COMPUTATIONAL FINANCE: 422

Mean-Variance Portfolio Theory

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This Lecture

- Asset returns
- Portfolio returns
- Variance as a risk measure
- Mean-variance diagrams
  - Feasible set
  - Minimum-variance set
  - Efficient frontier
- Markowitz problem
- Parameter estimation

Further reading:

- D.G. Luenberger: *Investment Science*, Chapters 6 & 8

Asset Return I

- Asset: investment instrument that can be bought/sold.
- If you buy an asset today at a price X_0 and sell it in 1 year at a price X_1, then the total return R on your investment is defined to be

$$R = \frac{X_1}{X_0}\,.$$

- Similarly, the rate of return r is defined as

$$r = \frac{X_1 - X_0}{X_0}\,.$$

- The expression 'return' is used both for the total return and the rate of return. The context should make clear which interpretation is meant.

Asset Return II

- By definition, we have

$$R = 1 + r$$

$$\Rightarrow \quad X_1 = (1 + r) X_0 \, .$$

Thus, the rate of return acts much like an interest rate.

- If X_1 is uncertain, then r must be uncertain, as well. In contrast, current interest rates are always certain.

Short Sales

- Sometimes it is possible to sell an asset that you do not own. This process is called short selling or shorting.
- How does it work?
  1. You borrow the asset from someone who owns it.
  2. You sell the asset to someone else at its current price X_0.
  3. At a later date, you buy the asset for X_1.
  4. You return the asset to your lender.
- Your overall profit is $X_0 - X_1 \Rightarrow$ short selling is profitable if the asset price declines.
- The potential loss of a short sale is unbounded!

  $\Rightarrow$ Short selling is often restricted or avoided.

Portfolios

- Suppose n different assets are available.
- We form a master asset or portfolio by apportioning an amount X_0 among the assets.
- We select amounts X_{0i}, $i = 1, 2, \ldots, n$, such that

$$\sum_{i=1}^{n} X_{0i} = X_0 \,,$$

where X_{0i} represents the amount invested in asset i.

- If short selling is allowed, some X_{0i}'s can be negative; otherwise we require $X_{0i} \geq 0$.
- The X_{0i} can be expressed as $X_{0i} = w_i X_0$, $i = 1, 2, \ldots, n$, where w_i is the weight of asset i in the portfolio.

Portfolio Return

- The asset weights sum to 1, that is, $\sum_{i=1}^{n} w_i = 1$.
- $R_i =$ total return of asset i. $\Rightarrow$ The amount of money generated at the end of the period by the ith asset is

$$R_i X_{0i} = R_i w_i X_0 \,.$$

- Thus, the total value of the portfolio after the period is

$$\sum_{i=1}^{n} R_i w_i X_0 \,.$$

$\Rightarrow$ The portfolio's total return and rate of return are

$$R = \frac{\sum_{i=1}^{n} w_i R_i X_0}{X_0} = \sum_{i=1}^{n} w_i R_i \quad \text{and} \quad r = \sum_{i=1}^{n} w_i r_i \,.$$

Describing a Portfolio

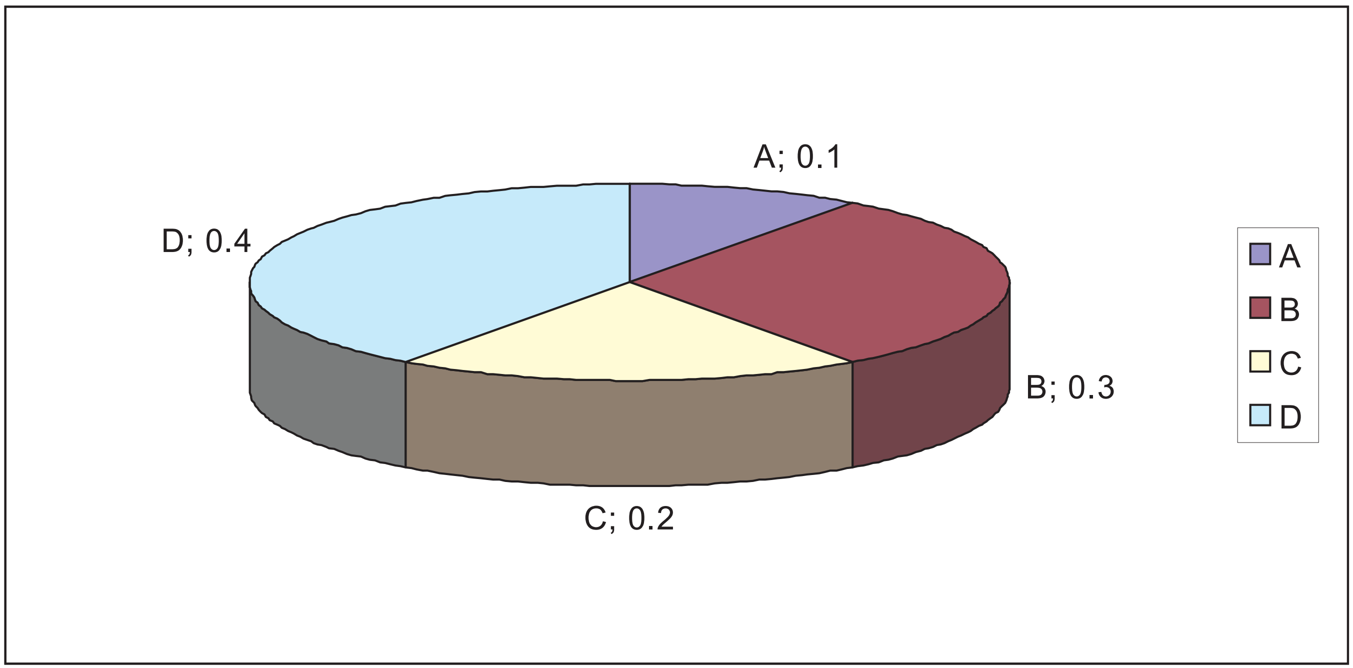
Assume that there are n assets in which you can invest.

