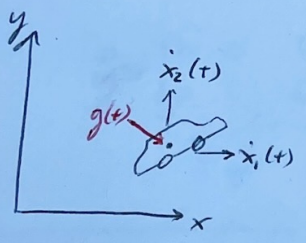
ECE6555, Hw #4

Tew Wilkening
Due: 11/10/22

#1 Kalman Filter in Action

→ use Jupyter notebooks + python (numpy + scipy)

State Space model of a vehicle subject to unknown forces
↑ single point w/ known mass, for simplicity



$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2 \quad \text{(vehicle moving in a plane)}$$

$$\vec{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \in \mathbb{R}^2 \quad \text{unknown force as a time-varying vector}$$

From Newtonian physics: $\vec{g}(t) = m \dfrac{\partial^2 \vec{x}(t)}{\partial t^2}$

"We choose to model $\vec{g}(t)/m$ as a 2D Gaussian white noise process with auto-correlation function $\delta(t)$" s.t. $R_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\frac{\partial^2 \vec{x}(t)}{\partial t^2} \triangleq \vec{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

Set $x_3(t) \triangleq \dfrac{\partial x_1(t)}{\partial t}$ and $x_4(t) \triangleq \dfrac{\partial x_2(t)}{\partial t}$ to consider augmented state of car:

$$\vec{z}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

Q1 Show the evolution of $\vec{z}(t)$ is governed by the matrix differential eqtn:

$$\frac{\partial \vec{z}(t)}{\partial t} = \underline{F}\,\vec{z}(t) + \underline{G}\,\vec{w}(t) \qquad (1)$$

$$\Rightarrow \frac{\partial \vec{z}(t)}{\partial t} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \partial^2 x_1/\partial t^2 \\ \partial^2 x_2/\partial t^2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ w_1(t) \\ w_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\triangleq \underline{F}} \vec{z} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\triangleq \underline{G}} \vec{w}$$

Q2] position of vehicle: measured through a noisy sensor $\vec{y} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \mathbb{R}^2$

which is a corrupted version of the true position by a 2D Gaussian white noise process $\vec{v}(t)$ w/ auto-correlation $\sigma^2\delta(t) \Rightarrow R_v = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$

Show: measurement equation given by $\vec{y} = \underline{H}\vec{z} + \vec{v}(t)$ & specify $\underline{H}$:

$$\vec{y}(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\triangleq \underline{H}} \vec{z}(t) + \vec{v}(t) \quad \left(= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \right) \qquad (2)$$

Simulations & filters run in discrete-time, need to sample at a period Δ.
Therefore only consider state of vehicle at times $t_k \in \{k\Delta : k \in \mathbb{N}\}$

Discretization: takes some care to get a meaningful time model.
One can show the solution of (1) satisfies for $k \geq 0$:

$$\vec{z}(t_{k+1}) = e^{\underline{F}\Delta}\vec{z}(t_k) + \underbrace{\int_0^\Delta \exp(\underline{F}(\Delta - \tau))\,\underline{G}\,\vec{w}(\tau + t_k)\,d\tau}_{\triangleq \vec{u}(t_k)} \qquad (3)$$

One can also show that the discrete-time process $\{\vec{u}(t_k)\}_{k \geq 0}$ is Gaussian and white with a covariance matrix $\underline{Q}$ that can be computed from $\underline{F}$, $\underline{G}$, Δ and the auto-correlation function of $\vec{w}(t)$. For $\Delta \ll 1$, one can show

$$\underline{Q} = \Delta \begin{bmatrix} \Delta^2/3 & 0 & \Delta/2 & 0 \\ 0 & \Delta^2/3 & 0 & \Delta/2 \\ \Delta/2 & 0 & 1 & 0 \\ 0 & \Delta/2 & 0 & 1 \end{bmatrix} \qquad (4)$$

Q3] writing $\vec{x}_k \triangleq \vec{z}(t_k)$ and $\vec{u}_k \triangleq \vec{u}(t_k)$ and using (3), show that the discretized state-space model when $\Delta \ll 1$ is:

$$\vec{x}_{k+1} = \begin{bmatrix} 1 & 0 & \Delta & 0 \\ 0 & 1 & 0 & \Delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{x}_k + \vec{u}_k \quad , \text{ where } \{\vec{u}_k\} \text{ is white with covariance } Q$$

→

Q3 cont'd

$$\underline{F}\Delta = \begin{bmatrix} 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \exp(\underline{F}\Delta) = e^{\underline{F}t}\Big|_{t=\Delta} = \mathcal{L}^{-1}\left\{(sI-\underline{F})^{-1}\right\}\Big|_{t=\Delta}$$

$$(sI-\underline{F})^{-1} = \begin{pmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix}^{-1} = \begin{pmatrix} 1/s & 0 & 1/s^2 & 0 \\ 0 & 1/s & 0 & 1/s^2 \\ 0 & 0 & 1/s & 0 \\ 0 & 0 & 0 & 1/s \end{pmatrix}$$

$$\mathcal{L}\left\{(sI-\underline{F})^{-1}\right\} = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathcal{L}\left\{(sI-\underline{F})^{-1}\right\}\Big|_{t=\Delta} = \begin{pmatrix} 1 & 0 & \Delta & 0 \\ 0 & 1 & 0 & \Delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \exp(\underline{F}\Delta)$$

$\Rightarrow$ Thus the discretized state-space model from (3) when $\Delta \ll 1$ is

$$\vec{x}_{k+1} = \begin{bmatrix} 1 & 0 & \Delta & 0 \\ 0 & 1 & 0 & \Delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{x}_k + \vec{w}_k \quad , \text{ where } \{\vec{w}_k\} \text{ is white w/ covariance } Q \text{ given above.}$$

Q4 See jupyter notebook for plot of data.

