

Bonus assignment

● Graded

Student

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Total Points

4 / 4 pts

Question 1

(no title)

2 / 2 pts

+ 2 pts Correct

+ 1 pt Not quite.

+ 0 pts Click here to replace this description.

Question 2

(no title)

1 / 1 pt

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Question 3

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1 / 1 pt

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ECE 6555 Midterm Bonus Problems

Tew Wilkinson
Due: 11/7/22

B1] A non-standard state-space model.

$$\begin{cases} \vec{x}_{i+1} = F_i \vec{x}_i + G_i \vec{u}_{i+1} & \text{for } i \geq 0 \\ \vec{y}_i = H_i \vec{x}_i + \vec{v}_i \end{cases} \quad (1)$$

Random variables $\vec{x}_0, \vec{u}_i, \vec{v}_i$ that satisfy:

$$\langle \cdot, \cdot \rangle = \| \cdot \|^2 = \sqrt{\langle \cdot, \cdot \rangle}$$

$$\langle x, y \rangle = E[x^T y]$$

$$\left\langle \begin{bmatrix} \vec{x}_0 \\ \vec{u}_i \\ \vec{v}_i \end{bmatrix}, \begin{bmatrix} \vec{x}_0 \\ \vec{u}_j \\ \vec{v}_j \end{bmatrix} \right\rangle = \begin{bmatrix} \langle \vec{x}_0, \vec{x}_0 \rangle & \langle \vec{x}_0, \vec{u}_j \rangle & \langle \vec{x}_0, \vec{v}_j \rangle & \langle \vec{x}_0, 0 \rangle \\ \langle \vec{u}_i, \vec{x}_0 \rangle & \langle \vec{u}_i, \vec{u}_j \rangle & \langle \vec{u}_i, \vec{v}_j \rangle & \langle \vec{u}_i, 0 \rangle \\ \langle \vec{v}_i, \vec{x}_0 \rangle & \langle \vec{v}_i, \vec{u}_j \rangle & \langle \vec{v}_i, \vec{v}_j \rangle & \langle \vec{v}_i, 0 \rangle \end{bmatrix}$$

assume $\{\vec{u}_i\}, \{\vec{v}_i\}$
are white noise.
per cross-correlation matrix
to right

$$= \begin{bmatrix} \Pi_0 & 0 & 0 & 0 \\ 0 & Q_i S_{ij} & S_i S_{ij} & 0 \\ 0 & S_i^+ S_{ij} & R_i S_{ij} & 0 \end{bmatrix} \quad (2)$$

Q1] a) Show that $\langle \vec{u}_i, \vec{x}_i \rangle = Q_i G_{i-1}^+$

Correlation vs. covariance ...

$$k_{xx} = R_{xx} - E[x] E[x]^T$$

$$k_{xy} = E[(x - \mu_x)(y - \mu_y)^T]$$

$$R_{xx} = E[x x^T]$$

$$\text{note tov: } \langle x, y \rangle = \underbrace{R_{xy}}_{E[xy^T]}$$

$$= \underbrace{E[xy^T]}_{\text{see Lec 06}}$$

pg 7&9

a) $\langle \vec{u}_i, \vec{x}_i \rangle = \langle \vec{u}_i, F_i \vec{x}_{i-1} + G_{i-1} \vec{u}_i \rangle$

(reference
Lec. 13) $= \langle \vec{u}_i, \vec{x}_{i-1} \rangle F_{i-1}^+ + \langle \vec{u}_i, \vec{u}_i \rangle G_{i-1}^+ \quad (5)$

$$\rightarrow \langle \vec{u}_i, \vec{x}_{i-1} \rangle = 0.$$

b/c $\vec{x}_{i-1} \in \text{Span}(\vec{x}_0, \vec{u}_1, \vec{u}_2, \vec{u}_3, \dots, \vec{u}_{i-1})$

that is, $\vec{x}_{i-1} = F_{i-2} \vec{x}_{i-2} + G_{i-2} \vec{u}_{i-1}$

$$= F_{i-2} (F_{i-3} \vec{x}_{i-3} + G_{i-3} \vec{u}_{i-2}) + G_{i-2} \vec{u}_{i-1}$$

$$= F_{i-2} (F_{i-4} \vec{x}_{i-4} + G_{i-4} \vec{u}_{i-3}) + G_{i-3} \vec{u}_{i-2} + G_{i-2} \vec{u}_{i-1}$$

→ $\boxed{1}$

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$$\begin{aligned}
\Rightarrow \vec{x}_{i-1} &= \cancel{F_{i-2} F_{i-3} \dots F_0 \vec{x}_0} + F_{i-2} (F_{i-3} (\dots (F_0 \vec{x}_0 + G_0 \vec{u}_1) + G_1 \vec{u}_2) + \dots) + G_{i-2} \vec{u}_{i-1} \\
&= \cancel{F_{i-2} F_{i-3} \dots F_0 \vec{x}_0} + F_{i-2} (F_{i-3} (\dots (F_1 G_0 \vec{u}_1) + \dots) + \\
\vec{x}_{i-1} &= F_{i-2} (F_{i-3} (F_{i-4} (\dots (F_i (F_0 \vec{x}_0 + G_0 \vec{u}_1) + G_1 \vec{u}_2) + \dots) + G_{i-2} \vec{u}_{i-1}) \\
&= F_{i-2} (F_{i-3} (F_{i-4} (\dots (F_i (F_0 \vec{x}_0 + G_0 \vec{u}_1) + G_1 \vec{u}_2) + G_2 \vec{u}_3) + \dots) + G_{i-2} \vec{u}_{i-1}) \\
&= F_{i-2} F_{i-3} \dots F_0 F_0 \vec{x}_0 + F_{i-2} F_{i-3} \dots F_1 G_0 \vec{u}_1 + F_{i-2} F_{i-3} \dots F_2 G_1 \vec{u}_2 \\
&\quad + F_{i-2} F_{i-3} \dots F_3 G_2 \vec{u}_3 + F_{i-2} F_{i-3} \dots F_4 G_3 \vec{u}_4 \\
&\quad + \dots + \cancel{B_2 G_2 \vec{u}_2} + F_{i-2} G_{i-3} \vec{u}_{i-2} + G_{i-2} \vec{u}_{i-1}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \langle \vec{u}_i, \vec{x}_{i-1} \rangle &= \langle \vec{u}_i, \vec{x}_0 \rangle F_0^T F_1^T \dots F_{i-3}^T F_{i-2}^T \\
&\quad + \langle \vec{u}_i, \vec{u}_1 \rangle F_1^T \dots F_{i-2}^T + \langle \vec{u}_i, \vec{u}_2 \rangle G_1^T F_2^T F_3^T \dots F_{i-2}^T \\
&\quad + \dots + \langle \vec{u}_i, \vec{u}_{i-2} \rangle G_{i-3}^T F_{i-2}^T + \langle \vec{u}_i, \vec{u}_{i-1} \rangle G_{i-2}^T
\end{aligned}$$

and per our stochastic model given in ②, we have that

$$\langle \vec{u}_i, \vec{x}_0 \rangle = 0 \quad \text{and} \quad \langle \vec{u}_i, \vec{u}_j \rangle = 0 \quad \text{for } i \neq j$$

$$\text{thus, } \langle \vec{u}_i, \vec{x}_{i-1} \rangle = 0 \quad \text{③}$$

if we substitute j for $i-1$ in \vec{x}_{i-1} then we have

$$\begin{aligned}
\langle \vec{u}_i, \vec{x}_j \rangle &= \langle \vec{u}_i, \vec{x}_0 \rangle F_0^T F_1^T \dots F_{j-2}^T F_{j-1}^T \\
&\quad + \langle \vec{u}_i, \vec{u}_1 \rangle F_1^T F_2^T \dots F_{j-1}^T + \langle \vec{u}_i, \vec{u}_2 \rangle G_1^T F_2^T F_3^T \dots F_{j-1}^T \\
&\quad + \dots + \langle \vec{u}_i, \vec{u}_{j-2} \rangle G_{j-3}^T F_{j-1}^T + \langle \vec{u}_i, \vec{u}_j \rangle G_{j-1}^T \quad \text{④}
\end{aligned}$$

b) and if $i > j$, then as shown above for ③, we have that

$$\langle \vec{u}_i, \vec{x}_j \rangle = 0 \quad \text{per the stochastic model given in ②.} \quad \rightarrow \boxed{\text{J2}}$$

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Thus, from ⑤ we now have:

$$\text{a) } \Rightarrow \boxed{\langle \vec{u}_i, \vec{x}_i \rangle = Q_i b_{i-1}^+} \quad (6)$$