Asset	£ invested	% invested	Return
1	X_{01}	$w_1 = X_{01}/X_0$	R_1
2	X_{02}	$w_2 = X_{02}/X_0$	R_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	X_{0n}	$w_n = X_{0n}/X_0$	R_n
Total:	$X_0 = \sum_{i=1}^{n} X_{0i}$	$1 = \sum_{i=1}^{n} w_i$	$R = \sum_{i=1}^{n} w_i R_i$

A portfolio can be described by £ invested or by portfolio weights. Using weights facilitates the calculation of the portfolio return.

Example

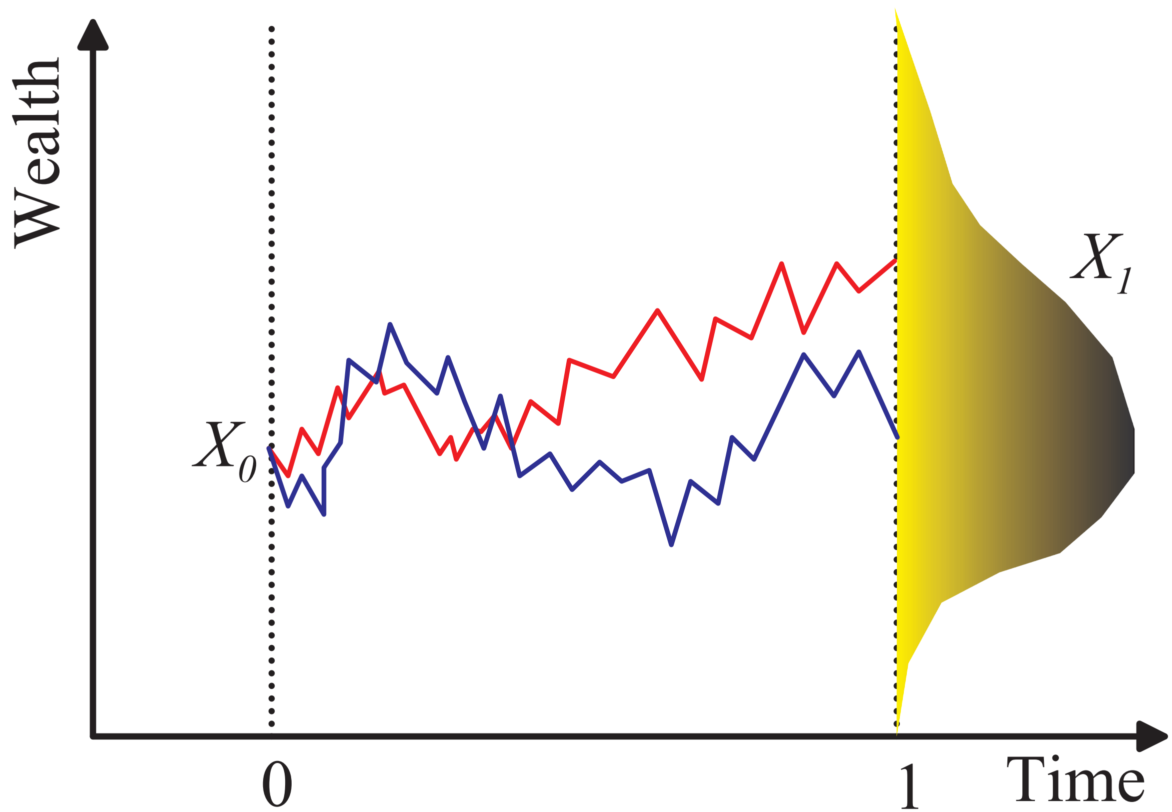
Security	£ Amount X_{0i}	% Weight w_i	Return r_i	$w_i\ r_i$
A	100	0.1	1.1	0.11
B	300	0.3	1.2	0.36
C	200	0.2	1.05	0.21
D	400	0.4	1.25	0.5
Total	1000	1		1.18



Randomness I

- For any asset, today's value X_0 is deterministic, while the future value X_1 is random.

$\Rightarrow$ The total return R and the rate of return r are random.

