ECE 6555 Midterm Bonus Problems

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Due: 11/7/22

#1) A non-standard state-space model.

$$\begin{cases} \vec{x}_{i+1} = \underline{F}_i \vec{x}_i + \underline{G}_i \vec{u}_{i+1} \\ \vec{y}_i = \underline{H}_i \vec{x}_i + \vec{v}_i \end{cases} \quad \text{for } i \geq 0 \qquad (1)$$

Random variables $\vec{x}_0, \vec{u}_i, \vec{v}_i$ that satisfy:

$\langle \cdot, \cdot \rangle = \|\cdot\|^2 = \sqrt{\langle \cdot, \cdot \rangle}$

$\langle x, y \rangle = E[x y^T]$

$$\left\langle \begin{bmatrix} \vec{x}_0 \\ \vec{u}_i \\ \vec{v}_i \end{bmatrix}, \begin{bmatrix} \vec{x}_0 \\ \vec{u}_j \\ \vec{v}_j \\ 0 \end{bmatrix} \right\rangle = \begin{bmatrix} \langle \vec{x}_0, \vec{x}_0 \rangle & \langle \vec{x}_0, \vec{u}_j \rangle & \langle \vec{x}_0, \vec{v}_j \rangle & \langle \vec{x}_0, 0 \rangle \\ \langle \vec{u}_i, \vec{x}_0 \rangle & \langle \vec{u}_i, \vec{u}_j \rangle & \langle \vec{u}_i, \vec{v}_j \rangle & \langle \vec{u}_i, 0 \rangle \\ \langle \vec{v}_i, \vec{x}_0 \rangle & \langle \vec{v}_i, \vec{u}_j \rangle & \langle \vec{v}_i, \vec{v}_j \rangle & \langle \vec{v}_i, 0 \rangle \end{bmatrix}$$

~~assume~~ $\{\vec{u}_i\}, \{\vec{v}_i\}$ are white noise. per cross-correlation matrix to right

$$= \begin{bmatrix} \Pi_0 & 0 & 0 & 0 \\ 0 & Q_i \delta_{ij} & S_i \delta_{ij} & 0 \\ 0 & S_i^+ \delta_{ij} & R_i \delta_{ij} & 0 \end{bmatrix} \qquad (2)$$

Q1) a) Show that $\langle \vec{u}_i, \vec{x}_i \rangle = Q_i \underline{G}_{i-1}^+$

b) $\langle \vec{u}_i, \vec{x}_j \rangle = 0$, $i > j$

c) $\langle \vec{v}_i, \vec{x}_j \rangle = 0$, $i > j$

Correlation vs. covariance...

$K_{xx} = R_{xx} - E[x]E[x]^T$

$K_{xx} = E[(x-\mu_x)(x-\mu_x)^T]$

$R_{xx} = E[xx^T]$

note too: $\langle x, y \rangle = \cancel{K_{xy}} R_{xy} = E[xy^T]$

see Lec 06 pg 7 & 9

a) $\langle \vec{u}_i, \vec{x}_i \rangle = \langle \vec{u}_i, F_{i-1}\vec{x}_{i-1} + G_{i-1}\vec{u}_i \rangle$

(reference Lec. 13)

$$= \langle \vec{u}_i, \vec{x}_{i-1} \rangle F_{i-1}^+ + \langle \vec{u}_i, \vec{u}_i \rangle G_{i-1}^+ \qquad (5)$$

$\rightarrow \langle \vec{u}_i, \vec{x}_{i-1} \rangle = 0$.

b/c $\vec{x}_{i-1} \in \text{Span}(\vec{x}_0, u_1, u_2, u_3, \ldots, u_{i-1})$

that is, $\vec{x}_{i-1} = F_{i-2}\vec{x}_{i-2} + G_{i-2}\vec{u}_{i-1}$

$$= F_{i-2}(F_{i-3}\vec{x}_{i-3} + G_{i-3}\vec{u}_{i-2}) + G_{i-2}\vec{u}_{i-1}$$

$$= F_{i-2}(F_{i-3}(F_{i-4}\vec{x}_{i-4} + G_{i-4}\vec{u}_{i-3}) + G_{i-3}\vec{u}_{i-2}) + G_{i-2}\vec{u}_{i-1} \rightarrow$$

$$\Rightarrow \vec{x}_{i-1} = \cancel{F_{i-2} F_{i-3} \cdots F_0 \vec{x}_0} + F_{i-2}(F_{i-3}(F_{i-4}(\cdots(F_0 \vec{x}_0 + G_0 \vec{u}_1) + G_1 \vec{u}_2) + G_2 \vec{u}_3) \cdots) + G_{i-2} \vec{u}_{i-1}$$

$$\cancel{= F_{i-2} F_{i-3} \cdots F_0 \vec{x}_0 + F_{i-2}(F_{i-3}(F_{i-4} \cdots (F_1 G_0 \vec{u}_1) +}$$

$$\vec{x}_{i-1} = F_{i-2}(F_{i-3}(F_{i-4}(\cdots(F_1(F_0 \vec{x}_0 + G_0 \vec{u}_1) + G_1 \vec{u}_2) + G_2 \vec{u}_3) \cdots) + G_{i-2} \vec{u}_{i-1}$$

$$= F_{i-2}(F_{i-3}(F_{i-4}(\cdots(F_2 F_1 F_0 \vec{x}_0 + F_2 F_1 G_0 \vec{u}_1 + F_2 G_1 \vec{u}_2) + G_2 \vec{u}_3) \cdots) + G_{i-2} \vec{u}_{i-1}$$

$$= F_{i-2} F_{i-3} \cdots F_1 F_0 \vec{x}_0 + F_{i-2} F_{i-3} \cdots F_1 G_0 \vec{u}_1 + F_{i-2} F_{i-3} \cdots F_2 G_1 \vec{u}_2$$
$$+ F_{i-2} F_{i-3} \cdots F_3 G_2 \vec{u}_3 + F_{i-2} F_{i-3} \cdots F_4 G_3 \vec{u}_4$$
$$+ \cdots + \cancel{F_{i-2} G_{i-2} \vec{u}_{i-1}} F_{i-2} G_{i-3} \vec{u}_{i-2} + G_{i-2} \vec{u}_{i-1}$$

$$\Rightarrow \langle \vec{u}_i, \vec{x}_{i-1} \rangle = \langle \vec{u}_i, \vec{x}_0 \rangle F_0^T F_1^T \cdots F_{i-3}^T F_{i-2}^T$$
$$+ \langle \vec{u}_i, \vec{u}_1 \rangle G_0^T F_1^T \cdots F_{i-2}^T + \langle \vec{u}_i, \vec{u}_2 \rangle G_1^T F_2^T F_3^T \cdots F_{i-2}^T$$
$$+ \cdots + \langle \vec{u}_i, \vec{u}_{i-2} \rangle G_{i-3}^T F_{i-2}^T + \langle \vec{u}_i, \vec{u}_{i-1} \rangle G_{i-2}^T$$

and per our stochastic model given in 2, we have that

$\langle u_i, \vec{x}_0 \rangle = 0$ and $\langle u_i, u_j \rangle = 0$ for $i \neq j$

thus, $\langle u_i, \vec{x}_{i-1} \rangle = 0$ (3)

if we substitute j for $i-1$ in $\vec{x}_{i-1}$ then we have

$$\langle \vec{u}_i, \vec{x}_j \rangle = \langle \vec{u}_i, \vec{x}_0 \rangle F_0^T F_1^T \cdots F_{j-2}^T F_{j-1}^T$$
$$+ \langle \vec{u}_i, \vec{u}_1 \rangle G_0^T F_1^T \cdots F_{j-1}^T + \langle \vec{u}_i, \vec{u}_2 \rangle G_1^T F_2^T F_3^T \cdots F_{j-1}^T$$
$$+ \cdots + \langle \vec{u}_i, \vec{u}_{j-1} \rangle G_{j-2}^T F_{j-1}^T + \langle \vec{u}_i, \vec{u}_j \rangle G_{j-1}^T \quad (4)$$

b) and if $i > j$, then as shown above for 3, we have that $\langle \vec{u}_i, \vec{x}_j \rangle = 0$ per the stochastic model given in 2.

