Numerical Methods - Integration

Supporting Code Available At: https://tinyurl.com/NumericalInt

Differential equations are the language of physics and engineering. We often assume the functions we work with are differentiable and integrable.

However, we have already seen that not all functions are easily integrable. Some functions cannot be analytically integrated at all. Sometimes we may need to integrate data without knowing the underlying function! While these problems cannot easily be solved analytical (if at all), they can be solved numerically.

Our strategy will be as follows:

1. Evaluate the function on a grid of points. (That is, if we know the function in question. If we are dealing with a set of measured data, then the sampling rate of the data provides our grid of points.)
2. Divide the grid into a set of small chunks
3. For each chunk, determine the polynomial that passes through each point
4. Integrate this polynomial for each chunk
5. Sum over all the chunks to compute the integral over the full range

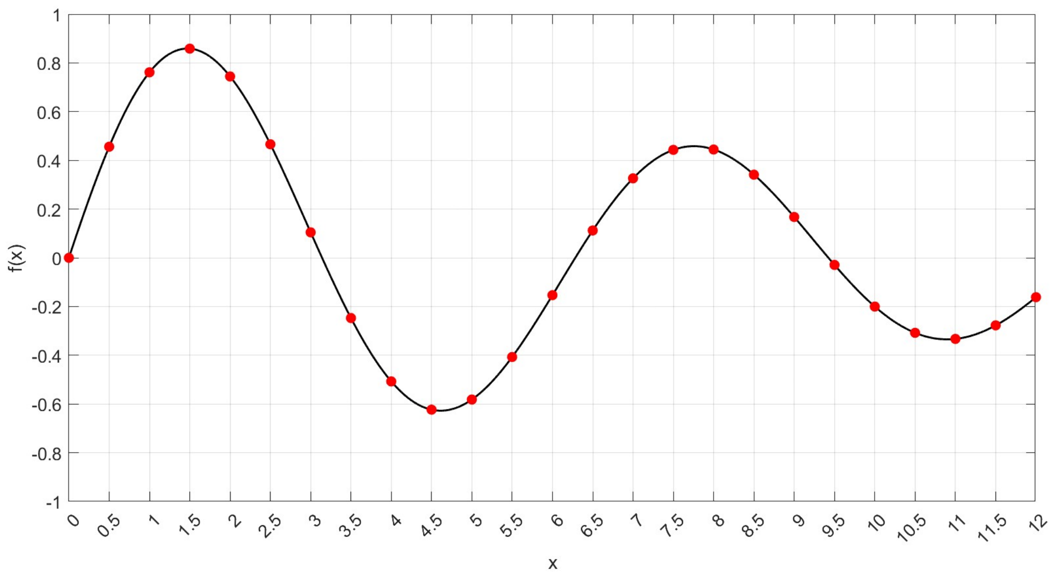
The precision of our calculation will depend on the grid of points and polynomial function used. We will see that using 0th and 1st order polynomials corresponds to a Reimann sum and the trapezoid rule, respectively. Using higher-order polynomials results in a more accurate approximation of the integral.

Learning objectives: After this lecture, you will be able to...

1. Recall the definition of an integral and apply this definition to numerically integrate an unknown function using the Riemann sum and trapezoid rule
2. Derive the trapezoid rule using a linear approximation
3. Numerically compute the integral using higher-order polynomials
4. Estimate the error associated with these numerical techniques
5. Implement these numerical techniques in MATLAB

Activity 1: Recall - The Definition of the Integral

Consider the below plot of the function $f(x)$. What feature of the curve is described by the integral?



Given this definition, can you think of a quick way to numerically approximate the integral over the interval shown? Write an equation for your approximation in terms of the points x_i and $f(x_i)$.

