Georgia Institute of Technology

ECE 6555 - Assignment 3

due Thursday October 13, 2022 - v1.1

- There are 3 problems over 5 pages (including the cover page).
- The problems are not necessarily in order of difficulty.
- Every question in a problem is worth 2 points, so problems with many questions are worth more than problems with few questions.
- Each question is graded as follows: no credit without meaningful work, half credit for partial work, full credit if essentially correct.
- Unless otherwise specified, you should concisely indicate your reasoning and show all relevant work.
- The grade on each question is based on our judgment of your level of understanding as reflected by what you have written. If we cannot read it, we cannot grade it.
- Please use a pen and not a pencil if you handwrite your solution.
- **You must submit your assignment on Gradescope**.

Problem 1: Using innovations

Consider the model $\mathbf{y}_i = \mathbf{x}_i + \mathbf{v}_i$ with $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = R_i\delta_{ij}$, $\langle \mathbf{v}_i, \mathbf{x}_j \rangle = 0$ for $i > j$, and $\langle \mathbf{v}_i, \mathbf{x}_i \rangle = D_i$. All random variables are zero mean. Let $\{\mathbf{e}_i\}$ denote the innovation process of $\{\mathbf{y}_i\}$ and let $\|\mathbf{e}_i\|^2 = R_{e,i}$ denote the variance of the innovation $\mathbf{e}_i$. The goal of this problem is to use innovations wisely to analyze $\hat{\mathbf{x}}_{i|i}$, the Linear Least Mean Square (LLMS) of $\mathbf{x}_i$ given $\{\mathbf{y}_j\}_{j=0}^{i}$. Note that this problem requires little calculations if done right.

[Q1] Show that $\langle \mathbf{v}_i, \hat{\mathbf{y}}_{i|i-1} \rangle = 0$ where $\hat{\mathbf{y}}_{i|i-1}$ is the LLMS of $\mathbf{y}_i$ given $\{\mathbf{y}_j\}_{j=0}^{i-1}$.

SOLUTION Note that $\hat{\mathbf{y}}_{i|i-1}$ takes the form $\hat{\mathbf{y}}_{i|i-1} \triangleq \sum_{j=0}^{i-1} \alpha_j \mathbf{y}_j$. Therefore,

$$\begin{aligned}
\langle \mathbf{v}_i, \hat{\mathbf{y}}_{i|i-1} \rangle &= \sum_{j=0}^{i-1} \alpha_j \langle \mathbf{v}_i, \mathbf{y}_j \rangle \\
&= \sum_{j=0}^{i-1} \alpha_j \underbrace{\langle \mathbf{v}_i, \mathbf{x}_j \rangle}_{\text{because } j<i} + \sum_{j=0}^{i-1} \alpha_j \underbrace{\langle \mathbf{v}_i, \mathbf{v}_j \rangle}_{=D_i\delta_{ij}} \\
&= 0
\end{aligned}$$

$\square$

[Q2] Show that $\langle \mathbf{v}_i, \mathbf{e}_i \rangle = D_i + R_i$ (*Hint:* use part **(a)** to express $\mathbf{e}_i$)

SOLUTION Note that

$$\langle \mathbf{v}_i, \mathbf{e}_i \rangle = \langle \mathbf{v}_i, \mathbf{y}_i - \hat{\mathbf{y}}_{i|i-1} \rangle = \langle \mathbf{v}_i, \mathbf{y}_i \rangle = \langle \mathbf{v}_i, \mathbf{x}_i \rangle + \langle \mathbf{v}_i, \mathbf{v}_i \rangle = D_i + R_i$$

$\square$

[Q3] Show that $\hat{\mathbf{x}}_{i|i} = \mathbf{y}_i - (D_i + R_i)R_{e,i}^{-1}\mathbf{e}_i$ (*Hint:* project the equation $\mathbf{y}_i = \mathbf{x}_i + \mathbf{v}_i$)

SOLUTION Note that

$$\hat{\mathbf{y}}_{i|i} = \hat{\mathbf{x}}_{i|i} + \hat{\mathbf{v}}_{i|i}$$

We know that $\hat{\mathbf{y}}_{i|i} = \mathbf{y}_i$. In addition,

$$\hat{\mathbf{v}}_{i|i} = \sum_{j=0}^{i} \langle \mathbf{v}_i, \mathbf{e}_j \rangle \|\mathbf{e}_j\|^{-2} e_j.$$

We have just proved in the previous question that $\langle \mathbf{v}_i, \mathbf{e}_i \rangle = D_i + R_i$ and for $j < i$ $\langle \mathbf{v}_i, \mathbf{e}_j \rangle = 0$. Hence the result follows. $\square$

[Q4] Show that $\|\mathbf{x}_i - \hat{\mathbf{x}}_{i|i}\|^2 = R_i - (D_i + R_i)R_{e,i}^{-1}(D_i + R_i)$

SOLUTION

$$\begin{aligned}
\|\mathbf{x}_i - \hat{\mathbf{x}}_{i|i}\|^2 &= \langle \mathbf{x}_i - \mathbf{y}_i - (D_i + R_i)R_{e,i}^{-1}\mathbf{e}_i, \mathbf{x}_i - \mathbf{y}_i - (D_i + R_i)R_{e,i}^{-1}\mathbf{e}_i \rangle \\
&= \langle \mathbf{v}_i - (D_i + R_i)R_{e,i}^{-1}\mathbf{e}_i, \mathbf{v}_i - (D_i + R_i)R_{e,i}^{-1}\mathbf{e}_i \rangle \\
&= R_i - 2(D_i + R_i)R_{e,i}^{-1}(D_i + R_i) + 2(D_i + R_i)R_{e,i}^{-1}(D_i + R_i) \\
&= R_i - (D_i + R_i)R_{e,i}^{-1}(D_i + R_i)
\end{aligned}$$

$\square$

Problem 2: Innovation for simple process

Consider the situation where $y_k = v_k + v_{k-1}$ for $k \geq 0$ and $\{v_i\}_{i \geq -1}$ is a white noise with unit variance.

