Emission, Propagation and Mode Conversion of Electron Bernstein Waves - Mathematical Supplement

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This document follows Chapter 1 Francesco Volpe's Thesis [1] in Section 2. I have included extra mathematical details and anything I think is relevant to understanding in section 1.

1 Plasma Definitions

1.1 Notation

Vectors will be denoted in bold i.e. $\mathbf{a}$. Vector components will be denoted a_x, a_y, a_z. a will refer to the magnitude of the vector $\sqrt{\mathbf{a}\cdot\mathbf{a}}$. Second order tensors will be noted using double underscore e.g. $\underline{\underline{T}}$. When referring to particle density n we mean the *number density* (particles per m^{-3}) not *mass density* (kgm^{-3}).

1.1.1 Constants

$$\epsilon_0 = 8.854 \times 10^{-12}\text{F.m}^{-1} \quad \text{(Permittivity of Free Space)}$$

$$\mu_0 = 4\pi \times 10^{-7}\text{H.m}^{-1} \quad \text{(Permeability of Free Space)}$$

$$c = (\epsilon_0\mu_0)^{-0.5} = 2.998 \times 10^{8}\text{m.s}^{-1} \quad \text{(Speed of Light)}$$

$$K = 1.381 \times 10^{-23}\text{J.K}^{-1} \quad \text{(Boltzmann's Constant)}$$

$$e = 1.609 \times 10^{-19}\text{C} \quad \text{(Charge of Electron)}$$

$$m_e = 9.109 \times 10^{-31}\text{kg} = 511\text{keV / c}^2 \quad \text{(Rest Mass of Electron)}$$

$$m_p = 1.672 \times 10^{-27}\text{kg} = 1836.15 m_e \quad \text{(Rest Mass of Proton)}$$

1.1.2 Variables

$\boldsymbol{E}$ = Electric Field (Vm^{-1})

$\boldsymbol{B}$ = Magnetic Field (T)

$\boldsymbol{j}$ = Current (A)

q = Particle Charge (C)

m = Particle Mass (kg)

$\boldsymbol{v}$ = Particle Velocity (ms^{-1})

v_T = Particle Thermal Velocity (ms^{-1})

β = Normalised Particle Thermal Velocity

$\|$ = Component of variable parallel to Magnetic Field

$\perp$ = Component of variable perpendicular to Magnetic Field

ω = Angular Frequency ($2\pi\times$Hz)

f = Frequency (Hz)

k = Wavenumber (m^{-1})

λ = Wavelength (m)

N = Refractive Index

n_e = Electron Density (m^{-3})

n = Plasma Density (m^{-3})

T_e = Electron Temperature (eV)

λ_D = Debye Radius (m)

ρ = Larmor Radius (m)

ω_{ce} = Electron Cyclotron (Angular) Frequency ($2\pi\times$Hz)

ω_{ci} = Ion Cyclotron (Angular) Frequency ($2\pi\times$Hz)

ω_{pe} = Electron Plasma (Angular) Frequency ($2\pi\times$Hz)

ω_{pi} = Ion Plasma (Angular) Frequency ($2\pi\times$Hz)

1.2 Equations

Plasmas are usually modelled as an electrically conducting fluid. In simple models we can use equations of fluid mechanics combined with the Maxwell equations. Beyond this we have to use kinetic theory and forms of the Boltzmann transport equation.

1.2.1 Electrostatic Equations

$$\boldsymbol{E} = \underline{\underline{\sigma}} \cdot \boldsymbol{j} \tag{1}$$

$$\boldsymbol{j} = \sum_{\text{species}} \boldsymbol{j_s} = \sum_{\text{species}} m_s q_s \boldsymbol{v_s} \tag{2}$$

(1) = Ohm's Law

(2) = Current Drift Equation

1.2.2 Maxwell Equations

In a vacuum, for an electric field $\boldsymbol{E}$, a magnetic field $\boldsymbol{B}$, a charge density ρ and current $\boldsymbol{j}$

$$\nabla \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{3}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{4}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{5}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{6}$$

(3) = Gauss's Law

(4) = Gauss's Law for Magnetism

(5) = Faraday-Maxwell Law

(6) = Ampère-Maxwell Law

Note these are the Heaviside form of the Maxwell equations. There are also Gaussian forms which have extra factors of c and π.

1.2.3 Fluid Equations

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{v}) = 0 \tag{7}$$

$$mn\frac{D\boldsymbol{v}}{dt} = n\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right) = -\nabla\cdot\underline{\underline{P}} + \boldsymbol{F} \tag{8}$$

$$n\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right) = -\nabla p + qn\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \tag{9}$$

where p is Pressure (Pa), $\underline{\underline{P}}$ is the Stress Tensor and $\boldsymbol{F}$ are external forces (gravity, electromagnetic, etc.)

(7) = Continuity Equation

(8) = Navier-Stokes Equation

(9) = Navier-Stokes for Maxwellian Plasma in Electromagnetic Field

1.3 Cyclotron Motion

In an electromagnetic field $\mathbf{E}$ and $\mathbf{B}$ a particle with charge q moving at velocity $\mathbf{v}$ experiences the Lorentz force $q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$. Writing the equation of motion $\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$:

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{10}$$

Define a co-ordinate system where WLOG $\mathbf{B} = B\hat{\boldsymbol{z}}$ and consider components of the equation of motion.

$$\frac{dv_x}{dt} = \frac{q}{m}(E_x + v_y B) \tag{11}$$

$$\frac{dv_y}{dt} = \frac{q}{m}(E_y - v_x B) \tag{12}$$

$$\frac{dv_z}{dt} = \frac{qE_z}{m} \tag{13}$$

The component parallel to the magnetic field is simple acceleration in an electric field. The two components perpendicular to the magnetic field form a 2D system of coupled Ordinary Differential Equations (ODEs). Assume the fields are static and constant, i.e.

$$\frac{d\mathbf{E}}{dt} = \frac{d\mathbf{B}}{dt} = 0 \tag{14}$$

$$\frac{d\mathbf{E}}{dx} = \frac{d\mathbf{B}}{dx_i} = 0 \quad i \in (1,2,3) \tag{15}$$

Take the derivatives of (11) and (12) with respect to time

$$\frac{d^2 v_x}{dt^2} = \frac{d}{dt}\left(\frac{q}{m}(E_x + v_y B)\right) = \frac{qB}{m}\frac{dv_y}{dt} \tag{16}$$

$$\frac{d^2 v_y}{dt^2} = \frac{d}{dt}\left(\frac{q}{m}(E_y - v_x B)\right) = \frac{qB}{m}\frac{dv_x}{dt} \tag{17}$$

Eliminate $\frac{dv_y}{dt}$ using (12) and $\frac{dv_x}{dt}$ using (11)

$$\frac{d^2 v_x}{dt^2} = \frac{q^2 B}{m^2}(E_y - v_x B) = \frac{q^2 B E_y}{m^2} - \left(\frac{qB}{m}\right)^2 v_x^2 = \left(\frac{qB}{m}\right)^2 \left(\frac{E_y}{B} - v_x^2\right) \tag{18}$$

$$\frac{d^2 v_y}{dt^2} = -\frac{q^2 B}{m^2}(E_x + v_y B) = \frac{-q^2 B E_x}{m^2} - \left(\frac{qB}{m}\right)^2 v_y^2 = \left(\frac{qB}{m}\right)^2 \left(\frac{-E_x}{B} - v_y^2\right) \tag{19}$$

Equations (18) and (19) have very similar forms apart from the sign of the electric field. They resemble the well known ODE $\frac{d^2 v}{dt^2} = -v^2$ which has solution $v(t) = \sin(t)$. We have an extra constant term so look for solutions of type $v_i(t) = \sin(\omega t) + v_{di}$ where v_{di} is a constant we will call the drift velocity

$$-\omega^2 sin(\omega t) = \left(\frac{qB}{m}\right)^2 \left(\frac{E_y}{B} - sin(\omega t) - v_{dx}\right) \tag{20}$$

$$-\omega^2 sin(\omega t) = \left(\frac{qB}{m}\right)^2 \left(-\frac{E_x}{B} - sin(\omega t) - v_{dy}\right) \tag{21}$$

By inspection we can find the angular frequency ω

$$\omega = \frac{|q|B}{m} := \omega_c \tag{22}$$

The frequency ω_c is the well known cyclotron frequency. Note this is an angular frequency. Also note for convenience we take the absolute value of the charge so the frequency is positive for both ions and electrons. We can do this so long as we remember ions and electrons gyrate in opposite directions.Commonly the electron and ion cyclotron frequencies are denoted using ω_{ce} and Ω_{ci} respectively, where i is a subscript for ion species.

A useful approximation for the electron cyclotron frequency is

$$f_{ce}[GHz] = 27.95 \times B[T] \tag{23}$$

We also can read off the components of the drift velocity $\boldsymbol{v_d}$

$$v_{dx} = \frac{E_y}{B} \quad v_{dy} = -\frac{E_x}{B} \tag{24}$$

These terms look similar to the terms we got from $\mathbf{v} \times \mathbf{B}$ in (11) and (12) except with $\mathbf{E}$ instead of $\mathbf{v}$:

$$\mathbf{E} \times \mathbf{B} = E_y B \hat{\boldsymbol{x}} - E_x B \hat{\boldsymbol{y}} \tag{25}$$

Dividing by B^2 we see the drift velocity $\boldsymbol{v_d}$ is the well known $E \times B$ drift

$$\boldsymbol{v_d} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} \tag{26}$$

Therefore particles in an electromagnetic field experience ballistic motion along the magnetic field ($v \sim t^2$), circular motion (gyration) perpendicular to the magnetic field and a constant drift perpendicular to both the electric and magnetic field.

