1 High Frequency Linear Waves in a Hot Uniform Plasma

This follows Section 2 from the thesis of Decker, referencing heavily Fundamentals of Plasma Physics by Paul Bellan [1].

1.1 Maxwell Equations

The Maxwell equations are (in Heaviside Form)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.1.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.1.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.1.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} - \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.1.4}$$

When considering waves its more convenient to Fourier transform the equations. The Fourier transformed Maxwell equations are

$$i\mathbf{k} \cdot \mathbf{E}_k = \frac{\rho_k}{\epsilon_0} \tag{1.1.5}$$

$$i\mathbf{k} \cdot \mathbf{B}_k = 0 \tag{1.1.6}$$

$$i\mathbf{k} \times \mathbf{E}_k = i\omega \mathbf{B}_k \tag{1.1.7}$$

$$i\mathbf{k} \times \mathbf{B}_k = \mu_0 \mathbf{j} - \epsilon_0 \mu_0 i\omega \mathbf{E}_k \tag{1.1.8}$$

1.2 Wave Equation

The wave equation for a linear wave in a weakly inhomogeneous medium, using the WKB approximation, is

$$\underline{\underline{\mathcal{D}}} \cdot \mathbf{E} = 0 \tag{1.2.1}$$

$$\underline{\underline{\mathcal{D}}} := \mathbf{N} \otimes \mathbf{N} - N^2 \underline{\underline{I}} + \underline{\underline{\epsilon}} \tag{1.2.2}$$

where $\mathbf{N}$ is the refractive index, $\otimes$ is the Tensor/Outer Product, $\underline{\underline{I}}$ is the Identity Tensor, $\underline{\underline{\epsilon}}$ is the Dielectric Tensor, $\mathbf{E}$ is the wave electric field and $\underline{\underline{\mathcal{D}}}$ is the Dispersion Tensor. The Dielectric Tensor is related to the Susceptibility Tensor $\underline{\underline{\chi}}$ and the Conductivity Tensor $\underline{\underline{\sigma}}$ by

$$\underline{\underline{\epsilon}} = \underline{\underline{I}} + \underline{\underline{\chi}} = \underline{\underline{I}} + \frac{i}{\omega \epsilon_0} \underline{\underline{\sigma}} \tag{1.2.3}$$

The conductivity tensor also appears in linear Ohm's law to relate $\mathbf{E}$ and $\mathbf{j}$

$$\mathbf{j} = \underline{\underline{\sigma}} \cdot \mathbf{E} \tag{1.2.4}$$

The condition (1.2.1) can be left multiplied by $\mathbf{E}$ to get a quadratic form

$$\mathcal{D} = \mathbf{E}^* \cdot \underline{\underline{\mathcal{D}}} \cdot \mathbf{E} = 0 \tag{1.2.5}$$

When there is weak absorption, $\underline{\underline{\epsilon}}$ and hence $\underline{\underline{\mathcal{D}}}$ are Hermitian. We define the Hermitian part $\underline{\underline{\mathcal{D}}}^H := \frac{1}{2}\left(\underline{\underline{\mathcal{D}}} + \underline{\underline{\mathcal{D}}}^\dagger\right)$ and anti-Hermitian part $\underline{\underline{\mathcal{D}}}^A := \frac{1}{2}\left(\underline{\underline{\mathcal{D}}} - \underline{\underline{\mathcal{D}}}^\dagger\right)$ such that $\underline{\underline{\mathcal{D}}} = \underline{\underline{\mathcal{D}}}^H + \underline{\underline{\mathcal{D}}}^A$. We can then define the condition (1.2.1) as

$$\mathcal{D} = \mathbf{E}^* \cdot \underline{\underline{\mathcal{D}}}^H \cdot \mathbf{E} + \mathbf{E}^* \cdot i\underline{\underline{\mathcal{D}}}^A \cdot \mathbf{E} = 0 \tag{1.2.6}$$

1.3 Energy Equation for Linear Waves

From the continuity equation we can define the conservation of energy for a weakly inhomogeneous, weakly dissipative plasma for the linear mode $\mathbf{E}_k$

$$\frac{\partial \omega_k}{\partial t} + \nabla \cdot \mathbf{s}_k = -P_k^{lin} \tag{1.3.1}$$

where ω_k is the time averaged energy density, $\mathbf{s}_k$ is the time averaged energy flow and P_k^{lin} is the density of power dissipated. This equation is a function of the real parts of ω and $\mathbf{k}$.

1.3.1 Poynting's Theorem

Poynting's Theorem describes the energy flow of electromagnetic waves. It is derived by first considering the work done by an electromagnetic field on a charged particle dw_i travelling at velocity $\mathbf{v}$ in a time dt

$$dw_i = \mathbf{F} \cdot d\mathbf{l} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \mathbf{v}dt = q\mathbf{E} \cdot \mathbf{v}dt \tag{1.3.2}$$

We get the total work done dw by summing over all particles and recognising $\mathbf{j} = \sum_i q\mathbf{v}$, where $\mathbf{j}$ is the current density.

$$dw = \sum_i \mathbf{E} \cdot q\mathbf{v}dt = \mathbf{E} \cdot \mathbf{j}dt \tag{1.3.3}$$

Re-arranging we get an expression for the rate of transfer of energy density from the electromagnetic field to the particles w_P, i.e. the kinetic energy associated with particles coherent motion in the electromagnetic field.

$$\frac{dw_P}{dt} = \mathbf{E} \cdot \mathbf{j} \tag{1.3.4}$$

We see this term is related to kinetic energy as

$$\mathbf{F} \cdot \mathbf{v} = m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{m}{2}\frac{dv^2}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) \tag{1.3.5}$$

$\mathbf{E} \cdot \mathbf{j}$ can be expressed using (1.1.4), re-arranged for $\mathbf{j}$, and dotting with $\mathbf{E}$

$$\begin{aligned} \mathbf{j} &= \frac{1}{\mu_0}\nabla \times \mathbf{B} - \epsilon_0\frac{\partial \mathbf{E}}{\partial t} \\ \mathbf{E} \cdot \mathbf{j} &= \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{1.3.6}$$

Now use the vector expression $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}$ allowing us to replace $\mathbf{E} \cdot (\nabla \times \mathbf{B})$ with $-\nabla \cdot (\mathbf{E} \times \mathbf{B}) + (\nabla \times \mathbf{E}) \cdot \mathbf{B}$

$$\mathbf{E} \cdot \mathbf{j} = -\frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) + (\nabla \times \mathbf{E}) \cdot \mathbf{B} - \epsilon_0\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \tag{1.3.7}$$

Use (1.1.3) to replace $\nabla \times \mathbf{E}$

$$\mathbf{E} \cdot \mathbf{j} = -\frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) - \frac{1}{\mu_0}\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{B} - \epsilon_0\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \tag{1.3.8}$$

Rearranging we get

$$\mathbf{E} \cdot \mathbf{j} + \epsilon_0\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0}\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{B} + \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) = 0 \tag{1.3.9}$$

This is Poynting's Theorem, which is really just a statement of the continuity equation for electromagnetic waves. We define the Poynting Vector $\mathbf{S}$

$$\mathbf{S} := \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{1.3.10}$$

which is the energy flux carried by the electromagnetic wave. Defining the total energy of the wave $\mathcal{U}$, we can rewrite Poynting's Theorem as

$$\frac{\partial\mathcal{U}}{\partial t} + \nabla\cdot\mathbf{S} = 0 \tag{1.3.11}$$

where

$$\frac{\partial\mathcal{U}}{\partial t} = \mathbf{j}\cdot\mathbf{E} + \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu_0}\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{B} \tag{1.3.12}$$

The total energy $\mathcal{U} = \mathcal{U}_P + \mathcal{U}_{EM}$, where $\mathcal{U}_P$ is the energy associated with coherent motion of charged particles within the field and $\mathcal{U}_{EM}$ is the energy associated with the electromagnetic field. Earlier we associated $\mathbf{E}\cdot\mathbf{j}$ with $\mathcal{U}_P$ so we can write

$$\frac{\partial\mathcal{U}_P}{\partial t} = \mathbf{E}\cdot\mathbf{j} \tag{1.3.13}$$

$$\frac{\partial\mathcal{U}_{EM}}{\partial t} = \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu_0}\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{B} \tag{1.3.14}$$

We can also write the energy of the wave $\mathcal{U}(t)$ using (1.3.12)

$$\mathcal{U}(t) = \mathcal{U}(-\infty) + \int_{-\infty}^{t}\left\langle\mathbf{j}\cdot\mathbf{E} + \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu_0}\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{B}\right\rangle dt' \tag{1.3.15}$$

where $\langle ... \rangle$ represents time averaging over a wave period. Time averaging will remove the rapidly varying component from the oscillation of the fields and keep the slowly varying component from the evolution of the wave, which is what we are interested in. Likewise we would need to take a time average of the Poynting Flux $\mathbf{S}$ to see evolution of the wave energy flow.

Also note the background energy term $\mathcal{U}(-\infty)$, indicating the integral term is the difference in energy between when the wave is present and when it is absent. This implies the existence of 'negative energy waves', i.e. the plasma is unstable to the creation of electromagnetic waves.

1.3.2 Time Averaged Poynting's Theorem

In order to make progress we need to take the time average of the terms in Poynting's Theorem over a wave period. This is complicated as some of the terms are the product of two complex oscillating fields, e.g. $\mathbf{E}\cdot\mathbf{j}$. As only the real parts correspond to physical quantities we need to be careful, this is covered in 4.1, to summarise we need to use (4.1.5) to correctly calculate time averages for products of complex oscillating fields.

We will let $\omega = \omega_R + i\omega_I$ and $\mathbf{k} = \mathbf{k}_R + i\mathbf{k}_I$, essentially allowing for wave dissipation in space and time. We only consider weak dissipation i.e. small ω_I and ω_I. We will define our time averaged energy density as w and the time averaged energy flow as $\mathbf{s}$. We also define their Fourier Transforms as w_k and $\mathbf{s}_k$, giving the respective energy density and energy flow for each mode $\mathbf{k}$.

The first thing to note is (4.1.5) implies all our time averaged quantities will be proportional to the factor $e^{2(\omega_I t - \mathbf{k}_I\cdot\mathbf{x})}$. Therefore our time averaged Poynting's Theorem will look like

$$\frac{\partial}{\partial t}\left(we^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})}\right) + \nabla\cdot\left(\mathbf{s}e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})}\right) = 0$$

$$2\omega_I we^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})} - 2\mathbf{k}_I\cdot\mathbf{s}e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})} = 0 \tag{1.3.16}$$

We will get rid of $e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})}$ factors in final results by assuming ω_I and $\mathbf{k}_I$ are so small $e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})}\approx 1$. However they are important as through the time derivative and divergence they introduce factors of $2\omega_I$ and $2\mathbf{k}_I$ we need to divide by to get the time averaged energy density and energy flow.