$\sqrt{2}$

Thus, from ⑤ we now have:

$$\langle \vec{u}_i, \vec{x}_i \rangle = \cancel{\langle \vec{u}_i, \vec{x}_{i-1} \rangle}^{\,0} F_{i-1}^{+} + \langle \vec{u}_i, \vec{u}_i \rangle G_{i-1}^{+}$$

a) $\Rightarrow \boxed{\langle \vec{u}_i, \vec{x}_i \rangle = Q_i G_{i-1}^{+}}$ ⑥

□

$$\langle \vec{v}_i, \vec{x}_j \rangle = \langle \vec{v}_i, F_{j-1}\vec{x}_{j-1} + G_{j-1}\vec{u}_j \rangle$$

$$\vec{x}_j = F_{j-1}\vec{x}_{j-1} + G_{j-1}\vec{u}_j$$

$$= F_{j-1}\left(F_{j-2}(\vec{x}_{j-2}) + G_{j-2}\vec{u}_{j-1}\right) + G_{j-1}\vec{u}_j$$

$$= F_{j-1}\left(F_{j-2}\left(\cdots\left(F_1(F_0\vec{x}_0 + G_0\vec{u}_1) + G_1\vec{u}_2\right) + \cdots\right) + G_{j-2}\vec{u}_{j-1}\right) + G_{j-1}\vec{u}_j$$

$$= F_{j-1}F_{j-2}\cdots F_2F_1F_0\vec{x}_0 + F_{j-1}F_{j-2}\cdots F_1G_0\vec{u}_1 + F_{j-1}F_{j-2}\cdots F_2G_1\vec{u}_2$$

$$+ F_{j-1}F_{j-2}\cdots F_3G_2\vec{u}_3 + \cdots + F_{j-1}F_{j-2}G_{j-3}\vec{u}_{j-2}$$

$$+ F_{j-1}G_{j-2}\vec{u}_{j-1} + G_{j-1}\vec{u}_j \quad ⑦$$

$$\Rightarrow \langle \vec{v}_i, \vec{x}_j \rangle = \langle \vec{v}_i, \vec{x}_0 \rangle F_0^T F_1^T \cdots F_{j-2}^T F_{j-1}^T$$

$$+ \langle \vec{v}_i, \vec{u}_1 \rangle G_0^T F_1^T \cdots F_{j-2}^T F_{j-1}^T$$

$$+ \langle \vec{v}_i, \vec{u}_2 \rangle G_1^T F_2^T \cdots F_{j-2}^T F_{j-1}^T$$

$$\vdots$$

$$+ \langle \vec{v}_i, \vec{u}_{j-1} \rangle G_{j-2}^T F_{j-1}^T$$

$$+ \langle \vec{v}_i, \vec{u}_j \rangle G_{j-1}^T \quad ⑧$$

from ② we have that $\langle \vec{v}_i, \vec{x}_0 \rangle = 0$ and $\langle \vec{v}_i, \vec{u}_j \rangle = S_i^+ \delta_{ij}$

Thus if ~~i≠0~~ $i > j$ then $\langle \vec{v}_i, \vec{u}_j \rangle = 0$

c) $\boxed{\text{and } \langle \vec{v}_i, \vec{x}_j \rangle = 0 \text{ for } i > j}$ ⑨

□

#1-Q2 a) show $\langle \vec{u}_i, \vec{y}_j \rangle = Q_i G_{i-1}^T H_i^T + S_i$

b) $\langle u_i, y_j \rangle = 0 \quad \forall\, i>j$

c) $\langle \vec{v}_i, y_j \rangle = 0 \quad \forall\, i>j$

$$\vec{y}_j = H_j \vec{x}_j + \vec{v}_j$$
$$= H_j (F_{j-1} \vec{x}_{j-1} + G_{j-1} u_j) + \vec{v}_j$$
$$\vdots$$

as shown earlier for $\vec{x}_j$

$$\Rightarrow \vec{y}_j = H_j\big(F_{j-1} F_{j-2} \cdots F_2 F_1 F_0 \vec{x}_0 + F_{j-1} F_{j-2} \cdots F_1 G_0 u_1 + F_{j-1} F_{j-2} \cdots F_2 G_1 u_2 + \cdots + F_{j-1} G_{j-2} u_{j-1} + G_{j-1} u_j \big) + \vec{v}_j \qquad (10)$$

Thus,
$$\langle \vec{u}_i, \vec{y}_j \rangle = \langle \vec{u}_i, \vec{x}_0 \rangle F_0^T F_1^T \cdots F_{j-1}^T H_j^T$$
$$+ \langle \vec{u}_i, \vec{u}_1 \rangle G_0^T F_1^T F_2^T \cdots F_{j-1}^T H_j^T$$
$$+ \langle \vec{u}_i, \vec{u}_2 \rangle G_1^T F_2^T \cdots F_{j-1}^T H_j^T$$
$$\vdots$$
$$+ \langle \vec{u}_i, \vec{u}_j \rangle G_{j-1}^T H_j^T$$
$$+ \langle \vec{u}_i, \vec{v}_j \rangle \qquad (11)$$

From (2) we are given that $\langle \vec{u}_i, \vec{x}_0 \rangle = 0$ and $\langle \vec{u}_i, \vec{u}_j \rangle = 0 \ \forall\, i>j$

and $\langle \vec{u}_i, \vec{v}_j \rangle = S_i \delta_{ij}$

b) $\boxed{\text{Thus, combining (2) and (11) we have that } \langle \vec{u}_i, \vec{y}_j \rangle = 0 \ \forall\, i>j}$ $\square$

$\longrightarrow$

Similarly,

$$\langle \vec{v}_i, \vec{y}_j \rangle = \langle \vec{v}_i, \vec{x}_0 \rangle F_0^T F_1^T \cdots F_{j-2}^T F_{j-1}^T H_j^T$$
$$+ \langle \vec{v}_i, \vec{u}_1 \rangle G_0^T F_1^T \cdots F_{j-1}^T H_j^T$$
$$\vdots$$
$$+ \langle \vec{v}_i, \vec{u}_j \rangle G_{j-1}^T H_j^T$$
$$+ \langle \vec{v}_i, \vec{v}_j \rangle \quad (12)$$

c) Combining the stochastic model of (2) with (12) we have that

$$\boxed{\langle \vec{v}_i, \vec{y}_j \rangle = 0 \quad \forall\, i > j}$$

$\square$

Now, if $i=j$ then from (11) we have:

$$\langle \vec{u}_i, \vec{y}_j \rangle = \langle \vec{u}_i, \vec{y}_i \rangle = \langle \vec{u}_i, \vec{x}_0 \rangle F_0^T F_1^T \cdots F_{j-1}^T H_j^T$$
$$+ \langle \vec{u}_i, \vec{u}_1 \rangle G_0^T F_1^T \cdots F_{i-1}^T H_i^T$$
$$\vdots$$
$$+ \langle \vec{u}_i, \vec{u}_i \rangle G_{i-1}^T H_i^T$$
$$+ \langle \vec{u}_i, \vec{v}_i \rangle \quad (13)$$

Combining (13) with the information of (2), we have that

a)
$$\boxed{\begin{aligned} \langle \vec{u}_i, \vec{y}_i \rangle &= Q_i G_{i-1}^T H_i^T + S_i \delta_{ij} \\ &= Q_i G_{i-1}^T H_i^T + S_i \end{aligned}} \quad (14)$$

$\square$

#2 A different State-Space Model

$$\begin{cases} x_{i+1} = Fx_i + Gu_i \\ y_i = H_0 x_i + H_1 x_{i-1} + v_i \end{cases}, \quad i \geq 0$$

u_i, v_i satisfy: $\left\langle \begin{bmatrix} u_i \\ v_i \end{bmatrix}, \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} Q_i \delta_{ij} & S_i \delta_{ij} & 0 \\ S_i^{+} \delta_{ij} & R_i & 0 \end{bmatrix}$

↑ i.e. 0-mean

define $z_0 = \begin{bmatrix} x_{i-1} \\ x_0 \end{bmatrix}$ where $\langle z_0, z_0 \rangle = \Pi_0$

$\langle z_0, u_i \rangle = 0$

$\langle z_0, v_i \rangle = 0$, $i \geq 0$

Find recursions for computing the innovations. →

See page 2-8 & 2-11 for a summary of results

1) Get properties of statespace ~~model~~ model:

First, re-write in terms of z: (let $\mu_i = \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}$)

$$\begin{cases} z_{i+1} = \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} Fx_{i-1} + Gu_{i-1} \\ Fx_i + Gu_i \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x_{i-1} \\ x_i \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix} \\ \quad = F_z z_i + G_z \mu_i \\ y_i = \cancel{\begin{bmatrix} H_0 & 0 \\ 0 & H_0 \end{bmatrix}} \begin{bmatrix} x_{i-1} \\ x_i \end{bmatrix} + v_i = H_z z_i + v_i \end{cases}$$

$[H_1 \quad H_0]$

Note: would have been much simpler to write $z_{i+1} = \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ 0 & F \end{bmatrix}}_{F_z} \begin{bmatrix} x_{i-1} \\ x_i \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}}_{G_z} \begin{bmatrix} 0 \\ u_i \end{bmatrix}$

μ_i

→ expectations would have been much easier to compute!?!

gah! should have watched OH...

now,

$\langle u_i, z_j \rangle = 0$?