There are additional drifts induced by gravity, gradients in magnetic field, etc.

Perpendicular to the field particles gyrate at frequency ω_c. Writing the magnitude of velocity perpendicular to the field as $v_\perp$ the radius of the gyration ρ (Larmor radius) can be calculated.

The angular frequency ω_c is related to the frequency f by $\omega_c = 2\pi f$. Travelling at velocity $v_\perp$ in one period of the orbit $T = \frac{1}{f}$, the particle travels a distance of $v_\perp T$. This is equal to the circumference of the orbit $2\pi\rho$. Therefore

$$\rho = \frac{v_\perp T}{2\pi} = \frac{v_\perp}{2\pi f} = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{eB} \tag{27}$$

1.4 Concept of Temperature

Temperature tries to quantify the kinetic energy and hence velocity of a gas. A general way to describe the state of a gas is to give the density of particles in space and the distribution of the particle velocities, all in time.

In 3D this gives the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ taking 3 components of space $\mathbf{x}$, 3 components of velocity $u\mathbf{u}$ and time t. Together $(\mathbf{x}, \mathbf{u}, t)$ form a 7D space called 'phase space'.

The distribution function has the property

$$\int \cdots \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x} d\mathbf{u} = N \tag{28}$$

where $d\mathbf{x}$ refers to integration over all components $dxdydz$ from $-\infty$ to ∞ and $d\mathbf{u}$ refers to integration over all components $du_x du_y du_z$ from 0 to ∞. N is the total number of particles.

In thermal equilibrium we get a special distribution of velocities, the Maxwellian distribution, allowing us to completely describe the distributions of velocity inside an infinitesimal element of 3D velocity space $d\mathbf{u}$ by a single number, the thermodynamic temperature T.

$$f(v)dv = 4\pi A v^2 \exp\left(-\frac{mv^2}{2KT}\right) dv \tag{29}$$

A is calculated by ensuring the integral of the Maxwellian over all velocity is 1. Using standard results for the Gaussian integral we get

$$A = \left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}} \tag{30}$$

This distribution is symmetric for all components of $\mathbf{u}$. However, it is possible for a plasma to be in different thermal equilibrium in different directions. For example, in a strongly magnetised plasma where collisions along the field lines occur freely, but collisions across the field are heavily restricted, there can be two different temperatures; $T_{\parallel}$ along the field and $T_{\perp}$ across it.

The 'average' speed has different values depending on the definition. We have the mean speed $\langle v \rangle$, the most probable speed v_p and the root mean square speed v_{rms}.

The most probable speed is easiest to calculate as the maximum of $f(\mathbf{u})$

$$\frac{df(v)}{dv} = 4\pi A v^2 \exp\left(-\frac{mv^2}{2KT}\right) = 4\pi A \exp\left(-\frac{mv^2}{2KT}\right)\left(2v - \frac{mv^3}{KT}\right) \tag{31}$$

$$\implies v_p^2 = 0 \quad \text{or} \quad 1 = \frac{mv_p^2}{2KT} \quad \text{or} \quad v_p^2 \to \infty \tag{32}$$

Discarding the minima at 0 and ∞ where $f \to 0$ we get

$$v_p = \sqrt{\frac{2KT}{m}} \tag{33}$$

A useful approximation is $v_p \approx 18.755 \times 10^6 \sqrt{T[\text{keV}]}$. The other two are calculated as more Gaussian integrals

$$\langle v \rangle = \int_0^\infty u f(u) du = \sqrt{\frac{8KT}{\pi m}} = \frac{2}{\sqrt{\pi}} v_p \tag{34}$$

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \int_0^\infty u^2 f(u) du = \sqrt{\frac{3KT}{m}} = \sqrt{\frac{3}{2}} v_p \tag{35}$$

Ignoring time, we see from (28) that we take the integral of f over all velocity and integrate over all space we get the total number of particles. We also know if we integrate the density over all space we get the total number of particles. Therefore the integral of f over velocity must be the density.

$$n(\boldsymbol{x}) = \int f(\boldsymbol{x}, \boldsymbol{u}) d\boldsymbol{u} \tag{36}$$

1.5 Debye Shielding

A plasma is usually defined as a quasineutral gas of charged and neutral particles capable of collective motion. As electrons are free and highly conductive compared to the ions they have their own local density n_e compared to the local ion density n_i. However, increases in n_e generate electric fields which accelerate electrons from regions where $n_e > n_i$ towards regions where $n_e < n_i$.

Gauss' Law describes the electric field $\mathbf{E}$ generated by a charge density ρ. The local charge density in a region (assuming ions have charge +1) is $-e(n_e(\mathbf{x}) - n_i(\mathbf{x}))$

$$\epsilon_0 \nabla \cdot \boldsymbol{E}(\boldsymbol{x}) = \rho = -e(n_e(\boldsymbol{x}) - n_i(\boldsymbol{x})) \tag{37}$$

The electric field $\mathbf{E}(\mathbf{x}) = -\nabla\phi(\mathbf{x})$ where $\phi(\mathbf{x})$ is electric potential. Substituting into (37) we get Poisson's equation

$$\epsilon_0 \nabla^2 \phi(\boldsymbol{x}) = -e(n_i(\boldsymbol{x}) - n_e(\boldsymbol{x})) \tag{38}$$

There is now a potential energy $q\phi$ which changes the Maxwellian distribution of the electron velocities

$$f(u) = A \exp\left(\frac{-1}{KT_e}\left(\frac{1}{2}mu^2 - e\phi\right)\right) = A \exp\left(\frac{e\phi}{KT_e}\right) \exp\left(\frac{-mu^2}{2KT_e}\right) \tag{39}$$

This is just a scaled version of the density where $\phi = 0$ i.e. far from this location. Let the density far from this local disturbance be n_i=n. Therefore the electron density is

$$n_e = n \exp\left(\frac{e\phi}{KT_e}\right) \tag{40}$$

Substituting into (38)

$$\epsilon_0 \nabla^2 \phi(\boldsymbol{x}) = en\left(\exp\left(\frac{e\phi}{KT_e}\right) - 1\right) \tag{41}$$

Assuming a small potential energy compared to the thermal energy ($e\phi \ll KT_e$) we can Taylor expand the exponential to linear order $e^x = 1 + x + \dots$

$$\epsilon_0 \nabla^2 \phi(\boldsymbol{x}) = en\left(\frac{e\phi(\boldsymbol{x})}{KT_e}\right) = \frac{e^2 n}{KT_e}\phi(\boldsymbol{x}) \tag{42}$$

$$\implies \nabla^2 \phi = \left(\frac{e^2 n}{\epsilon_0 KT_e}\right)\phi := \frac{1}{\lambda_D^2}\phi \quad \lambda_D = \sqrt{\frac{\epsilon_0 KT_e}{e^2 n}} \tag{43}$$

Here we introduce the Debye length λ_D. Now assume the charge density perturbation is a point charge. This means $\phi(\boldsymbol{x})$ is radially symmetric. Switching to spherical co-ordinates (r, θ, Φ), as $\phi = \phi(r)$ only we can write out the Laplacian ∇^2 and discard everything but the first term

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \underbrace{\frac{\partial \phi}{\partial \theta}}_{=0}\right) + \frac{1}{r^2 \sin^2\theta}\underbrace{\frac{\partial^2 \phi}{\partial \Phi^2}}_{=0} \tag{44}$$

Therefore (43) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = \frac{\phi}{\lambda_D^2} \tag{45}$$

The solution to this can be calculated using Green's functions, instead we will cheat and substitute in the correct answer to prove it's a solution

$$\phi(r) = \frac{1}{r}\exp\left(\frac{-r}{\lambda_D}\right) \tag{46}$$

$$\implies \frac{\partial \phi(r)}{\partial r} = \frac{-1}{r^2}\exp\left(\frac{-r}{\lambda_D}\right) - \frac{1}{r\lambda_D}\exp\left(\frac{-r}{\lambda_D}\right) \tag{47}$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi(r)}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-\exp\left(\frac{-r}{\lambda_D}\right) - \frac{r}{\lambda_D}\exp\left(\frac{-r}{\lambda_D}\right)\right) \tag{48}$$

$$= \frac{1}{r^2}\left(\frac{1}{\lambda_D}\exp\left(\frac{-r}{\lambda_D}\right) - \frac{1}{\lambda_D}\exp\left(\frac{-r}{\lambda_D}\right) + \frac{r^2}{\lambda_D^2}\exp\left(\frac{-r}{\lambda_D}\right)\right) \tag{49}$$

$$= \frac{1}{r\lambda_D^2} \exp\left(\frac{-r}{\lambda_D}\right) \tag{50}$$

Therefore the potential from a perturbation in charge density is exponentially screened, known as Debye shielding. This means strong electric fields can only exist over a length scale λ_D. This also means the electron density must equal the ion density over scales larger than λ_D, called quasi-neutrality.

1.6 Plasma Oscillations

If electrons in a plasma are displaced from a uniform background density of ions, electric fields are generated which restore the electrons to their original positions. As electrons have inertia, they overshoot their original positions. As this repeats you get a periodic motion called Plasma Oscillations at a characteristic frequency called the Plasma Frequency ω_p.

As the electrons are very mobile compared to the ions assume the ions form a uniform background with density n. To make things easy also assume no magnetic field, no thermal motion (cold plasma or $T = 0$), infinite plasma and oscillations only occur in the $\hat{\boldsymbol{x}}$ direction. This means

$$\nabla = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} \quad \boldsymbol{E} = E\hat{\boldsymbol{x}} \quad \nabla \times \boldsymbol{E} = 0 \quad \boldsymbol{E} = \nabla\phi \tag{51}$$

We therefore have no oscillating magnetic field ($\nabla \times \boldsymbol{E} = 0$) i.e. this is electrostatic.

