$$\hat{x} = \mathbb{E}[x] + \frac{(\mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y])}{\mathbb{E}[y^2] - \mathbb{E}[y]^2}(y - \mathbb{E}[y])$$

See page 7-12 for further comments

$\Rightarrow$ MATLAB

$$\Rightarrow \hat{x} = \frac{y\mu^2 - \mu + \lambda}{\lambda^2 + \mu^2}$$

no exponentials

does not match → this is the LLSE for part $\boxed{Q4}$

↑ comment: ~~similar form..~~

~~$\hat{x} = -$~~

~~what went wrong?? How to get $e^{-\lambda y}$ and $e^{-\mu y}$ into $\hat{x}$??~~

~~$\hat{x} = \mu_x + R_{xy} R_y^{-1}(y - \mu_y)$~~

~~$\mu_x = \frac{1}{\lambda}$~~

~~$\mu_y = \mathbb{E}[y] = \mathbb{E}[x+v] = \frac{1}{\lambda} + \frac{1}{\mu}$~~

~~$R_{xy} = \mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y] = \mathbb{E}[x(x+v)] - \mathbb{E}[x]\mathbb{E}[x+v]$~~

~~$= \mathbb{E}[x^2] + \mathbb{E}[xv] - \mathbb{E}[x]\mathbb{E}[x] - \mathbb{E}[x]\mathbb{E}[v]$~~

~~$= \cancel{\frac{1}{\lambda^2}} + \cancel{\frac{1}{\lambda}\left(\frac{1}{\mu}\right)} - \cancel{\frac{1}{\lambda^2}} - \cancel{\frac{1}{\lambda}\frac{1}{\mu}} = 0$~~

$\boxed{\#7\text{-}Q3}$

#7-Q3 Start on page 7-8

~~Non-linear, optimal estimate:~~

~~let $h^* = \underset{h}{\operatorname{argmin}}\, \mathbb{E}[(x-h(y))(x-h(y))^T]$~~

~~since $\mathbb{E}[(x-h(y))(x-h(y))^T] \geq 0$ by definition,~~

~~then if we can find $h(y)$ s.t. $\mathbb{E}_{xh} = 0$, $h(y) = h^*$ since~~

~~$\mathbb{E}_{xh}$ cannot be any lower value.~~

~~calculation →~~

Let $h(y) = \hat{x} = \frac{1}{\lambda-\mu} - \left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right) y = \frac{1}{\lambda-\mu} - \left(\frac{e^{-\lambda(x+v)}}{e^{-\mu(x+v)} - e^{-\lambda(x+v)}}\right)(x+v)$

$\Rightarrow \mathbb{E}[(x-\hat{x})(x-\hat{x})^T] = \mathbb{E}[(x-\hat{x})^2] = \mathbb{E}[x^2] - \mathbb{E}[x\hat{x}] - \mathbb{E}[\hat{x}x] + \mathbb{E}[\hat{x}^2]$

$\mathbb{E}[x^2] = \frac{1}{\lambda^2}$

$\mathbb{E}[x\hat{x}] = \mathbb{E}\left[\frac{x}{\lambda-\mu} - \left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right) xy\right] \Rightarrow$ MATLAB

$= \frac{1}{\lambda(\lambda-\mu)} -$

$\mathbb{E}[\hat{x}x] = \mathbb{E}[x\hat{x}]$ (scalar) $= \mathbb{E}\left[\frac{x}{\lambda-\mu} - x(x+v)\left(\frac{e^{-\lambda x}e^{-\lambda v}}{e^{-\mu x}e^{-\mu v} - e^{-\lambda x}e^{-\lambda v}}\right)\right]$

$\mathbb{E}[\hat{x}^2] =$

$= \frac{1}{\lambda^2}\left(\frac{1}{\lambda-\mu}\right) - \mathbb{E}\left[(x^2+xv)\left(\frac{e^{-\lambda x}e^{-\lambda v}}{e^{-\mu x}e^{-\mu v} - e^{-\lambda x}e^{-\lambda v}}\right)\right]$

too long...