Q5 Kalman Filter equations

From class: Time Update: (prediction)

$$\begin{cases} \hat{x}_{i|i-1} = F_{i-1}\,\hat{x}_{i-1|i-1} \\ P_{i|i-1} = F_{i-1}\,P_{i-1|i-1}\,F_{i-1}^T + G_{i-1}Q_{i-1}\,G_{i-1}^T \end{cases}$$

2) Measurement Update (correction)

$$\begin{cases} \hat{x}_{i|i} = \hat{x}_{i|i-1} + K_{f,i}\,(y_i - H_i\,\hat{x}_{i|i-1}) \\ K_{f,i} = P_{i|i-1}\,H_i^T\,(H_i\,P_{i|i-1}\,H_i^T + R_i)^{-1} \\ P_{i|i} = P_{i|i-1} - K_{f,i}\,H_i\,P_{i|i-1} = (I - K_{f,i}\,H_i)\,P_{i|i-1} \end{cases}$$

To write out the equations further:

1) Time Update:

$$\hat{x}_{i|i-1} = \underline{F}_{\text{no dependence on time}}\, \hat{x}_{i-1|i-1}$$

$$P_{i|i-1} = \underline{F} P_{i-1|i-1} \underline{F}^T + G Q_{i-1} G^T$$

~~Note~~:

2) Measurement Update:

$$\hat{x}_{i|i} = \hat{x}_{i|i-1} + K_{f,i} \quad \text{... same as previous actually}$$

Q6| Implement Kalman Filter

(try both update first & prediction first)

State Space Model:

$$\vec{x}_{k+1} = \underbrace{\begin{bmatrix} 1 & 0 & \Delta & 0 \\ 0 & 1 & 0 & \Delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{F} \vec{x}_k + G\vec{u}_k \qquad G = \text{Identity}(4)$$

$$\vec{y}_k = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{H} \vec{x}_k + \vec{v}_k$$

$$K_{f,i} = P_{i|i-1} H_i^T \left(H_i P_{i|i-1} H_i^T + R_i\right)^{-1}$$

$$(4\times4)(4\times2)\left(2\times\cancel{4}\,\cancel{(4\times4)}\,(4\times2) + 2\times2\right)^{-1}$$

$$(4\times\cancel{4}\times\cancel{4}\times\cancel{2}\,\cancel{(2\times2)})^{-1}$$

$$= (4\times2)$$

✓

See Jupyter Notebook for code

Markov chain ✓ (uses cond. indep.)
Conditional Independence. ✓

Q9

given our discrete-time probabilistic state-space model

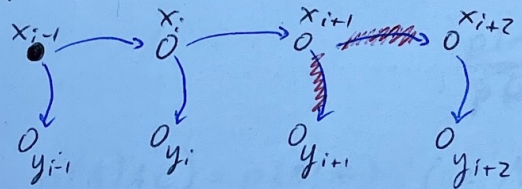
$$x_{i+1} \sim p(x_{i+1} | x_{0:i}\, y_{0:i}) \quad \text{and} \quad y_i \sim p(y_i | x_{0:i}\, y_{0:i-1})$$

that is Markovian & satisfies the conditional independent property of measurements:

$$p(x_{i+1} | x_{0:i}\, y_{0:i}) = p(x_{i+1} | x_i) \quad \text{and} \quad p(y_i | x_{0:i}\, y_{0:i-1}) = p(y_i | x_i)$$

show that $p(x_i | x_{i+1}, y_{0:n}) = p(x_i | x_{i+1}, y_{0:i})$

1) sub-graph of our probabilistic state-space model:



2) Condition on x_{i+1} shown by breaking the edges coming out of x_{i+1} (in red)

This shows us visually the conditional independent relationships.

As we can see, by breaking the edges out of x_{i+1}, we have that $y_{i+1:n}$ are no longer connected to x_i by any edge. Thus, by the Fractional Dependence Graph theorem and definition given in class,

$$p(x_i | x_{i+1}, y_{0:n}) = p(x_i | x_{i+1}, y_{0:i})$$

□

Q10] Using Bayes' rule and the properties of the state space model show that:

$$p(x_i | x_{i+1}, y_{0:n}) = \frac{p(x_{i+1} | x_i)\, p(x_i | y_{0:i})}{p(x_{i+1} | y_{0:i})}$$

Bayes' Rule: $P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$, $p(A,B) = P(A|B)\,P(B)$

---

let $A = x_i$, $B = x_{i+1}$, $C = y_{0:n}$

then we have: $p(A|B,C) = \frac{p(B|A)\,p(A|C)}{p(B|C)}$ ~~are~~ as desired relationship.

~~Using Bayes Rule~~ ~~$p(A|B,C) = \frac{p(BC|A)\,p(A)}{p(BC)}$~~

$= \frac{p(ABC)}{p(BC)}$ ~~(Ok, but not where we want to go)~~ $=$

Note: $p(B|AC) = p(B|A)$ (i.e. Markovian Property)

s.t. $p(A|BC) = \overset{*}{\frac{p(ABC)}{p(BC)}} = \overset{*}{\frac{p(AC)\,p(B|AC)}{p(BC)}} = \frac{p(A|C)\,p(C)\,p(B|AC)}{p(C)\,p(B|C)}$

$p(B|AC) = \frac{p(ABC)}{p(AC)}$ $\quad * =$ Bayes' Rule used $\quad = \frac{p(B|AC)\,p(A|C)}{p(B|C)}$

(Bayes' Rule)

(using conditional independent property of Markovian) $\rightarrow = \frac{p(B|C)\,p(A|C)}{p(B|C)}$

$$= \frac{p(x_{i+1}|x_i)\,p(x_i|y_{0:i})}{p(x_{i+1}|y_{0:i})} \quad \square$$

Q11 Show that the joint distribution of x_i and x_{i+1} given $y_{0:n}$ is given by

$$p(x_i, x_{i+1} | y_{0:n}) = \frac{p(x_{i+1}|x_i)\, p(x_i|y_{0:i})\, p(x_{i+1}|y_{0:n})}{p(x_{i+1}|y_{0:i})}$$

So, $p(x_i, x_{i+1}|y_{0:n}) = \frac{p(x_i, x_{i+1}, y_{0:n})}{p(y_{0:n})} = \frac{p(x_{i+1}|x_i, y_{0:n})\, p(x_i, y_{0:n})}{p(x_{i+1}|y_{0:i})}$

did not get us where we wanted. Looking first at results from Q9 & Q10:

$$p(x_i | x_{i+1}, y_{0:n}) = p(x_i | x_{i+1}, y_{0:i}) \quad (11\text{-}1)$$

$$p(x_i | x_{i+1}, y_{0:n}) = \frac{p(x_{i+1}|x_i)\, p(x_i|y_{0:i})}{p(x_{i+1}|y_{0:i})} \quad (11\text{-}2)$$

noting that 11-2 has the denominator we are looking for, we use Bayes' Theorem to get the term $p(x_i | x_{i+1}, y_{0:n})$ from our initial distribution. (11-3)

$$p(x_i, x_{i+1} | y_{0:n}) \quad (11\text{-}3)$$

$$\Rightarrow p(x_i, x_{i+1} | y_{0:n}) = \frac{p(x_i, x_{i+1}, y_{0:n})}{p(y_{0:n})} = \frac{p(x_i | x_{i+1}, y_{0:n})\, p(x_{i+1}, y_{0:n})}{p(y_{0:n})}$$

