ECE 6555; HW #3

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#1) $\vec{y}_i = \vec{x}_i + \vec{v}_i$, $\langle \vec{v}_i, \vec{v}_j \rangle = R_i \delta_{ij}$, $\langle \vec{v}_i, \vec{x}_j \rangle = 0$ for $i > j$

$\langle \vec{v}_i, \vec{x}_i \rangle = D_i$

also random variables are 0-mean

Let $\{\vec{e}_i\}$ denote the innovation of $\{\vec{y}_i\}$, $\underbrace{\|\vec{e}_i\|^2 = R_{e,i}}_{\text{variance of innovation } \vec{e}_i}$

Goal: use innovations wisely to analyze $\vec{x}_{i|i}$

(i.e. LLMS of $\vec{x}_i$ given $\{\vec{y}_j\}_{j=0}^{i}$)

Note: problem requires few calculations if done right

1-Q1) Show: $\langle \vec{v}_i, \hat{y}_{i|i-1} \rangle = 0$ where $\hat{y}_{i|i-1}$ is the LLMS of $\vec{y}_i$ given $\{y_j\}_{j=0}^{i-1}$

note: $\hat{y}_{i|i-1} = \sum_{j=0}^{i-1} \langle \vec{y}_i, e_j \rangle \|\vec{e}_j\|^{-2} e_j$, $e_0 = y_0$

also: $\hat{y}_{i|i-1} = \hat{x}_{i|i-1} + \underbrace{\hat{v}_{i|i-1}}_{=0 \text{ b/c noise at } \vec{v}_i \text{ is uncorrelated with noise at all other times}}$ (per $\vec{y}_i = \vec{x}_i + \vec{v}_i$)

$\hat{x}_{i|i-1} = ?? = \sum_{j=0}^{i-1} K_{i,j} \, y_j$ (per linear modeling assumptions)

$= \sum_{j=0}^{i-1} K_{i,j} (\vec{x}_j + \vec{v}_j)$

note: example from Lec 10, pg 7:
$\hat{x}_m = \sum_{j=0}^{m} \langle x_m, e_j \rangle \|e_j\|^{-2} e_j$
need to use?

$\Rightarrow \langle \vec{v}_i, \hat{y}_{i|i-1} \rangle = \langle \vec{v}_i, \sum_{j=0}^{i-1} K_{i,j} (\vec{x}_j + \vec{v}_j) \rangle$

$= \underbrace{\langle \vec{v}_i, \sum_{j=0}^{i-1} K_{i,j} \vec{x}_j \rangle}_{=0 \text{ per } \langle \vec{v}_i, \vec{x}_j \rangle = 0 \ \forall i > j} + \underbrace{\langle \vec{v}_i, \sum_{j=0}^{i-1} K_{i,j} \vec{v}_j \rangle}_{=0 \text{ per } \langle \vec{v}_i, \vec{v}_j \rangle = 0 \ \forall i \neq j}$

$\Rightarrow \boxed{\langle \vec{v}_i, \hat{y}_{i|i-1} \rangle = 0}$

□

#1-Q2) Show $\langle \vec{v}_i , \vec{e}_i \rangle = D_i + R_{e,i}$

$$\vec{e}_i = \vec{y}_i - \hat{y}_{i|i-1} \quad , \quad \hat{y}_{i|i-1} = \left[\sum_{j=0}^{i-1} \langle \vec{y}_i , \vec{e}_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j \right] = \hat{x}_{i|i-1} + \cancel{\hat{v}_{i|i-1}} \xrightarrow{} 0$$

$$\vec{y}_i = \vec{x}_i + \vec{v}_i$$

$$\Rightarrow \vec{e}_i = \vec{x}_i + \vec{v}_i - \cancel{\hat{x}_{i|i-1}}\ \hat{y}_{i|i-1}$$

$$\Rightarrow \langle \vec{v}_i , \vec{e}_i \rangle = \langle \vec{v}_i , \vec{x}_i + \vec{v}_i - \hat{y}_{i|i-1} \rangle$$

$$= \langle \vec{v}_i , \vec{x}_i \rangle + \langle \vec{v}_i , \vec{v}_i \rangle - \cancel{\langle \vec{v}_i , \hat{y}_{i|i-1} \rangle} \rightarrow 0 \text{ per (\#1-Q1)}$$

$$\boxed{= D_i + R_{e,i}}$$

$\square$

#1-Q3)