Fourier Transforming and letting $e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})}\approx 1$ we get a generalised form of Poynting's Theorem

$$2\omega_I w_k - 2\mathbf{k}_I\cdot\mathbf{s}_k = 0 \tag{1.3.17}$$

Now we will calculate the time averaged quantities. To start with Fourier Transform (1.3.9)

$$\mathbf{E}_k\cdot\mathbf{j}_k + \epsilon_0\mathbf{E}_k\cdot(-i\omega\mathbf{E}_k) + \frac{1}{\mu_0}\mathbf{B}_k\cdot(-i\omega\mathbf{B}_k) + \frac{1}{\mu_0}i\mathbf{k}\cdot(\mathbf{E}_k\times\mathbf{B}_k) = 0 \tag{1.3.18}$$

It's convenient to split this calculation into 3 parts. To start with we will calculate $\langle\mathbf{E}_k\cdot\mathbf{j}_k + \epsilon_0\mathbf{E}_k\cdot(-i\omega\mathbf{E}_k)\rangle$. Firstly we note using the definition of the dielectric tensor (1.2.3) and Ohm's Law (1.2.4)

$$\mathbf{E}_k \cdot \mathbf{j}_k + \epsilon_0 \mathbf{E}_k \cdot (-i\omega \mathbf{E}_k) = \mathbf{E}_k \cdot \left(\underline{\underline{\sigma}} - i\omega\epsilon_0 \underline{\underline{I}}\right) \cdot \mathbf{E}_k$$

$$= \mathbf{E}_k \cdot (-i\omega\epsilon_0) \left(\frac{i}{\omega\epsilon_0}\underline{\underline{\sigma}} + \underline{\underline{I}}\right) \cdot \mathbf{E}_k = \mathbf{E}_k \cdot \left(-i\omega\epsilon_0 \underline{\underline{\epsilon}}\right) \cdot \mathbf{E}_k$$

$$\implies \langle \mathbf{E}_k \cdot \mathbf{j}_k + \epsilon_0 \mathbf{E}_k \cdot (-i\omega \mathbf{E}_k) \rangle = \left\langle \mathbf{E}_k \cdot \left(-i\omega\epsilon_0 \underline{\underline{\epsilon}}\right) \cdot \mathbf{E}_k \right\rangle \tag{1.3.19}$$

Using (4.1.5) and substituting in $i\omega^* = i\omega_R + \omega_I$ and $-i\omega = -i\omega_R + \omega_I$

$$\left\langle \mathbf{E}_k \cdot \left(-i\omega\epsilon_0 \underline{\underline{\epsilon}}\right) \cdot \mathbf{E}_k \right\rangle = \frac{1}{4}\left[\mathbf{E}_k^* \cdot \left(-i\omega\epsilon_0 \underline{\underline{\epsilon}}\right) \cdot \mathbf{E}_k + \mathbf{E}_k \cdot \left(i\omega^*\epsilon_0 \underline{\underline{\epsilon}}^*\right) \cdot \mathbf{E}_k^*\right] e^{2(\omega_I t - \mathbf{k}_I \cdot \mathbf{x})}$$

$$= \frac{\epsilon_0}{4}\left[\mathbf{E}_k^* \cdot (-i\omega_R + \omega_I)\, \underline{\underline{\epsilon}} \cdot \mathbf{E}_k + \mathbf{E}_k \cdot (i\omega_R + \omega_I)\, \underline{\underline{\epsilon}}^* \cdot \mathbf{E}_k^*\right] e^{2(\omega_I t - \mathbf{k}_I \cdot \mathbf{x})}$$

$$= \frac{\epsilon_0}{4}\left[i\omega_R \left(\mathbf{E}_k \cdot \underline{\underline{\epsilon}}^* \cdot \mathbf{E}_k^* - \mathbf{E}_k^* \cdot \underline{\underline{\epsilon}} \cdot \mathbf{E}_k\right) + \omega_I \left(\mathbf{E}_k \cdot \underline{\underline{\epsilon}}^* \cdot \mathbf{E}_k^* + \mathbf{E}_k^* \cdot \underline{\underline{\epsilon}} \cdot \mathbf{E}_k\right)\right] e^{2(\omega_I t - \mathbf{k}_I \cdot \mathbf{x})} \tag{1.3.20}$$

We can now use a quadratic form identity to replace $\mathbf{E}_k \cdot \underline{\underline{\epsilon}}^* \cdot \mathbf{E}_k^*$ with $\mathbf{E}_k^* \cdot \underline{\underline{\epsilon}}^\dagger \cdot \mathbf{E}_k$, where $\dagger$ is the conjugate transpose.

$$\mathbf{E} \cdot \underline{\underline{\epsilon}}^* \cdot \mathbf{E}^* = \sum_{p,q} E_p \epsilon_{pq}^* E_q^* = \sum_{p,q} E_p \epsilon_{qp}^\dagger E_q^* = \mathbf{E}^* \cdot \underline{\underline{\epsilon}}^\dagger \cdot \mathbf{E} \tag{1.3.21}$$

Applying this identity above we get, also taking the opportunity to assume $e^{2(\omega_I t - \mathbf{k}_I \cdot \mathbf{x})} \approx 1$

$$\langle \dots \rangle = \frac{\epsilon_0}{4}\left[i\omega_R \left(\mathbf{E}_k^* \cdot \left(\underline{\underline{\epsilon}}^\dagger - \underline{\underline{\epsilon}}\right) \cdot \mathbf{E}_k\right) + \omega_I \left(\mathbf{E}_k^* \cdot \left(\underline{\underline{\epsilon}}^\dagger + \underline{\underline{\epsilon}}\right) \cdot \mathbf{E}_k\right)\right] \tag{1.3.22}$$

Now we use the fact ω_I and $\mathbf{k}_I$ are small to Taylor expand $\underline{\underline{\epsilon}}$ about ω_R and $\mathbf{k}_R$

$$\underline{\underline{\epsilon}}\left(\omega_R + i\omega_I, \mathbf{k}_R + \mathbf{k}_I\right) \approx \underline{\underline{\epsilon}}\left(\omega_R, \mathbf{k}_R\right) + i\omega_I \left.\frac{\partial \underline{\underline{\epsilon}}}{\partial \omega}\right|_{\omega_R, \mathbf{k}_R} + i\mathbf{k}_I \cdot \left.\frac{\partial \underline{\underline{\epsilon}}}{\partial \mathbf{k}}\right|_{\omega_R, \mathbf{k}_R} \tag{1.3.23}$$

$$\left[\underline{\underline{\epsilon}}\left(\omega_R + i\omega_I, \mathbf{k}_R + \mathbf{k}_I\right)\right]^\dagger \approx \underline{\underline{\epsilon}}\left(\omega_R, \mathbf{k}_R\right) - i\omega_I \left.\frac{\partial \underline{\underline{\epsilon}}}{\partial \omega}\right|_{\omega_R, \mathbf{k}_R} - i\mathbf{k}_I \cdot \left.\frac{\partial \underline{\underline{\epsilon}}}{\partial \mathbf{k}}\right|_{\omega_R, \mathbf{k}_R} \tag{1.3.24}$$

Recognising that $\underline{\underline{\epsilon}}\left(\omega_R, \omega_I\right) = \underline{\underline{\epsilon}}^H$ as it is the small imaginary components of ω and $\mathbf{k}$ which make $\underline{\underline{\epsilon}}$ non-hermitian, we have

$$\underline{\underline{\epsilon}}^H = \frac{1}{2}\left(\underline{\underline{\epsilon}} + \underline{\underline{\epsilon}}^\dagger\right) = \underline{\underline{\epsilon}}\left(\omega_R, \mathbf{k}_R\right) \tag{1.3.25}$$

$$\underline{\underline{\epsilon}}^A = \frac{1}{2}\left(\underline{\underline{\epsilon}} - \underline{\underline{\epsilon}}^\dagger\right) = i\left[\omega_I \frac{\partial \underline{\underline{\epsilon}}}{\partial \omega} + \mathbf{k}_I \cdot \frac{\partial \underline{\underline{\epsilon}}}{\partial \mathbf{k}}\right]_{\omega_R, k_R} \tag{1.3.26}$$

Substituting in these expressions and cancelling $-i^2$ we get

$$\langle \dots \rangle = \frac{\epsilon_0}{2}\left[\omega_R \left(\mathbf{E}_k^* \cdot \left(\omega_I \frac{\partial \underline{\underline{\epsilon}}}{\partial \omega} + \mathbf{k}_I \cdot \frac{\partial \underline{\underline{\epsilon}}}{\partial \mathbf{k}}\right) \cdot \mathbf{E}_k\right) + \omega_I \left(\mathbf{E}_k^* \cdot \underline{\underline{\epsilon}} \cdot \mathbf{E}_k\right)\right]\Bigg|_{\omega_R, \mathbf{k}_R} \tag{1.3.27}$$

Gathering terms containing ω_I and terms containing $\mathbf{k}_I$

$$\langle \dots \rangle = \frac{\epsilon_0}{2}\left[\omega_I \mathbf{E}^* \cdot \left(\omega_R \frac{\partial \underline{\underline{\epsilon}}}{\partial \omega} + \underline{\underline{\epsilon}}\right) \cdot \mathbf{E} + \omega_R \mathbf{E}^* \cdot \left(\mathbf{k}_I \cdot \frac{\partial \underline{\underline{\epsilon}}}{\partial \mathbf{k}}\right) \cdot \mathbf{E}\right]\Bigg|_{\omega_R, \mathbf{k}_R} \tag{1.3.28}$$

Using the product rule and the fact we are evaluating all terms at $\omega_R, \mathbf{k}_R$ we can replace ω_R with ω and write

$$\omega_R \frac{\partial \underline{\underline{\epsilon}}}{\partial \omega} + \underline{\underline{\epsilon}} = \frac{\partial\left(\omega \underline{\underline{\epsilon}}\right)}{\partial \omega} \tag{1.3.29}$$

Applying this substitution, we also rewrite to make comparison with (1.3.17)

$$\langle \dots \rangle = 2\omega_I \left(\frac{\epsilon_0}{4} \mathbf{E}^* \cdot \frac{\partial\left(\omega \underline{\underline{\epsilon}}\right)}{\partial \omega} \cdot \mathbf{E}\right) - 2\mathbf{k}_I \cdot \left(-\frac{\omega \epsilon_0}{4} \mathbf{E}^* \cdot \frac{\partial \underline{\underline{\epsilon}}}{\partial \mathbf{k}} \cdot \mathbf{E}\right) \tag{1.3.30}$$

Notice we've picked up an energy flow term despite time averaging terms we've associated with energy density. This flow is associated with particle motion due to the wave.

Now time average the third term in (1.3.18)

$$\left\langle \frac{1}{\mu_0} \mathbf{B}_k \cdot \left(-i\omega \mathbf{B}_k\right)\right\rangle = \frac{1}{4\mu_0}\left[\mathbf{B}_k \cdot \left(i\omega^*\right) \cdot \mathbf{B}^* - \mathbf{B}_k^* \cdot (i\omega) \cdot \mathbf{B}\right] e^{2(\omega_I t - \mathbf{k}_I \cdot \mathbf{x})} \tag{1.3.31}$$

Again assuming $e^{2(\omega_I t - \mathbf{k}_I \cdot \mathbf{x})} \approx 1$ and noting $\mathbf{B}^* \cdot \mathbf{B} = \|\mathbf{B}\|^2$

$$\left\langle \frac{1}{\mu_0} \mathbf{B}_k \cdot \left(-i\omega \mathbf{B}_k\right)\right\rangle = \frac{i\|\mathbf{B}\|^2}{4\mu_0}\left(\omega^* - \omega\right) = 2\omega_I \frac{\|\mathbf{B}_k\|^2}{4\mu_0} \tag{1.3.32}$$

Now we will time average the Poynting Flux term $\frac{1}{\mu_0}\nabla \cdot \langle(\mathbf{E}\times\mathbf{B})\rangle$. We can use the standard method to write

$$\begin{aligned}\frac{1}{\mu_0}\nabla \cdot \langle \mathbf{E}\times\mathbf{B}\rangle &= \frac{1}{\mu_0}\nabla \cdot \frac{1}{2}\mathcal{R}\left[\mathbf{E}^*\times\mathbf{B}\right]e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})} \\ &= -2\mathbf{k}_I\cdot\frac{1}{2\mu_0}\mathcal{R}\left[\mathbf{E}^*\times\mathbf{B}\right]e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})} \qquad (1.3.33)\end{aligned}$$