□

$$\langle \vec{v}_i, \vec{x}_i \rangle = \langle \vec{v}_i, \vec{F}_{i-1} \vec{x}_{i-1} + b_i \vec{u}_i \rangle$$

$$\vec{x}_{i-1} = \vec{F}_{j-1} \vec{x}_{j-1} + b_{j-1} \vec{u}_j$$

$$= \vec{F}_{j-1} (\vec{F}_{j-2} (\vec{x}_{j-2}) + b_{j-2} \vec{u}_{j-1}) + b_{j-1} \vec{u}_j$$

$$= \vec{F}_{j-1} (\vec{F}_{j-2} (\dots (\vec{F}_1 (\vec{F}_0 \vec{x}_0 + b_0 \vec{u}_1) + b_1 \vec{u}_2) + \dots) + b_{j-2} \vec{u}_{j-1}) + b_{j-1} \vec{u}_j$$

$$= \vec{F}_{j-1} \vec{F}_{j-2} \dots \vec{F}_2 \vec{F}_1 \vec{x}_0 + \vec{F}_{j-1} \vec{F}_{j-2} \dots \vec{F}_1 b_0 \vec{u}_1 + \vec{F}_{j-1} \vec{F}_{j-2} \dots \vec{F}_2 b_1 \vec{u}_2$$

$$+ \vec{F}_{j-1} \vec{F}_{j-2} \dots \vec{F}_3 b_2 \vec{u}_3 + \dots + \vec{F}_{j-1} \vec{F}_{j-2} b_{j-3} \vec{u}_{j-2}$$

$$+ \vec{F}_{j-1} b_{j-2} \vec{u}_{j-1} + b_{j-1} \vec{u}_j \quad (7)$$

$$\Rightarrow \langle \vec{v}_i, \vec{x}_i \rangle = \langle \vec{v}_i, \vec{x}_0 \rangle \vec{F}_0^T \vec{F}_1^T \dots \vec{F}_{j-2}^T \vec{F}_{j-1}^T$$

$$+ \langle \vec{v}_i, \vec{u}_1 \rangle b_0^T \vec{F}_1^T \dots \vec{F}_{j-2}^T \vec{F}_{j-1}^T$$

$$+ \langle \vec{v}_i, \vec{u}_2 \rangle b_1^T \vec{F}_2^T \dots \vec{F}_{j-2}^T \vec{F}_{j-1}^T$$

$$\vdots \\ + \langle \vec{v}_i, \vec{u}_{j-1} \rangle b_{j-2}^T \vec{F}_{j-1}^T$$

$$+ \langle \vec{v}_i, \vec{u}_j \rangle b_{j-1}^T$$

⑧

from ② we have that $\langle \vec{v}_i, \vec{x}_0 \rangle = 0$ and $\langle \vec{v}_i, \vec{u}_j \rangle = S_i^+ S_{ij}$

Thus if ~~i > j~~ $i > j$ then $\langle \vec{v}_i, \vec{u}_j \rangle = 0$

a) $\boxed{\text{and } \langle \vec{v}_i, \vec{x}_j \rangle = 0 \text{ for } i > j} \quad (9)$

□

Question assigned to the following page: [1](#)

#1-Q2] a) show $\langle \vec{u}_i, \vec{y}_j \rangle = Q_i G_{i-1}^+ H_i^+ + S_i$

b) $\langle \vec{u}_i, \vec{y}_j \rangle = 0 \quad \forall i > j$

c) $\langle \vec{v}_i, \vec{y}_j \rangle = 0 \quad \forall i > j$

$$\begin{aligned}\vec{y}_j &= H_j \vec{x}_j + \vec{v}_j \\ &= H_j (F_{j-1} \vec{x}_{j-1} + G_{j-1} u_j) + \vec{v}_j\end{aligned}$$

as shown earlier for \vec{x}_j

$$\Rightarrow \vec{y}_j = H_j (F_{j-1} F_{j-2} \cdots F_2 F_1 F_0 \vec{x}_0 + F_{j-1} F_{j-2} \cdots F_1 G_0 u_1 + F_{j-1} F_{j-2} \cdots F_2 G_1 u_2 + \cdots + F_{j-1} G_{j-2} u_{j-1} + G_{j-1} u_j) + \vec{v}_j \quad (10)$$

$$\begin{aligned}\text{Thus, } \langle \vec{u}_i, \vec{y}_j \rangle &= \langle \vec{u}_i, \vec{x}_0 \rangle F_0^T F_1^T \cdots F_{j-1}^T H_j^T \\ &\quad + \langle \vec{u}_i, \vec{u}_1 \rangle G_0^T F_1^T \cdots F_{j-1}^T H_j^T \\ &\quad + \langle \vec{u}_i, \vec{u}_2 \rangle G_1^T F_2^T \cdots F_{j-1}^T H_j^T \\ &\quad \vdots \\ &\quad + \langle \vec{u}_i, \vec{u}_j \rangle G_{j-1}^T H_j^T \\ &\quad + \langle \vec{u}_i, \vec{v}_j \rangle\end{aligned} \quad (11)$$

From (10) we are given that $\langle \vec{u}_i, \vec{x}_0 \rangle = 0$ and $\langle \vec{u}_i, \vec{u}_j \rangle = 0 \quad \forall i > j$

and $\langle \vec{u}_i, \vec{v}_j \rangle = S_i S_{ij}$

b) Thus, combining (10) and (11) we have that

$$\langle \vec{u}_i, \vec{y}_j \rangle = 0 \quad \forall i > j$$



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Similarly,

$$\begin{aligned}\langle \vec{v}_i, \vec{y}_j \rangle &= \langle \vec{v}_i, \vec{x}_0 \rangle F_0^T F_1^T \cdots F_{j-2}^T F_{j-1}^T H_j^T \\ &\quad + \langle \vec{v}_i, \vec{u}_i \rangle G_0^T F_1^T \cdots F_{j-1}^T H_j^T \\ &\quad + \langle \vec{v}_i, \vec{u}_j \rangle G_{j-1}^T H_j^T \\ &\quad + \langle \vec{v}_i, \vec{v}_j \rangle\end{aligned}\quad (12)$$

a) Combining the stochastic model of (2) with (12) we have that

$$\langle \vec{v}_i, \vec{y}_j \rangle = 0 \quad \forall i > j$$

□

Now, if $i=j$ then from (11) we have:

$$\begin{aligned}\langle \vec{u}_i, \vec{y}_i \rangle &= \langle \vec{u}_i, \vec{y}_i \rangle = \langle \vec{u}_i, \vec{x}_0 \rangle F_0^T F_1^T \cdots F_{i-2}^T F_{i-1}^T H_i^T \\ &\quad + \langle \vec{u}_i, \vec{u}_i \rangle G_0^T F_1^T \cdots F_{i-1}^T H_i^T \\ &\quad + \langle \vec{u}_i, \vec{u}_i \rangle G_{i-1}^T H_i^T \\ &\quad + \langle \vec{u}_i, \vec{v}_i \rangle\end{aligned}\quad (13)$$

Combining (13) with the information of (2), we have that

$$\begin{aligned}\langle \vec{u}_i, \vec{y}_i \rangle &= Q_i G_{i-1}^T H_i^T + S_i \vec{s}_{ij} \\ &= Q_i G_{i-1}^T H_i^T + S_i\end{aligned}\quad (14)$$

□

Question assigned to the following page: [2](#)

#2 | A different State-Space Model

$$\begin{cases} x_{i+1} = Fx_i + Bu_i \\ y_i = H_0 x_i + H_1 x_{i-1} + v_i \end{cases}, i \geq 0$$

U_i, V_i satisfy: $\begin{pmatrix} U_i \\ V_i \end{pmatrix}, \begin{pmatrix} U_i \\ V_i \end{pmatrix}^T = \begin{pmatrix} Q_i S_{ij} & S_i S_{ij} \\ S_i^T S_{ij} & R_i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

↑ i.e. 0-mean

define $\beta_0 = \begin{bmatrix} x_{i-1} \\ x_0 \end{bmatrix}$ where $\langle \beta_0, \beta_0 \rangle = \pi_0$

$$\langle \beta_0, U_i \rangle = 0$$

$$\langle \beta_0, V_i \rangle = 0, i \geq 0$$

Find recursions for computing the innovations. \rightarrow

See page 2-8 & 2-11
for a summary of results

1) Get properties of statespace model:

First, re-write in terms of β : (let $U_i = \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}$)

$$\begin{cases} \beta_{i+1} = \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} Fx_{i-1} + Bu_{i-1} \\ Fx_i + Bu_i \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x_{i-1} \\ x_i \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix} \\ = F_2 \beta_i + B_2 U_i \\ y_i = \underbrace{\begin{bmatrix} H_0 & \cancel{H_1} \\ \cancel{H_1} & H_0 \end{bmatrix}}_{[H_1 \quad H_0]} \begin{bmatrix} x_{i-1} \\ x_i \end{bmatrix} + v_i = H_2 \beta_i + v_i \end{cases}$$

Now,

$$\langle U_i, x_j \rangle = ?$$

$$\langle U_i, \beta_j \rangle = ?$$

$$\langle U_i, y_j \rangle = ?$$

$$\langle V_i, y_j \rangle = ?$$

Question assigned to the following page: [2](#)

$$\langle u_i, \beta_j \rangle = \langle u_i, F_2 \beta_{j-1} + b_2 u_i \rangle = \langle u_i, \beta_{j-1} \rangle F_2 + \langle u_i, u_i \rangle b_2$$

$$\text{where } \langle u_i, \beta_{j-1} \rangle = \langle u_i, \begin{bmatrix} x_{j-2} \\ x_{j-1} \end{bmatrix} \rangle = \begin{bmatrix} \langle u_i, x_{j-2} \rangle \\ \langle u_i, x_{j-1} \rangle \end{bmatrix}$$

