# Advanced Algorithms Assignment VI

Matthew Bennett

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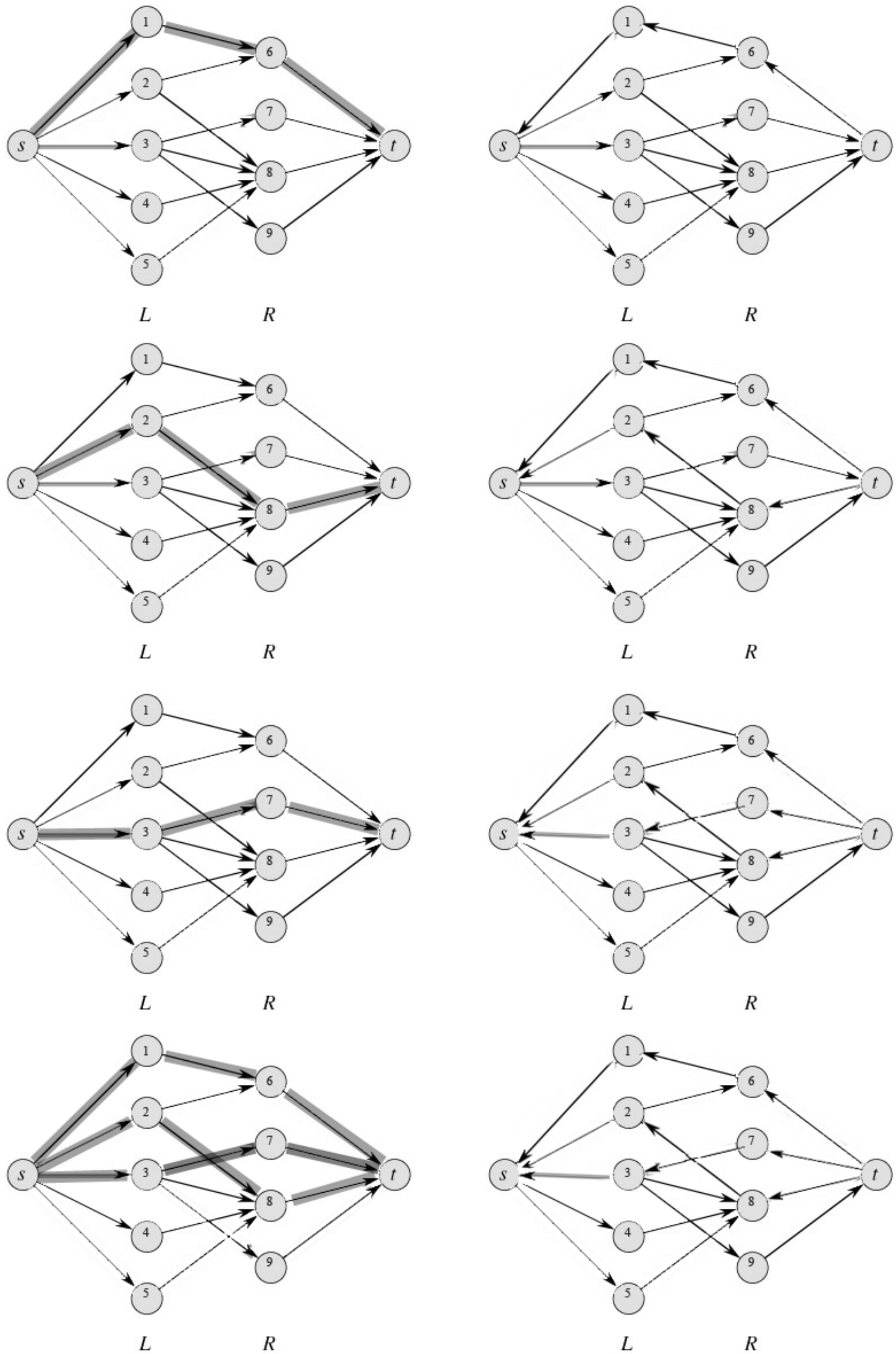
## Exercise 26.3-1

**Run the Ford-Fulkerson algorithm on the flow network in Figure 26.8(b) and show the residual network after each flow augmentation. Number the vertices in L top to bottom from 1 to 5 and in R top to bottom from 6 to 9. For each iteration, pick the augmenting path that is lexicographically smallest.**

Please note that as given in the book, each edge has unit capacity **1**. Answer on next page ☺.