Using the assumption $e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})}\approx 1$ and rewriting we get

$$\frac{1}{\mu_0}\nabla \cdot \langle \mathbf{E}\times\mathbf{B}\rangle = -2\mathbf{k}_I\cdot\frac{1}{2\mu_0}\mathcal{R}\left[\mathbf{E}^*\times\mathbf{B}\right] \qquad (1.3.34)$$

Combining (1.3.30), (1.3.32) and (1.3.34) we get

$$2\omega_I\left(\frac{\epsilon_0}{4}\mathbf{E}^*\cdot\frac{\partial\left(\omega\underline{\underline{\epsilon}}\right)}{\partial\omega}\cdot\mathbf{E}+\frac{\|\mathbf{B}_k\|^2}{4\mu_0}\right)-2\mathbf{k}_I\cdot\left(-\frac{\omega\epsilon_0}{4}\mathbf{E}^*\cdot\frac{\partial\underline{\underline{\epsilon}}}{\partial\mathbf{k}}\cdot\mathbf{E}+\frac{1}{2\mu_0}\mathcal{R}\left[\mathbf{E}^*\times\mathbf{B}\right]\right)=0 \qquad (1.3.35)$$

Comparing with (1.3.17) we see immediately

$$w_k = \frac{\epsilon_0}{4}\mathbf{E}^*\cdot\frac{\partial\left(\omega\underline{\underline{\epsilon}}\right)}{\partial\omega}\cdot\mathbf{E}+\frac{\|\mathbf{B}_k\|^2}{4\mu_0} \qquad (1.3.36)$$

$$\mathbf{s}_k = -\frac{\omega\epsilon_0}{4}\mathbf{E}^*\cdot\frac{\partial\underline{\underline{\epsilon}}}{\partial\mathbf{k}}\cdot\mathbf{E}+\frac{1}{2\mu_0}\mathcal{R}\left[\mathbf{E}^*\times\mathbf{B}\right] \qquad (1.3.37)$$

1.3.3 Normalised Energy Density and Energy Flow

We define the normalised time averaged energy density Σ_k and the normalised time averaged energy flow Φ_k for mode $\mathbf{k}$ as

$$w_k = \frac{\epsilon_0}{2}\|\mathbf{E}_k\|^2\Sigma_k \qquad (1.3.38)$$

$$\mathbf{s}_k = \frac{\epsilon_0 c}{2}\|\mathbf{E}_k\|^2\Phi_k \qquad (1.3.39)$$

Define the Polarisation vector $\mathbf{e}_k$

$$\mathbf{e}_k = \frac{\mathbf{E}_k}{\|\mathbf{E}_k\|} \qquad (1.3.40)$$

Starting with Σ_k, using (1.3.36) and (1.3.38) we can write

$$\Sigma_k = \frac{1}{2}\mathbf{e}_k \cdot \frac{\partial\left(\omega\underline{\underline{\epsilon}}\right)}{\partial\omega} \cdot \mathbf{e}_k + \frac{c^2}{2}\frac{\|B_k\|^2}{\|\mathbf{E}_k\|^2} \tag{1.3.41}$$

Taking the norm of (1.1.7) we get

$$\|i\mathbf{k}\times\mathbf{E}\| = i\omega\|\mathbf{B}_k\|$$

$$\implies \frac{\|\mathbf{B}_k\|}{\|\mathbf{E}_k\|} = \frac{\|\frac{\mathbf{k}}{\omega}\times\mathbf{E}_k\|}{\|E_k\|} = \frac{1}{c}\|\mathbf{N}\times\mathbf{e}_k\| \tag{1.3.42}$$

Substituting in we get

$$\Sigma_k = \frac{1}{2}\mathbf{e}_k \cdot \frac{\partial\left(\omega\underline{\underline{\epsilon}}\right)}{\partial\omega} \cdot \mathbf{e}_k + \frac{1}{2}\|\mathbf{N}\times\mathbf{e}_k\|^2 \tag{1.3.43}$$

Using the vector identity $\|\mathbf{N}\times\mathbf{e}_k\|^2 = \mathbf{e}_k^* \cdot \left(N^2\underline{\underline{I}} - \mathbf{N}\otimes\mathbf{N}\right)\cdot\mathbf{e}_k$ (see 4.2.1) we get

$$\Sigma_k = \frac{1}{2}\mathbf{e}_k^* \cdot \left(\frac{\partial\left(\omega\underline{\underline{\epsilon}}\right)}{\partial\omega} + N^2\underline{\underline{I}} - \mathbf{N}\otimes\mathbf{N}\right)\cdot\mathbf{e}_k \tag{1.3.44}$$

We want to move the last 2 terms inside the ω derivative. We can do this by seeing

$$\mathbf{N} = \frac{c\mathbf{k}}{\omega} \implies \frac{\partial}{\partial\omega}\left(\omega^2\mathbf{N}^2\right) = 0$$

$$\implies \frac{\partial}{\partial\omega}\left[\omega^2\left(\mathbf{N}\otimes\mathbf{N} - N^2\underline{\underline{I}}\right)\right] = 0 \tag{1.3.45}$$

Splitting this term into $\omega\times\omega\left(\mathbf{N}\otimes\mathbf{N} - N^2\underline{\underline{I}}\right)$ and using the Product rule we get

$$\omega\left(\mathbf{N}\otimes\mathbf{N} - N^2\underline{\underline{I}}\right) + \omega\frac{\partial}{\partial\omega}\left[\omega\left(\mathbf{N}\otimes\mathbf{N} - N^2\underline{\underline{I}}\right)\right] = 0$$

$$\implies \left(N^2\underline{\underline{I}} - \mathbf{N}\otimes\mathbf{N}\right) = \frac{\partial}{\partial\omega}\left[\omega\left(\mathbf{N}\otimes\mathbf{N} - N^2\underline{\underline{I}}\right)\right] \tag{1.3.46}$$

Substituting in we get, using the definition of $\mathcal{D}$ (1.2.2)

$$\Sigma_k = \frac{1}{2}\mathbf{e}_k^* \cdot \frac{\partial}{\partial\omega}\left[\omega\left(\underline{\underline{\epsilon}} + \mathbf{N}\otimes\mathbf{N} - N^2\underline{\underline{I}}\right)\right]\cdot\mathbf{e}_k$$

$$= \frac{1}{2}\mathbf{e}_k^* \cdot \frac{\partial}{\partial \omega}\left(\omega \mathcal{D}\right) \cdot \mathbf{e}_k \tag{1.3.47}$$

Expanding the derivative we get

$$\Sigma_k = \frac{1}{2}\mathbf{e}_k^* \cdot \mathcal{D} \cdot \mathbf{e}_k + \frac{\omega}{2}\mathbf{e}_k^* \cdot \frac{\partial \mathcal{D}}{\partial \omega} \cdot \mathbf{e}_k \tag{1.3.48}$$

Using $\mathbf{e}_k^* \cdot \mathcal{D} \cdot \mathbf{e}_k = 0$ in the limit of weak dissipation we get the final expression

$$\Sigma_k = \frac{\omega}{2}\mathbf{e}_k^* \cdot \frac{\partial \mathcal{D}}{\partial \omega} \cdot \mathbf{e}_k \tag{1.3.49}$$

Now we move on to Φ_k. Using (1.3.37) and (**??**) we can write

$$\begin{aligned} \Phi_k &= -\frac{\omega}{2c}\mathbf{e}_k^* \cdot \frac{\partial \underline{\underline{\epsilon}}}{\partial \mathbf{k}} \cdot \mathbf{e}_k + \frac{1}{\|\mathbf{E}_k\|^2}\mathcal{R}\left[\mathbf{E}_k^* \times \mathbf{B}_k\right] \\ &= -\frac{1}{2}\mathbf{e}_k^* \cdot \frac{\partial \underline{\underline{\epsilon}}}{\partial \mathbf{N}} \cdot \mathbf{e}_k + \frac{1}{\|\mathbf{E}_k\|^2}\mathcal{R}\left[\mathbf{E}_k^* \times \mathbf{B}_k\right] \end{aligned} \tag{1.3.50}$$

Taking $\mathbf{E}_k^* \times$ (1.1.7) we get

$$\begin{aligned} \mathbf{E}_k^* \times \left(i\mathbf{k} \times \mathbf{E}_k\right) &= i\omega \mathbf{E}_k^* \times \mathbf{B}_k \\ \implies \|\mathbf{E}_k^* \times \left(\frac{\mathbf{k}}{\omega} \times \mathbf{E}_k\right)\| &= \|\mathbf{E}_k^* \times \mathbf{B}_k\| \end{aligned} \tag{1.3.51}$$

Dividing through by $\|\mathbf{E}_k\|^2$ and using the definition of $\mathbf{N}$ we get

$$\frac{1}{\|\mathbf{E}_k\|^2}\mathcal{R}\left[\mathbf{E}_k^* \times \mathbf{B}_k\right] = \frac{1}{c}\mathcal{R}\left[\mathbf{e}_k^* \times \left(\mathbf{N} \times \mathbf{e}_k\right)\right] \tag{1.3.52}$$

We can expand the double cross product as

$$\mathbf{e}_k^* \times \left(\mathbf{N} \times \mathbf{e}_k\right) = \left(\mathbf{e}_k^* \cdot \mathbf{e}_k\right)\mathbf{N} - \left(\mathbf{e}_k^* \cdot \mathbf{N}\right)\mathbf{e}_k \tag{1.3.53}$$

Using the vector identity $\mathbf{N} - \mathcal{R}\left[\left(\mathbf{e}_k^* \cdot \mathbf{N}\right)\mathbf{e}_k\right] = -\frac{1}{2}\mathbf{e}_k^* \cdot \frac{\partial\left(\mathbf{N}\otimes\mathbf{N} - N^2\underline{\underline{I}}\right)}{\partial \mathbf{N}} \cdot \mathbf{e}_k$ (see 4.2.2) we can rewrite the expression for Φ_k

$$\begin{aligned} \Phi_k &= -\frac{1}{2}\mathbf{e}_k^* \cdot \frac{\partial \underline{\underline{\epsilon}}}{\partial \mathbf{N}} \cdot \mathbf{e}_k - \frac{1}{2}\mathbf{e}_k^* \cdot \frac{\partial\left(\mathbf{N}\otimes\mathbf{N} - N^2\underline{\underline{I}}\right)}{\partial \mathbf{N}} \cdot \mathbf{e}_k \\ &= -\frac{1}{2}\mathbf{e}_k^* \cdot \left(\underline{\underline{\epsilon}} + \mathbf{N}\otimes\mathbf{N} - N^2\underline{\underline{I}}\right) \cdot \mathbf{e}_k \end{aligned} \tag{1.3.54}$$

Again using the definition for $\mathcal{D}$ (1.2.2) we get

$$\Phi_k = -\frac{1}{2}\mathbf{e}_k^* \cdot \frac{\partial \mathcal{D}}{\partial \mathbf{N}} \cdot \mathbf{e}_k \tag{1.3.55}$$

1.3.4 Group Velocity

We can get a short definition of the group velocity in terms of our normalised energy flows. Using the standard definition of group velocity

$$\mathbf{v}_g = \frac{\partial \mathbf{k}}{\partial \omega} = \frac{\frac{\partial \mathcal{D}}{\partial \omega}}{\frac{\partial \mathcal{D}}{\partial \mathbf{k}}} = \frac{c\Phi_k}{\Sigma_k} \tag{1.3.56}$$

1.4 Absorption Coefficient

Poynting's Theorem covers the energy density and flow in both the wave and the charged particles. However, energy flow from the wave fields to the particles is the definition of wave absorption. This term appeared from the $\mathbf{E} \cdot \mathbf{j}$ term. If we were to rearrange Poynting's Theorem to isolate this term we get

$$\epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \frac{1}{\mu_0}\nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j} \tag{1.4.1}$$

