# HW #5, ECE 6555

Theodore B. Wearing
Due: 12/16/22

Goal: Compare EKF w/ Particle Filter.

Scalar, non-linear model:

$$x_k = \frac{1}{2} x_{k-1} + \frac{25 x_{k-1}}{1 + x_{k-1}^2} + 8\cos(1.2(k-1)) + u_k \qquad (1)$$

$$y_k = \frac{1}{20} x_k^2 + v_k \qquad (2)$$

$\{u_k\}$, $\{v_k\}$ are white, Gaussian noise sequences w/ unit variance.

$Q_k = 1$, $R_k = 1$

initial state: $x_0 = 0.1$, $\hat{x}_0 = 0$, $\hat{\sigma}_0 = 2$

note: both state & measurement processes are non-linear.

$\boxed{\text{Q1) Plot the state process via python (See python code for plot)}}$

Now for the Particle Filter

method:
1) draw $n$ samples from the prior $x_0^{(i)} \sim p(x_0)$ $\quad i = 1 \ldots n$
   and set $w^{(i)} = 1/n$ for $i = 1 \ldots n$
2) For each $k = 1 \ldots T$
   a) draw samples $x_k^{(i)}$ from importance distributions
   $$x_k^{(i)} \sim \pi(x_k \mid x_{0:k-1}^{(i)}\, y_{0:k}) \qquad i = 1 \ldots n$$
   b) compute new weights
   $$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(y_k \mid x_k^{(i)})\, p(x_k^{(i)} \mid x_{k-1}^{(i)})}{\pi(x_k^{(i)} \mid x_{0:k-1}^{(i)}\, y_{0:k})}$$
   and normalize
3) If the effective # of particle is too low, re-sample and reset to uniform weights, where $n_{eff} \approx \frac{1}{\sum_{j=1}^{n} (w_k^{(j)})^2}$ (i.e. the # of effective particles at a particular time-step)

But, in class we did not specify how to choose the importance distribution and how to perform the sampling.

(basically, try to get as close to real distr. as possible, and be Markovian)

Now, we choose

$$\pi(x_k^{(i)} \mid x_{0:k-1}^{(i)}\, y_{1:k}) \triangleq p(x_k^{(i)} \mid x_{k-1}^{(i)}) \qquad (3)$$

that is, we only push the current particle $x_{k-1}^{(i)}$ through a process update.

Q2] Given a set of particles $\{x_{k-1}^{(i)}\}$, show that sampling from the importance distribution $\pi(x_k \mid x_{0:k-1}\, y_{1:k})$ then reduces to computing:

$$\frac{1}{2} x_{k-1}^{(i)} + \frac{25\, x_{k-1}^{(i)}}{1+(x_{k-1}^{(i)})^2} + 8\cos(1.2(k-1)) \qquad (4)$$

and adding the realization of the 0-mean white Gaussian noise.

→

Q2 cont'd |

from (3) we have

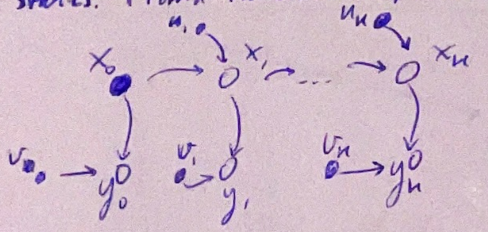
$$\pi(x_k \mid x_{0:k-1}^{(i)}, y_{1:k}) \triangleq p(x_k \mid x_{k-1}^{(i)})$$

from eqtn (1) $\longrightarrow$

$$\Rightarrow p(x_k \mid x_{k-1}^{(i)}) = p\left(\tfrac{1}{2}x_{k-1} + \frac{25 x_{k-1}}{1+x_{k-1}^2} + 8\cos(1.2(k-1)) + u_k \,\Big|\, x_{k-1}^{(i)}\right)$$

$$= \frac{1}{2}x_{k-1}^{(i)} + \frac{25 x_{k-1}^{(i)}}{1+(x_{k-1}^{(i)})^2} + 8\cos(1.2(k-1)) + p(u_k \mid x_{k-1}^{(i)})$$

Since $u_k$ is Gaussian white noise, it is independent of past and present states. From a Functional Dependence Graph perspective, we see that indeed




$u_k$ is independent of $x_{k-1}$.

