde{v}^2 + \beta^2 u^2 + \gamma^2 u^4 - \beta u \tilde{v} - \beta u \tilde{v} - 2\gamma u^2 \tilde{v} + 2\beta u^3 \gamma]$$

$$= \mathbb{E}[\tilde{v}^2] + \beta^2(1) + \gamma^2(3) - 2\beta R_{u\tilde{v}} - 2\gamma \mathbb{E}(u^2\tilde{v}) + 2\beta\gamma \cancelto{0}{\mathbb{E}(u^3)}$$

$$= R_{\tilde{v}} + \beta^2 + 3\gamma^2 - 2\beta R_{uv} - 2\gamma \mathbb{E}(u^2 \tilde{v})$$

$$\frac{\partial \rho}{\partial \alpha} = 0$$

$$\frac{\partial \rho}{\partial \beta} = 2\beta - 2R_{uv} = 0 \Rightarrow \boxed{\beta = R_{uv}}$$

$$\frac{\partial \rho}{\partial \gamma} = 6\gamma - 2\mathbb{E}[u^2\tilde{v}] = 0 \Rightarrow \gamma = \frac{1}{3}\mathbb{E}[u^2\tilde{v}] = \frac{1}{3}\mathbb{E}[u^2(v - \mu_v)] = \frac{1}{3}(\mathbb{E}(u^2v) - \mu_v)$$

$$MSE = \mathbb{E}[\cancel{(v-\hat{v})(v-\hat{v})}] = \mathbb{E}[(v - (\hat{\tilde{v}} + \overset{\alpha}{\mu_v}))(v - (\hat{\tilde{v}} + \overset{\alpha}{\mu_v}))]$$

$$= \mathbb{E}[(v-\hat{v})(v-\hat{v})] = \mathbb{E}[v^2] - 2\mathbb{E}[v\hat{v}] + \mathbb{E}[\hat{v}^2]$$

note: $\mathbb{E}[u^3] = 0$

$\mathbb{E}[u^4] = 3$

where

$$\mathbb{E}[v\hat{v}] = \mathbb{E}[v(\beta u + \gamma u^2 + \alpha)] = \beta R_{uv} + \gamma\mathbb{E}[u^2 v] + \mu_v^2$$

$$\mathbb{E}[\hat{v}^2] = \mathbb{E}[(\mu_v + \beta u + \gamma u^2)^2] = \mathbb{E}[\mu_v^2 + \beta^2 u^2 + \gamma^2 u^4 + 2\beta\cancelto{0}{\mu_v u} + 2\mu_v \gamma u^2 + 2\beta\cancelto{0}{\gamma u^3}]$$

$$= \mu_v^2 + \beta^2 + 3\gamma^2 + 2\mu_v\gamma$$

thus,

$$\boxed{\begin{aligned} MSE &= R_v - 2(\beta R_{uv} + \gamma\mathbb{E}(u^2 v) + \mu_v^2) + (\mu_v^2 + \beta^2 + 3\gamma^2 + 2\mu_v\gamma) \\ \beta &= R_{uv} = \mathbb{E}[u|u|] = \mathbb{E}[u\sqrt{u^2}] \qquad \mathbb{E}[u^2 v] = \mathbb{E}[u^2\sqrt{u^2}] \\ \gamma &= \tfrac{1}{3}(\mathbb{E}[u^2\sqrt{u^2}] - \mathbb{E}[\sqrt{u^2}]) \\ R_v &= \mathbb{E}[(\sqrt{u^2})^2] \end{aligned}}$$

estimate $\rightarrow$

to numerically estimate, generate $n = 1e3$ or $1e4$ points per $U \sim N(0,1)$ and then actually calculate the MSE for each problem!

MSE summary:

$\alpha = \mu_v$

$\beta = R_{uv} = R_{vu}$

$\gamma = \frac{1}{3}\left(\mathbb{E}[u^2 v] - \mu_v\right) \rightarrow$ let $\emptyset = \mathbb{E}[u^2 v]$

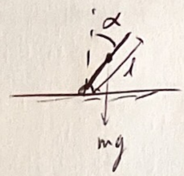
$\Rightarrow \gamma = \frac{1}{3}(\emptyset - \mu_v)$

$$MSE = R_v - 2\left(\beta R_{vu} + \gamma\emptyset + \mu_v^2\right) + \left(\mu_v^2 + \beta^2 + 3\gamma^2 + 2\mu_v\gamma\right)$$