Writing the time average $\langle \mathbf{E} \cdot \mathbf{j} \rangle$, we can use Ohm's Law (1.2.4) and the standard method for time averaging

$$\begin{aligned} \left\langle \mathbf{E} \cdot \underline{\underline{\sigma}} \cdot \mathbf{E} \right\rangle &= \frac{1}{4}\left[\mathbf{E}^* \cdot \underline{\underline{\sigma}} \cdot \mathbf{E} + \mathbf{E} \cdot \underline{\underline{\sigma}}^* \cdot \mathbf{E}^*\right] e^{2(\omega_I t - \mathbf{k}_I \cdot \mathbf{x})} \\ &= \frac{1}{4}\left[\mathbf{E}^* \cdot \left(\underline{\underline{\sigma}}^\dagger + \underline{\underline{\sigma}}\right) \cdot \mathbf{E}\right] e^{2(\omega_I t - \mathbf{k}_I \cdot \mathbf{x})} \end{aligned} \tag{1.4.2}$$

Assuming $e^{2(\omega_I t - \mathbf{k}_I \cdot \mathbf{x})} \approx 1$ and recognising $\underline{\underline{\sigma}}^\dagger + \underline{\underline{\sigma}} = 2\underline{\underline{\sigma}}^H$ we get

$$\left\langle \mathbf{E} \cdot \underline{\underline{\sigma}} \cdot \mathbf{E} \right\rangle = \frac{1}{2}\mathbf{E}^* \cdot \underline{\underline{\sigma}}^H \cdot \mathbf{E} \tag{1.4.3}$$

We can associate this with the power dissipation. Taking the Fourier transform we get the power dissipation P_k^{lin} associated with mode k

$$P_k^{lin} = \frac{1}{2}\mathbf{E}_k^* \cdot \underline{\underline{\sigma}}^H \cdot \mathbf{E}_k \tag{1.4.4}$$

Consider the anti-hermitian part of the susceptibility tensor $\underline{\underline{\chi}}$

$$\underline{\underline{\chi}}^A = \frac{1}{2}\left(\underline{\underline{\chi}} - \underline{\underline{\chi}}^\dagger\right) = \frac{i}{\omega\epsilon_0}\frac{1}{2}\left(\underline{\underline{\sigma}} + \underline{\underline{\sigma}}^\dagger\right) = \frac{i}{\omega\epsilon_0}\underline{\underline{\sigma}}^H$$

$$\implies \underline{\underline{\sigma}}^H = -i\omega\epsilon_0\underline{\underline{\chi}}^A \tag{1.4.5}$$

Substituting in, we can rewrite P_k^{lin}

$$P_k^{lin} = -\frac{i\omega\epsilon_0}{2}\|\mathbf{E}_k\|^2\mathbf{e}_k^* \cdot \underline{\underline{\chi}}^A \cdot \mathbf{e}_k \tag{1.4.6}$$

We almost have the normalisation constant between $\mathbf{s}_k$ and Φ_k from (1.3.39). Substituting in we get

$$P_k^{lin} = -\frac{i\omega}{c}\frac{\|\mathbf{s}_k\|}{\|\Phi_k\|}\mathbf{e}_k^* \cdot \underline{\underline{\chi}}^A \cdot \mathbf{e}_k \tag{1.4.7}$$

Defining the absorption coefficient α_k^{lin}

$$\alpha_k^{lin} = \frac{P_k^{lin}}{\|\mathbf{s}_k\|} = -\frac{i\omega}{c}\frac{1}{\|\Phi_k\|}\mathbf{e}_k^* \cdot \underline{\underline{\chi}}^A \cdot \mathbf{e}_k \tag{1.4.8}$$

2 Hot Plasma Dielectric Tensor

This follows Richard Fitzpatrick's explanation [2] except I've filled in some of the gaps. The aim is to derive a non-relativistic form of the dielectric tensor accounting for finite temperature and hence finite Larmor radius.

Start by considering small amplitude waves propagating through a plasma in a uniform magnetic field $\mathbf{B} = B\hat{z}$. Consider the collisionless Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{2.0.1}$$

We linearise by defining $\mathbf{E} = \mathbf{E}_1$, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ and $f\left(\mathbf{x}, \mathbf{v}, t\right) = f_0\left(\mathbf{v}\right) + f_1\left(\mathbf{x}, \mathbf{v}, t\right)$. As f_0 is a solution of (2.0.1) but the time and space derivatives vanish we get

$$\left(\mathbf{v} \times \mathbf{B}_0 \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0 \tag{2.0.2}$$

The cross product expands as $v_\perp B_0 \sin\theta \hat{\mathbf{x}} - v_\perp B_0 \sin\theta \hat{\mathbf{y}}$. We want to transform to polar co-ordinates (r, θ, z), so use $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$ and $\hat{\theta} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}$. Under this transformation the cross product becomes $v_\perp B_0 \hat{\theta}$ so (2.0.2) implies

$$\frac{\partial f_0}{\partial \theta} = 0 \tag{2.0.3}$$

i.e. $f_0 = f_0\left(v_\perp, v_\parallel\right)$ only.

Linearising (2.0.1) and re-arranging we get

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \nabla f_1 + \frac{e}{m} \left(\mathbf{v} \times \mathbf{B}_0 \right) \cdot \frac{\partial f_1}{\partial \mathbf{v}} = -\frac{e}{m} \left(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1 \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} \tag{2.0.4}$$

We can recognise the LHS as the total rate of change of f_1 following the unperturbed particle trajectories. Therefore

$$\frac{Df_1}{Dt} = -\frac{e}{m} \left(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1 \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} \tag{2.0.5}$$

Assuming f_1 vanishes at $t = -\infty$ we can write (dropping the 1 subscript as all vacuum fields are now on the LHS)

$$f_1\left(\mathbf{r}, \mathbf{v}, t\right) = -\frac{e}{m} \int_{-\infty}^{t} \left[\mathbf{E}\left(\mathbf{r}', t'\right) + \mathbf{v} \times \mathbf{B}\left(\mathbf{r}', t'\right) \right] \cdot \frac{\partial f_0\left(\mathbf{v}'\right)}{\partial \mathbf{v}} dt' \tag{2.0.6}$$

where $(\mathbf{r}', \mathbf{v}')$ is the unperturbed trajectory that passes through $(\mathbf{r}, \mathbf{v})$ when $t' = t$. We know the dielectric tensor $\underline{\underline{\epsilon}}$ is defined as

$$\underline{\underline{\epsilon}} \cdot \mathbf{E} = \mathbf{E} + \frac{i}{\omega\epsilon_0}\mathbf{j} = \mathbf{E} + \frac{i}{\omega\epsilon_0}\sum_s e_s \int \mathbf{v} f_{1s} d^3\mathbf{v} \tag{2.0.7}$$

relating the current $\mathbf{j}$ to the moments of f_1. Here s refers to a sum over species we will drop for now to keep things simpler. So we need to do some work to evaluate f_1, and then calculate the time and velocity integrals to get the dielectric tensor elements.

The Cartesian components of the velocity are

$$\mathbf{v} = \left(v_\perp \cos\theta, v_\perp \sin\theta, v_\parallel\right) \tag{2.0.8}$$

which implies (Ω_s is the gyrofrequency for species s)

$$\mathbf{v} = \left(v_\perp \cos\left[\Omega_s\left(t - t'\right) + \theta\right], v_\perp \sin\left[\Omega_s\left(t - t'\right) + \theta\right], v_\parallel\right) \tag{2.0.9}$$

This expression is a bit long so define $\chi = \Omega_s\left(t - t'\right) + \theta$ so

$$\mathbf{v} = \left(v_\perp \cos\chi, v_\perp \sin\chi, v_\parallel\right) \tag{2.0.10}$$

We can integrate in time to get the electron position

$$x' - x = -\frac{v_\perp}{\Omega}\left(\sin\chi - \sin\theta\right) \tag{2.0.11}$$

$$y' - y = \frac{v_\perp}{\Omega}\left(\cos\chi - \cos\theta\right) \tag{2.0.12}$$

$$z' - z = v_\parallel\left(t' - t\right) \tag{2.0.13}$$

Both $v_\perp$ and $v_\parallel$ are constants of the motion, implying $f_0\left(\mathbf{v}'\right) = f_0\left(\mathbf{v}\right)$. Using $v_\perp = \sqrt{v_x^2 + v_y^2}$ we get

$$\frac{\partial f_0}{\partial v_x'} = \frac{\partial v_\perp}{\partial v_x'}\frac{\partial f_0}{\partial v_\perp} = \frac{v_x'}{\partial v_\perp}\frac{\partial f_0}{\partial v_\perp} = \cos\chi\frac{\partial f_0}{\partial v_\perp} \tag{2.0.14}$$

$$\frac{\partial f_0}{\partial v_y'} = \frac{\partial v_\perp}{\partial v_y'}\frac{\partial f_0}{\partial v_\perp} = \frac{v_y'}{\partial v_\perp}\frac{\partial f_0}{\partial v_\perp} = \sin\chi\frac{\partial f_0}{\partial v_\perp} \tag{2.0.15}$$

$$\frac{\partial f_0}{\partial v_z'} = \frac{\partial f_0}{\partial v_\parallel} \tag{2.0.16}$$

If we expand the integrand in (2.0.4) we get

$$[\mathbf{E}(\mathbf{r}',t') + \mathbf{v}\times\mathbf{B}(\mathbf{r}',t')]\cdot\frac{\partial f_0(\mathbf{v}')}{\partial\mathbf{v}} = \left(E_x + v_y'B_z - v_z'B_y\right)\frac{\partial f_0}{\partial v_x'}$$

$$+\left(E_xy + v_z'B_x - v_x'B_z\right)\frac{\partial f_0}{\partial v_y'} + \left(E_z + v_x'B_y - v_y'B_x\right)\frac{\partial f_0}{\partial v_y'} \tag{2.0.17}$$

Using (2.0.14), (2.0.15) and (2.0.16)

$$= \left(E_x - v_\parallel B_y\right)\cos\chi\frac{\partial f_0}{\partial v_\perp} + \left(E_y + v_\parallel B_x\right)\sin\chi\frac{\partial f_0}{\partial v_\perp} +$$

$$+\left(v_y'\cos\chi - v_x'\sin\chi\right)B_z\frac{\partial f_0}{\partial v_\perp} + \left(E_z + v_\perp\cos\chi B_y - v_\perp\sin\chi B_x\right)\frac{\partial f_0}{\partial v_\parallel} \tag{2.0.18}$$

Using $v_y'\cos\chi - v_x'\sin\chi = v_\perp\cos\chi\sin\chi - v_\perp\cos\chi\sin\chi = 0$ the third term vanishes. Now assuming all linearised terms have a $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ dependence the integrand becomes

$$= \left[\left(E_x - v_\parallel B_y\right)\cos\chi\frac{\partial f_0}{\partial v_\perp} + \left(E_y + v_\parallel B_x\right)\sin\chi\frac{\partial f_0}{\partial v_\perp} +\right.$$

$$\left.+\left(E_z + v_\perp\cos\chi B_y - v_\perp\sin\chi B_x\right)\frac{\partial f_0}{\partial v_\parallel}\right] e^{i(\mathbf{k}\cdot\mathbf{x}'-\omega t')}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{2.0.19}$$

where the first complex exponential is from the perturbed electric and magnetic field and the second is from the perturbed f_1. We can combine these to get a term like

$$\exp\left[i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})\right]\exp\left[-i\omega\left(t'-t\right)\right] \tag{2.0.20}$$