Thus, $p(u_k \mid x_{k-1}^{(i)}) = p(u_k) = 0$

In summary,

$$\pi(x_k \mid x_{0:k-1}^{(i)}, y_{1:k}) \triangleq p(x_k \mid x_{k-1}^{(i)}) = \frac{1}{2}x_{k-1}^{(i)} + \frac{25 x_{k-1}^{(i)}}{1+(x_{k-1}^{(i)})^2} + 8\cos(1.2(k-1)) + p(u_k)$$

so that when we sample from the importance distribution, given $\{x_{k-1}^{(i)}\}$, then the resultant sample is eqtn (1) plus the realization of $\{u_k\}$.

---

[Q3] Explicit expression for computation of weights $w_k^{(i)}$ as a function of $w_{k-1}^{(i)}$. Make the subsequent normalization to avoid unnecessary computations. $\longrightarrow$

Q3 cont'd

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(y_k|x_k^{(i)})\,p(x_k^{(i)}|x_{k-1}^{(i)})}{\pi(x_k^{(i)}|x_{0:k-1}^{(i)}, y_{1:k})}$$

~~how to deal w/ this? is it the same as $\pi(x_k|x_{0:k}^{(i)}, y_{1:k})$? I don't think so... there's a comma there... ↳ it is the same → should be a comma always!~~

where $\pi(x_k^{(i)}|x_{0:k-1}^{(i)}\, y_{0:k})$

$= p(x_k^{(i)}|x_{k-1}^{(i)})$

thus,

$$w_{k-1}^{(i)} \frac{p(y_k|x_k^{(i)})\, p(\cancel{x_k^{(i)}|x_{k-1}^{(i)}})}{\pi(\cancel{x_k^{(i)}|x_{0:k-1}^{(i)}}\, y_{0:k})} = w_{k-1}^{(i)}\, p(y_k|x_k^{(i)})$$

$\rightarrow p(\cancel{x_k^{(i)}|x_{k-1}^{(i)}})$

and $\overset{\mathbb{E}}{p}(y_k|x_k^{(i)}) = \overset{\mathbb{E}}{p}(\frac{1}{20}x_k^2 + v_k | x_k^{(i)}) = \overset{\mathbb{E}}{p}(\frac{1}{20}x_k^2|x_k^{(i)}) + \overset{\mathbb{E}}{p}(v_k|x_k^{(i)})$

note: $\overset{\mathbb{E}}{p}(v_k|x_k^{(i)}) = \overset{\mathbb{E}}{p}(v_k)$ b/c $v_k$ is independent of $x_k^{(i)}$

and ~~thus~~ $\overset{\mathbb{E}}{p}(v_k) = 0$, thus;

$$\boxed{\overset{\mathbb{E}}{p}(y_k|x_k^{(i)}) = \overset{\mathbb{E}}{p}(\tfrac{1}{20}x_k^2|x_k^{(i)}) = \tfrac{1}{20}(x_k^{(i)})^2} \qquad (5)$$

~~such that~~

$$\cancel{w_k^{(i)} \propto w_{k-1}^{(i)}\, p(y_k|x_k^{(i)}) = \frac{w_{k-1}^{(i)}\,(x_{k-1}^{(i)})^2}{20}} \rightarrow$$

$$\cancel{\Rightarrow \boxed{w_k^{(i)} \propto \frac{1}{20} w_{k-1}^{(i)} (x_{k-1}^{(i)})^2} \qquad (6)}$$