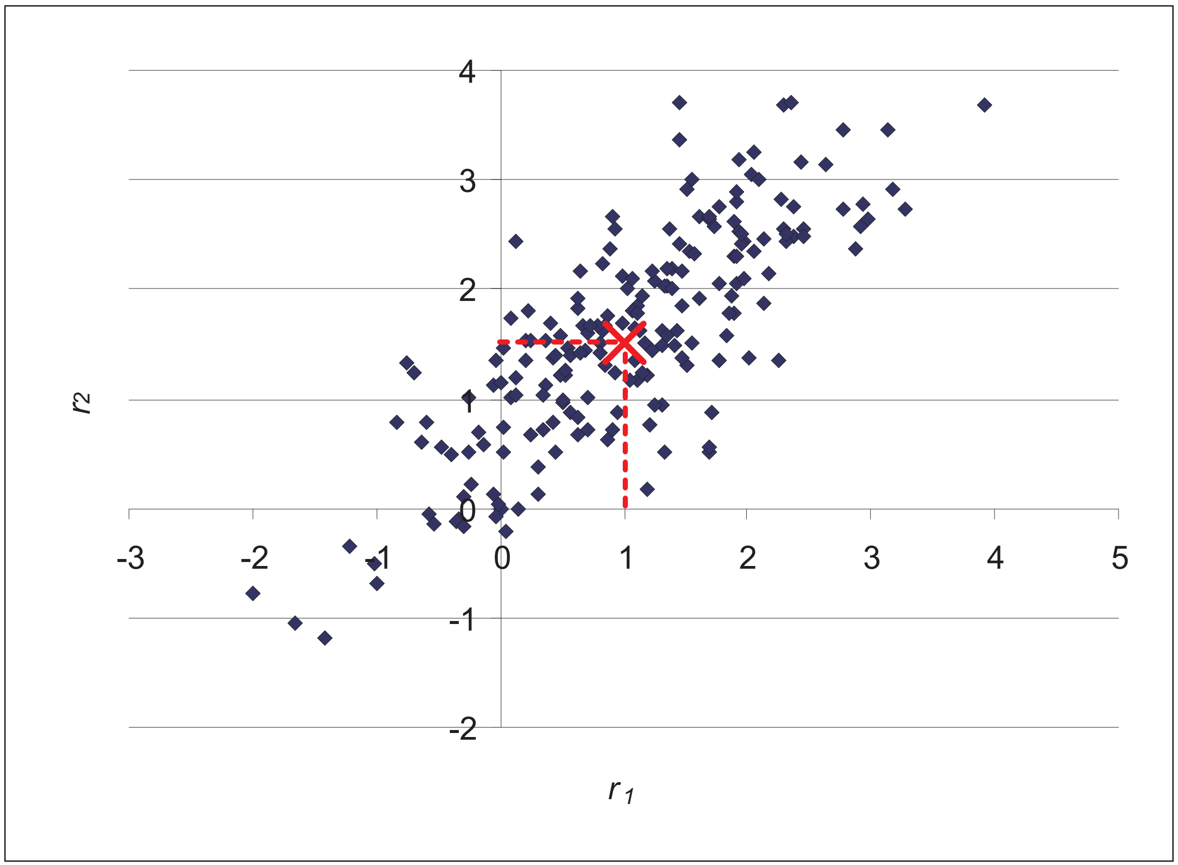


Randomness II

- Suppose there are n assets with random rates of return $r_1, r_2, \ldots, r_n$.
- These have expected values $\mathrm{E}(r_1) = \bar{r}_1, \mathrm{E}(r_2) = \bar{r}_2, \ldots$ $\ldots, \mathrm{E}(r_n) = \bar{r}_n$.
- We denote the variance of r_i by σ_i^2 and the covariance of r_i with r_j by σ_{ij} ($\Rightarrow \sigma_{ii} = \sigma_i^2$).
- Otherwise, we make no assumptions about the (joint) distribution of $r_1, r_2, \ldots, r_n$.

Randomness III

- Scatter plot of two jointly normally distributed returns r_1 and r_2 with $\bar{r}_1 = 1$, $\bar{r}_2 = 1.5$, $\sigma_1^2 = \sigma_2^2 = 1$, and $\sigma_{12} = 0.8$.



Mean and Variance of Portfolio Return

- The return of a portfolio is given by $r = \sum_{i=1}^{n} w_i r_i$.
- The expected (or mean) return of a portfolio is given by