$\langle u_i, z_j \rangle = ?$

$\langle u_i, y_j \rangle = ?$

$\langle v_i, y_j \rangle = ?$

$$\langle u_i, z_j \rangle = \langle u_i, F_2 z_{i-1} + G_2 \mu_i \rangle = \langle u_i, z_{i-1} \rangle F_2^{+} + \langle u_i, \mu_i \rangle G_2$$

where $\langle u_i, z_{i-1} \rangle = \langle u_i, \begin{bmatrix} x_{j-2} \\ x_{j-1} \end{bmatrix} \rangle = \begin{bmatrix} \langle u_i, x_{j-2} \rangle \\ \langle u_i, x_{j-1} \rangle \end{bmatrix}$

<u>Note:</u>

$$z_i = F_2 z_{i-1} + G_2 \mu_{i-1} = F_2 (F_2 z_{i-2} + G_2 \mu_{i-2}) + G_2 \mu_{i-1}$$

$$= \overset{i-1}{F_2} (\overset{i-2}{F_2} (\overset{i-3}{F_2} z_{i-3} + G_2 \mu_{i-3}) + G_2 \mu_{i-2}) + G_2 \mu_{i-1}$$

$$= \overset{i-1}{F_2} (\overset{i-2}{F_2} (\cdots (\overset{2}{F_2} (\overset{1}{F_2} (\underbrace{\overset{0}{F_2} z_0 + G_2 \mu_0}_{z_1}) + G_2 \mu_1) + G_2 \mu_2) + \cdots) + G_2 \mu_{i-2}) + G_2 \mu_{i-1}$$

(with $F_2 (\ldots) + G_2 \mu_1$ labelled z_2)

$$= \overset{i-1}{F_2} \overset{i-2}{F_2} \cdots \overset{2}{F_2} \overset{1}{F_2} \overset{0}{F_0} z_0 + \overset{i-1}{F_2} \overset{i-2}{F_2} \cdots \overset{2}{F_2} \overset{1}{F_2} G_2 \mu_0 + \overset{i-1}{F_2} \overset{i-2}{F_2} \cdots \overset{2}{F_2} G_2 \mu_1$$

$$+ \overset{i-1}{F_2} \overset{i-2}{F_2} \cdots \overset{3}{F_2} G_2 \mu_2 + \cdots + \overset{i-1}{F_2} G_2 \mu_{i-2} + G_2 \mu_{i-1}$$

$\Rightarrow$ $\cancel{\langle u_i, x_{j-2} \rangle = 0 \ \forall i \geq j-3}$

~~Similarly~~, thus:

$$x_i = \overset{i-1}{F} \overset{i-2}{F} \cdots \overset{2}{F} \overset{1}{F} \overset{0}{F} x_0 + \overset{i-1}{F} \overset{i-2}{F} \cdots \overset{2}{F} \overset{1}{F} G_2 u_0 + \overset{i-1}{F} \overset{i-2}{F} \cdots \overset{2}{F} G_2 u_1$$

$$+ \overset{i-1}{F} \overset{i-2}{F} \cdots \overset{3}{F} G_2 u_2 + \cdots + F G_2 u_{i-2} + G_2 u_{i-1}$$

$$x_{i-1} = \text{''} \quad x_{-1} + \text{''} \quad G_2 \overset{\text{D.N.E.}}{u_{-1}} \to 0 + \text{''} \quad u_0$$

$$+ \text{''} \quad G_2 u_1 + \cdots + F G_2 u_{i-3} + G_2 u_{i-2}$$

$\Rightarrow \langle u_i, x_j \rangle = 0, \quad i \geq j$

$\Rightarrow \langle u_i, \underbrace{\begin{bmatrix} x_{j-1} \\ x_j \end{bmatrix}}_{z_j} \rangle = 0, \quad i \geq j$

clearly written: $\boxed{\langle u_i, z_j \rangle = 0, \quad i \geq j}$

$\cancel{<\ldots} \; \langle v_i, z_j \rangle = \langle v_i, \begin{bmatrix} x_{j-1} \\ x_j \end{bmatrix} \rangle = \underset{i \geq j}{0}$ b/c $x_{j-1} \overset{\cancel{\in \text{Span}}}{=} f(x_{-1}, u_{-1}, u_0, \ldots, u_{j-2})$

$x_j = f(x_0, u_0, u_1, \ldots, u_{j-1})$

$\boxed{\langle v_i, z_j \rangle = 0} \; \boxed{i \geq j}$

$\Rightarrow z_j = f(x_{-1}, x_0, u_0, u_1, \ldots, u_{j-1})$

$\langle u_i, y_j \rangle = \langle u_i, H_2 z_j + v_j \rangle$ $\boxed{= 0, \; i > j}$

$= \cancel{\langle u_i, z_j \rangle}^{0} H_2^+ + \langle u_i, v_j \rangle = \boxed{\cancel{S_i \delta_{ij} = \langle u_i, y_j \rangle}}$

$\boxed{S_i = \langle u_i, y_j \rangle \text{ if } i = j}$

$\langle v_i, y_j \rangle = \langle v_i, H_2 z_j + v_j \rangle$

$= \langle v_i, z_j \rangle H_2^+ + \langle v_i, v_j \rangle =$

<u>if $i > j$</u>

$\langle v_i, y_j \rangle = 0 + 0 = 0$

<u>if $i = j$</u>

$\langle v_i, y_j \rangle = 0 + R_i = R_i$

Step 2; <u>Recursions for innovations</u>

$e_i = y_i - \hat{y}_{i|i-1} = y_i - \sum_{j=0}^{i-1} \langle y_i, e_j \rangle \|e_j\|^{-2} e_j$

$y_i = H_2 z_i + v_i \Rightarrow e_i = y_i - \sum_{j=0}^{i-1} \langle H_2 z_i + v_i, e_j \rangle \|e_j\|^{-2} e_j$

for $i > j$: $\langle H_2 z_i + v_i, e_j \rangle = H_2 \langle z_i, e_j \rangle + \cancel{\langle v_i, e_j \rangle}^{0}$

$e_j \in \text{Span}(y_0, y_1, \ldots, y_j)$

$= H_2 \langle z_i, e_j \rangle$

thus, $e_i = y_i - \sum_{j=0}^{i-1} \langle y_i, e_j \rangle \|e_j\|^{-2} e_j$

$\Rightarrow \boxed{e_i = y_i - H_2 \sum_{j=0}^{i-1} \langle z_i, e_j \rangle \|e_j\|^{-2} e_j}$

$\boxed{= y_i - H_2 \hat{z}_{i|i-1}}$

$\rightarrow$

$$\hat{z}_{i+1|i} = \sum_{j=0}^{i} \langle z_{i+1}, e_j \rangle \|e_j\|^{-2} e_j = \underbrace{\langle z_{i+1}, e_i \rangle \|e_i\|^{-2}}_{K_{p,i}} e_i + \sum_{j=0}^{i-1} \langle z_{i+1}, e_j \rangle \|e_j\|^{-2} e_j$$

note: $z_{i+1} = F_2 z_i + G_2 \mu_i$

$$\Rightarrow \hat{z}_{i+1|i} = K_{p,i} e_i + \sum_{j=0}^{i-1} \langle F_2 z_i + G_2 \mu_i, e_j \rangle \|e_j\|^{-2} e_j$$

where for $i>j$: $\langle F_2 z_i + G_2 \mu_i, e_j \rangle = F_2 \langle z_i, e_j \rangle + G_2 \langle \mu_i, e_j \rangle$

note $e_j \in \text{Span}(y_0, y_1, \cdots y_j) \Rightarrow \langle \mu_i, e_j \rangle \cancel{\Rightarrow 0} = \langle \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}, e_j \rangle = \begin{cases} 0 & j \le i-1 \\ \begin{bmatrix} S_{i-1} \\ 0 \end{bmatrix} & j = i-1 \end{cases}$

then $\langle F_2 z_i + G_2 \mu_i, e_j \rangle = F_2 \langle z_i, e_j \rangle$, $i>j$

$= F_2 \langle z_i, e_j \rangle + G_2 \begin{bmatrix} S_{i-1} \\ 0 \end{bmatrix}$, $i-1=j$

and we have that

$$\hat{z}_{i+1|i} = K_{p,i} e_i + \sum_{j=0}^{i-1} F_2 \langle z_i, e_j \rangle \|e_j\|^{-2} e_j + \begin{bmatrix} G S_{i-1} \\ 0 \end{bmatrix}$$

since $i>j$ for the whole summation

$$\boxed{\Rightarrow \hat{z}_{i+1|i} = K_{p,i} e_i + F_2 \hat{z}_{i|i-1} + G_2 \begin{bmatrix} S_{i-1} \\ 0 \end{bmatrix}}$$

Recursion for $K_{p,i}$ ...