#7-Q3 on page 7-8

#7 - Q3) Show that the optimal (non-linear) LMS estimate of x given y is:

$$\hat{x} = \frac{1}{\lambda-\mu} - \left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right) y$$

$p(x) = \lambda e^{-\lambda x}$ $\lambda > 0$

$p(v) = \mu e^{-\mu v}$ $\mu > 0$

$y = x + v$ $x, v \in \mathbb{R}$, random, independent

$\Rightarrow R_{xv} = R_{vx} = 0$

$P_{xv}(x,v) = p(x)p(v)$

note: $p(x,y) = \lambda\mu e^{-(\lambda-\mu)x} e^{-\mu y}$

and;

$$p(y) = \frac{\lambda\mu}{\lambda-\mu}\left(e^{-\mu y} - e^{-\lambda y}\right)$$

~~$\mathbb{E}(x|y) = ? = \int_{-\infty}^{\infty} x\, p(x,y)\, dy$ (?)~~

note: marginal density from a joint density:

$$p(y) = \int_{-\infty}^{\infty} p(x,y)\, dx$$

in our case: $x \geq 0$, $x \leq y$

$$\Rightarrow p(y) = \int_0^y p(x,y)\, dx = \frac{\lambda\mu}{\lambda-\mu}\left(e^{-\mu y} - e^{-\lambda y}\right) \checkmark$$

note: expectation of a variable $\mathbb{E}(x) = \int_{-\infty}^{\infty} x\, p_x(x)\, dx$

note: the conditional pdf is: $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

note: conditional expectation:

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

$$\Rightarrow \mathbb{E}[X|Y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\cancel{\lambda\mu}\left(e^{-(\lambda-\mu)x} e^{-\mu y}\right)}{\frac{\cancel{\lambda\mu}}{(\lambda-\mu)}\left(e^{-\mu y} - e^{-\lambda y}\right)} \rightarrow$$

#7 - Q3 cont'd

$$f_{X|Y}(x|y) = (\lambda-\mu)\frac{\left(e^{-(\lambda-\mu)\cancel{y}x}e^{-\mu y}\right)}{e^{-\mu y}-e^{-\lambda y}}$$

$$\mathbb{E}[X|Y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dy$$ where $x \geq 0$, $x \leq y$ due to the constituent pdf's

$$= \int_0^y x(\lambda-\mu)\frac{\left(e^{-(\lambda-\mu)\cancel{x}}e^{-\mu y}\right)}{e^{-\mu y}-e^{-\lambda y}}\,d\cancel{x}$$

$$= \frac{(\lambda-\mu)e^{-\mu y}}{e^{-\mu y}-e^{-\lambda y}}\int_0^y x e^{-(\lambda-\mu)x}\,dx$$

Integration by parts:

$$\int f\,dg = fg - \int g\,df$$

$f = x$ $\quad dg = e^{-(\lambda-\mu)x}dx$

$df = dx$ $\quad g = \frac{-e^{-(\lambda-\mu)x}}{\lambda-\mu}$

$$\Rightarrow -\frac{xe^{-(\lambda-\mu)x}}{(\lambda-\mu)} + \int \frac{e^{-(\lambda-\mu)x}}{\lambda-\mu}dx$$

$$+\frac{(-1)e^{-(\lambda-\mu)x}}{(\lambda-\mu)^2}$$

$$= \left.\frac{-x(\lambda-\mu)e^{-(\lambda-\mu)x} - e^{-(\lambda-\mu)x}}{(\lambda-\mu)^2}\right|_0^y$$

$$\Rightarrow \int_0^y xe^{-(\lambda-\mu)x}dx = \frac{-y(\lambda-\mu)e^{-(\lambda-\mu)y} - e^{-(\lambda-\mu)y}}{(\lambda-\mu)^2} + \frac{1}{(\lambda-\mu)^2}$$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{\cancel{(\lambda-\mu)}e^{-\mu y}}{e^{-\mu y}-e^{-\lambda y}}\left[\left(\frac{1}{(\lambda-\mu)^{\cancel{2}}}\right)\left(-y(\lambda-\mu)e^{-(\lambda-\mu)y} - e^{-(\lambda-\mu)y} + 1\right)\right]$$

$$= \left(\frac{e^{-\mu y}}{e^{-\mu y}-e^{-\lambda y}}\right)\left[-ye^{-(\lambda-\mu)y} - \frac{e^{-(\lambda-\mu)y}}{\lambda-\mu} + \frac{1}{\lambda-\mu}\right]$$