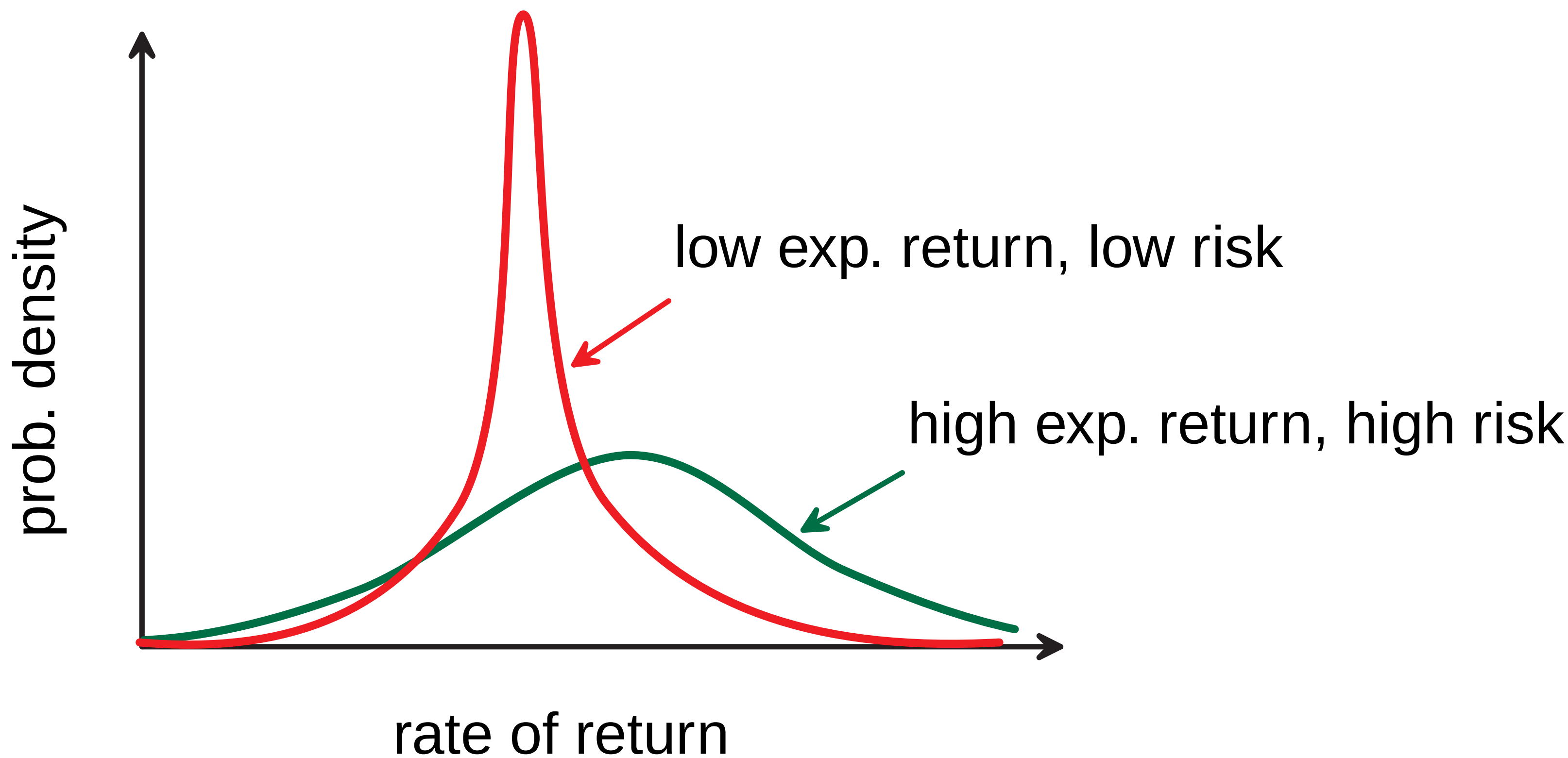
$$\bar{r} = \mathrm{E}(r) = \mathrm{E}\Big(\sum_{i=1}^{n} w_i r_i\Big) = \sum_{i=1}^{n} w_i \mathrm{E}(r_i) = \sum_{i=1}^{n} w_i \bar{r}_i \,.$$

- The variance of the return of a portfolio is given by

$$\begin{aligned} \sigma^2 &= \mathrm{var}(r) = \mathrm{E}\left[(r - \bar{r})^2\right] = \mathrm{E}\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \bar{r}_i\right)^2\right] \\ &= \mathrm{E}\left[\Big(\sum_{i=1}^{n} w_i (r_i - \bar{r}_i)\Big)\Big(\sum_{j=1}^{n} w_j (r_j - \bar{r}_j)\Big)\right] \\ &= \mathrm{E}\left[\sum_{i,j=1}^{n} w_i w_j (r_i - \bar{r}_i)(r_j - \bar{r}_j)\right] = \sum_{i,j=1}^{n} w_i \sigma_{ij} w_j \,. \end{aligned}$$

Variance as a Risk Measure

- The variance of the return can be interpreted as a measure of the risk associated with an asset/portfolio.



Diversification

- Q: Why should we form portfolios?

  A: Portfolios can reduce risk (variance) w/o sacrificing mean return.

- Example: Consider n assets with independent and identically distributed (iid) returns, that is,

$$\bar{r}_i = \bar{r} \quad \text{and} \quad \sigma_i^2 = \sigma^2 \quad \forall\, i = 1, 2, \ldots, n\,.$$

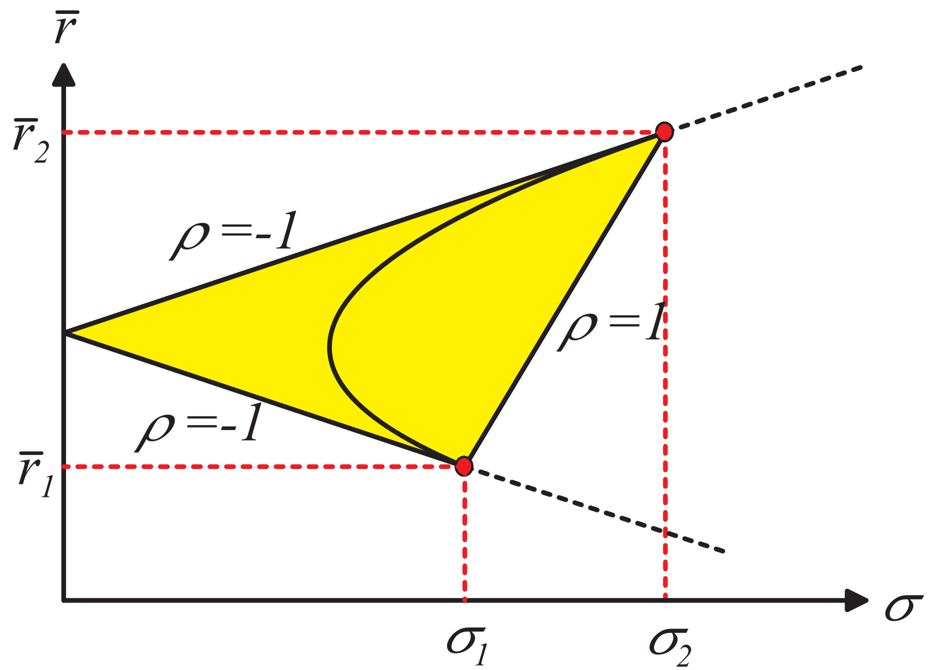
  What are the mean and variance $\bar{r}_{\mathrm{P}}$ and σ_{P}^2 of the portfolio with $w_1 = w_2 = \cdots = w_n = 1/n$?

$$\begin{aligned} \bar{r}_{\mathrm{P}} &= \textstyle\sum_{i=1}^n w_i \bar{r}_i = \sum_{i=1}^n \frac{1}{n}\bar{r} = \bar{r} \\ \sigma_{\mathrm{P}}^2 &= \textstyle\sum_{i=1}^n w_i^2 \sigma_i^2 = \sum_{i=1}^n \frac{1}{n^2}\sigma^2 = \frac{\sigma^2}{n} \end{aligned}$$

  $\Rightarrow$ Portfolios reduce risk!

