Aérodynamique II
Ecoulements compressibles

Partie II

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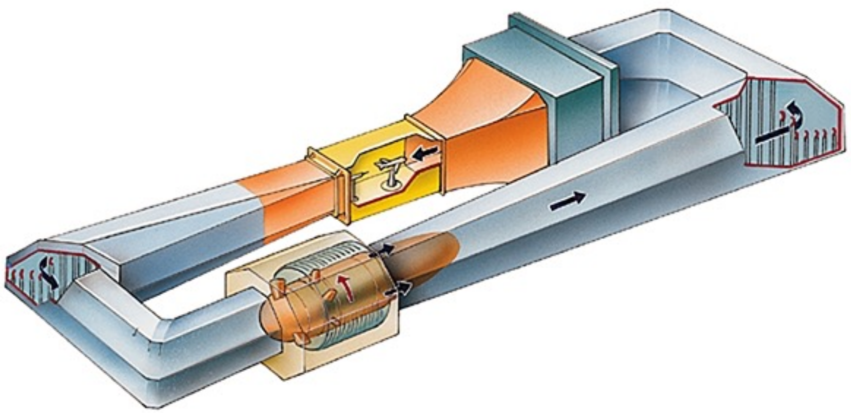
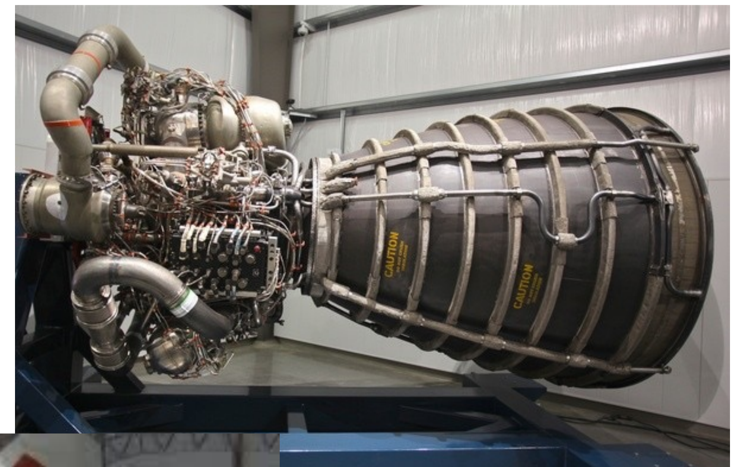
Internal aerodynamics

▷ Internal aerodynamics is dedicated to the study of fluid flows in ducts.

▷ <u>Applications</u>: wind tunnel (subsonic and supersonic), rocket engine, air intake.



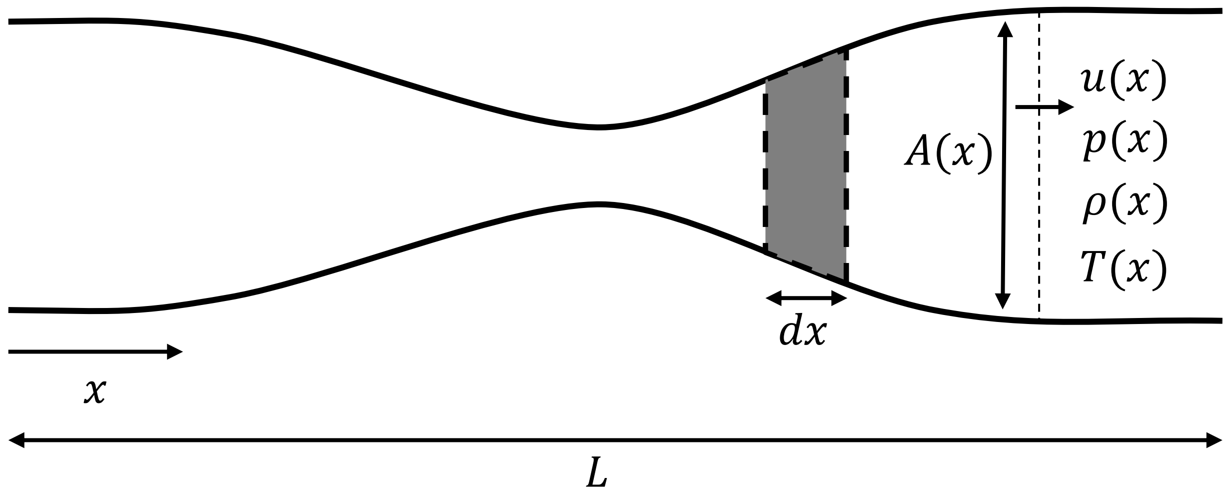


▷ Tools for the calculation of the mass flow rate q_m through a duct, the velocity/Mach number, the static and stagnation pressure along the duct from known upstream and downstream conditions.

1. Conservation laws for quasi-1D flow

Quasi-monodimensional hypothesis

▷ Compressible inviscid steady adiabatic flow



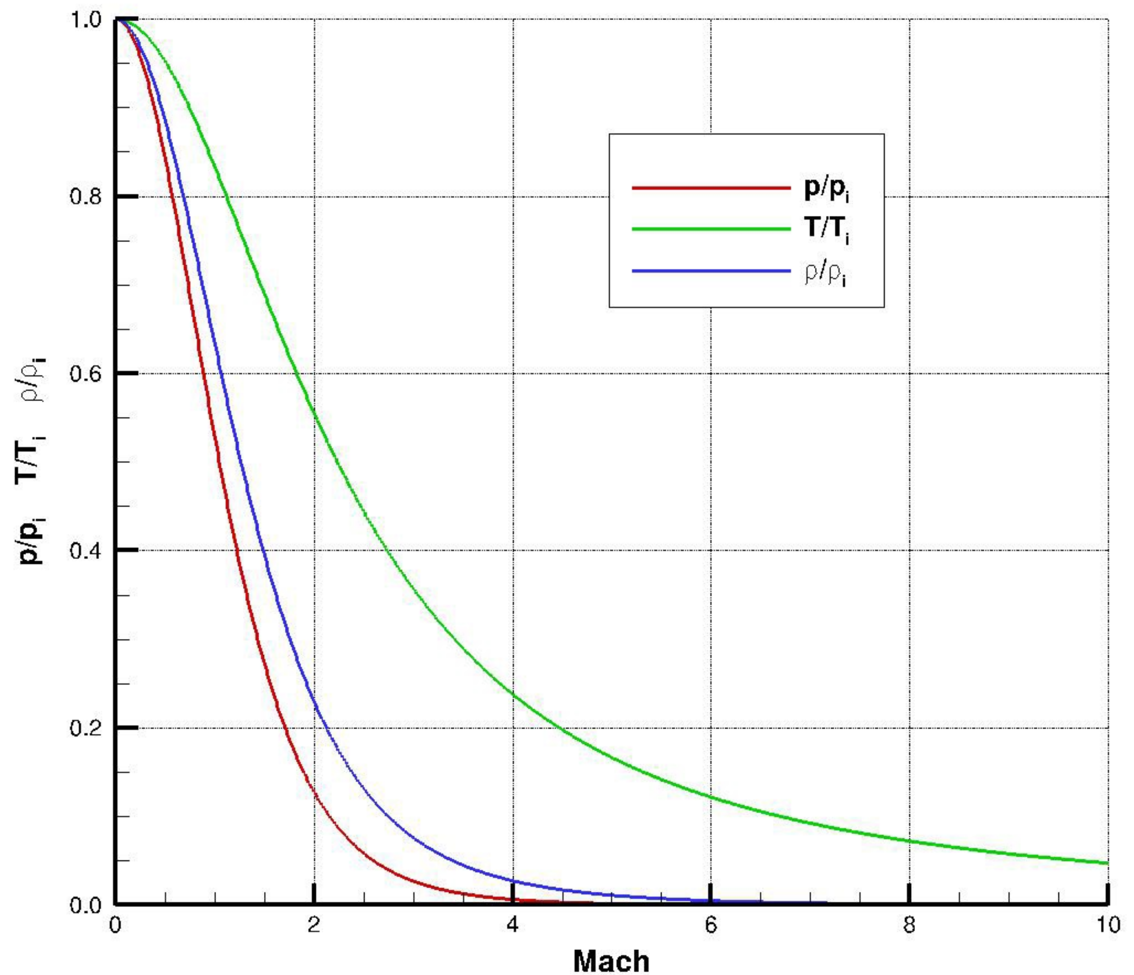
▷ Quasi-monodimensional hypothesis: all flow properties are uniform across any given cross section A of the flow → cross section area variation dA/dx must be low ($A/L \ll$).