Show $\hat{x}_{i|i} = \vec{y}_i - (D_i + R_{e,i}) R_{e,i}^{-1} \vec{e}_i$

~~No~~ we have that $\hat{y}_{i|i-1} = \sum_{j=0}^{i-1} \langle \vec{y}_i , \vec{e}_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j$

Linear modeling assumption:

$$\hat{x}_{i|i} = \sum_{j=0}^{i} K_{i,j} \vec{y}_j$$

$$\vec{y}_i = \vec{x}_i + \vec{v}_i (= e_i + \hat{y}_{i|i-1})$$

$$\hat{y}_{i|i-1} = \sum_{j=0}^{i-1} \langle \vec{x}_i + \vec{v}_j \rangle R_{e,j}^{-1} \vec{e}_j = \hat{x}_{i|i-1} + \hat{v}_{i|i-1}$$

project $\vec{y}_i$ onto $\vec{e}_i$ gives us $\hat{y}_{i|i} = \langle \vec{y}_i , \vec{e}_i \rangle$ (orthogonal components dropout).

$$\hat{x}_{i|i} = \underbrace{(e_i + \hat{y}_{i|i-1})}_{y_i} - (\langle \vec{v}_i , \vec{e}_i \rangle + \langle \vec{e}_i , \vec{e}_i \rangle) \|\vec{e}_i\|^{-2} e_i$$

$\rightarrow$

$$\cancel{\hat{x}_{i|i} = (e_i + \hat{y}_{i|i-1}) - (\langle \vec{v}_i, \vec{e}_i \rangle + \|\vec{e}_i\|^2)\|\vec{e}_i\|^{-2} e_i}$$

$$\cancel{= (\cancel{e_i} + \hat{y}_{i|i-1}) - (\langle \vec{v}_i, \vec{e}_i \rangle \|\vec{e}_i\|^{-2} e_i + \cancel{e_i})}$$

$$\cancel{= \hat{y}_{i|i-1} - \langle \vec{v}_i, \vec{e}_i \rangle \|\vec{e}_i\|^{-2} e_i}$$

$$(y_i) = \hat{y}_{i|i} = \sum_{j=0}^{i} \langle \vec{y}_i, \vec{e}_j \rangle \|e_j\|^{-2} e_j$$

$$= \sum_{j=0}^{i} \langle \vec{x}_i, \vec{e}_j \rangle \|e_j\|^{-2} e_j + \sum_{j=0}^{i} \langle \vec{v}_i, \vec{e}_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j$$

$$y_i = \hat{x}_{i|i} + \underbrace{\langle \vec{v}_i, \vec{e}_i \rangle R_{e,i}^{-1} \vec{e}_i}_{=(D_i + R_i) R_{e,i}^{-1} \vec{e}_i} + \underbrace{\sum_{j=0}^{i-1} \langle \vec{v}_i, \vec{e}_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j}$$

$e_j \in \mathrm{Span}(y_0, y_1, \dots, y_j) \in \mathrm{Span}(x_0, x_1, \dots x_j, v_0, v_1, \dots v_j)$

$\langle \vec{v}_i, \vec{x}_j \rangle = \langle \vec{v}_i, \vec{v}_j \rangle = 0 \quad \forall i > j$

$\Rightarrow \langle \vec{v}_i, \vec{e}_j \rangle = 0 \quad \forall i > j$

$\Rightarrow \sum_{j=0}^{i-1} \langle \vec{v}_i, \vec{e}_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j = 0$

(can also re-use arguments from #1-Q1 and #1-Q2)

$$\Rightarrow \boxed{\hat{x}_{i|i} = \vec{y}_i - (D_i + R_i) R_{e,i}^{-1} \vec{e}_i}$$

