HW #5, ECE 6555

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Due: 12/16/22

Goal: Compare EKF w/ Particle Filter.

Scalar, non-linear model:

$$x_k = \frac{1}{2} x_{k-1} + \frac{25 x_{k-1}}{1 + x_{k-1}^2} + 8\cos(1.2(k-1)) + u_k \quad (1)$$

$$y_k = \frac{1}{20} x_k^2 + v_k \quad (2)$$

$\{u_k\}$, $\{v_k\}$ are white, Gaussian noise sequences w/ unit variance.

$Q_k = 1$, $R_k = 1$

initial state: $x_0 = 0.1$, $\hat{x}_0 = 0$, $\hat{\sigma}_0 = 2$

note: both state & measurement processes are non-linear.

(a1) Plot the state process via python (See python code for plot)

Now for the Particle Filter

method:

1) draw n samples from the prior $x_0^{(i)} \sim p(x_0)$ $i = 1 \dots n$
and set $w^{(i)} = \frac{1}{n}$ for $i = 1 \dots n$

2) For each $k = 1 \dots T$

a) draw samples $x_k^{(i)}$ from importance distributions

$$x_k^{(i)} \sim \pi(x_k \mid x_{0:k-1}^{(i)}, y_{0:k}) \qquad i = 1 \dots n$$

b) compute new weights

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(y_k \mid x_k^{(i)})\, p(x_k^{(i)} \mid x_{k-1}^{(i)})}{\pi_i(x_k^{(i)} \mid x_{0:k-1}^{(i)}, y_{0:k})}$$

and normalize

3) If the effective # of particle is too low, re-sample and reset to uniform weights, where $n_{eff} \approx \frac{1}{\sum_{j=1}^{n} (w_k^{(j)})^2}$ (i.e. the # of effective particles at a particular time-step)

But, in class we did not specify how to choose the importance distribution and how to perform the sampling.

↑ (basically, try to get as close to real distr. as possible, and be Markovian)

Now, we choose

$$\pi(x_k^{(i)} \mid x_{0:k-1}^{(i)}, y_{1:k}) \triangleq p(x_k^{(i)} \mid x_{k-1}^{(i)}) \qquad (3)$$

that is, we only push the current particle $x_{k-1}^{(i)}$ through a process update.

Q2] Given a set of particles $\{x_{k-1}^{(i)}\}$, show that sampling from the importance distribution $\pi(x_k \mid x_{0:k-1}^{(i)}, y_{1:k})$ then reduces to computing:

$$\frac{1}{2} x_{k-1}^{(i)} + \frac{25\, x_{k-1}^{(i)}}{1 + (x_{k-1}^{(i)})^2} + 8 \cos(1.2(k-1)) \qquad (4)$$

and adding the realization of the 0-mean white Gaussian noise.

→

Q2 cont'd

from (3) we have

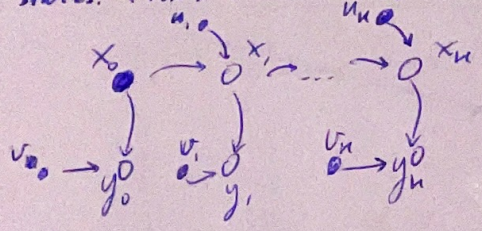
$$\pi(x_k \mid x_{0:k-1}^{(i)}, y_{1:k}) \triangleq p(x_k \mid x_{k-1}^{(i)})$$

From eqtn (1) $\rightarrow$

$$\Rightarrow p(x_k \mid x_{k-1}^{(i)}) = p\left(\tfrac{1}{2}x_{k-1} + \frac{25\,x_{k-1}}{1+x_{k-1}^2} + 8\cos(1.2(k-1)) + u_k \mid x_{k-1}^{(i)}\right)$$

$$= \frac{1}{2}x_{k-1}^{(i)} + \frac{25\,x_{k-1}^{(i)}}{1+(x_{k-1}^{(i)})^2} + 8\cos(1.2(k-1)) + p(u_k \mid x_{k-1}^{(i)})$$

Since u_k is Gaussian white noise, it is independent of past and present states. From a Functional Dependence Graph perspective, we see that indeed



u_k is independent of x_{k-1}.

Thus, $p(u_k \mid x_{k-1}^{(i)}) = p(u_k) = 0$

In summary,

$$\pi(x_k \mid x_{0:k-1}^{(i)}, y_{1:k}) \triangleq p(x_k \mid x_{k-1}^{(i)}) = \frac{1}{2}x_{k-1}^{(i)} + \frac{25\,x_{k-1}^{(i)}}{1+(x_{k-1}^{(i)})^2} + 8\cos(1.2(k-1)) + p(u_k)$$

so that when we sample from the importance distribution, given $\{x_{k-1}^{(i)}\}$, then the resultant sample is eqtn (1) plus the realization of $\{u_k\}$.

[Q3] Explicit expression for computation of weights $w_k^{(i)}$ as a function of $w_{k-1}^{(i)}$. Make the subsequent normalization to avoid unnecessary computations.

$\longrightarrow$

Q3 cont'd

$$w_n^{(i)} \propto w_{n-1}^{(i)} \frac{p(y_n | x_n^{(i)}) p(x_n^{(i)} | x_{n-1}^{(i)})}{\pi(x_n^{(i)} | x_{0:n-1}^{(i)}, y_{1:n})}$$

how to deal w/ this? is it the same as $\pi(x_n | x_{0:n-1}^{(i)}, y_{1:n})$?

I don't think so... there's a comma there...