▷ This is the area change dA that causes the flow properties to vary as a function of x.

▷ Application of the three conservation laws (mass, momentum and energy) on a control volume to derive the relations between the area and the flow properties.

Reminder: Stagnation quantities

$$\frac{p}{p_i} = \omega(M) = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{-\gamma}{\gamma - 1}} \qquad \frac{T}{T_i} = \Theta(M) = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1} \qquad \frac{\rho}{\rho_i} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{-1}{\gamma - 1}}$$



Continuity equation

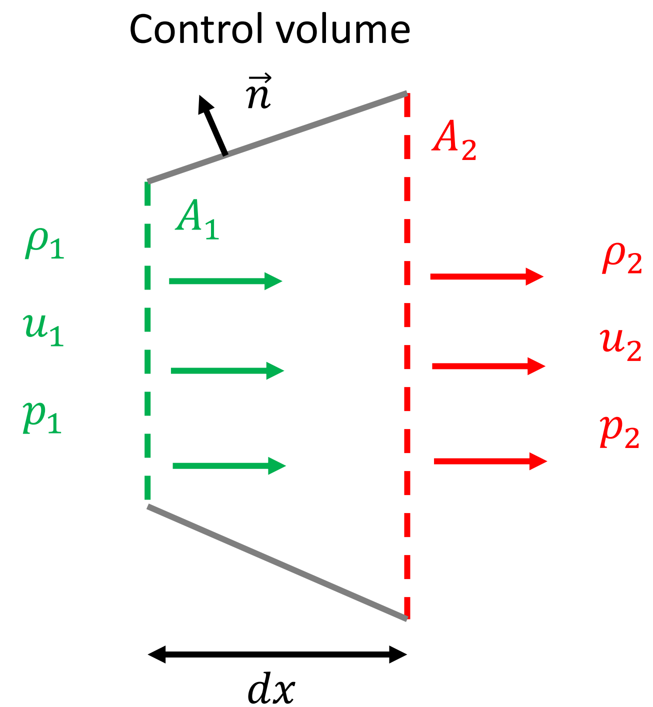
▷ Integral form: $\int_S \rho \vec{U} \cdot \vec{n}\, dS = 0$

▷ Application to the control volume:

$$-\rho_1 u_1 A_1 + \rho_2 u_2 A_2 + 0 = 0$$

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

$$\rho u A = Cst$$



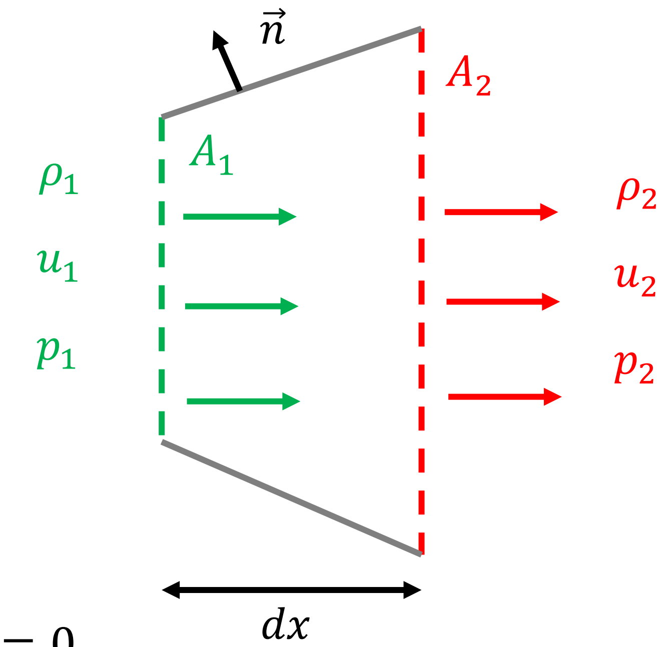
▷ Differential form on the control volume:

$$d(\rho u A) = U A \mathrm{d}\rho + \rho A \mathrm{d}u + \rho u \mathrm{d}A$$

$$\frac{d(\rho u A)}{\rho u A} = \frac{1}{\rho U A}(U A \mathrm{d}\rho + \rho A \mathrm{d}u + \rho u \mathrm{d}A) \longrightarrow \boxed{\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}A}{A} = 0}$$

Momentum equation (1)

Control volume



▷ Integral form: $$\int_S \left[\rho\vec{U}(\vec{U}\cdot\vec{n}) + p\vec{n}\right] dS = 0$$

▷ Application on the control volume (x-component):

$$-(\rho_1 u_1^2 + p_1)A_1 + (\rho_2 u_2^2 + p_2)A_2 + \int_{S_p} p n_x dx = 0$$

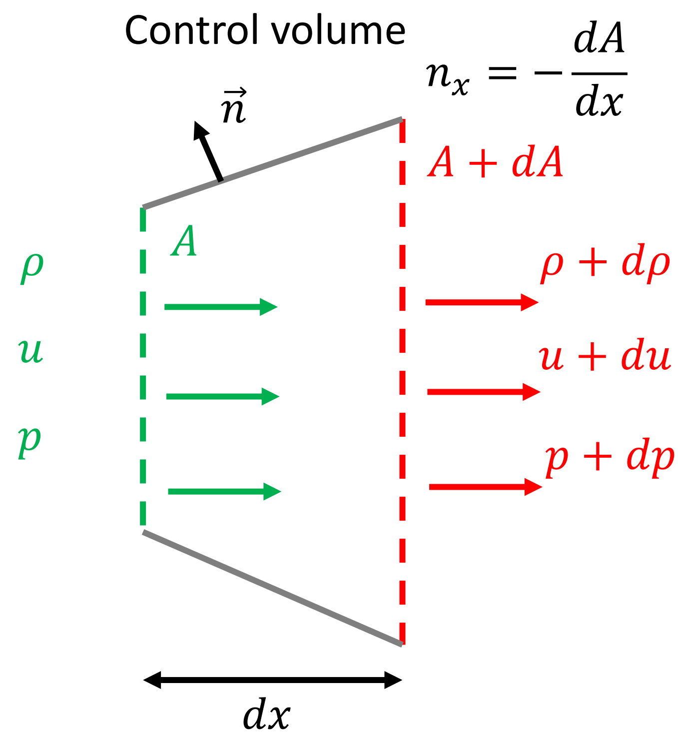
$$(\rho_1 u_1^2 + p_1)A_1 - \int_{S_p} p n_x dx = (\rho_2 u_2^2 + p_2)A_2$$

$$\underbrace{(\rho_2 u_2^2 + p_2)}_{D_2} A_2 - \underbrace{(\rho_1 u_1^2 + p_1)}_{D_1} A_1 = -F_x = P$$

[Dynalpy theorem]

Momentum equation (2)

$$(\rho_1 u_1^2 + p_1)A_1 - \int_{S_p} p n_x dx = (\rho_2 u_2^2 + p_2)A_2$$



▷ Differential form on the control volume:

$$pA + \rho u^2 A + pdA$$
$$= (p + dp)(A + dA) + (\rho + d\rho)(u + du)^2 (A + dA)$$

$$Adp + Au^2 d\rho + \rho u^2 dA + 2\rho u A du = 0$$

Differential form of mass conservation

$$d(\rho u A) = \rho u dA + \rho A du + Au d\rho = 0$$

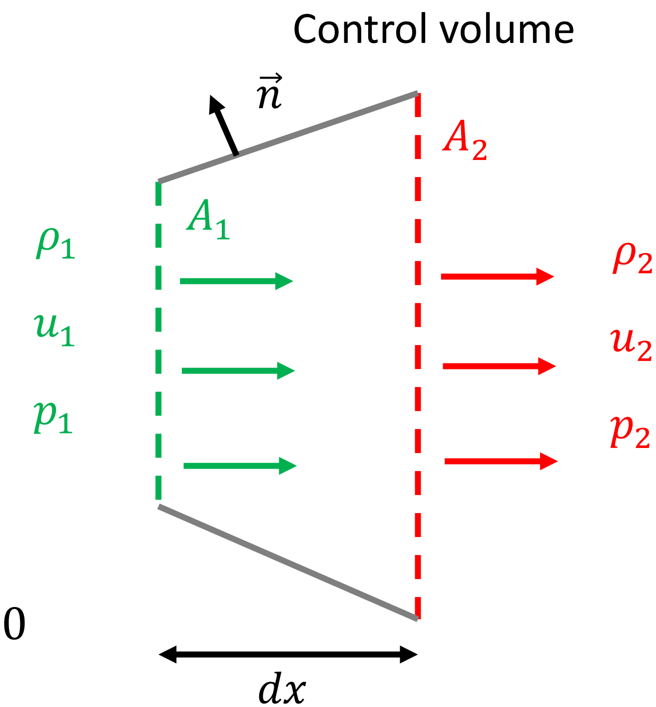
$\times u$

$$\rho u^2 dA + \rho u A du + Au^2 d\rho = 0$$

$$\boxed{dp = -\rho u \, du}$$ [Euler's equation]