Taking $\mathbf{k}$ to lie in the $x-z$ plane we can use (2.0.11) and (2.0.13) to get

$$\exp\left[i\frac{k_\perp v_\perp}{\Omega}\left(\sin\chi - \sin\theta\right) + ik_\parallel v_\parallel\left(t'-t\right)\right]\exp\left[-i\omega\left(t'-t\right)\right]$$

$$= \exp\left[i\mu\sin\chi\right]\exp\left[-i\mu\sin\theta\right]\exp\left[i\left(k_\parallel v_\parallel - \omega\right)\left(t'-t\right)\right] \tag{2.0.21}$$

where we define $\mu = \frac{k_\perp v_\perp}{\Omega}$. We can use the complex exponential dependence of $\mathbf{E}$ and $\mathbf{B}$ to eliminate $\mathbf{B}$ using the (1.1.7)

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B} \implies \mathbf{B} = \frac{\mathbf{k}\times\mathbf{E}}{\omega} \tag{2.0.22}$$

Again using the fact $\mathbf{k}$ lies in the $x-z$ plane to get

$$B_x = -\frac{k_\parallel E_y}{\omega} \tag{2.0.23}$$

$$B_y = \frac{k_\parallel E_x - k_\perp E_z}{\omega} \tag{2.0.24}$$

Using these expressions we can write

$$E_x - v_\parallel B_y = \frac{\omega - v_\parallel k_\parallel}{\omega} E_x + \frac{k_\perp v_\parallel}{\omega} E_z \tag{2.0.25}$$

$$E_y + v_\parallel B_x = \frac{\omega - k_\parallel v_\parallel}{\omega} E_y \tag{2.0.26}$$

Substituting into (2.0.19) we get

$$\begin{aligned}
&\frac{1}{\omega}\Bigg(\left[\left(\omega - k_\parallel v_\parallel\right) E_x + k_\perp v_\parallel E_z\right] \cos\chi \frac{\partial f_0}{\partial v_\perp} + \left(\omega - k_\parallel v_\parallel\right) \sin\chi \frac{\partial f_0}{\partial v_\perp} E_y \\
&\quad + \left[\omega E_z + k_\parallel v_\perp \cos\chi E_c - k_\perp v_\perp \cos\chi E_z + k_\parallel v_\perp \sin\chi E_y\right] \frac{\partial f_0}{\partial v_\parallel}\Bigg) \times \\
&\quad \exp\left[i\mu\sin\chi\right] \exp\left[-i\mu\sin\theta\right] \exp\left[i\left(k_\parallel v_\parallel - \omega\right)(t' - t)\right]
\end{aligned} \tag{2.0.27}$$

In a little bit we will use moments of this to find the elements of the dielectric tensor, i.e. the coefficients of E_x, E_y and E_z so it is convenient to group these coefficients now. Doing this we obtain

$$\begin{aligned}
&\frac{1}{\omega}\Bigg(\left[\left(\omega - k_\parallel v_\parallel\right)\frac{\partial f_0}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_0}{\partial v_\parallel}\right] \cos\chi E_x + \left[\left(\omega - k_\parallel v_\parallel\right)\frac{\partial f_0}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_0}{\partial v_\parallel}\right] \sin\chi E_y \\
&\quad + \left[\omega \frac{\partial f_0}{\partial v_\parallel} + \left(k_\perp v_\parallel \frac{\partial f_0}{\partial v_\perp} - k_\perp v_\perp \frac{\partial f_0}{\partial v_\parallel}\right)\cos\chi\right] E_z\Bigg) \times \\
&\quad \exp\left[i\mu\sin\chi\right] \exp\left[-i\mu\sin\theta\right] \exp\left[i\left(k_\parallel v_\parallel - \omega\right)(t' - t)\right]
\end{aligned} \tag{2.0.28}$$

This expression is quite cumbersome so we will introduce some new variables P and Q

$$P = \left(\omega - k_\parallel v_\parallel\right)\frac{\partial f_0}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_0}{\partial v_\parallel} \tag{2.0.29}$$

$$Q = k_\perp v_\parallel \frac{\partial f_0}{\partial v_\perp} - k_\perp v_\perp \frac{\partial f_0}{\partial v_\parallel} \tag{2.0.30}$$

Making these substitutions we can write our expression for f_1 with our rearranged integrand

$$f_1 = \frac{-e}{m\omega}\int_{-\infty}^{t}\left(P\cos\chi E_x + P\sin\chi E_y + \left[\omega\frac{\partial f_0}{\partial v_\parallel} + Q\cos\chi\right]E_z\right)\times$$

$$\exp\left[i\mu\sin\chi\right]\exp\left[-i\mu\sin\theta\right]\exp\left[i\left(k_\parallel v_\parallel - \omega\right)\left(t'-t\right)\right]dt' \tag{2.0.31}$$

To perform this time integration we can expand these terms using the Jacobi-Anger expansion

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} J_n\left(z\right)e^{in\theta} \tag{2.0.32}$$

where J_n is the Bessel function of the first kind. We can do some manipulation to get 2 more useful identities

$$\cos\theta e^{iz\sin\theta} = \frac{1}{iz}\frac{\partial}{\partial\theta}e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty}\frac{nJ_n\left(z\right)}{z}e^{in\theta} \tag{2.0.33}$$

$$\sin\theta e^{iz\sin\theta} = \frac{1}{i}\frac{\partial}{\partial z}e^{iz\sin\theta} = -i\sum_{n=-\infty}^{\infty}J_n'\left(z\right)e^{in\theta} \tag{2.0.34}$$

Using these identities we can replace $\cos\chi e^{i\mu\sin\chi}$, $\sin\chi e^{i\mu\sin\chi}$, and $e^{i\mu\sin\chi}$ terms

$$f_1 = \frac{-e}{m\omega}\int_{-\infty}^{t}\sum_{n=-\infty}^{\infty}\left(\frac{nJ_n\left(\mu\right)}{\mu}PE_x - iJ_n'\left(\mu\right)PE_y + \left[\omega\frac{\partial f_0}{\partial v_\parallel}J_n\left(\mu\right) + \frac{nJ_n\left(\mu\right)}{\mu}Q\right]E_z\right)\times$$

$$\exp\left[-i\mu\sin\theta\right]\exp\left[in\theta\right]\exp\left[i\left(n\Omega_s + k_\parallel v_\parallel - \omega\right)\left(t'-t\right)\right]dt' \tag{2.0.35}$$

where we have split $\exp\left[in\chi\right] = \exp\left[in\theta\right]\exp\left[in\Omega_s\left(t'-t\right)\right]$ and combined some terms. From now on we'll drop the argument of the Bessel functions for brevity. We can rewrite the coefficient for E_z as

$$\begin{aligned}\omega\frac{\partial f_0}{\partial v_\parallel}J_n + \frac{nJ_n}{\mu}\left(k_\perp v_\parallel\frac{\partial f_0}{\partial v_\perp} - k_\perp v_\perp\frac{\partial f_0}{\partial v_\parallel}\right)\\ = \left[\frac{n\Omega_s v_\parallel}{v_\perp}\frac{\partial f_0}{\partial v_\perp} + \left(\omega - n\Omega_s\right)\frac{\partial f_0}{\partial v_\parallel}\right]J_n\end{aligned} \tag{2.0.36}$$

We can define a new simpler variable R to replace Q

$$R = \frac{n\Omega v_\parallel}{v_\perp}\frac{\partial f_0}{\partial v_\perp} + (\omega - n\Omega_s)\frac{\partial f_0}{\partial v_\parallel} \tag{2.0.37}$$

allowing us to write

$$f_1 = \frac{-e}{m\omega}\int_{-\infty}^{t}\sum_{n=-\infty}^{\infty}\left(\frac{nJ_n}{\mu}PE_x - iJ_n'PE_y + J_nRE_z\right)\times$$
$$\exp\left[-i\mu\sin\theta\right]\exp\left[in\theta\right]\exp\left[i\left(n\Omega_s + k_\parallel v_\parallel - \omega\right)(t'-t)\right]dt' \tag{2.0.38}$$

We now the only time dependence in our integrand comes from the final complex exponential term, so we can write

$$f_1 = \frac{-e}{m\omega}\sum_{n=-\infty}^{\infty}\left(\frac{nJ_n}{\mu}PE_x - iJ_n'PE_y + J_nRE_z\right)\times$$
$$\exp\left[-i\mu\sin\theta\right]\int_{-\infty}^{t}\exp\left[in\theta\right]\exp\left[i\left(n\Omega_s + k_\parallel v_\parallel - \omega\right)(t'-t)\right]dt' \tag{2.0.39}$$

This is an easy integral

$$\int_{-\infty}^{t}\exp\left[i\left(n\Omega + k_\parallel v_\parallel - \omega\right)(t'-t)\right]dt' = \frac{i}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.40}$$

where we make some arguments using delta functions about how the complex exponential evaluates to 0 at $-\infty$. So we get our simplest form for f_1

$$f_1 = \frac{-e}{m\omega}\sum_{n=-\infty}^{\infty}\left(\frac{nJ_n}{\mu}PE_x - iJ_n'PE_y + J_nRE_z\right)\times$$
$$\exp\left[-i\mu\sin\theta\right]\exp\left[in\theta\right]\frac{i}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.41}$$

Now we need to take the first velocity moment f_1 to get our currents

$$\mathbf{j}_i = e\int \mathbf{v}_i f_1 d^3\mathbf{v} = e\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\mathbf{v}_i f_1 2\pi v_\perp dv_\perp d\theta dv_\parallel \tag{2.0.42}$$

where we have expressed the velocity in cylindrical co-ordinates, taking $v_\perp \in [0,\infty]$ and $\theta \in [0,2\pi]$ for the components of the velocity perpendicular to the magnetic field and $v_\parallel \in [-\infty, \infty]$ for the component of velocity parallel to the magnetic field. Note these integral limits are important! We already saw $v_x = v_\perp \cos\theta$, $v_y = v_\parallel \sin\theta$ and $v_z = v_\parallel$ so we get these expressions for the currents

$$j_x = \frac{-e^2 v_\perp}{m\omega} \sum_{n=-\infty}^{\infty} \left(\frac{nJ_n}{\mu} PE_x - iJ_n' PE_y + J_n RE_z \right) \times$$

$$\cos\theta \exp\left[-i\mu \sin\theta\right] \exp\left[in\theta\right] \frac{i}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.43}$$

$$j_y = \frac{-e^2 v_\perp}{m\omega} \sum_{n=-\infty}^{\infty} \left(\frac{nJ_n}{\mu} PE_x - iJ_n' PE_y + J_n RE_z \right) \times$$

$$\sin\theta \exp\left[-i\mu \sin\theta\right] \exp\left[in\theta\right] \frac{i}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.44}$$

$$j_z = \frac{-e^2 v_\parallel}{m\omega} \sum_{n=-\infty}^{\infty} \left(\frac{nJ_n}{\mu} PE_x - iJ_n' PE_y + J_n RE_z \right) \times$$

$$\exp\left[-i\mu \sin\theta\right] \exp\left[in\theta\right] \frac{i}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.45}$$

which are all very similar apart from factors of $v_\perp$ or $v_\parallel$ and more expressions $\propto \exp\left[-i\mu \sin\theta\right]$ we can replace using (2.0.32), (2.0.33) and (2.0.34) where we use the expression $e^{-i\mu \sin\theta} = e^{i\mu \sin(-\theta)}$. Doing these replacements we get

$$j_x = \frac{-e^2 v_\perp}{m\omega} \sum_{n=-\infty}^{\infty} \left(\frac{nJ_n}{\mu} PE_x - iJ_n' PE_y + J_n RE_z \right) \times$$

$$\sum_{m=-\infty}^{\infty} \frac{mJ_m}{\mu} \exp\left[i\left(n-m\right)\theta\right] \frac{i}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.46}$$

$$j_y = \frac{-e^2 v_\perp}{m\omega} \sum_{n=-\infty}^{\infty} \left(\frac{nJ_n}{\mu} PE_x - iJ_n' PE_y + J_n RE_z \right) \times$$

$$\sum_{m=-\infty}^{\infty} -iJ_m' \exp\left[i\left(n-m\right)\theta\right] \frac{i}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.47}$$

$$j_z = \frac{-e^2 v_\parallel}{m\omega} \sum_{n=-\infty}^{\infty} \left(\frac{nJ_n}{\mu} PE_x - iJ_n' PE_y + J_n RE_z \right) \times$$
$$\sum_{m=-\infty}^{\infty} J_m \exp\left[i\left(n-m\right)\theta \right] \frac{i}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.48}$$