1) First find $\|e_i\|^{-2}$: note $e_i = y_i - H_2 \hat{z}_{i|i-1} = H_2 z_i + v_i - H_2 \hat{z}_{i|i-1}$

let $\tilde{z}_i = z_i - \hat{z}_{i|i-1}$, $\hat{z}_i \triangleq \hat{z}_{i|i-1}$, $P_i \triangleq \langle \tilde{z}_i, \tilde{z}_i \rangle$

then, $\boxed{e_i = H_2 \tilde{z}_i + v_i}$

so that $\|e_i\|^{-2} = \langle e_i, e_i \rangle = \langle H_2 \tilde{z}_i + v_i, H_2 \tilde{z}_i + v_i \rangle$

$= H_2 P_i H_2^{+} + H_2 \langle \tilde{z}_i, v_i \rangle + \langle v_i, \tilde{z}_i \rangle H_2^{+} + R_i$

Now, $\langle \tilde{z}_i, v_i \rangle = \langle z_i - \hat{z}_i, v_i \rangle = \overset{0}{\cancel{\langle z_i, v_i \rangle}} - \langle \hat{z}_i, v_i \rangle$

note: $\hat{z}_i = \sum_{j=0}^{i-1} \langle z_i, e_j \rangle \|e_j\|^{-2} e_j$, which is the projection of z_i onto the Span$(y_0, y_1, \ldots, y_j)$

and since $\langle v_i, y_j \rangle = 0$, $i>j$ then $\langle \hat{z}_i, v_i \rangle = 0$

therefore we have that $\boxed{\langle \tilde{z}_i, v_i \rangle = \langle z_i, v_i \rangle - \langle \hat{z}_i, v_i \rangle = 0}$

→

also, $\langle \tilde{\xi}_i, v_i \rangle = (\langle v_i, \tilde{\xi}_i \rangle)^*$ (in $\mathbb{C}^n$)

thus $0 = (\langle v_i, \tilde{\xi}_i \rangle)^* \Rightarrow \boxed{\langle v_i, \tilde{\xi}_i \rangle = 0}$

therefore, $\|e_i\|^{-2} = H_2 P_i H_2^+ + H_2 \cancelto{0}{\langle \tilde{\xi}_i, v_i \rangle} + \cancelto{0}{\langle v_i, \tilde{\xi}_i \rangle} H_2^+ + R_i$

$\Rightarrow \boxed{\|e_i\|^{-2} = H_2 P_i H_2^+ + R_i}$

2) Recall $K_{p,i} = \langle \xi_{i+1}, e_i \rangle \|e_i\|^{-2}$

So now let's inspect $\langle \xi_{i+1}, e_i \rangle = \langle F_2 \xi_i + G_2 \mu_i, e_i \rangle$

$= F_2 \langle \xi_i, e_i \rangle + G_2 \langle \mu_i, e_i \rangle$

A) $\langle \xi_i, e_i \rangle = \langle \xi_i, H_2 \tilde{\xi}_i + v_i \rangle$

$= \langle \xi_i, \tilde{\xi}_i \rangle H_2^+ + \cancelto{0}{\langle \xi_i, v_i \rangle}$ per earlier (pg 2-3)

$= \langle \xi_i - \hat{\xi}_i + \hat{\xi}_i, \tilde{\xi}_i \rangle H_2^+ = \langle \tilde{\xi}_i + \hat{\xi}_i, \tilde{\xi}_i \rangle H_2^+$

$= P_i H_2^+ + \cancelto{0}{\langle \hat{\xi}_i, \tilde{\xi}_i \rangle} H_2^+$

prediction is $\perp$ to error

thus $\boxed{\langle \xi_i, e_i \rangle = P_i H_2^+}$

B) $\langle \mu_i, e_i \rangle = \begin{bmatrix} \langle u_{i-1}, e_i \rangle \\ \langle u_i, e_i \rangle \end{bmatrix}$

B-1) $\langle u_{i-1}, H_2 \tilde{\xi}_i + v_i \rangle = \langle u_{i-1}, \tilde{\xi}_i \rangle H_2^+ + \cancelto{0}{\langle u_{i-1}, v_i \rangle}$

$= \langle u_{i-1}, \xi_i - \hat{\xi}_i \rangle H_2^+ = \boxed{\langle u_{i-1}, \xi_i \rangle H_2^+ - \langle u_{i-1}, \hat{\xi}_i \rangle H_2^+ = \langle u_{i-1}, e_i \rangle}$

B-1.1) $\langle u_{i-1}, \xi_i \rangle = \langle u_{i-1}, F_2 \xi_{i-1} + G_2 \mu_{i-1} \rangle = \cancelto{0}{\langle u_{i-1}, \xi_{i-1} \rangle} F_2^+ + \langle u_{i-1}, \begin{bmatrix} u_{i-2} \\ u_{i-1} \end{bmatrix} \rangle G_2^+$

$= \begin{bmatrix} \langle u_{i-1}, u_{i-2} \rangle \\ \langle u_{i-1}, u_{i-1} \rangle \end{bmatrix}^+ G_2^+ = \begin{bmatrix} 0 \\ S_{i-1} \end{bmatrix}^+ G_2^+ \boxed{= [0 \quad S_{i-1}^+] \, G_2^+ = \langle u_{i-1}, \xi_i \rangle}$

B-1.2) $\langle u_{i-1}, \hat{\xi}_i \rangle = \langle u_{i-1}, \underbrace{\sum_{j=0}^{i-1} \langle \xi_i, e_j \rangle \|e_j\|^{-2} e_j}_{\text{projection of } \xi_i \text{ onto Span}(y_0, y_1, \ldots, y_{i-1})} \rangle$

$= \langle u_{i-1}, \langle \xi_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1} + \sum_{j=0}^{i-2} \langle \xi_i, e_j \rangle \|e_j\|^{-2} e_j \rangle \longrightarrow$

B-1.2 cont'd)

$$\langle u_{i-1}, \hat{z}_i \rangle = \langle u_{i-1}, \langle z_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1} + \underbrace{\hat{z}_{i|i-2}}_{\text{projection of } z_i \text{ onto Span}(y_0, y_1, \ldots, y_{i-2})} \rangle$$

note: $u_{i-1} \perp \text{Span}(y_0, y_1, \ldots y_{i-2})$

$$\Rightarrow \langle u_{i-1}, \hat{z}_i \rangle = \langle u_{i-1}, \langle z_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1} \rangle + \cancelto{0}{\langle u_{i-1}, \hat{z}_{i|i-2} \rangle}$$

where $\langle z_i, e_{i-1} \rangle = \langle z_i, H_2 \tilde{z}_{i-1} + v_{i-1} \rangle$

$$= \langle z_i, H_2 z_{i-1} + v_{i-1} - H_2 \hat{z}_{i-1} \rangle$$

$$= \langle z_i, \tilde{z}_{i-1} \rangle H_2^+ + \langle z_i, v_{i-1} \rangle$$

and $\langle z_i, v_{i-1} \rangle = \langle F_2 z_{i-1} + G_2 u_{i-1}, v_{i-1} \rangle$

$$= F_2 \cancelto{0}{\langle z_{i-1}, v_{i-1} \rangle} + G_2 \langle u_{i-1}, v_{i-1} \rangle$$

$$\Rightarrow \boxed{\langle z_i, v_{i-1} \rangle = G_2 \langle \begin{bmatrix} u_{i-2} \\ u_{i-1} \end{bmatrix}, v_{i-1} \rangle = G_2 \begin{bmatrix} 0 \\ S_{i-1} \end{bmatrix}}$$

and $\langle z_i, \tilde{z}_{i-1} \rangle H_2^+ =$

$$= \langle z_i, z_{i-1} \rangle H_2^+ - \langle z_i, \hat{z}_{i-1} \rangle H_2^+$$

$=$

~~$\langle u_i, e_i \rangle = \langle u_i, y_i - \underset{\in \text{Span}(y_0, y_1, \cdots y_{i-1})}{\hat{y}_i} \rangle = \langle u_i, y_i \rangle - \cancelto{0}{\langle u_i, \hat{y}_i \rangle} = S_i$~~

~~$\langle u_{i-1}, e_{i-1} \rangle = \langle u_{i-1}, y_i - H_2 \hat{z}_i \rangle$~~ $\hat{z}_{i|i-1} = F_2 \hat{z}_{i-1}$

~~$\langle u_{i-1}, \hat{z}_{i|i-1} \rangle = ?$~~ ~~$\hat{z}_{i|i-1} = \langle z_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1} + \hat{z}_{i|i-2}$~~

~~$\rightarrow \langle u_{i-1}, \underbrace{\langle z_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1}}_{\text{projection of } z_i \text{ onto } e_{i-1}} \rangle$~~ note: $\langle z_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} = K_{p,i-1}$

$=$

$e_{i-1} = y_{i-1} - \hat{y}_{i-1|i-2}$

~~$\Rightarrow \langle u_{i-1}, e_{i-1} \rangle K_{p,i-1}^+ = S_{i-1} K_{p,i-1}^+$~~

B-1.2 cont'd)

$$\langle u_{i-1}, \hat{z}_i \rangle = \langle u_{i-1}, \langle z_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_{i-1} + \hat{z}_{i|i-2} \rangle$$

$$= \langle u_{i-1}, \langle z_i, e_{i-1} \rangle \|e_{i-1}\|^{-2} e_i \rangle + \langle u_{i-1}, \hat{z}_{i|i-2} \rangle \to 0$$