#1-Q4

show $\|\vec{x}_i - \hat{x}_{i|i}\|^2 = R_i + (D_i + R_i) R_{e,i}^{-1} (D_i + R_i)^+$

$$\|\vec{x}_i - \hat{x}_{i|i}\|^2 = \langle \vec{x}_i - \hat{x}_{i|i}, \vec{x}_i - \hat{x}_{i|i} \rangle$$

$$= \langle \vec{x}_i, \vec{x}_i \rangle - \langle \vec{x}_i, \hat{x}_{i|i} \rangle - \langle \hat{x}_{i|i}, \vec{x}_i \rangle + \langle \hat{x}_{i|i}, \hat{x}_{i|i} \rangle$$

$$\vec{x}_i - \hat{x}_{i|i} = (\cancel{\vec{y}_i} - \vec{v}_i) - (\cancel{\vec{y}_i} - (D_i + R_i) R_{e,i}^{-1} \vec{e}_i)$$

$$= -\vec{v}_i + (D_i + R_i) R_{e,i}^{-1} \vec{e}_i$$

$$\Rightarrow \|\vec{x}_i - \hat{x}_{i|i}\|^2 = \|-\vec{v}_i + (D_i + R_i) R_{e,i}^{-1} \vec{e}_i\|^2$$

$$= \langle \vec{v}_i, \vec{v}_i \rangle - \langle \vec{v}_i, (D_i + R_i) R_{e_i}^{-1} \vec{e}_i \rangle - \langle (D_i + R_i) R_{e_i}^{-1} \vec{e}_i, \vec{v}_i \rangle$$

$$+ \langle (D_i + R_i) R_{e_i}^{-1} \vec{e}_i, (D_i + R_i) R_{e_i}^{-1} \vec{e}_i \rangle$$

$$\langle \vec{v}_i, \vec{v}_i \rangle = R_i$$

$$\langle \vec{v}_i, (D_i + R_i) R_{e_i}^{-1} \vec{e}_i \rangle = \langle \vec{v}_i, \vec{e}_i \rangle R_{e_i}^{-+} (D_i + R_i)^+ = (D_i + R_i) R_{e_i}^{-+} (D_i + R_i)^+$$

$$\langle (D_i + R_i) R_{e_i}^{-1} \vec{e}_i, \vec{v}_i \rangle = (D_i + R_i) R_{e_i}^{-1} (D_i + R_i)^+$$

$$\langle (D_i + R_i) R_{e_i}^{-1} \vec{e}_i, (D_i + R_i) R_{e_i}^{-1} \vec{e}_i \rangle = (D_i + R_i) \cancel{R_{e_i}^{-1}} \underbrace{\cancel{\langle \vec{e}_i, \vec{e}_i \rangle}}_{R_{e_i}} R_{e_i}^{-+} (D_i + R_i)^+$$

$$= (D_i + R_i) R_{e_i}^{-+} (D_i + R_i)^+$$

$$R_{e_i} = \begin{bmatrix} R(e_0, e_0)(i) & R(e_0, e_1)_i & \cdots \\ R(e_1, e_0)(i) & \ddots & \\ \vdots & & \ddots \\ & & R(e_{n-1}, e_{n-1})_i \end{bmatrix}$$

$$= \begin{bmatrix} R(e_0, e_0)_i & & 0 \\ & \ddots & \\ 0 & & R(e_{n-1}, e_{n-1})_i \end{bmatrix}$$ ← due to orthogonality of $e_0, e_1, \ldots, e_{n-1}$ $\Rightarrow R_{e_i} = R_{e,i}^T$

→

$R_{e,i} = R_{e,i}^T$

$\Rightarrow \langle \vec{v}_i , (D_i + R_i) R_{e_i}^{-1} \vec{e}_i \rangle = (D_i + R_i) R_{e_i}^{-T} (D_i + R_i)^+ = (D_i + R_i) R_{e,i}^{-1} (D_i + R_i)^+$

$\langle (D_i + R_i) R_{e_i}^{-1} \vec{e}_i , \vec{v}_i \rangle = (D_i + R_i) R_{e,i}^{-1} (D_i + R_i)^+$

$\langle (D_i + R_i) R_{e_i}^{-1} \vec{e}_i , (D_i + R_i) R_{e,i}^{-1} \vec{e}_i \rangle = (D_i + R_i) R_{e_i}^{-1} (D_i + R_i)^+$

$$\boxed{\Rightarrow \| \vec{x}_i - \hat{\vec{x}}_{i|i} \|^2 = R_i - (D_i + R_i) R_{e,i}^{-1} (D_i + R_i)^+}$$

