In order to obtain a limit cycle and thus oscillatory behavior, we look for parameters at which a Hopf bifurcation occurs. Such a parametrization must produce a pair of purely imaginary complex conjugate eigenvalues for the linearizaton matrix around the fixed point.

The original equations are

$$
\begin{aligned}
\tau_{E1}\frac{dr_E}{dt} &= -r_E + F_E(I_E) \\
\tau_{I1}\frac{dr_I}{dt} &= -r_I + F_I(I_I) \\
\frac{dS_E}{dt} &= -\frac{S_E}{\tau_E} + (1-S_E)\alpha r_E \\
\frac{dS_I}{dt} &= -\frac{S_I}{\tau_I} + r_I \\
F_E(I_E) &= \frac{c_E I_E - I_{th(E)}}{1 - e^{-g(c_E I_E - I_{th(E)})} + \frac{\tau_{re}}{1000}(c_E I_E - I_{th(E)})} \\
F_I(I_I) &= \frac{c_I I_I - I_{th(I)}}{1 - e^{-g(c_I I_I - I_{th(I)})} + \frac{\tau_{ri}}{1000}(c_I I_I - I_{th(I)})} \\
I_E &= J_{EE}S_E - J_{IE}S_I + I_{stim} + I_o \\
I_I &= J_{EI}S_E - J_{II}S_I + I_o
\end{aligned}
$$

Assuming the firing dynamics are fast, τ_{E1} and τ_{I1} are small, thus

$$
\begin{aligned}
\frac{dS_E}{dt} &= -\frac{S_E}{\tau_E} + (1-S_E)\alpha F_E(I_E) \\
\frac{dS_I}{dt} &= -\frac{S_I}{\tau_I} + F_I(I_I)
\end{aligned}
$$

The Jacobian matrix

$$
\begin{pmatrix}
-\frac{1}{\tau_E} - \alpha F_E(I_E) + \alpha(1-S_E)\frac{\partial F_E(I_E)}{\partial S_E} & \alpha(1-S_E)\frac{\partial F_E(I_E)}{\partial S_I} \\
\frac{\partial F_I(I_I)}{\partial S_E} & -\frac{1}{\tau_I} + \frac{\partial F_I(I_I)}{\partial S_I}
\end{pmatrix}
$$

where

$$
\frac{\partial F_E}{\partial S_I} = -\frac{(-I_{th(E)} + c_E(I_o + I_{stim} + J_{EE}S_E - J_{IE}S_I))(c_E e^{-g_E(-I_{th(E)} + c_E(I_o + I_{stim} + J_{EE}S_E - J_{IE}S_I))} g_E J_{EE} + \frac{c_E J_{EE}\tau_{re}}{1000}}{(1 - e^{-g_E(-I_{th(E)} + c_E(I_o + I_{stim} + J_{EE}S_E - J_{IE}S_I))} + (-I_{th(E)} + c_E(I_o + I_{stim} + J_{EE}S_E - J_{IE}S_I))\frac{\tau_{re}}{1000})^2} + \frac{c_E J_{EE}}{1 - e^{-g_E(-I_{th(E)} + c_E(I_o + I_{stim} + J_{EE}S_E - J_{IE}S_I))} + (-I_{th(E)} + c_E(I_o + I_{stim} + J_{EE}S_E - J_{IE}S_I))\frac{\tau_{re}}{1000}}
$$

$$\frac{\partial F_E}{\partial S_I} = -\frac{((-I_{th(I)} + c_I(I_o + J_{EI}S_E - J_{II}S_I))(-c_I e^{-g_I(-I_{th(I)}+c_I(I_o+J_{EI}S_E-J_{II}S_I))} g_I J_{II} - \frac{c_I J_{II}\tau_{ri}}{1000}))}{(1 - e^{-g_I(-I_{th(I)}+c_I(I_o+J_{EI}S_E-J_{II}S_I))} + ((-I_{th(I)} + c_I(I_o + J_{EI}S_E - J_{II}S_I))\frac{\tau_{ri}}{1000})^2} - \frac{c_I J_{II}}{1 - e^{-g_I(-I_{th(I)}+c_I(I_o+J_{EI}S_E-J_{II}S_I))} + ((-I_{th(I)} + c_I(I_o + J_{EI}S_E - J_{II}S_I))\frac{\tau_{ri}}{1000}}$$

The characteristic polynomial of the Jacobian matrix is

$$(-\frac{1}{\tau_E} - \alpha F_E + \alpha(1 - S_E)\frac{\partial F_E}{\partial S_E} - \lambda)(-\frac{1}{\tau_I} + \frac{\partial F_I}{\partial S_I} - \lambda) + \frac{\partial F_I}{\partial S_E}(\alpha(1 - S_E)\frac{\partial F_E(I_E)}{\partial S_I})$$

The root of which gives the eigenvalues. We then substitute for S_E and S_I at steady state, where both $\frac{dS_E}{dt}$ and $\frac{dS_I}{dt}$ are zero:

$$S_E = \frac{\alpha F_E(I_E)}{\frac{1}{\tau_E} + \alpha F_E(I_E)}$$

$$S_I = \tau_I F_I(I_I)$$

Therefore, any (S_E, S_I) pair at steady state that satisfies the condition that the roots of the characteristic polynomial are a pair of purely imaginary complex conjugates gives rise to a Hopf bifurcation.