Algorithms

Notes on Wavelet transforms

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Discrete cosine transform

- A discrete cosine transform (DCT) is a Fourier-related transform similar to the discrete Fourier transform (DFT), but using only real numbers.
- DCTs are equivalent to DFTs of roughly twice the length, operating on real data with even symmetry (since the Fourier transform of a real and even function is real and even), where in some variants the input and/or output data are shifted by half a sample.

DFT and DCT

- Recall: DFT is defined as

$$y_k = \sum_{j=0}^{n-1} a_j \omega_n^{kj}$$

or expressed in terms of trigonometric functions:

$$y_k = \sum_{j=0}^{n-1} a_j \left(\cos(2\pi kj/n) + i\sin(2\pi kj/n)\right)$$

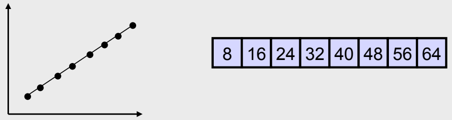
- DCT is defined only using cosine terms

$$y_k = \sum_{j=0}^{n-1} a_j \cos(2\pi kj/n)$$

Discrete cosine transform

- Like the discrete Fourier transform (DFT) the discrete cosine transforms (DCT):
  - express a function or a signal in terms of a sum of sinusoids with different frequencies and amplitudes.
  - operates on a function at a finite number of discrete data points.
- But:
  - The obvious distinction between a DCT and a DFT is that the former uses only cosine functions, while the latter uses both cosines and sines (in the form of complex exponentials).
- However, this visible difference is merely a consequence of a deeper distinction: a DCT implies different boundary conditions than the DFT or other related transforms.

Comparison between DFT and DCT



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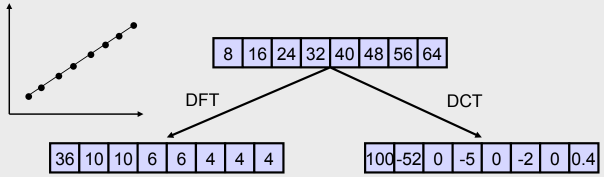

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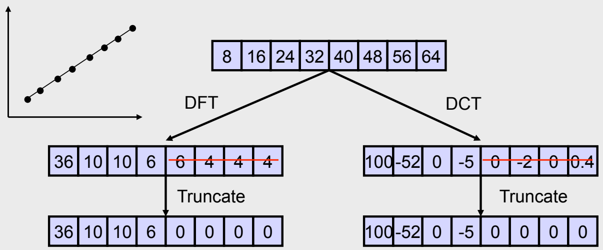

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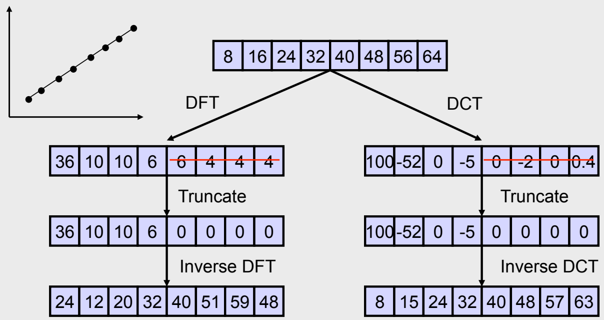

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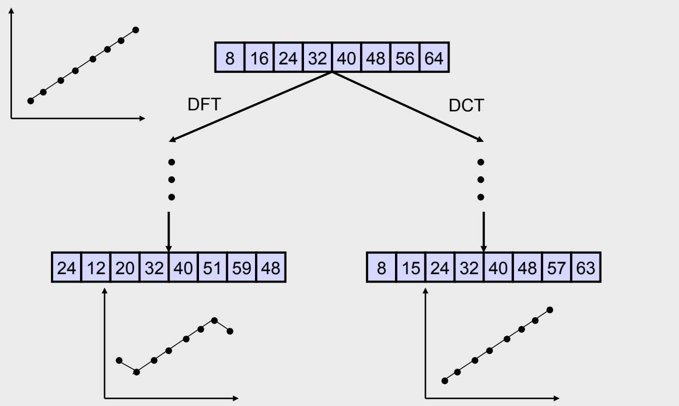