#2 Innovation for simple process

$y_n = v_n + v_{n-1}$, $n \geq 0$; $\{v_i\}_{i\geq -1}$, $\mu_v = 0$

$\sigma_{v_j} = 1$

#2-Q1 show $e_j = v_j + \frac{(-1)^{j-1}}{j+1}\left(\sum_{l=-1}^{j-1} (-1)^l v_l\right)$ & $\|e_j\|^2 = \frac{j+2}{j+1}$

by inspection show true for $j = 0, 1$:

$$e_0 = v_0 + \frac{(-1)^{-1}}{1} \sum_{l=-1}^{-1} (-1)^l v_l = v_0 + -1\left((-1) v_{-1}\right) = v_0 + v_{-1} \checkmark$$

$(y_0 = v_0 + v_{-1}) = e_0$

$$e_1 = v_1 + \frac{(-1)^0}{1+1} \sum_{l=-1}^{0} (-1)^l v_l = v_1 + \frac{1}{2}(-v_{-1} + v_0) \quad \text{(eq 2.1)}$$

$$= y_1 - \hat{y}_{1|0} = v_1 + v_0 - \left(\sum_{i=0}^{0} \langle y_1, e_i\rangle \|e_i\|^{-2} e_i\right)$$

$$= v_1 + v_0 - \left(\langle y_1, e_0\rangle \|e_0\|^{-2} e_0\right) \quad ; \quad \langle y_1, e_0\rangle = \langle v_1 + v_0, v_0 + v_{-1}\rangle = 1$$

$$= v_1 + v_0 - \left((1)\left(\tfrac{1}{2}\right)(v_0 + v_{-1})\right) \qquad \|e_0\|^2 = \langle e_0, e_0\rangle = \langle v_0 + v_{-1}, v_0 + v_{-1}\rangle = 2$$

$\Rightarrow \|e_0\|^{-2} = \frac{1}{2}$

$$= v_1 - \tfrac{1}{2} v_{-1} + v_0 - \tfrac{1}{2} v_0$$

$$= v_1 + \tfrac{1}{2}(-v_{-1} + v_0) \checkmark \quad \text{(eq 2.2)}$$

equation (2.1) equals equation (2.2) ✓

→

Now assume holds true for ~~e_j~~ e_{j-1}

then: $e_j = y_j - \hat{y}_{j|j-1} = v_j + v_{j-1} - \sum_{i=0}^{j-1} \langle y_i, e_i \rangle \|e_i\|^{-2} e_i$ (2.3)

$$\sum_{i=0}^{j-1} \langle v_j + v_{j-1}, e_i \rangle \|e_i\|^{-2} e_i = \sum_{i=0}^{j-1} \Big(\overset{j>j-1}{\cancel{\langle v_j, e_i \rangle}} + \underbrace{\langle v_{j-1}, e_i \rangle}_{\substack{=0 \text{ for } i<j-1 \\ =1 \text{ for } j-1=i}} \Big) \|e_i\|^{-2} e_i \quad (2.4)$$

$= \cancel{\$}$ ~~(1)~~

$$\Rightarrow \quad = (1) \|e_{j-1}\|^{-2} e_{j-1}$$

e_{j-1} is given: $e_{j-1} = v_{j-1} + \frac{(-1)^{j-2}}{j} \sum_{l=1}^{j-2} (-1)^l v_l$

$$\|e_{j-1}\|^2 = \langle v_{j-1}, v_{j-1} \rangle + \underbrace{0}_{\substack{\langle v_{j-1}, v_i \rangle \\ i \le j-2 \\ \text{(white noise)}}} + \frac{\overset{1}{\cancel{(-1)^{j-2} (-1)^{j-2}}}}{j^2} \underbrace{\Big\langle \sum_{l=1}^{j-2} (-1)^l v_l, \sum_{k=1}^{j-2} (-1)^k v_k \Big\rangle}_{\substack{\langle v_l, v_k \rangle = \delta_{lk}, \left[(-1)^l (-1)^k = (-1)^{2l} \atop l=k \quad = 1 \right] \\ \Rightarrow = j}}$$

$$= 1 + \frac{1}{j} = \frac{j+1}{j} \quad \boxed{\left(\Rightarrow \|e_j\|^{-2} = \frac{j+1}{j+2} \right)}, \boxed{\|e_j\| = \frac{j+2}{j+1}}$$