Q3 cont'd)

each $y_n$ is Gaussian distributed around the mean, due to the Gaussian noise $\{v_n\}$. Thus,

$$y_n \sim \mathcal{N}(E[y_n], 1) \quad \text{since } v_n \sim \mathcal{N}(0,1)$$

note: $K_y = E[(y_n - \mu_y)(y_n - \mu_y)^T|_{x_n}] = E\left[\left(\cancel{\tfrac{1}{20}x_n^{(i)}} + v_n - \cancel{\tfrac{1}{20}x_n^{(i)}}\right)\left(\cancel{\tfrac{1}{20}x_n^{(i)}} + v_n - \cancel{\tfrac{1}{20}x_n^{(i)}}\right)\right]$

$$= E(v_n^2) = R_n = 1$$

Now, $p_y(y_n|x_n^{(i)}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\frac{y_n - \mu_y}{\sigma^2}\right)\right]$

$$\Rightarrow \boxed{p_{y_n}(y_n | x_n^{(i)}) = \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\left(y_n - \tfrac{1}{20}x_n^{(i)}\right)\right]} \qquad (7)$$

Such that,

$$w_n^{(i)} \propto w_{n-1}^{(i)} p(y_n|x_n^{(i)})$$

$$\Rightarrow \boxed{w_n^{(i)} \propto w_{n-1}^{(i)}\left(\frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\left(y_n - \tfrac{1}{20}x_n^{(i)}\right)\right]\right)} \qquad (8)$$

**Q4** Implement particle filter (without) the resampling step. Use 200 part.

4.1) Provide a graph w/ at least mean & var. of filter superposed to data.

4.2) Plot the $n_{eff}$ as a function of time index $k$.

$\boxed{\rightarrow \text{see python code}}$

(recall PF method on page 1, given in problem statement)

1) draw $n$ samples from the prior $x_0^{(i)} \sim p(x_0)$ $i = 1 \dots n$

and set $w^{(i)} = \frac{1}{n}$ for $i = 1 \dots n$.

↳ note: $x_0 \sim (0, 2)$ from problem statement: $[x_0 = 0.1,\ \hat{x}_0 = 0,\ \sigma_0 = 2]$

2) For each $k = 1 \dots T$

a) draw samples $x_k^{(i)}$ from importance distributions

$$x_k^{(i)} \sim \pi(x_k^{(i)} / x_{0:k-1}^{(i)}\ y_{0:k})$$

$$\Rightarrow \quad x_k^{(i)} \sim p(x_k^{(i)} / x_{k-1}^{(i)})$$

where $p(x_k^{(i)} / x_{k-1}^{(i)}) = \frac{1}{2} x_{k-1}^{(i)} + \frac{25 x_{k-1}^{(i)}}{1 + (x_{k-1}^{(i)})^2} + 8 \cos(1.2(k-1)) + p(u_k)$

and $u_k \sim (0, 1)$

b) compute new weights, and normalize:

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(y_k | x_k^{(i)})\, p(x_k^{(i)} / x_{k-1}^{(i)})}{\pi(x_k^{(i)} / x_{0:k-1}^{(i)}\ y_{0:k}^{(i)})} \Rightarrow w_k^{(i)} \propto \cancel{\frac{1}{20} w_{k-1}^{(i)} (x_{k-1}^{(i)})^2}$$

$$w_{k-1}^{(i)} \left( \frac{1}{\sqrt{2\pi}} \exp\left[ -\frac{1}{2} \left( y_k - \frac{x_{k-1}^{(i)}}{20} \right) \right] \right)$$

then, $\hat{w}_k^{(i)} = \dfrac{w_k^{(i)}}{\cancel{\| w_k^{(i)} \|} \sum_{i=1}^{n} w_k^{(i)}}$ ← per lecture, $i = 1 \dots n$, $\cancel{\| w_k^{(i)} \| = \sqrt{\sum_{i=1}^{n} (w_k^{(i)})^2}}$

want only that $\sum w_k^{(i)} = 1$, to prepare for re-sampling & to standardize weights

3) skip for this part/problem

Q4 cont'd

weighted mean: $\mu_w = \dfrac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$

biased weighted variance: $\hat{\sigma}_w^2 = \dfrac{\sum_{i=1}^{n} w_i (x_i - \mu_w)^2}{\sum_{i=1}^{n} w_i}$