Wavelets

- Sine and cosine functions used in Fourier analysis are:
  - very smooth (infinitely differentiable)
  - very broad (nonzero almost everywhere on real line).
  - not good for representing functions that change abruptly or have highly localized support.
- Like DFTs and DCTs, wavelet transforms are
  - linear operations transforming the input vector (of length 2^n) into an output vector (of equal length)
  - invertible and orthogonal

Wavelets

- Q: What are wavelets?
- A: A mathematical tool for hierarchically decomposing function:
  - coarse representation of overall shape of the function
  - coarse-to-fine representation of details of the function
- Applications:
  - compression
  - computer graphics
  - image processing
- We will focus on a simple class of wavelets called Haar wavelets

Wavelets in one dimension

- Basic idea is to represent function in terms of basis functions
- Haar basis function is the simplest basis function

Haar wavelet transform - Example

- Assume a one-dimensional function F (e.g. signal or image) discretely sampled into y_k coefficients

[9 7 3 5]

Haar wavelet transform - Example

- Assume a one-dimensional function F (e.g. signal or image) discretely sampled into y_k coefficients

[9 7 3 5]

- Start by computing a new lower resolution image by pairwise averaging of values

[9 7 3 5]

Averaging

[8 4]

Haar wavelet transform - Example

- Averaging process destroys information. To recover this information we need some detail coefficients, in this example

[1 -1]

- Why?

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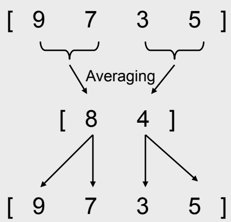

Haar wavelet transform - Example

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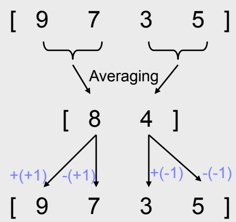

Haar wavelet transform - Example

Resolution	Averages	Detail coefficients
4	[9 7 3 5]	
2	[8 4]	[1 -1]

Haar wavelet transform - Example

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Haar wavelet transform - Example

Resolution	Averages	Detail coefficients
4	[9 7 3 5]	
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1	[6]	[2]

- Thus, wavelet transform of F is given by:

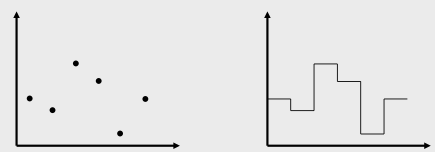
[6 2 1 -1]

Wavelet transform

- Wavelet transform is computed recursively by averaging and differencing
- No information is lost or gained, e.g. the original sampled function was represented by four values, the transformed representation also consists of four values
- Original function can be reconstructed from its wavelet transform

Some notation

- So far we have considered sampled functions (e.g. signals, images) as a set of coefficients y_k
- Alternatively think of sampled function as piecewise constant function on the interval [0,1)



Some more notation

- A function represented by a single coefficient can be viewed as a constant function on [0,1).
  - V^0 defines the corresponding vector space of all these functions

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- A function represented by two coefficients can be viewed as two constant functions on [0,1/2) and [1/2, 1).
  - V^1 defines the corresponding vector space of all these functions

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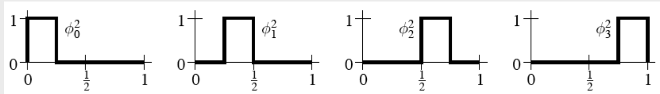
- A function represented by two coefficients can be viewed as two constant functions on [0,1/2) and [0,1/2).
  - V^1 defines the corresponding vector space of all these functions
- And so on...



Even more notation

- V^j includes all piecewise constant functions on the interval [0,1) with constant value for each of 2^j subintervals:
  - a one-dimensional signal or image with 2^j pixels is a vector in V^j
  - every vector in V^j is also contained in V^{j+1}, e.g. we can represent a piecewise constant function with two intervals as a piecewise constant function with four intervals
  - Spaces are nested: $V^0 \subset V^1 \subset V^2 \subset \cdots$
- Basis for V^j:

$$\phi_i^j(x) = \phi(2^j x - i) \quad \text{where} \quad \phi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Wavelets

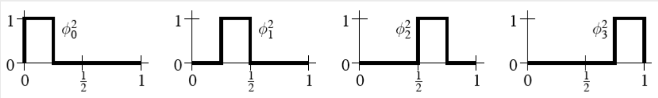
- Inner product:

$$\langle f \mid g \rangle = \int_0^1 f(x) g(x) \quad \text{for any } f, g \in V^j$$