We can replace $\exp\left[i\left(n-m\right)\theta\right] = 2\pi\delta\left(n-m\right)$ where δ is the Dirac delta. To get rid of these terms we'll just integrate over gyrophase θ now which just sets $n = m$ and we pick up a factor of 2π. This collapses the double sum to a single sum, allowing us to write an expression which almost fits on a single line!

$$\int_0^{2\pi} j_x d\theta = \frac{-2\pi i e^2 v_\perp}{m\omega} \sum_{n=-\infty}^{\infty} \frac{\left(\frac{nJ_n}{\mu}\right)^2 PE_x - i\frac{nJ_nJ_n'}{\mu} PE_y + \frac{nJ_n^2}{\mu} RE_z}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.49}$$

$$\int_0^{2\pi} j_y d\theta = \frac{2\pi e^2 v_\perp}{m\omega} \sum_{n=-\infty}^{\infty} \frac{\frac{nJ_nJ_n'}{\mu} PE_x - i\left(J_n'\right)^2 PE_y + J_nJ_n' RE_z}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.50}$$

$$\int_0^{2\pi} j_z d\theta = \frac{-2i\pi e^2 v_\parallel}{m\omega} \sum_{n=-\infty}^{\infty} \frac{\frac{nJ_n^2}{\mu} PE_x - iJ_nJ_n' PE_y + J_n^2 RE_z}{\omega - n\Omega_s - k_\parallel v_\parallel} \tag{2.0.51}$$

We can now use the definition of the dielectric tensor (2.0.7) to write

$$\epsilon_{ij} = \delta_{ij} + \sum_s X_s \sum_{n=-\infty}^{\infty} \frac{1}{n_s} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{S_{ij}}{\omega - n\Omega_s - k_\parallel v_\parallel} 2\pi v_\perp dv_\perp dv_\parallel \tag{2.0.52}$$

where

$$S_{ij} = \begin{bmatrix} v_\perp \left(\frac{nJ_n}{\mu}\right)^2 P & iv_\perp \frac{nJ_nJ_n'}{\mu} P & v_\perp \frac{nJ_n^2}{\mu} R \\ -iv_\perp \frac{nJ_nJ_n'}{\mu} P & v_\perp \left(J_n'\right)^2 P & -iv_\perp J_nJ_n' R \\ v_\parallel \frac{nJ_n^2}{\mu} P & iv_\parallel J_nJ_n' P & v_\parallel J_n^2 R \end{bmatrix} \tag{2.0.53}$$

$$P = \left(\omega - k_\parallel v_\parallel\right) \frac{\partial f_{0s}}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_0}{\partial v_\parallel}$$
$$R = \frac{n\Omega_s v_\parallel}{v_\perp} \frac{\partial f_{0s}}{\partial v_\perp} + \left(\omega - n\Omega_s\right) \frac{\partial f_0}{\partial v_\parallel}$$

and the arguments of the Bessel functions are $\mu = \frac{k_\perp v_\perp}{\Omega_s}$. We pick up a factor of $2\pi v_\perp$ from the integration area element in polar co-ordinates. We have also used the expression

$$\frac{e_s^2}{\omega^2 \epsilon_0 m_s} = \frac{1}{n_s}\frac{1}{\omega^2}\frac{n_s e_s^2}{\epsilon_0 m_s} = \frac{1}{n_s}\frac{\omega_{ps}^2}{\omega^2} = \frac{X_s}{n_s}$$

to simplify the factor in front of the sum over harmonics n. We make further progress by assuming the equilibrium distribution function f_{0s} is a Maxwellian

$$f_{0s} = \frac{n_s}{\pi^{\frac{3}{2}} v_{Ts}^3} \exp\left[-\frac{v_\perp^2 + v_\parallel^2}{v_{Ts}^2}\right] \tag{2.0.54}$$

where $v_{Ts} = \sqrt{\frac{2kT_s}{2m_s}}$ is the thermal velocity. Hence we can write the derivatives of f_{0s}

$$\frac{\partial f_0}{\partial v_\perp} = \frac{-2v_\perp}{v_{Ts}^2} f_{0s} \tag{2.0.55}$$

$$\frac{\partial f_0}{\partial v_\parallel} = \frac{-2v_\parallel}{v_{Ts}^2} f_{0s} \tag{2.0.56}$$

Substituting into the expression for P

$$P = \left(\omega - k_\parallel v_\parallel\right) \frac{-2v_\perp}{v_{Ts}^2} f_{0s} + k_\parallel v_\perp \frac{-2v_\parallel}{v_{Ts}^2} f_{0s}$$

$$= -\frac{2f_{0s}}{v_{Ts}^2}\left[\left(\omega - k_\parallel v_\parallel\right) v_\perp + k_\parallel v_\parallel v_\perp\right] = -\frac{2\omega v_\perp}{v_{Ts}^2} f_{0s} \tag{2.0.57}$$

Substituting into the expression for R

$$R = \frac{n\Omega_s v_\parallel}{v_\perp}\left(-\frac{2v_\perp f_{0s}}{v_{Ts}^2}\right) + \left(\omega - n\Omega_s\right)\left(-\frac{2v_\parallel f_{0s}}{v_{Ts}^2}\right)$$

$$= -\frac{2f_{0s}}{v_{Ts}^2}\left[n\Omega_s v_\parallel + \left(\omega - n\Omega_s\right) v_\parallel\right] = -\frac{2\omega v_\parallel}{v_{Ts}^2} f_{0s} \tag{2.0.58}$$

Now we can make some progress on the double integral in (2.0.75) which I denote

$$I_{ij} = \frac{1}{n_s}\int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\pi S_{ij}}{\omega - n\Omega_s - k_\parallel v_\parallel} dv_\perp dv_\parallel \tag{2.0.59}$$

We see the only difference now between P and R is the component of v so we can write

$$I_{ij} = -\frac{4\pi\omega}{n_s}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{S_{ij}f_{0s}}{\omega - n\Omega_s - k_\parallel v_\parallel}\frac{v_\perp dv_\perp dv_\parallel}{v_{Ts}^2}$$

$$= \frac{4\pi\omega}{n_s}\frac{n_s}{\pi^{\frac{3}{2}}v_{Ts}^3}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{S_{ij}}{k_\parallel v_\parallel + n\Omega_s - \omega}\exp\left[-\frac{v_\perp^2 + v_\parallel^2}{v_{Ts}^2}\right]\frac{v_\perp dv_\perp dv_\parallel}{v_{Ts}^2} \quad (2.0.60)$$

where

$$S_{ij} = \begin{bmatrix} v_\perp^2\left(\frac{nJ_n}{\mu}\right)^2 & iv_\perp^2\frac{nJ_nJ_n'}{\mu} & v_\perp v_\parallel\frac{nJ_n^2}{\mu} \\ -iv_\perp^2\frac{nJ_nJ_n'}{\mu} & v_\perp^2\left(J_n'\right)^2 & -iv_\perp v_\parallel J_nJ_n' \\ v_\perp v_\parallel\frac{nJ_n^2}{\mu} & iv_\perp v_\parallel J_nJ_n' & v_\parallel^2 J_n^2 \end{bmatrix} \quad (2.0.61)$$

Now we change co-ordinates to normalised velocity $u_\perp$ and $u_\parallel$

$$u_\perp = \frac{v_\perp}{v_{Ts}} \implies du_\perp = \frac{dv_\perp}{v_{Ts}}$$

$$u_\parallel = \frac{v_\parallel}{v_{Ts}} \implies du_\parallel = \frac{dv_\parallel}{v_{Ts}}$$

So now we can write

$$I_{ij} = \frac{4\omega}{\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{S_{ij}}{k_\parallel v_{Ts}u_\parallel + n\Omega_s - \omega}u_\perp e^{-\left(u_\perp^2 + u_\parallel^2\right)}du_\perp du_\parallel \quad (2.0.62)$$

where

$$S_{ij} = \begin{bmatrix} u_\perp^2\left(\frac{nJ_n}{\mu}\right)^2 & iu_\perp^2\frac{nJ_nJ_n'}{\mu} & u_\perp u_\parallel\frac{nJ_n^2}{\mu} \\ -iu_\perp^2\frac{nJ_nJ_n'}{\mu} & u_\perp^2\left(J_n'\right)^2 & -iu_\perp u_\parallel J_nJ_n' \\ u_\perp u_\parallel\frac{nJ_n^2}{\mu} & iu_\perp u_\parallel J_nJ_n' & u_\parallel^2 J_n^2 \end{bmatrix} \quad (2.0.63)$$

We also need to make this substitution to the arguments of the Bessel functions

$$\mu = \frac{k_\perp v_\perp}{\Omega_s} = \frac{k_\perp v_{Ts}}{\Omega}u_\perp = \sqrt{2\lambda_s}u_\perp \quad (2.0.64)$$

where we have defined another convenient parameter

$$\lambda_s := \frac{k_\perp^2 v_{Ts}^2}{2\Omega_s^2} \quad (2.0.65)$$

We have cheated and used future knowledge that having these factors of $\sqrt{2\lambda_s}$ in the Bessel functions will simplify some standard integrals.

Now we have to evaluate 6 terms $I_{xx}, I_{xy}, I_{xz}, I_{yy}, I_{yz}$ and I_{zz} to completely define the dielectric tensor, taking advantage of the symmetries $I_{yx} = -I_{xy}$, $I_{zx} = I_{xz}$ and $I_{zy} = -I_{yz}$. Starting with I_{xx}, using the definition of λ_s so $\frac{u_\perp}{\mu} = \frac{1}{\sqrt{2\lambda_s}}$

$$I_{xx} = \frac{4\omega}{\sqrt{\pi}}\frac{n^2}{2\lambda_s}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{-\left(u_\perp^2+u_\parallel^2\right)}}{k_\parallel v_{Ts}u_\parallel + n\Omega_s - \omega}u_\perp J_n\left(\sqrt{2\lambda_s}u_\perp\right)^2 du_\perp du_\parallel$$

$$= \frac{4n^2}{2\lambda_s}\left[\frac{\omega}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-u_\parallel^2}}{k_\parallel v_{Ts}u_\parallel + n\Omega_s - \omega}du_\parallel\right]\left[\int_0^{\infty} J_n\left(\sqrt{2\lambda_s}u_\perp\right)^2 u_\perp e^{-u_\perp^2} du_\perp\right] \tag{2.0.66}$$

We can simplify the first integral by taking out a factor of $k_\parallel v_{Ts}$ to get

$$\frac{\omega}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-u_\parallel^2}}{k_\parallel v_{Ts}u_\parallel + n\Omega_s - \omega}du_\parallel = \frac{\omega}{k_\parallel v_{Ts}}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-u_\parallel^2}}{u_\parallel - \zeta_n}du_\parallel \tag{2.0.67}$$

where we define another useful parameter

$$\zeta_{ns} := \frac{\omega - n\Omega_s}{k_\parallel v_{Ts}} = \frac{1 - nY_s}{N_\parallel \beta_{Ts}} \tag{2.0.68}$$

where $\beta_{Ts} = \frac{v_{Ts}}{c}$. ζ_{ns} gives a measure of the distance from the cyclotron harmonics accounting for Doppler shift. We also note that $\zeta_{0s} = \frac{\omega}{k_\parallel v_{Ts}}$ which was the factor we extracted from the first integral. Hence we get

$$\frac{1}{k_\parallel v_{Ts}}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-u_\parallel^2}}{u_\parallel - \zeta_n}du_\parallel = \zeta_{0s} Z\left(\zeta_{ns}\right)$$

where we define the Plasma Dispersion Function Z as

$$Z\left(\zeta_{ns}\right) := \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-u_\parallel^2}}{u_\parallel - \zeta_{ns}}du_\parallel \tag{2.0.69}$$