Energy equation

▷ Integral form: $\int_S (\rho E + p)\vec{U} \cdot \vec{n} dS = 0$



▷ Application to the control volume:

$$-(\rho_1 E_1 u_1 + p_1 u_1)A_1 + (\rho_2 E_2 u_2 + p_2 u_2)A_2 + 0 = 0$$

$E = e + \frac{u^2}{2}$

$$p_1 u_1 A_1 + \rho_1 u_1 A_1 \left(e_1 + \frac{u_1^2}{2}\right) = p_2 u_2 A_2 + \rho_2 u_2 A_2 \left(e_2 + \frac{u_2^2}{2}\right)$$

$\rho u A = Cst$

$$\frac{p_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + e_2 + \frac{u_2^2}{2}$$

$h = e + \frac{p}{\rho}$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$\boxed{h_{i,1} = h_{i,2}}$$

▷ Differential form on the control volume:

$$d\left(h + \frac{u^2}{2}\right) = 0 \qquad \boxed{dh + udu = 0}$$

2. Velocity-area relation

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Area-velocity relation (1)

Differential form of mass conservation

$$\boxed{\frac{\mathrm{d}\rho}{\rho}} + \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}A}{A} = 0$$

Differential form of momentum conservation

$$\mathrm{d}p = -\rho u \,\mathrm{d}u$$

$$\frac{\mathrm{d}p}{\rho} = \frac{\mathrm{d}p}{\mathrm{d}\rho}\boxed{\frac{\mathrm{d}\rho}{\rho}} = -u \,\mathrm{d}u$$

Speed of sound

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2$$

$$a^2 \frac{\mathrm{d}\rho}{\rho} = -u \,\mathrm{d}u$$

$$\frac{\mathrm{d}\rho}{\rho} = -\frac{u}{a^2}\,\mathrm{d}u = -\frac{u^2}{a^2}\frac{\mathrm{d}u}{u} = -M^2 \frac{\mathrm{d}u}{u}$$

$$-M^2 \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}A}{A} = 0$$

$$\boxed{\frac{\mathrm{d}A}{A} = (M^2 - 1)\frac{\mathrm{d}u}{u}}$$

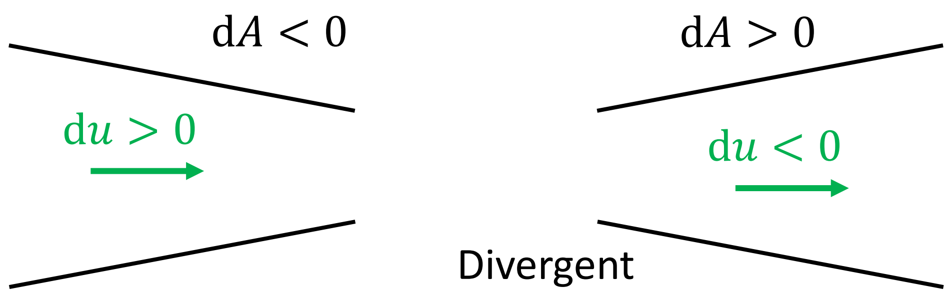
Area-velocity relation (2)

$$\boxed{\frac{\mathrm{d}A}{A} = (M^2 - 1)\frac{\mathrm{d}u}{u}}$$ [Area-velocity relation]

▷ For $M = 0$ (incompressible flow) $\quad \frac{\mathrm{d}A}{A} = -\frac{\mathrm{d}u}{u} \longrightarrow \boxed{Au = Cst}$ [Venturi effect]

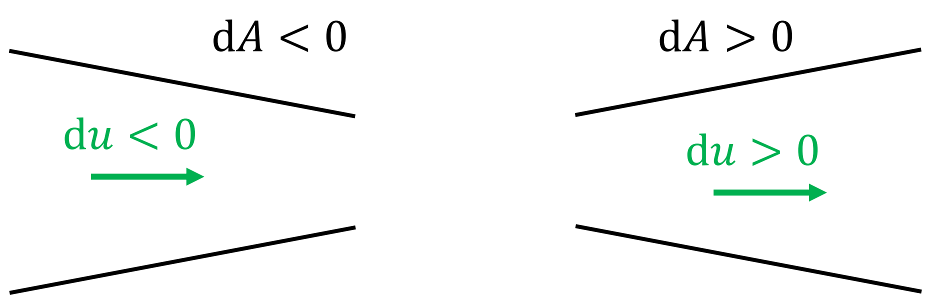
▷ For $M < 1$ (subsonic flow): $\quad \mathrm{d}u \propto -\mathrm{d}A$

Convergent



An increase in velocity is associated with a decrease in area and vice versa.

▷ For $M > 1$ (supersonic flow): $\quad \mathrm{d}u \propto \mathrm{d}A$

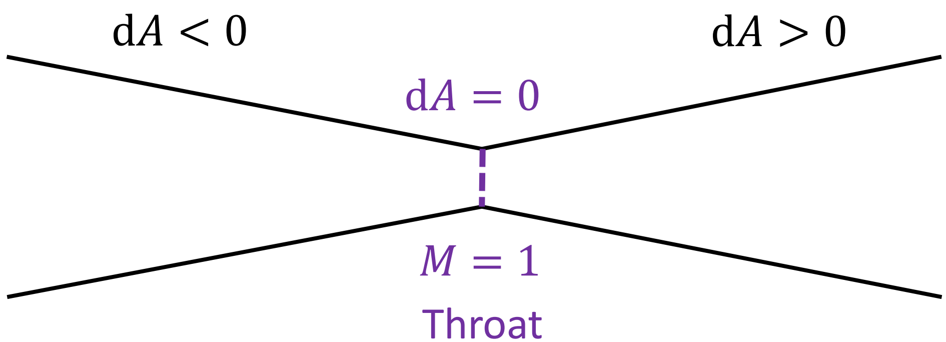


An increase in velocity is associated with an increase in area and vice versa.

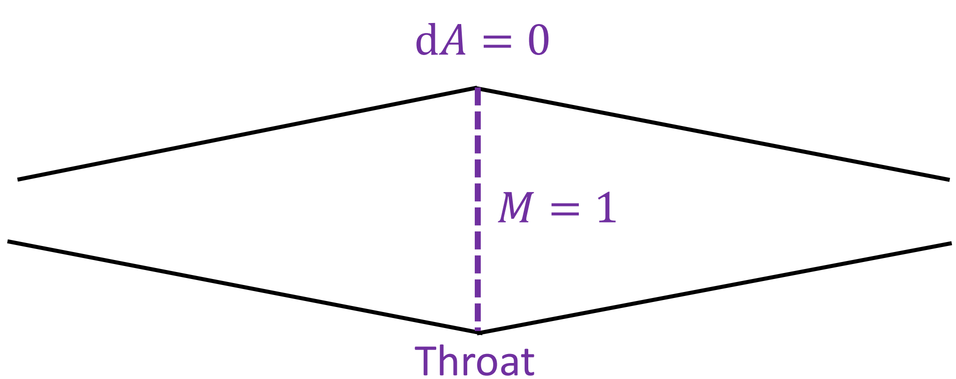
Area-velocity relation (3)

$$\boxed{\frac{\mathrm{d}A}{A} = (M^2 - 1)\frac{\mathrm{d}u}{u}}$$ [Area-velocity relation]