Portfolio Diagrams

Two assets in a mean-standard deviation diagram:



The portfolio with $w_1 = \alpha$ and $w_2 = 1 - \alpha$ has:

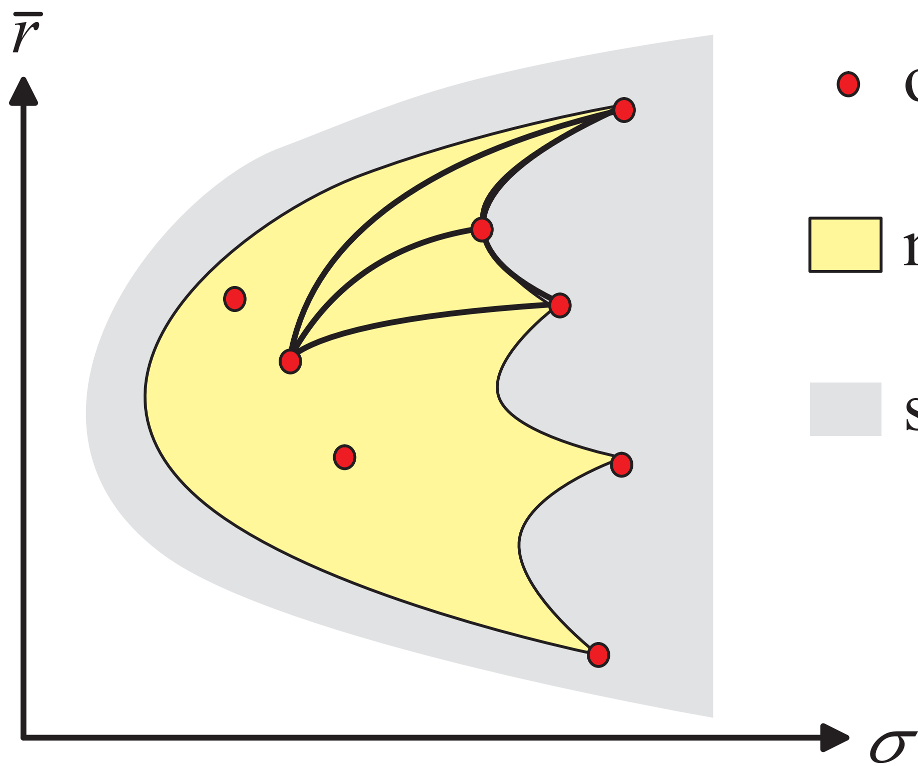
mean return: $\bar{r}_\mathrm{P} = \alpha \bar{r}_1 + (1-\alpha)\bar{r}_2$

variance: $\sigma_\mathrm{P}^2 = \alpha^2 \sigma_1^2 + 2\alpha(1-\alpha)\sigma_{12} + (1-\alpha)^2\sigma_2^2$

standard deviation: $\sigma_\mathrm{P} = \sqrt{\alpha^2\sigma_1^2 + 2\alpha(1-\alpha)\rho\sigma_1\sigma_2 + (1-\alpha)^2\sigma_2^2}$

The Feasible Set

Given n assets, what does the set of all possible portfolios look like in the $(\sigma, \bar{r})$ plane?