From (9) and (7) give

$$mn_e\left(\frac{\partial \boldsymbol{v_e}}{\partial t} + (\boldsymbol{v_e} \cdot \nabla)\,\boldsymbol{v_e}\right) = -en_e\boldsymbol{E} \tag{52}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e\boldsymbol{v_e}) = 0 \tag{53}$$

We have a local deviation from quasi-neutrality so use (3) to find the $\boldsymbol{E}$

$$\epsilon\nabla \cdot \boldsymbol{E} = \epsilon_0\frac{\partial \boldsymbol{E}}{\partial x} = -e(n_e - n) \tag{54}$$

To solve this we *linearise*, assuming we have small deviations from equilibrium values, discarding all terms except those linear in the deviation and then solving. Write quantities which will change (n_e, v_e, $\boldsymbol{E}$) as sum of an equilibrium part x_0 and a deviation x_1

$$n_e = n_{e0} + n_{e1} = n + n_{e1} \quad \boldsymbol{v_e} = \boldsymbol{v_{e0}} + \boldsymbol{v_{e1}} = \boldsymbol{v_{e1}} \quad \boldsymbol{E} = \boldsymbol{E_0} + \boldsymbol{E_1} = \boldsymbol{E_1} \tag{55}$$

We simplify by knowing in equilibrium the electrons are stationary and there is no Electric field hence $v_{e0} = 0$ and $\boldsymbol{E_1} = 0$. Also we can write the equilibrium electron density $n_{e0} = n$ due to quasi-neutrality. As we have assumed static ions $n_i = n$. The time derivatives of all equilibrium components are also 0. Finally as we have a uniform background density $\nabla n = 0$. Substituting in the linearised quantities

$$m\left(\frac{\partial \boldsymbol{v_{e1}}}{\partial t} + \underbrace{\left(\boldsymbol{v_{e1}} \cdot \nabla\right)\boldsymbol{v_{e1}}}_{\text{quadratic}}\right) = -e\boldsymbol{E_1} \tag{56}$$

$$\implies m\frac{\partial \boldsymbol{v_{e1}}}{\partial t} = -e\boldsymbol{E_1} \tag{57}$$

$$\underbrace{\frac{\partial n_{e0}}{\partial t}}_{=0} + \frac{\partial n_{e1}}{\partial t} + n\nabla \cdot \boldsymbol{v_{e1}} + \boldsymbol{v_{e1}}\underbrace{\nabla \cdot n}_{=0} = 0 \tag{58}$$

$$\implies \frac{\partial n_{e1}}{\partial t} + n\nabla \cdot \boldsymbol{v_{e1}} = 0 \tag{59}$$

$$\frac{\partial \boldsymbol{E_1}}{\partial x} = \frac{-en_{e1}}{\epsilon_0} \tag{60}$$

Now also assume all deviations are plane waves

$$\boldsymbol{v_{e1}} = v_{e1}e^{i(kx-\omega t)}\hat{\boldsymbol{x}} \quad n_{e1} = n_{e1}e^{i(kx-\omega t)} \quad \boldsymbol{E_1} = E_1 e^{i(kx-\omega t)}\hat{\boldsymbol{x}} \tag{61}$$

This allows us to replace $\frac{\partial}{\partial t}$ with $-iw$ and $\frac{\partial}{\partial x}$ with $-ik$. This is the same as taking the Fourier Transform

$$-i\omega v_{e1} = -eE_1 \quad -i\omega n_{e1} = -iknv_{e1} \quad ik\epsilon_0 E_1 = -en_1 \tag{62}$$

We can eliminate E_1 and n_{e1}

$$n_{e1} = \frac{nkv_{e1}}{\omega} \therefore ik\epsilon_0 E_1 = -e\frac{nkv_{e1}}{\omega} \therefore E_1 = \frac{-e}{ik\epsilon}\left(\frac{nkv_{e1}}{\omega}\right) = \frac{ienv_{e1}}{\epsilon_0\omega} \tag{63}$$

$$im\omega v_{e1} = \frac{ie^2nv_{e1}}{\epsilon_0\omega} \implies \omega^2 = \frac{ne^2}{m\epsilon_0} := \omega_{pe}^2 \tag{64}$$

ω_{pe} is the electron plasma frequency. A useful approximation is

$$\omega_{pe} \approx 18\pi\sqrt{n[\mathrm{m}{-3}]} \tag{65}$$

As the group velocity is zero this wave cannot propagate. It can propagate when $T_e > 0$.

2 Cold Plasma Dispersion

Discussing EBWs requires discussing electromagnetic waves propagating in a plasma. This is most easily done in a collision free data ($T = 0$).

2.1 Eikonal Ansatz and WKB Approximation

In a space and time varying medium the displacement vector is related to the electric field by $\boldsymbol{D} = \underline{\underline{\epsilon}} \cdot \mathbf{E}$. $\underline{\underline{\epsilon}}$ is the dielectric tensor which captures the effect of the medium. As the plasma in a tokamak or stellerator is stationary over a microwave period, we can ignore any special relativistic effects and retarded potentials

To start, take the curl of the Faraday-Maxwell equation (5)

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\nabla \times \frac{\partial \boldsymbol{B}}{\partial t} \underset{B \in C^2}{=} \frac{-(\nabla \times \boldsymbol{B})}{\partial t} \tag{66}$$

We can exchange curl and partial time derivatives as all partial derivatives of $\boldsymbol{B}$ are continuous ($\boldsymbol{B} \in C^2$). Eliminate $\nabla \times \boldsymbol{B}$ using (6)

$$\nabla \times (\nabla \times \boldsymbol{E}) = -mu_0 \frac{\partial \boldsymbol{j}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{67}$$

Now take the Fourier Transform with respect to time. We saw when calculating plasma oscillations this is equivalent to exchanging $\frac{\partial}{\partial t} \to -i\omega$. As we only transform with respect to time we leave all space derivatives

$$\mathcal{F}[\nabla \times (\nabla \times \boldsymbol{E})] = -(-i\omega) mu_0 \boldsymbol{j} - \frac{(-i\omega)^2}{c^2} \boldsymbol{E} = imu_0 \omega \boldsymbol{j} + \frac{\omega^2 \boldsymbol{E}}{c^2}$$

$$\implies \nabla \times (\nabla \times \boldsymbol{E}(\boldsymbol{x}, \omega)) = imu_0 \omega \boldsymbol{j}(\boldsymbol{x}, \omega) + \frac{\omega^2 \boldsymbol{E}(\boldsymbol{x}, \omega)}{c^2} \tag{68}$$

We can write $\boldsymbol{E}$ as a phasor using the Eikonal representation. This is completely general and is just a re-arrangement

$$\boldsymbol{E}(\boldsymbol{x}, \omega) = \boldsymbol{a}(\boldsymbol{x}, \omega) e^{iS(\boldsymbol{x}, \omega)} \quad \boldsymbol{k} := \nabla S \tag{69}$$

To substitute into (68) we need to calculate $\nabla \times (\nabla \times \boldsymbol{E})$. We need a vector identity and the gradient of e^{iS}

$$\nabla \times (\Psi \boldsymbol{A}) = \Psi (\nabla \times \boldsymbol{A}) + (\nabla \Psi) \times \boldsymbol{A} \tag{70}$$

$$\nabla e^{iS} = i \nabla S e^{iS} = i\boldsymbol{k} e^{iS} \tag{71}$$

Substitute (69) into (70)

$$\nabla \times \left(\boldsymbol{a}e^{iS}\right) = e^{iS}\left(\nabla \times \boldsymbol{a}\right) + \nabla e^{iS} \times \boldsymbol{a}$$

$$\implies e^{iS}\left(\nabla \times \boldsymbol{a}\right) + i\boldsymbol{k}e^{iS} = \left(\nabla + i\boldsymbol{k}\right) \times \boldsymbol{a}e^{iS} \tag{72}$$

To calculate the curl of this we can use (70) again

$$\begin{aligned}\nabla \times \left[\left(\nabla + i\boldsymbol{k}\right) \times \boldsymbol{a}\right] e^{iS} &= \nabla \times \left[\left(\nabla + i\boldsymbol{k}\right) \times \boldsymbol{a}\right] e^{iS} + \nabla e^{iS} \times \left[\left(\nabla + i\boldsymbol{k}\right) \times \boldsymbol{a}\right] \\ &= \nabla \times \left[\left(\nabla + i\boldsymbol{k}\right) \times \boldsymbol{a}\right] e^{iS} + i\boldsymbol{k}e^{iS} \times \left[\left(\nabla + i\boldsymbol{k}\right) \times \boldsymbol{a}\right] \\ &= \left[\left(\nabla + i\boldsymbol{k}\right) \times \left(\left(\nabla + i\boldsymbol{k}\right) \times \boldsymbol{a}\right)\right]\end{aligned} \tag{73}$$

We can now substitute into (70)

$$\left[\left(\nabla + i\boldsymbol{k}\right) \times \left(\left(\nabla + i\boldsymbol{k}\right) \times \boldsymbol{a}\right)\right] e^{iS} = \frac{\omega^2}{c^2}\boldsymbol{a}e^{iS} + i\mu_0\omega\boldsymbol{j} \tag{74}$$

Dividing by e^{iS} we get our final amplitude equation

$$\left(\nabla + i\boldsymbol{k}\right) \times \left(\left(\nabla + i\boldsymbol{k}\right) \times \boldsymbol{a}\right) = \frac{\omega^2}{c^2}\boldsymbol{a} + i\mu_0\omega e^{-iS}\boldsymbol{j} \tag{75}$$

In a weakly inhomogeneous medium with inhomogeneity scale $L \gg \lambda$, we can look for solutions which deviate slightly from a plane wave. We can therefore assume the magnitude of the gradient of plasma quantities (density, temperature, magnetic field, etc.) $|\nabla \boldsymbol{x}| \sim \frac{\boldsymbol{x}}{L}$.