$\in \text{Span}(y_0, y_1, \ldots y_{i-2})$ which is $\perp u_{i-1}$

where $\langle z_i, e_{i-1} \rangle = \langle z_i, y_{i-1} - \hat{y}_{i-1} \rangle$

$$= \langle z_i, H_2 z_{i-1} + v_{i-1} - H_2 \hat{z}_{i-1} \rangle = \langle z_i, v_{i-1} \rangle + \langle z_i, \tilde{z}_{i-1} \rangle H_2^+$$

$$\langle z_i, v_{i-1} \rangle = b_2 \left\langle \begin{bmatrix} u_{i-2} \\ u_{i-1} \end{bmatrix}, v_{i-1} \right\rangle = b_2 \begin{bmatrix} 0 \\ S_{i-1} \end{bmatrix}$$

$$\langle z_i, \tilde{z}_{i-1} \rangle H_2^+ = \langle F_2 z_{i-1} + b_2 u_{i-1}, \tilde{z}_{i-1|i-2} \rangle$$

$$= \langle F_2 z_{i-1}, \tilde{z}_{i-1} \rangle + b_2 \left\langle \begin{bmatrix} u_{i-2} \\ u_{i-1} \end{bmatrix}, \tilde{z}_{i-1} \right\rangle$$

$$\langle z_i, e_{i-1} \rangle = F_2 \langle z_{i-1}, e_{i-1} \rangle + b_2 \langle u_{i-1}, e_{i-1} \rangle$$

$$= F_2 P_{i-1} H_2^+ + b_2 \langle u_{i-1}, e_{i-1} \rangle$$

---

again: $\langle u_{i-1}, \hat{z}_i \rangle = \langle u_{i-1}, \underbrace{\langle z_i, e_{i-1} \rangle \|e_{i-1}\|^{-2}}_{K_{p,i-1}} e_{i-1} \rangle + \langle u_{i-1}, \hat{z}_{i|i-2} \rangle$

$\in \text{Span}(y_0, y_1, \ldots y_{i-2})$

$u_{i-1} \perp \hat{z}_{i|i-2}$

$$= \langle u_{i-1}, e_{i-1} \rangle K_{p,i-1}^+$$

$$= \langle u_{i-1}, y_{i-1} - \hat{y}_{i-1} \rangle K_{p,i-1}^+ \quad , \quad \hat{y}_{i-1} \in \text{Span}(y_0, y_1, \ldots y_{i-2})$$

$u_{i-1} \perp \hat{y}_{i-1}$

$$= \langle u_{i-1}, y_{i-1} \rangle K_{p,i-1}^+$$

$$\boxed{\langle u_{i-1}, \hat{z}_i \rangle = S_{i-1} K_{p,i-1}^+}$$

$\longrightarrow$

Thus,

B-1) $$\boxed{\begin{aligned}\langle u_{i-1}, e_i\rangle &= \langle u_{i-1}, z_i\rangle H_2^+ - \langle u_{i-1}, \hat{z}_i\rangle H_2^+ \\ &= [0 \;\; S_{i-1}^+] G_2^+ H_2^+ - S_{i-1} K_{p,i-1}^+ H_2^+\end{aligned}}$$

B-2) $$\langle u_i, e_i\rangle = \langle u_i, y_i - \hat{y}_i\rangle = \langle u_i, y_i\rangle - \cancelto{0}{\langle u_i, \hat{y}_i\rangle}$$

$\leftarrow \in \text{Span}(y_0, y_1, \cdots y_{i-1})$

$$\Rightarrow \boxed{\langle u_i, e_i\rangle = S_i}$$

$$\Rightarrow \text{B)} \quad \langle \mu_i, e_i\rangle = \begin{bmatrix} \langle u_{i-1}, e_i\rangle \\ \langle u_i, e_i\rangle \end{bmatrix} = \begin{bmatrix} [0 \;\; S_i^+] G_2^+ H_2^+ - S_{i-1} K_{p,i-1}^+ H_2^+ \\ S_i \end{bmatrix}$$

So that:

$$\begin{aligned} K_{p,i} &= \langle z_{i+1}, e_i\rangle \|e_i\|^{-2} \\ &= (F_2 \langle z_i, e_i\rangle + G_2 \langle \mu_i, e_i\rangle) \|e_i\|^{-2} \end{aligned}$$

$$\Rightarrow \boxed{K_{p,i} = \left(F_2 P_i H_2^+ + G_2 \begin{bmatrix} [0 \;\; S_i^+] G_2^+ H_2^+ - S_{i-1} K_{p,i-1}^+ H_2^+ \\ S_i \end{bmatrix}\right) \|e_i\|^{-2}}$$

Recall: (to summarize)

$$\|e_i\|^{-2} = H_2 P_i H_2^+ + R_i$$

$$\hat{z}_{i+1|i} = K_{p,i} e_i + F_2 \hat{z}_{i|i-1} + G_2 \begin{bmatrix} S_{i-1} \\ 0 \end{bmatrix}$$

$$e_i = H_2 \tilde{z}_i + v_i$$

$K_{p,i}$ = (as shown above)

Now to solve for P_{i+1} $\longrightarrow$

(Final result on pg 2-11)

P_{i+1} is a function of $\tilde{z}_{i+1}$, so let's look at $\tilde{z}_{i+1}$:

$$\tilde{z}_{i+1} = z_{i+1} - \hat{z}_{i+1} = F_2 z_i + G_2 \mu_i - \left(F_2 \hat{z}_{i|i-1} + K_{p,i} e_i + G_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix} \right)$$

$$= F_2 \tilde{z}_i + G_2 \mu_i - K_{p,i} (H_2 \tilde{z}_i + v_i) - G_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}$$

$$= [F_2 - K_{p,i} H_2] \tilde{z}_i + G_2 \mu_i - K_{p,i} v_i - G_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}$$

let $(F_2 - K_{p,i} H_2) = F_{p,i}$

$$\Rightarrow \tilde{z}_{i+1} = F_{p,i} \tilde{z}_i + G_2 \left(\mu_i - \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix} \right) - K_{p,i} v_i$$

~~So that: $P_{i+1} = \langle \tilde{z}_{i+1}, \tilde{z}_{i+1} \rangle$~~

$$\cancel{= F_{p,i} P_i F_{p,i}^{+} + G_2 \left\langle \mu_i - \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}, \mu_i - \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix} \right\rangle G_2^{+} + K_{p,i} R_i K_{p,i}^{+}}$$

$$\Rightarrow \tilde{z}_{i+1} = F_{p,i} \tilde{z}_i + [G_2 \quad -K_{p,i}] \begin{bmatrix} \mu_i \\ v_i \end{bmatrix} - G_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix}$$

$$\Rightarrow P_{i+1} = \langle \tilde{z}_{i+1}, \tilde{z}_{i+1} \rangle$$ (assume $\langle z_i, 1 \rangle = 0$, i.e. 0-mean)

$$= F_{p,i} P_i F_{p,i}^{+} + [G_2 \quad -K_{p,i}] \begin{bmatrix} \langle \mu_i, \mu_i \rangle & \langle \mu_i, v_i \rangle \\ \langle v_i, \mu_i \rangle & \langle v_i, v_i \rangle \end{bmatrix} \begin{bmatrix} G_2^{+} \\ -K_{p,i}^{+} \end{bmatrix} + G_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix} [s_{i-1}^{+} \quad 0] G_2^{+}$$

$$+ F_{p,i} \begin{bmatrix} \langle \tilde{z}_i, \mu_i \rangle & \langle \tilde{z}_i, v_i \rangle \end{bmatrix} \begin{bmatrix} G_2^{+} \\ -K_{p,i}^{+} \end{bmatrix} + F_{p,i} \ldots$$ ~~and so on.~~ $\longrightarrow$

$\Longrightarrow$

$$- F_{p,i} \overset{0}{\cancel{\langle \tilde{z}_i, 1 \rangle}} [s_{i-1}^{+} \quad 0] G_2^{+} - G_2 \begin{bmatrix} s_{i-1} \\ 0 \end{bmatrix} \overset{0}{\cancel{\langle 1, \tilde{z}_i \rangle}} F_{p,i}^{+}$$

$$+ [G_2 \quad -K_{p,i}] \begin{bmatrix} \langle \mu_i, \tilde{\xi}_i \rangle \\ \langle v_i, \tilde{\xi}_i \rangle \end{bmatrix} F_{p,i}^+ \quad - [G_2 \quad -K_{p,i}] \begin{bmatrix} \cancelto{0}{\langle \mu_i, 1 \rangle} \\ \cancel{\langle v_i, 1 \rangle} \end{bmatrix} [S_{i-1}^+ \quad 0] G_2^+$$

$$- G_2 \begin{bmatrix} S_{i-1} \\ 0 \end{bmatrix} [\cancelto{0}{\langle 1, \mu_i \rangle} \quad \cancelto{0}{\langle 1, v_i \rangle}] \begin{bmatrix} G_2^+ \\ -K_{p,i}^+ \end{bmatrix}$$

$$\Rightarrow P_{i+1} = F_{p,i} P_i F_{p,i}^+ + [G_2 \quad -K_{p,i}] \begin{bmatrix} \langle \mu_i, \mu_i \rangle & \langle \mu_i, v_i \rangle \\ \langle v_i, \mu_i \rangle & \langle v_i, v_i \rangle \end{bmatrix} \begin{bmatrix} G_2^+ \\ -K_{p,i} \end{bmatrix}$$

$$+ G_2 \begin{bmatrix} S_{i-1} \\ 0 \end{bmatrix} [S_{i-1}^+ \quad 0] G_2^+ + F_{p,i} [\langle \tilde{\xi}_i, \mu_i \rangle \quad \langle \tilde{\xi}_i, v_i \rangle] \begin{bmatrix} G_2^+ \\ -K_{p,i} \end{bmatrix}$$

$$+ [G_2 \quad -K_{p,i}] \begin{bmatrix} \langle \mu_i, \tilde{\xi}_i \rangle \\ \langle v_i, \tilde{\xi}_i \rangle \end{bmatrix} F_{p,i}^+$$