Sub in (11-2):

$$= \frac{p(x_{i+1}|x_i)\, p(x_i|y_{0:i})\, p(x_{i+1}, y_{0:n})}{p(x_{i+1}|y_{0:i})\, p(y_{0:n})}$$

Use Bayes' Rule:

$$= \frac{p(x_{i+1}|x_i)\, p(x_i|y_{0:i})\, p(x_{i+1}|y_{0:n})\, \cancel{p(y_{0:n})}}{p(x_{i+1}|y_{0:i})\, \cancel{p(y_{0:n})}} = \boxed{\frac{p(x_{i+1}|x_i)\, p(x_i|y_{0:i})\, p(x_{i+1}|y_{0:n})}{p(x_{i+1}|y_{0:i})}}$$

□

Q-12]

Assume $x_{i-1}|y_{0:i-1}$ is distributed according to Gaussian distribution

$$\mathcal{N}(\hat{x}_{i-1|0:i-1}, P_{i-1|0:i-1})$$

Show that joint distribution of x_{i-1} and x_i given $y_{0:i-1}$ is:

$$\mathcal{N}\left(\begin{bmatrix} \hat{x}_{i-1|0:i-1} \\ \hat{x}_{i|0:i-1} \end{bmatrix}, \begin{bmatrix} P_{i-1|0:i-1} & P_{i-1|0:i-1}F^T \\ FP_{i-1|0:i-1} & P_{i|0:i-1} \end{bmatrix}\right)$$

where $\hat{x}_{i|0:i-1} = F\hat{x}_{i-1|0:i-1}$

$$P_{i|0:i-1} = FP_{i-1|0:i-1}F^T + Q$$

Gauss-Markov model: $x_{i+1} = Fx_i + u_i$, $y_i = Hx_i + v_i$

$R_u = Q\delta_{ij}$, $R_v = R\delta_{ij}$, $\{u_i\}, \{v_i\}$ are white & Gaussian

from page 8 of Lecture 18 we have the following Lemma:

Lemma: let $x \sim \mathcal{N}(\mu_x, \Sigma_x)$ and $y|x \sim \mathcal{N}(Hx, R)$ (eq. $y = Hx + v$)

Then $\begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_x \\ H\mu_x \end{bmatrix}, \begin{bmatrix} \Sigma_x & \Sigma_x H^T \\ H\Sigma_x & H\Sigma_x H^T + R \end{bmatrix}\right)$

we are given the distribution of $x_{i-1}|y_{0:i-1}$ and the state-space model $x_{i+1} = Fx_i + u_i$. Thus if we set $x = x_{i-1}|y_{0:i-1}$ and $y = x_i|y_{0:i-1}$ then we have $y = Fx + u$, which matches the form given in the example of the Lemma.

To prove the result to ourselves, we will actually calculate the joint distribution:

let $\mathcal{N}(\hat{x}_{i-1|0:i-1}, P_{i-1|0:i-1}) = \mathcal{N}(\mu_x, \Sigma_x)$ i.e. $\mu_x \triangleq \hat{x}_{i-1|0:i-1}$

$\Sigma_x \triangleq P_{i-1|0:i-1}$

First, we calculate the mean of $x_i|y_{0:i-1}$:

$$\hat{x}_{i|0:i-1} = \mathbb{E}(x_i|y_{0:i-1}) = \mathbb{E}(Fx_{i-1} + u_i|y_{0:i-1}) = F\hat{x}_{i-1|0:i-1} + \underbrace{\mathbb{E}(u_i|y_{0:i-1})}_{\to 0} = F\hat{x}_{i-1|0:i-1} = F\mu_x$$

process noise is ~~independent~~ uncorrelated of past measurements & $\mathbb{E}(u_i) = 0$

Q4.2 cont'd

Covariance of $x_i | y_{0:i-1}$;

$$\mathbb{E}\left((x_i - \mathbb{E}(x_i|y_{0:i-1}))(x_i - \mathbb{E}(x_i|y_{0:i-1}))^T | y_{0:i-1}\right)$$

$$= \mathbb{E}\left((x_i - F\mu_x)(x_i - F\mu_x)^T | y_{0:i-1}\right)$$

$$= \mathbb{E}\left(x_i x_i^T - x_i \mu_x^T F^T - F\mu_x x_i^T + F\mu_x \mu_x^T F^T | y_{0:i-1}\right)$$

$$= \mathbb{E}\left((Fx_{i-1} + u_{i-1})(Fx_{i-1} + u_{i-1})^T | y_{0:i-1}\right) - F\mu_x\mu_x^T F^T - F\mu_x\mu_x^T F^T + F\mu_x\mu_x^T F^T$$

$$= F R_x F^T + F\,\mathbb{E}[x_{i-1} u_{i-1}^T | y_{0:i-1}] + \mathbb{E}[u_{i-1} x_{i-1}^T | y_{0:i-1}] F^T + \mathbb{E}[u_{i-1} u_{i-1}^T | y_{0:i-1}] - F\mu_x\mu_x^T F^T$$

(uncorrelated)

$$= F\,\mathbb{E}(x_{i-1}x_{i-1}^T | y_{0:i-1}) + F\,\mathbb{E}[x_{i-1}|y_{0:i-1}]\,\mathbb{E}[u_{i-1}|y_{0:i-1}]^T + \mathbb{E}[u_{i-1}|y_{0:i-1}]\,\mathbb{E}[x_{i-1}^T|y_{0:i-1}] F^T + Q - F\mu_x\mu_x F^T$$

