Remarks on a rumor propagation model

1 author:

Alberto Ragagnin
University of Udine

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University of Udine

Department of Mathematics, Computer Science and Physics

Via delle Scienze 206, 33100 Udine, Italy

e-mail: alberto.ragagnin.85@gmail.com

Abstract

This short note contains a few comments and corrections about some recent models for the spread of rumors in a population. We consider a system of ordinary differential equations which describes the evolution of Ignorant-Spreaders-Stiflers in time. State of the art of analytical understanding of those equations is based on studying asymptotic solutions of the rumor spreading equations. In this work we find a First Integral of these differential equations. We qualitatively discuss the evolution of the system in the light of those new more precise solutions.

1 Introduction

The study of the propagation of rumors in a population has become a research topic of increasing interest in the recent years. Motivations for these investigations come from different perspectives, such as social sciences, economy, informatics and military interests.

The first rumor propagation models considered in the literature have been adapted from the famous SIR model by Kermack-McKendrik, whose history is well described in [1]. However, in modelling rumor propagation, the mechanism differs basically from that governing the spread of an epidemic. Interesting pioneering works were made by Daley and Kendall [3] who proposed a stochastic, random-walk, model and by Maki and Thompson [7] who treat a deterministic discrete model. In [7] the authors did not write down explicitly the assumptions for a continuous-time model, but left this task as an exercise for the reader [7, Ch. 9, p.388]. From this point of view, the sources that we have found in literature are somehow a little confusing, since usually one refers to as Maki-Thompson models also models based on differential equations.

In this paper we focus our attention on two differential equation models investigated by J.R. Piqueira in the recent article [9] and S. Belen and C. Pearce in [2], respectively.

The set of population could be partitioned into three subsets of sub-populations: the *I-Ignorant*, namely the individuals who ignore the rumor (who play the same role as the susceptible of the SIR model), the *S-Spreaders* who disseminate the rumor (and play the same role as the infected of the SIR model), and the *R-Stiflers* who do not spread the rumor after receiving it (who play a similar role as the recovered of the SIR model).

2 Remarks on Piqueira's Model

We have obtained a different result and a different conclusion for the dynamical system proposed in the original paper [9] by J.R. Piqueira. Let us briefly recall Piqueira's model. The functions $I(t)$, $S(t)$ and $R(t)$ are continuously differentiable and represent the number of individuals of the three sub-populations at the time t. We suppose that along all the time interval in which we study the model, the total population is constant, that is

$$I(t) + S(t) + R(t) = N, \quad \forall\, t. \tag{2.1}$$

It will be not restrictive to suppose $N = 1$. The dynamics of the triplet $(I(t), S(t), R(t))$ is supposed to be governed by the nonlinear ODE:

$$\begin{cases} I' = -\rho_2 \mu I S \\ S' = \rho_2 \mu I S - \rho_1 \mu S(S + R) \\ R' = \rho_1 \mu S(S + R) \end{cases} \tag{2.2}$$

where the positive parameters ρ_1, ρ_2 and μ are assigned as follows: ρ_1 is the probability that a Spreader meets another Spreader causing their silencing, ρ_2 is the probability that an Ignorant converts into a Spreader after heard the rumor and μ is the average number of contacts for every individual (see

[9]). According to [9], this system is inspired by the Daley-Kendall model, following also [8]. Concerning equation (2.2), we notice that the parameter μ is practically useless from the point of view of the *qualitative analysis* and, therefore, we could omit it by posing $\mu = 1$. Accordingly, system (2.2) takes the form

$$\begin{cases} I' = -\rho_2 IS \\ S' = S(\rho_2 I - \rho_1(S+R)) \\ R' = \rho_1 S(S+R) \end{cases} \tag{2.3}$$

2.1 Comparison with Asymptotic Stability

From now on, we will focus our attention to the qualitative study of the trajectories for which numerical simulations show a behavior like that described in Figure 1, below.