$\rightarrow$ see python for code to estimate

$\rightarrow$ $\boxed{MSE \approx 0.1815}$

Problem 2



$l = 1$

$m = 3.5\,kg$

$g = 9.8\ m/s^2$

EOM:

$$\frac{d^2\alpha}{dt^2} = -\frac{g}{l}\sin\alpha + w(t)$$

define state as: $x \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq \begin{bmatrix} \alpha \\ \frac{d\alpha}{dt} \end{bmatrix}$

Q1 State Space model:

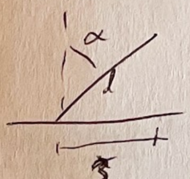
$$\dot{x} = \begin{bmatrix} \frac{d\alpha}{dt} \\ \frac{d^2\alpha}{dt} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin\alpha + w(t) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

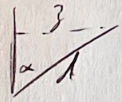
$\Rightarrow$ form: $\dot{x} = \begin{bmatrix} f_1(x_2) \\ f_2(x_1) \end{bmatrix} + G\,w(t)$ $\Rightarrow$

$$\boxed{\begin{aligned} f_1(x_2) &= x_2 \\ f_2(x_1) &= -\frac{g}{l}\sin(x_1) \\ G &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}}$$

Q2 Sensor model:

show $y = l\sin x_1 + v(t)$ i.e. tracks horizontal position + additive noise





$\sin\alpha = \frac{z}{l} \Rightarrow z = l\sin\alpha = l\sin x_1$

thus, $\boxed{y = l\sin x_1 + v(t)}$

Q3 Discretize:

Discretization of step Δt s.t. $\frac{dx}{dt} = \frac{x(t+\Delta t) - x(t)}{\Delta t}$

let $x_k \triangleq x(k\Delta t)$ so that $\left.\frac{dx}{dt}\right|_{\Delta t = k\Delta t} \approx \frac{x_{k+1} - x_k}{\Delta t}$

Show the discretized model takes the following form:

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} + x_{2,k}\Delta t \\ x_{2,k} - g\Delta t \sin(x_{1,k}) \end{bmatrix} + q_{k+1}$$

$$y_{k+1} = \sin x_{1,k+1} + v_{k+1}$$

note: (instead of a_{k+1}, use v_{k+1} for consistency)

---

process:

$$\dot{x}_1 = x_2 \Rightarrow \frac{x_{1,k+1} - x_{1,k}}{\Delta t} = x_{2,k} \Rightarrow x_{1,k+1} = x_{1,k} + \Delta t\, x_{2,k}$$

$$\dot{x}_2 = \cancel{x_2} - g/l\,(\sin(x_1)) + w(t) \qquad (l = 1)$$

$$\Rightarrow \frac{x_{2,k+1} - x_{2,k}}{\Delta t} = -g\sin(x_{1,k}) + w(t) \qquad w(\Delta t k) = w_k$$

$$\Rightarrow x_{2,k+1} = x_{2,k} - g\Delta t \sin(x_{1,k}) + \Delta t\, w_{k}$$

then,

$$\boxed{\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} + x_{2,k}\Delta t \\ x_{2,k} - g\sin(x_{1,k})\Delta t \end{bmatrix} + q_k, \quad q_k = \begin{bmatrix} 0 \\ \Delta t\, w_k \end{bmatrix}}$$

$$q_k = \begin{bmatrix} 0 \\ \Delta t\, w_k \end{bmatrix}$$

measurement:

$$y(t) = l\sin(x_1) + v(t), \quad l = 1$$

$$\Rightarrow y(\Delta t k) = \sin(x_1(\Delta t k)) + v(\Delta t k) \Rightarrow y_k = \sin(x_{1,k}) + v_k$$

same as:

$$\boxed{y_{k+1} = \sin(x_{1,k+1}) + v_{k+1}}$$

note: if $w(t)$ is white Gaussian, then q_{k+1} is white Gaussian w/ covariance matrix:

$$Q \triangleq \sigma_p^2 \begin{bmatrix} \frac{\Delta t^3}{3} & \frac{\Delta t^2}{2} \\ \frac{\Delta t^2}{2} & \Delta t \end{bmatrix}$$

note 2: v_{k+1} is assumed white Gaussian w/ variance σ_m^2

Q4] ~~linearize~~ for EKF

Form of model: $x_{k+1} = f(x_k) + q_k$, $y_{k+1} = h(x_{k+1}) + v_{k+1}$

Show that the linearization of functions f and h around a state

$x^* \triangleq \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ take the form:

$$f(x) \approx f(x^*) + \begin{bmatrix} 1 & \Delta t \\ -g\cos(x_1^*)\Delta t & 1 \end{bmatrix} (x - x^*)$$

$$h(x) \approx h(x^*) + [\cos(x_1^*) \quad 0] (x - x^*)$$

note: $f(x) \approx f(x^*) + F(x^*)(x - x^*)$ where $F(x^*) \triangleq \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x = x^*}$