Using this we can make some assumptions which simplify (75)

$$\frac{|\nabla \times \boldsymbol{a}|}{|i\boldsymbol{k} \times \boldsymbol{a}|} \sim \frac{|\boldsymbol{a}|}{L|\boldsymbol{a}||\boldsymbol{k}|} = \frac{1}{L|\boldsymbol{k}|} = \frac{\lambda}{2\pi L} \approx 0 \implies \nabla + i\boldsymbol{k} \approx i\boldsymbol{k} \tag{76}$$

∇ captures the medium dependence while $i\boldsymbol{k}$ captures the wave dependence. As we assume the medium is weakly inhomogeneous we are saying we can ignore medium dependence compared to wave dependence. This is called the Wentzel-Kramer-Brillouin (WKB) approximation.

If we apply this approximation to (75), to be consistent we must use a Local Ohm's law for the RHS (i.e. keeping zeroeth order in $\frac{\nabla}{i\boldsymbol{k}}$)

$$\boldsymbol{j}(\boldsymbol{x}) = \underline{\underline{\sigma}} \cdot \boldsymbol{E} + \boldsymbol{j}_{\text{ext}} = \underline{\underline{\sigma}} \cdot \boldsymbol{a}e^{iS} + \boldsymbol{j}_{\text{ext}} \tag{77}$$

Here $\underline{\underline{\sigma}}$ is the conductivity tensor. This Ohm's law applies only for small electric field amplitudes so $\boldsymbol{j}$ is proportional to $\boldsymbol{E}$. $\boldsymbol{j}_{\text{ext}}$ is not a response of the medium to $\boldsymbol{E}$ but accounts for spontaneous emission or absorption. Substituting into (75)

$$i\boldsymbol{k} \times (i\boldsymbol{k} \times \boldsymbol{a}) = \frac{\omega^2}{c^2}\boldsymbol{a} + i\mu_0\omega e^{-iS}\left(\underline{\underline{\sigma}} \cdot \boldsymbol{a}e^{iS} + \boldsymbol{j}_{\text{ext}}\right)$$

$$= \left(\frac{\omega^2}{c^2}\underline{\underline{I}} + i\mu_0\omega\underline{\underline{\sigma}}\right)\boldsymbol{a} + i\mu_0\omega e^{-iS}\boldsymbol{j}_{\text{ext}} = \frac{\omega^2}{c^2}\underline{\underline{\epsilon}} \cdot \boldsymbol{a} + i\mu_0\omega e^{-iS}\boldsymbol{j}_{\text{ext}} \tag{78}$$

where $\underline{\underline{I}}$ is the identity. Here we introduce $\underline{\underline{\epsilon}}$ the *local* dielectric tensor

$$\underline{\underline{\epsilon}} := \underline{\underline{I}} + \frac{i\mu_0 c^2}{\omega}\underline{\underline{\sigma}} = \underline{\underline{I}} + \frac{i}{\epsilon_0\omega}\underline{\underline{\sigma}} \quad \text{or} \quad \epsilon_{ij} = \delta_{ij} + \frac{i}{\epsilon_0\omega}\sigma_{ij} \tag{79}$$

where δ_{ij} is the Kronecker delta. Dividing through by $\frac{\omega^2}{c^2}$ we can redefine in terms of the refractive index $\boldsymbol{N} = \frac{c\boldsymbol{k}}{\omega}$ (multiplying the i^2 as -1 on the LHS)

$$-\boldsymbol{N} \times (\boldsymbol{N} \times \boldsymbol{a}) = \underline{\underline{\epsilon}} \cdot \boldsymbol{a} + \frac{i\mu_0 c^2}{\omega}e^{iS}\boldsymbol{j}_{\text{ext}} \tag{80}$$

We can evaluate $\boldsymbol{N} \times (\boldsymbol{N} \times \boldsymbol{a})$

$$\begin{pmatrix} N_x \\ N_y \\ N_z \end{pmatrix} \times \begin{pmatrix} N_x \\ N_y \\ N_z \end{pmatrix} \times \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times = \begin{pmatrix} N_x \\ N_y \\ N_z \end{pmatrix} \times \begin{pmatrix} N_y a_z - a_y N_z \\ N_z a_x - a_z N_x \\ N_x a_y - a_x N_y \end{pmatrix}$$

$$\begin{pmatrix} N_x N_y a_z - N_y^2 a_x - N_z^2 a_x + N_x N_z a_z \\ N_y N_z a_y - N_z^2 a_y - N_x^2 a_y + N_x N_y a_y \\ N_x N_z a_x - N_x^2 a_z - N_y^2 a_z + N_y N_z a_y \end{pmatrix} = \begin{pmatrix} -N_y^2 - N_z^2 & N_x N_y & N_x N_z \\ N_x N_y & -N_x^2 - N_z^2 & N_y N_z \\ N_x N_z & N_y N_z & -N_x^2 - N_y^2 \end{pmatrix} \cdot \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

$$\begin{pmatrix} N_x^2 - N^2 & N_x N_y & N_x N_z \\ N_x N_y & N_y^2 - N^2 & N_y N_z \\ N_x N_z & N_y N_z & N_z^2 - N^2 \end{pmatrix} \cdot \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

$$\left(N^2\underline{\underline{I}} - \boldsymbol{N} \otimes \boldsymbol{N}\right) \cdot \boldsymbol{a} \tag{81}$$

where $\otimes$ is the outer product. Therefore $-\boldsymbol{N} \times (\boldsymbol{N} \times \boldsymbol{a}) = \boldsymbol{N} \otimes \boldsymbol{N} - N^2\underline{\underline{I}}$. Substituting in

$$0 = \left(\underline{\underline{\epsilon}} - \boldsymbol{N} \otimes \boldsymbol{N} + N^2\underline{\underline{I}}\right) \cdot \boldsymbol{a} + \frac{i\mu_0 c^2}{\omega}e^{iS}\boldsymbol{j}_{\text{ext}}$$

$$\implies \underline{\underline{\Lambda}} \cdot \boldsymbol{a} = -\frac{i\mu_0 c^2}{\omega}e^{iS}\boldsymbol{j}_{\text{ext}} \quad \underline{\underline{\Lambda}} := \underline{\underline{\epsilon}} - \boldsymbol{N} \otimes \boldsymbol{N} + N^2\underline{\underline{I}} \tag{82}$$

We can split Λ into Hermititan and Anti-Hermitian parts $\underline{\underline{\Lambda}} = \underline{\underline{\Lambda}}^h + i\underline{\underline{\Lambda}}^a$ where $\underline{\underline{\Lambda}}^h = \frac{1}{2}\left(\underline{\underline{\Lambda}} + \underline{\underline{\Lambda}}^\dagger\right)$ and $\underline{\underline{\Lambda}}^a = \frac{1}{2i}\left(\underline{\underline{\Lambda}} - \underline{\underline{\Lambda}}^\dagger\right)$. Substituting into (75)

$$\underline{\underline{\Lambda}}^h \cdot \boldsymbol{a} = -i\underline{\underline{\Lambda}}^a \cdot \boldsymbol{a} - \frac{i\mu_0 c^2}{\omega} e^{iS} \boldsymbol{j}_{\text{ext}} \tag{83}$$

The first time on the RHS describes absorption and stimulated emission while the second term is spontaneous emission. Far from emitting and absorbing locations this reduces to

$$\underline{\underline{\Lambda}}^h \cdot \boldsymbol{a} = 0 \tag{84}$$

which describes non-dissipative wave propagation. It's solvability condition is

$$\text{Det}\underline{\underline{\Lambda}} = \mathcal{D}\left(\boldsymbol{k}, \omega\left(\boldsymbol{k}, \boldsymbol{x}, t\right), \boldsymbol{x}, t\right) = 0 \tag{85}$$

2.2 Deriving the Conductivity Tensor

In order to evaluate (84) we need to calculate $\underline{\underline{\epsilon}}$ which involves calculating $\underline{\underline{\sigma}}$. We will use the Navier Stokes equations (8) to calculate the velocity of each species s, calculate current using the Drift Equation (2) then use Ohm's law (1) to find the components of $\underline{\underline{\sigma}}$. We already linearised the Navier Stokes equations for a cold plasma when calculating plasma oscillations. We need to add in a static background magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$

$$\frac{\partial \boldsymbol{v}_s}{\partial t} = \frac{q}{m_s}\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) \tag{86}$$

We can Fourier transform in time and write out the components

$$-i\omega v_{sx} = \frac{q}{m_s}\left(E_x + v_{sy}B\right) \implies v_{sx} = \frac{iq}{\omega m_s}\left(E_x + v_{sy}B\right) \tag{87}$$

$$-i\omega v_{sy} = \frac{q}{m_s}\left(E_y - v_{sx}B\right) \implies v_{sy} = \frac{iq}{\omega m_s}\left(E_y - v_{sx}B\right) \tag{88}$$

$$-i\omega v_{sz} = \frac{q}{m_s}E_z \implies v_{sz} = \frac{iq}{\omega m_s}E_z \tag{89}$$

