ECE 6555 - Optimal estimation of exponential distribution

Set $y \triangleq x + v$ w/ $x \sim Exp(\lambda)$ $v \sim Exp(\nu)$ independent

① $p_{xy}(x,y) = p_v(y-x)\, p_x(x)\, \mathbb{1}\{y \geq x\}$ by indep. of v and x

$$= \mu e^{-\mu(y-x)} \lambda e^{-\lambda x} \mathbb{1}\{y \geq x\}$$

$$= \lambda\mu\, e^{-(\lambda-\mu)x} e^{-\mu y} \mathbb{1}\{y \geq x\}$$

② $p_y(y) = \int_0^{+\infty} p_{xy}(x,y)\, dx = \lambda\mu e^{-\mu y} \int_0^y e^{-(\lambda-\mu)x}$

$$= \frac{\lambda\mu}{\lambda-\mu} e^{-\mu y} \times -\left[e^{-(\lambda-\mu)y} - 1\right]$$

Hence $p_y(y) = \frac{\lambda\mu}{\lambda-\mu}\left[e^{-\mu y} - e^{-\lambda y}\right]$

③ We know that $\hat{x} = \mathbb{E}(x|y)$

Here, $p(x|y) = (\lambda-\mu)\frac{e^{-(\lambda-\mu)x} e^{-\mu y}}{e^{-\mu y} - e^{-\lambda y}} \mathbb{1}\{y \geq x\}$

Hence $\mathbb{E}(x|y) = \frac{(\lambda-\mu) e^{-\mu y}}{e^{-\mu y} - e^{-\lambda y}} \int_0^y x\, e^{-(\lambda-\mu)x} dx$

$$= \frac{(\lambda-\mu) e^{-\mu y}}{e^{-\mu y} - e^{-\lambda y}} \left\{\left[\frac{-x}{\lambda-\mu} e^{-(\lambda-\mu)x}\right]_0^y + \frac{1}{\lambda-\mu}\int_0^y e^{-(\lambda-\mu)x} dx\right\}$$

$$= \frac{(\lambda-\mu) e^{-\mu y}}{e^{-\mu y} - e^{-\lambda y}} \left[-y\frac{e^{-(\lambda-\mu)y}}{\lambda-\mu} - \frac{1}{(\lambda-\mu)^2}\left(e^{-(\lambda-\mu)y} - 1\right)\right]$$

$$= -y\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}} + \frac{1}{\lambda-\mu}$$

④ Note that $\mathbb{E}(x) = \lambda^{-1}$ $\mathbb{E}(y) = \lambda^{-1} + \mu^{-1}$

$$\mathbb{E}(xy) = \int_0^\infty dx \int_x^\infty dy\; xy\, \lambda\mu\, e^{-(\lambda-\mu)x} e^{-\mu y}$$

$$= \int_0^\infty dx\; x\lambda\mu\, e^{-(\lambda-\mu)x} \underbrace{\int_x^{+\infty} dy\; y e^{-\mu y}}$$

$$= \left[-\frac{y}{\mu} e^{-\mu y}\right]_x^\infty + \int_x^\infty \frac{1}{\mu} e^{-\mu y} dy$$

$$= \frac{x}{\mu} e^{-\mu x} + \frac{1}{\mu^2} e^{-\mu x}$$

$$= \int_0^\infty dx \left(x^2 \lambda e^{-\lambda x} + x\frac{\lambda}{\mu} e^{-\lambda x}\right)$$

$$= 2\lambda^{-2} + \lambda^{-1}\mu^{-1} = \lambda^{-1}(\lambda^{-1} + \mu^{-1})$$

The fast way

$$\mathbb{E}(xy) = \mathbb{E}(x(x+v)) = 2\lambda^{-2} + \lambda^{-1}\mu^{-1}$$

$$\mathbb{E}(y^2) = \mathbb{E}(x^2 + 2vx + v^2) = 2\lambda^{-2} + 2\lambda^{-1}\mu^{-1} + 2\mu^{-2} = (\lambda^{-1} + \mu^{-1})^2 + \lambda^{-2} + \mu^{-2}$$

Hence, if $\tilde{x} \triangleq x - \mathbb{E}(x)$ and $\tilde{y} \triangleq y - \mathbb{E}(y)$

$$\mathbb{E}(\tilde{x}\tilde{y}) = \mathbb{E}(xy) - \mathbb{E}(x)\mathbb{E}(y) = \lambda^{-2}$$

$$\mathbb{E}(\tilde{y}^2) = \mathbb{E}(\tilde{y}^2) - \mathbb{E}(y)^2 = \lambda^{-2} + \mu^{-2}$$

so that $\boxed{\hat{x}_{LLMSE} = \lambda^{-1} + \frac{\lambda^{-2}}{\lambda^{-2} + \mu^{-2}}\left(y - (\lambda^{-1} + \mu^{-1})\right)}$

$$= \frac{\lambda - \mu}{\lambda^2 + \mu^2} + \frac{\mu^2}{\lambda^2 + \mu^2} y$$