$$\Rightarrow (2.4) = \|e_{j-1}\|^{-2} e_{j-1} = \frac{j}{j+1} \left(v_{j-1} + \frac{(-1)^{j-2}}{j} \sum_{l=1}^{j-2} (-1)^l v_l \right)$$

$$\Rightarrow (2.3) \Rightarrow e_j = v_j + v_{j-1} - \frac{j}{j+1} (v_{j-1}) - \frac{(-1)^{j-2}}{j+1} \sum_{l=1}^{j-2} (-1)^l v_l$$

$$= v_j + v_{j-1} \left(1 - \frac{j}{j+1} \right) - \quad ''$$

$\longrightarrow$

$(-1)(-1)^{j-2} = (-1)^{j-1}$

$$\Rightarrow e_j = v_j + v_{j-1}\left(\frac{\cancel{j+1}-j}{j+1}\right) - \frac{(-1)^{j-2}}{j+1}\sum_{l=-1}^{j-2}(-1)^l v_l$$

note: $(-1)^{j-1}(-1)^{j-1} = 1$ ✓

$$\Rightarrow e_j = v_j + v_{j-1}\left(\frac{(-1)^{j-1}(-1)^{j-1}}{j+1}\right) + \frac{(-1)^{j-1}}{j+1}\sum_{l=-1}^{j-2}(-1)^l v_l$$

$$\Rightarrow \boxed{e_j = v_j + \frac{(-1)^{j-1}}{j+1}\sum_{l=-1}^{j-1}(-1)^l v_l} \checkmark$$

Then by induction, we have proven

$$e_j = v_j + \frac{(-1)^{j-1}}{j+1}\sum_{l=-1}^{j-1}(-1)^l v_l \quad \text{and} \quad \|e_j\|^2 = \frac{j+2}{j+1}$$

□

#2-Q2 Prove that $\hat{y}_{k+1|k} = \frac{k+1}{k+2} e_k$

by inspection: $\hat{y}_{1|0} = \sum_{i=0}^{0} \langle y_1, e_i\rangle \|e_i\|^{-2} e_i = \langle y_1, e_0\rangle^{1} \|e_0\|^{-2}{}^{\frac{1}{2}} e_0$ (per #2-Q1)

$= \frac{1}{2}(v_0 + v_{-1}) = \frac{1}{2} e_0$

and $\hat{y}_{1|0} = \frac{0+1}{0+2} e_0 = \frac{1}{2} e_0$ ✓

assume holds for $\hat{y}_{k|k-1}$, then check $\hat{y}_{k+1|k}$

$$\Rightarrow \hat{y}_{k+1|k} = \sum_{i=0}^{k} \langle y_{k+1}, e_i\rangle \|e_i\|^{-2} e_i$$

$$= \sum_{i=0}^{k}\left(\underbrace{\langle v_{k+1}, e_i\rangle}_{0 \;\; k+1>k} + \underbrace{\langle v_k, e_i\rangle}_{=0 \; k>i,\; =1 \; k=i}\right)\|e_i\|^{-2} e_i$$

$$\boxed{\Rightarrow \hat{y}_{k+1|k} = (1)\|e_k\|^{-2} e_k = \frac{k+1}{k+2} e_k \quad \text{by induction}}$$

□

#3-a1) A Useful Relation with innovations

$\{y_i\}_{i\geq 0}$, innovation = $\{e_i\}_{i\geq 0}$ Let $\vec{y} \triangleq \begin{bmatrix} \vec{y}_0 \\ \vdots \\ \vec{y}_{n-1} \end{bmatrix}$

Show: $\vec{y}^{+} R_y^{-1} \vec{y} = \sum_{i=0}^{n-1} \vec{e}_i^{+} \|\vec{e}_i\|^{-2} \vec{e}_i$

note: assume R_y non-singular. (Recall $\vec{y} = L\vec{e}$ for some well-chosen lower triangular matrix L w/ unit diagonal.)