▷ For $M = 1$ (sonic flow): $\mathrm{d}A = 0$ Extremum of $A(x)$



The sonic Mach number can be reached only at the throat of a duct



Impossible case: flow cannot be accelerated to $M = 1$ at the highest cross-section location

3. Converging-Diverging Nozzle

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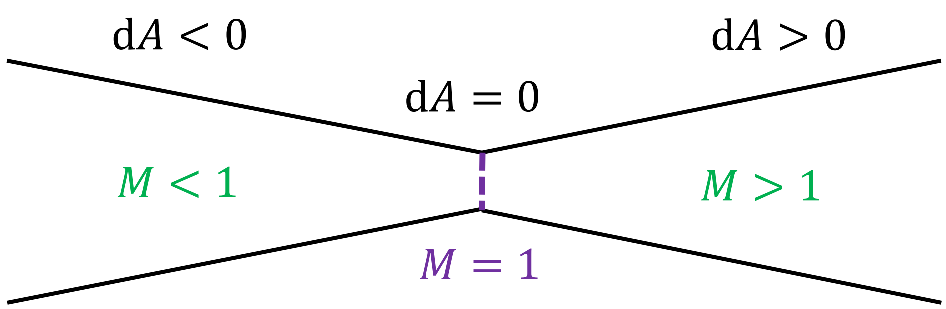
De Laval Nozzle

How to generate a supersonic stream ?



G. De Laval
(1845-1913)

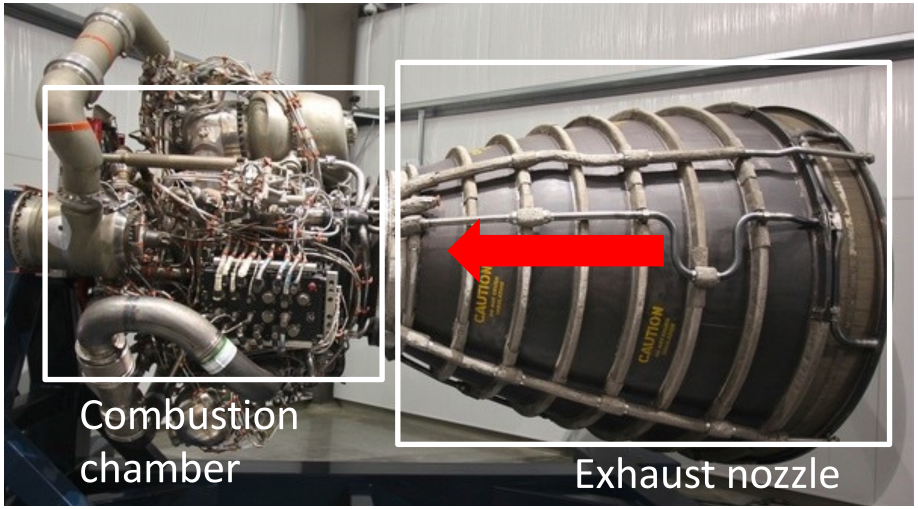
▷ A supersonic stream can be generated from quiescent flow by:

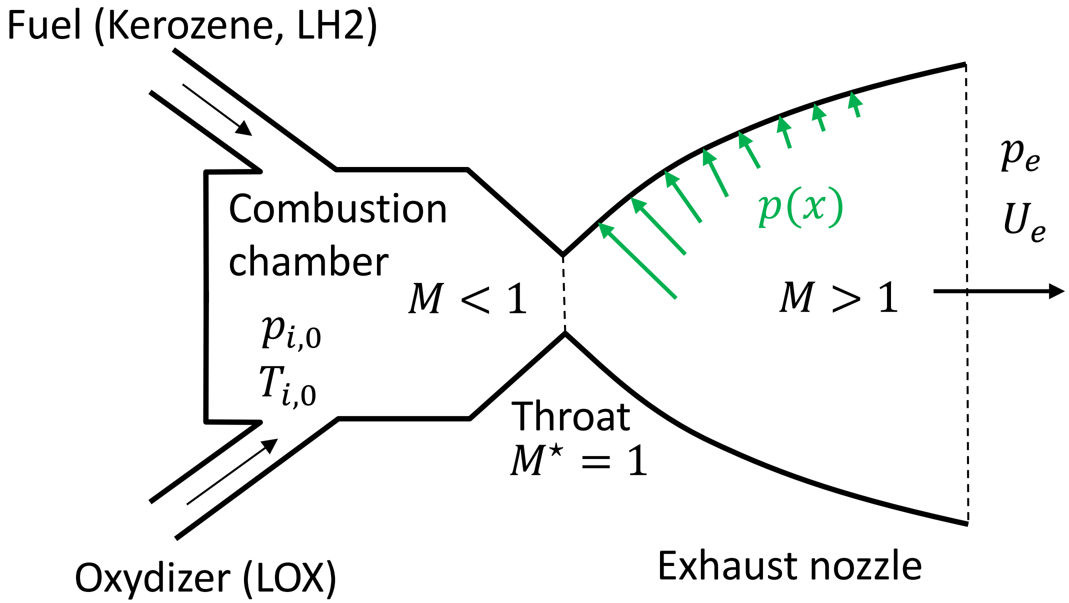
1. Accelerating it in a convergent till sonic condition are reached at the throat;
2. Further accelerating it in a divergent.

▷ Principle of a supersonic wind tunnel and rocket-engine.

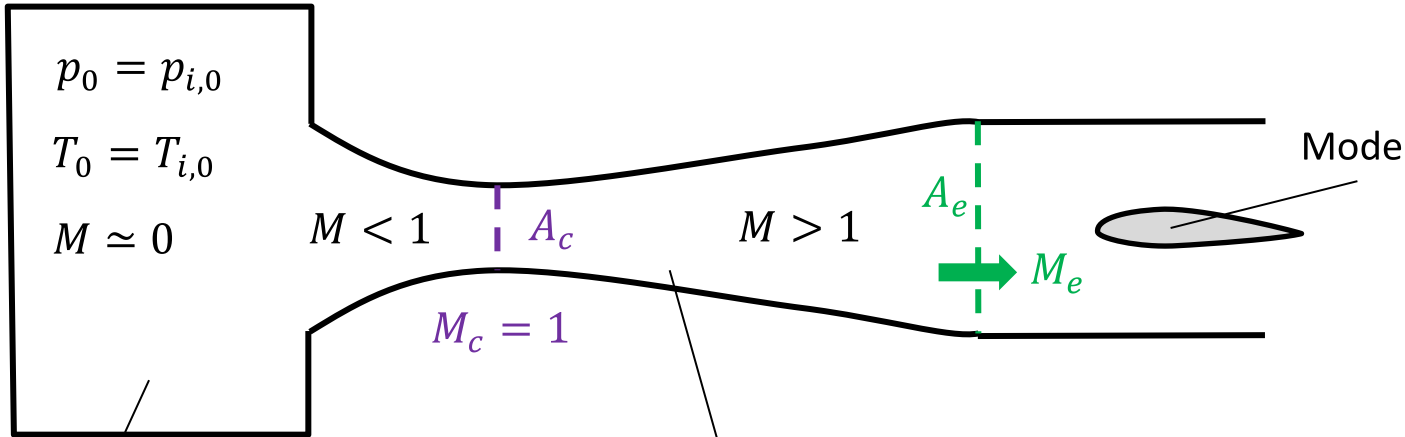
▷ When $M = 1$ at the throat the nozzle is said to be choked. In that case, the mass flow rate q_m is fixed by the throat cross section area A_c.