Recall: $\langle v_i, \tilde{\xi}_i \rangle = \langle \tilde{\xi}_i, v_i \rangle^* = 0$

$$\langle \mu_i, \tilde{\xi}_i \rangle = \langle \mu_i, \xi_i \rangle - \langle \mu_i, \hat{\xi}_i \rangle = \left\langle \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}, \xi_i \right\rangle - \left\langle \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}, \hat{\xi}_i \right\rangle$$

$$= \begin{bmatrix} [0 \quad S_{i-1}^+] G_2^+ \\ 0 \end{bmatrix} - \begin{bmatrix} S_{i-1} K_{p,i-1}^+ \\ 0 \end{bmatrix} = \begin{bmatrix} [0 \quad S_{i-1}^+] G_2^+ - S_{i-1} K_{p,i-1}^+ \\ 0 \end{bmatrix}$$

$$\langle \tilde{\xi}_i, \mu_i \rangle = \left\langle \xi_i, \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix} \right\rangle - \left\langle \hat{\xi}_i, \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix} \right\rangle = [\langle \xi_i, u_{i-1} \rangle \quad 0] - [\langle \hat{\xi}_i, u_{i-1} \rangle \quad 0]$$

where $\langle \xi_i, u_{i-1} \rangle = \left\langle \cancelto{0}{F_2 \xi_{i-1}} + G_2 \begin{bmatrix} u_{i-2} \\ u_{i-1} \end{bmatrix}, u_{i-1} \right\rangle = G_2 \begin{bmatrix} 0 \\ S_i \end{bmatrix}$

$$\langle \hat{\xi}_i, u_{i-1} \rangle = \Big\langle \underbrace{\langle \xi_i, e_{i-1} \rangle \|e_{i-1}\|^{-2}}_{K_{p,i-1}} e_{i-1} + \sum_{j=0}^{i-2} \cancelto{0}{\langle \xi_i, e_j \rangle \|e_j\|^{-2} e_j}, u_{i-1} \Big\rangle$$

$$= K_{p,i-1} \langle e_{i-1}, u_{i-1} \rangle = K_{p,i-1} \langle y_{i-1} - \cancelto{0}{\hat{y}_{i-1}}, u_{i-1} \rangle = K_{p,i-1} S_{i-1}^+$$

$$\Rightarrow \langle \xi_i, \mu_i \rangle = \begin{bmatrix} G_2 \begin{bmatrix} 0 \\ S_i \end{bmatrix} - K_{p,i-1} S_{i-1}^+ & 0 \\ \cancel{0} & \end{bmatrix}$$

$$\langle \mu_i, \mu_i \rangle = \left\langle \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix}, \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix} \right\rangle = \begin{bmatrix} Q_{i-1} & 0 \\ 0 & Q_i \end{bmatrix}$$

$$\langle \mu_i, v_i \rangle = \begin{bmatrix} \langle u_{i-1}, v_i \rangle \\ \langle u_i, v_i \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ S_i \end{bmatrix}$$

$$\langle v_i, \mu_i \rangle = \begin{bmatrix} \langle v_i, u_{i-1} \rangle & \langle v_i, u_i \rangle \end{bmatrix} = \begin{bmatrix} 0 & S_i^+ \end{bmatrix}$$

Therefore:

$$P_{i+1} = F_{p,i} P_i F_{p,i}^+ + \begin{bmatrix} G_2 & -K_{p,i} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} Q_{i-1} & 0 \\ 0 & Q_i \end{bmatrix} & \begin{bmatrix} 0 \\ S_i \end{bmatrix} \\ \begin{bmatrix} 0 & S_i^+ \end{bmatrix} & R_i \end{bmatrix} \begin{bmatrix} G_2^+ \\ -K_{p,i}^+ \end{bmatrix}$$

$$+ G_2 \begin{bmatrix} S_{i-1} \\ 0 \end{bmatrix} \begin{bmatrix} S_{i-1}^+ & 0 \end{bmatrix} G_2^+ + F_{p,i} \begin{bmatrix} G_2 \begin{bmatrix} 0 \\ S_i \end{bmatrix} - K_{p,i-1} S_{i-1}^+ & 0 & 0 \end{bmatrix} \begin{bmatrix} G_2^+ \\ -K_{p,i} \end{bmatrix}$$

$$+ \begin{bmatrix} G_2 & -K_{p,i} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & S_{i-1}^+ \end{bmatrix} G_2^+ - S_{i-1} K_{p,i-1}^+ \\ 0 \\ 0 \end{bmatrix} F_{p,i}^+$$

#3 A modified State Space model

$$\begin{cases} x_{i+1} = Fx_i + G_1 u_i + G_2 u_{i+1} \\ y_i = Hx_i + v_i \end{cases} \qquad i \geq 0$$

Conditions: $\left\langle \begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix}, \begin{bmatrix} x_0 \\ u_j \\ v_j \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & S_i\delta_{ij} & 0 \\ 0 & S_i^*\delta_{ij} & R_i\delta_{ij} & 0 \end{bmatrix}$

Find recursive equations for $\hat{x}_{i|i-1}$ and $P_{i|i-1}$ $\boxed{\text{(pg 3-3 \& 3-4 for summary of eqtns.)}}$

so $\hat{x}_{i|i-1}$ would be in terms of y_{i-1}, $\hat{x}_{i-1|i-2}$, etc.

Step 1: get properties of SS model

~~(To-do: compute properties & double-check recursion calcs agree with props)~~

DONE

$\langle u_i, x_j \rangle = ?$
$\langle v_i, x_j \rangle = ?$
$\langle u_i, y_j \rangle = ?$
$\langle v_i, y_j \rangle = ?$

(see pages 3-5 and 3-6)

let $\langle x_i, x_i \rangle = \Pi_i$, ~~$\Pi_{i+1} = ?$~~

~~$\langle x_i, x_j \rangle = ?$~~

~~$\langle y_i, y_j \rangle = ?$~~

Step 2:

$$e_i = y_i - \hat{y}_{i|i-1} = y_i - \sum_{j=0}^{i-1} \langle Hx_i + \cancelto{0}{v_i}, e_j \rangle \|e_j\|^{-2} e_j$$

(uncorrelated) b/c $j = 0, 1, \ldots i-1$

$$= y_i - H\hat{x}_{i|i-1}$$

$$\hat{x}_{i+1|i} = \sum_{j=0}^{i} \langle x_{i+1}, e_j \rangle \|e_j\|^{-2} e_j = \underbrace{\langle x_{i+1}, e_i \rangle \|e_i\|^{-2}}_{K_{p,i}} e_i + \sum_{j=0}^{i-1} \langle x_{i+1}, e_j \rangle \|e_j\|^{-2} e_j$$

$$x_{i+1} = Fx_i + G_1 u_i + G_2 u_{i+1}$$

$(i > j)$ $(i > j)$

$$\Rightarrow \langle x_{i+1}, e_j \rangle = F\langle x_i, e_j \rangle + G_1 \cancelto{0}{\langle u_i, e_j \rangle} + G_2 \cancelto{0}{\langle u_{i+1}, e_j \rangle} = F\langle x_i, e_j \rangle$$

$$\Rightarrow \hat{x}_{i+1|i} = K_{p,i} e_i + F\hat{x}_{i|i-1}$$

$\longrightarrow$

(see pg 3-7)

~~check this w/ prop.~~ uncorrelated w/ meas. noise

$\|e_i\|^{-2} = ?$, $e_i = y_i - H\hat{x}_{i|i-1} = Hx_i + v_i - H\hat{x}_{i|i-1}$

let $\tilde{x}_i = x_i - \hat{x}_{i|i-1}$, $P_i = \langle \tilde{x}_i, \tilde{x}_i \rangle$, $\hat{x}_i \triangleq \hat{x}_{i|i-1}$

then $e_i = H\tilde{x}_i + v_i$, $\boxed{\|e_i\|^{-2} = \langle e_i, e_i \rangle = HP_iH^* + H\cancel{\langle \tilde{x}_i, v_i \rangle}^{0} + \cancel{\langle v_i, \tilde{x}_i \rangle}^{0} H^T + R_i = HP_iH^* + R_i}$