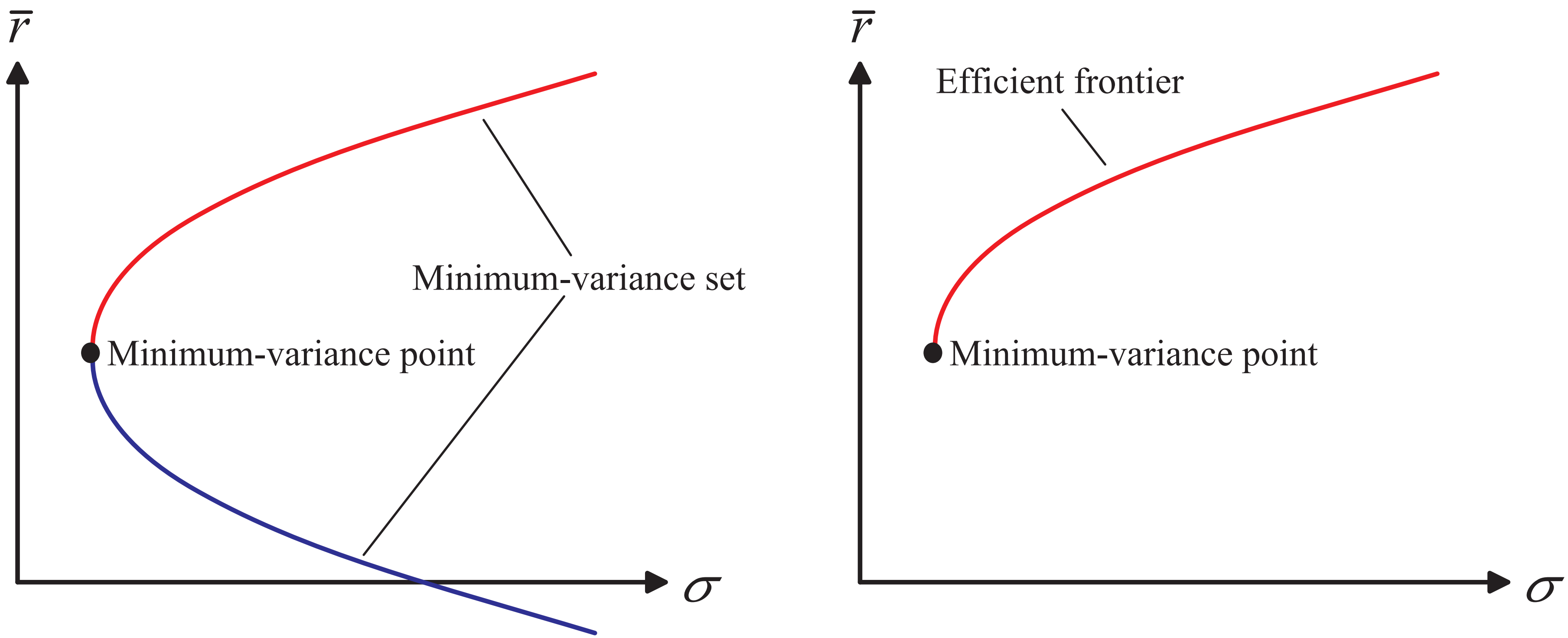
The feasible set defined with short selling allowed contains the one defined without short selling.

Minimum-Variance Set

- The left boundary of the feasible set is called the minimum variance set.
- The point with lowest possible variance is called the minimum variance point.
- Decision criteria:
  - Given 2 portfolios with the same mean return, a risk-averse investor will prefer the one with the smaller risk (variance).
  - Given 2 portfolios with the same risk (variance), a greedy investor will prefer the one with the higher mean return.

$\Rightarrow$ Only the upper half of the minimum variance set is of interest to investors. This set is termed efficient frontier.

Efficient Frontier



- The minimum-variance set is obtained by minimizing the risk/variance for any given mean return.
- The efficient frontier is the top portion of the minimum-variance set.

Harry Markowitz

- Harry Max Markowitz (born August 24, 1927) won the Nobel Prize in Economics in 1990 for his pioneering work on portfolio theory.

The Markowitz Model I

- Markowitz formulated the problem to determine the efficient frontier as a mathematical optimization problem.

- Assume there are n risky assets with
  - mean returns $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n$
  - covariances σ_{ij} for $i, j = 1, 2, \ldots, n$.

- The portfolio with weights $w_1, w_2, \ldots, w_n$ has
  - mean return $\bar{r}_{\mathrm{P}} = \sum_{i=1}^{n} w_i \bar{r}_i$
  - variance $\sigma_{\mathrm{P}}^2 = \sum_{i,j=1}^{n} w_i \sigma_{ij} w_j$.

The Markowitz Model II

$$\begin{array}{llll} \text{minimize} & \displaystyle\frac{1}{2}\sum_{i,j=1}^{n} w_i \sigma_{ij} w_j & = & \tfrac{1}{2}\sigma_{\mathrm{P}}^2 \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}_{\mathrm{P}} & = & \text{exp. return target} \\ & \displaystyle\sum_{i=1}^{n} w_i = 1 & = & \text{weights sum to } 1 \end{array}$$

- In this formulation, short selling is allowed.
- The solution of the problem depends on the return target parameter $\bar{r}_{\mathrm{P}}$.
- The minimum-variance set is obtained by plotting the minimal σ_{P}^2 for different parameter values $\bar{r}_{\mathrm{P}}$.

Solution of the Markowitz Model I

$$
\begin{array}{llcc}
\text{minimize} & \displaystyle \frac{1}{2}\sum_{i,j=1}^{n} w_i \sigma_{ij} w_j & & \text{Lagrange multipliers:} \\
\text{subject to} & \displaystyle \sum_{i=1}^{n} w_i \bar{r}_i - \bar{r}_{\mathrm{P}} = 0 & \longleftarrow & \lambda \\
& \displaystyle \sum_{i=1}^{n} w_i - 1 = 0 & \longleftarrow & \mu
\end{array}
$$

The associated Lagrangian function L is given by

$$
L = \frac{1}{2}\sum_{i,j=1}^{n} w_i \sigma_{ij} w_j - \lambda \left(\sum_{i=1}^{n} w_i \bar{r}_i - \bar{r}_{\mathrm{P}} \right) - \mu \left(\sum_{i=1}^{n} w_i - 1 \right) .
$$

Solution of the Markowitz Model II

Differentiate the Lagrangian w.r.t. $w_1, w_2, \ldots, w_n$, λ, and μ, and set all derivatives $= 0$:

$$w_i: \qquad \sum_{j=1}^{n} \sigma_{ij} w_j - \lambda \bar{r}_i - \mu = 0 \qquad \text{for } i = 1, 2, \ldots, n$$

$$\lambda: \qquad \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}_{\mathrm{P}}$$

$$\mu: \qquad \sum_{i=1}^{n} w_i = 1$$

$\Rightarrow n + 2$ equations for $n + 2$ unknowns $w_1, w_2, \ldots, w_n, \lambda, \mu$.

These equations characterize the efficient portfolios!