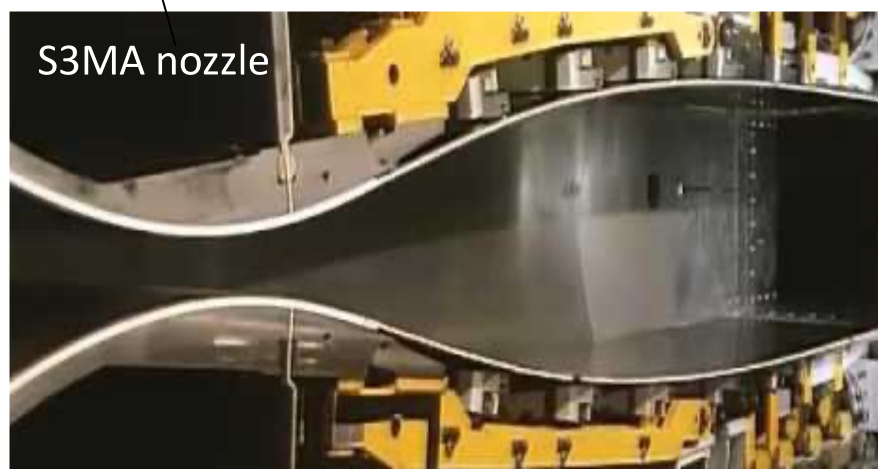
Rocket-engine



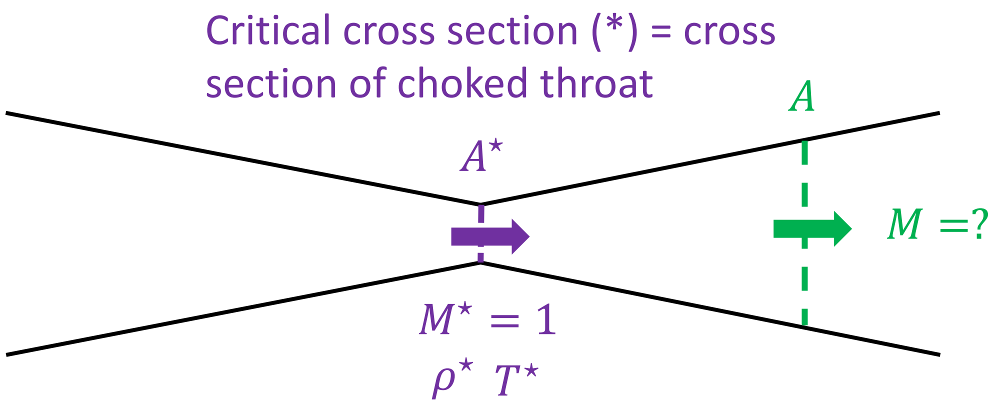


Supersonic wind tunnel





Mach-area relation (1)

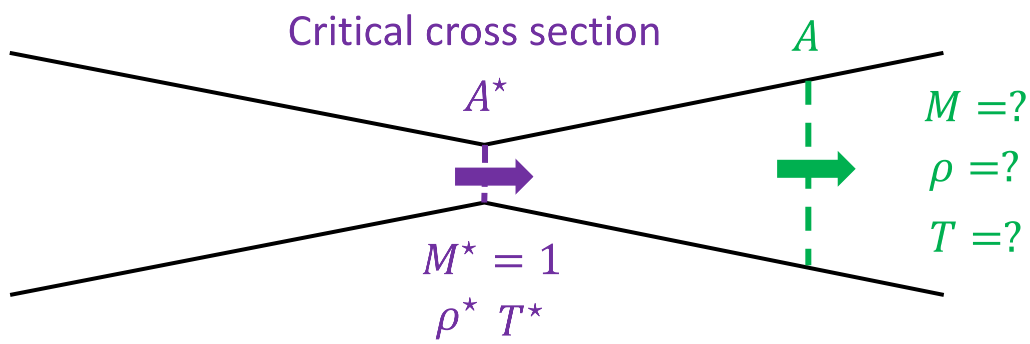


$$\frac{A}{A^\star} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = \Sigma(M)$$

▷ The Mach-area relation allows to compute the Mach number M at the location of section A from the ratio $A/A^\star$.

Mach-area relation (2)

Demonstration



[Integral form of mass conservation] $\rho^\star u^\star A^\star = \rho u A$

$\boxed{u^\star = a^\star}$

$$\rho^\star a^\star A^\star = \rho u A$$

$$\frac{A}{A^\star} = \frac{\rho^\star}{\rho}\frac{a^\star}{u} = \frac{\rho^\star}{\rho_i}\frac{\rho_i}{\rho}\frac{a^\star}{a}\frac{a}{u} = \frac{\rho^\star}{\rho_i}\frac{\rho_i}{\rho}\sqrt{\frac{T^\star}{T}}\frac{1}{M} = \frac{\rho^\star}{\rho_i}\frac{\rho_i}{\rho}\sqrt{\frac{T^\star}{T_i}\frac{T_i}{T}}\frac{1}{M}$$

$$\frac{\rho_i}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

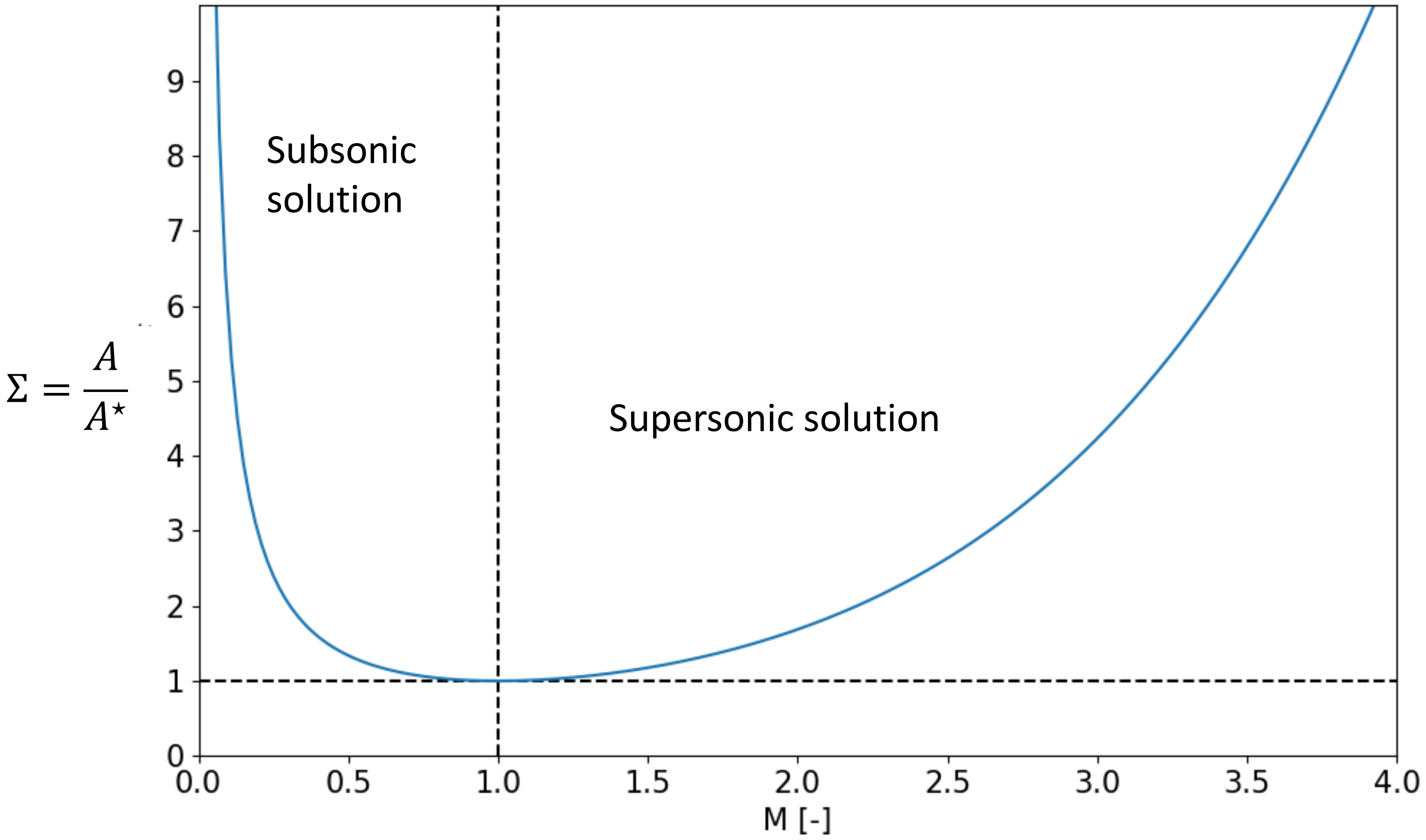
$$\frac{\rho^\star}{\rho_i} = \left(\frac{\gamma + 1}{2}\right)^{\frac{-1}{\gamma-1}}$$

$$\frac{T_i}{T} = 1 + \frac{\gamma - 1}{2}M^2$$

$$\frac{T^\star}{T_i} = \left(\frac{\gamma + 1}{2}\right)^{-1}$$

$$\boxed{\frac{A}{A^\star} = \frac{1}{M}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = \Sigma(M)}$$