Eliminate v_{sx} and v_{sy}

$$v_{sx} = \frac{iqE_x}{\omega m_s} + \frac{iqBv_{sy}}{\omega m_s} = \frac{iqE_x}{\omega m_s} + \frac{iqB}{\omega m_s}\left(\frac{iqE_y}{\omega m_s} - \frac{iqBv_{sx}}{\omega m_s}\right) \tag{90}$$

$$= \frac{iq}{m_s}\frac{E_x}{\omega} - \frac{q}{m_s}\frac{qB}{m_s}\frac{E_y}{\omega^2} + \frac{q^2B^2}{m_s^2}\frac{v_{sx}^2}{\omega} \tag{91}$$

$$v_{sy} = \frac{iqE_y}{\omega m_s} - \frac{iqBv_{sx}}{\omega m_s} = \frac{iqE_y}{\omega m_s} - \frac{iqB}{\omega m_s}\left(\frac{iqE_x}{\omega m_s} + \frac{iqBv_{sy}}{\omega m_s}\right) \tag{92}$$

$$= \frac{iqE_y}{\omega m_s} + \frac{q}{m_s}\frac{qB}{m_s}\frac{E_x}{\omega^2} + \frac{q^2B^2}{m_s^2}\frac{v_{sy}^2}{\omega} \tag{93}$$

We recognise the species cyclotron frequency $\omega_{cs} = \frac{qB}{m_s}$. Gather like terms to write

$$\left(1 - \frac{\omega_{cs}^2}{\omega^2}\right) v_{sx} = \frac{iqE_x}{\omega m_s} - \frac{q}{m_s\omega}\frac{\omega_{cs}}{\omega}E_y \tag{94}$$

$$\left(1 - \frac{\omega_{cs}^2}{\omega^2}\right) v_{sy} = \frac{iqE_y}{\omega m_s} + \frac{q}{m_s\omega}\frac{\omega_{cs}}{\omega}E_x \tag{95}$$

We can use the drift equation to write out the components of the current $j_{si} = nqv_{si}$

$$j_{sx} = nqv_{sx} = \left(1 - \frac{\omega_{cs}^2}{\omega^2}\right)^{-1}\left(\frac{inq^2E_x}{\omega m_s^2} - \frac{nq^2}{m_s^2\omega}\frac{\omega_{cs}}{\omega}E_y\right) \tag{96}$$

$$j_{sy} = nqv_{sy} = \left(1 - \frac{\omega_{cs}^2}{\omega^2}\right)^{-1}\left(\frac{inq^2E_y}{\omega m_s^2} - + \frac{nq^2}{m_s^2\omega}\frac{\omega_{cs}}{\omega}E_x\right) \tag{97}$$

$$j_{sz} = \frac{inq^2E_z}{\omega m_s^2} \tag{98}$$

We recognise we almost have the species plasma frequency $\omega_{ps}^2 = \frac{nq^2}{\epsilon_0 m_s^2}$. Substituting in $\frac{nq^2}{m_s^2} = \epsilon_0\omega_{ps}^2$

$$j_{sx} = \left(1 - \frac{\omega_{cs}^2}{\omega^2}\right)^{-1}\left(\frac{\epsilon_0\omega_{ps}^2}{\omega}E_x - \frac{\epsilon_0\omega_{ps}^2}{\omega}\frac{\omega_{cs}}{\omega}E_y\right) \tag{99}$$

$$j_{sy} = \left(1 - \frac{\omega_{cs}^2}{\omega^2}\right)^{-1}\left(\frac{\epsilon_0\omega_{ps}^2}{\omega}E_y + \frac{\epsilon_0\omega_{ps}^2}{\omega}\frac{\omega_{cs}}{\omega}E_x\right) \tag{100}$$

$$j_{sz} = \frac{\epsilon_0\omega_{ps}^2}{\omega}E_z \tag{101}$$

Now we have the current $\boldsymbol{j}$ in terms of components of the Electric field $\boldsymbol{E}$. We can read off the components of the conductivity tensor $\underline{\underline{\sigma}}$ using (1)

$$\boldsymbol{j} = \underline{\underline{\sigma}} \cdot \boldsymbol{E} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \cdot \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx}j_x + \sigma_{xy}j_y + \sigma_{xz}j_z \\ \sigma_{yx}j_x + \sigma_{yy}j_y + \sigma_{yz}j_z \\ \sigma_{zx}j_x + \sigma_{zy}j_y + \sigma_{zz}j_z \end{pmatrix} \tag{102}$$

$$= \sum_s \begin{pmatrix} \sigma_{xx} j_{sx} + \sigma_{xy} j_{sy} + \sigma_{xz} j_{sz} \\ \sigma_{yx} j_{sx} + \sigma_{yy} j_{sy} + \sigma_{yz} j_{sz} \\ \sigma_{zx} j_{sx} + \sigma_{zy} j_{sy} + \sigma_{zz} j_{sz} \end{pmatrix} \tag{103}$$

We'll cheat a bit and take a factor of $\epsilon_0\omega$ out the from

$$\sigma_{xx} = \sigma yy = \epsilon_0 \omega \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{1}{1 - \frac{\omega_{cs}^2}{\omega^2}} \tag{104}$$

$$\sigma_{zz} = \epsilon_0 \omega \sum_s \frac{\omega_{ps}^2}{\omega^2} \tag{105}$$

$$\sigma_{xy} = -\sigma_{yx} = -i\epsilon_0 \omega \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega_{cs}}{\omega} \tag{106}$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = 0 \tag{107}$$

We can now write out the cold plasma dielectric tensor using (79). Fortunately we can cancel $\epsilon_0\omega$

$$\underline{\underline{\epsilon}} = \begin{pmatrix} 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{1}{1 - \frac{\omega_{cs}^2}{\omega^2}} & -i\sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega_{cs}}{\omega} & 0 \\ i\sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{\omega_{cs}}{\omega} & 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \frac{1}{1 - \frac{\omega_{cs}^2}{\omega^2}} & 0 \\ 0 & 0 & 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \end{pmatrix} \tag{108}$$

This is rather cumbersome so we'll introduce some useful Stix Parameters X and Y

$$X_s = \frac{\omega_{ps}^2}{\omega^2} \quad Y_s = \frac{\omega_{cs}}{\omega} \tag{109}$$

Note at constant frequency ω $X \propto n$ and $Y \propto B$, so changing X and Y represents changes in density and magnetic field.

$$\underline{\underline{\epsilon}} = \begin{pmatrix} 1 - \sum_s \frac{X_s}{1 - Y_s^2} & -i\sum_s \frac{X_s Y_s}{1 - Y_s^2} & 0 \\ -i\sum_s \frac{X_s Y_s}{1 - Y_s^2} & 1 - \sum_s \frac{X_s}{1 - Y_s^2} & 0 \\ 0 & 0 & 1 - \sum_s X_s \end{pmatrix} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \tag{110}$$

We simplify again by introducing some more Stix Parameters S, D and P

$$S = 1 - \sum_s \frac{X_s}{1 - Y_s^2} = 1 - \sum_s \frac{\omega_{ps}}{\omega^2 - \omega_{ce}^2} \tag{111}$$

$$D = \sum_s \frac{X_s Y_s}{1 - Y_s^2} = \sum_s \frac{\omega_{ce}}{\omega} \frac{\omega_{ps}^2}{\omega^2 - \omega_{ce}^2} \tag{112}$$

$$P = 1 - \sum_s X_s = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \tag{113}$$

There are 2 more Stix Parameters we'll introduce as they'll be useful: $R = S + D$ and $L = S - D$

$$\begin{aligned} R &= 1 - \sum_s \frac{X_s}{1 - Y_s^2} + \sum_s \frac{X_s Y_s}{1 - Y_s^2} = 1 - \sum_s \frac{X_s \left(1 - Y_s\right)}{1 - Y_s^2} \\ &= 1 - \sum_s \frac{X_s}{1 + Y_s} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega \left(\omega - \omega_{cs}\right)} \end{aligned} \tag{114}$$

$$\begin{aligned} L &= 1 - \sum_s \frac{X_s}{1 - Y_s^2} - \sum_s \frac{X_s Y_s}{1 - Y_s^2} = 1 - \sum_s \frac{X_s \left(1 + Y_s\right)}{1 - Y_s^2} \\ &= 1 - \sum_s \frac{X_s}{1 - Y_s} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega \left(\omega + \omega_{cs}\right)} \end{aligned} \tag{115}$$

2.3 Appleton Hartree Equation

To find the dispersion relation we substitute the expression for $\underline{\underline{\epsilon}}$ into (84). Without loss of generality we define a co-ordinate system with $\hat{\boldsymbol{z}}$ parallel to $\boldsymbol{B}$. As we have cylindrical symmetry we can set $N_y = 0$ so $N^2 = N_x^2 - N_z^2$. In the general case for oblique propagation (neither parallel nor perpendicular to $\boldsymbol{B}$), defining θ to be the angle between $\boldsymbol{k}$ and $\boldsymbol{B}$ we can write $N_x = N \sin\theta$ and $N_z = N\cos\theta$. We will end up with an equation for N^2 as a function of X, Y, ω and θ called the Appleton-Hartree Equation.