$\longrightarrow$

#7 - Q3 con'd

$$= \left[\left(\frac{1}{\lambda-\mu}\right)\left(1 - e^{-(\lambda-\mu)y}\right) - y e^{-(\lambda-\mu)y} \right] \left(\frac{e^{-\mu y}}{e^{-\mu y} - e^{-\lambda y}} \right)$$

$$= \left(\frac{1}{\lambda-\mu}\right)\left(\frac{\left(1 - e^{-(\lambda-\mu)y}\right)e^{-\mu y}}{e^{-\mu y} - e^{-\lambda y}}\right) - y\,\frac{e^{(-\lambda+\mu-\mu)y}}{e^{-\mu y} - e^{-\lambda y}}$$

$$= \left(\frac{1}{\lambda-\mu}\right)\left(\frac{\cancel{e^{-\mu y} - e^{-\lambda y}}}{\cancel{e^{-\mu y} - e^{-\lambda y}}}\right) - y\left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right)$$

$$\mathbb{E}[X|Y] = \boxed{\frac{1}{\lambda-\mu} - y\left(\frac{e^{-\lambda y}}{e^{-\mu y} - e^{-\lambda y}}\right)} \quad \square$$

#7-Q4)

let $\hat{x} = Ky$, $P(K) = \cancel{E(x_x^T)}\ E((x-\hat{x})(x-\hat{x})^T)$,

$= E((x-Ky)(x-Ky))$, $y = x+v$

$= E((x-K(x+v))(x-K(x+v)))$

$= E(x^2 - 2K(x+v) + K^2(x+v)^2)$

$= E(x^2 - 2K(x+v) + K^2(x^2+2xv+v^2))$

$= E(x^2 - 2Kx - 2Kv + K^2x^2 + 2K^2xv + K^2v^2)$

$= R_x - 2KE(x) - 2KE(v) + K^2R_x + 2K^2R_{xv}^{\,0} + K^2R_v$

$= \frac{1}{\lambda^2} - 2K\frac{1}{\lambda} - 2K\frac{1}{\mu} + K^2\frac{1}{\lambda^2} + 2K^2(0) + K^2\frac{1}{\mu^2}$ → $P(K)$ is (P.S.D by definition)

$p(x) = \lambda e^{-\lambda x}$

$E(x) = \frac{1}{\lambda}$

$R_x = \frac{1}{\lambda^2}$

$p(v) = \mu e^{-\mu v}$

$\cancel{E}_v = \frac{1}{\mu}$

$\sigma_v^2 = \frac{1}{\mu^2}$

independent ⇒ uncorrelated ⇓ $R_{xv} = 0$

$\frac{\partial P(K)}{\partial K} = 0 - \frac{2}{\lambda} - \frac{2}{\mu} + \frac{2K}{\lambda^2} + \frac{2K}{\mu^2} = 0$ ← set equal to 0 to find ~~minimum~~ since $P(K)$ is P.S.D. and is 2nd order function of K

$\Rightarrow \cancel{2}K\left(\frac{1}{\lambda^2} + \frac{1}{\mu^2}\right) = \cancel{2}\left(\frac{1}{\lambda} + \frac{1}{\mu}\right)$

$K\left(\frac{\mu^2+\lambda^2}{\mu^2\lambda^2}\right) = \left(\frac{\mu+\lambda}{\lambda\mu}\right)$

$\Rightarrow \boxed{K_0 = \left(\frac{\mu+\lambda}{\cancel{\mu\lambda}}\right)\left(\frac{\mu^{\cancel{2}}\lambda^{\cancel{2}}}{\mu^2+\lambda^2}\right) = \frac{(\mu+\lambda)\mu\lambda}{\mu^2+\lambda^2}}$

for a linear equation of $y = x+v$, a LMS estimate (per class notes Lec. 6) is $\hat{x} = K_0 y$,

$K_0 = R_{xy}R_y^{-1}$

⇒ plug K_0 back in to $P(K)$:

$P(K_0) = \frac{1}{\lambda^2} - \frac{2K_0}{\lambda} - \frac{2K_0}{\mu} + \frac{K_0^2}{\lambda^2} + \frac{K_0^2}{\mu^2}$

⇒ MATLAB

⇒ $P(K_0) =$

→

$\cancel{\hat{x} = Ky}$, $K = R_{xy} R_y^{-1}$ per Lec. 6 notes

$\hat{x} = \mu_x + K_0 (y - \mu_y)$, affine solution

$$K_0 = R_{xy} R_y^{-1}$$
$$= \mathbb{E}(xy^T)\, \mathbb{E}(y^2)^{-1}$$
$$= \frac{\mathbb{E}[x(x+v)]}{\mathbb{E}[\cancel{x+v}]} \left(\mathbb{E}[(x+v)^2]\right)^{-1}$$
$$= \mathbb{E}[x^2 + xv^{\to 0}] \left(\cancel{\frac{1}{\lambda^2} + 2\dots\frac{1}{\mu^2}}\right)^{-1} = \left(\frac{1}{\lambda^2} + 0\right)\left(\frac{\mu^2+\lambda^2}{\mu^2\lambda^2}\right)^{-1} = \frac{\mu^2(\lambda^2)}{\lambda^2(\mu^2+\lambda^2)} = \frac{\mu^2}{\mu^2+\lambda^2}$$