[Q1] Show that the innovations are given by

$$e_j = v_j + \frac{(-1)^{j-1}}{j+1} \sum_{\ell=-1}^{j-1} (-1)^\ell v_\ell$$

and that $\|e_j\|^2 = \frac{j+2}{j+1}$. (*Hint:* use a proof by induction)

$\boxed{\text{SOLUTION}}$ Let our induction hypothesis be

$$H_j : \forall j \in [0; j] \quad e_j = v_j + \frac{(-1)^{j-1}}{j+1} \sum_{\ell=-1}^{j-1} (-1)^\ell v_\ell.$$

For $k = 0$, we have $e_0 = y_0 = v_0 + v_{-1} = v_0 + \frac{(-1)^{0-1}}{0+1}(-1)^{-1} v_{-1}$. Hence H_0 holds.

Assume now that H_k holds. Then

$$\begin{aligned}
e_{k+1} &= y_{k+1} - \sum_{j=0}^{k} \langle y_{k+1}, e_j \rangle \|e_j\|^{-2} e_j \\
&= v_{k+1} + v_k - \sum_{j=0}^{k} \langle v_{k+1} + v_k, e_j \rangle \|e_j\|^{-2} e_j \\
&= v_{k+1} + v_k - \sum_{j=0}^{k} \underbrace{\langle v_k, e_j \rangle}_{=\delta_{jk}} \|e_j\|^{-2} e_j \\
&= v_{k+1} + v_k - \frac{k+1}{k+2} \left(v_k + \frac{(-1)^{k-1}}{k+1} \sum_{\ell=-1}^{k-1} (-1)^\ell v_\ell \right) \\
&= v_{k+1} + \frac{1}{k+2} v_k + \frac{(-1)^k}{k+2} \sum_{\ell=-1}^{k-1} (-1)^\ell v_\ell \\
&= v_{k+1} + \frac{(-1)^k}{k+2} \sum_{\ell=-1}^{k} (-1)^\ell v_\ell
\end{aligned}$$

Hence H_{k+1} holds.

Therefore, our assumption hypothesis holds for any $k \geq 0$.

Finally, note that

$$\begin{aligned}
\|e_j\|^2 &= \left\langle v_j + \frac{(-1)^{j-1}}{j+1} \sum_{\ell=-1}^{j-1} (-1)^\ell v_\ell, v_j + \frac{(-1)^{j-1}}{j+1} \sum_{\ell=-1}^{j-1} (-1)^\ell v_\ell \right\rangle \\
&= 1 + \sum_{\ell=-1}^{j-1} \frac{1}{(j+1)}^2 \\
&= 1 + \frac{1}{j+1} \\
&= \frac{j+2}{j+1}.
\end{aligned}$$

$\square$

[Q2] Prove that $\hat{y}_{k+1|k} = \frac{k+1}{k+2} e_k$.

SOLUTION Note that

$$\begin{aligned}
\hat{y}_{k+1|k} &= \sum_{j=0}^{k} \langle y_{k+1}, e_k \rangle \|e_k\|^{-2} e_k \\
&= \sum_{j=0}^{k} \langle v_{k+1} + v_k, e_k \rangle \|e_k\|^{-2} e_k \\
&= \sum_{j=0}^{k} \langle v_{k+1} + v_k, e_k \rangle \|e_k\|^{-2} e_k \\
&= \langle v_k, v_k \rangle \|e_k\|^{-2} e_k \\
&= \frac{k+1}{k+2} e_k
\end{aligned}$$

where we have used the characterization of the innovation from the previous question and the properties of the process $\{v_k\}_{k\geq 1}$. $\square$

Problem 3: A useful relation with innovations

Consider a process $\{y_i\}_{i\geq 0}$ and its corresponding innovations sequence $\{e_i\}_{i\geq 0}$. Letting $\mathbf{y} \triangleq \begin{bmatrix} y_0^\mathsf{T} & \cdots & y_{n-1}^\mathsf{T} \end{bmatrix}^\mathsf{T}$, show that

$$\mathbf{y}^\dagger R_y^{-1} \mathbf{y} = \sum_{i=0}^{n-1} e_i^\dagger \|e_i\|^{-2} e_i.$$

You can assume that R_y is not singular. (*Hint:* recall that we wrote $\mathbf{y} = \mathbf{Le}$ for some well chosen lower triangular matrix L with unit diagonal)

SOLUTION We let $\mathbf{e} \triangleq \begin{bmatrix} e_0^\mathsf{T} & \cdots & e_{n-1}^\mathsf{T} \end{bmatrix}^\mathsf{T}$. Recall that $\mathbf{y} = \mathbf{Le}$ where $R_y = \mathbf{LDL}^\dagger$. Hence,

$$\begin{aligned}
\mathbf{y}^\dagger R_y^{-1} \mathbf{y} &= \mathbf{y}^\dagger \mathbf{L}^{-\dagger} \mathbf{D}^{-1} \mathbf{L}^{-1} \\
&= \mathbf{e}^\dagger \mathbf{D}^{-1} \mathbf{e}
\end{aligned}$$

The diagonal elements of $\mathbf{D}$ are exactly the $\|e_i\|^2$, hence the results follows. $\square$