So; $K_{p,i} = \langle x_{i+1}, e_i \rangle \|e_i\|^{-2} = \langle Fx_i + G_1u_i + G_2u_{i+1}, e_i \rangle \|e\|^{-2}$

1) $\langle x_i, e_i \rangle = \langle x_i, H\tilde{x}_i + \cancel{v_i} \rangle = \langle x_i, \tilde{x}_i \rangle H^* + \langle x_i, v_i \rangle = \langle x_i - \hat{x}_i + \hat{x}_i, \tilde{x}_i \rangle H^* + \underbrace{\langle Fx_{i-1} + G_1u_{i-1} + G_2u_i, v_i \rangle}_{G_2 S_i}$

$= \langle \tilde{x}_i + \hat{x}_i, \tilde{x}_i \rangle H^* = P_iH^* + \cancel{\langle \hat{x}_i, \tilde{x}_i \rangle}^{0} H^* + G_2S_i$

prediction is $\perp$ to error!

$$\boxed{\langle x_i, e_i \rangle = P_iH^* + G_2S_i}$$

2) $\langle u_i, e_i \rangle = \langle u_i, H\tilde{x}_i + \cancel{v_i} \rangle = \langle u_i, \tilde{x}_i \rangle H^* + S_i$

$\tilde{x}_i = x_i - \hat{x}_i = x_i - \underbrace{\sum_{j=0}^{i-1} \langle x_i, e_j \rangle \|e_j\|^{-2} e_j}_{\text{projection onto Span}(y_0, \dots y_{i-1})}$

$\langle u_i, y_{i-1} \rangle = \langle u_i, Hx_{i-1} + \cancel{v_{i-1}}^{0} \rangle$

$= \langle u_i, Fx_{i-2} + G_1u_{i-2} + G_2u_{i-1} \rangle H^T + 0$

$= 0$

$\Rightarrow \langle u_i, x_i \rangle = \cancel{0}$, $\langle u_i, \hat{x}_i \rangle = 0$

$= \langle u_i, Fx_{i-1} + G_1u_{i-1} + G_2u_i \rangle$

$= Q_iG_2^*$

$$\boxed{\langle u_i, \tilde{x}_i \rangle = Q_iG_2^*}$$

$\Rightarrow \langle u_i, e_i \rangle = \langle u_i, \tilde{x}_i \rangle H^* + S_i = \langle u_i, x_i - \hat{x}_i \rangle H^* + S_i$

$= (\langle u_i, x_i \rangle - \cancel{\langle u_i, \hat{x}_i \rangle}^{0}) H^* + S_i$

$$\boxed{\langle u_i, e_i \rangle = Q_iG_2^*H^* + S_i}$$

$\longrightarrow$

3) $\langle u_{i+1}, e_i \rangle = \langle u_{i+1}, H\tilde{x}_i + v_i \rangle = \langle u_{i+1}, \tilde{x}_i \rangle H^+ + \cancelto{0}{\langle u_{i+1}, v_i \rangle}$

$\langle u_{i+1}, \tilde{x}_i \rangle = \langle u_{i+1}, x_i - \hat{x}_i \rangle = \langle u_{i+1}, x_i \rangle - \cancelto{0}{\langle u_{i+1}, \hat{x}_i \rangle}$

$\in \text{Span}(y_0, \cdots y_{i-1})$

and $\langle u_{i+1}, y_j \rangle = \langle u_{i+1}, Hx_j + \cancelto{0}{v_j} \rangle = \langle u_{i+1}, \cancelto{0}{Fx_{j-1}} + \cancelto{0}{G_1 u_{j-1}} + \cancelto{0}{G_2 u_j} \rangle \quad \forall\, i > j$

$\Rightarrow \langle u_{i+1}, x_i \rangle = \langle u_{i+1}, \cancelto{0}{Fx_{i-1}} + \cancelto{0}{G_1 u_{i-1}} + \cancelto{0}{G_2 u_i} \rangle$

$\Rightarrow \boxed{\langle u_{i+1}, \tilde{x}_i \rangle = 0}$

$\Rightarrow \boxed{\langle u_{i+1}, e_i \rangle = 0}$

Now, $K_{p,i} = \langle x_{i+1}, e_i \rangle \|e_i\|^{-2}$

$= \langle Fx_i + G_1 u_i + G_2 u_{i+1}, e_i \rangle \|e_i\|^{-2}$

$= [FP_i H^+ + G_1 Q_i G_2^+ H^+ + G_1 S_i + G_2 (0)] (HP_i H^+ + R_i)$

$+G_2 S_i$

$$\boxed{K_{p,i} = (FP_i H^+ + G_1 Q_i G_2^+ H^+ + G_1 S_i) \|e_i\|^{-2}}$$

$+G_2 S_i$

So we have,

$e_i = y_i - H\hat{x}_{i|i-1}$ (P_{i+1} on page 3-4)

$\hat{x}_{i+1|i} = K_{p,i} e_i + F\hat{x}_{i|i-1}$

$+G_2 S_i$

$K_{p,i} = \langle x_{i+1}, e_i \rangle \|e_i\|^{-2} = (FP_i H^+ + G_1 Q_i G_2^+ H^+ + G_1 S_i) \|e_i\|^{-2}$

$\|e_i\|^{-2} = HP_i H^+ + R_i + G_2 S_i + S_i^+ G_2^+$

Now to solve for P_i recursively:

P_{i+1} is a function of $\tilde{x}_{i+1}$, so let's look @ $\tilde{x}_{i+1}$:

$\tilde{x}_{i+1} = x_{i+1} - \hat{x}_{i+1} = Fx_i + G_1 u_i + G_2 u_{i+1} - (\cancel{F\hat{x}_{i|i-1} + G_1 \hat{u}_{i|i-1} + G_2}$

$(F\hat{x}_{i|i-1} + K_{p,i} e_i)$

→

$$\rightarrow \tilde{x}_{i+1} = Fx_i + G_1 u_i + G_2 u_{i+1} - (F\hat{x}_{i|i-1} + K_{p,i} e_i)$$

$$- K_{p,i}(y_i - H\hat{x}_i) - F\hat{x}_i$$

$$- K_{p,i}(Hx_i + v_i - H\hat{x}_i) - F\hat{x}_i$$

$$= F\tilde{x}_i - K_{p,i} H\tilde{x}_i + G_1 u_i + G_2 u_{i+1} - K_{p,i} v_i$$

$$\begin{bmatrix} u_i \\ u_{i+1} \\ v_i \end{bmatrix} \begin{bmatrix} u_i & u_{i+1} & v_i \end{bmatrix}$$

$$\tilde{x}_{i+1} = F_{p,i}\tilde{x}_i + \begin{bmatrix} G_1 & G_2 & -K_{p,i} \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \\ v_i \end{bmatrix}$$

$$P_{i+1} = \langle \tilde{x}_{i+1}, \tilde{x}_{i+1} \rangle = F_{p,i} P_i F_{p,i}^T + \begin{bmatrix} G_1 & G_2 & -K_{p,i} \end{bmatrix} \begin{bmatrix} Q_i & 0 & S_i \\ 0 & Q_{i+1} & 0 \\ S_i^+ & 0 & R_i \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ -K_{p,i} \end{bmatrix}$$

$$+ \underbrace{F_{p,i} \left\langle \tilde{x}_i, \begin{bmatrix} u_i \\ u_{i+1} \\ v_i \end{bmatrix} \right\rangle \begin{bmatrix} G_1 \\ G_2 \\ -K_{p,i} \end{bmatrix}} + \begin{bmatrix} G_1 & G_2 & -K_{p,i} \end{bmatrix} \left\langle \begin{bmatrix} u_i \\ u_{i+1} \\ v_i \end{bmatrix}, \tilde{x}_i \right\rangle F_{p,i}^T$$

$$F_{p,i} \begin{bmatrix} \underbrace{\langle \tilde{x}_i, u_i \rangle}_{Q_i G_2^+} & \underbrace{\langle \tilde{x}_i, u_{i+1} \rangle}_{=0} & \underbrace{\langle \tilde{x}_i, v_i \rangle}_{G_2 S_i} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ -K_{p,i} \end{bmatrix}$$

$$= F_{p,i} G_2 Q_i \cancel{G_2^+} G_1 - G_2 S_i K_{p,i}$$

$$\begin{bmatrix} \langle u_i, \tilde{x}_i \rangle \\ \langle u_{i+1}, \tilde{x}_i \rangle \\ \langle v_i, \tilde{x}_i \rangle \end{bmatrix} = \begin{bmatrix} Q_i G_2^+ \\ 0 \\ S_i^+ G_2^+ \end{bmatrix} \Rightarrow \begin{bmatrix} G_1 & G_2 & -K_{p,i} \end{bmatrix} \begin{bmatrix} Q_i G_2^+ \\ 0 \\ S_i^+ G_2^+ \end{bmatrix} F_{p,i}^+$$