(x_{i-1} & u_{i-1} independent; $\{u_i\}$ white noise, 0-mean)

$$= F\left(\mathbb{E}(x_{i-1}x_{i-1}^T | y_{0:i-1}) - \mu_x\mu_x^T\right) F^T + Q$$

$$= \Sigma_x = \mathbb{E}\left((x_{i-1} - \mu_x)(x_{i-1} - \mu_x)^T | y_{0:i-1}\right) = P_{i-1|0:i-1}$$

$$= F P_{i-1|0:i-1} F^T + Q$$

$\triangleq P_{i|0:i-1}$ as given in problem statement. □

Similarly, cross-covariance of $x_{i-1}|y_{0:i-1}$ and $x_i|y_{0:i-1}$;

$$\mathbb{E}\left((x_{i-1} - \mu_x)(x_i - F\mu_x)^T | y_{0:i-1}\right) = \mathbb{E}\left(x_{i-1}x_i^T + \mu_x\mu_x^T F^T - \mu_x x_i^T - x_{i-1}\mu_x^T F^T | y_{0:i-1}\right)$$

$$= \mathbb{E}\left(x_{i-1}(Fx_{i-1} + u_{i-1})^T | y_{0:i-1}\right) - \mu_x\,\mathbb{E}(Fx_{i-1} + u_{i-1}|y_{0:i-1})^T - \mu_x\mu_x^T F^T + \mu_x\mu_x^T F^T$$

$$= \mathbb{E}\left(x_{i-1}x_{i-1}^T F^T | y_{0:i-1}\right) + \mu_x\,\mathbb{E}[u_{i-1}^T | y_{0:i-1}] - \mu_x\left((F\mu_x)^T + \mathbb{E}[u_{i-1}|y_{0:i-1}]^T\right)$$

(u_{i-1} uncorrelated & independent of $y_{0:i-1}$ ($\{u_i\}, \{v_i\}$ white, Gaussian); 0-mean)

$$= \mathbb{E}\left(x_{i-1}x_{i-1}^T | y_{0:i-1}\right) F^T + 0 - \mu_x\mu_x^T F^T = \left(\mathbb{E}(x_{i-1}x_{i-1}^T | y_{0:i-1}) - \mu_x\mu_x\right) F^T$$

$$= \Sigma_x F^T$$ □

Since $R_{xy} = R_{yx}^T$, $\mathbb{E}\left((x_{i-1} - F\mu_x)(x_{i-1} - \mu_x)^T | y_{0:i-1}\right) = (\Sigma_x F^T)^T = F\Sigma_x$ □

Therefore,

$$\begin{bmatrix} x_{i-1} | y_{0:i-1} \\ x_i | y_{0:i-1} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_x \\ F\mu_x \end{bmatrix}, \begin{bmatrix} \Sigma_x & \Sigma_x F^T \\ F\Sigma_x & F\Sigma_x F^T + Q \end{bmatrix} \right)$$

where $\Sigma_x = P_{i-1|0:i-1}$, $\mu_x = \hat{x}_{i-1|0:i-1}$

□

Q13] Using the result of Q-12, show that the distribution of x_{i-1} given x_i and $y_{0:n}$, which is equal to distribution of x_{i-1} given x_i and $y_{0:i-1}$ by Q-9, is

$$x_{i-1} | x_i, y_{0:n} \sim \mathcal{N}(\tilde{x}_{i-1|i-1}, \tilde{P}_{i-1|i-1})$$

where $\tilde{x}_{i-1|i-1} = \hat{x}_{i-1|0:i-1} + k_i (x_i - F\hat{x}_{i-1|0:i-1})$

$$\tilde{P}_{i-1|i-1} = P_{i-1|0:i-1} - k_i (F P_{i-1|0:i-1} F^T + Q) k_i^T$$

$$k_i = P_{i-1|0:i-1} F^T (F P_{i-1|0:i-1} F^T + Q)^{-1}$$

---

mean: $\mathbb{E}(x_{i-1} | x_i, y_{0:n}) = \mathbb{E}(x_{i-1} | x_i, y_{0:i-1})$ by Q-9 (independence)

$$\mathbb{E}(x_{i-1} | x_i, y_{0:i-1}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_{i-1} \, p(x_{i-1} | x_i, y_{0:i-1}) \, d\vec{x}_{i-1}$$

note from Q-10: $p(x_{i-1} | x_i, y_{0:n}) = \dfrac{p(x_i | x_{i-1}) \, p(x_{i-1} | y_{0:i-1})}{p(x_i | y_{0:i-1})}$

$$\hat{x}_{i-1|0:i-1} = \mathbb{E}(x_{i-1} | y_{0:i-1}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_{i-1} \, p(x_{i-1} | y_{0:i-1}) \, d\vec{x}_{i-1}$$

From Lecture #18, pg 7: if $\begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} R_x & R_{xy} \\ R_{yx} & R_y \end{bmatrix}\right)$

then $x \sim \mathcal{N}(\mu_x, R_x)$, $y \sim \mathcal{N}(\mu_y, R_y)$ and

$$x|y \sim \mathcal{N}(\mu_x + R_{xy} R_y^{-1} (y - \mu_y), \; R_x - R_{xy} R_y^{-1} R_{yx}) \quad \leftarrow \text{use this}$$

$$y|x \sim \mathcal{N}(\mu_y + R_{yx} R_x^{-1} (x - \mu_x), \; R_y - R_{yx} R_x^{-1} R_{xy})$$

---

let $x_{i-1} | y_{0:i-1} = x$ and $x_i | y_{0:i-1} = y$ s.t.

$\hat{x}_{i-1|0:i-1} = \mu_x$ $\quad P_{i-1|0:i-1} = R_x$ $\quad P_{i-1|0:i-1} F^T = R_{xy}$

$F\hat{x}_{i-1|0:i-1} = \hat{x}_{i|0:i-1} = \mu_y$ $\quad F P_{i-1|0:i-1} F^T + Q = R_y$ $\quad F P_{i-1|0:i-1} = R_{yx}$

} from Q-12 →

and $x_{i-1} | x_i\, y_{0:n} = x_{i-1} | x_i\, y_{0:i-1} = x|y$

thus!