Is it is the same → should be a comma always!

where $\pi(x_n^{(i)} | x_{0:n-1}^{(i)} y_{0:n}) = p(x_n^{(i)} | x_{n-1}^{(i)})$

thus,

$$w_{n-1}^{(i)} \frac{p(y_n | x_n^{(i)}) p(x_n^{(i)} | x_{n-1}^{(i)})}{\pi(x_n^{(i)} | x_{0:n-1}^{(i)} y_{0:n})} = w_{n-1}^{(i)} p(y_n | x_n^{(i)})$$

$\pi(x_n^{(i)} | x_{0:n-1}^{(i)} y_{0:n}) \rightarrow p(x_n^{(i)} | x_{n-1}^{(i)})$

and $\mathbb{E}\, p(y_n | x_n^{(i)}) = \mathbb{E}\left(\frac{1}{20} x_n^2 + v_n | x_n^{(i)}\right) = \mathbb{E}\left(\frac{1}{20} x_n^2 | x_n^{(i)}\right) + \mathbb{E}(v_n | x_n^{(i)})$

note: $\mathbb{E}(v_n | x_n^{(i)}) = \mathbb{E}(v_n)$ b/c v_n is independent of $x_n^{(i)}$

and thus $\mathbb{E}(v_n) = 0$, thus:

$$\mathbb{E}(y_n | x_n^{(i)}) = \mathbb{E}\left(\frac{1}{20} x_n^2 | x_n^{(i)}\right) = \frac{1}{20}\left(x_n^{(i)}\right)^2 \qquad (5)$$

such that

$$w_n^{(i)} \propto w_{n-1}^{(i)} p(y_n | x_n^{(i)}) = w_{n-1}^{(i)} \frac{\left(x_{n-1}^{(i)}\right)^2}{20}$$

$$\Rightarrow \boxed{w_n^{(i)} \propto \frac{1}{20} w_{n-1}^{(i)} \left(x_{n-1}^{(i)}\right)^2} \qquad (6)$$

Q3 cont'd)

each y_n is Gaussian distributed around the mean, due to the Gaussian noise $\{v_n\}$. Thus,

$$y_n \sim \mathcal{N}(\mathbb{E}[y_n], 1) \qquad \text{since } v_n \sim \mathcal{N}(0,1)$$

Note: $K_y = \mathbb{E}[(y_n - \mu_y)(y_n - \mu_y)^T] = \mathbb{E}\left[\left(\frac{1}{20}x_n^{(i)} + v_n - \frac{1}{20}x_n^{(i)}\right)\left(\frac{1}{20}x_n^{(i)} + v_n - \frac{1}{20}x_n^{(i)}\right)\right]$

$$= \mathbb{E}(v_n^2) = K_n = 1$$

Now, $p_y(y_n | x_n^{(i)}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y_n - \mu_y}{\sigma^2}\right)\right]$

$$\Rightarrow \boxed{p_{y_n}(y_n | x_n^{(i)}) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(y_n - \frac{1}{20}x_n^{(i)}\right)\right]} \qquad (7)$$

Such that,

$$w_n^{(i)} \propto w_{n-1}^{(i)} p(y_n | x_n^{(i)})$$

$$\Rightarrow \boxed{w_n^{(i)} \propto w_{n-1}^{(i)}\left(\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(y_n - \frac{1}{20}x_n^{(i)}\right)\right]\right)} \qquad (8)$$

Q4 Implement particle filter (without) the resampling step. Use 200 part.

4.1) Provide a graph w/ at least mean & var. of filter superposed to data.

4.2) Plot the n_{eff} as a function of time index k.

→ see python code

(recall PF method on page 1, given in problem statement)

1) draw n samples from the prior $x_0^{(i)} \sim p(x_0)$ $i = 1 \dots n$

and set $w^{(i)} = \frac{1}{n}$ for $i = 1 \dots n$.

↳ note: $x_0 \sim (0, 2)$ from problem statement: [$x_0 = 0.1$, $\hat{x}_0 = 0$, $\sigma_0 = 2$]

2) For each $k = 1 \dots T$

a) draw samples $x_k^{(i)}$ from importance distributions

$$x_k^{(i)} \sim \pi(x_k^{(i)} | x_{0:k-1}^{(i)}, y_{0:k})$$

$$\Rightarrow x_k^{(i)} \sim p(x_k^{(i)} | x_{k-1}^{(i)}) \quad *$$

where $p(x_k^{(i)} | x_{k-1}^{(i)}) = \frac{1}{2} x_{k-1}^{(i)} + \frac{25 x_{k-1}^{(i)}}{1 + (x_{k-1}^{(i)})^2} + 8 \cos(1.2(k-1)) + p(u_k)$

and $u_k \sim (0, 1)$

b) compute new weights, and normalize:

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(y_k | x_k^{(i)}) \, p(x_k^{(i)} | x_{k-1}^{(i)})}{\pi(x_k^{(i)} | x_{0:k-1}^{(i)}, y_{0:k})} \Rightarrow w_k^{(i)} \propto \frac{1}{20} w_{k-1}^{(i)} (x_{k-1}^{(i)})^2$$

$w_{k-1}^{(i)} \left(\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(y_k - \frac{x_{k-1}^{(i)}}{20} \right) \right] \right)$

then, $\hat{w}_k^{(i)} = \frac{w_k^{(i)}}{\| w_k^{(i)} \|}$, $i = 1 \dots n$, $\| w_k^{(i)} \| = \sqrt{\sum_{i=1}^{n} (w_k^{(i)})^2}$

$\| w_k^{(i)} \| \hat{\sum}_{i=1}^{n} w_k^{(i)}$ ← per lecture

want only that $\sum w_k^{(i)} = 1$, to prepare for re-sampling & to standardize weights ✓5

3) skip for this part/problem

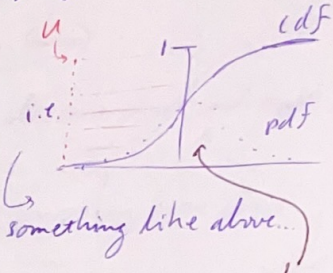
Q4 cont'd

weighted mean: $\mu_w = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$

biased weighted variance: $\hat{\sigma}_w^2 = \frac{\sum_{i=1}^{n} w_i (x_i - \mu_w)^2}{\sum_{i=1}^{n} w_i}$

Q5] Let X be a random variable with PDF p_x and CDF F. Let U be a random variable uniformly distributed in $[0,1]$. Show that the variable $F^{-1}(U)$ is distributed according to p_x. s.t. we can sample from $\sum_{i=1}^{n} w_n^{(i)} \delta[x - x_n^{(i)}]$ (CDF)