$$= G_1 Q_i G_2^+ F_{p,i}^+ - K_{p,i} S_i^+ G_2^+ F_{p,i}^+$$

$$\boxed{\Rightarrow P_{i+1} = F_{p,i} P_i F_{p,i}^+ + \begin{bmatrix} G_1 & G_2 & -K_{p,i} \end{bmatrix} \begin{bmatrix} Q_i & 0 & S_i \\ 0 & Q_{i+1} & 0 \\ S_i^+ & 0 & R_i \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ -K_{p,i} \end{bmatrix} \cancel{+ F_{p,i} Q_i G_2^+ G_1} + G_1 Q_i G_2^+ F_{p,i}^+ - K_{p,i} S_i^+ G_2^+ F_{p,i}^+}$$

$$+ F_{p,i} G_2 Q_i G_1 - G_2 S_i K_{p,i}$$

Properties of SS model for Q-3: (see Lec. 13 pg 4 for example derivations)

$$\langle u_i, x_j \rangle = \langle u_i, Fx_{j-1} + G_1 u_{j-1} + G_2 u_j \rangle$$

$$x_{j-1} \in \text{Span}(x_0, u_0, u_1, \dots u_{j-1}) \Rightarrow$$

$$x_j = Fx_{j-1} + G_1 u_{j-1} + G_2 u_j$$

$$= F(Fx_{j-2} + G_1 u_{j-2} + G_2 u_{j-1}) + G_1 u_{j-1} + G_2 u_j$$

$$= F(F(Fx_{j-3} + G_1 u_{j-3} + G_2 u_{j-2}) + G_1 u_{j-2} + G_2 u_{j-1}) + G_1 u_{j-1} + G_2 u_j$$

$$\vdots$$

$$\cancel{= F(F(F(\cdots(Fx_0 + G_1 u_0 + G_2 u_1)}$$

$$= F(F(\cdots(\underbrace{F_2(\underbrace{F_1(\underbrace{F_0 x_0 + G_1 u_0 + G_2 u_1}_{x_1}) + G_1 u_1 + G_2 u_2}_{x_2}) + G_1 u_2 + G_2 u_3}_{x_3}) + \cancel{G_1 u_3 + G_2 u_4} + \dots) + G_1 u_{j-2} + G_2 u_{j-1}) + G_1 u_{j-1} + G_2 u_j$$

$$= F_{j-1} F_{j-2} \cdots F_2 F_1 F_0 (x_0) + F_{j-1} \cdots F_1 F_0 G_1 u_0 + F_{j-1} \cdots F_1 F_0 G_2 u_1$$
$$+ F_{j-1} \cdots F_2 G_1 u_1 + F_{j-1} \cdots F_2 G_2 u_2$$
$$+ F_{j-1} \cdots F_3 G_1 u_2 + F_{j-1} \cdots F_3 G_2 u_3$$
$$\vdots$$
$$+ F_{j-1} G_1 u_{j-2} + F_{j-1} G_2 u_{j-1}$$
$$+ G_1 u_{j-1} + G_2 u_j$$

$$= F^j x_0 + F^{j-1} G_1 u_0 + F^{j-1} G_2 u_1$$
$$+ F^{j-2} G_1 u_0 + F^{j-2} G_2 u_2$$
$$\vdots$$
$$+ F G_1 u_{j-2} + F G_2 u_{j-1}$$
$$+ G_1 u_{j-1} + G_2 u_j$$

$\rightarrow \langle u_i, x_0 \rangle = 0$ per conditions of SS model

$\langle u_i, u_j \rangle = 0 \;\; \forall \, i \neq j$

$\langle u_i, u_i \rangle = Q_i$

$\longrightarrow$

thus, $\boxed{\begin{array}{l} \langle u_i, x_j \rangle = 0 \quad, i > j \\ \langle u_i, x_i \rangle = Q_i G_2^{+} \quad, i = j \end{array}}$

and similarly: $\boxed{\begin{array}{l} \langle v_i, x_j \rangle = 0 \quad, i > j \\ \langle v_i, x_i \rangle = \langle v_i, u_i \rangle G_2^{+} = S_i^{+} G_2^{+} \quad, i = j \end{array}}$

$\langle u_i, y_j \rangle = ?$

$\Rightarrow \langle u_i, y_j \rangle = \langle u_i, H x_j + v_j \rangle = \langle u_i, x_j \rangle H^{+} + \langle u_i, v_j \rangle$

$\boxed{\begin{array}{l} \text{if } \underline{i > j} \Rightarrow \langle u_i, y_j \rangle = 0 H^{+} + 0 = \boxed{0} \\ \text{if } i = j \Rightarrow \langle u_i, y_i \rangle = \boxed{Q_i G_2^{+} H^{+} + S_i} \end{array}}$

$\langle v_i, y_j \rangle = \langle v_i, H x_j + v_j \rangle = \langle v_i, x_j \rangle H^{+} + \langle v_i, v_j \rangle$

$\boxed{\begin{array}{l} \underline{i > j} \Rightarrow \langle v_i, y_j \rangle = 0 + 0 = 0 \\ \underline{i = j} \Rightarrow \langle v_i, y_i \rangle = S_i G_2^{+} H^{+} + R_i \end{array}}$

$$\|e_i\|^{-2} = \langle H\tilde{x}_i + v_i, H\tilde{x}_i + v_i\rangle = HP_iH^+ + R_i + H\langle \tilde{x}_i, v_i\rangle + \langle v_i, \tilde{x}_i\rangle H^+$$

where $\langle \tilde{x}_i, v_i\rangle = \langle x_i - \hat{x}_i, v_i\rangle = \langle x_i, v_i\rangle - \overset{0}{\cancel{\langle \hat{x}_i, v_i\rangle}}$

$\hat{x}_i \in \text{Span}(y_0, y_1, \cdots y_{i-1})$

$$= \langle Fx_{i-1} + G_1u_{i-1} + G_2u_i, v_i\rangle$$

$$= G_2\langle u_i, v_i\rangle = \boxed{G_2 S_i = \langle \tilde{x}_i, v_i\rangle}$$

and $\langle v_i, \tilde{x}_i\rangle = \langle v_i, x_i - \hat{x}_i\rangle = \langle v_i, x_i\rangle - \overset{0}{\cancel{\langle v_i, \hat{x}_i\rangle}}$

$$= \langle v_i, Fx_{i-1} + G_1u_{i-1} + G_2u_i\rangle$$

$$= \langle v_i, u_i\rangle G_2^+ = \boxed{S_i^+G_2^+ = \langle v_i, \tilde{x}_i\rangle}$$

$$\Rightarrow \boxed{\|e_i\|^{-2} = HP_iH^+ + R_i + G_2S_i + S_i^+G_2^+}$$

$$e_i = y_i - \hat{y}_{i|i-1} = y_i - \sum_{j=0}^{i-1}\langle y_i, e_j\rangle \|e_j\|^{-2} e_j$$

where $\langle y_i, e_j\rangle = \langle Hx_i + v_i, e_j\rangle = H\langle x_i, e_j\rangle + \langle v_i, y_j - \hat{y}_{j|j-1}\rangle$

$e_j \in$ $\qquad\qquad$ $\hat{y}_{j|j-1} \in \text{Span}(y_0, \ldots y_{j-1})$

for $i > j$:

$$\langle y_i, e_j\rangle = H\langle x_i, e_j\rangle + \overset{0}{\cancel{\langle v_i, y_j - \hat{y}_{j|j-1}\rangle}}$$

$$= H\langle x_i, e_j\rangle$$

thus, $e_i = y_i - H\sum_{j=0}^{i-1}\langle x_i, e_j\rangle \|e_j\|^{-2} e_j$

$$\Rightarrow \boxed{e_i = y_i - H\hat{x}_i} \quad ✓$$

$$\hat{x}_{i+1|i} = \sum_{j=0}^{i}\langle x_{i+1}, e_j\rangle \|e_j\|^{-2} e_j = \underbrace{\langle x_{i+1}, e_i\rangle \|e_i\|^{-2}}_{K_{p,i}} e_i + \sum_{j=0}^{i-1}\langle x_{i+1}, e_j\rangle \|e_j\|^{-2} e_j$$

$$x_{i+1} = Fx_i + G_1u_i + G_2u_{i+1}$$

$\Rightarrow \langle x_{i+1}, e_j\rangle = \langle Fx_i + G_1u_i + G_2u_{i+1}, e_j\rangle = F\langle x_i, e_i\rangle + G_1\langle u_i, e_i\rangle + G_2\overset{0}{\cancel{\langle u_{i+1}, e_i\rangle}}$

where $\langle u_i, e_i\rangle = \langle u_i, y_j - \overset{0}{\hat{y}_{j|j-1}}\rangle = \langle u_i, y_j\rangle = Q_iG_2^+H^+ + S_i \quad (i = j)$

$\langle u_i, e_j\rangle =$ $\qquad\qquad$ $\boxed{= 0 \quad (i > j)}$

$$\Rightarrow \boxed{\hat{x}_{i+1|i} = K_{p,i}\,e_i + F\hat{x}_{i|i-1}} \quad ✓$$