$\vec{e}_i = \vec{y}_i - \hat{y}_{i|i-1} \Rightarrow \vec{y}_i = \vec{e}_i + \hat{y}_{i|i-1}$, $\hat{y}_{i|i-1} = \sum_{j=0}^{i-1} \langle \vec{y}_i, \vec{e}_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j$

$\vec{y}^{+} R_y^{-1} \vec{y} = \vec{e}^{+} L^{+} R_y^{-1} L \vec{e}$

$R_y^{-1} = (\mathbb{E}[y y^{+}])^{-1} = (\langle \vec{y}, \vec{y} \rangle)^{-1}$

$\langle \vec{y}, \vec{y} \rangle =$ ~~$\vec{e}_s$~~

$\vec{y} = \begin{bmatrix} \vec{y}_0 \\ \vdots \\ \vec{y}_{n-1} \end{bmatrix} = \begin{bmatrix} \vec{e}_0 + \hat{y}_0 \\ \vec{e}_1 + \hat{y}_{1|0} \\ \vdots \\ \vec{e}_{n-1} + \hat{y}_{n-1|n-2} \end{bmatrix} = \vec{e} + \underbrace{\begin{bmatrix} 0 \\ \hat{y}_{1|0} \\ \vdots \\ \hat{y}_{n-1|n-2} \end{bmatrix}}_{\underline{Y}}$

$\langle \vec{y}, \vec{y} \rangle = \langle \vec{e} + \underline{Y}, \vec{e} + \underline{Y} \rangle$
$= \langle \vec{e}, \vec{e} \rangle + \langle \vec{e}, \underline{Y} \rangle + \langle \underline{Y}, \vec{e} \rangle + \langle \underline{Y}, \underline{Y} \rangle$

$\langle \vec{e}, \underline{Y} \rangle = \mathbb{E}(\vec{e}\, \underline{Y}^{+}) = \mathbb{E}(\vec{e}\, [\hat{y}_{1|0}^{T} \cdots \hat{y}_{n-1|n-2}^{T}])$

$= \mathbb{E} \begin{bmatrix} 0 & \vec{e}_0 \hat{y}_{1|0}^{T} & \cdots & \vec{e}_0 \hat{y}_{n-1|n-2}^{T} \\ 0 & \vec{e}_1 \hat{y}_{1|0}^{T} & \ddots & \vdots \\ \vdots & & & \\ 0 & \hat{e}_{n-1} \hat{y}_{1|0}^{T} & \cdots & \hat{e}_{n-1} \hat{y}_{n-1|n-2}^{T} \end{bmatrix}$

by definition, $\mathbb{E}(\vec{e}_i\, \hat{y}_{i|i-1}^{T}) = \langle \vec{e}_i, \hat{y}_{i|i-1} \rangle = 0$ (orthogonal)

$$\hat{y}_{1|0} = \langle \vec{y}_1, \vec{e}_0 \rangle \|\vec{e}_0\|^{-2} \vec{e}_0$$

$$\Rightarrow \langle \vec{e}_1, \hat{y}_{1|0} \rangle = \langle \vec{e}_1, \langle \vec{y}_1, \vec{e}_0 \rangle \|\vec{e}_0\|^{-2} \vec{e}_0 \rangle = \overbrace{\langle \vec{e}_1, \vec{e}_0 \rangle}^{0} (\langle \vec{y}_1, \vec{e}_0 \rangle \|e_0\|^{-2})^+$$

(orthogonal)

$$\Rightarrow \left[\langle \vec{e}_j, \hat{y}_{i|i-1} \rangle = 0 \quad \forall j \geq i \right]$$

$$\langle \vec{e}_0, \hat{y}_{1|0} \rangle = \langle \vec{e}_0, \langle \vec{y}_1, \vec{e}_0 \rangle \|\vec{e}_0\|^{-2} \vec{e}_0 \rangle = \langle \vec{e}_0, \vec{e}_0 \rangle (\langle \vec{y}_1, \vec{e}_0 \rangle \|e_0\|^{-2})^+$$

$$\langle \vec{e}_0, \hat{y}_{n-1|n-2} \rangle = \langle \vec{e}_0, \sum_{j=0}^{n-1-1} \langle y_{n-1}, e_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j \rangle = \langle \vec{e}_0, \vec{e}_0 \rangle (\langle y_{n-1}, e_0 \rangle \|e_0\|^{-2})^+$$

by orthogonality.