Let $F = \sum_{i=1}^{n} w_n^{(i)} \delta(x - x_n^{(i)})$



$[P(F^{-1}(U) \leq x) = F(x)$

$F^{-1}(\alpha)$ will give the value x s.t. $F(x) = \alpha \; (= P(X \leq x))$

So, the probability that $F^{-1}(U) \leq x$ is $P(F^{-1}(U) \leq x) =$

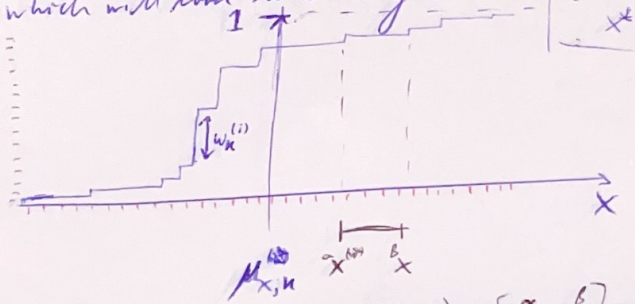
$F^{-1}(U) \leq x \Rightarrow U \leq F(x) \Rightarrow P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x)$

since U is a uniform distribution, any value in $[0,1]$ is equally possible, s.t. the probability that $U < F(x)$ is simply the value of $F(x)$.

$\Rightarrow$ $F^{-1}(U) \sim p_x$ (distributed according to p_x)

So, $F(x) = \sum_{i=1}^{n} w_n^{(i)} \delta(x - x_n^{(i)})$

which will look something like: $P(x^* \leq x) = F(x)$, $x^* \sim p_x$



So, $F^{-1}(0.9) \equiv [x^a, x^b]$

So have to figure out how to split and sample. if $F^{-1}(U_i)$ is between two values, choose the lower one. Then within the range of values that it is $[x^a, x^b]$, ~~random~~ randomly select a point within between. (w/ uniform distr.)

Q6| Plot the PF with re-sampling if $n_{eff} < 20$ (10% of ~~samples~~ particles)

method for resampling:

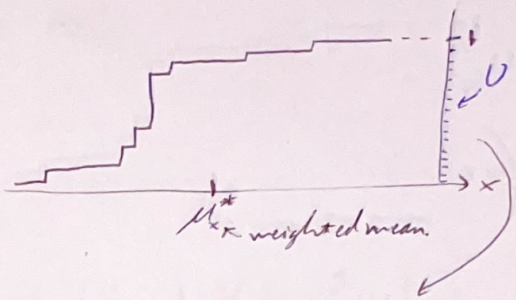
at step k, if $n_{eff}[k-1] < 20$:

draw new samples according to:

$F(x) = \sum_{i=1}^{n} w_k^{(i)} \delta(x - x_k^{(i)})$ →

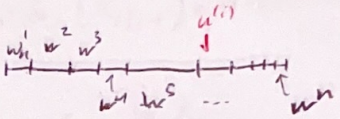
~~bins~~ bins = iterative sum of $w^{(i)}$

U = random (uniform, 200)



for each $u^{(i)}$:

for each bin: (in reverse)



if $u^{(i)} \geq$ bin :

$x_{new}^{(i)} = x_k^{(i)} + \left(x_k^{(i)} + x_k^{(i+1)}\right) * \text{random (uniform)}$

↳ do this so not the same point each time

break/continue

and continue algorithm...

See python code for implemented PF and algorithm

(Q7) →

Q7 Derive a linearized version of the non-linear system:

System:

$$y_k = \frac{1}{20} x_k^2 + v_k$$

$$x_k = \frac{1}{2} x_{k-1} + \frac{25 x_{k-1}}{1 + x_{k-1}^2} + 8\cos(1.2(k-1)) + u_k$$

---

per Lec 17,

$$x_i = f_{i-1}(x_{i-1}) + u_{i-1} \quad , \quad y_i = h_i(x_i) + v_i$$

$$f_i(x_i) \approx f_i(\hat{x}_{i|i}) + F_i(x_i - \hat{x}_{i|i}) \quad , \quad h_i(x_i) \approx h_i(\hat{x}_{i|i-1}) + H_i(x_i - \hat{x}_{i|i-1})$$

where $F_i = \begin{bmatrix} \frac{\partial f_{i,1}}{\partial x_{i,1}} & \cdots & \frac{\partial f_{i,1}}{\partial x_{i,n}} \\ \vdots & & \\ \frac{\partial f_{i,n}}{\partial x_{i,1}} & \cdots & \frac{\partial f_{i,n}}{\partial x_{i,n}} \end{bmatrix}$ and similar for H_i's

let $z_k = x_k - 8\cos(1.2(k-1))$

$$\Rightarrow z_k = \frac{1}{2} x_{k-1} + \frac{25 x_{k-1}}{1 + x_{k-1}^2} + u_k \Rightarrow \text{let } \boxed{F_{k-1}(x_{k-1}) = \frac{1}{2} x_{k-1} + \frac{25 x_{k-1}}{1 + x_{k-1}^2}} \quad (7.1)$$

$$\Rightarrow \boxed{F_k = \frac{\partial f_k}{\partial x_k} = \frac{1}{2} + 25(1 + x_k^2)^{-1} + 25 x_k (-1)(2x_{k+1})^{-2}} \quad (7.2)$$

s.t. $z_{k+1} \approx F_k(\hat{x}_{k|k}) + F_k(\hat{x}_{k|k})(x_k - \hat{x}_{k|k}) + u_k$

$$\Rightarrow \boxed{x_{k+1} \approx F_k(\hat{x}_{k|k}) + F_k(\hat{x}_{k|k})(x_k - \hat{x}_{k|k}) + 8\cos(1.2(k-1)) + u_k} \quad (7.3)$$

let $\boxed{h_k(x_k) = \frac{1}{20} x_k^2 \Rightarrow h_k(x_k) \approx h_k(\hat{x}_{k|k-1}) + H_k(\hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})} \quad (7.4)$

s.t. $\boxed{y_k = h_k(\hat{x}_{k|k-1}) + H_k(\hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1}) + v_k}$ (7.5)

$$\boxed{H_k(x_k) = \frac{\partial h_k}{\partial x_k} = \frac{1}{10} x_k} \quad (7.6)$$