Here we assume electrons as the only species (i.e. ions form uniform background) so we drop subscripts on X and Y. Using the definition of $\underline{\underline{\Lambda}}$ from (82)

$$\mathcal{D} = \begin{vmatrix} S - N_z^2 & -iD & N_x N_z \\ iD & S - N^2 & 0 \\ N_x N_z & 0 & P - N_x^2 \end{vmatrix}$$

$$= \begin{vmatrix} S - N^2\cos^2\theta & -iD & N^2\cos\theta\sin\theta \\ iD & S - N^2 & 0 \\ N^2\cos\theta\sin\theta & 0 & P - N^2\sin^2\theta \end{vmatrix} \tag{116}$$

Solving the determinant of a 3D matrix is well known, we'll use the bottom row as the leading coefficients

$$\mathcal{D} = N^2\cos\theta\sin\theta \begin{vmatrix} -iD & N^2\cos\theta\sin\theta \\ s - N^2 & 0 \end{vmatrix}$$

$$+\left(P - N^2\sin^2\theta\right) \begin{vmatrix} S - N^2\cos^2\theta & -iD \\ iD & S - N^2 \end{vmatrix}$$

$$= N^2\cos\theta\sin\theta\left[-\left(S - N^2\right)N^2\cos\theta\sin\theta\right]$$

$$+\left(P - N^2\sin^2\theta\right)\left[\left(S - N^2\cos^2\theta\right)\left(S - N^2\right) - D^2\right]$$

$$= N^4\sin^2\theta\cos^2\theta\left(N^2 - S\right)$$

$$+\left(P - N^2\sin^2\theta\right)\left(S^2 - N^2\cos^2\theta\sin^2\theta - SN^2 + N^4\cos^2\theta - D^2\right)$$

$$= N^6\sin^2\theta\cos^2\theta - SN^4\cos^2\theta\sin^2\theta + PS^2 - PSN^2\cos^2\theta - PSN^2$$

$$+PN^4\cos^2\theta - D^2P - N^2S^2\sin^2\theta + N^4S\sin^2\theta\cos^2\theta + N^4S\sin^2\theta - N^6\sin^2\theta\cos^2\theta + N^2D^2\sin^2\theta$$

$$= \left(S\sin^2\theta + P\cos^2\theta\right)N^4$$

$$+\left(D^2\sin^2\theta - S^2\sin^2\theta - PS\cos^2\theta - PS\right)N^2 + PS^2 - PD^2$$

This is a quadratic equation of the form $AN^4 - B^2 + C$ with coefficients

$$A = S\sin^2\theta + P\cos^2\theta \tag{117}$$

$$B = \left(S^2 - D^2\right)\sin^2\theta + PS\left(1 + \cos^2\theta\right) \tag{118}$$

$$C = P\left(S^2 - D^2\right) \tag{119}$$

We can simplify $S^2 - D^2$

$$S^2 - D^2 = \frac{1}{4}\left(R + L\right)^2 - \frac{1}{4}\left(R - L\right)^2$$

$$\frac{1}{4}\left(R^2 + 2RL + L^2 - R^2 + 2RL - L^2\right) = RL \tag{120}$$

So we get the a simpler form of $AN^4 - B^2 + C$ with coefficients

$$A = S\sin^2\theta + P\cos^2\theta \tag{121}$$

$$B = RL\sin^2\theta + PS\left(1 + \cos^2\theta\right) \tag{122}$$

$$C = PRL \tag{123}$$

This is a quadratic in N^2 so we can solve using the quadratic formula. We need to evaluate the discriminant

$$F^2 = B^2 - 4AC = \left(RL\sin^2\theta + PS\left(1+\cos^2\theta\right)\right)^2 - 4PRL\left(S\sin^2\theta + P\cos^2\theta\right)$$

$$= R^2L^2\sin^4\theta + P^2S^2\left(1+2\cos^2\theta+\cos^4\theta\right) + 2PSRL\sin^2\theta\left(1+\cos^2\theta\right)$$

$$-4PSRL\sin^2\theta - 4P^2RL\cos^2\theta$$

$$= R^2L^2\sin^4\theta + P^2S^2\left(1+2\cos^2\theta+\cos^4\theta\right) - 2PSRL\underbrace{\sin^2\theta\left(1-\cos^2\theta\right)}_{-\sin^4\theta} - 4P^2\underbrace{RL}_{S^2-D^2}\cos^2\theta$$

$$= R^2L^2\sin^4\theta + P^2S^2\underbrace{\left(1+2\cos^2\theta+\cos^4\theta\right)}_{4\cos^2\theta+\sin^4\theta} + 2PSRL\sin^4\theta - 4P^2S^2\cos^2\theta + 4P^2D^2\cos^2\theta$$

$$= R^2L^2\sin^4\theta + P^2S^2\left(4\cos^2\theta+\sin^4\theta\right) + 2PSRL\sin^4\theta - 4P^2S^2\cos^2\theta + 4P^2D^2\cos^2\theta$$

$$= R^2L^2\sin^4\theta + P^2S^2\sin^4\theta + 2PSRL\sin^4 + 4P^2D^2\cos^2\theta$$

$$= (RL-PS)^2\sin^4\theta + 4P^2D^2\cos^2\theta$$

Traditionally this has a solution $N^2 = \frac{1}{2A}(B \pm F)$, however will use some different forms. Firstly we can divide the $AN^2 - BN^2 + C$ by $\cos^2\theta$

$$\left(S\tan^2\theta + P\right)N^4 - \left(RL\tan^2\theta + PS\left(1+\frac{1}{\cos^2\theta}\right)\right)N^2 + \frac{PRL}{\cos^2\theta}$$

Use the vector identity

$$\frac{1}{\cos^2\theta} = \frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta} = 1+\tan^2\theta \qquad (124)$$

$$\left(S\tan^2\theta + P\right)N^4 - \left(RL\tan^2\theta + PS\left(2+\tan^2\theta\right)\right)N^2 + PRL\left(1+\tan^2\theta\right)$$

$$= \left(SN^4 - RLN^2 - PSN^2 + PRL\right)\tan^2\theta + PN^4 - 2PSN^2 + PRL$$

$$= \left(SN^2 - RL\right)\left(N^2-P\right)\tan^2\theta + P\left(N^4 - \underbrace{2S}_{R+L}N^2 + RL\right)$$

$$= \left(SN^2 - RL\right)\left(N^2-P\right)\tan^2\theta + P\left(N^2-R\right)\left(N^2-L\right)$$

$$\implies \tan^2\theta = \frac{P\left(N^2-R\right)\left(N^2-L\right)}{\left(SN^2-RL\right)\left(N^2-P\right)} \qquad (125)$$

We can look at this in 2 limits: $\theta = 0$ and $\theta = \frac{\pi}{2}$.

For $\theta = 0$ (Parallel to $\boldsymbol{B}$), $\tan\theta = 0$ giving 3 solutions $P = 0$, $N^2 = R$ and $N^2 = L$. $P = 0$ is simply electrostatic plasma oscillations. The other two solutions correspond to electromagnetic waves.

For $\theta = \frac{\pi}{2}$ (Perpendicular to $\boldsymbol{B}$), $\tan\theta \to \infty$ giving 2 solutions $N^2 = \frac{RL}{S}$ and $N^2 = P$. The first is the extraordinary wave (X Mode) while the second is the ordinary wave (O Mode).

Returning to the dispersion relation $AN^4 - BN^2 + C$ we can look for solutions $N^2 = 1 - x$

$$A(1-x)^2 - B(1-x) + C = A\left(1 - 2x + x^2\right) - B + Bx + C$$

$$= Ax^2 + (B - 2A)x + A - B + C$$

This is also a quadratic, calculate the discriminant

$$\tilde{F}^2 = (B - 2A)^2 - 4A(A - B + C)$$

$$= 4A^2 - 4AB + B^2 - 4A^2 + 4AB - 4AC$$

$$= B^2 - 4AC = F^2$$

Now use the alternate form of the quadratic formula. Dividing ax^2+bx+c by x^2

$$a + \frac{b}{x} + \frac{c}{x^2} = 0$$

$$\frac{1}{x} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c} \implies x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$$

Combining this all together

$$N^2 = 1 - \frac{2(A - B + C)}{2A - B \pm F} \quad F^2 = B^2 - 4AC \tag{126}$$

Now we need to evaluate the terms on the RHS in terms of the Stix Parameters. Start with the numerator

$$A - B + C = S\sin^2\theta + P\cos^2\theta - RL\sin^2\theta - PS\left(1 + \cos^2\theta\right) + PRL$$

$$= (S - RL)\sin^2\theta + P(1 - S)\cos^2\theta + P(RL - S) \tag{127}$$

Evaluate the coefficients

$$P(1 - S) = (1 - X)\left(\frac{X}{1 - Y^2}\right) \tag{128}$$

$$RL = S^2 - D^2 = \left(1 - \frac{X}{1 - Y^2}\right)^2 - \left(\frac{XY}{1 - Y^2}\right)^2$$

$$= 1 - \frac{2X}{1-Y^2} + \frac{X^2}{\left(1-Y^2\right)^2} - \frac{X^2Y^2}{1-Y^2}$$

$$= 1 - \frac{2X}{1-Y^2} + \frac{X^2\left(1-Y^2\right)}{\left(1-Y^2\right)^2} = 1 + (X-2)\frac{X}{1-Y^2} \tag{129}$$

$$S - RL = 1 - \frac{X}{1-Y^2} - 1 - (X-2)\frac{X}{1-Y^2} = (1-X)\frac{X}{1-Y^2} \tag{130}$$