$$\Rightarrow \left[\langle e_j, \hat{y}_{i|i-1} \rangle = \langle e_j, e_j \rangle (\langle y_i, e_j \rangle \|e_j\|^{-2})^+ \quad \forall i > j \right]$$

$$\langle \underline{y}, \vec{e} \rangle = \mathbb{E}[\underline{y}\, \vec{e}^{\,+}] = \mathbb{E}[\underline{y} [\vec{e}_0^{\,+} \ \vec{e}_2^{\,+} \cdots \vec{e}_{n-1}^{\,+}]]$$

$$= \mathbb{E} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \hat{y}_{1|0} \vec{e}_0 & \hat{y}_{1|0} \vec{e}_1 & \cdots & \hat{y}_{1|0} \vec{e}_{n-1} \\ \vdots & & & \\ \hat{y}_{n-1|n-2} \vec{e}_0 & & \cdots & \hat{y}_{n-1|n-2} \vec{e}_{n-1} \end{bmatrix}$$

$$\cancel{\hat{y}_{1|0} =} \ \hat{y}_{i|i-1} = \sum_{j=0}^{i-1} \langle \vec{y}_i, \vec{e}_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j$$

$$\Rightarrow \langle \hat{y}_{i|i-1}, \vec{e}_k \rangle = 0 \quad \forall \ k > i-1 \quad \Rightarrow \mathbb{E}[\underline{y}\, \vec{e}^{\,+}] \text{ upper traiangle} = 0$$

$$\langle \hat{y}_{1|0}, \vec{e}_0 \rangle = \langle \langle y_1, \vec{e}_0 \rangle \|\vec{e}_0\|^{-2} \vec{e}_0, \vec{e}_0 \rangle = \langle \vec{y}_1, \vec{e}_0 \rangle \|e\|^{-2} \langle \vec{e}_0, \vec{e}_0 \rangle$$

$$\langle \hat{y}_{n-1|n-2}, \vec{e}_0 \rangle = \langle \sum_{j=0}^{n-2} \langle \vec{y}_{n-1}, \vec{e}_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j, \vec{e}_0 \rangle$$

$$= \langle \langle \vec{y}_{n-1}, \vec{e}_0 \rangle \|\vec{e}_0\|^{-2} \vec{e}_0, \vec{e}_0 \rangle \quad \text{by orthogonality}$$

try:

$$R_y^{-1} = (\langle \vec{y}, \vec{y} \rangle)^{-1} = (\langle L\vec{e}, L\vec{e} \rangle)^{-1}$$
$$= (\mathbb{E}(L\vec{e}\vec{e}^{+}L^{+}))^{-1} =$$
$$= \cancel{\mathbb{E}(L^{-+}\vec{e}^{-+}\vec{e}^{-1}L^{-1})}$$

$L = ?$

$$\vec{y} = \begin{bmatrix} \vec{y}_0 \\ \vec{y}_1 \\ \vdots \\ \vec{y}_{n-1} \end{bmatrix} \qquad \vec{y}_i = \vec{e}_i + \sum_{j=0}^{i-1} \langle \vec{y}_i, \vec{e}_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j, \; i \geq 1, \quad \vec{e}_0 = \vec{y}_0$$

$$\Rightarrow \vec{y} = \begin{bmatrix} \vec{y}_0 = \vec{e}_0 \\ \vec{e}_1 + \langle \vec{y}_1, \vec{e}_0 \rangle \|\vec{e}_0\|^{-2} \vec{e}_0 \\ \vec{e}_2 + \langle \vec{y}_2, \vec{e}_0 \rangle \|\vec{e}_0\|^{-2} \vec{e}_0 + \langle \vec{y}_2, \vec{e}_1 \rangle \|\vec{e}_1\|^{-2} \vec{e}_1 \\ \vdots \\ \vec{e}_{n-1} + \sum_{j=0}^{n-2} \langle \vec{y}_{n-1}, \vec{e}_j \rangle \|\vec{e}_j\|^{-2} \vec{e}_j \end{bmatrix}$$