EKF equations:

$$\hat{x}_{i+1|i} = f_i(\hat{x}_{i|i})$$

$$\hat{x}_{i|i} = \hat{x}_{i|i-1} + K_{f,i}\left(y_i - h_i(\hat{x}_{i|i-1})\right)$$

$$K_{f,i} = P_{i|i-1} H_{i|i-1}^T \left(H_{i|i-1} P_{i|i-1} H_{i|i-1}^T + R_i\right)^{-1}$$

$$P_{i|i} = \left(I - K_{f,i} H_{i|i-1}\right) P_{i|i-1}$$

$$P_{i+1|i} = F_{i|i} P_{i|i} F_{i|i}^T + G_i Q_i G_i^T \quad , \quad G_i = I_{\text{identity}}$$

Table of Contents

ECE6555 HW5

Author: Teo Wilkening Due Date: 2022-12-16

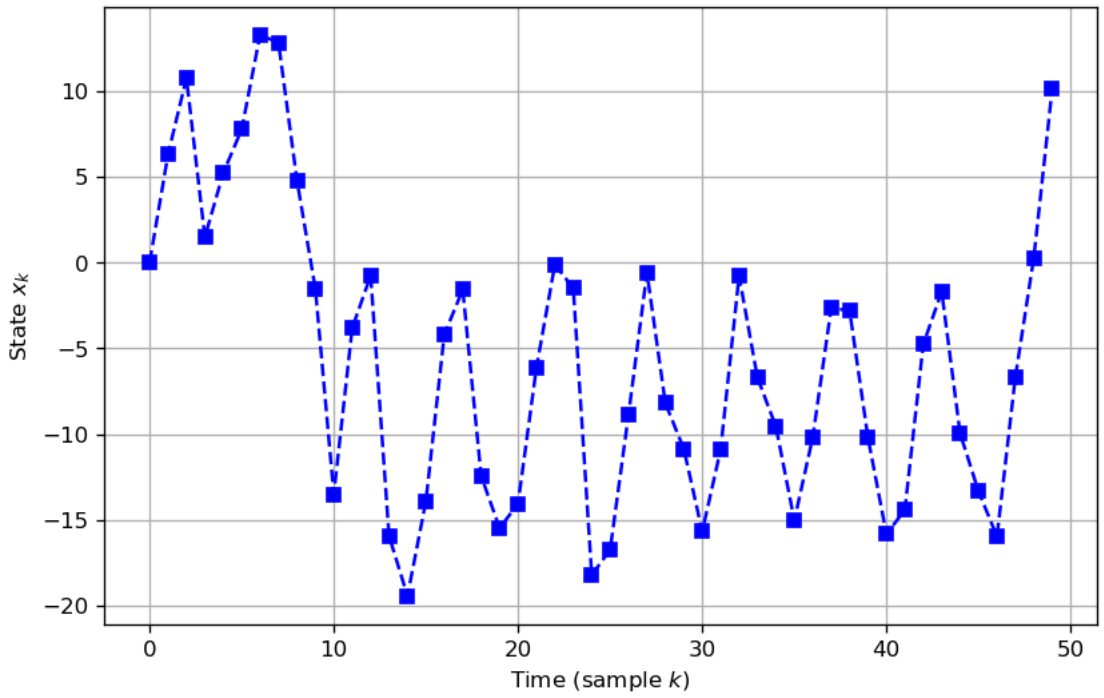
1 [Q1] Make a pot of the trajectory. This will serve as a reference throughout the problem

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt

# Create the trajectory
np.random.seed(202212)
NumSteps = 50
TimeScale = np.arange(1,NumSteps,1)
x0=0
sigma=1

x = [x0]
y = [0]
for k in TimeScale:
    xk = 0.5*x[-1]+25*x[-1]/(1+x[-1]**2)+8*np.cos(1.2*(k-1))+np.random.randn()
    yk = 1/20*xk**2+np.random.randn()
    x.append(xk)
    y.append(yk)

```

In [2]:

```
# Plot the trajectory
fig, ax = plt.subplots(figsize=(8,5), dpi=120)

ax.plot(np.insert(TimeScale,0,0),x,'b--')
ax.grid(True)
ax.plot(np.insert(TimeScale,0,0),x,'bs',markersize=6)
#plt.legend(['line','markers'])
ax.set_ylabel(r'State $x_k$')
ax.set_xlabel(r'Time (sample $k$)')

plt.show()
```



2 [Q4] Implement the Particle Filter without the resampling step

Use 200 particles.

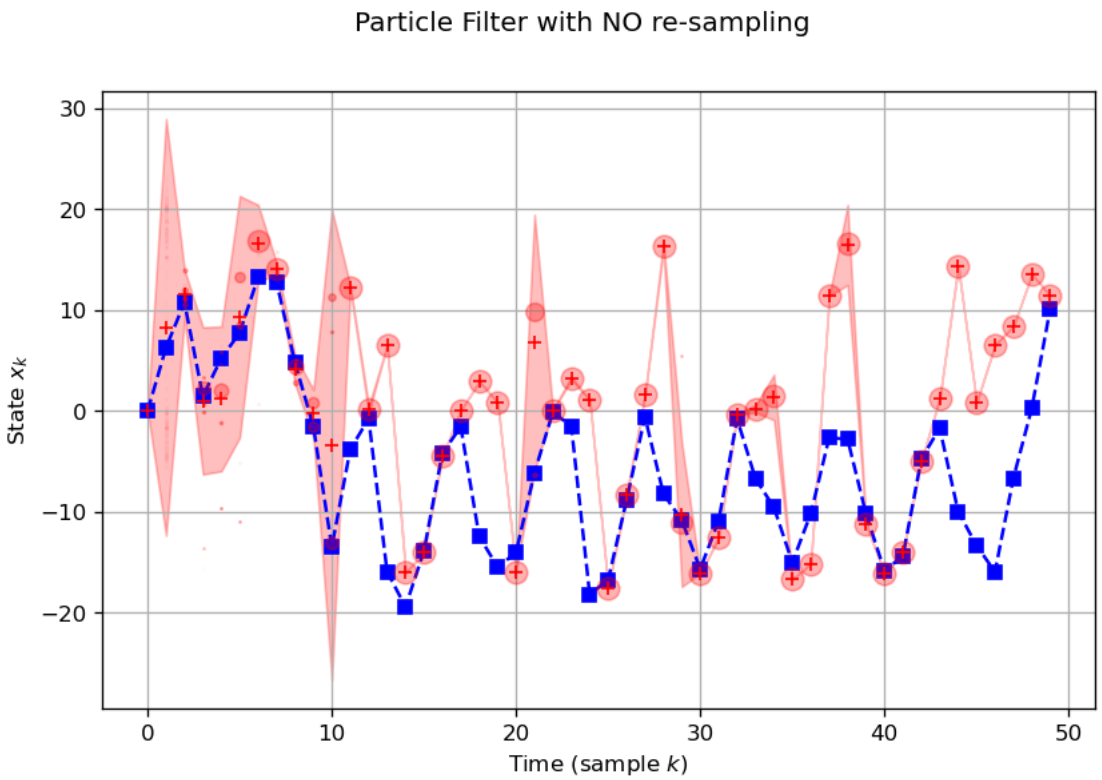
In [12]:

```
# 1) draw n samples from the prior
# 2) for each k = 1...T
#     a) draw samples x_k(i) from the importance distribution
#     b) compute the new weights
#     c) normalize the new weights

# initialize x^i_k and w^i_k matrices to keep track of state estimation distributions and weights
n = 200 # number of particles
xki = np.zeros((NumSteps,n),dtype=float)
wki = np.zeros((NumSteps,n),dtype=float)

# 1) draw n samples from the prior
x0_mu, x0_sigma = 0, np.sqrt(2)
x0 = np.random.normal(x0_mu, x0_sigma, n)
w0 = 1/n*np.ones(n)

# insert the samples from the prior into our matrices for keeping track of things
xki[0,:] = x0
wki[0,:] = w0

# initialize noise Gaussian parameters
u_mu, u_sigma = 0, 1
v_mu, v_sigma = 0, 1

# 2) for each k = 1...T
mean = np.zeros(NumSteps) # keep track of the mean of the particles
var = np.zeros(NumSteps) # keep track of the variance of the particles at each step
neff = np.zeros(NumSteps)

for k in np.arange(1,NumSteps,1):
    # a) draw samples x_k(i) from the importance distribution
    xki[k,:] = 1/2*xki[k-1,:] + 25*xki[k-1,:]/(1 + xki[k-1,:]**2) + 8*np.cos(1.2*(k-1)) + \
                np.random.normal(u_mu, u_sigma,n)
    # print(sum(xki[k,:]))
    # b) compute the new weights
    wki[k,:] = wki[k-1,:]*1/np.sqrt(2*np.pi)*np.exp(-0.5*(y[k] - 1/20*(xki[k-1,:]**2))**2)
    # c) normalize the new weights
    wki[k,:] = wki[k,:]/sum(wki[k,:])
    mean[k] = np.average(xki[k,:],weights=wki[k,:])
    var[k] = np.average((xki[k,:] - mean[k])**2,weights=wki[k,:])
    #var[k] = np.average((xki[k,:])**2,weights=wki[k,:]) - (mean[k])**2
    neff[k] = 1/sum(wki[k,:]**2)

# track mean for later analysis
mean_pf = mean
```

2.1 [Q4.1] Provide a graph with at least the mean and variance of the filter superposed to the data.

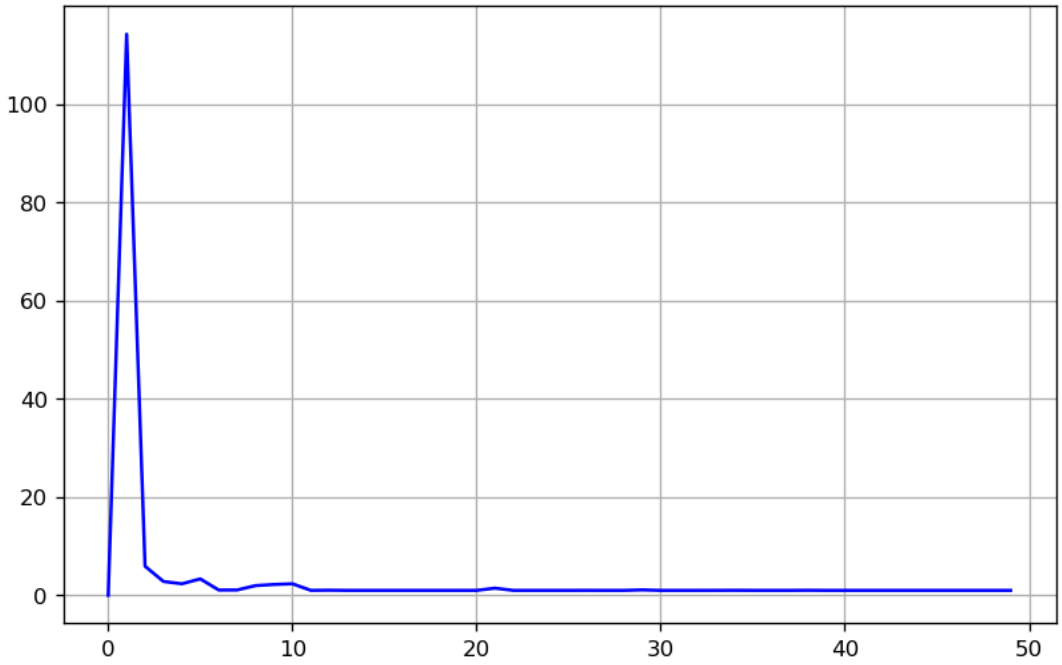
In [4]:

```
# Plot the trajectory
fig, ax = plt.subplots(figsize=(8,5), dpi=120)

ax.plot(np.insert(TimeScale,0,0),x,'b--')
ax.grid(True)
ax.plot(np.insert(TimeScale,0,0),x,'bs',markersize=6)
#plt.legend(['line','markers'])
ax.set_ylabel(r'State $x_k$')
ax.set_xlabel(r'Time (sample $k$)')
for k in np.arange(1,NumSteps,1):
    for i in np.arange(n):
        if wki[k,i] > 1e-3:
            ax.plot(k,xki[k,i],'ro',markersize=10*wki[k,i],alpha=0.3)
ax.plot(mean,'r+')
ax.fill_between(np.arange(NumSteps), mean-2*np.sqrt(var), mean+2*np.sqrt(var), alpha=0.25, color='r')
fig.suptitle('Particle Filter with NO re-sampling')

plt.show()
```



2.2 [Q4.2] Plot the n_{eff} as a function of time index k.

In [5]:
```
# Plot the neff
fig, ax = plt.subplots(figsize=(8,5), dpi=120)

ax.plot(np.insert(TimeScale,0,0),neff,'b')
ax.grid(True)
plt.show()
```

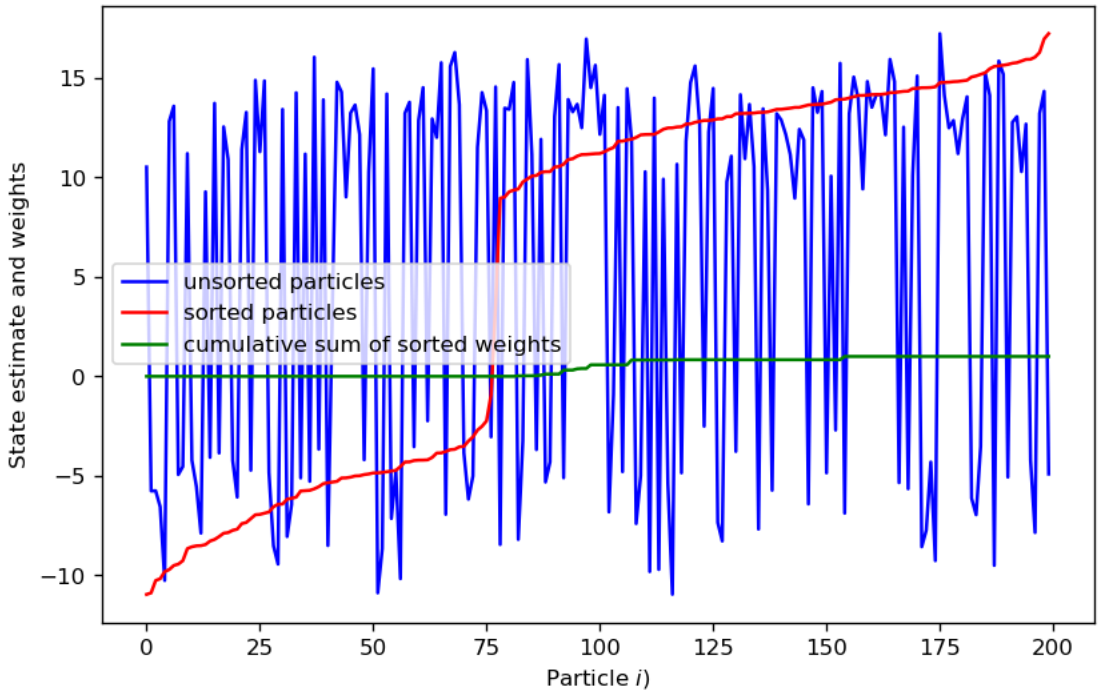


In [ ]:
```
1
```

3 [Q6] Implement the Particle Filter with the resampling step

3.1 Illustration of sorted particles and weights at a single step

In [6]:
```
fig, ax = plt.subplots(figsize=(8,5), dpi=120)


ax.plot(xki[2,:],'b')
ax.plot(np.sort(xki[2,:]),'r')
ax.plot(np.cumsum(np.take_along_axis(wki[2,:],np.argsort(xki[2,:]),axis=0)),'g')
#plt.legend(['line','markers'])
ax.set_ylabel(r'State estimate and weights')
ax.set_xlabel(r'Particle $i$)')
ax.legend(['unsorted particles','sorted particles','cumulative sum of sorted weights'])

plt.show()
```



3.2 [Q6.1] The Particle Filter algorithm (with resampling step)

In [13]:

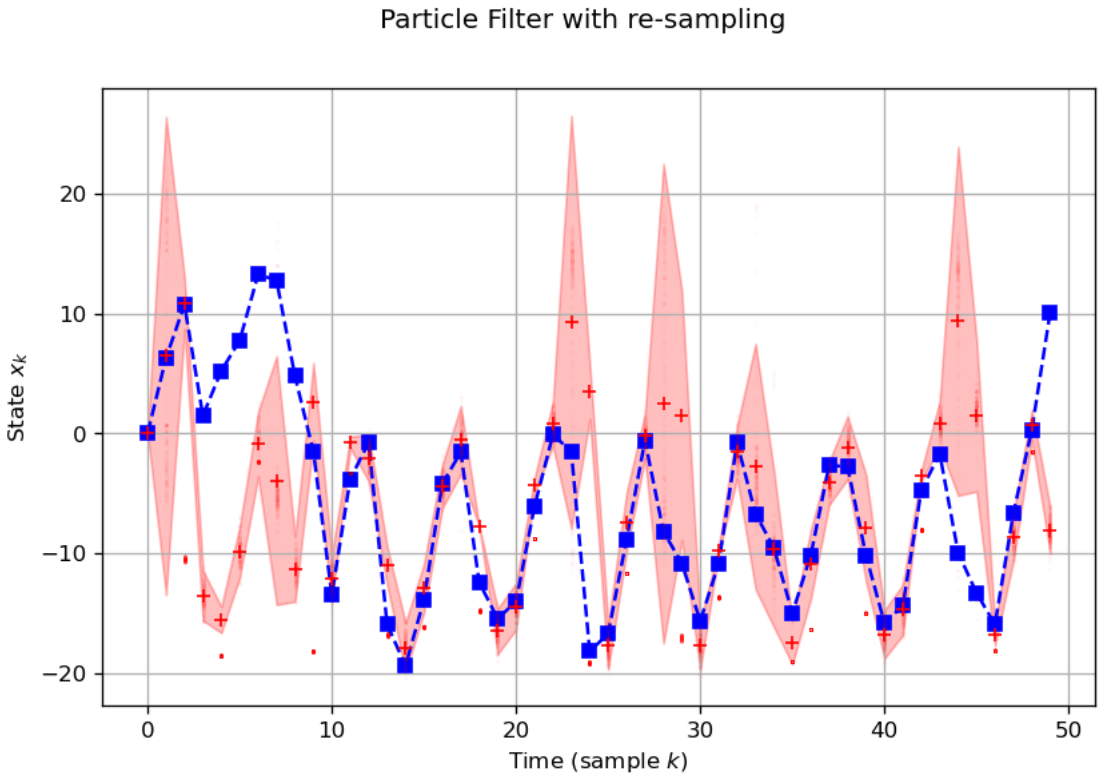
```
# 1) draw n samples from the prior
# 2) for each k = 1...T
#     a) draw samples x_k(i) from the importance distribution
#     b) compute the new weights
#     c) normalize the new weights