$= \hat{x}$ $= \mu_x + R_{xy} R_y^{-1} (y - \mu_y)$

$= \mathbb{E}(x_{i-1} | x_i\, y_{0:i-1})$

$= \mathbb{E}(x|y)$

$= \hat{x}_{i-1|0:i-1} + \underbrace{P_{i-1|0:i-1} F^T (F P_{i-1|0:i-1} F^T + Q)^{-1}}_{K_i} (x_i | y_{0:i-1} - \hat{x}_{i|0:i-1})$

$x_i | y_{0:i-1} = x_i$ (Markovian)

$= F\hat{x}_{i-1|0:i-1}$

$= \hat{x}_{i-1|0:i-1} + K_i (x_i - F\hat{x}_{i-1|0:i-1})$

$= \tilde{x}_{i-1|i-1}$ □

Now for the covariance:

$Cov(x|y) = R_x - R_{xy} R_y^{-1} R_{yx}$

$\Rightarrow \tilde{P}_{i-1|i-1} = P_{i-1|0:i-1} - \underbrace{P_{i-1|0:i-1} F^T (F P_{i-1|0:i-1} F^T + Q)^{-1}}_{K_i} F P_{i-1|0:i-1}$

$= P_{i-1|0:i-1} - K_i F P_{i-1|0:i-1}$

$= P_{i-1|0:i-1} - K_i (F P_{i-1|0:i-1} F^T + Q)(F P_{i-1|0:i-1} F^T + Q)^{-1} F P_{i-1|0:i-1}$

Note: $K_i^T = (F P_{i-1|0:i-1} F^T + Q)^{-T} F P_{i-1|0:i-1}$

$= (F P_{i-1|0:i-1}^T F^T + Q^T)^{-1} F P_{i-1|0:i-1}$

$= (F P_{i-1|0:i-1} F^T + Q)^{-1} F P_{i-1|0:i-1}$

where $P_{i-1|0:i-1} = P_{i-1|0:i-1}^T$

self-covariance matrices are symmetric

and $Q^T = Q = R_n$ (symmetric)

$\Rightarrow \tilde{P}_{i-1|i-1} = P_{i-1|0:i-1} - K_i (F P_{i-1|0:i-1} F^T + Q) K_i^T$ □

Q-14] Implement the Kalman Filter for smoothing

Summary:

(predict)

$\hat{x}_{i|0:i-1} = F\hat{x}_{i-1|0:i-1}$

$P_{i|0:i-1} = FP_{i-1|0:i-1}F^T + Q$

(update)

$\hat{x}_{i|0:i} = \hat{x}_{i|0:i-1} + K_{f,i}(y_i - H_i\hat{x}_{i|0:i-1})$

$K_{f,i} = P_{i|0:i-1}H_i^T(H_iP_{i|0:i-1}H_i^T + R_i)^{-1}$

$P_{i|0:i} = P_{i|0:i-1} - K_{f,i}H_iP_{i|0:i-1}$

} Kalman Filter

$\hat{x}_{i-1|i-1} = \hat{x}_{i-1|x_i,0:i-1} = \hat{x}_{i-1|0:i-1} + K_i(x_i - F\hat{x}_{i-1|0:i-1})$

$K_i = P_{i-1|0:i-1}F^T(FP_{i-1|0:i-1}F^T + Q)^{-1}$

$\tilde{P}_{i-1|i-1} = P_{i-1|i-1} - K_i(FP_{i-1|0:i-1}F^T + Q)K_i^T$

$\hat{x}_{i-1|0:n} = \hat{x}_{i-1|0:i-1} + K_i(\hat{x}_{i|0:n} - F\hat{x}_{i-1|0:i-1})$

$P_{i-1|0:n} = K_iP_{i|0:n}K_i^T + \tilde{P}_{i-1|i-1}$

} Kalman Smoother

So, in Q-6 we produced $\hat{x}_{i|0:i}$ and $P_{i|0:i}$, now we will use them to go backwards and produce:

K_i, $\hat{x}_{i-1|0:n}$

See code in Jupyter

Q-15] Plot the smoothed trajectory → see Jupyter

Q-16] Compute RMS for smoothed vs. true trajectory: $\sqrt{\frac{1}{n}\sum_{i=1}^{n}\|\hat{x}_{i|0:n} - x_i\|_2^2}$

& How does this compare? See Jupyter code

Table of Contents

ECE6555 HW4

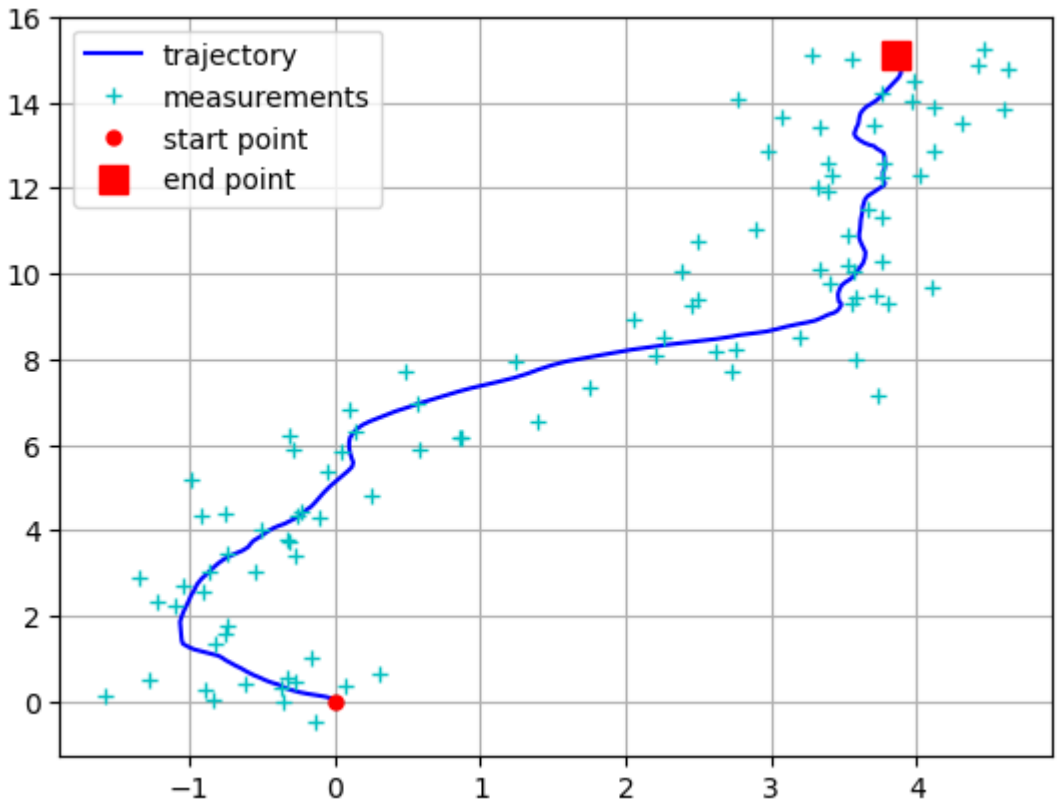
Author: Teo Wilkening Due Date: 2022/11/10