Combining it all together

$$A - B + C = (1-X)\frac{X\sin^2\theta}{1-Y^2} + (1-X)\frac{X\cos^2\theta}{1-Y^2} - (1-X)^2\frac{X}{1-Y^2}$$

$$= (1-X)\frac{X}{1-Y^2}\left[\sin^2\theta + \cos^2\theta - 1 + X\right]$$

$$= X(1-X)\frac{X}{1-Y^2} \tag{131}$$

The first term on the denominator

$$2A - B = 2P\cos^2\theta + 2S\sin^2\theta - RL\sin^2\theta - PS\left(1+\cos^2\theta\right)$$

$$= (2S - RL)\sin^2\theta + P(2-S)\cos^2\theta - PS \tag{132}$$

Evaluate the coefficients

$$2S - RL = 2\left(1 - \frac{X}{1-Y^2}\right) - 1 - (X-2)\frac{X}{1-Y^2}$$

$$= 2 - \frac{2X}{1-Y^2} - 1 - \frac{X^2}{1-Y^2} + \frac{2X}{1-Y^2} = 1 - \frac{X^2}{1-Y^2} \tag{133}$$

$$P(2-S) = (1-X)\left(1 + \frac{X}{1-Y^2}\right)$$

$$= 1 - X + \frac{X}{1-Y^2} - \frac{X^2}{1-Y^2} \tag{134}$$

$$PS = (1-X)\left(1 - \frac{X}{1-Y^2}\right)$$

$$= 1 - X - \frac{X}{1-Y^2} + \frac{X^2}{1-Y^2} \tag{135}$$

Substituting into (132)

$$2A - B = \left(1 - \frac{X^2}{1-Y^2}\right)\sin^2\theta + \left(1 - X + \frac{X}{1-Y^2} - \frac{X^2}{1-Y^2}\right)\cos^2\theta$$

$$-1+X+\frac{X}{1-Y^2}-\frac{X^2}{1-Y^2}$$

$$=\sin^2\theta-\frac{X^2\sin^2\theta}{1-Y^2}+\cos^2\theta-X\cos^2\theta+\frac{X\cos^2\theta}{1-Y^2}-\frac{X^2\cos^2\theta}{1-Y^2}-1+X+\frac{X}{1-Y^2}-\frac{X^2}{1-Y^2}$$

$$=\left(\sin^2\theta+\cos^2\theta-1\right)+X\left(1-\cos^2\theta\right)+\frac{X}{1-Y^2}\left(-X\sin^2\theta+\cos^2\theta-X\cos^2\theta+1-X\right)$$

$$=\frac{X\sin^2\theta}{1-Y^2}\left(1-Y^2\right)+\frac{X}{1-Y^2}\left(1-2X+\cos^2\theta\right)$$

$$=\frac{X}{1-Y^2}\left(\sin^2\theta-Y^2\sin^2\theta+1-2X+\cos^2\theta\right)$$

$$=\frac{X}{1-Y^2}\left(2-2X-Y^2\sin^2\theta\right)=\frac{X}{1-Y^2}\left(2\left(1-X\right)-Y^2\sin^2\theta\right) \tag{136}$$

Finally the discriminant

$$F^2=B^2-4AC=\left(RL-PS\right)^2\sin^4\theta+4P^2D^2\cos^2\theta$$

Expand the coefficient

$$RL-PS=1+\left(X-2\right)\frac{X}{1-Y^2}-\left(1-X-\frac{X}{1-Y^2}+\frac{X^2}{1-Y^2}\right)$$

$$=\frac{X}{1-Y^2}\left(X-2\right)-\frac{X}{1-Y^2}\left(1-Y^2+1-X\right)=\frac{-XY^2}{1-Y^2}$$

$$=-X\left(1+X\right)\frac{Y^2}{1-Y^2} \tag{137}$$

Substituting in

$$F^2=\frac{X^2Y^4}{\left(1-Y^2\right)^2}\sin^4\theta+4\left(1-X\right)^2\left(\frac{X^2Y^2}{1-Y^2}\right)\cos^2\theta$$

$$=\left(\frac{X}{1-Y^2}\right)^2\left(Y^4\sin^4\theta+4\left(1-X\right)^2Y^2\cos^2\theta\right) \tag{138}$$

Now substitute all terms into (126)

$$N^2=1-\frac{2X\left(1-X\right)\frac{X}{1-Y^2}}{\left(2\left(1-X\right)-Y^2\sin^2\theta\right)\frac{X}{1-Y^2}\pm\sqrt{\left(\frac{X}{1-Y^2}\right)^2\left(Y^4\sin^4\theta+4\left(1-X\right)^2Y^2\cos^2\theta\right)}} \tag{139}$$

We can cancel a factor of $\frac{X}{1-Y^2}$ on the top and bottom to arrive at the final result called the Appleton-Hartree equation

$$N^2=1-\frac{2X\left(1-X\right)}{2\left(1-X\right)-Y^2\sin^2\theta\pm\sqrt{Y^4\sin^4\theta+4\left(1-X\right)^2Y^2\cos^2\theta}} \tag{140}$$

3 Electron Bernstein Waves

Electron Bernstein waves are electrostatic waves propagating across the magnetic field in a *hot* plasma. They are sustained by electron cyclotron motion and can be thought of as fronts of electron rarefraction and compression perpendicular to the magnetic field. They therefore have wavelength of order of the electron Larmor radius ρ:

$$\lambda \sim \rho \tag{141}$$

We clearly need to consider a finite ρ which implies a finite temperature. In a cold plasma electrons are tight to the field lines and only allow space charge waves along the field lines (plasma oscillations).

If we have $\rho \sim \lambda$ and $\omega \sim \omega_{ce}$

4 Absorption of Electron Bernstein Waves

4.1 Doppler Shifted Resonance Condition

Resonant absorption of electromagnetic waves in a magnetised plasma occurs when the oscillation of the wave electric field is synchronised with the gyration of the charged particles, in our case electrons. However, the finite value of the Larmor radius leads to absorption at higher harmonics. Bornatici has this to say on the matter [2]:

On purely classical grounds one would expect that here is a coupling to the electromagnetic field, i.e. emission and absorption, only at a wave frequency of $\omega = \omega_{ce}$. However, even if the electron dynamics is taken to be non-relativistic, there appear emission and absorption also around the harmonics $n\omega_{ce}$ because of the finiteness of the gyration (Larmor) radius of the electrons ρ and of the speed of light, or, more precicely, of the quantity

$$k_\perp^2 \rho^2 = \left(\frac{v_\perp}{c} \frac{\omega}{\omega_{ce}} N \sin\theta \right)^2 \tag{142}$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{B_0}$. In fact the finiteness of $k_\perp \rho$ has the consequence that the electromagnetic field generated by an electron gyrating with frequency ω_{ce}, although periodic with a period $\frac{2\pi}{\omega_{ce}}$, is not simply harmonic, implying that higher Fourier components appear. Hence, this framework, although commonly termed 'non-relativistic', actually contains a kind of relativistic effect already, due to the fact that Maxwell equations for the electromagnetic field are, indeed, relativistic equations. On the other hand, for emission and absorption of waves propagating close to perpendicularly to the magnetic field, it is just the relativistic electron dynamics that *provides the dominant kinetic line-broadening mechanism, as for these directions the longitudinal Doppler effect tends to become small. Therefore, to correctly describe radiation at all angles a fully relativistic description is required to begin with, even when the electron velocity $v \ll c$.*

Consider an electromagnetic wave propagating at frequency ω with wavevector $\boldsymbol{k}$. Taking the magnetic field along the $\boldsymbol{z}$ axis, the phase velocity of the wave parallel to the magnetic field $v_{\varphi,\|}$ is

$$v_{\varphi,\|} = \frac{\omega}{k_\|} \tag{143}$$

We can write the wave electric field along the magnetic field $\boldsymbol{E}_\|$, which is just a plane wave with wavevector $k_\|$ and frequency ω

$$\boldsymbol{E}_\| = E \sin\left(k_\| z - \omega t\right) \tag{144}$$

In order to get resonance, the frequency of the electric field needs to match the period of the gyration. The components of the electron motion are

$$x(t) = \rho \cos\left(-\omega_{ce} t\right) \tag{145}$$

$$y(t) = \rho \sin\left(-\omega_{ce} t\right) \tag{146}$$

$$z(t) = v_\| t \tag{147}$$

where we include the minus sign to ensure $\omega_{ce} > 0$. The electric field in the frame of the particle is

$$\boldsymbol{E}_\| = E \sin\left(k_\| v_\| t - \omega t\right) = E \sin\left(\left(k_\| v_\| - \omega\right) t\right) \tag{148}$$

To be in resonance the frequency of the electric field must be a harmonic of the cyclotron frequency $-n\omega_{ce}$ where again we include the minus sign. We therefore get the resonance condition

$$k_\| v_\| - \omega = -n\omega_{ce} \implies \omega = n\omega_{ce} + k_\| v_\| \tag{149}$$

4.2 Doppler Broadened Absorption Coefficient

As particles have finite temperature they have a distribution of velocities. This means the Doppler shifted resonance condition can be satisfied by different parts of the distribution at different frequencies. A simple formula can be derived for this starting with the Doppler shifted resonance condition at the n-th harmonic, given wave perpendicular wavevector $k_\|$ and electron parallel velocity $v_\|$

$$\omega = n\omega_{ce} + k_\| v_\| \tag{150}$$

As we have finite temperature $v_\parallel$ is distributed according to a 1D Maxwell-Boltzmann distribution (Gaussian) centred on a velocity v_0 with width the electron thermal velocity V_T

$$f\left(v_\parallel\right) = \left(\frac{1}{\pi v_T^2}\right)^{0.5} \exp\left(-\left(\frac{v_\parallel - v_0}{v_T}\right)^2\right) \quad V_T = \sqrt{\frac{KT_e}{2m}} \tag{151}$$

Now define the 'bulk' of the distribution as a velocity range where almost all of the particles are found. Standard results from Normal distributions tell us 99.7% of particles are located within $3\sigma = 3v_T$ of v_0. This distinction is somewhat arbitrary but it does successfully divide the distribution into a 'bulk' part containing almost all particles ($|v_\parallel - v_0| \leq 3v_T$) and a 'tail' containing almost no particles ($|v_\parallel - v_0| > 3v_T$).