Note:

$$\beta_i = F_2 \beta_{i-1} + b_2 u_{i-1} = F_2 (F_2 \beta_{i-2} + b_2 u_{i-2}) + b_2 u_{i-1}$$

$$= F_2^2 (F_2 (\dots (F_2 (\underbrace{F_2 \beta_0 + b_2 u_0}_{\beta_1} + b_2 u_1) + b_2 u_2) + \dots) + b_2 u_{i-2}) + b_2 u_{i-1}$$

$$= F_2^{i-1} \underbrace{F_2^2 \dots F_2^2}_{\beta_2} \beta_0 + F_2^{i-1} \underbrace{F_2^2 \dots F_2^2}_{\beta_2} b_2 u_0 + F_2^{i-1} \underbrace{F_2^2 \dots F_2^2}_{\beta_2} b_2 u_1 + \dots + F_2^{i-1} \underbrace{F_2^2 \dots F_2^2}_{\beta_2} b_2 u_{i-2} + b_2 u_{i-1}$$

$$= F_2^{i-1} F_2^2 \dots F_2^2 F_0 \beta_0 + F_2^{i-1} F_2^2 \dots F_2^2 b_2 u_0 + F_2^{i-1} F_2^2 \dots F_2^2 b_2 u_1$$

$$+ F_2^{i-1} F_2^2 \dots F_2^2 b_2 u_2 + \dots + F_2^{i-1} F_2^2 \dots F_2^2 b_2 u_{i-2} + b_2 u_{i-1}$$

$$\Rightarrow \langle u_i, x_{j-2} \rangle = 0 \quad i \geq j-3$$

Similarly, then:

$$x_i = F_2^{i-1} F_2^2 \dots F_2^2 F_0 x_0 + F_2^{i-1} F_2^2 \dots F_2^2 b_2 u_0 + F_2^{i-1} F_2^2 \dots F_2^2 b_2 u_1$$

$$+ F_2^{i-1} F_2^2 \dots F_2^2 b_2 u_2 + \dots + F_2^{i-1} F_2^2 \dots F_2^2 b_2 u_{i-2} + b_2 u_{i-1}$$

$$x_{i-1} = " \quad x_{-1} + " \quad \underbrace{b_2 u_{-1}}_{\stackrel{\text{D.N.E.}}{0}} + " \quad u_0$$

$$+ " \quad b_2 u_1 + \dots + F_2^{i-1} F_2^2 \dots F_2^2 b_2 u_{i-3} + b_2 u_{i-2}$$

$$\Rightarrow \langle u_i, x_j \rangle = 0, \quad i \geq j$$

$$\Rightarrow \langle u_i, \underbrace{\begin{bmatrix} x_{j-1} \\ x_j \end{bmatrix}}_{\beta_j} \rangle = 0, \quad i \geq j$$

β_j

clearly written: $\boxed{\langle u_i, \beta_j \rangle = 0, \quad i \geq j}$

Question assigned to the following page: [2](#)

$$\langle v_i, \beta_j \rangle = \langle v_i, \begin{bmatrix} x_{j-1} \\ x_j \end{bmatrix} \rangle = 0 \quad \text{if } i \geq j$$

b/c $x_{j-1} = f(x_1, u_1, u_2, \dots, u_{j-2})$
 $x_j = f(x_0, u_0, u_1, \dots, u_{j-1})$

$$\Rightarrow \beta_j = f(x_1, x_0, u_0, u_1, \dots, u_{j-1})$$

$$\begin{aligned} \langle u_i, y_j \rangle &= \langle u_i, H_2 \beta_j + v_j \rangle \\ &= \langle u_i, \beta_j \rangle^0 H_2 + \langle u_i, v_j \rangle = \boxed{\begin{array}{l} \text{if } i > j \\ S_i, S_j = \langle u_i, y_j \rangle \end{array}} \\ \langle v_i, y_j \rangle &= \langle v_i, H_2 \beta_j + v_j \rangle \\ &= \langle v_i, \beta_j \rangle^+ H_2 + \langle v_i, v_j \rangle = \boxed{\begin{array}{l} \text{if } i > j \\ v_i, v_j = 0 \\ \text{if } i = j \\ S_i = \langle v_i, y_j \rangle \end{array}} \end{aligned}$$

Step 2; Recursions for innovations

$$e_i = y_i - \hat{y}_i|_{i-1} = y_i - \sum_{j=0}^{i-1} \langle y_i, e_j \rangle \|e_j\|^{-2} e_j$$

$$y_i = H_2 \beta_i + v_i \Rightarrow e_i = y_i - \sum_{j=0}^{i-1} \langle H_2 \beta_i + v_i, e_j \rangle \|e_j\|^{-2} e_j$$

$$\begin{aligned} \text{for } i > j: \quad \langle H_2 \beta_i + v_i, e_j \rangle &= H_2 \langle \beta_i, e_j \rangle + \langle v_i, e_j \rangle^0 \\ &= H_2 \langle \beta_i, e_j \rangle \end{aligned}$$

$\wedge e_j \in \text{span}(y_0, y_1, \dots, y_j)$

$$\text{thus, } e_i = y_i - \sum_{j=0}^{i-1} \langle y_i, e_j \rangle \|e_j\|^{-2} e_j$$

$$\Rightarrow \boxed{e_i = y_i - H_2 \sum_{j=0}^{i-1} \langle \beta_i, e_j \rangle \|e_j\|^{-2} e_j}$$

$$\boxed{= y_i - H_2 \hat{\beta}_i|_{i-1}}$$



Question assigned to the following page: [2](#)

$$\hat{\beta}_{i+1|i} = \sum_{j=0}^i \langle \beta_{i+1}, e_j \rangle \|e_j\|^{-2} e_j = \underbrace{\langle \beta_{i+1}, e_i \rangle \|e_i\|^{-2} e_i}_{k_{p,i}} + \sum_{j=0}^{i-1} \langle \beta_{i+1}, e_j \rangle \|e_j\|^{-2} e_j$$

note: $\beta_{i+1} = F_2 \beta_i + G_2 \mu_i$

$$\Rightarrow \hat{\beta}_{i+1|i} = k_{p,i} e_i + \sum_{j=0}^{i-1} \langle F_2 \beta_i + G_2 \mu_i, e_j \rangle \|e_j\|^{-2} e_j$$

where for $i > j$: $\langle F_2 \beta_i + G_2 \mu_i, e_j \rangle = F_2 \langle \beta_i, e_j \rangle + G_2 \langle \mu_i, e_j \rangle$

note $e_j \in \text{Span}(y_0, y_1, \dots, y_j) \Rightarrow \langle \mu_i, e_j \rangle \geq 0 = \langle \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}, e_j \rangle = \begin{cases} 0 & i < j \\ s_{i-1} & i = j \\ 0 & i > j \end{cases}$

thus $\langle F_2 \beta_i + G_2 \mu_i, e_j \rangle = F_2 \langle \beta_i, e_j \rangle$, $i > j$

and we have that $= F_2 \langle \beta_i, e_j \rangle + G_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}$, $i-1=j$

$$\hat{\beta}_{i+1|i} = k_{p,i} e_i + \sum_{j=0}^{i-1} F_2 \langle \beta_i, e_j \rangle \|e_j\|^{-2} e_j + \begin{bmatrix} G_2 s_{i-1} \\ 0 \end{bmatrix} \quad \text{since } i > j \text{ for the whole summation}$$

$$\Rightarrow \hat{\beta}_{i+1|i} = k_{p,i} e_i + F_2 \hat{\beta}_{i|i-1} + \begin{bmatrix} G_2 s_{i-1} \\ 0 \end{bmatrix}$$

Recursion for $k_{p,i}$...