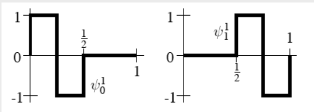
- We can define a new vector space W^j as the orthogonal complement of V^j in V^{j+1}
- Or: W^j is the space of all functions in V^{j+1} that are orthogonal to all functions V^j in under given the inner product
- Wavelets are a collection of linearly independent functions $\psi_i^j(x)$ which span W^j

Basis functions and wavelets

- Basis functions for V^2



- Haar wavelets for W^1



Wavelets

- Wavelets have a number of important properties:
  1. The basis functions $\psi_i^j(x)$ of W^j, together with the basis functions $\phi_i^j(x)$ of V^j form a basis for V^{j+1}.
  2. Every basis function $\psi_i^j(x)$ of W^j is orthogonal to every basis function $\phi_i^j(x)$ of V^j for a chosen inner product.

Haar wavelets

- For the box basis the corresponding wavelets are called Haar wavelets:

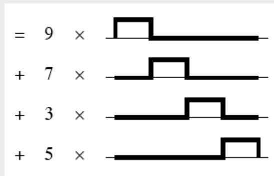
$$\psi_i^j(x) = \psi(2^j x - i)$$

$$\psi(x) = \begin{cases} +1 & \text{if } 0 \le x < 1/2 \\ -1 & \text{if } 1/2 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Example

- Original data: [9 7 3 5]
  can be expressed as a linear combination of box basis functions in V^2:

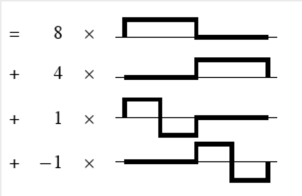
$$f(x) = c_0^2\phi_0^2(x) + c_1^2\phi_1^2(x) + c_2^2\phi_2^2(x) + c_3^2\phi_3^2(x)$$



Example

- We can rewrite this expression in terms of basis functions in V^1 and W^1:

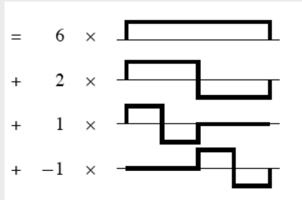
$$f(x) = c_0^1\phi_0^1(x) + c_1^1\phi_1^1(x) + d_0^1\psi_0^1(x) + d_1^1\psi_1^1(x)$$



Example

- We can rewrite this expression in terms of basis functions in V^0, W^0 and W^1:

$$f(x) = c_0^0\phi_0^0(x) + d_0^0\psi_0^0(x) + d_0^1\psi_0^1(x) + d_1^1\psi_1^1(x)$$



Normalization

- We can normalize wavelets so that

$$\langle u \mid u \rangle = 1$$

- Normalized Haar basis

$$\phi_i^j(x) = 2^{j/2}\phi(2^j x - i)$$

$$\psi_i^j(x) = 2^{j/2}\psi(2^j x - i)$$

Decomposition

```
DecompositionStep(c, n){
  for (i = 1; i <= n/2; i++){
      b[i]     = (c[2*i-1] + c[2*i])/sqrt(2);
      b[n/2+i] = (c[2*i-1] – c[2*i])/sqrt(2);
  }
  for (i = 1; i <= n; i++) c[i] = b[i];
}
Decomposition(c, n){
  for (i = 1; i <= n; i++) c[i] = c[i]/sqrt(n);
  while (n > 1){
      DecompositionStep(c, n);
      n = n/2;
  }
}
```

Reconstruction

```
ReconstructionStep(c, n){
  for (i = 1; i <= n/2; i++){
      b[2*i-1] = (c[i] + c[n/2+i])/sqrt(2);
      b[2*i]   = (c[i] - c[n/2+i])/sqrt(2);
  }
  for (i = 1; i < n; i++) c[i] = b[i];
}
Reconstruction(c, n){
  i = 2;
  while (i < n+1){
      ReconstructionStep(c, i);
      i = 2*i;
  }
  for (i = 1; i <= n; i++) c[i] = c[i]*sqrt(n);
}
```