$\cancel{F(x^*)} f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$, $\cancel{\frac{\partial f_1}{\partial x_1}} =$ $f_1(x_k) = x_{1,k} + x_{2,k}\Delta t$

$f_2(x_k) = x_{2,k} - g\sin(x_{1,k})\Delta t$

$$\left.\begin{aligned} \frac{\partial f_1}{\partial x_1} &= 1 \\ \frac{\partial f_1}{\partial x_2} &= \Delta t \\ \frac{\partial f_2}{\partial x_1} &= -g\Delta t\cos(x_{1,k}) \\ \frac{\partial f_2}{\partial x_2} &= 1 \end{aligned}\right\} \Rightarrow F(x^*) = \begin{bmatrix} 1 & \Delta t \\ -g\Delta t\cos(x_{1,k}^*) & 1 \end{bmatrix}$$

thus, ~~the~~ 1st order approximation of $f(x_k)$ is:

$$\boxed{f(x_k) \approx f(x_k^*) + \begin{bmatrix} 1 & \Delta t \\ -g\Delta t\cos(x_{1,k}^*) & 1 \end{bmatrix} (x_k - x_k^*)}$$

note: $f(x_k^*) = \begin{bmatrix} x_{1,k}^* + x_{2,k}^*\Delta t \\ x_{2,k}^* - g\sin(x_{1,k}^*)\Delta t \end{bmatrix}$

for the measurement equation;

$$\cancel{h(x)} \; h(x_k) \approx h(x_k^*) + H(x_k^*)(x_k - x_k^*) \quad , \quad h(x_k) = \sin(x_{1,k})$$

$$h(x_k^*) = \sin(x_{1,k}^*)$$

$$H(x_k^*) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{bmatrix} \quad , \quad \frac{\partial h}{\partial x_1} = \cos(x_{1,k})$$

$$\Rightarrow H(x_k^*) = \begin{bmatrix} \cos(x_{1,k}^*) & 0 \end{bmatrix}$$

$$\frac{\partial h}{\partial x_2} = 0$$

then,

$$\boxed{h(x_k) \approx \sin(x_{1,k}^*) + \begin{bmatrix} \cos(x_{1,k}^*) & 0 \end{bmatrix}(x - x_k^*)}$$

Implement the EKF!

parameters: $\sigma_p = 0.1$, $\sigma_m = 0.3$, $\Delta t = 20ms$

ground truth.npy → true trajectory sampled at 1ms

measurements.npy → noisy measurements, ~~sampled~~ sampled at 20ms

Q5 EKF equations of the code: (from class lecture, Lec 17, pg 6)

$$\hat{x}_{k+1|k} = f_k(\hat{x}_{k|k})$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{f,k}\left(y_k - h_k(\hat{x}_{k|k-1})\right)$$

$$K_{f,k} = P_{k|k-1} H(\hat{x}_{k|k-1})^T \left(H(\hat{x}_{k|k-1}) P_{k|k-1} H(\hat{x}_{k|k-1})^T + R_k\right)^{-1}$$

$$P_{k|k} = \left(I - K_{f,k} H(\hat{x}_{k|k-1})\right) P_{k|k-1}$$

$$P_{k+1|k} = F(\hat{x}_{k|k}) P_{k|k} F(\hat{x}_{k|k})^T + Q_k$$

→

$R_k = \mathbb{E}[v_k^2] = \sigma_m^2$ (variance of v_k)

$$Q_k = \sigma_p^2 \begin{bmatrix} \Delta t^3/3 & \Delta t^2/2 \\ \Delta t^2/2 & \Delta t \end{bmatrix}$$

$\sigma_m = 0.3$

$\sigma_p = 0.1$

$\Delta t = 20\text{ms}$

$f_k(x_k) = f(x_k)$

$h_k(x_k) = h(x_k)$

$$f_k(\hat{x}_{k|k}) = \begin{bmatrix} \hat{x}_{1,k|k} + \hat{x}_{2,k|k}\,\Delta t \\ \hat{x}_{2,k|k} - g\,\Delta t \sin(\hat{x}_{1,k|k}) \end{bmatrix}$$

$$F(\hat{x}_{k|k}) = \begin{bmatrix} 1 & \Delta t \\ -g\,\Delta t \sin(\hat{x}_{1,k|k}) & 1 \end{bmatrix}$$

$$h_k(\hat{x}_{k|k-1}) = \sin(\hat{x}_{1,k|k-1})$$

$$H(\hat{x}_{k|k-1}) = \begin{bmatrix} \cos(\hat{x}_{1,k|k-1}) & 0 \end{bmatrix}$$