$$= L\vec{e}$$

$$= \begin{bmatrix} 1 & & & & 0 \\ \langle \vec{y}_1, \vec{e}_0 \rangle \|\vec{e}_0\|^{-2} & 1 & & & \\ \langle \vec{y}_2, \vec{e}_0 \rangle \|\vec{e}_0\|^{-2} & \langle \vec{y}_2, \vec{e}_1 \rangle \|\vec{e}_1\|^{-2} & 1 & & \\ \vdots & & & \ddots & \\ \langle \vec{y}_{n-1}, \vec{e}_0 \rangle \|\vec{e}_0\|^{-2} & \langle \vec{y}_{n-1}, \vec{e}_1 \rangle \|\vec{e}_1\|^{-2} & \cdots & \langle \vec{y}_{n-1}, \vec{e}_{n-2} \rangle \|\vec{e}_{n-2}\|^{-2} & 1 \end{bmatrix} \begin{bmatrix} \vec{e}_0 \\ \vec{e}_1 \\ \vdots \\ \vec{e}_{n-1} \end{bmatrix}$$

$\Rightarrow L$ is constant

$$\Rightarrow R_y^{-1} = \cancel{\mathbb{E}(L^{-+}\vec{e}^{-+}\vec{e}^{-1}L^{-1}) = L^{-+}\mathbb{E}(\vec{e}^{+}\vec{e})^{-1}}$$
$$= \mathbb{E}(L\vec{e}\vec{e}^{+}L^{+})^{-1} = [L\mathbb{E}(\vec{e}\vec{e}^{+})L^{+}]^{-1} = L^{-+}\mathbb{E}(\vec{e}\vec{e}^{+})^{-1}L^{-1}$$

$$\Rightarrow \vec{y}^{+} R_y^{-1} \vec{y} = \vec{e}^{+} \underbrace{\cancel{L^{+}} \cancel{L^{-+}}}_{I} \mathbb{E}(\vec{e}\vec{e}^{+})^{-1} \underbrace{\cancel{L^{-1}} \cancel{L}}_{I} \vec{e}$$
$$= \vec{e}^{+} \|\vec{e}\|^{-2} \vec{e}$$

$\longrightarrow$

$$\underline{\vec{e}^+ \|\vec{e}\|^{-2} \vec{e}} \quad = ?$$

$$\|\vec{e}\|^{-2} = (\langle \vec{e}, \vec{e} \rangle)^{-1} = \mathbb{E}(\vec{e}\vec{e}^+)^{-1} = \left[\mathbb{E}\begin{bmatrix} \vec{e}_0\vec{e}_0 & \vec{e}_0\vec{e}_1 & \cdots & \vec{e}_0\vec{e}_{n-1} \\ \vec{e}_1\vec{e}_0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ \vec{e}_{n-1}\vec{e}_0 & & & \vec{e}_{n-1}\vec{e}_{n-1} \end{bmatrix}\right]^{-1}$$

$$= \begin{bmatrix} \|e_0\|^{-2} & & & 0 \\ & \|e_1\|^{-2} & & \\ & & \ddots & \\ 0 & & & \|e_{n-1}\|^{-2} \end{bmatrix}$$

by orthogonality i.e. $\mathbb{E}[\vec{e}_j \vec{e}_i^+]^{-1} = \langle e_j, e_i \rangle^{-1} \delta_{ij}$

$$\Rightarrow \|\vec{e}\|^{-2}\vec{e} = \begin{bmatrix} \|\vec{e}_0\|^{-2}\vec{e}_0 \\ \|\vec{e}_1\|^{-2}\vec{e}_1 \\ \vdots \\ \|\vec{e}_{n-1}\|^{-2}\vec{e}_{n-1} \end{bmatrix}$$

$$\Rightarrow \vec{e}^+\|\vec{e}\|^{-2}\vec{e} = \vec{e}_0^+\|\vec{e}_0\|^{-2}\vec{e}_0 + \vec{e}_1^+\|\vec{e}_1\|^{-2}\vec{e}_1 + \ldots + \vec{e}_{n-1}^+\|\vec{e}_{n-1}\|^{-2}\vec{e}_{n-1}$$

$$= \sum_{i=0}^{n-1} \vec{e}_i^+ \|e_i\|^{-2} \vec{e}_i$$

$$\Rightarrow \boxed{\text{thus, } \quad \vec{y}^+ R_y^{-1} \vec{y} = \sum_{i=0}^{n-1} \vec{e}_i^T \|\vec{e}_i\|^{-2} \vec{e}_i}$$

$\square$