Q-4 Trajectory + Measurements Plot

In [1]:
```
import numpy as np
from scipy import signal

np.random.seed(1992)
NumSteps = 201
TimeScale = np.linspace(0,10,NumSteps)
DeltaSim = np.diff(TimeScale)[0]
SigmaInput = 1
SigmaNoise = 0.5

F = np.array([[1,0,DeltaSim,0],[0,1,0,DeltaSim],[0,0,1,0],[0,0,0,1]])
Q = SigmaInput**2 * np.array([[DeltaSim**3/3,0,DeltaSim**2/2,0],
                              [0,DeltaSim**3/3,0,DeltaSim**2/2],
                              [DeltaSim**2/2,0,DeltaSim,0],
                              [0,DeltaSim**2/2,0,DeltaSim]])
H = np.array([[1,0,0,0],[0,1,0,0]])
R = SigmaNoise**2 * np.identity(2)

State = np.zeros((4,NumSteps))
NoisyMeasurements = np.zeros((2,NumSteps))

for t in np.arange(1,NumSteps):
    ProcessNoise = np.squeeze(np.matmul(np.linalg.cholesky(Q),np.random.randn(4,1)))
    State[:,t] = np.matmul(F,State[:,t-1]) + ProcessNoise
    MeasurementNoise = SigmaNoise * np.squeeze(np.random.randn(2))
    NoisyMeasurements[:,t] = np.matmul(H,State[:,t]) + MeasurementNoise
```

```
StateX1 = State[0,:]
StateX2 = State[1,:]

DownSampling=2
NoisyMeasurements = NoisyMeasurements[:,::DownSampling]
MeasurementY1 = NoisyMeasurements[0,:]
MeasurementY2 = NoisyMeasurements[1,:]
```

In [2]:
```
import matplotlib.pyplot as plt

# plt.figure(figsize=(3,3), dpi = 180)
plt.figure(1)
plt.plot(StateX1,StateX2,'b')
plt.plot(MeasurementY1,MeasurementY2,'c+')
plt.grid(True)
plt.plot(StateX1[0],StateX2[0],'r.',markersize=10)
plt.plot(StateX1[-1],StateX2[-1],'rs',markersize=10)
plt.legend(['trajectory','measurements','start point','end point'])

plt.rcParams['figure.figsize'] = [6, 6]
plt.rcParams['figure.dpi'] = 120 # 200 e.g. is really fine, but slower
plt.show()
```



Q-5 Kalman Filter Equations

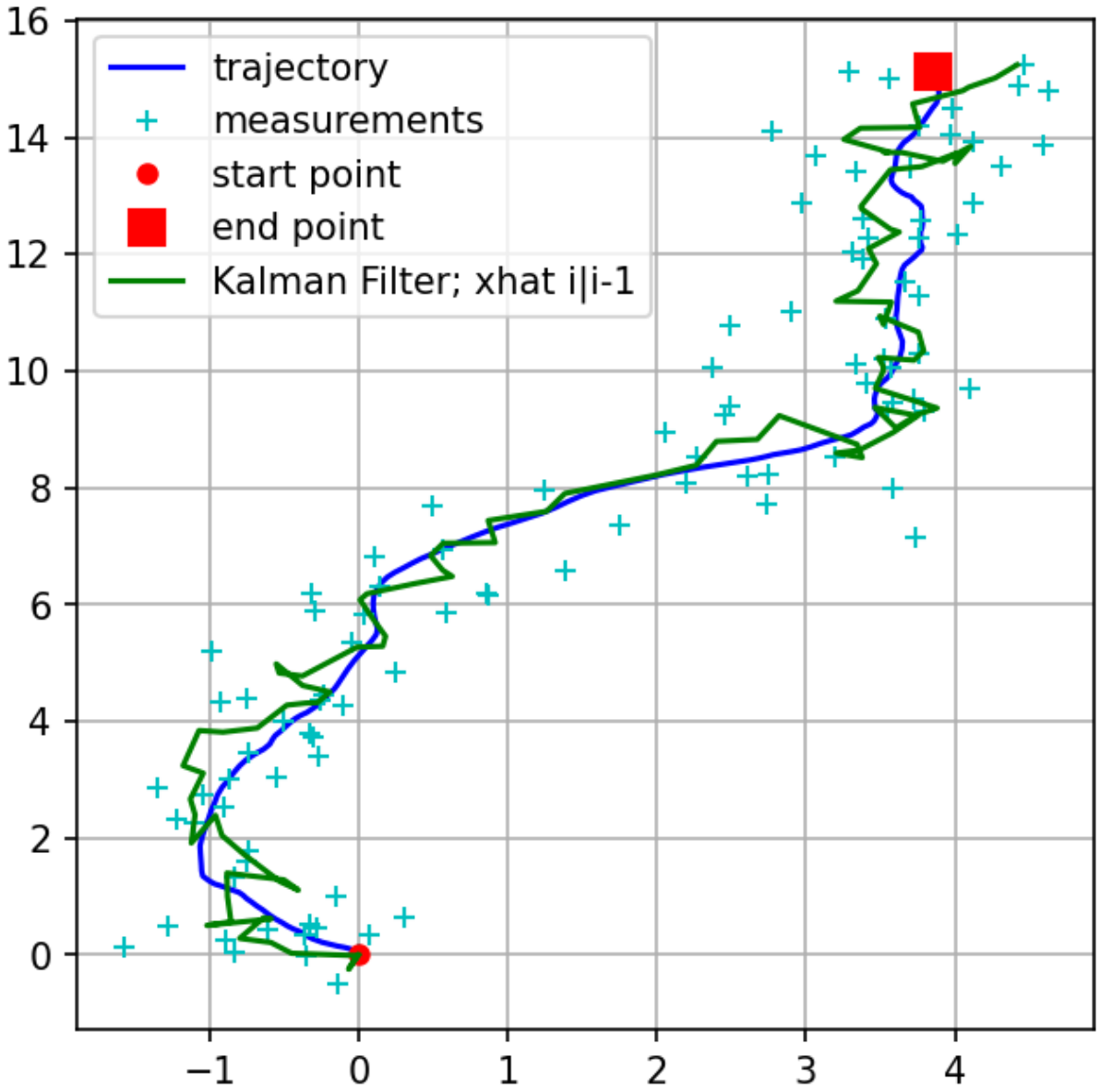
(see written work)