1) First find $\|e_i\|^{-2}$: note $e_i = y_i - H_2 \hat{\beta}_{i|i-1} = H_2 \beta_i + v_i - H_2 \hat{\beta}_{i|i-1}$

let $\tilde{\beta}_i = \beta_i - \hat{\beta}_{i|i-1}$, $\tilde{\beta}_i \triangleq \hat{\beta}_{i|i-1}$, $P_i \triangleq \langle \tilde{\beta}_i, \tilde{\beta}_i \rangle$

then, $|e_i = H_2 \tilde{\beta}_i + v_i|$

so that $\|e_i\|^{-2} = \langle e_i, e_i \rangle = \langle H_2 \tilde{\beta}_i + v_i, H_2 \tilde{\beta}_i + v_i \rangle$

$$= H_2 P_i H_2^T + H_2 \tilde{\beta}_i^T v_i + \langle v_i, \tilde{\beta}_i \rangle H_2^T + R_i$$

Now, $\langle \tilde{\beta}_i, v_i \rangle = \langle \beta_i - \hat{\beta}_{i|i-1}, v_i \rangle = \langle \beta_i, v_i \rangle - \langle \hat{\beta}_{i|i-1}, v_i \rangle$

note: $\hat{\beta}_i = \sum_{j=0}^{i-1} \langle \beta_i, e_j \rangle \|e_j\|^{-2} e_j$, which is the projection of β_i onto the $\text{Span}(y_0, y_1, \dots, y_j)$

and since $\langle v_i, y_j \rangle = 0$, $i > j$ then $\langle \hat{\beta}_i, v_i \rangle = 0$

therefore we have that $|\langle \tilde{\beta}_i, v_i \rangle = \langle \beta_i, v_i \rangle - \langle \hat{\beta}_i, v_i \rangle = 0|$

→

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also, $\langle \tilde{\beta}_i, v_i \rangle = (\langle v_i, \tilde{\beta}_i \rangle)^*$ (in C^n)

$$\text{thus } 0 = (\langle v_i, \tilde{\beta}_i \rangle)^* \Rightarrow \underline{\langle v_i, \tilde{\beta}_i \rangle = 0}$$

$$\text{therefore, } \|e_i\|^{-2} = H_2 P_i H_2^+ + H_2 \langle \tilde{\beta}_i, v_i \rangle^0 + \langle v_i, \tilde{\beta}_i \rangle^0 H_2^+ + R_i$$

$$\Rightarrow \boxed{\|e_i\|^{-2} = H_2 P_i H_2^+ + R_i}$$

$$2) \text{ Recall } K_{p,i} = \langle \tilde{\beta}_{i+1}, e_i \rangle \|e_i\|^{-2}$$

$$\begin{aligned} \text{so now let's inspect } \langle \tilde{\beta}_{i+1}, e_i \rangle &= \langle F_2 \tilde{\beta}_i + b_2 u_i, e_i \rangle \\ &= F_2 \langle \tilde{\beta}_i, e_i \rangle + b_2 \langle u_i, e_i \rangle \end{aligned}$$

A)

$$\begin{aligned} \langle \tilde{\beta}_i, e_i \rangle &= \langle \tilde{\beta}_i, H_2 \tilde{\beta}_i + v_i \rangle \\ &= \langle \tilde{\beta}_i, \tilde{\beta}_i \rangle^0 H_2^+ + \langle \tilde{\beta}_i, v_i \rangle^0 \quad \text{per earlier (pg 2-3)} \\ &= \langle \tilde{\beta}_i - \tilde{\beta}_i + \tilde{\beta}_i, \tilde{\beta}_i \rangle^0 H_2^+ = \langle \tilde{\beta}_i + \tilde{\beta}_i, \tilde{\beta}_i \rangle^0 H_2^+ \\ &= P_i H_2^+ + \langle \tilde{\beta}_i, \tilde{\beta}_i \rangle^0 H_2^+ \\ &\quad \checkmark \end{aligned}$$

$$\text{thus } \boxed{\langle \tilde{\beta}_i, e_i \rangle = P_i H_2^+} \quad \text{prediction is } \perp \text{ to error}$$

$$B) \langle u_i, e_i \rangle = \begin{bmatrix} \langle u_{i-1}, e_i \rangle \\ \langle u_i, e_i \rangle \end{bmatrix}$$

$$B-1) \langle u_{i-1}, H_2 \tilde{\beta}_i + v_i \rangle = \langle u_{i-1}, \tilde{\beta}_i \rangle^0 H_2^+ + \langle u_{i-1}, v_i \rangle^0$$

$$= \langle u_{i-1}, \tilde{\beta}_i - \tilde{\beta}_i \rangle^0 H_2^+ = \boxed{\langle u_{i-1}, \tilde{\beta}_i \rangle^0 H_2^+ \langle u_{i-1}, \tilde{\beta}_i \rangle^0 H_2^+ = \langle u_{i-1}, e_i \rangle}$$

$$B-1.1) \langle u_{i-1}, \tilde{\beta}_i \rangle = \langle u_{i-1}, F_2 \tilde{\beta}_{i-1} + b_2 u_{i-1} \rangle = \langle u_{i-1}, \tilde{\beta}_{i-1} \rangle^0 F_2^+ + \langle u_{i-1}, \begin{bmatrix} u_{i-2} \\ u_{i-1} \end{bmatrix} \rangle^0 b_2^+$$

$$= \begin{bmatrix} \langle u_{i-1}, u_{i-2} \rangle \\ \langle u_{i-1}, u_{i-1} \rangle \end{bmatrix}^0 b_2^+ = \begin{bmatrix} 0 \\ s_{i-1}^+ \end{bmatrix}^0 b_2^+ = \boxed{\begin{bmatrix} 0 & s_{i-1}^+ \end{bmatrix} b_2^+ = \langle u_{i-1}, \tilde{\beta}_i \rangle}$$

$$B-1.2) \langle u_{i-1}, \tilde{\beta}_i \rangle = \langle u_{i-1}, \underbrace{\sum_{j=0}^{i-1} \langle \tilde{\beta}_i, e_j \rangle \|e_j\|^{-2} e_j}_{\text{projection of } \tilde{\beta}_i \text{ onto } \text{Span}(y_0, y_1, \dots, y_{i-1})} \rangle$$

$$= \langle u_{i-1}, \langle \tilde{\beta}_i, e_i \rangle \|e_i\|^{-2} e_i \rangle + \sum_{j=0}^{i-2} \langle \tilde{\beta}_i, e_j \rangle \|e_j\|^{-2} e_j \rangle \rightarrow$$

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B-1.2 cont'd)

$$\langle u_i, \hat{g}_i \rangle = \langle u_{i-1}, \langle \beta_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1} + \underbrace{\hat{\beta}_{i|i-2}}_{\text{projection of } g_i \text{ onto } \text{Span}(y_0, y_1, \dots, y_{i-2})} \rangle$$

note: $u_{i-1} \perp \text{Span}(y_0, y_1, \dots, y_{i-2})$

$$\Rightarrow \langle u_{i-1}, \hat{g}_i \rangle = \langle u_{i-1}, \langle \beta_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1} \rangle + \underbrace{\langle u_{i-1}, \hat{\beta}_{i|i-2} \rangle}_0$$

$$\text{where } \langle \beta_i, e_{i-1} \rangle = \langle \beta_i, H_2 \hat{\beta}_{i-1} + v_{i-1} \rangle$$

$$= \langle \beta_i, H_2 \hat{\beta}_{i-1} + v_{i-1} \rangle \xrightarrow{H_2 \hat{\beta}_{i-1}} \langle \beta_i, v_{i-1} \rangle$$

$$= \langle \beta_i, \hat{\beta}_{i-1} \rangle H_2 + \langle \beta_i, v_{i-1} \rangle$$

$$\text{and } \langle \beta_i, v_{i-1} \rangle = \langle F_2 \beta_{i-1} + G_2 M_{i-1}, v_{i-1} \rangle$$

$$= F_2 \langle \beta_{i-1}, v_{i-1} \rangle + G_2 \langle M_{i-1}, v_{i-1} \rangle$$

$$\Rightarrow \langle \beta_i, v_{i-1} \rangle = G_2 \langle \begin{bmatrix} u_{i-2} \\ u_{i-1} \end{bmatrix}, v_{i-1} \rangle = G_2 \begin{bmatrix} 0 \\ s_{i-1} \end{bmatrix}$$

$$\text{and } \langle \beta_i, \hat{\beta}_{i-1} \rangle H_2 =$$

$$= \langle \beta_i, \beta_{i-1} \rangle H_2 - \langle \beta_i, \hat{\beta}_{i-1} \rangle H_2$$



=

$$\langle u_i, e_i \rangle = \langle u_i, y_i - \hat{g}_i \rangle = \langle u_i, y_i \rangle - \langle u_i, \hat{g}_i \rangle \xrightarrow{\text{in } \text{Span}(y_0, y_1, \dots, y_{i-1})} = s_i$$

$$\langle u_{i-1}, e_{i-1} \rangle = \langle u_{i-1}, y_i - H_2 \hat{\beta}_i \rangle \quad \hat{\beta}_{i|i-1} = F_2 \hat{\beta}_{i-1}$$

$$\langle u_{i-1}, \hat{\beta}_{i|i-1} \rangle = ? \quad \hat{\beta}_{i|i-1} = \langle \beta_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1} + \hat{\beta}_{i-1|i-2}$$

$$\Rightarrow \langle u_{i-1}, \langle \beta_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1} \rangle \quad \text{note: } \langle \beta_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} = K_{p,i-1}$$

projection of β_i onto e_{i-1} =

$$e_{i-1} = y_{i-1} - \hat{g}_{i-1|i-2}$$

$$\Rightarrow \langle u_{i-1}, e_{i-1} \rangle K_{p,i-1} = s_{i-1} K_{p,i-1}$$

Question assigned to the following page: [2](#)

$B - 1.2 \text{ unit } d$)

$$\langle u_{i-1}, \hat{g}_i \rangle = \langle u_{i-1}, \underbrace{\langle g_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1}}_{k_{p,i-1}} + \hat{g}_{i-1} \rangle$$

$$= \langle u_{i-1}, \langle g_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1} \rangle + \langle u_{i-1}, \hat{g}_{i-1} \rangle$$