For a linear model w/ $y = x + v$

$\Rightarrow H = I$, ... $K_0 = (R_x^{-1} + H^T R_v^{-1} H)^{-1} H^T R_v^{-1}$

$$= (\lambda^2 + \mu^2)^{-1} \mu^2 = \frac{\mu^2}{\lambda^2 + \mu^2}$$

match ✓

affine solution:

$$\Rightarrow \hat{x} = \mu_x + K_0 (y - \mu_y) \qquad \mathbb{E}(y) = \mathbb{E}(x+v) = \frac{1}{\lambda} + \frac{1}{\mu} = \frac{\mu+\lambda}{\lambda\mu} \checkmark$$

$$= \frac{1}{\lambda} + \frac{\mu^2}{(\mu^2+\lambda^2)}\left(y - \frac{\mu+\lambda}{\lambda\mu}\right)$$

$$= \frac{1}{\lambda} - \frac{\mu(\mu+\lambda)}{(\mu^2+\lambda^2)\lambda} + \frac{\mu^2 y}{\mu^2+\lambda^2} = \boxed{\frac{\lambda-\mu}{\mu^2+\lambda^2} + \frac{\mu^2}{\mu^2+\lambda^2} y = \hat{x}}$$

$$\frac{\mu^2+\lambda^2 - \mu^2 - \mu\lambda}{\lambda(\mu^2+\lambda^2)}$$

$\cancel{\lambda\mu^2 + \lambda^3 - \mu\lambda^2 - \lambda^2\mu}$

both $\hat{x}_{\text{non-LMSE}}$ and $\hat{x}_{\text{LLMS}}$ have a offset term, but the coefficient for the y in $\hat{x}_{\text{LLMS}}$ is constant, whereas $\hat{x}_{\text{non-LMS}}$ changes based on y.

also $\hat{x}_{\text{LLMS}}$ has ~~not~~ exponential terms ~~of~~ with y.

#8 - ~~Q1~~ | Optimal Nonlinear estimator for binary signals

observations: $y_i = x + v_i$, x & v_i are independent, real-valued, random

$\{v_i\}_{i\geq 0} \equiv V$ is a white-noise Gaussian process

$R_v = 1$, $\mathbb{E}(V) = \mu_v = 0$

#8-Q1 | let $x = \pm 1$ w/ equal probability

Show the optimal nonlinear LMS estimator of x given n obs $\{y_i\}_{i=0}^{n-1}$ is

$$\hat{x}_n = \tanh\left(\sum_{i=0}^{n-1} y_i\right)$$

note: expectation of a discrete variable: $\mathbb{E}(X) = \sum_{i=0}^{n-1} x_i\, p(x_i)$ for n possible states

note: joint probability mass function (pmf):

$$p_{X,Y}(x,y) = P(X=x \text{ and } Y=y) = P(X \cap Y)$$

~~in~~ in terms of conditional distributions:

$$p_{X,Y}(x,y) = P(Y=y \mid X=x)\cdot P(X=x) = P(X=x \mid Y=y)\cdot P(Y=y)$$ (chain rule of probability)

note: conditional pmf:

$$f(x|X) = P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$$

example: if we want to know the prob. that two dice = 3, and we know one die = 3, then $p(x|y) = \frac{1}{6}$ (and not $\frac{1}{36}$)

note: conditional expectation of a pmf:

$$\mathbb{E}[X|Y] = \sum_{X} x \, f_{X|Y}(x|Y)$$

$$= \sum_{X} x \, \frac{P(X,Y)}{P(Y)}$$

→

optimal estimator as shown in class is $\mathbb{E}[X|Y]$, thus:

$$\mathbb{E}[X|Y] = \sum_i x_i \frac{P(X_i, Y)}{P(Y)}$$

$$= \cancel{(\sum)}\, X_1 \frac{P(X_1, Y)}{P(Y)} + X_2 \frac{P(X_2, Y)}{P(Y)}$$

$P(y) = ?$

n observations: $\{y_i\}_{i=0}^{n-1}$, $P(Y) = \frac{1}{n}$? $x = y_i - v_i$

~~Similar to #7:~~

$y = x + v$, x, v independent. $\Rightarrow f_{XY}(x,y) = \dfrac{f_{XV}(x,v)}{\left|\frac{\partial(x,y)}{\partial(x,v)}\right|} = \dfrac{f_X(x) f_V(v)}{\left|\frac{\partial(x,y)}{\partial(x,v)}\right|}$