Q-6 & Q-7 Implementing + Plot the Kalman Filter


In [3]:
```
# initialize all of the variables that we're going to need; F, H, G, Q, R
Delta = 0.1
Fk = np.array([[1,0,Delta,0],[0,1,0,Delta],[0,0,1,0],[0,0,0,1]])
Gk = np.identity(4)
Qk = np.array([[Delta**3/3,0,Delta**2/2,0],
               [0,Delta**3/3,0,Delta**2/2],
               [Delta**2/2,0,Delta,0],
               [0,Delta**2/2,0,Delta]])
Hk = np.array([[1,0,0,0],[0,1,0,0]])
Sigma = 0.5
Rk = Sigma**2 * np.identity(2)
```

In [10]:
```
# the initial guesses of x and P
xhat_init = np.zeros((4,1))
P_init = np.identity(4)
NumSteps = len(MeasurementY1)

# initializing the matrices for computing Kalman filter state evolution over time
xhat_i_pred = np.zeros((4,NumSteps))
xhat_i_curr = np.zeros((4,NumSteps))
P_i_pred = np.zeros((NumSteps,4,4))
P_i_curr = np.zeros((NumSteps,4,4))
Kfi_curr = np.zeros((NumSteps,4,2))

# start running the Kalman Filter, using the NoisyMeasurements
for t in np.arange(0,NumSteps):
    # make the prediction update (time update)
    if t == 0:
        # use the initial guesses
        xhat_i_pred[:,t] = (Fk @ xhat_init).reshape(1,4)
        P_i_pred[t,:,:] = Fk @ P_init @ Fk.T + Gk @ Qk @ Gk.T
    elif t > 0:
        # then use the time-update (prediction) calculation
        # x i|i-1
        xhat_i_pred[:,t] = (Fk @ xhat_i_curr[:,t-1]).reshape(1,4)
        # P i|i-1
        P_i_pred[t,:,:] = Fk @ P_i_curr[t-1,:,:] @ Fk.T + Gk @ Qk @ Gk.T

    # make the measurement update
    # K f,i
    Kfi_curr[t,:,:] = P_i_pred[t,:,:] @ Hk.T @ np.linalg.inv(Hk @ P_i_pred[t,:,:] @
                                                             Hk.T + Rk)
    # P i|i
    P_i_curr[t,:,:] = (np.identity(4) - Kfi_curr[t,:,:] @ Hk) @ P_i_pred[t,:,:]
    # x i|i
    xhat_i_curr[:,t] = xhat_i_pred[:,t] + Kfi_curr[t,:,:] @ (NoisyMeasurements[:,t] -
                                                             Hk @ xhat_i_pred[:,t])


# plot the results
plt.figure()
plt.plot(StateX1,StateX2,'b')
plt.plot(MeasurementY1,MeasurementY2,'c+')
plt.grid(True)
plt.plot(StateX1[0],StateX2[0],'r.',markersize=10)
```

```
plt.plot(StateX1[-1],StateX2[-1],'rs',markersize=10)
plt.plot(xhat_i_pred[0,:],xhat_i_pred[1,:],'g')
plt.legend(['trajectory','measurements','start point',
            'end point','Kalman Filter; xhat i|i-1'])

plt.rcParams['figure.figsize'] = [4, 4]
plt.rcParams['figure.dpi'] = 120 # 200 e.g. is really fine, but slower
plt.show()
```



Q-8 RMS Error

In [5]:
```
StateX = State[:,::DownSampling][0:2,:]
error_measure = NoisyMeasurements - StateX
rms_measure = np.linalg.norm( np.linalg.norm( error_measure, ord=2, axis=0 ) ,ord=2)
error_kalman_f = xhat_i_pred[0:2,:] - StateX
rms_kalman_f = np.linalg.norm( np.linalg.norm( error_kalman_f, ord=2, axis=0), ord=2)
```

```
print(f'''RMS of the noisy measurements: {rms_measure}''')
print(f'''RMS of the Kalman filter estimates: {rms_kalman_f}''')
```

```
RMS of the noisy measurements: 7.085174167758084
RMS of the Kalman filter estimates: 3.889252921005451
```

RMS Conclusion

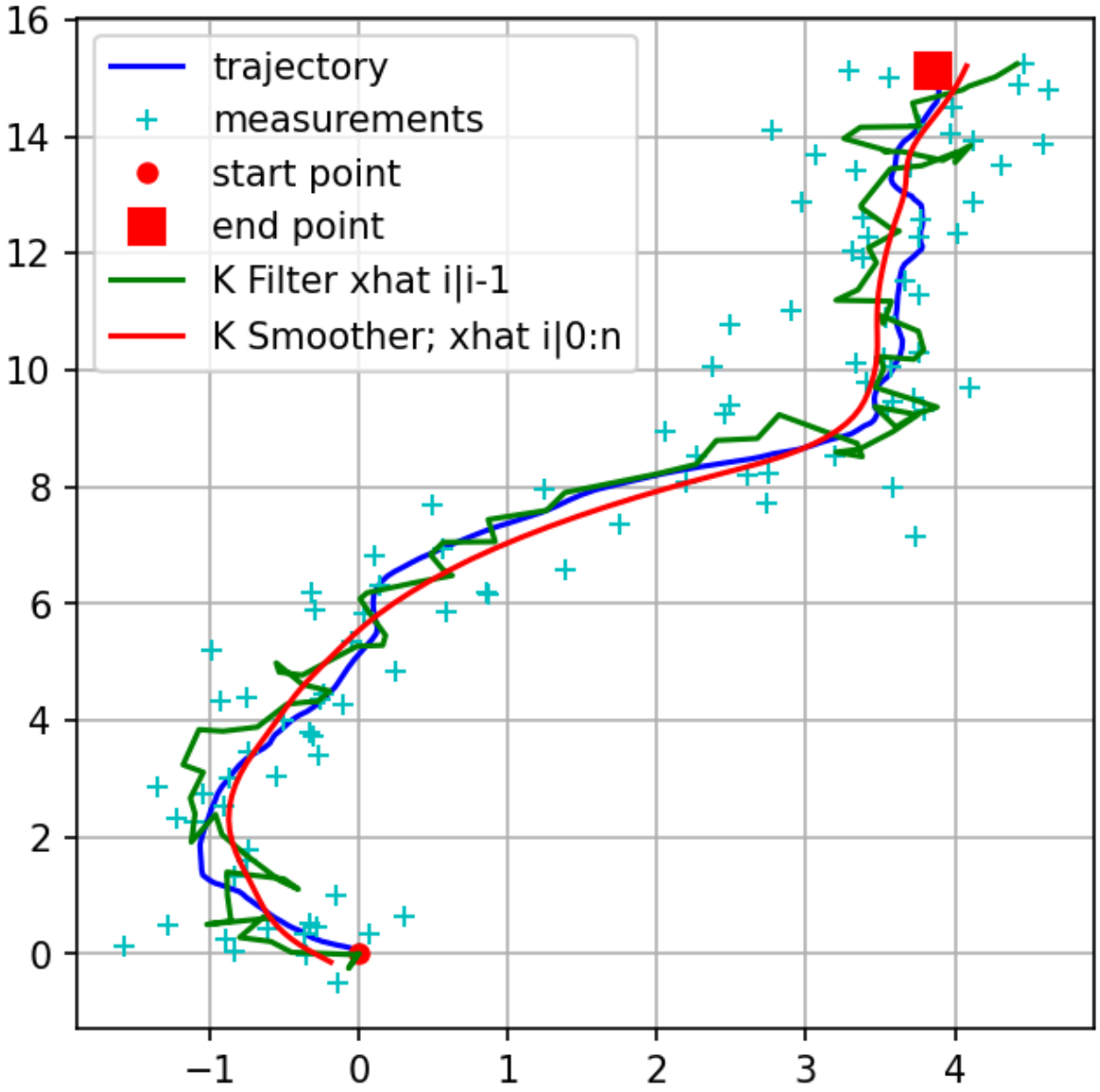
Based on the above RMS calculation, the RMS of the Kalman filter is less than the RMS of the Noisy Measurements. This is indeed expected since we are using the Kalman filter to "filter" out some of the effects of the noise on our measurements.