$\hookrightarrow \text{span}(y_0, y_1, \dots, y_{i-2}) \perp u_{i-1}$

where $\langle g_i, e_{i-1} \rangle = \langle g_i, y_{i-1} - \hat{g}_{i-1} \rangle$

$$= \langle g_i, b_2 \hat{g}_{i-1} + \cancel{b_2 \hat{g}_{i-1}} + H_2 \hat{g}_{i-1} \rangle = \langle g_i, v_{i-1} \rangle + \langle g_i, \hat{g}_{i-1} \rangle H_2^+$$

$$\langle g_i, v_{i-1} \rangle = b_2 \langle \begin{bmatrix} u_{i-2} \\ u_{i-1} \end{bmatrix}, v_{i-1} \rangle = b_2 \begin{bmatrix} 0 \\ s_{i-1} \end{bmatrix}$$

$$\langle g_i, \hat{g}_{i-1} \rangle H_2^+ = \langle F_2 g_{i-1} + b_2 M_{i-1} \hat{g}_{i-1}, \hat{g}_{i-1} \rangle H_2^+$$

$$= \langle F_2 g_{i-1}, \hat{g}_{i-1} \rangle + b_2 \langle \begin{bmatrix} u_{i-2} \\ u_{i-1} \end{bmatrix}, \hat{g}_{i-1} \rangle$$

$$\langle g_i, e_{i-1} \rangle = \cancel{F_2 \langle g_{i-1}, e_{i-1} \rangle} + b_2 \langle u_{i-1}, e_{i-1} \rangle$$

$$= F_2 P_{i-1} H_2^+ + b_2 \langle M_{i-1}, e_{i-1} \rangle$$

again: $\langle u_{i-1}, \hat{g}_i \rangle = \langle u_{i-1}, \underbrace{\langle g_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1}}_{k_{p,i-1}} + \langle u_{i-1}, \hat{g}_{i-1} \rangle$

$U_{i-1} \perp \hat{g}_{i-1}$

$$= \langle u_{i-1}, e_{i-1} \rangle k_{p,i-1}^+$$

$$= \cancel{s_{i-1} \langle u_{i-1}, y_{i-1} - \hat{g}_{i-1} \rangle} k_{p,i-1}^+, \quad \hat{g}_{i-1} \in \text{span}(y_0, y_1, \dots, y_{i-2})$$

$U_{i-1} \perp \hat{g}_{i-1}$

$$= \langle u_{i-1}, y_{i-1} \rangle k_{p,i-1}^+$$

$$\boxed{\langle u_{i-1}, \hat{g}_i \rangle = s_{i-1} k_{p,i-1}^+}$$



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Thus,

$$\text{B-1) } \begin{aligned} \langle u_{i+1}, e_i \rangle &= \langle u_{i+1}, \tilde{\beta}_i \rangle H_2^+ - \langle u_{i+1}, \tilde{\beta}_i \rangle H_2^+ \\ &= [0 \ s_i^+] G_2^+ H_2^+ - s_{i+1} k_{p,i+1}^+ H_2^+ \end{aligned}$$

$$\text{B-2) } \begin{aligned} \langle u_i, e_i \rangle &= \langle u_i, y_i - \tilde{y}_i \rangle = \langle u_i, y_i \rangle - \langle u_i, \tilde{y}_i \rangle^0 \\ &\Rightarrow \langle u_i, e_i \rangle = s_i \end{aligned}$$

$$\Rightarrow \text{B) } \langle u_i, e_i \rangle = \begin{bmatrix} \langle u_{i+1}, e_i \rangle \\ \langle u_i, e_i \rangle \end{bmatrix} = \begin{bmatrix} [0 \ s_i^+] G_2^+ H_2^+ - s_{i+1} k_{p,i+1}^+ H_2^+ \\ s_i \end{bmatrix}$$

S_o + Lat:

$$\begin{aligned} K_{p,i} &= \langle \tilde{\beta}_{i+1}, e_i \rangle \|e_i\|^{-2} \\ &= (F_2 \langle \tilde{\beta}_i, e_i \rangle + G_2 \langle u_i, e_i \rangle) \|e_i\|^{-2} \\ \Rightarrow K_{p,i} &= \left(F_2 P_i H_2^+ + G_2 \begin{bmatrix} [0 \ s_i^+] G_2^+ H_2^+ - s_{i+1} k_{p,i+1}^+ H_2^+ \\ s_i \end{bmatrix} \right) \|e_i\|^{-2} \end{aligned}$$

Recall: (to summarize)

$$\|e_i\|^{-2} = H_2 P_i H_2^+ + R_i$$

$$\tilde{\beta}_{i+1|i} = K_{p,i} e_i + F_2 \tilde{\beta}_{i|i-1} + G_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}$$

$$e_i = H_2 \tilde{\beta}_i + v_i$$

$$K_{p,i} = (\text{as shown above})$$

Now to solve for P_{i+1} \longrightarrow

(Final result on pg. 2-11)

Question assigned to the following page: [2](#)

P_{i+1} is a function of $\tilde{\beta}_{i+1}$, so let's look at $\tilde{\beta}_{i+1}$:

$$\star \tilde{\beta}_{i+1} = \beta_{i+1} - \hat{\beta}_{i+1} = F_2 \beta_i + b_2 \mu_i - (F_2 \tilde{\beta}_{i+1} + k_{p,i} \ell_i + b_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix})$$

$$= F_2 \beta_i + b_2 \mu_i - k_{p,i} (b_2 \tilde{\beta}_i + v_i) - b_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}$$

$$= (F_2 - k_{p,i} b_2) \tilde{\beta}_i + b_2 \mu_i - k_{p,i} v_i - b_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}$$

$$\text{let } (F_2 - k_{p,i} b_2) = F_{p,i}$$

$$\Rightarrow \tilde{\beta}_{i+1} = F_{p,i} \tilde{\beta}_i + b_2 (\mu_i - \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}) - k_{p,i} v_i$$

~~So that~~

$$\text{So that: } P_{i+1} = \langle \tilde{\beta}_{i+1}, \tilde{\beta}_{i+1} \rangle$$

$$= F_{p,i} P_i F_{p,i}^T + b_2 \langle \mu_i - \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}, \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix} \rangle b_2^T + k_{p,i} R_i k_{p,i}^T$$

$$\Rightarrow \tilde{\beta}_{i+1} = F_{p,i} \tilde{\beta}_i + [b_2 - k_{p,i}] \begin{bmatrix} \mu_i \\ v_i \end{bmatrix} - b_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}$$

$$\Rightarrow P_{i+1} = \langle \tilde{\beta}_{i+1}, \tilde{\beta}_{i+1} \rangle \quad (\text{assume } \langle \beta_i, \beta_i \rangle = 0, \text{i.e. 0-mean})$$

$$= F_{p,i} P_i F_{p,i}^T + [b_2 - k_{p,i}] \begin{bmatrix} \langle \mu_i, \mu_i \rangle & \langle \mu_i, v_i \rangle \\ \langle v_i, \mu_i \rangle & \langle v_i, v_i \rangle \end{bmatrix} \begin{bmatrix} b_2^T \\ -k_{p,i}^T \end{bmatrix} + b_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix} [s_{i-1}^T \ 0] b_2^T$$

$$+ F_{p,i} \begin{bmatrix} \langle \tilde{\beta}_i, \mu_i \rangle & \langle \tilde{\beta}_i, v_i \rangle \\ \langle \tilde{\beta}_i, \mu_i \rangle & \langle \tilde{\beta}_i, v_i \rangle \end{bmatrix} \begin{bmatrix} b_2^T \\ -k_{p,i}^T \end{bmatrix} + F_{p,i} \dots \text{and so on.} \rightarrow$$

$$- F_{p,i} \underbrace{\langle \tilde{\beta}_i, I \rangle}_{\begin{bmatrix} s_{i-1}^T & 0 \end{bmatrix}} b_2^T - b_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix} \underbrace{\langle I, \tilde{\beta}_i \rangle}_{F_{p,i}^T} b_2^T$$

