Analytical Models Assignment I

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(1) Solve the system of linear equations:

(a) $\pi_1 = .7\pi_1 + .3\pi_2 + .2\pi_3$

(b) $\pi_2 = .2\pi_1 + .5\pi_2 + .6\pi_3$

(c) $\pi_3 = .1\pi_1 + .2\pi_2 + .2\pi_3$

(d) $1 = \pi_1 + \pi_2 + \pi_3$

Since the above equations are all linearly independent, (c) will not be used to solve the unknowns. Using the Gauss-Jordan method, our initial matrix is as follows:

$$\begin{pmatrix} -\frac{3}{10} & \frac{3}{10} & \frac{2}{10} & 0 \\ \frac{2}{10} & -\frac{5}{10} & \frac{6}{10} & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Swap row 1 and row 3:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{2}{10} & -\frac{5}{10} & \frac{6}{10} & 0 \\ -\frac{3}{10} & \frac{3}{10} & \frac{2}{10} & 0 \end{pmatrix}$$

$Row2 \leftarrow -\frac{2}{10}Row1 + Row2$
$Row3 \leftarrow \frac{3}{10}Row1 + Row3$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -\frac{7}{10} & \frac{4}{10} & -\frac{2}{10} \\ 0 & \frac{6}{10} & \frac{5}{10} & \frac{3}{10} \end{pmatrix}$$

$Row2 \leftarrow -\frac{10}{7}Row2$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{4}{7} & \frac{2}{7} \\ 0 & \frac{6}{10} & \frac{5}{10} & \frac{3}{10} \end{pmatrix}$$

$Row3 \leftarrow -\frac{6}{10}Row2 + Row3$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{4}{7} & \frac{2}{7} \\ 0 & 0 & \frac{59}{70} & \frac{9}{70} \end{pmatrix}$$

$Row3 \leftarrow \frac{70}{59} Row3$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{4}{7} & \frac{2}{7} \\ 0 & 0 & 1 & \frac{9}{59} \end{pmatrix}$$

$Row1 \leftarrow -Row3 + Row1$
$Row2 \leftarrow \frac{7}{4} Row3 + Row2$

$$\begin{pmatrix} 1 & 1 & 0 & -\frac{9}{59} \\ 0 & 1 & 0 & \frac{22}{59} \\ 0 & 0 & 1 & \frac{9}{59} \end{pmatrix}$$

$Row1 \leftarrow -Row2 + Row1$
$Row2 \leftarrow \frac{7}{4} Row3 + Row2$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{28}{59} \\ 0 & 1 & 0 & \frac{22}{59} \\ 0 & 0 & 1 & \frac{9}{59} \end{pmatrix} \text{ Done.}$$

$$\pi_1 = \frac{28}{59}, \pi_2 = \frac{22}{59}, \pi_3 = \frac{9}{59} \tag{1}$$

(2) Explain how these values are derived:

"the steady state probability of a sunny day is .47,"
From the array obtained in the previous problem, we know that the steady state probability for a sunny day is 28/59, which is approximately 0.4745762711864406779661016949 1525. Marsan rounded down.

"the mean recurrence time of sunny days is 2.1 days,"
Mean recurrence time for state j is given by the following formula (2.26) for an ergodic system

$$\pi_j = \frac{1}{M_j} \tag{2}$$

Therefore, $M_j = \frac{1}{\pi_j} = \frac{59}{28} \approx 2.1$.

"the average number of sunny days between two consecutive rainy days is 3.11,"
This should be the same as (2.34 and Bayes' Rule) $\frac{\pi 1}{\pi 3} \approx 3.11$

"if today the sun is shining, we can expect 2.33 more rainy days before the weather changes,"
The average number of steps spent in state i before going to another state is described by a geometric process where the expected wait time to leave state i (2.36) is

$$E[W_i] = \frac{1}{1 - p_{ii}} \tag{3}$$

. One Sunny day has already passed, so the number of sunny days remaining until a non-Sunny say is $E[W_{sunny}] = \frac{1}{1-0.7} - 1 \approx 2.33$

(3) Give a new piece of information.

The fourth-step transition probability matrix is

$$P^4 = P^2 P^2 = \begin{pmatrix} .4917 & .3598 & .1485 \\ .4612 & .3831 & .1557 \\ .4540 & .3886 & .1574 \end{pmatrix}$$

Therefore, the probability that the fourth day after a sunny day is 0.4917, and other facts can be garnished from that matrix.