There are no more augmenting paths in the residual network, so our bipartite matching is maximal.

## Problem 26-1

**An $n \times n$ grid is an undirected graph consisting of $n$ rows and $n$ columns of vertices, as shown in Figure 26.11. We denote the vertex in the $i^{th}$ row and the $j^{th}$ column by $(i, j)$. All vertices in a grid have exactly four neighbors, except for the boundary vertices, which are the points $(i, j)$ for which $i = 1, i = n, j = 1,$ or $j = n$.**

**Given $m \leq n^2$ starting points $(x_1, y_1), (x_2, y_2), ..., (x_m, y_m)$ in the grid, the escape problem is to determine whether or not there are $m$ vertex-disjoint paths from the starting points to any $m$ different points on the boundary. For example, the grid in Figure 26.11(a) has an escape, but the grid in Figure 26.11(b) does not.**

**a. Consider a flow network in which vertices, as well as edges, have capacities. That is, the total positive flow entering any given vertex is subject to a capacity constraint. Show that determining the maximum flow in a network with edge and vertex capacities can be reduced to an ordinary maximum-flow problem on a flow network of comparable size.**

I like to think of this as a "plumbing problem," where each edge represents a pipe and each vertex represents a joint.

The vertex capacity limits the flow through a single vertex. So, the total flow in and out of a single vertex cannot exceed the newly introduced vertex capacity. If the vertex capacity *for all vertices* is assigned so that the capacity exceeds the sum of the incoming edge capacities, then the vertex capacities limit the maximum flow algorithm in no way whatsoever.

This case is trivial, and because we are not in control of the vertex capacities, I assume that a graph transformation is appropriate. Because the vertices act like edges in so far as they have flow constraints, why not transform all constraint vertices into constraint edges, introducing two new traditional vertices to bound each one? It works, but is it "comparable size"?

Graph: $\rightarrow_5$ (10) $\rightarrow_{15}$ $\dashrightarrow$ becomes $\dashrightarrow$ Graph: $\rightarrow_5$ ($\infty$) $\rightarrow_{10}$ ($\infty$) $\rightarrow_{15}$

Each constraint vertex splits into two traditional (non-constraint) vertices, which means that the size is bounded by a constant factor of two. $O(2V) = O(V)$. Additionally, a new edge is introduced for each vertex, which means that the graph grows by a factor up to $|V| = O(V) \leq O(E)$ for any connected graph. So, the size of the original graph with vertex constraints is $|V| + |E| = O(V + E)$, and the size of the new graph is $3|V| + |E| = O(V + E)$. Asymptotic equality is assumed to mean "comparable size."

**b. Describe an efficient algorithm to solve the escape problem. Analyze its running time.**

The ***escape problem*** can be modeled by a maximum flow network in which the total number of matchings must equal the number of starting points $m$. The super-source $s$ must be connected to all $m$ starting points, and the super-sink $t$ must be connected to all of the boundary points. Additionally, since the paths must be vertex-disjoint (blocking), they must all have unit capacity. (Taking a hint from (a), we split the unit vertex capacities into new unit capacity edges with traditional vertex pairs.) Each matching in the maximum flow network corresponds to an "escape route" or vertex-disjoint path between the starting vertices and the border vertices.

If the maximum flow algorithm results in $m$ matchings, then the graph is said to be escapable. Otherwise, the graph is not escapable for all $m$ starting vertices. This should be clear from the definition of a vertex-disjoint path and the fact that all edge capacities are initialized to 1.

From (a) we have that the graph is of upper bound $|V| \leq 2|v|$ (splitting) and $|E| \leq |e| + |v|$. Assume that the maximum flow algorithm used is Edmunds-Karp. The running time of Edmunds-Karp as given in the book is $O(V\ E^2)$. By a simple replacement, we see that the running time is $O(2v\ (e + v)^2) = O(v\ (e^2 + 2ev + v^2)) = O(e^2 v + ev^2 + v^3))$, a 3rd-degree polynomial of the original graph's $e$ edges, and the original graph's $v$ vertices.