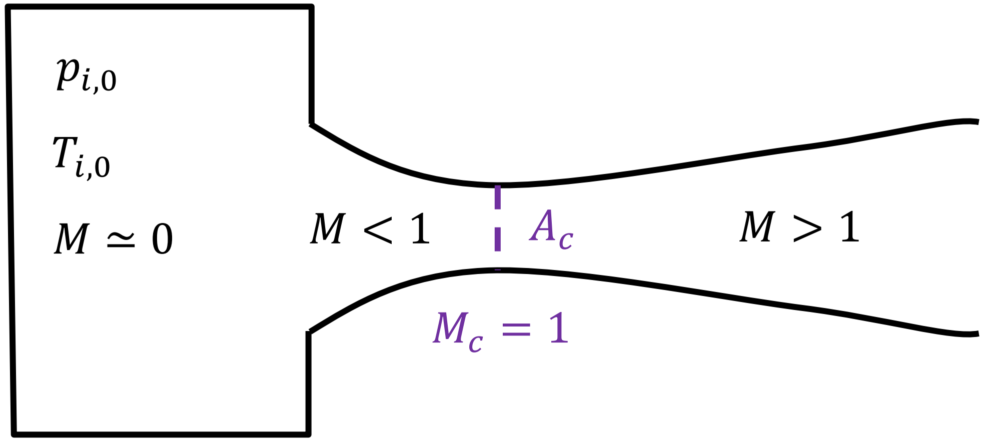
Mach-area relation (3)



▷ For a given $\Sigma = \frac{A}{A^{\star}}$, there are two solutions:

1. A subsonic one.
2. A supersonic one.

Mass flow rate



$$q_m = \rho^\star a^\star A^\star = \frac{p^\star}{rT^\star}\sqrt{\gamma r T^\star}\, A^\star = \sqrt{\frac{\gamma}{r}}\frac{p^\star}{p_{i,0}}\sqrt{\frac{T_{i,0}}{T^\star}}\frac{p_{i,0}A^\star}{\sqrt{T_{i,0}}}$$

$$q_m = \sqrt{\frac{\gamma}{r}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}\frac{p_{i,0}A^\star}{\sqrt{T_{i,0}}} = \delta(\gamma, r)\frac{p_{i,0}A^\star}{\sqrt{T_{i,0}}}$$

The mass flow rate is fixed by the choked throat cross section $A^\star$

Conservation of the mass flow rate → $\boxed{p_i A^\star = C^{te}}$ along the duct

Dynalpy

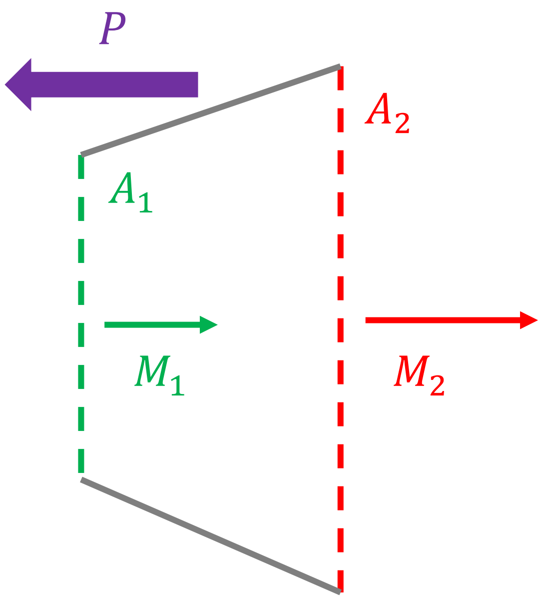
$$D = (p + \rho u^2)\, A$$

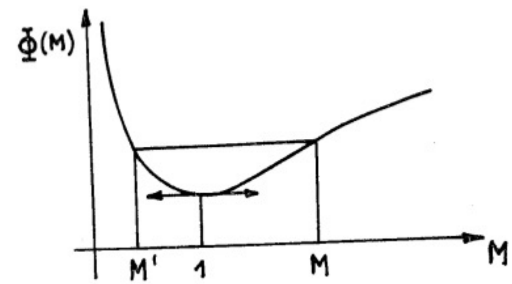
$$= (1 + \gamma M^2)\, p\, A$$

$$= p_i\, A^\star\, \Phi(M)$$

$$\Phi(M) = \varpi(M) \times (1 + \gamma M^2) \times \Sigma(M)$$

Dynalpy function



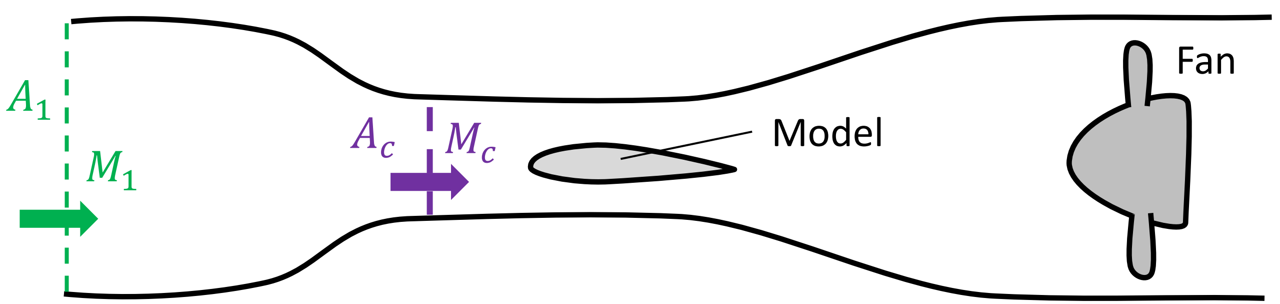


Thrust

$$\boxed{P = D_2 - D_1 = [\Phi(M_2) - \Phi(M_1)]\, p_i A^\star}$$

Application 6 : Subsonic wind tunnel

Subsonic wind tunnel (1)



▷ What is the contraction ratio A_1/A_c required to reach a given Mach number M_c at the throat ?

$$\frac{A_1}{A_c} = \frac{A_1}{A^\star}\frac{A^\star}{A_c} = \frac{\Sigma(M_1)}{\Sigma(M_c)}$$

Critical cross section area $A^\star$ is virtual (nowhere in the duct the flow reaches sonic conditions)

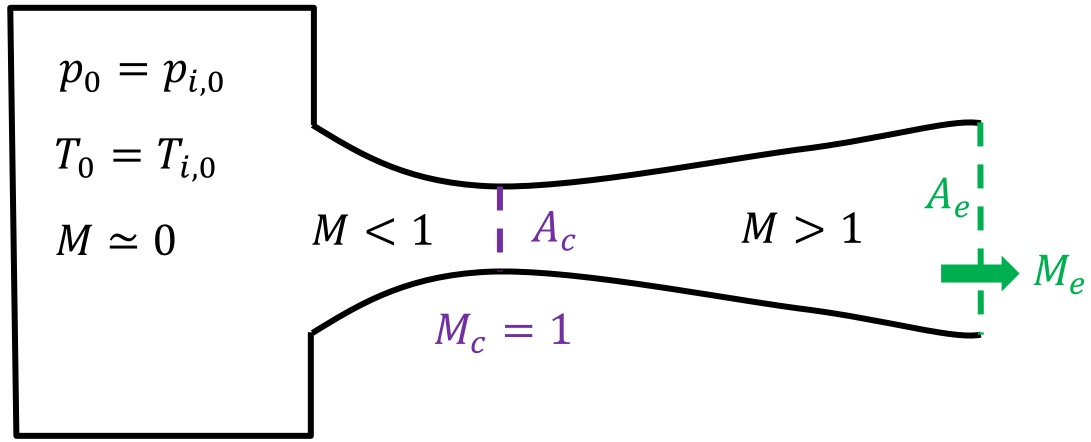
▷ Calculation of the static pressure:

$$\frac{p_1}{p_c} = \frac{p_1}{p_{i,1}}\frac{p_{i,c}}{p_c}\frac{p_{i,1}}{p_{i,c}} = \frac{\omega(M_1)}{\omega(M_c)}$$

$p_{i,1} = p_{i,c}$ (Flow is isentropic)

Application 7 : Rocket nozzle

Rocket nozzle



▷ What is the contraction ratio A_e/A_c required to reach a given Mach number M_e at the divergent exit ?