This function has been studied in the famous paper by Fried and Conte [reference?]. Some similar integrals will appear later so it is convenient to state some results derived in the appendix (4.4.6), (4.4.9)

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{u_\parallel e^{-u_\parallel^2}}{u_\parallel - \zeta_{ns}}du_\parallel = Z'\left(\zeta_{ns}\right) = 1 + \zeta_{ns} Z\left(\zeta_{ns}\right)$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{u_\parallel^2 e^{-u_\parallel^2}}{u_\parallel-\zeta_{ns}}du_\parallel=\zeta_{ns}Z'\left(\zeta_{ns}\right)=\zeta_{ns}\left(1+\zeta_{ns}Z\left(\zeta_{ns}\right)\right)$$

Making this substitution, evaluating the second integral using (4.3.8), we get

$$I_{xx}=\frac{n^2\zeta_{0s}}{\lambda_s}Z\left(\zeta_{ns}\right)I_n(\lambda_s)e^{-\lambda_s}$$

Now let's evaluate I_{xy}. Proceeding similarly to before, using (4.3.9) to evaluate the $u_\perp$ integral and the definition of Z for the $u_\parallel$ integral, we get

$$I_{xy}=in\left[\frac{\zeta_{0s}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-u_\parallel^2}}{u_\parallel-\zeta_{ns}}du_\parallel\right]\left[\frac{4}{\sqrt{2\lambda_s}}\int_0^\infty J_n\left(\sqrt{2\lambda_s}u_\perp\right)J_n'\left(\sqrt{2\lambda_s}u_\perp\right)u_\perp^2e^{-u_\perp^2}du_\perp\right]$$

$$=in\zeta_{0s}Z\left(\zeta_{ns}\right)\left[I_n'\left(\lambda_s\right)-I_n\left(\lambda_s\right)\right]e^{-\lambda_s} \tag{2.0.70}$$

We proceed similarly for I_{xz}, using (4.3.8) again to evaluate the $u_\perp$ integral and using (4.4.6) to evaluate the $u_\parallel$ integral

$$I_{xz}=n\sqrt{\frac{2}{\lambda_s}}\left[\frac{\zeta_{0s}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{u_\parallel e^{-u_\parallel^2}}{u_\parallel-\zeta_{ns}}du_\parallel\right]\left[2\int_0^\infty J_n\left(\sqrt{2\lambda_s}u_\perp\right)^2u_\perp e^{-u_\perp^2}du_\perp\right]$$

$$=n\zeta_{0s}\sqrt{\frac{2}{\lambda_s}}\left[1+\zeta_{ns}Z\left(\zeta_{ns}\right)\right]I_n\left(\lambda_s\right)e^{-\lambda_s} \tag{2.0.71}$$

For I_{yy} we use (4.3.10) to evaluate the $u_\perp$ integral and the definition of Z to evaluate the $u_\parallel$ integral

$$I_{yy}=\left[\frac{\zeta_{0s}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{u_\parallel e^{-u_\parallel^2}}{u_\parallel-\zeta_{ns}}du_\parallel\right]\left[4\int_0^\infty J_n'\left(\sqrt{2\lambda_s}u_\perp\right)^2u_\perp^3e^{-u_\perp^2}du_\perp\right]$$

$$=\zeta_{0s}Z\left(\zeta_{ns}\right)\left[\frac{n^2I_n\left(\lambda_s\right)}{\lambda_s}+2\lambda_sI_n\left(\lambda_s\right)-2\lambda_sI_n'\left(\lambda_s\right)\right]e^{-\lambda_s} \tag{2.0.72}$$

For I_{yz} we use (4.3.9) to evaluate the $u_\perp$ integral and (4.4.6) to evaluate the $u_\parallel$ integral

$$I_{yz}=-i\left[\frac{\zeta_{0s}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{u_\parallel e^{-u_\parallel^2}}{u_\parallel-\zeta_{ns}}du_\parallel\right]\left[4\int_0^\infty J_n\left(\sqrt{2\lambda_s}u_\perp\right)J_n'\left(\sqrt{2\lambda_s}u_\perp\right)u_\perp^2e^{-u_\perp^2}du_\perp\right]$$

$$= -i\zeta_{0s}\left[1+\zeta_{ns}Z\left(\zeta_{ns}\right)\right]\sqrt{2\lambda_s}\left[I'_n\left(\lambda_s\right)-I_n\left(\lambda_s\right)\right]e^{-\lambda_s} \tag{2.0.73}$$

Finally for I_{zz} we use (4.3.8) to evaluate the $u_\perp$ integral and (4.4.9) to evaluate the $u_\parallel$ integral

$$I_{zz} = 2\left[\frac{\zeta_{0s}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{u_\parallel^2 e^{-u_\parallel^2}}{u_\parallel-\zeta_{ns}}du_\parallel\right]\left[2\int_0^\infty J_n\left(\sqrt{2\lambda_s}u_\perp\right)^2 u_\perp e^{-u_\perp^2}du_\perp\right]$$

$$= 2\zeta_{0s}\zeta_{ns}\left[1+\zeta_{ns}Z\left(\zeta_{ns}\right)\right]I_n\left(\lambda_s\right)e^{-\lambda_s} \tag{2.0.74}$$

Bringing this all together we get our final expression for the warm plasma dielectric tensor

$$\epsilon_{ij} = \delta_{ij} + \sum_s X_s\zeta_{0s}e^{-\lambda_s}\sum_{n=-\infty}^{\infty}T_{ij} \tag{2.0.75}$$

where

$$\begin{gathered}
T_{xx} = \frac{n^2}{\lambda_s}I_n\left(\lambda_s\right)Z\left(\zeta_{ns}\right)\\
T_{xy} = in\left[I'_n\left(\lambda_s\right)-I_n\left(\lambda_s\right)\right]Z\left(\zeta_{ns}\right)\\
T_{xz} = n\sqrt{\frac{2}{\lambda_s}}\left[1+\zeta_{ns}Z\left(\zeta_{ns}\right)\right] = \frac{-inZ'\left(\zeta_{ns}\right)}{\sqrt{2\lambda_s}}\\
T_{yx} = -T_{xy}\\
T_{yy} = \left[\frac{n^2 I_n\left(\lambda_s\right)}{\lambda_s}+2\lambda_s I_n\left(\lambda_s\right)-2\lambda_s I'_n\left(\lambda_s\right)\right]Z\left(\zeta_{ns}\right)\\
T_{yz} = -i\sqrt{2\lambda_s}\left[I'_n\left(\lambda_s\right)-I_n\left(\lambda_s\right)\right]\left[1+\zeta_{ns}Z\left(\zeta_{ns}\right)\right] = i\sqrt{\frac{\lambda_s}{2}}\left[I'_n\left(\lambda_s\right)-I_n\left(\lambda_s\right)\right]Z'\left(\zeta_{ns}\right)\\
T_{zx} = -T_{xz}\\
T_{zy} = -T_{yz}\\
T_{zz} = 2\zeta_{ns}I_n\left(\lambda_s\right)\left[1+\zeta_{ns}Z\left(\zeta_{ns}\right)\right] = -\zeta_{ns}I_n\left(\lambda_s\right)Z'\left(\zeta_{ns}\right)
\end{gathered} \tag{2.0.76}$$

3 Application to EBWs

The previous section was completely general. Now we'll apply these formulas to EBWs.

3.0.1 Electrostatic Approximation

We can break down the wave electric field $\mathbf{E}_k$ into a transverse component $\mathbf{E}_{kT}$ and a longitudinal component $\mathbf{E}_{kL}$

4 Appendix

4.1 Complex Exponential Quantities

Based on section 7.4 in [1]. A useful mathematical notation is to represent an oscillating physical quantity as a phasor, with the understanding the actual physical quantity is the real part, i.e.

$$\psi(t) = \mathcal{R}\left[\tilde{\psi}e^{i\mathbf{k}\cdot\mathbf{x}-\mathbf{i}\omega\mathbf{t}}\right] = \frac{1}{2}\left(\psi + \psi^*\right) \tag{4.1.1}$$

For linear relationships taking the real is done implicitly as it has no effect. However, for non-linear relationships we must do this to get the correct answer. For example, the product of two oscillating quantities $\psi\left(t\right) = \tilde{\psi}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ and $\chi(t) = \tilde{\chi}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ must be written as

$$\psi(t)\chi(t) = \mathcal{R}\left[\tilde{\psi}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] \times \mathcal{R}\left[\tilde{\chi}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right] \tag{4.1.2}$$

Expanding this out, assuming $\mathbf{k} = \mathbf{k}_R + i\mathbf{k}_I$ and $\omega = \omega_R + i\omega_I$

$$\begin{aligned}
\psi(t)\chi(t) = &\frac{1}{4}\left(\tilde{\psi}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \tilde{\psi}^* e^{i(\mathbf{k}^*\cdot\mathbf{x}-\omega^* t)}\right)\left(\tilde{\chi}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \tilde{\chi}^* e^{i(\mathbf{k}^*\cdot\mathbf{x}-\omega^* t)}\right) \\
&= \frac{1}{4}\left(\tilde{\psi}\tilde{\chi}e^{2i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \tilde{\psi}\tilde{\chi}^* e^{i[(\mathbf{k}-\mathbf{k}^*)\cdot\mathbf{x}-(\omega-\omega^*)t]}\right. \\
&\left.+\tilde{\psi}^*\tilde{\chi}e^{i[(\mathbf{k}-\mathbf{k}^*)\cdot\mathbf{x}-(\omega-\omega^*)t]} + \tilde{\psi}^*\tilde{\chi}^* e^{2i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)
\end{aligned} \tag{4.1.3}$$

As $-i\left(\omega-\omega^*\right) = 2\omega_I$ and $i\left(\mathbf{k}-\mathbf{k}^*\right) = -2\mathbf{k}_R$ we can rewrite this as

$$\psi(t)\chi(t) = \frac{1}{4}\left[\tilde{\psi}\tilde{\chi} + \tilde{\psi}^*\tilde{\chi}^*\right] e^{2i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \frac{1}{4}\left[\tilde{\psi}^*\tilde{\chi} + \tilde{\psi}\tilde{\chi}^*\right] e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})} \tag{4.1.4}$$

The first term is oscillatory so if we time average over a period of the oscillation they will vanish, while the second term will remain as it is non-oscillatory. Assuming ω_I and $\mathbf{k}_R$ are very small and so $e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})}$ is constant over a single wave period, the time averaged quantity $\langle\psi(t)\chi(t)\rangle$ is

$$\langle\psi(t)\chi(t)\rangle = \frac{1}{4}\left(\tilde{\psi}\tilde{\chi}^* + \tilde{\psi}^*\tilde{\chi}\right) e^{2(\omega_I t-\mathbf{k}_I\cdot\mathbf{x})} \tag{4.1.5}$$

The term inside the brackets can be rewritten as the real part of a complex number using (4.1.1)

$$\langle\psi(t)\chi(t)\rangle = \frac{1}{2}\mathcal{R}\left[\tilde{\psi}\tilde{\chi}^*\right] e^{2(\omega_I t - \mathbf{k}_I\cdot\mathbf{x})} \tag{4.1.6}$$

4.2 Vector Identities

4.2.1 $\|\mathbf{N}\times\mathbf{e}\|^2 = \mathbf{e}_k^*\cdot\left(N^2\underline{\underline{I}} - \mathbf{N}\otimes\mathbf{N}\right)\cdot\mathbf{e}_k$

Start at the end and work backwards

$$\mathbf{e}_k^*\cdot\left(N^2\underline{\underline{I}} - \mathbf{N}\otimes\mathbf{N}\right)\cdot\mathbf{e}_k =$$

$$\begin{pmatrix} e_x & e_y & e_z \end{pmatrix}\begin{pmatrix} N_y^2+N_z^2 & -N_xN_y & -N_xN_z \\ -N_xN_y & N_x^2+N_z^2 & -N_yN_z \\ -N_xN_z & -N_yN_z & N_y^2+N_z^2 \end{pmatrix}\begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}$$

$$= \left(N_y^2+N_z^2\right)e_x^2 + \left(N_x^2+N_z^2\right)e_y^2 + \left(N_x^2+N_y^2\right)e_z^2 - 2N_xN_ye_xe_y - 2N_xN_ze_xe_z - 2N_yN_ze_ye_z$$

$$= \left(N_ye_z - N_ze_y\right)^2 + \left(N_ze_x - N_xe_z\right)^2 + \left(N_xe_y - N_ye_x\right)^2 \tag{4.2.1}$$

These terms are the components of a cross product, hence we see this is equal to $\|\mathbf{N}\times\mathbf{e}\|^2$.