Question assigned to the following page: [2](#)

$$+ [G_2 - k_{p,i}] \begin{bmatrix} \langle u_i, \tilde{\beta}_i \rangle \\ \langle v_i, \tilde{\beta}_i \rangle \end{bmatrix} F_{p,i}^+ - [G_2 - k_{p,i}] \begin{bmatrix} \langle u_i, \tilde{\beta}_i \rangle \\ \cancel{\langle v_i, \tilde{\beta}_i \rangle} \end{bmatrix} \begin{bmatrix} s_{i-1}^+ & 0 \end{bmatrix} G_2^+$$

$$- G_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix} \begin{bmatrix} \cancel{\langle v_i, u_i \rangle}^0 & \cancel{\langle v_i, v_i \rangle}^0 \end{bmatrix} \begin{bmatrix} G_2^+ \\ -k_{p,i}^+ \end{bmatrix}$$

$$\Rightarrow P_{i+1} = F_{p,i} P_i F_{p,i}^+ + [G_2 - k_{p,i}] \begin{bmatrix} \langle u_i, u_i \rangle & \langle u_i, v_i \rangle \\ \langle v_i, u_i \rangle & \langle v_i, v_i \rangle \end{bmatrix} \begin{bmatrix} G_2^+ \\ -k_{p,i}^+ \end{bmatrix}$$

$$+ G_2 \begin{bmatrix} s_{i-1}^+ & 0 \end{bmatrix} G_2^+ + F_{p,i} \begin{bmatrix} \langle \tilde{\beta}_i, u_i \rangle & \langle \tilde{\beta}_i, v_i \rangle \end{bmatrix} \begin{bmatrix} G_2^+ \\ -k_{p,i}^+ \end{bmatrix}$$

$$+ [G_2 - k_{p,i}] \begin{bmatrix} \langle u_i, \tilde{\beta}_i \rangle \\ \langle v_i, \tilde{\beta}_i \rangle \end{bmatrix} F_{p,i}^+$$

Recall: $\langle v_i, \tilde{\beta}_i \rangle = \langle \hat{\beta}_i, v_i \rangle^* = 0$

$$\langle u_i, \tilde{\beta}_i \rangle = \langle u_i, \beta_i \rangle - \langle u_i, \hat{\beta}_i \rangle = \langle \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}, \beta_i \rangle - \langle \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}, \hat{\beta}_i \rangle$$

$$= \begin{bmatrix} 0 & s_{i-1}^+ \\ 0 & 0 \end{bmatrix} G_2^+ - \begin{bmatrix} s_{i-1} k_{p,i-1}^+ \\ 0 \end{bmatrix} = \begin{bmatrix} [0 & s_{i-1}^+] G_2^+ - s_{i-1} k_{p,i-1}^+ \\ 0 \end{bmatrix}$$

$$\langle \hat{\beta}_i, u_i \rangle = \langle \hat{\beta}_i, \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix} \rangle - \langle \hat{\beta}_i, \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix} \rangle = [\langle \hat{\beta}_i, u_{i-1} \rangle \quad 0] - [\langle \hat{\beta}_i, u_i \rangle \quad 0]$$

where $\langle \hat{\beta}_i, u_{i-1} \rangle = \langle F_3 \hat{\beta}_{i-1}^0 + G_2 \begin{bmatrix} u_{i-2} \\ u_{i-1} \end{bmatrix}, u_{i-1} \rangle = G_2 \begin{bmatrix} 0 \\ s_i \end{bmatrix}$

$$\langle \hat{\beta}_i, u_i \rangle = \underbrace{\langle \hat{\beta}_i, e_{i-1} \rangle}_{k_{p,i-1}} \|e_{i-1}\|^2 + \sum_{j=0}^{i-2} \underbrace{\langle \hat{\beta}_i, e_j \rangle}_{\cancel{\langle \hat{\beta}_i, e_j \rangle}} \cancel{\|e_j\|^2} e_j^T, u_{i-1} \rangle$$

$$= k_{p,i-1} \langle e_{i-1}, u_{i-1} \rangle = k_{p,i-1} \langle y_{i-1} - \hat{y}_{i-1}^0, u_{i-1} \rangle = k_{p,i-1} s_{i-1}^+$$

Question assigned to the following page: [2](#)

$$\Rightarrow \langle \tilde{s}_i, M_i \rangle = \left[\begin{matrix} b_2 \begin{bmatrix} 0 \\ s_i \end{bmatrix} - k_{p,i-1} s_{i-1}^+ & 0 \end{matrix} \right]$$

$$\langle M_i, M_i \rangle = \left[\begin{matrix} u_{i-1} \\ u_i \end{matrix} \right], \left[\begin{matrix} u_{i-1} \\ u_i \end{matrix} \right] = \begin{bmatrix} \alpha_{i-1} & 0 \\ 0 & \alpha_i \end{bmatrix}$$

$$\langle M_i, v_i \rangle = \begin{bmatrix} \langle u_{i-1}, v_i \rangle \\ \langle u_i, v_i \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ s_i \end{bmatrix}$$

$$\langle V_i, M_i \rangle = [\langle v_i, u_{i-1} \rangle \quad \langle v_i, u_i \rangle] = [0 \quad s_i^+]$$

Therefore:

$$\begin{aligned} P_{i+1} &= F_{p,i} P_i F_{p,i}^+ + [b_2 \quad -k_{p,i}] \begin{bmatrix} \alpha_{i-1} & 0 \\ 0 & \alpha_i \end{bmatrix} \begin{bmatrix} 0 \\ s_i \end{bmatrix} \begin{bmatrix} b_2^+ \\ k_{p,i}^+ \end{bmatrix} \\ &\quad + b_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix} [s_{i-1}^+ \quad 0] b_2^+ + F_{p,i} \left[\begin{bmatrix} b_2 \begin{bmatrix} 0 \\ s_i^+ \end{bmatrix} - k_{p,i-1} s_{i-1}^+ & 0 \end{bmatrix} \right. \\ &\quad \left. \begin{bmatrix} b_2^+ \\ -k_{p,i} \end{bmatrix} \right] \\ &\quad + [b_2 \quad -k_{p,i}] \left[\begin{bmatrix} [0 \ s_{i-1}^+] b_2^+ - s_{i-1} k_{p,i-1}^+ \\ 0 \\ 0 \end{bmatrix} \right] F_{p,i}^+ \end{aligned}$$

Question assigned to the following page: [3](#)

#3 | A modified State Space model

$$\begin{cases} x_{i+1} = Fx_i + b_1 u_i + b_2 u_{i+1} & i \geq 0 \\ y_i = Hx_i + v_i \end{cases}$$

Conditions: $\left\langle \begin{bmatrix} x_i \\ u_i \\ v_i \end{bmatrix}, \begin{bmatrix} x_j \\ u_j \\ v_j \end{bmatrix} \right\rangle = \begin{bmatrix} \pi_0 & 0 & 0 & 0 \\ 0 & Q_i S_{ij} & S_i S_{ij} & 0 \\ 0 & S_i^T S_{ij} & R_i S_{ij} & 0 \end{bmatrix}$

Find recursive equations for $\hat{x}_{i|i-1}$ and $P_{i|i-1}$

(pg 3-3 & 3-4 For
summary of eqtns.)

so $\hat{x}_{i|i-1}$ would be in terms of y_{i-1} , $\hat{x}_{i-1|i-2}$, etc.

Step 1: get properties of SS model

(To-do: compute properties & double-check
recursion calc agree with props)

$$\begin{aligned} \langle u_i, x_i \rangle &= ? \\ \langle v_i, x_i \rangle &= ? \\ \langle u_i, y_i \rangle &= ? \\ \langle v_i, y_i \rangle &= ? \end{aligned} \quad \left. \begin{array}{l} \text{(see pages} \\ 3-5 \text{ and } 3-6 \end{array} \right)$$

DONE

let $\langle x_i, x_i \rangle = \pi_i$, $\pi_{i+1} = ?$

$\langle x_i, x_j \rangle = ?$

$\langle y_i, y_j \rangle = ?$

Step 2!

$$e_i = y_i - \hat{y}_{i|i-1} = y_i - \sum_{j=0}^{i-1} \langle Hx_i + v_i, e_j \rangle \|e_j\|^{-2} e_j \quad \text{(uncorrelated) b/c } j=0, 1, \dots, i-1$$

$= y_i - H \hat{x}_{i|i-1}$

$$\hat{x}_{i+1|i} = \cancel{\sum_{j=0}^i} \langle x_{i+1}, e_j \rangle \|e_j\|^{-2} e_j = \underbrace{\langle x_{i+1}, e_i \rangle \|e_i\|^{-2} e_i}_{K_{P,i}} + \sum_{j=0}^{i-1} \langle x_{i+1}, e_j \rangle \|e_j\|^{-2} e_j$$

$$x_{i+1} = Fx_i + b_1 u_i + b_2 u_{i+1} \quad (i>j) \quad \Rightarrow \langle x_{i+1}, e_j \rangle = F \langle x_i, e_j \rangle + b_1 \cancel{\langle u_i, e_j \rangle} + b_2 \cancel{\langle u_{i+1}, e_j \rangle} = F \langle x_i, e_j \rangle$$

$$\Rightarrow \hat{x}_{i+1|i} = K_{P,i} e_i + F \hat{x}_{i|i-1}$$

→ $\boxed{3-1}$

Question assigned to the following page: [3](#)

$$\|\epsilon_i\|^2 = ? \quad , \quad \epsilon_i = y_i - H\hat{x}_{i|i-1} = Hx_i + v_i - H\hat{x}_{i|i-1}$$

$$let \quad \tilde{x}_i = x_i - \hat{x}_{i|i-1}, \quad p_i = \langle x_i, \tilde{x}_i \rangle, \quad \hat{x}_i \triangleq \hat{x}_{i|i-1}$$