→ See python code for implementation

RMS error: $\sqrt{\dfrac{\Sigma(\hat{x}_{1,k|k} - x)^2}{n}} \approx \cancel{4.03}\ 0.180$

$\sqrt{\dfrac{\Sigma(y-x)^2}{n}} = 0.311$

Particle Filter →

Particle Filter:

from 2-Q3), we have the discretized non-linear model:

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} + x_{2,k}\Delta t \\ x_{2,k} - g\Delta t \sin(x_{1,k}) \end{bmatrix} + q_k \quad \text{where } q_k = \begin{bmatrix} 0 \\ \Delta t\, w_k \end{bmatrix}$$

$$y_{k+1} = \sin(x_{1,k+1}) + v_{k+1}$$

$\{v_k\}$ and $\{w_k\}$ are white, Gaussian, $R_v = \sigma_m^2$, $\sigma_m = 0.3$

$\{q_k\}$ is white, Gaussian w/ $R_q = Q_k \triangleq \sigma_p^2 \begin{bmatrix} \Delta t^3/3 & \Delta t^2/2 \\ \Delta t^2/2 & \Delta t \end{bmatrix}$, $\sigma_p = 0.1$

with some a-priori knowledge of the statistics of the system, set

$x_0^{(i)} \sim N(y_0, 0.5)$ for $n = 200$ particles
$i = 1 \ldots n$

$w^{(i)} = 1/n$

then we have step 1) of a PF: draw n samples from the prior and set weights to $1/n$

now

2) For each $k = 1 \ldots T$

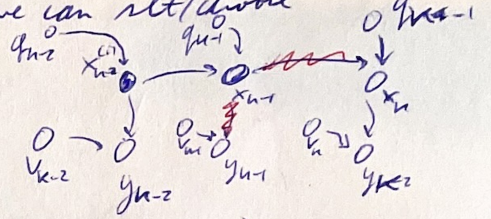
a) draw samples $x_k^{(i)}$ from importance distribution

$$x_k^{(i)} \sim \pi(x_k \mid x_{0:k-1}^{(i)}, y_{0:k}) \qquad i = 1 \ldots n$$

b) compute new weights

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(y_i \mid x_k^{(i)})\, p(x_k^{(i)} \mid x_{k-1}^{(i)})}{\pi(x_k^{(i)} \mid x_{0:k-1}^{(i)}, y_{0:k})}$$ and normalize

So, b/c we have the state-space model, we can set/choose



$$\pi(x_k^{(i)} | x_{0:k-1}^{(i)}\, y_{0:k}) \triangleq p(x_k^{(i)} | x_{k-1}^{(i)})$$

(i.e. only push the current particle through the process update)

$$\text{thus, } p(x_k^{(i)} | x_{k-1}^{(i)}) = p\left(\begin{bmatrix} x_{1,k-1} + x_{2,k-1} \Delta t \\ x_{2,k-1} - g \Delta t \sin(x_{1,k-1}) \end{bmatrix} + q_{k-1} \;\middle|\; x_{k-1}^{(i)} \right)$$

$$= \underbrace{\begin{bmatrix} x_{1,k-1}^{(i)} + x_{2,k-1}^{(i)} \Delta t \\ x_{2,k-1}^{(i)} - g \Delta t \sin(x_{1,k-1}^{(i)}) \end{bmatrix}}_{(PF-2)} + \underbrace{p(q_{k-1} | x_{k-1}^{(i)})}$$

$= p(q_{k-1})$ (noise is independant of past & current states)

$= 0$ ← white noise Gaussian when ~~realized~~

$$\boxed{\Rightarrow p(x_k^{(i)} | x_{k-1}^{(i)}) = \begin{bmatrix} x_{1,k-1}^{(i)} + x_{2,k-1}^{(i)} \Delta t \\ x_{2,k-1}^{(i)} - g \Delta t \sin(x_{1,k-1}^{(i)}) \end{bmatrix} + p(q_{k-1})}$$

(PF-1)

therefore, when we sample from the importance distribution, given $x_{k-1}^{(i)}$, the result is (PF-2) plus the realization of $\{q_k\}$

Now compute expression for the update of the weights:

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(y_i | x_k^{(i)})\, \cancel{p(x_k^{(i)} | x_{k-1}^{(i)})}}{\underbrace{\cancel{\pi(x_k^{(i)} | x_{0:k-1}^{(i)}\, y_{0:k})}}_{\triangleq \cancel{p(x_k^{(i)} | x_{k-1}^{(i)})}}} = w_{k-1}^{(i)}\, p(y_i | x_k^{(i)})$$

→

to solve for $p(y_n|x_{n-1}^{(i)})$, we note that y_n is Gaussian due to the Gaussian noise $\{v_n\}$. thus, $v_n \sim N(0, \sigma_m^2)$

$$\cancel{y_n \sim N(\mathbb{E}[y_n]}, \quad y_n|x_n^{(i)} \sim N(\mathbb{E}[y_n|x_n^{(i)}], K_v)$$

$$\boxed{\mathbb{E}[y_n|x_n^{(i)}] = E[\sin(x_n) + v_n | x_n^{(i)}] = \sin(x_n^{(i)}) \triangleq \mu_y} \quad \text{(PF-3)}$$

$$K_y = \mathbb{E}[(y-\mu_y)(y-\mu_y)|x_n^{(i)}] = \mathbb{E}[y^2 - 2\mu_y y + \mu_y^2 | x_n^{(i)}]$$

$$= \mathbb{E}[(\sin(x_n)+v_n)^2] - \cancel{2}\mu_y^2 + \cancel{\mu_y^2}$$

$$= \mathbb{E}[\sin^2(x_n) + 2\sin(x_n)v_n + v_n^2 | x_n^{(i)}] - \mu_y^2$$

$$= \cancel{\sin^2(x_n^{(i)})} + 2\sin(x_n^{(i)})\cancel{\mathbb{E}[v_n|x_n^{(i)}]}^{0} + R_v - \cancel{\sin^2(x_n^{(i)})}$$

$$\Rightarrow \boxed{K_y = R_v = \sigma_m^2} \quad \text{(PF-4)}$$

thus, $$\boxed{y_n|x_n^{(i)} \sim N(\sin(x_n^{(i)}), \sigma_m^2)} \quad \text{(PF-5)}$$

and $$\boxed{p_{y_n}(y_n|x_n^{(i)}) = \frac{1}{\sigma_m\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(y_n - \sin(x_n^{(i)}))^2}{\sigma_m^2}\right)} \quad \text{(PF-6)}$$

Such that

$$w_n^{(i)} \propto w_{n-1}^{(i)} p(y_n|x_n^{(i)})$$

$$\Rightarrow \boxed{w_n^{(i)} \propto w_{n-1}^{(i)} \left[\frac{1}{\sigma_m\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(y_n - \sin(x_n^{(i)}))^2}{\sigma_m^2}\right)\right]} \quad \text{(PF-7)}$$

then, $w_n^{(i)} = \dfrac{w_n^{(i)}}{\sum_{i=1}^{n} w_n^{(i)}}$ to normalize

PF part 3) re-sampling:

Re-sample if $n_{eff} < 20$ (i.e. 10%)

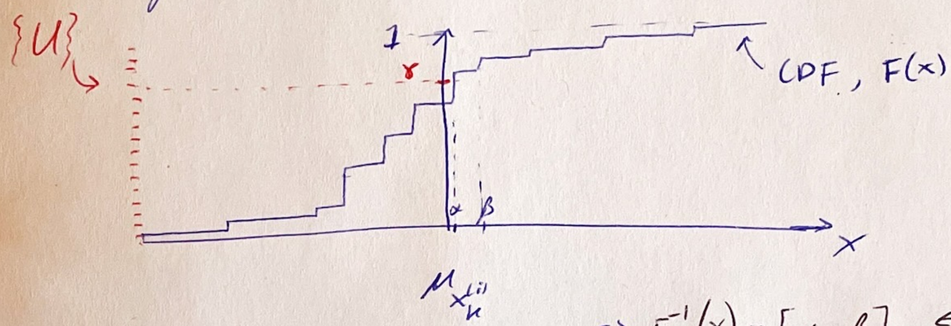
let $n_{eff} \approx \frac{1}{\sum_{j=1}^{n} (w_k^{(j)})^2}$

per HW#5 of ECE 6555, we can re-sample according $F^{-1}(U)$

where U is a uniform distribution of particles $[0,1]$ and

$F = \sum_{i=1}^{n} w_k^{(i)} \delta(x - x_k^{(i)})$ where $x_k^{(i)}$ are sorted (~~and~~ low → high.

basically this would look something like:



$\Rightarrow F^{-1}(\gamma) = [\alpha, \beta]$ ← in code, I will sample from this range.

| See python code for implementation |

note: did not know how to deal w/ 2-D state model... code will <u>not</u> function