$v = y - x$

$$\left|\frac{\partial(x,y)}{\partial(x,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \Rightarrow f_{XY}(x,y) = f_X(x) f_V(y-x)$$

OK...

$$\boxed{f_X(x) = \begin{cases} 0.5 & \text{if } x = \pm 1 \\ 0 & \text{else} \end{cases}}$$

$$f_V(v) = \frac{1}{\sigma_v \sqrt{2\pi}} \exp\left(\frac{-(v-\mu)^2}{2\sigma_v}\right), \quad \mu_v = 0, \ \sigma_v = 1, \ \sigma_v^2 = 1 \Rightarrow f_V(v) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-v^2}{2}\right)$$

Gaussian pdf

$$\Rightarrow f_V(y-x) = \frac{\exp\left(\frac{-(y-x)^2}{2}\right)}{\sqrt{2\pi}}$$

$$\Rightarrow f_{X,Y}(x,y) = f_X(x) \left(\frac{1}{\sqrt{2\pi}}\right) \exp\left(\frac{-(y-x)^2}{2}\right) \longrightarrow$$

#8-Q1 cont'd

$$f_y = \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dx$$

$$= \sum_{x_i} f_x(x_i) \frac{\exp\left(\frac{-(y-x_i)^2}{2}\right)}{\sqrt{2\pi}} \qquad \{x\} = \{1, -1\}$$

$$f_y = 0.5\,\frac{\exp(-(y-1)^2/2)}{\sqrt{2\pi}} + 0.5\,\frac{\exp\left(\frac{-(y+1)^2}{2}\right)}{\sqrt{2\pi}}$$

$$\Rightarrow \frac{f_{x,y}(x,y)}{f_y(y)} = \frac{f_x(x)\left(\cancel{\frac{1}{\sqrt{2\pi}}}\right)\exp\left(\frac{-(y-x)^2}{2}\right)}{\frac{0.5}{\cancel{\sqrt{2\pi}}}\left[\exp\left(\frac{-(y-1)^2}{2}\right) + \exp\left(\frac{-(y+1)^2}{2}\right)\right]}$$

$$= \frac{2 f_x(x) \exp\left(\frac{-(y-x)^2}{2}\right)}{\left(\exp\left(\frac{-(y-1)^2}{2}\right) + \exp\left(\frac{-(y+1)^2}{2}\right)\right)}$$

$$\Rightarrow E[X|Y] = \sum_{x_i} x_i \frac{f_{x,y}(x_i,y)}{f(y)}$$

$$= 1\left(\frac{\cancel{2(0.5)}\exp\left(\frac{-(y-1)^2}{2}\right)}{\exp\left(\frac{-(y-1)^2}{2}\right) + \exp\left(\frac{-(y+1)^2}{2}\right)}\right) + (-1)\left(\frac{\cancel{2(0.5)}\exp\left(\frac{-(y+1)^2}{2}\right)}{\exp\left(\frac{-(y-1)^2}{2}\right) + \exp\left(\frac{-(y+1)^2}{2}\right)}\right)$$

$$= \frac{\exp\left(\frac{-(y-1)^2}{2}\right) - \exp\left(\frac{-(y+1)^2}{2}\right)}{\exp\left(-\frac{(y-1)^2}{2}\right) + \exp\left(-\frac{(y+1)^2}{2}\right)} \cdot \frac{e^2}{e^2}$$

$\boxed{??}$

$$e^{(-(y-1)^2)(\frac{1}{2})} = \sqrt{e^{(-(y+1)^2)}}$$

$$\frac{(y-1)^2}{2} + x = y$$

$$\frac{y^2-2y+1}{2} + x = y$$

$$x = 2y - \frac{(y)^2+1}{2}$$

$$\rightarrow \sum_{i=0}^{n-1} y_i$$

treat the sum as one variable?

So I'm almost there... just need to figure out how to get $\underline{f_{x,y}(x,y) = f_x(x)\exp\left(\sum_{i=0}^{n-1} y_i\right)}$

treat the sum as a single variable?...

hopefully other terms cancel...?

$$x^{2-3} = x^{\cancel{\Sigma}-1}?$$

??...

$$\frac{x^2}{x^3} = x^{-1}$$

8-3

#8-Q1 (cont'd)

let $Y = \sum_{i=0}^{n-1} y_i = \sum_{i=0}^{n-1} x + v_i = xn + \sum_{i=0}^{n-1} v_i$, let $V = \sum_{i=0}^{n-1} v_i$

note: b/c v_i is a white noise Gaussian process w/ 0 mean & unit variance,

$\Rightarrow E(V) = 0$, $\cancel{E(VV^T) = R_v = I}$

$\Rightarrow E[V^2] = n\sigma_v^2$ $E(v_i v_j) = 0$ b/c white noise i, j are uncorrelated