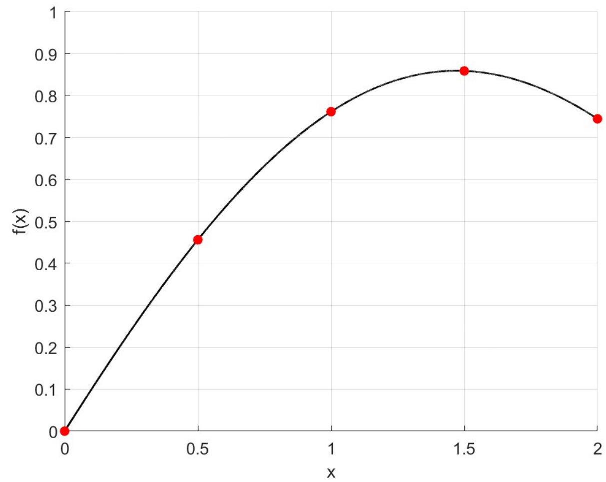
Bonus: If time permits, load the provided MATLAB file and implement your technique to integrate the function provided. How does your approximation compare to the analytical solution?

Activity 2: Use a Linear Approximation

In the previous example, we divided the area under the curve into thin slices, and then approximated the area of each slice using a rectangle or a trapezoid. This is a nice starting point, especially because it's easy to find the area of those shapes.

However, there's another (entirely equivalent) to think about this. Instead of approximating the area of a slice, we can approximate the *function* itself over each intervals using a polynomial, and then integrate this polynomial.

Note that we did this implicitly in the previous activity - what function did you use to approximate the function?



Now, let's try this using a line.

1. Consider points x_0 and x_1, which have corresponding function values $f(x_0)$ and $f(x_1)$. Write the equation of the line which connects these points.

2. Compute the integral of this linear approximation. This describes the approximate area under the curve between x_0 and x_1.

3. Finally, generalize your result to arbitrary points x_i and x_{i+1}. Write an expression to find the integral over the full interval.

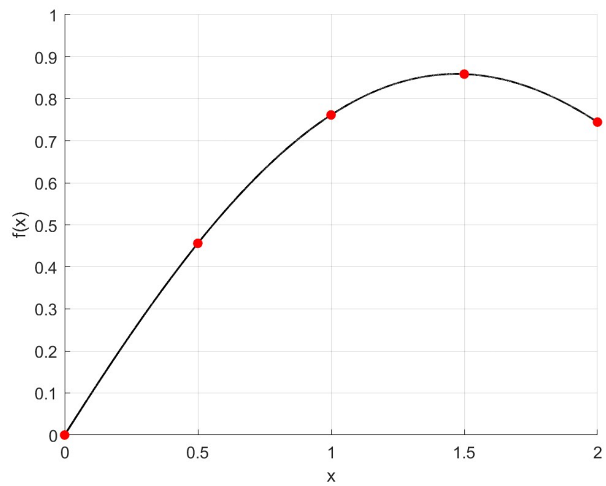
4. Compare your result with the method discussed previously. How do the two methods compare?

Bonus: Estimate the error associated with this approximation. It may help to represent the integrand useing a Taylor series.

Activity 3: Using a Higher-Order Approximation

The previous technique easily generalizes to use higher-order polynomials. Using higher-order polynomials allows us better approximate the curvature of the integrand, resulting (hopefully) in a better approximation.

Let's repeat the previous process, this time approximating each section of the function using a quadratic curve.



1. Consider points x_0, x_1, and x_2, which have corresponding function values $f(x_0)$, $f(x_1)$, and $f(x_2)$. Write the equation of the quadratic which connects these points.

2. Compute the integral of this quadratic approximation. This describes the approximate area under the curve between x_0 and x_2.

3. Finally, generalize your result to arbitrary points x_i, x_{i+1}, and x_{i+2}. Write an expression to find the integral over the full interval.

Bonus: If time permits, implement this technique in MATLAB on the function provided. How does this accuracy of this approximation compare to the earlier techniques?

Another bonus: Increasing the order of our polynomial improves the accuracy of our approximation. Should we use an arbitrarily high-order polynomial to arbitrarily increase the accuracy of our integral? Why or why not?

One-Minute Paper

Tear this off and submit it before leaving today.

Your Name: ____________________________________

Name of people you worked with today:

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- Do you have any questions that came up during today's class that we didn't cover?

- What single topic left you **most confused** after today's class?

- Any other comments or reflections on today's class?

- Draw a short cartoon about what you learned today (if you'd like)