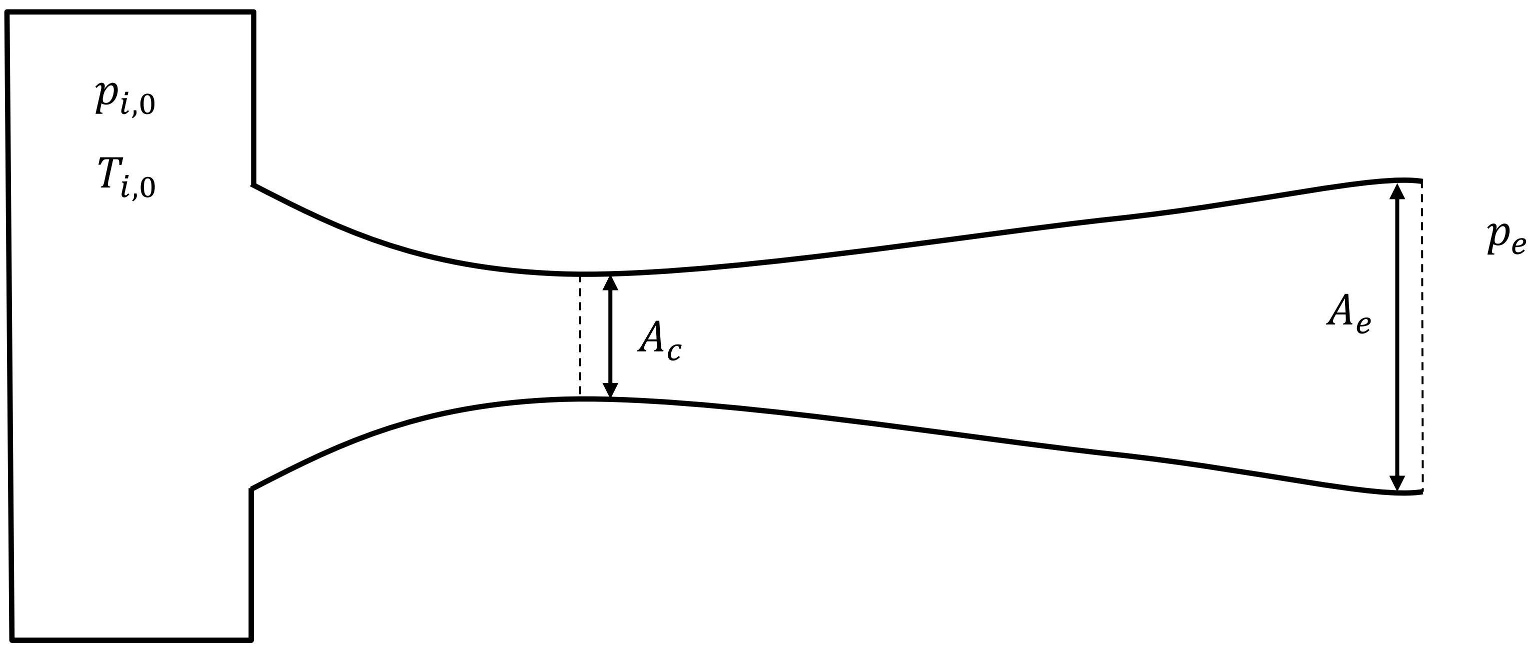
The throat is sonic (choked): $A_c = A^\star$ $\longrightarrow$ $\boxed{\dfrac{A_e}{A_c} = \dfrac{A_e}{A^\star} = \Sigma(M_e)}$

Application 8 : Nozzle regimes

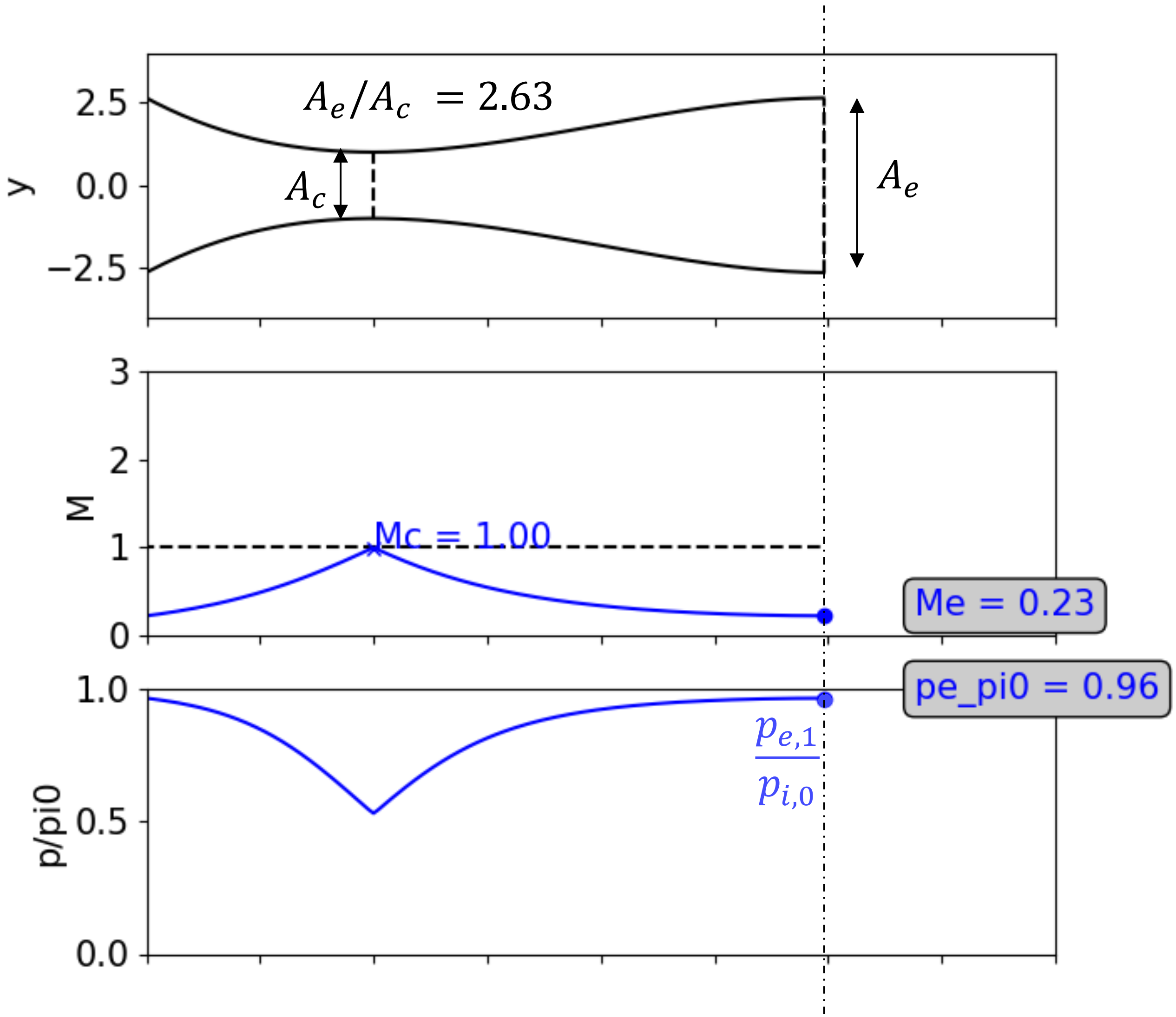
Flow regimes

▷ Consider a converging-diverging nozzle (A_e/A_c fixed) connected upstream to an reservoir of total pressure $p_{i,0}$ and downstream to the ambient pressure p_e (called backpressure).

What happens when backpressure p_e is progressively reduced ?