Vector Notation

Define

- $\boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ vector of portfolio weights;
- $\bar{\boldsymbol{r}} = (\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n) \in \mathbb{R}^n$ vector of exp. asset returns;
- $\boldsymbol{e} = (1, 1, \ldots, 1) \in \mathbb{R}^n$ vector of 1's;
- $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$ vector of 0's;
- covariance matrix of asset returns

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Markowitz Revisited

In vectorial notation, the Markowitz problem reads:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} \\ \text{subject to} & \boldsymbol{w}^\top \bar{\boldsymbol{r}} - \bar{r}_{\mathrm{P}} = 0 \\ & \boldsymbol{w}^\top \boldsymbol{e} - 1 = 0 \end{array}$$

The associated Lagrangian function can be rewritten as

$$L(\boldsymbol{w}, \lambda, \mu) = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} - \lambda \left(\boldsymbol{w}^\top \bar{\boldsymbol{r}} - \bar{r}_{\mathrm{P}} \right) - \mu \left(\boldsymbol{w}^\top \boldsymbol{e} - 1 \right) ,$$

while the optimality conditions become

$$\Sigma \boldsymbol{w} - \lambda \bar{\boldsymbol{r}} - \mu \boldsymbol{e} = \boldsymbol{0} , \quad \bar{\boldsymbol{r}}^\top \boldsymbol{w} = \bar{r}_{\mathrm{P}} \quad \text{and} \quad \boldsymbol{e}^\top \boldsymbol{w} = 1 .$$

Solution of Optimality Conditions

The optimality conditions

$$\Sigma \boldsymbol{w} - \lambda \bar{\boldsymbol{r}} - \mu \boldsymbol{e} = \boldsymbol{0}\,, \quad \bar{\boldsymbol{r}}^\top \boldsymbol{w} = \bar{r}_{\mathrm{P}} \quad \text{and} \quad \boldsymbol{e}^\top \boldsymbol{w} = 1$$

can be written as one vectorial equation

$$\begin{pmatrix} \Sigma & -\bar{\boldsymbol{r}} & -\boldsymbol{e} \\ -\bar{\boldsymbol{r}}^\top & 0 & 0 \\ -\boldsymbol{e}^\top & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ -\bar{r}_{\mathrm{P}} \\ -1 \end{pmatrix}.$$

This is solvable if Σ has full rank and $\bar{\boldsymbol{r}}$ is not a multiple of $\boldsymbol{e}$.

$$\Rightarrow \quad \begin{pmatrix} \boldsymbol{w} \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \Sigma & -\bar{\boldsymbol{r}} & -\boldsymbol{e} \\ -\bar{\boldsymbol{r}}^\top & 0 & 0 \\ -\boldsymbol{e}^\top & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{0} \\ -\bar{r}_{\mathrm{P}} \\ -1 \end{pmatrix}.$$

Markowitz Model w/o Short Selling

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{2}\sum_{i,j=1}^{n} w_i \sigma_{ij} w_j \\
\text{subject to} \quad & \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}_{\mathrm{P}} \\
& \sum_{i=1}^{n} w_i = 1 \\
& w_i \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, n
\end{aligned}
$$

- This problem cannot be reduced to the solution of a set of linear equations. It is termed a quadratic program.
- Such problems can be solved via special computer programs (use e.g. the function ‘quadprog’ in Matlab).

Parameter Estimation

- Means, variances, and covariances of the asset returns must be estimated from historical data.
- Select a basic period length p (e.g. $p = 1/12$ for monthly periods).
- For a given asset, assume that we have n samples of returns $r_1, r_2, \ldots, r_n$ corresponding to non-overlapping periods of length p.
- Assume that these returns are
  - independent and
  - identically distributed with common mean value $\bar{r}$ and variance σ^2.

Estimation of $\bar{r}$

- The best estimate $\hat{\bar{r}}$ of the (unknown) mean rate of return $\bar{r}$ is obtained by averaging the samples:

$$\hat{\bar{r}} = \frac{1}{n}\sum_{i=1}^{n} r_i \, .$$

- Note: the value $\hat{\bar{r}}$ is itself random! If we used a different set of n samples, we would obtain a different $\hat{\bar{r}}$.
- What are the mean and variance of $\hat{\bar{r}}$?
  - $\mathrm{E}(\hat{\bar{r}}) = \mathrm{E}(\frac{1}{n}\sum_{i=1}^{n} r_i) = \bar{r}$
  - $\mathrm{var}(\hat{\bar{r}}) = \mathrm{E}[(\hat{\bar{r}} - \bar{r})^2] = \mathrm{E}[\frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r})]^2 = \frac{1}{n}\sigma^2$

$\Rightarrow$ $\hat{\bar{r}}$ is an unbiased estimator for $\bar{r}$.

Estimation of σ^2 I

- An estimate $\hat{\sigma}^2$ of the (unknown) variance of the rate of return σ^2 is given by the sample variance:

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{\bar{r}})^2 \, .$$

- Note that this formula uses the sample mean $\hat{\bar{r}}$ as an input.

- The value of $\hat{\sigma}^2$ is also random!