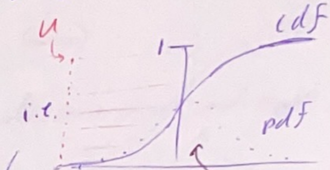
---

Q5 Let $X$ be a random variable width PDF $p_X$ and CDF $F$. Let $U$ be a random variable uniformly distributed in $[0,1]$. Show that the variable $F^{-1}(U)$ is distributed according to $p_X$. s.t. we can sample from

$\sum_{i=1}^{n} w_k^{(i)} \delta[x - x_k^{(i)}]$ **(CDF)**

Let $F = \sum_{i=1}^{n} w_k^{(i)} \delta(x - x_k^{(i)})$

i.e.




something like above...

higher concentration of points around mean for $F^{-1}(U)$

$[\, P(F^{-1}(U) \le x) = F(x)$

$F^{-1}(\alpha)$ ~~is the~~ will give the value $x$ s.t. $F(x) = \alpha\ (= P(X \le x))$

So, the probability that $F^{-1}(U)$ ~~is~~ $\le x$ is $P(F^{-1}(U) \le x) =$

$F^{-1}(U) \le x \Rightarrow U \le F(x) \Rightarrow P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x)$

$\Rightarrow \boxed{F^{-1}(U) \propto p_X}$

(distributed according to $p_X$)

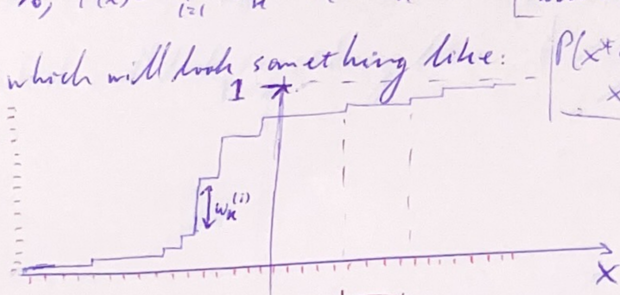
since $U$ is a uniform distribution, any value in $[0,1]$ is equally possible, s.t. the probability that $U < F(x)$ is simply the value of $F(x)$.

So, $F(x) = \sum_{i=1}^{n} w_k^{(i)} \delta(x - x_k^{(i)})$

$P(x^* \le x) = F(x)$

$x^* \sim p_X$

which will look something like:




So, $F^{-1}(0.9) = [x^\alpha, x^\beta]$

So have to figure out how to split and sample. if $F^{-1}(U_i)$ is between two values, choose the lower one. Then within the range of values that it is $[x^\alpha, x^\beta]$, ~~random~~ radomly select a position in between. (w/ uniform distr.)

6

## Q6 | Plot the PF with re-sampling if $n_{eff} < 20$ (10% of ~~samples~~ particles)

method for resampling:

~~?~~ at step $k$, if $n_{eff}[k-1] < 20$ :

draw new samples according to:

$F(x) = \sum_{i=1}^{n} w_k^{(i)} \delta(x - x_k^{(i)}) \rightarrow$

~~...~~ ~~b~~bins = iterative sum of $w^{(i)}$

$U$ = random (uniform, 200)

~~?~~foreach $U^{(i)}$:

~~...~~

foreach bin: (in reverse)

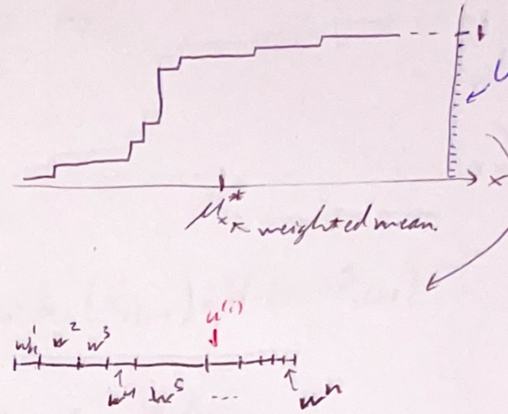
if $U^{(i)} \geq$ bin:

$x_{new}^{(i)} = x_k^{(i)} + \underbrace{(x_k^{(i)} + x_k^{(i+1)}) * \text{random(uniform)}}_{}$

break/continue

do this so not the same point each time

And continue algorithm...




| See python code for implemented PF and algorithm |

(Q7) →

Q7| Derive a linearized version of the non-linear system:

System:

$$y_k = \frac{1}{20} x_k^2 + v_k$$

$$x_k = \frac{1}{2} x_{k-1} + \frac{25 x_{k-1}}{1 + x_{k-1}^2} + 8\cos(1.2(k-1)) + u_k$$

per Lec 17,

$$x_i = f_{i-1}(x_{i-1}) + u_{i-1} \quad , \quad y_i = h_i(x_i) + v_i$$

$$f_i(x_i) \approx f_i(\hat{x}_{i|i}) + F_i(x_i - \hat{x}_{i|i}) \quad , \quad h_i(x_i) \approx h_i(\hat{x}_{i|i-1}) + H_i(x_i - \hat{x}_{i|i-1})$$

where $F_i = \begin{bmatrix} \frac{\partial f_{i,1}}{\partial x_{i,1}} & \cdots & \frac{\partial f_{i,1}}{\partial x_{i,n}} \\ \vdots & & \\ \frac{\partial f_{i,n}}{\partial x_{i,1}} & \cdots & \frac{\partial f_{i,n}}{\partial x_{i,n}} \end{bmatrix}$ and similar for $H_i$

let $z_k = x_k - 8\cos(1.2(k-1))$

$$\Rightarrow z_k = \frac{1}{2} x_{k-1} + \frac{25 x_{k-1}}{1 + x_{k-1}^2} + u_k \Rightarrow \text{let } \boxed{f_{k-1}(x_{k-1}) = \frac{1}{2} x_{k-1} + \frac{25 x_{k-1}}{1 + x_{k-1}^2}} \quad (7.1)$$

$$\Rightarrow \boxed{F_k = \frac{\partial f_k}{\partial x_k} = \frac{1}{2} + 25(1 + x_k^2)^{-1} + 25 x_k (-1)(2x_k)^{-2}} \quad (7.2)$$

s.t. $z_{k+1} \approx f_k(\hat{x}_{k|k}) + F_k(\hat{x}_{k|k})(x_k - \hat{x}_{k|k}) + u_k$

$$\Rightarrow \boxed{x_{k+1} \approx f_k(\hat{x}_{k|k}) + F_k(\hat{x}_{k|k})(x_k - \hat{x}_{k|k}) + 8\cos(1.2(k-1)) + u_k} \quad (7.3)$$

let $\boxed{h_k(x_k) = \frac{1}{20} x_k^2 \Rightarrow h_k(x_k) \approx h_k(\hat{x}_{k|k-1}) + H_k(\hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})} \quad (7.4)$

s.t. $\boxed{y_k = h_k(\hat{x}_{k|k-1}) + H_k(\hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1}) + v_k} \quad (7.5)$

$$\boxed{H_k(x_k) = \frac{\partial h_k}{\partial x_k} = \frac{1}{10} x_k} \quad (7.6)$$

EKF equations:

$$
\begin{cases}
\hat{x}_{i+1|i} = f_i(\hat{x}_{i|i}) \\
\hat{x}_{i|i} = \hat{x}_{i|i-1} + K_{f,i}\left(y_i - h_i(\hat{x}_{i|i-1})\right) \\
K_{f,i} = P_{i|i-1} H_{i|i-1}^T \left(H_{i|i-1} P_{i|i-1} H_{i|i-1}^T + R_i\right)^{-1} \\
P_{i|i} = (I - K_{f,i} H_{i|i-1}) P_{i|i-1} \\
P_{i+1|i} = F_{i|i} P_{i|i} F_{i|i}^T + G_i Q_i G_i^T
\end{cases}
$$

$G_i$ = Identity