Compression using wavelets

- Assume we have a function $f(x)$ expressed as a weighted sum of basis functions:

$$f(x) = \sum_{i=1}^{m} c_i u_i(x)$$

- The data we would like to compress are the coefficients $c_1, \ldots, c_m$
- Goal: Find a function $\tilde{f}(x)$ which approximates $f(x)$
  - requiring few coefficients
  - using a different basis (we will assume a fixed-basis)

$$\tilde{f}(x) = \sum_{i=1}^{\tilde{m}} \tilde{c}_i \tilde{u}_i(x)$$

Compression using wavelets

- Idea: Sort coefficients $c_1, \ldots, c_m$ in such a way that for every $\tilde{m} < m$ the first $\tilde{m}$ elements give the best approximation $\tilde{f}(x)$ to $f(x)$:

$$\tilde{f}(x) = \sum_{i=1}^{m} c_{\sigma(i)} u_{\sigma(i)}(x)$$

- Best approximation is defined via the L_2 metric

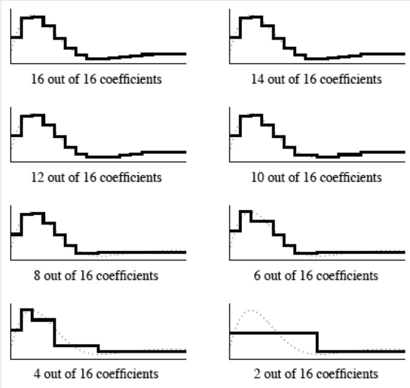
$$\left\| f(x) - \tilde{f}(x) \right\|_2^2$$

Compression using wavelets

$$\begin{aligned}
\left\| f(x) - \tilde{f}(x) \right\|_2^2 &= \left\langle f(x) - \tilde{f}(x) \middle| f(x) - \tilde{f}(x) \right\rangle \\
&= \left\langle \sum_{i=\tilde{m}+1}^{m} c_{\sigma(i)} u_{\sigma(i)}(x) \middle| \sum_{j=\tilde{m}+1}^{m} c_{\sigma(j)} u_{\sigma(j)}(x) \right\rangle \\
&= \sum_{i=\tilde{m}+1}^{m} \sum_{j=\tilde{m}+1}^{m} c_{\sigma(i)} c_{\sigma(j)} \left\langle u_{\sigma(i)} \middle| u_{\sigma(j)} \right\rangle \\
&= \sum_{j=\tilde{m}+1}^{m} \left(c_{\sigma(i)} \right)^2
\end{aligned}$$

- Thus, the best choice for σ is the permutation that sorts the coefficients in decreasing magnitude, e.g. $\left|c_{\sigma(1)}\right| \geq \cdots \geq \left|c_{\sigma(m)}\right|$

Compression using wavelets: Example



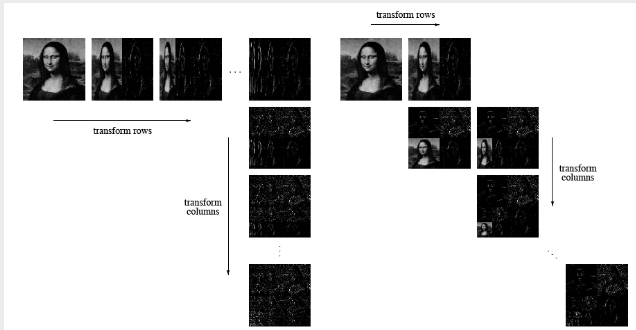
2D wavelet transforms: Method 1

```
StandardDecomposition(c, w, h){
  for (i = 1; i <= h; i++)
      Decomposition(row(c, i));
  for (i = 1; i <= w; i++)
      Decomposition(column(c, i));
}
```

2D wavelet transforms: Method 2

```
NonStandardDecomposition(c, n){
  for (i = 1; i <= n; i++){
      c[i] = c[i]/n;
  }
  while (n > 1){
      for (i = 0; i <= n; i++)
           DecompositionStep(row(c, i));
      for (i = 0; i <= n; i++)
           DecompositionStep(column(c, i));
      n = n/2;
  }
}
```

2D wavelet transforms



Standard decomposition

Non-standard decomposition

Image compression using wavelets

- Compute coefficients $c_1,\ldots,c_m$ representing an image in a normalized two-dimensional Haar basis.
- Sort the coefficients in order of decreasing magnitude to produce the sequence $c_{\sigma(1)},\ldots,c_{\sigma(m)}$.
- Starting with $\tilde{m} = m$, find the smallest $\tilde{m}$ for which $\sum_{i=\tilde{m}+1}^{m} \left(c_{\sigma(i)}\right)^2 \le \varepsilon^2$ where ε is the allowable L_2 error.