Estimation of σ^2 II

$\hat{\sigma}^2$ is an unbiased estimator of σ^2:

$$
\begin{aligned}
\mathrm{E}(\hat{\sigma}^2) &= \mathrm{E}\left(\tfrac{1}{n-1}\sum_{i=1}^{n}[r_i - \tfrac{1}{n}\sum_{j=1}^{n} r_j]^2\right) \\
&= \mathrm{E}\left(\tfrac{1}{n-1}\sum_{i=1}^{n}[(r_i - \bar{r}) - \tfrac{1}{n}\sum_{j=1}^{n}(r_j - \bar{r})]^2\right) \\
&= \mathrm{E}\left(\tfrac{1}{n-1}\sum_{i=1}^{n}[(r_i - \bar{r})^2 - \tfrac{2}{n}\sum_{j=1}^{n}(r_i - \bar{r})(r_j - \bar{r})\right. \\
&\quad \left. + \tfrac{1}{n^2}\sum_{j,k=1}^{n}(r_j - \bar{r})(r_k - \bar{r})]\right) \\
&= \tfrac{1}{n-1}\left(\sum_{i=1}^{n}\mathrm{E}[(r_i - \bar{r})^2] - \tfrac{1}{n}\sum_{i,j=1}^{n}\mathrm{E}[(r_i - \bar{r})(r_j - \bar{r})]\right) \\
&= \tfrac{1}{n-1}\left(n\sigma^2 - \tfrac{1}{n}n\sigma^2\right) \\
&= \sigma^2
\end{aligned}
$$

If the returns are normally distributed, it can be shown that

$$\mathrm{var}(\hat{\sigma}^2) = \frac{2\sigma^4}{n-1}$$

Estimation of Covariances

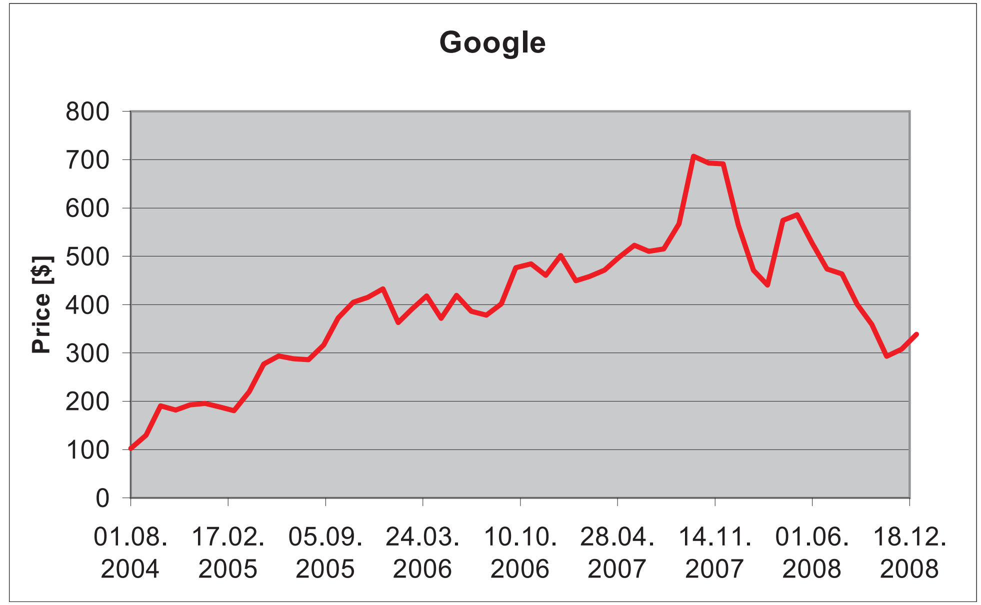
- Assume that $r_{A,1}, r_{A,2}, \ldots, r_{A,n}$ and $r_{B,1}, r_{B,2}, \ldots, r_{B,n}$ are the returns of assets A and B over non-overlapping periods of length p.
- An estimate $\hat{\sigma}_{AB}$ of the (unknown) covariance σ_{AB} is given by the sample covariance:

$$\hat{\sigma}_{AB} = \frac{1}{n-1} \sum_{i=1}^{n} (r_{A,i} - \hat{\bar{r}}_A)(r_{B,i} - \hat{\bar{r}}_B)\,.$$

- Note that this formula uses the sample means $\hat{\bar{r}}_A$ and $\hat{\bar{r}}_B$ as inputs.
- The value of $\hat{\sigma}_{AB}$ is random!
- It can be shown that $\mathrm{E}(\hat{\sigma}_{AB}) = \sigma_{AB}$.

Monthly Returns of Google Stock Price

Date	Close	Return	Squared Err.
Feb 09	340.45		
Jan 09	338.53	5.67E-03	5.97E-04
Dec-08	307.65	1.00E-01	4.94E-03
Nov 08	292.96	5.01E-02	4.02E-04
Oct-08	359.36	-1.85E-01	4.62E-02
Sep 08	400.52	-1.03E-01	1.77E-02
Aug 08	463.29	-1.35E-01	2.74E-02
Jul 08	473.75	-2.21E-02	2.72E-03
Jun 08	526.42	-1.00E-01	1.69E-02
May-08	585.8	-1.01E-01	1.73E-02
Apr 08	574.29	2.00E-02	1.01E-04
Mar-08	440.47	3.04E-01	7.49E-02
Feb 08	471.18	-6.52E-02	9.08E-03
Jan 08	564.3	-1.65E-01	3.81E-02
Dec-07	691.48	-1.84E-01	4.58E-02
Nov 07	693	-2.19E-03	1.04E-03
Oct-07	707	-1.98E-02	2.49E-03
Sep 07	567.27	2.46E-01	4.67E-02
Aug 07	515.25	1.01E-01	5.02E-03
Jul 07	510	1.03E-02	3.92E-04
Jun 07	522.7	-2.43E-02	2.96E-03



Sample average	3.01E-02
Sample variance	1.68E-02
Sample std. deviation	1.30E-01