(b/c $E[v_i v_j] = 0$ b/c white noise

$\Rightarrow \sigma_V = \sqrt{n}\,\sigma_v = \sqrt{n}$

$(\sigma_V^2 = n\sigma_v^2) = n$

$\cancel{f_V(V) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(V)^2}{2R_v}\right)}$

pdf of V: still Gaussian:

$$f_V(V) = \frac{1}{\sigma_V\sqrt{2\pi}} \exp\left(\frac{-(V-\cancel{\mu_V})^2}{2\sigma_V^2}\right)$$

$$= \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{(V-\cancel{\mu_v}\sqrt{n})^2}{2\sigma_v^2\sqrt{n}^2}\right)$$

$$= \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{(V-\cancel{0})^2}{2\cancel{\sqrt{\pi}}\,n}\right) = \frac{1}{\sqrt{2\pi n}} \exp\left(\frac{-V^2}{2\cancel{\sqrt{\pi}}\,n}\right)$$

$$f_{x,Y}(x,Y) = \frac{f_{xV}(x,V)}{\left|\frac{\partial(x,Y)}{\partial(x,V)}\right|} = \frac{f_x(x) f_V(V)}{\left|\frac{\partial(x,Y)}{\partial(x,V)}\right|} , \quad \left|\frac{\partial(x,Y)}{\partial(x,V)}\right| = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial V} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ n & 1 \end{vmatrix} = 1$$

$\boxed{\text{blue is OK}}$

green is best

$\cancel{V = y - x}$?

$$= f_x(x) f_V(V) \cancel{/n} \qquad \cancel{f_V = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-V^2}{2}\right)}$$

$$f_Y(Y) = \int_{-\infty}^{\infty} f_{x,Y}(x,Y)\,dx = \sum_{x_i} f_x(x) \frac{\exp\left(-V^2/2\sqrt{n}\right)}{\sqrt{2\pi n}} = \sum_{x_i} f_x(x) \frac{\exp\left(-\frac{(Y-xn)^2}{2\sqrt{n}}\right)}{\sqrt{2\pi n}}$$

$$= 0.5\,\frac{\exp\left(-\frac{(Y-n)^2}{2\sqrt{n}}\right)}{\sqrt{2\pi n}} + 0.5\,\frac{\exp\left(\frac{-(Y+n)^2}{2\sqrt{n}}\right)}{\sqrt{2\pi n}}$$

$$\frac{f_{x,Y}(x,Y)}{f_Y(Y)} = \frac{f_x(x) \exp\left(-\frac{(Y-xn)^2}{n2\sqrt{n}}\right) / \sqrt{2\pi n}}{0.5\left[\exp\left(-\frac{(Y-n)^2}{n2\sqrt{n}}\right) + \exp\left(-\frac{(Y+n)^2}{n2\sqrt{n}}\right)\right] / \sqrt{2\pi n}}$$

$\longrightarrow$

$$\mathbb{E}[X|Y] = \sum_{x_i} x_i \frac{f_{x,y}(x_i, Y)}{f_y(Y)}$$

$$= (1)\frac{\overset{0.5}{\cancel{f_x(1)}} \exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right)}{0.5\left(\exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right) + \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)\right)} + \frac{(-1)\cancel{(0.5)} \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)}{0.5(\quad " \quad)}$$

$$= \frac{\exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right) - \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)}{\exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right) + \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)}$$

$$\frac{-(Y-n)^2}{n2\sqrt{n}} + \alpha = Y$$

$$\frac{-(Y^2 - 2Yn + n^2)}{n2\sqrt{n}} + \alpha = Y$$

$$\Rightarrow \alpha = Y + \frac{Y^2 - 2Yn + n^2}{n2\sqrt{n}}$$

$$= Y + \frac{Y^2 + n^2}{n2\sqrt{n}} - \frac{2Yn}{n2\sqrt{n}}$$

$$\frac{-(Y+n)^2}{n2\sqrt{n}} + \alpha_2 = Y$$

$$\frac{-(Y^2 + 2Yn + n^2)}{n2\sqrt{n}} + \alpha_2 = Y$$

$$\alpha_2 = Y + \frac{(Y^2 + 2Yn + n^2)}{n2\sqrt{n}}$$

---

multiply by $\exp\left(\frac{Y^2+n^2}{n2\sqrt{n}}\right) / \exp\left(\frac{Y^2+n^2}{n2\sqrt{n}}\right)$:

$$\mathbb{E}[X|Y] = \frac{\exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right) - \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)}{\exp\left(\frac{-(Y-n)^2}{n2\sqrt{n}}\right) + \exp\left(\frac{-(Y+n)^2}{n2\sqrt{n}}\right)} \left(\frac{\exp\left(\frac{Y^2+n^2}{n2\sqrt{n}}\right)}{\exp\left(\frac{Y^2+n^2}{n2\sqrt{n}}\right)}\right)$$

$$\frac{-(Y-n)^2}{2\sqrt{n}} \rightarrow \frac{-(\cancel{Y^2+n^2} - 2Yn)}{n2\sqrt{n}} + \frac{\cancel{Y^2+n^2}}{n2\sqrt{n}} = \frac{2Yn}{n2\sqrt{n}} = Y\sqrt{n}$$

$$\Rightarrow$$

$$\frac{-(Y+n)^2}{n2\sqrt{n}} + \frac{(Y^2+n^2)}{n2\sqrt{n}} = \frac{-(\cancel{Y^2+n^2} + 2Yn)}{n2\sqrt{n}} + \frac{\cancel{Y^2+n^2}}{n2\sqrt{n}} = -Y\sqrt{n}$$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{\exp(Y\cancel{\sqrt{n}}) - \exp(-Y\cancel{\sqrt{n}})}{\exp(Y\cancel{\sqrt{n}}) + \exp(-Y\cancel{\sqrt{n}})} = \tanh(\cancel{\sqrt{n}}Y)$$

$Y\sqrt{n} + \beta = Y$

$\beta = Y - Y\sqrt{n}$

$Y = \sum_{i=0}^{n-1} y_i$

$$\mathbb{E}[V^2] = \mathbb{E}[(\Sigma v_i)^2] = \mathbb{E}[(v_0 + v_1 + \dots + v_n)(v_0 + v_1 + \dots + v_n)]$$

$, \quad \mathbb{E}[v_i v_j] = 0$

$\mathbb{E}[v_i^2] = \sigma_v^2$

$$= \mathbb{E}[\Sigma v_i^2] \quad , \; i = 0, 1, \dots n$$

$$= n\sigma_v^2$$

$$\Rightarrow \sigma_V = \sqrt{n}\,\sigma_v = \sqrt{n} \quad \checkmark, \; \sigma_V^2 = n$$

$$\hat{x}_n = \boxed{\mathbb{E}[X|Y] = \frac{\exp(Y) - \exp(-Y)}{\exp(Y) + \exp(-Y)} = \tanh(Y) = \tanh\left(\sum_{i=0}^{n-1} y_i\right)}$$

$\square$

#8-Q2) Now assume x takes values of 1 w/ probability p, and -1 w/ probability $(1-p)$

from #8-Q1) we have that: $y_i = x + v_i \Rightarrow \sum_{i=0}^{n-1} y_i = xn + \sum_{i=0}^{n-1} v_i$

$\Rightarrow Y = xn + V$

$$f_V(v) = \cancel{f_x(x) f_V(v)} = \frac{1}{\sqrt{2\pi n}} \exp\left(\frac{-V^2}{2n}\right)$$

$$f_{xY}(x,Y) = f_x(x) f_V(V) = f_x(x) \exp\left(\frac{-(Y-xn)^2}{2n}\right) \Big/ \sqrt{2\pi n}$$

$$f_Y(Y) = \sum_{x_i} \frac{f_x(x_i) \exp\left(\frac{-(Y-x_i n)^2}{2n}\right)}{\sqrt{2\pi n}}$$

and $\mathbb{E}[X|Y] = \sum_i x_i \frac{P(x_i, Y)}{P(Y)} = \sum_i x_i \frac{f_{xY}(x_i, Y)}{f_Y(Y)}$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{(+1)(p)\left(\frac{1}{\cancel{\sqrt{2\pi n}}}\right)\left(\exp\left(\frac{-(Y-1n)^2}{2n}\right)\right)}{\frac{(p)\exp\left(\frac{-(Y-n)^2}{2n}\right)}{\cancel{\sqrt{2\pi n}}} + \frac{(1-p)\exp\left(\frac{-(Y+n)^2}{2n}\right)}{\cancel{\sqrt{2\pi n}}}}$$