4.2.2 $\mathbf{N} - \mathcal{R}\left[\left(\mathbf{e}_k^*\cdot\mathbf{N}\right)\mathbf{e}_k\right] = -\frac{1}{2}\mathbf{e}_k^*\cdot\frac{\partial\left(\mathbf{N}\otimes\mathbf{N}-N^2\underline{\underline{I}}\right)}{\partial\mathbf{N}}\cdot\mathbf{e}_k$

4.3 Bessel Functions

4.3.1 Bessel's Differential Equation

$$J_{-n}(x) = (-1)^n J_n(x)) \tag{4.3.1}$$

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos\left(n\theta - x\sin\theta\right)d\theta = \frac{1}{\pi}\int_0^\pi \cos\left(x\sin\theta - n\theta\right)d\theta \tag{4.3.2}$$

4.3.2 Bessel Function Recursion Relations

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x}J_n(x) \tag{4.3.3}$$

$$J_n'(x) = \frac{1}{2}\left(J_{n+1}(x) - J_{n-1}(x)\right) \tag{4.3.4}$$

$$I_{n+1}(x) - I_{n-1}(x) = -\frac{2n}{x} I_n(x) \tag{4.3.5}$$

$$I_n'(x) = \frac{1}{2}\left(I_{n+1}(x) + I_{n-1}(x)\right) \tag{4.3.6}$$

$$\lim_{\mathbf{x}\to\infty} I_n(x) = \left(\frac{x}{2}\right)^{|n|} \tag{4.3.7}$$

4.3.3 Bessel Function Integrals

$$\int_0^\infty \frac{J_n(x)}{x} dx = \frac{1}{n} \tag{4.3.8}$$

$$2\int_0^\infty J_n^2\left(x\sqrt{2b}\right) xe^{-x^2} dx = I_n(b)e^{-b} \tag{4.3.9}$$

$$4\int_0^\infty J_n\left(x\sqrt{2b}\right) J_n'\left(x\sqrt{2b}\right) x^2 e^{-x^2} dx = \sqrt{2b}\left(I_n'(b) - I_n(b)\right) e^{-b} \tag{4.3.10}$$

$$4\int_0^\infty \left(J_n'\left(x\sqrt{2b}\right)\right)^2 x^3 e^{-x^2} dx = \left[\frac{n^2}{b} I_n(b) + 2b\left(I_n(b) - I_n'(b)\right)\right] e^{-b} \tag{4.3.11}$$

These equations are derived from Weber's second integral with q=1 [reference?]

$$\int_0^\infty J_n(\alpha x) J_n(\beta x) xe^{-x^2} dx = \frac{1}{2} e^{-\left(\frac{\alpha^2+\beta^2}{4}\right)} I_n\left(\frac{\alpha\beta}{2}\right) \tag{4.3.12}$$

Setting $\alpha = \beta = \sqrt{2b}$ so $\alpha^2 + \beta^2 = 4b$ and $\alpha\beta = 2b$ we get (4.3.8). To get the next integral we take a derivative of (4.3.11) with respect to α. The left hand side is

$$\frac{\partial}{\partial\alpha}\int_0^\infty J_n(\alpha x) J_n(\beta x) xe^{-x^2} dx = \int_0^\infty \left[xJ_n'(\alpha x)\right] J_n(\beta x) xe^{-x^2} dx$$

$$= \int_0^\infty J_n'(\alpha x) J_n(\beta x) x^2 e^{-x^2} dx \tag{4.3.13}$$

The right hand side gives

$$\frac{\partial}{\partial\alpha}\frac{1}{2} e^{-\left(\frac{\alpha^2+\beta^2}{4}\right)} I_n\left(\frac{\alpha\beta}{2}\right) = \frac{1}{2}\left[\frac{-2\alpha}{4} I_n\left(\frac{\alpha\beta}{2}\right) + \frac{\beta}{2} I_n'\left(\frac{\alpha\beta}{2}\right)\right] e^{-\left(\frac{\alpha^2+\beta^2}{4}\right)}$$

$$= \frac{1}{4}\left[\beta I_n'\left(\frac{\alpha\beta}{2}\right) - \alpha I_n\left(\frac{\alpha\beta}{2}\right)\right] e^{-\left(\frac{\alpha^2+\beta^2}{4}\right)} \tag{4.3.14}$$

Equating and letting $\alpha = \beta = \sqrt{2b}$ gives (4.3.9). To get the final expression we take another derivative, this time with respect to β. The left hand side gives

$$\frac{\partial}{\partial\beta}\int_0^\infty J_n'(\alpha x)\, J_n(\beta x)\, x^2 e^{-x^2} dx = \int_0^\infty J_n'(\alpha x)\left[x J_n'(\beta x)\right] x^2 e^{-x^2} dx$$

$$= \int_0^\infty J_n'(\alpha x)\, J_n'(\beta x)\, x^3 e^{-x^2} dx \tag{4.3.15}$$

The right hand side gives

$$\frac{\partial}{\partial\beta}\frac{1}{4}\left[\beta I_n'\left(\frac{\alpha\beta}{2}\right) - \alpha I_n\left(\frac{\alpha\beta}{2}\right)\right] e^{-\left(\frac{\alpha^2+\beta^2}{4}\right)}$$

$$= \frac{1}{4}\left[I_n' + \frac{\alpha\beta}{2} I_n'' - \frac{\alpha^2}{2} I_n' - \frac{\beta}{2}\left(\beta I_n' - \alpha I_n\right)\right] e^{-\left(\frac{\alpha^2+\beta^2}{4}\right)}$$

$$= \frac{1}{4}\left[\frac{\alpha\beta}{2} I_n'' + \left(1 - \frac{\alpha^2}{2} - \frac{\beta^2}{2} I_n'\right) + \frac{\alpha\beta}{2} I_n\right] e^{-\left(\frac{\alpha^2+\beta^2}{4}\right)} \tag{4.3.16}$$

Letting $\alpha = \beta = \sqrt{2b}$ and substituting gives

$$\frac{1}{4}\left[b I_n'' + (1-2b)\, I_n' + b I_n\right] e^{-b} \tag{4.3.17}$$

We can eliminate I_n'' by knowing Modified Bessel Functions satisfy the Modified Bessel Equation

$$b^2 I_n''(b) + b I_n'(b) - (b^2+n^2) I_n(b) = 0 \tag{4.3.18}$$

$$\implies b I_n'' = \left(b + \frac{n^2}{b}\right) I_n - I_n' \tag{4.3.19}$$

Substituting in we get

$$\frac{1}{4}\left[\left(b + \frac{n^2}{b}\right) I_n - I_n' + I_n' - 2b I_n' + b I_n\right] e^{-b}$$

$$= \frac{1}{4}\left[\frac{n^2}{b} I_n + 2b\left(I_n - I_n'\right)\right] \tag{4.3.20}$$

Equating with the left side with $\alpha = \beta = \sqrt{2b}$ gives (4.3.10).

4.4 Plasma Dispersion Function

The Plasma Dispersion Function Z appears in the derivation of the hot plasma dielectric tensor defined as

$$Z(\zeta) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x-\zeta} dx \tag{4.4.1}$$

Two similar integrals also appear, the first being

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{xe^{-x^2}}{x-\zeta} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{(x-\zeta+\zeta)\, e^{-x^2}}{x-\zeta} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\zeta e^{-x^2}}{x-\zeta} dx = 1 + \zeta \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x-\zeta} dx \tag{4.4.2}$$

using Gaussian integral is $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Hence we get

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{xe^{-x^2}}{x-\zeta} dx = 1 + \zeta Z\left(\zeta\right) \tag{4.4.3}$$

We can write this in terms of the derivative Z', given by

$$Z'\left(\zeta\right) = \frac{\zeta}{\partial \zeta} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x-\zeta} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{\left(x-\zeta\right)^2} dx \tag{4.4.4}$$

Integrating by parts as follows allows us to express Z' in terms of Z

$$u = e^{-x^2} \implies du = -2xe^{-x^2}$$

$$dv = \frac{1}{\left(x-\zeta\right)^2} \implies v = \frac{-1}{x-\zeta}$$

$$Z'\left(\zeta\right) = \frac{1}{\sqrt{\pi}} \left[\left. \frac{-e^{-x^2}}{x-\zeta} \right|_{-\infty}^{\infty} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2xe^{-x^2}}{x-\zeta} dx \right]$$

$$= \frac{-2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{xe^{-x^2}}{x-\zeta} dx = -2\left(1 + \zeta Z\left(\zeta\right)\right) \tag{4.4.5}$$

Hence

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{xe^{-x^2}}{x-\zeta} dx = 1 + \zeta Z\left(\zeta\right) = -\frac{Z'\left(\zeta\right)}{2} \tag{4.4.6}$$

The second integral is

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{x^2e^{-x^2}}{x-\zeta}dx \tag{4.4.7}$$

Again integrating by parts we get

$$u=\frac{x}{x-a} \implies du=\frac{x-a-x}{\left(x-a\right)^2}=fraca(x-a)^2$$

$$dv=xe^{-x^2} \implies v=-\frac{e^{-x^2}}{2}$$

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{x^2e^{-x^2}}{x-\zeta}dx &= \frac{1}{\sqrt{\pi}}\left.\frac{-xe^{-x^2}}{2\left(x-a\right)}\right|_{-\infty}^{\infty}-\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\zeta e^{-x^2}}{\left(x-\zeta\right)^2}dx\\ &= -\frac{\zeta}{2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{\left(x-\zeta\right)^2}dx \end{aligned}\tag{4.4.8}$$

We recognise the integral equals $Z'\left(\zeta\right)$. Hence

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{x^2e^{-x^2}}{x-\zeta}dx=\frac{-\zeta}{2}Z'\left(\zeta\right)=\zeta\left(1+\zeta Z\left(\zeta\right)\right) \tag{4.4.9}$$

In the limit $\zeta_{ns}\to\infty$

$$Z\to-\frac{1}{\zeta_{ns}}\quad Z'\to\frac{1}{\zeta_{ns}^2} \tag{4.4.10}$$

References

[1]P. M. Bellan, *Fundamentals of plasma physics* (Cambridge University Press, 2008).

[2]R. Fitzpatrick, *Plasma physics: an introduction* (Crc Press, 2014).