# initialize x^i_k and w^i_k matrices to keep track of state estimation distributions and weights
n = 200 # number of particles
xki = np.zeros((NumSteps,n),dtype=float)
wki = np.zeros((NumSteps,n),dtype=float)

# 1) draw n samples from the prior
x0_mu, x0_sigma = 0, np.sqrt(2)
x0 = np.random.normal(x0_mu, x0_sigma, n)
w0 = 1/n*np.ones(n)

# insert the samples from the prior into our matrices for keeping track of things
xki[0,:] = x0
wki[0,:] = w0

# initialize noise Gaussian parameters
u_mu, u_sigma = 0, 1
v_mu, v_sigma = 0, 1

# 2) for each k = 1...T
mean = np.zeros(NumSteps) # keep track of the mean of the particles
var = np.zeros(NumSteps) # keep track of the variance of the particles at each step
neff = np.zeros(NumSteps)

for k in np.arange(1,NumSteps,1):
    # a) draw samples x_k(i) from the importance distribution
    xki[k,:] = 1/2*xki[k-1,:] + 25*xki[k-1,:]/(1 + xki[k-1,:]**2) + 8*np.cos(1.2*(k-1)) + \
                np.random.normal(u_mu, u_sigma,n)
    # print(sum(xki[k,:]))
    # b) compute the new weights
    wki[k,:] = wki[k-1,:]*1/np.sqrt(2*np.pi)*np.exp(-0.5*(y[k] - 1/20*(xki[k-1,:]**2))**2)
    # c) normalize the new weights
    wki[k,:] = wki[k,:]/sum(wki[k,:])
    mean[k] = np.average(xki[k,:],weights=wki[k,:])
    var[k] = np.average((xki[k,:] - mean[k])**2,weights=wki[k,:])
    neff[k] = 1/sum(wki[k,:]**2)
    # draw new samples if the number of effective weights is < 20
    if neff[k] < 20:
        print(f"""Effective particles < 20 for step {k}""")
        ind = np.argsort(xki[k,:]) # index sort of the particles
        xki[k,:] = np.take_along_axis(xki[k,:],ind,axis=0)
        wki[k,:] = np.take_along_axis(wki[k,:],ind,axis=0) # sort the weights according to the particles
        bins = np.cumsum(wki[k,:]) # bins from which we are going to sample; cumulative sum of the weights
        uni = np.random.uniform(0,1,n) # uniform distribution used for re-sampling
        uni2 = np.random.uniform(0,1,n) # secondary random sampling for within bins
        for i in np.arange(0,n):
            for j in np.arange(n-1,-1,-1):
                if uni[i] >= bins[j]:
                    xki[k,i] = xki[k,j] + (xki[k,j+1] - xki[k,j])*uni2[i]
        # and reset the weights:
        wki[k,:] = w0

# track mean for later analysis
mean_pf_resamp = mean
```

```
Effective particles < 20 for step 2
Effective particles < 20 for step 4
Effective particles < 20 for step 8
Effective particles < 20 for step 9
Effective particles < 20 for step 11
Effective particles < 20 for step 13
Effective particles < 20 for step 16
Effective particles < 20 for step 19
Effective particles < 20 for step 21
Effective particles < 20 for step 24
Effective particles < 20 for step 26
Effective particles < 20 for step 29
Effective particles < 20 for step 31
Effective particles < 20 for step 35
Effective particles < 20 for step 36
Effective particles < 20 for step 39
Effective particles < 20 for step 42
Effective particles < 20 for step 46
Effective particles < 20 for step 48
```

3.3 [Q6.2] Plot mean and variance superposed to trajectory

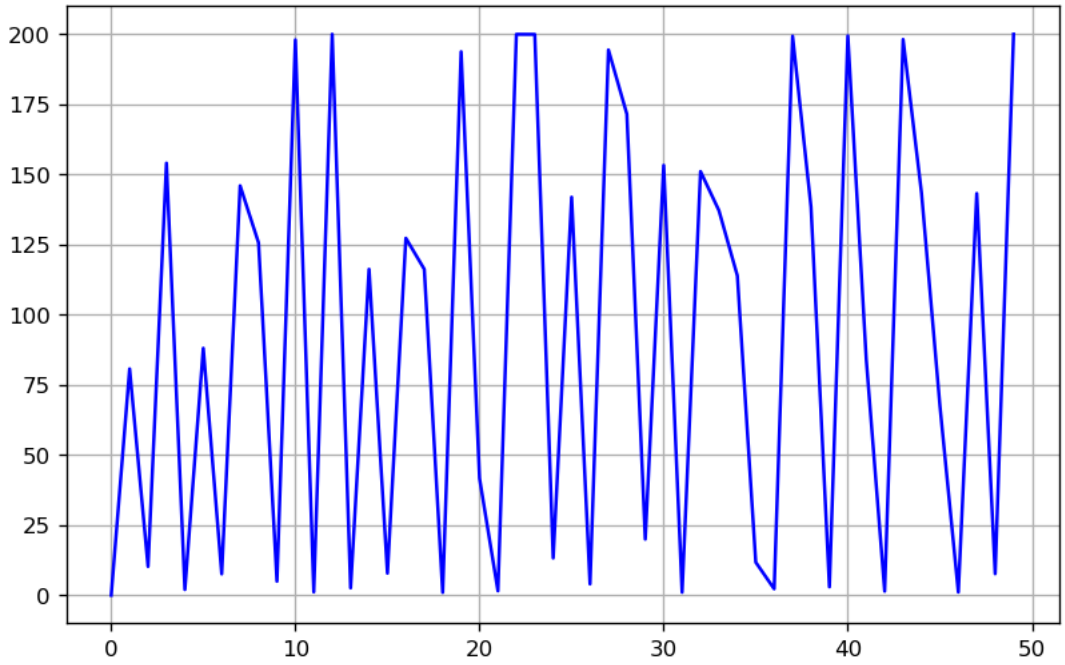
In [8]:

```
# Plot the trajectory
fig, ax = plt.subplots(figsize=(8,5), dpi=120)

ax.plot(np.insert(TimeScale,0,0),x,'b--')
ax.grid(True)
ax.plot(np.insert(TimeScale,0,0),x,'bs',markersize=6)
#plt.legend(['line','markers'])
ax.set_ylabel(r'State $x_k$')
ax.set_xlabel(r'Time (sample $k$)')
for k in np.arange(1,NumSteps,1):
    for i in np.arange(n):
        if wki[k,i] > 1e-3:
            ax.plot(k,xki[k,i],'ro',markersize=10*wki[k,i],alpha=0.3)
ax.plot(mean,'r+')
ax.fill_between(np.arange(NumSteps), mean-2*np.sqrt(var), mean+2*np.sqrt(var), alpha=0.25, color='r')
fig.suptitle('Particle Filter with re-sampling')

plt.show()
```



3.4 [Q6.3] Plot the n_{eff} as a function of time step k

In [9]:
```
# Plot the neff
fig, ax = plt.subplots(figsize=(8,5), dpi=120)

ax.plot(np.insert(TimeScale,0,0),neff,'b')
ax.grid(True)
plt.show()
```

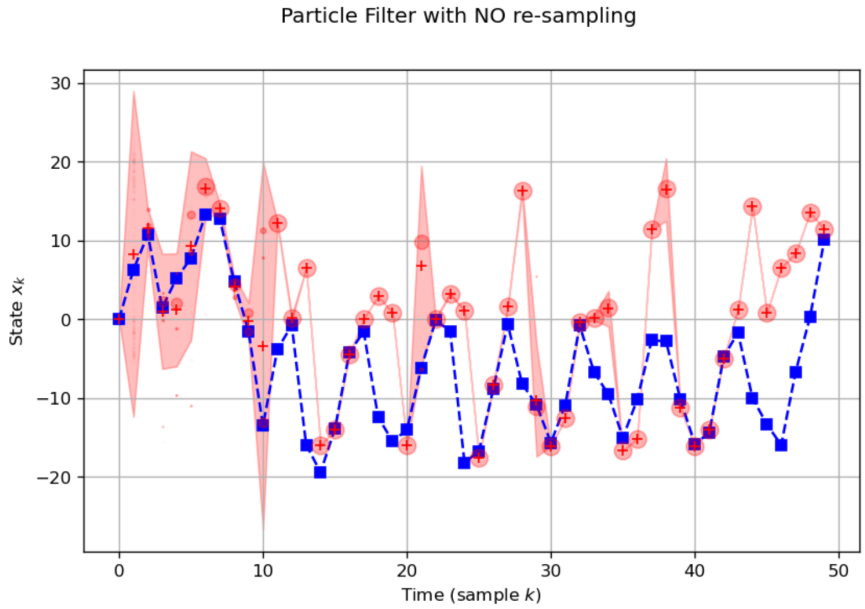


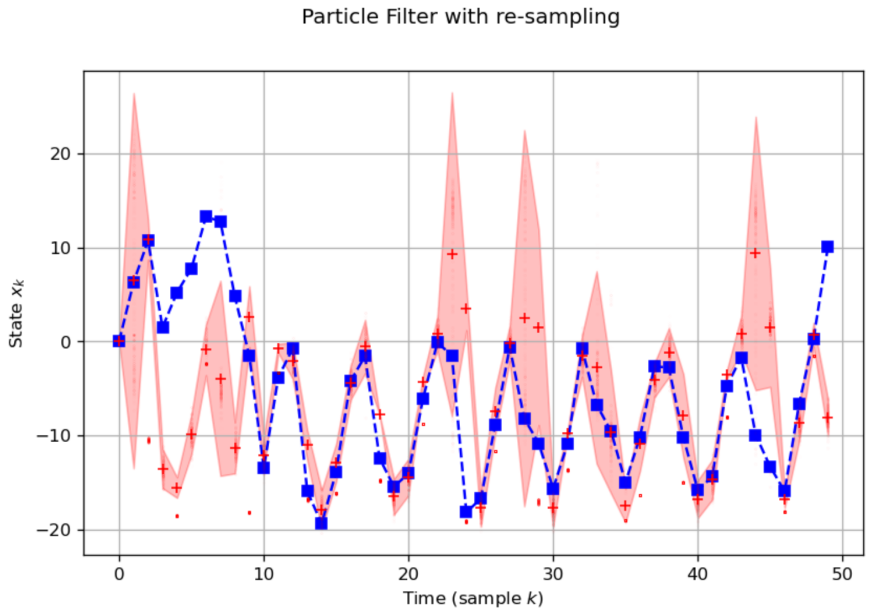
3.5 [Q6.4] Discussion of Results

As we can see from the below plots (re-captured from the above code), the particle filter with re-sampling maintains a better tracking on the variance of the estimation, and a slightly better tracking per the MSE calculation below.

In [18]:
```
mse_pf = np.sum((mean_pf - x)**2)/NumSteps
mse_pf_resamp = np.sum((mean_pf_resamp - x)**2)/NumSteps
print(f"""MSE of the PF without re-sampling: {mse_pf}""")
print(f"""MSE of the PF with re-sampling: {mse_pf_resamp}""")
```

```
MSE of the PF without re-sampling: 107.36649647973412
MSE of the PF with re-sampling: 76.75540576179377
```





NOTES:

In [10]:
```
a = np.array([1, 2, 3, 4])
b = np.ones(4) + 1
a - b
a * b
j = np.arange(5)
2**(j + 1) - j
```

Out[10]: array([2, 3, 6, 13, 28])

In [11]:
```
list = [0, 1, 2, 3, 4]
display([0, list])
len(x)
```

```
[0, [0, 1, 2, 3, 4]]
```

Out[11]: 50