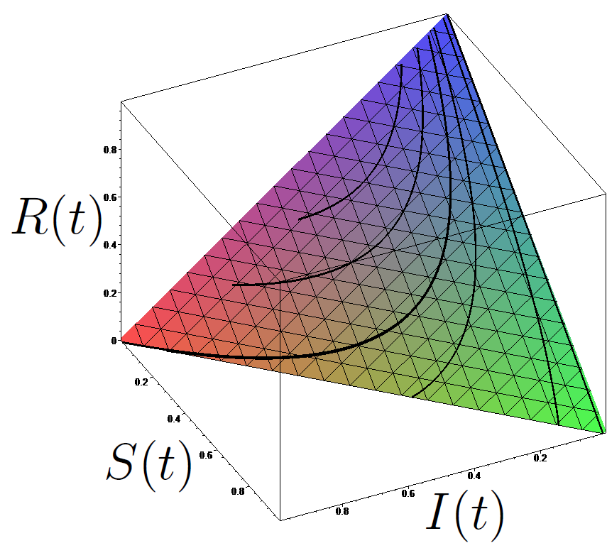

Figure 1: Typical behavior of the trajectories of system (2.2) for different initial values. Notice that in this model the population of the spreaders tends toward the extinction. The dynamics under the constraint $I + S + R = 1$.

First of all, we look for the equilibrium points, that is the zeros of the vector field

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(I,S,R) = (\rho_2 IS, \rho_2 IS - \rho_1 S(S+R), \rho_1 S(S+R)).$$

We also denote by ∇f the corresponding Jacobian matrix [1] which is defined as

$$\nabla f(I,S,R) = \begin{pmatrix} \frac{\partial I'}{\partial I} & \frac{\partial I'}{\partial S} & \frac{\partial I'}{\partial R} \\ \frac{\partial S'}{\partial I} & \frac{\partial S'}{\partial S} & \frac{\partial S'}{\partial R} \\ \frac{\partial R'}{\partial I} & \frac{\partial R'}{\partial S} & \frac{\partial R'}{\partial R} \end{pmatrix} = \begin{pmatrix} -\rho_2 & 0 & 0 \\ \rho_2 S & \rho_2 I + 2\rho_1 S - \rho_1 R & -\rho_1 S \\ 0 & 2\rho_1 S + \rho_1 R & \rho_1 S \end{pmatrix}.$$

We restrict the study of the vector field to the domain

$$D := \{(I,S,R) : I,S,R \geq 0,\ I+S+R = 1\}.$$

By the nature of the constraints defining the domani D, the only possible equilibrium points are with $S = 0$ and then $f(I,0,R) = 0$ when $I + R = 1$. As shown by Figures 1, 2 and 3, we have the extinction of the Spreaders when the time tends to the $+\infty$ since the trajectories tends to equilibrium points which stay on the hyperplane $I + R = 1$.

[1] We have fixed a minor misprint found in the original paper [9, Ch. 2, p.3], where the variable S is missing in system (2.2) and there is a wrong column in the Jacobian matrix.

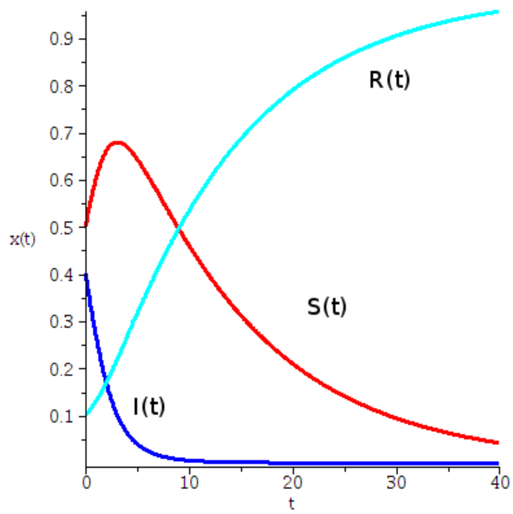

Figure 2: The behavior of the three populations for $\rho_1 = 0.1$, $\rho_2 = 0.9$, $\mu = 0.8$ and initial data $I(0) = 0.4$, $S(0) = 0.5$, $R(0) = 0.1$. The blue, red and cyan lines represent, respectively, the Ignorant, Spreaders and Stiflers at the time t.