Q-14 & Q-15 Implementation + Plot of Kalman Smoother

In [12]:

```
# Recall the initial setup for computing Kalman filter
# xhat_i_pred = np.zeros((4,NumSteps))          # xhat i|i-1
# xhat_i_curr = np.zeros((4,NumSteps))          # xhat i|i
# P_i_pred = np.zeros((NumSteps,4,4))           # P i|i-1
# P_i_curr = np.zeros((NumSteps,4,4))           # P i|i
# Kfi_curr = np.zeros((NumSteps,4,2))           # K f,i

# initialize new variables to hold the Kalman Smoother data
xhat_i_min_1_given_n = np.zeros((4,NumSteps))   # xhat i-1|0:n
Ki = np.zeros((NumSteps,4,4))                   # K_i

## Implement the Kalman Smoothing algorithm, given the Kalman Filtered data
# for xhat n|0:n, use the last step of the Kalman Filter
xhat_i_min_1_given_n[:,NumSteps-1] = xhat_i_curr[:,NumSteps-1]

# now for each time step compute the Kalman Smoothed state evolution
for t in np.arange(NumSteps-1,0,-1):
    # K_i
    Ki[t,:,:] = P_i_curr[t-1,:,:] @ Fk.T @ np.linalg.inv(Fk @ P_i_curr[t-1,:,:] @
                                                         Fk.T + Qk)
    # xhat i-1|0:n
    xhat_i_min_1_given_n[:,t-1] = (xhat_i_curr[:,t-1] +
                                   Ki[t,:,:] @ (xhat_i_min_1_given_n[:,t] -
                                                Fk @ xhat_i_curr[:,t-1]))


# plot the results
plt.figure()
plt.plot(StateX1,StateX2,'b')
plt.plot(MeasurementY1,MeasurementY2,'c+')
plt.grid(True)
plt.plot(StateX1[0],StateX2[0],'r.',markersize=10)
plt.plot(StateX1[-1],StateX2[-1],'rs',markersize=10)
plt.plot(xhat_i_pred[0,:],xhat_i_pred[1,:],'g')
plt.plot(xhat_i_min_1_given_n[0,:],xhat_i_min_1_given_n[1,:],'r')
plt.legend(['trajectory','measurements','start point','end point',
            'K Filter xhat i|i-1','K Smoother; xhat i|0:n'])
```

```
plt.rcParams['figure.figsize'] = [6,6]
plt.rcParams['figure.dpi'] = 150 # 200 e.g. is really fine, but slower
plt.show()
```



Q-16 RMS Error of Kalman Smoother

In [14]:
```
StateX = State[:,::DownSampling][0:2,:]
error_measure = NoisyMeasurements - StateX
rms_measure = np.linalg.norm( np.linalg.norm( error_measure, ord=2, axis=0 ) ,ord=2)
error_kalman_f = xhat_i_pred[0:2,:] - StateX
rms_kalman_f = np.linalg.norm( np.linalg.norm( error_kalman_f, ord=2, axis=0), ord=2)
error_kalman_f_curr = xhat_i_curr[0:2,:] - StateX
rms_kalman_f_curr = np.linalg.norm( np.linalg.norm( error_kalman_f_curr,
                                                    ord=2, axis=0), ord=2)
error_kalman_s = xhat_i_min_1_given_n[0:2,:] - StateX
rms_kalman_s = np.linalg.norm( np.linalg.norm( error_kalman_s, ord=2, axis=0), ord=2)
```

```
print(f'''RMS of the noisy measurements: {rms_measure}''')
print(f'''RMS of the Kalman Filter xhat i|i-1 estimates: {rms_kalman_f}''')
print(f'''RMS of the Kalman Filter xhat i|i estimates: {rms_kalman_f_curr}''')
print(f'''RMS of the Kalman Smoother xhat i|0:n estimates: {rms_kalman_s}''')
print(f'''RMS Kalman Smoother/RMS Kalman Filter: {rms_kalman_s/rms_kalman_f}''')
```

```
RMS of the noisy measurements: 7.085174167758084
RMS of the Kalman Filter xhat i|i-1 estimates: 3.889252921005451
RMS of the Kalman Filter xhat i|i estimates: 3.3022335707201216
RMS of the Kalman Smoother xhat i|0:n estimates: 2.2990220257280076
RMS Kalman Smoother/RMS Kalman Filter: 0.5911217584516627
```

RMS Conclusion

Based on the above RMS calculation, the RMS of the Kalman Smoother is ~41% less than the RMS of the Kalman Filter. This is good news since we expect our RMS to improve when using all of the measurments for each step.