$$then \quad \epsilon_i = H\tilde{x}_i + v_i, \quad \boxed{\|\epsilon_i\|^2 = \langle \epsilon_i, \epsilon_i \rangle = H P_i H^T + H \langle \tilde{x}_i, v_i \rangle + \langle v_i, \tilde{x}_i \rangle H^T + R_i = H P_i H^T + R_i}$$

$$so; \quad K_{p,i} = \langle x_{i+1}, \epsilon_i \rangle / \|\epsilon_i\|^2 = \langle Fx_i + b_1 u_i + b_2 u_{i+1}, \epsilon_i \rangle / \|\epsilon_i\|^2$$

$$1) \quad \langle x_i, \epsilon_i \rangle = \langle x_i, H\tilde{x}_i + v_i \rangle = \langle x_i, \tilde{x}_i \rangle H^T + \cancel{\langle x_i, v_i \rangle} = \langle x_i - \hat{x}_i + \hat{x}_i, \tilde{x}_i \rangle H^T + \cancel{\langle Fx_i + b_1 u_i + b_2 u_{i+1}, v_i \rangle} \\ = \langle \tilde{x}_i + \hat{x}_i, \tilde{x}_i \rangle H^T = P_i H^T + \cancel{\langle \hat{x}_i, \tilde{x}_i \rangle H^T} + b_2 s_i \\ \checkmark \quad prediction \text{ is } \perp \text{ to error!}$$

$$\boxed{\langle x_i, \epsilon_i \rangle = P_i H^T + b_2 s_i}$$

$$2) \quad \langle u_i, \epsilon_i \rangle = \langle u_i, H\tilde{x}_i + v_i \rangle = \langle u_i, \tilde{x}_i \rangle H^T + s_i$$

$$\tilde{x}_i = x_i - \hat{x}_i = x_i - \sum_{j=0}^{i-1} \frac{\langle x_i, e_j \rangle \|e_j\|^2 e_j}{\text{projection onto Span}(y_0, \dots, y_{i-1})}$$

$$\begin{aligned} \langle u_i, y_{i+1} \rangle &= \langle u_i, Hx_{i+1} + \tilde{x}_i \rangle \\ &= \langle u_i, Fx_{i+2} + b_1 u_{i+2} + b_2 u_{i+1} \rangle H^T + \cancel{s_i} = 0 \\ &= 0 \end{aligned}$$

$$\Rightarrow \langle u_i, x_i \rangle = 0, \quad \langle u_i, \tilde{x}_i \rangle = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \boxed{\langle u_i, \tilde{x}_i \rangle = Q_i b_2^T}$$

$$\begin{aligned} &= \langle u_i, Fx_{i+1} + b_1 u_{i+1} + b_2 u_i \rangle \\ &= Q_i b_2^T \end{aligned}$$

$$\Rightarrow \langle u_i, \epsilon_i \rangle = \langle u_i, \tilde{x}_i \rangle H^T + s_i = \langle u_i, x_i - \hat{x}_i \rangle H^T + s_i \\ = (\langle u_i, x_i \rangle - \langle u_i, \tilde{x}_i \rangle) H^T + s_i$$

$$\boxed{\langle u_i, \epsilon_i \rangle = Q_i b_2^T H^T + s_i}$$

(see pg 3-7)
Check this w/ me.
Un correlated w/ meas noise



Question assigned to the following page: [3](#)

$$3) \langle u_{i+1}, e_i \rangle = \langle u_{i+1}, H\tilde{x}_i + v_i \rangle = \langle u_{i+1}, \tilde{x}_i \rangle H^+ + \langle u_{i+1}, v_i \rangle^0$$

$$\langle u_{i+1}, \tilde{x}_i \rangle = \langle u_{i+1}, x_i - \hat{x}_i \rangle = \langle u_{i+1}, x_i \rangle - \langle u_{i+1}, \hat{x}_i \rangle^0$$

$$\Rightarrow \langle u_{i+1}, x_i \rangle = \langle u_{i+1}, F\tilde{x}_{i-1}^0 + b_1 u_{i-1}^0 + b_2 u_i^0 \rangle$$

$$\Rightarrow \boxed{\langle u_{i+1}, \tilde{x}_i \rangle = 0}$$

$$\Rightarrow \boxed{\langle u_{i+1}, e_i \rangle = 0}$$

$\in \text{Span}(y_0, \dots, y_{i-1})$

$$\text{and } \langle u_{i+1}, y_j \rangle^0$$

$$= \langle u_{i+1}, Hx_j + y_j \rangle^0$$

$$= \langle u_{i+1}, F\tilde{x}_{j-1}^0 + b_1 u_{j-1}^0 + b_2 u_j^0 \rangle^0$$

$\forall i > j$

$$\text{Now, } k_{p,i} = \langle x_{i+1}, e_i \rangle / \|e_i\|^{-2}$$

$$= \langle Fx_i + b_1 u_i + b_2 u_{i+1}, e_i \rangle / \|e_i\|^{-2}$$

$$= [FP_i H^+ + G_1 Q_i G_2^+ H^+ + G_1 S_i + G_2(0)] (HP_i H^+ + R_i)$$

$$\quad \quad \quad + G_2 S_i$$

$$\boxed{k_{p,i} = (FP_i H^+ + G_1 Q_i G_2^+ H^+ + G_1 S_i) / \|e_i\|^{-2}}$$

So we have,

$$e_i = y_i - H\tilde{x}_{i-1} \quad (P_{i+1} \text{ on page 3-4})$$

$$\hat{x}_{i+1|i} = k_{p,i} e_i + F\tilde{x}_{i-1}^0 \quad + G_2 S_i$$

$$k_{p,i} = \langle x_{i+1}, e_i \rangle / \|e_i\|^{-2} = (FP_i H^+ + G_1 Q_i G_2^+ H^+ + G_1 S_i) / \|e_i\|^{-2}$$

$$\|e_i\|^{-2} = HP_i H^+ + R_i + G_2 S_i + G_2^+ H^+$$

Now to solve for P_i recursively:

P_{i+1} is a function of \tilde{x}_{i+1} , so let's look @ \tilde{x}_{i+1} :

$$\tilde{x}_{i+1} = x_{i+1} - \hat{x}_{i+1} = Fx_i + b_1 u_i + b_2 u_{i+1} - (F\tilde{x}_{i-1}^0 + b_1 \hat{u}_{i+1} + b_2 \hat{u}_i)$$

$$(F\tilde{x}_{i-1}^0 + k_{p,i} e_i)$$

→

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$$\rightarrow \hat{x}_{i+1} = F_{x_i} + b_1 u_i + b_2 u_{i+1} - (F \hat{x}_{i+1} + k_{p,i} e_i) \\ - k_{p,i} (y_i - H \hat{x}_i) - F \hat{x}_i \\ - k_{p,i} (H \hat{x}_i + v_i - H \hat{x}_i) - F \hat{x}_i$$

$$= F \hat{x}_i - k_{p,i} H \hat{x}_i + b_1 u_i + b_2 u_{i+1} - k_{p,i} v_i$$

$$\hat{x}_{i+1} = F_{p,i} \hat{x}_i + [b_1 \quad b_2 \quad -k_{p,i}] \begin{bmatrix} u_i \\ u_{i+1} \\ v_i \end{bmatrix}$$

$$P_{i+1} = \langle \hat{x}_{i+1}, \hat{x}_{i+1} \rangle = F_{p,i} R F_{p,i}^T + [b_1 \quad b_2 \quad -k_{p,i}] \begin{bmatrix} Q_i & 0 & S_i \\ 0 & Q_{i+1} & 0 \\ S_i^T & 0 & R_i \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ -k_{p,i} \end{bmatrix}$$

$$+ F_{p,i} \langle \hat{x}_{i+1} \begin{bmatrix} u_i \\ u_{i+1} \\ v_i \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ -k_{p,i} \end{bmatrix} \rangle + [b_1 \quad b_2 \quad -k_{p,i}] \langle \begin{bmatrix} u_i \\ u_{i+1} \\ v_i \end{bmatrix}, \hat{x}_i \rangle F_{p,i}^T$$

$$F_{p,i} \underbrace{\langle \hat{x}_i, b_1 \rangle}_{Q_i b_1^+} \underbrace{\langle \hat{x}_i, u_{i+1} \rangle}_{=0} \underbrace{\langle \hat{x}_i, v_i \rangle}_{G_2 S_i^+} \begin{bmatrix} b_1 \\ b_2 \\ -k_{p,i} \end{bmatrix}$$

$$= F_{p,i} b_1 Q_i b_1^+ b_1 - G_2 S_i K_{p,i}$$

$$\begin{bmatrix} \langle u_i, \hat{x}_i \rangle \\ \langle u_{i+1}, \hat{x}_i \rangle \\ \langle v_i, \hat{x}_i \rangle \end{bmatrix} = \begin{bmatrix} Q_i b_1^+ \\ 0 \\ Q S_i^+ b_2^+ \end{bmatrix} \Rightarrow [b_1 \quad b_2 \quad -k_{p,i}] \begin{bmatrix} Q_i b_1^+ \\ 0 \\ Q S_i^+ b_2^+ \end{bmatrix} F_{p,i}^+$$

$$= b_1 Q_i b_1^+ F_{p,i} + -k_{p,i} S_i^+ b_2^+ F_{p,i}^+$$

$$\Rightarrow P_{i+1} = F_{p,i} P_i F_{p,i}^T + [b_1 \quad b_2 \quad -k_{p,i}] \begin{bmatrix} Q_i & 0 & S_i \\ 0 & Q_{i+1} & 0 \\ S_i^T & 0 & R_i \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ -k_{p,i} \end{bmatrix}$$

$$+ \cancel{F_{p,i} Q_i b_1^+ b_1} + b_1 Q_i b_1^+ F_{p,i}^+ - k_{p,i} S_i^+ b_2^+ F_{p,i}^+$$

$$+ F_{p,i} b_1 Q_i b_1^+ - G_2 S_i K_{p,i}$$

Question assigned to the following page: [3](#)