If we compute the Jacobian on these equilibrium points, we obtain:

$$\nabla f(I, 0, R) = \begin{pmatrix} -\rho_2 & 0 & 0 \\ 0 & \rho_2 I - \rho_1 R & 0 \\ 0 & \rho_1 R & 0 \end{pmatrix} \tag{2.4}$$

The corresponding eigenvalues are: $\lambda_1 = 0, \lambda_2 = -\rho_2\mu, \lambda_3 = \rho_2\mu I - \rho_1\mu R$. Thus we conclude that λ_2 is always negative, while $\lambda_3 < 0$ if and only if $(\rho_1 + \rho_2)I - \rho_1 < 0$, that is

$$I < \sigma := \frac{\rho_1}{\rho_1 + \rho_2}.$$

As observed in [9], the constant σ plays the role of a threshold. The conclusion in [9, p.3] is the following

(i) if $0 < I < \sigma$, the equilibrium point is asymptotically stable;

(ii) if $\sigma < I < 1$, the equilibrium point is unstable.

Unfortunately, the conclusion in (i) does not seem completely correct [cf. Figures 4 and 5]. We recall that an equilibrium point P is *asymptotically*

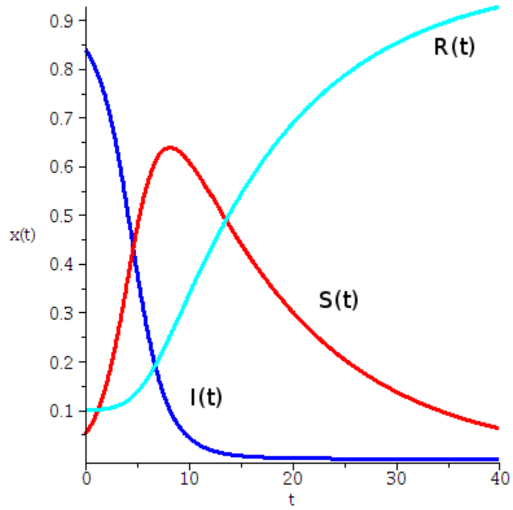

Figure 3: The behavior of the three populations for $\rho_1 = 0.1$, $\rho_2 = 0.9$, $\mu = 0.8$ and initial data $I(0) = 0.84$, $S(0) = 0.05$, $R(0) = 0.1$. The blue, red and cyan lines represent, respectively, the Ignorant, Spreaders and Stiflers at the time t.

stable if it is stable and there is a neighborhood $\mathcal{U}$ of P such that for each point z_0 in $\mathcal{U}$ the solution departing from z_0 at the time $t = 0$ tends to P as $t \to +\infty$. We claim that the correct conclusion for (i) would be that for $0 < I < \sigma$, the equilibrium point is only stable.

2.2 First Integral

A search for stable, non asymptotically, solutions lead us to the search of a First Integral. By here we show how to proceed with another method that permits to simplify the analysis and also gives, to our opinion, a better explanation of the results. Using condition (2.1) in the normalized form with $N = 1$, we can set

$$S(t) = 1 - I(t) - R(t)$$

and remove the second equation from system (2.3). In this manner, the original system can be downgraded to a planar system in the two variables $I(t)$ and $R(t)$ that we write as

$$\begin{cases} R' = \rho_1(1-I-R)(1-I) \\ I' = -\rho_2 I(1-I-R). \end{cases} \tag{2.5}$$

