Advanced Linear Algebra Solutions

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1 Friday Jun 02, 2006

4.1.1 Determine which subsets of $\mathfrak{R}^n$ are subspaces.

(a) $\{\mathbf{x}|x_i \geq 0\}$ No. It is not closed under scalar multiplication by a negative number.

(b) $\{\mathbf{x}|x_1 = 0\}]$ Yes.

(c) $\{\mathbf{x}|x_1 x_2 = 0\}$ No. Let $a_1 = 0, b_1 \neq 0$. Then $\mathbf{a} + \mathbf{b} \notin \{\mathbf{x}|x_1 x_2 = 0\}$

(d) $\{\mathbf{x}|\sum_{i=1}^{n} x_i = 0\}]$ Yes.

(e) $\{\mathbf{x}|\sum_{i=1}^{n} x_i = 0\}]$ No. It is not closed under scalar multiplication.

(f) $\{\mathbf{x}|\mathbf{Ax} = \mathbf{b}$, and $\mathbf{A}_{m\times n} \neq \mathbf{0}$ and $\mathbf{b}_{m\times 1} \neq \mathbf{0}\}]$
No (Why? Also, What is this Notation?)
• Unresolved.

4.1.2 Determine which subsets of $\mathfrak{R}^{n\times n}$ are subspaces.

(a) The Symmetric Matrices?
Yes.

(b) The Diagonal Matrices?
Yes.

(c) The Nonsingular Matrices?
No, because it is not closed under addition.

(d) The Singular Matrices?
No, not closed under addition.

(e) The Triangular Matrices?
No, because $\mathbf{U} + \mathbf{L}$ is not triangular.

(f) The Upper-Triangular Matrices?
Yes.

(g) All Commuting Matrices of $\mathbf{A}$?
Yes.

(h) All matrices $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{A}$
No, because it is not closed under scalar multiplication. (However, because of the pythagorean theorem it is closed under addition.)

(i) All matrices **A** such that $trace(\mathbf{A}) = 0$
Yes. (Proof?) If that is true, then

$$\sum_{i=1}^{n} x_{ii} = 0$$

Therefore, it is closed under addition. Also, scalar multiplication, since it is also true that

$$\sum_{i=1}^{n} \alpha x_{ii} = \alpha \sum_{i=1}^{n} x_{ii} = 0$$

4.1.3 If vector space $\mathcal{X}$ is a plane passing through the origin in $\Re^3$ and $\mathcal{Y}$ is the line through the origin that is perpendicular to $\mathcal{X}$, then what is $\mathcal{X} + \mathcal{Y}$?

Answer: $\Re^3$

4.1.4 Why must a real or complex nonzero vector field contain an infinite number of vectors?

Answer: A vector space is defined as all scalar multiples of any vectors contained, else it would not be closed under scalar multiplication. The only counterexample is the trivial vector space defined on $\emptyset$.

4.1.5 Describe the subspace defined by the given column-vector matrices

$$\left(\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} \begin{pmatrix} -3 \\ -9 \\ -6 \end{pmatrix} \right)$$

Column 3 is an elementary transformation of column 1. Therefore, the subspace defined should be a line, since only two column vectors are linearly independent.

$$\left(\begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right)$$

These vectors are all linearly independant. The space should be a plane, since the third component has a coefficient of zero in all vectors.

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

These vectors are both independent and span all of $\Re^3$

2 Wed June 7, 2006

4.2.1 Determine spanning sets for

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix}$$

Answer: Gauss Elimination yields a rank-2 matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R(\mathbf{A}) = span\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right)$$

Since the range space of **A** is the same as the space spanned by the columns of **A**. Similarly,

$$R(\mathbf{A}^\top) = span\left(\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \\ 8 \end{pmatrix} \right)$$

$$N(\mathbf{A}) = \left(\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$N(\mathbf{A}^\top) =$$