Properties of SS model for $Q=3$: (see Lec. 13 pg 4 for example derivations)

$$\langle u_i, x_j \rangle = \langle u_i, Fx_{j-1} + b_1 u_{j-1} + b_2 u_j \rangle$$

$$x_{j-1} \in \text{Span}(x_0, u_0, u_1, \dots, u_{j-1}) \Rightarrow$$

$$x_j = Fx_{j-1} + b_1 u_{j-1} + b_2 u_j$$

$$= F(Fx_{j-2} + b_1 u_{j-2} + b_2 u_{j-1}) + b_1 u_{j-1} + b_2 u_j$$

$$= F(F(Fx_{j-3} + b_1 u_{j-3} + b_2 u_{j-2}) + b_1 u_{j-2} + b_2 u_{j-1}) + b_1 u_{j-1} + b_2 u_j$$

⋮

$$= F(F(F(\dots(F(x_0 + b_1 u_0 + b_2 u_1)$$

$$= F(F(\dots(F(F(Fx_0 + b_1 u_0 + b_2 u_1) + b_1 u_1 + b_2 u_2) + b_1 u_2 + b_2 u_3) + b_1 u_3 + b_2 u_4)$$

$$\underbrace{\quad\quad\quad}_{b_1} \quad \underbrace{x_1}_{x_2} \quad \underbrace{\quad\quad\quad}_{b_2} \quad \underbrace{\quad\quad\quad}_{b_1} \quad \underbrace{\quad\quad\quad}_{b_2} \quad \dots \quad \underbrace{\quad\quad\quad}_{b_1} \quad \underbrace{\quad\quad\quad}_{b_2}$$

$$+ \dots) + b_1 u_{j-2} + b_2 u_{j-1}) \\ + b_1 u_{j-1} + b_2 u_j$$

$$= F_{j-1} F_{j-2} \dots F_2 F_1 F_0(x_0) + F_{j-1} \dots F_1 b_1 u_0 + F_{j-1} \dots F_1 b_2 u_1$$

$$+ F_{j-1} \dots F_2 b_1 u_1 + F_{j-1} \dots F_2 b_2 u_2$$

$$+ F_{j-1} \dots F_3 b_1 u_2 + F_{j-1} \dots F_3 b_2 u_3$$

⋮

$$+ F_{j-1} b_1 u_{j-2} + F_{j-1} b_2 u_{j-1}$$

$$+ b_1 u_{j-1} + b_2 u_j$$

$$= F^j x_0 + F^{j-1} b_1 u_0 + F^{j-1} b_2 u_1$$

$$+ F^{j-2} b_1 u_0 + F^{j-2} b_2 u_2$$

⋮

$\rightarrow \langle u_i, x_0 \rangle = 0$ per conditions of SS model

$$\langle u_i, u_j \rangle = 0 \quad \forall i > j$$

$$+ F b_1 u_{j-2} + F b_2 u_{j-1}$$

$$\langle u_i, u_i \rangle = Q_i$$

$$+ b_1 u_{j-1} + b_2 u_j$$

→

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thus, $\begin{cases} \langle u_i, x_j \rangle = 0, i > j \\ \langle u_i, x_i \rangle = Q_i b_2^+, i = j \end{cases}$

and similarly: $\begin{cases} \langle v_i, x_j \rangle = 0, i > j \\ \langle v_i, x_i \rangle = \langle v_i, u_i \rangle b_2^+ = S_i^+ b_2^+, i = j \end{cases}$
 $\langle u_i, y_j \rangle = ?$

$\Rightarrow \langle u_i, y_j \rangle = \langle u_i, Hx_j + v_j \rangle = \langle u_i, x_j \rangle H^+ + \langle u_i, v_j \rangle$

$\begin{cases} \text{if } i > j \Rightarrow \langle u_i, y_j \rangle = 0 H^+ + 0 = 0 \\ \text{if } i = j \Rightarrow \langle u_i, y_i \rangle = Q_i b_2^+ H^+ + S_i \end{cases}$

$\langle v_i, y_j \rangle = \langle v_i, Hx_j + v_j \rangle = \langle v_i, x_j \rangle H^+ + \langle v_i, v_j \rangle$

$\begin{cases} i > j \Rightarrow \langle v_i, y_j \rangle = 0 + 0 = 0 \\ i = j \Rightarrow \langle v_i, y_i \rangle = S_i b_2^+ H^+ + R_i \end{cases}$

Question assigned to the following page: [3](#)

$$\|e_i\|^2 = \langle H\tilde{x}_i + v_i, H\tilde{x}_i + v_i \rangle = HP_iH^+ + R_i + H\langle \tilde{x}_i, v_i \rangle + \langle v_i, \tilde{x}_i \rangle H^+$$

where $\langle \tilde{x}_i, v_i \rangle = \langle x_i - \hat{x}_i, v_i \rangle = \langle x_i, v_i \rangle - \langle \hat{x}_i, v_i \rangle^0$
 $\hat{x}_i \in \text{Span}(y_0, y_1, \dots, y_{i-1})$

$$= \langle Fx_{i-1} + b_1 u_{i-1} + b_2 u_i, v_i \rangle$$

$$= b_2 \langle u_i, v_i \rangle = \boxed{b_2 s_i = \langle \tilde{x}_i, v_i \rangle}$$

and $\langle v_i, \tilde{x}_i \rangle = \langle v_i, x_i - \hat{x}_i \rangle = \langle v_i, x_i \rangle - \langle v_i, \hat{x}_i \rangle^0$

$$= \langle v_i, Fx_{i-1} + b_1 u_{i-1} + b_2 u_i \rangle$$

$$= \langle v_i, u_i \rangle b_2^+ = \boxed{s_i^+ b_2^+ = \langle v_i, \tilde{x}_i \rangle}$$

$$\Rightarrow \|e_i\|^2 = HP_iH^+ + R_i + b_2 s_i + s_i^+ b_2^+$$

$$e_i = y_i - \hat{y}_{i|i-1} = y_i - \sum_{j=0}^{i-1} \langle y_j, e_j \rangle \|e_j\|^2 e_j$$

where $\langle y_j, e_j \rangle = \langle y_j, \underbrace{Hx_i + v_i, e_j \rangle}_{\in \text{Span}(y_0, \dots, y_{i-1})} = H\langle x_i, e_j \rangle + \langle v_i, y_j - \hat{y}_{j|i-1} \rangle$
for $i > j$:

$$\langle y_j, e_j \rangle = H\langle x_i, e_j \rangle + \langle v_i, y_j - \hat{y}_{j|i-1} \rangle^0$$

$$= H\langle x_i, e_j \rangle$$

thus, $e_i = y_i - H \sum_{j=0}^{i-1} \langle x_i, e_j \rangle \|e_j\|^2 e_j$

$$\Rightarrow \boxed{e_i = y_i - H\tilde{x}_i} \quad \checkmark$$

$$\tilde{x}_{i+1|i} = \sum_{j=0}^i \langle x_{i+1}, e_j \rangle \|e_j\|^2 e_j = \underbrace{\langle x_{i+1}, e_i \rangle \|e_i\|^2}_{k_{p,i}} e_i + \sum_{j=0}^{i-1} \langle x_{i+1}, e_j \rangle \|e_j\|^2 e_j$$

$$x_{i+1} = Hx_i + b_1 u_i + b_2 u_{i+1}$$

$$\rightarrow \langle x_{i+1}, e_j \rangle = \langle Hx_i + b_1 u_i + b_2 u_{i+1}, e_j \rangle = F\langle x_i, e_i \rangle + b_1 \langle u_i, e_i \rangle + b_2 \langle u_{i+1}, e_i \rangle^0$$

where $\langle u_i, e_i \rangle = \langle u_i, y_i - \hat{y}_{i|i-1} \rangle^0 = \langle u_i, y_i \rangle = Q_i G_2^+ H^+ + S_i \quad (i > j)$

$$\langle u_i, e_j \rangle =$$

$$= 0 \quad (i > j)$$

$$\Rightarrow \boxed{\tilde{x}_{i+1|i} = k_{p,i} e_i + F\tilde{x}_{i|i-1}} \quad \checkmark$$