$$+ \frac{(-1)(1-p)\left(\frac{1}{\cancel{\sqrt{2\pi n}}}\right)\exp\left(\frac{-(Y+n)^2}{2n}\right)}{\frac{(p)\exp\left(\frac{-(Y-n)^2}{2n}\right) + (1-p)\exp\left(\frac{-(Y+n)^2}{2n}\right)}{\cancel{\sqrt{2\pi n}}}}$$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{p\exp\left(\frac{-(Y-n)^2}{2n}\right) - (1-p)\exp\left(\frac{-(Y+n)^2}{2n}\right)}{p\exp\left(\frac{-(Y-n)^2}{2n}\right) + (1-p)\exp\left(\frac{-(Y+n)^2}{2n}\right)} \left(\frac{\exp\left(\frac{Y^2+n^2}{2n}\right)}{\exp\left(\frac{Y^2+n^2}{2n}\right)}\right)$$

$\longrightarrow$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{p\exp(Y) - (1-p)\exp(-Y)}{p(\exp(Y)) + (1-p)\exp(-Y)}$$

$y = \ln x$

$\Leftrightarrow x = e^{y}$

note: $p\exp(Y) = \exp(Y + \ln(p))$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{\exp(Y + \ln p) - \exp(-Y + \ln(1-p))}{\exp(Y + \ln(p)) + \exp(-Y + \underbrace{\ln(1-p)}_{-(-\ln(1-p))})}$$

note:

$$\ln(p) - \frac{1}{2}\ln\left(\frac{p}{1-p}\right) = \ln(p) - \frac{1}{2}(\ln(p) - \ln(1-p)) = \frac{\ln(p)}{2} - \frac{\ln(\cancel{p}\,1-p)}{2} = \frac{1}{2}\ln\frac{(p)}{\cancel{\ln}(1-p)}$$

$$+\ln(1-p) - \frac{1}{2}\ln\left(\frac{p}{1-p}\right) = +\ln(1-p) - \frac{1}{2}(\ln(p) - \ln(1-p))$$

$$= \cancel{? - \frac{1}{2}(\ln(p)}$$

$$\ln(1-p) + \alpha = -\frac{1}{2}\ln\left(\frac{p}{1-p}\right) = -\frac{1}{2}(\ln(p) - \ln(1-p)) = -\frac{1}{2}\ln(p) + \frac{1}{2}\ln(1-p)$$

$$\left[\Rightarrow \alpha = -\frac{1}{2}\ln(p) - \frac{1}{2}\ln(1-p) = \underbrace{-\frac{1}{2}\ln\left(\frac{p}{1-p}\right)}\right] \qquad (-\cancel{\tfrac{1}{2}})(-\tfrac{3}{2})$$

$$\cancel{\ln(1-p) - \frac{1}{2}(\ln(p) - \ln(1-p)) = \ln(1-p) - \frac{1}{2}\ln(p) + \frac{1}{2}\ln(1-p) = -\frac{1}{2}\ln(p) + \frac{3}{2}\ln(1-p)}$$

$$\cancel{= \ln(1-p) - \frac{1}{2}\ln(p) + \ln(1-p)^{1/2} = -\frac{1}{2}\left(\ln(p) + \ln(1-p)^{3/2}\right)}$$

$$\cancel{= -\left(-\ln(1-p) + \frac{1}{2}\ln(p)^{1/2} + \ln(1-p)^{1/2}\right)}$$

$$\cancel{= -\left(\ln\left(\frac{p^{1/2}}{(1-p)(1-p)^{1/2}}\right)\right)} \ ?$$

$$\Rightarrow \mathbb{E}[X|Y] = \frac{\exp(Y + \ln(p)) - \exp(-Y + \ln(1-p))}{\exp(Y + \ln(p)) + \exp(-Y + \ln(1-p))}\left(\frac{\exp\left(-\frac{1}{2}\ln\left(\frac{p}{1-p}\right)\right)}{\exp\left(-\frac{1}{2}\ln\left(\frac{p}{1-p}\right)\right)}\right)$$

$$\boxed{\Rightarrow \mathbb{E}[X|Y] = \hat{x}_n = \frac{\exp\left(\frac{1}{2}\ln\left(\frac{p}{1-p}\right) + Y\right) - \exp\left(-\frac{1}{2}\ln\left(\frac{p}{1-p}\right) - Y\right)}{\exp\left(\frac{1}{2}\ln\left(\frac{p}{1-p}\right) + Y\right) + \exp\left(-\frac{1}{2}\ln\left(\frac{p}{1-p}\right) - Y\right)} = \tanh\left(\frac{1}{2}\ln\left(\frac{p}{1-p}\right) + \sum_{i=0}^{n-1} y_i\right) \quad \square}$$