We denote by $g(R,I)$ the corresponding vector field related to (2.5). The analysis of system (2.5) will be performed in the set Ω, defined as

$$\Omega := \{(R,I) : 0 \le R \le 1, 0 \le I \le 1, R+I \le 1\}.$$

A simple investigation of the vector field on the boundary of Ω shows that on the segment $\{(R,0) : 0 \le R \le 1\}$, we have $R' \ge 0$ and $I' = 0$, while, on the segment $\{(0,I) : 0 \le I \le 1\}$. we have $R' \ge 0$ and $I' = -\rho_2 I(1-I)$. Furthermore, all the points of the segment

$$\mathscr{S} := \{(R,I) : 0 \le R \le 1, 0 \le I \le 1, R+I = 1\}$$

are equilibrium points. By the uniqueness of the solutions for the initial value problems associated to (2.5), we conclude that the interior of Ω is a positively invariant set, that is $(I(t),R(t)) \in \mathrm{int}\Omega$ for all $t \ge 0$, whenever $(I(0),R(0)) \in \mathrm{int}\Omega$. The special feature of system (2.5) is that there is a *continuum* of equilibrium points for the equation. Indeed, as previously observed, the set of equilibrium points contained in the domain Ω is given by the segment $\mathscr{S}$. Clearly, such points may be stable or unstable, but they can never be asymptotically stable (since any neighborhood of an equilibrium point contains infinitely many other equilibria, that is constant solutions).
We consider now the Jacobian matrix ∇g associated to the two dimensional vector field g and computed at a generic equilibrium point such that $R+I = 1$. A standard computation yields to the following

$$\nabla g(R,I) = \begin{pmatrix} \rho_1 I - \rho_1 & \rho_1 I - \rho_1 \\ \rho_2 I & \rho_2 I \end{pmatrix}$$

Clearly, one of the eigenvalues is zero (this is obvious). The other one is give by

$$\tau := (\rho_1 + \rho_2)I - \rho_1.$$

Thus we get the conclusion that if $0 < I < \sigma$ then equilibrium point (R,I) is stable, while, if $\sigma < I < 1$ the equilibrium point (R,I) is unstable.

In order to provide a more precise description of the global dynamics, we observe that system (2.5) possess a first integral, which can be found via the following steps. First of all, we write equation (2.5) as

$$\begin{cases} \frac{R'}{\rho_1} = (1-I-R)(1-I) \\ \frac{I'}{I\rho_2} = -(1-I-R), \end{cases}$$

then, we multiply by $1-I)$ the second equation and obtain

$$\begin{cases} \frac{R'}{\rho_1} = (1-I-R)(1-I) \\ \frac{I'}{I\rho_2}(1-I) = -(1-I-R)(1-I). \end{cases}$$

Finally, summing up the two equations, we obtain $\frac{I'}{I\rho_2}(1-I) + \frac{R'}{\rho_1} = 0$, from which we find that

$$\frac{d}{dt}\mathcal{H}(R(t), I(t)) = 0, \qquad \text{for } \mathcal{H}(R, I) := \frac{R}{\rho_1} + \frac{log(I)}{\rho_2} - \frac{I}{\rho_2}. \tag{2.6}$$

. We thus conclude that the solutions of (2.5) lie on the level lines of the Hamiltonian function $\mathcal{H}$, that is any solution satisfied the relation

$$\frac{R(t)}{\rho_1} + \frac{log(I(t))}{\rho_2} - \frac{I(t)}{\rho_2} = k = \mathcal{H}(R(0), I(0)), \quad \forall t.$$