## Problem 26-3

**Professor Spock is consulting for NASA, which is planning a series of space shuttle flights and must decide which commercial experiments to perform and which instruments to have on board each flight. For each flight, NASA considers a set $E = \{E_1, E_2, ..., E_m\}$ of experiments, and the commercial sponsor of experiment $E_j$ has agreed to pay NASA $p_j$ dollars for the results of the experiment. The experiments use a set $I = \{I_1, I_2, ..., I_n\}$ of instruments; each experiment $E_j$ requires all the instruments in a subset $R_j \subseteq I$. The cost of carrying instrument $I_k$ is $c_k$ dollars. The professor's job is to find an efficient algorithm to determine which experiments to perform and which instruments to carry for a given flight in order to maximize the net revenue, which is the total income from experiments performed minus the total cost of all instruments carried.**

**Consider the following network $G$: The network contains a source vertex $s$, instrument vertices $I_1, I_2, ..., I_n$, experiment vertices $E_1, E_2, ..., E_m$, and a sink vertex $t$. For $k = 1, 2..., n$, there is an edge $(s, I_k)$ of capacity $c_k$, and for $j = 1, 2, ..., m$, there is an edge $(E_j, t)$ of capacity $p_j$. For $k = 1, 2, ..., n$ and $j = 1, 2, ..., m$, if $I_k \epsilon R_j$ , then there is an edge $(I_k, E_j)$ of infinite capacity.**

*Didn't we do this problem in class? I wish I had written it down, now. I remember something similar.*

**a. Show that if $E_j \epsilon T$ for a finite-capacity cut $(S, T)$ of $G$, then $I_k \epsilon T$ for each $I_k \epsilon R_j$.**

Assume $I_k \ni T$ for each $I_k \epsilon R_j$ Therefore, $I_k \epsilon S$, because $(S, T)$ is a cut. We know $I_k \epsilon R_j \rightarrow c(I_k, E_j) = \infty$ spans the cut. However, the cut must be of *finite* capacity ($\Leftrightarrow$). Therefore, our assumption is incorrect, and $I_k \epsilon T$.

**b. Show how to determine the maximum net revenue from the capacity of the minimum cut of $G$ and the given $p_j$ values.**

Intuitively, the answer is given by the sum $\sum_j \{p_j | E_j \text{ experiment done}\} - \sum_k \{c_k | I_k \text{ instrument used}\}$. We must find an equivalent way of expressing this in terms of the capacity of the minimum cut, $c(S, T)$.

Calculate $\sum_j p_j$, the total profit for all possible experiments. Subtract the total capacity of the cut. This should yield the net revenue $R = \sum_j p_j - c(S, T)$. This works because the capacity of the *minimum* cut represents the profits for experiments *NOT* performed plus instrument expenses *that ARE* incurred for an optimal solution. The difference is the optimal net revenue.

**c. Give an efficient algorithm to determine which experiments to perform and which instruments to carry. Analyze the running time of your algorithm in terms of $m, n$, and $r = \sum_{j=1}^{m} | R_j |$.**

1. Use Edmunds-Karp to find a *minimum cut / maximum flow.*
2. Calculate net revenue as given in (b).
3. For all I and E values involved in the cut, add each instrument and experiment to the list of involved items. The set of instruments unused and experiments not performed is the Universe minus those listed here.

The book gives a running time of $O(VE^2)$ for Edmunds-Karp. Because of the super-source and super-sink, $|V| = 1 + 1 + n + m = O(n + m)$ and $|E| = n + m + r$. Doing a simple substitution, we have $O((n+m)(n+m+r)^2)$, which expands as $O(m^3 + 3m^2n + 3mn^2 + n^3 + 2m^2r + 4mnr + 2n^2r + mr^2 + nr^2)$. That is the running time for part (1) only.

The net revenue can be calculated in $O(m + n)$ time, since $m$ experiments and $n$ instruments need to be checked and tallied, and the profit vector $P$ is only $n$ units long. Similarly, the third step is also $O(m + n)$. Overall efficiency is therefore $O(m^3 + m^2n + mn^2 + n^3 + m^2r + mnr + n^2r + mr^2 + nr^2)$, where $r = \sum_{j=1}^{m} | R_j |$. Proving that this algorithm is of optimal efficiency is left as an exercise to the reader ☺.

This class was fun! See you in comprehensive exams.