We can write the cold Doppler shifted resonant frequency $\omega_0 = n\omega_{ce} + k_\parallel v_0$. Assuming there is no absorption outside the bulk, we can write the minimum and maximum frequencies we expect absoption ω_1 and ω_2

$$\omega_1 = n\omega_{ce} + k_\parallel\left(v_0 - 3v_T\right) = \omega_0 - 3k_\parallel v_T \tag{152}$$

$$\omega_2 = n\omega_{ce} + k_\parallel\left(v_0 + 3v_T\right) = \omega_0 + 3k_\parallel v_T \tag{153}$$

This can be combined into a single equation for the minimum and maximum absorption frequency ω due to Doppler broadening relative to the cold Doppler shifted frequency ω_0. Divide by ω and rearrange

$$\omega = \omega_0 \pm 3k_\parallel v_T \implies 1 = \frac{\omega_0}{\omega} \pm \frac{3k_\parallel v_T}{\omega} = \frac{\omega_0}{\omega} \pm \frac{3k_\parallel c}{\omega}\frac{v_T}{c} = \frac{\omega_0}{\omega} \pm 3N_\parallel \beta \tag{154}$$

where we introduce the normalised thermal velocity β. A useful approximation is $\beta \approx 9.953 \times 10^{-3}\sqrt{T_e[\mathrm{eV}]}$. Cancel the factors of 2π to convert from angular frequency to frequency and rearrange. Take the inverse to get a formula for f.

$$\frac{\omega_0}{\omega} = \frac{f_0}{f} = 1 \mp 3N_\parallel\beta \implies f = \frac{f_0}{1 \mp 3N_\parallel\beta} \tag{155}$$

Here we stress the Doppler **broadened** frequency band $f \in \left[\frac{f_0}{1-3N_\parallel\beta}, \frac{f_0}{1+3N_\parallel\beta}\right]$ is relative to the Doppler **shifted** frequency f_0 **not** the harmonic frequencies nf_{ce}, i.e. we have a shift on a shift. Calculating f_0 compared to f_{ce} requires an estimate of the average electron parallel velocity v_0.

We can play a similar game using the cold Doppler shifted resonance condition, replacing $k_\parallel$ with $\frac{\omega_0 N_\parallel}{c}$

$$\omega_0 = n\omega_{ce} + \frac{\omega_0 N_\parallel v_0}{c} = n\omega_{ce} + \omega_0 N_\parallel \beta_0 \quad \beta_0 := \frac{v_0}{c} \tag{156}$$

We can once again cancel factors of 2π to convert to frequency. Rearranging we get an expression for ω_0

$$f_0\left(1 - N_{\|}\beta_0\right) = nf_{ce} \implies f_0 = \frac{nf_{ce}}{1 - N_{\|}\beta_0} \tag{157}$$

Substituting for f_0 in (155) using (156) we arrive the final expression for the Doppler shifted and Doppler broadened absorption frequencies limits around each cyclotron harmonic

$$f = \frac{nf_{ce}}{\left(1 - N_{\|}\beta_0\right)\left(1 \mp 3N_{\|}\beta\right)} \tag{158}$$

The first bracket on the denominator is the Doppler shift term due to the electrons having relative velocity v_0 to the wave. If $v_0 = 0$ then $\beta_0 = 0$ we return to the cyclotron harmonics. The second bracket on the denominator is the Doppler broadening term due to finite temperature. Likewise if $T_e = 0$ then $\beta = 0$ and we once again get the cyclotron harmonics.

4.3 Absorption using a Complex Dispersion Relation

Consider the characteristic decay of the intensity of an EBW $\boldsymbol{I}_{\mathrm{EBW}}$. This has a characteristic decay rate per unit length along the ray trajectory s. We will call it α_ω

$$I \sim I_0 \exp\left(-\alpha_\omega s\right) \tag{159}$$

As $\boldsymbol{I} \sim \boldsymbol{E}^2$ we can write the decay rate of the electric field

$$E \sim E_0 \exp\left(\frac{-\alpha_\omega \boldsymbol{s}}{2}\right) \tag{160}$$

where E_0 is the electric field strength at the origin of the ray. We have already assumed the electric field has a plane wave form, so we can write

$$E = E_0 \exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - \frac{\alpha_\omega s}{2}\right) \tag{161}$$

This is equivalent to generalising the wavevector $\boldsymbol{k} \to \boldsymbol{k} + \frac{i\alpha_\omega}{2}\boldsymbol{s}$, where $\boldsymbol{s}$ is the unit vector pointing in the direction of the group velocity $\boldsymbol{v}_g$ i.e. $\boldsymbol{s} = \frac{\boldsymbol{v}_s}{|\boldsymbol{v}_s|}$ so the ray path length $s = \boldsymbol{s}\cdot\boldsymbol{x}$.

$$i\boldsymbol{k}\cdot\boldsymbol{x} - \frac{\alpha_\omega s}{2} = i\boldsymbol{k}\cdot\boldsymbol{x} + i^2\frac{\alpha_\omega}{2}\boldsymbol{s}\cdot\boldsymbol{x} = i\left(\boldsymbol{k}\cdot\boldsymbol{x} + \frac{i\alpha_\omega}{2}\boldsymbol{s}\right)\cdot\boldsymbol{x} \tag{162}$$

To justify imaginary wavevectors we need to include the anti-Hermitian part of the dispersion tensor $\underline{\underline{\Lambda}}\left(\boldsymbol{k},\omega\right)$. Our wave equation is

$$\underline{\underline{\Lambda}}^h\left(\boldsymbol{k},\omega\right)\cdot\boldsymbol{a}+i\underline{\underline{\Lambda}}^a\left(\boldsymbol{k},\omega\right)\cdot\boldsymbol{a}=0 \tag{163}$$

The real $\boldsymbol{k}$ solves the wave equation, i.e. $\underline{\underline{\Lambda}}^h\cdot\boldsymbol{a}=0$. We can left multiply this expression by $\boldsymbol{a}$ to get a scalar

$$\boldsymbol{a}^*\cdot\underline{\underline{\Lambda}}^h\left(\boldsymbol{k},\omega\right)\cdot\boldsymbol{a}:=\Lambda^h_{aa}\left(\boldsymbol{k},\omega\right)=0 \tag{164}$$

Similarly we define $\Lambda^h_{aa}\left(\boldsymbol{k},\omega\right):=\boldsymbol{a}^*\cdot\underline{\underline{\Lambda}}^a\left(\boldsymbol{k},\omega\right)\cdot\boldsymbol{a}$. Our wave equation is therefore

$$\Lambda^h_{aa}\left(\boldsymbol{k}+\frac{i\alpha_\omega\boldsymbol{s}}{2}\right)+i\Lambda^a_{aa}\left(\boldsymbol{k}+\frac{i\alpha_\omega\boldsymbol{s}}{2}\right)=0 \tag{165}$$

If $\frac{\alpha_\omega}{2k}\ll 1$ we can Taylor expand to first order (making the evaulation at ω implicit)

$$\Lambda^h_{aa}\left(\boldsymbol{k}+\frac{i\alpha_\omega\boldsymbol{s}}{2}\right)=\Lambda^h_{aa}\left(\boldsymbol{k}\right)+\frac{i\alpha_\omega\boldsymbol{s}}{2}\cdot\partial_{\boldsymbol{k}}\Lambda^h_{aa}\left(\boldsymbol{k}\right) \tag{166}$$

$$i\Lambda^a_{aa}\left(\boldsymbol{k}+\frac{i\alpha_\omega\boldsymbol{s}}{2}\right)=i\Lambda^a_{aa}\left(\boldsymbol{k}\right)-\frac{\alpha_\omega\boldsymbol{s}}{2}\cdot\partial_{\boldsymbol{k}}\Lambda^a_{aa}\left(\boldsymbol{k}\right) \tag{167}$$

where $\partial_{\boldsymbol{k}}=\left(\frac{\partial}{\partial k_x},\frac{\partial}{\partial k_y},\frac{\partial}{\partial k_z}\right)$ is the partial derivatives over components of $\boldsymbol{k}$. Applying this to (165), understanding Λ^h_{aa} and Λ^a_{aa} are evaulated at $(\boldsymbol{k},\omega)$

$$\Lambda^h_{aa}+\frac{i\alpha_\omega\boldsymbol{s}}{2}\cdot\partial_{\boldsymbol{k}}\Lambda^h_{aa}+i\Lambda^a_{aa}=0 \tag{168}$$

Immediately $\Lambda^h_{aa}=0$ by definition from (164). The first order Taylor expansion term of Λ^a_{aa} has been omitted as it is real. If incorporated into α_ω it would be an imaginary term, i.e. it's just a phase shift of $\boldsymbol{E}$. The second order Taylor expansion term would be real, however we are not expanding to that order (Note: This argument needs some clarification!). Rearranging we get an equation for α_ω

$$\alpha_\omega=\frac{-2\Lambda^a_{aa}}{\boldsymbol{s}\cdot\partial_{\boldsymbol{k}}\Lambda^h_{aa}} \tag{169}$$

At high N we are in the pure electrostatic regime. These modes are purely longitudinal, characterised by $\boldsymbol{a}\parallel\boldsymbol{k}$. This implies $\boldsymbol{a}=\alpha\boldsymbol{k}$ where $\alpha\in\Re$ is some scale factor. This also implies $\Lambda^i_{aa}=\alpha^2\Lambda^i_{kk}$. Multiplying (169) by α^2 top and bottom allows us to rewrite as

$$\alpha_\omega=\frac{-2\Lambda^a_{kk}}{\boldsymbol{s}\cdot\partial_{\boldsymbol{k}}\Lambda^h_{kk}} \tag{170}$$

References

[1] F. Volpe, "Electron bernstein emission diagnostic of electron temperature profile at w7-as stellarator", PhD thesis (Ernst Moritz Arndt Universitaet Greifswald, 2003).

[2] M. Bornatici, R. Cano, O. De Barbieri, and F. Engelmann, "Electron cyclotron emission and absorption in fusion plasmas", Nuclear Fusion **23**, 1153 (1983).