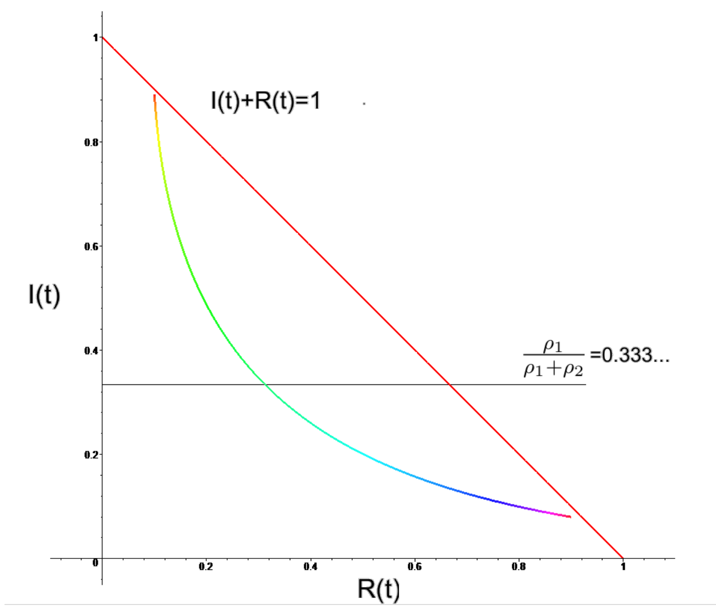

Figure 4: The present figure shows the effect of the threshold value σ. The equilibrium points above the line $I = \sigma$ are of repulsive type. The trajectories move from above σ and tend asymptotically to some point below the level σ. Lying on the level line of the first integral. The simulation has been performed for $\rho_1 = 0.4$, $\rho_2 = 0.8$ and $\mu = 1$.

2.3 Other Remarks

The approach we have described above in the study of Piqueira's model, can be easily adapted to investigate other rumor transmission models considered by different authors. For instance, we can apply our considerations to a model by Belen and Pearce in [2], where the Authors introduced the differential system:

$$\begin{cases} I' = -IS \\ S' = -S(1-2I) \\ R' = S(1-I) \end{cases}$$

With $I(0) = \alpha$, $S(0) = \beta$, $R(0) = \gamma$, $\alpha + \beta + \gamma = 1$, $\alpha, \beta > 0$ and $\gamma \geq 0$. We observe that is possible remove the equation $R' = S(1-I)$ and study

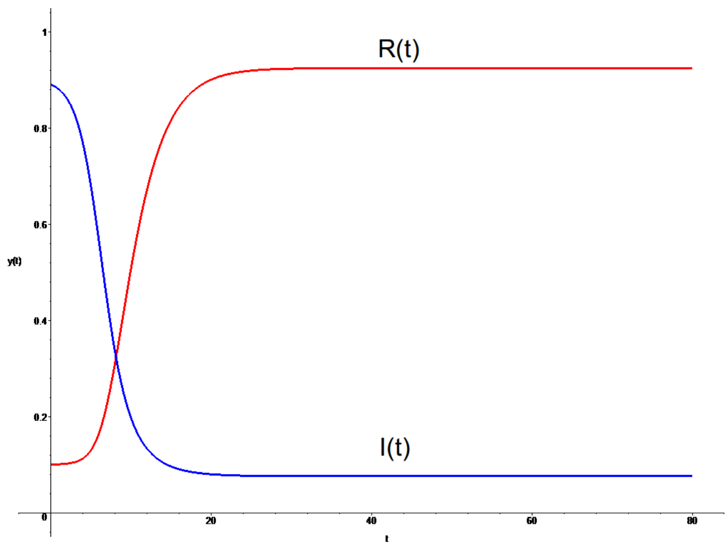

Figure 5: It the same setting of the previous figure, we show the behavior of the solutions for initial data $I(0) = 0.89$ and $R(0) = 0.11$. The blue line represents the Ignorant at the time t and the red line represents the Stiflers at the time t. The simulation shows that the two populations quickly stabilize in time.

directly the planar system

$$\begin{cases} I' = -IS \\ S' = -S(1-2I). \end{cases}$$

Passing to the equivalent system

$$\begin{cases} \frac{I'}{I} = -S \\ \frac{S'}{S} = (1-2I), \end{cases}$$

we can easily find a first integral of the form $\mathcal{H}(I, S) := \log(I) - 2I + S$.

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