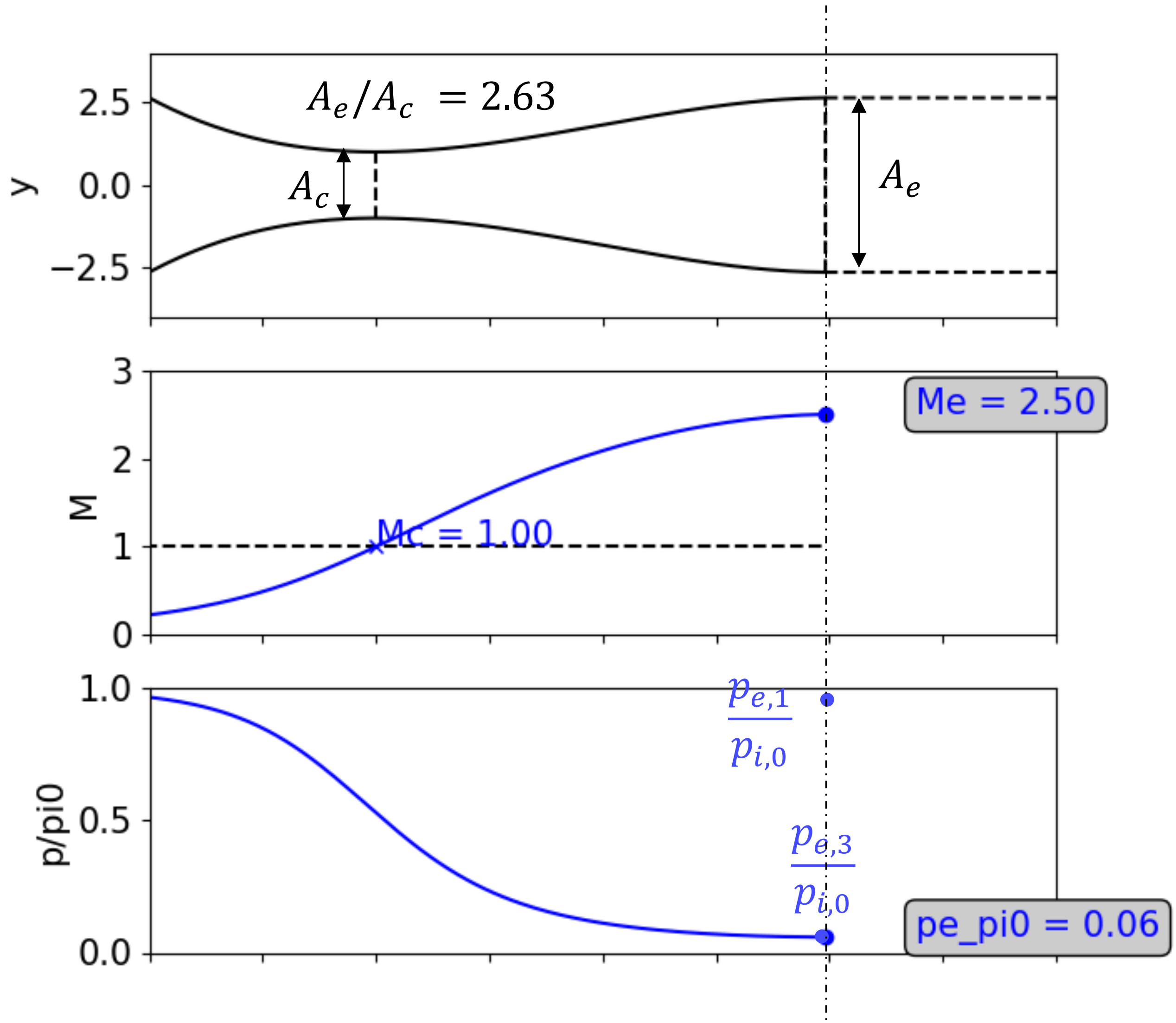
Choked nozzle: subsonic solution



$$\frac{A_e}{A_c} = \frac{A_e}{A^\star} = \Sigma(M_{e,1})$$

$$\frac{p_{e,1}}{p_{i,0}} = \omega(M_{e,1})$$

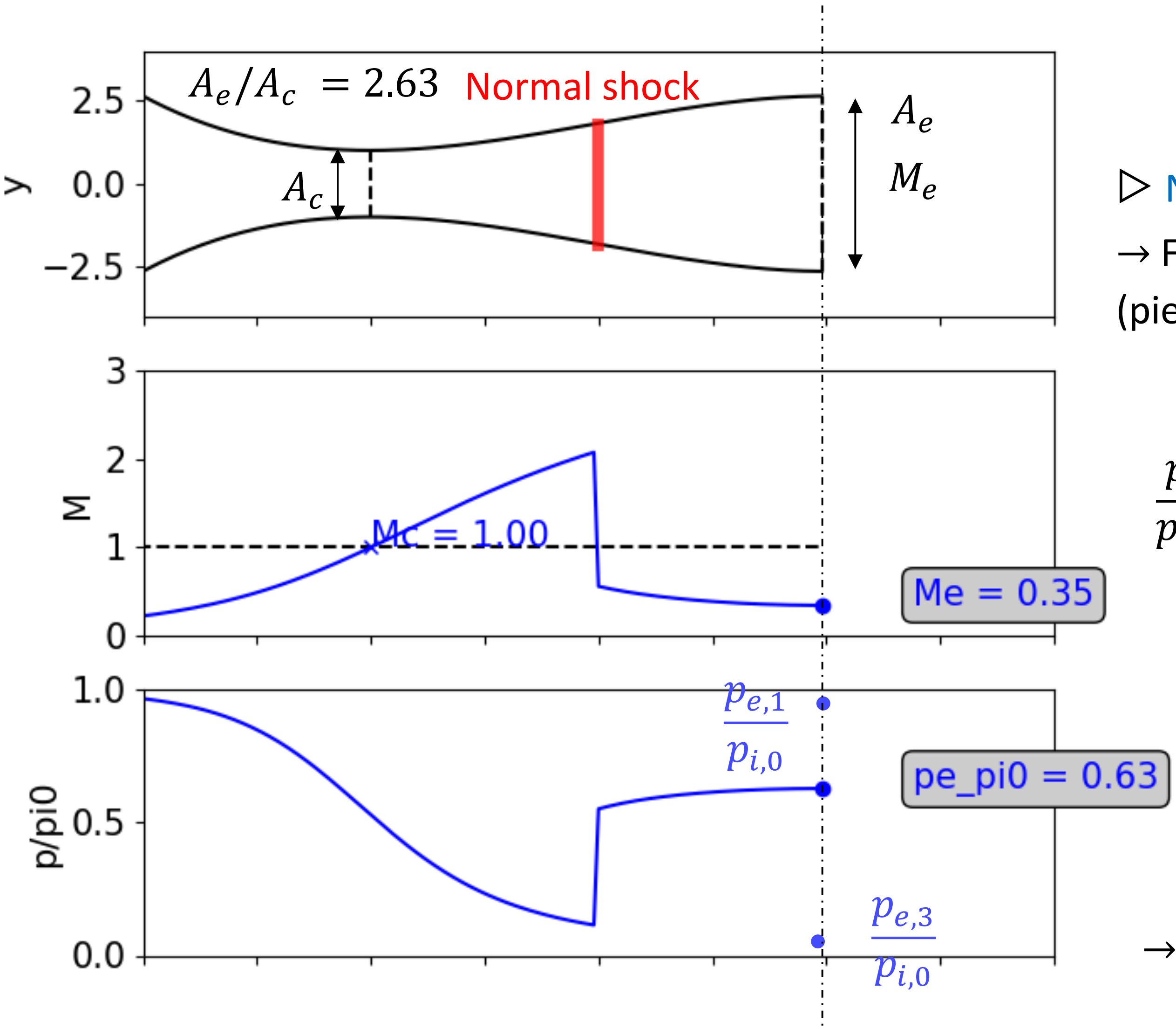
Choked nozzle: supersonic solution



$$\frac{A_e}{A_c} = \frac{A_e}{A^\star} = \Sigma(M_{e,3})$$

$$\frac{p_{e,3}}{p_{i,0}} = \omega(M_{e,3})$$

Over-expanded nozzle: normal shock (1)



$$\frac{p_{e,1}}{p_{i,0}} > \frac{p_e}{p_{i,0}} > \frac{p_{e,3}}{p_{i,0}}$$

▷ Normal shock in the divergent

→ Flow is not isentropic (piecewise isentropic)

$$p_{i,0} \neq p_{i,e}$$

$$\frac{p_e}{p_{i,0}} = \frac{p_e}{p_{i,e}}\frac{p_{i,e}}{p_{i,0}} = \omega(M_e)\frac{p_{i,e}}{p_{i,0}}$$

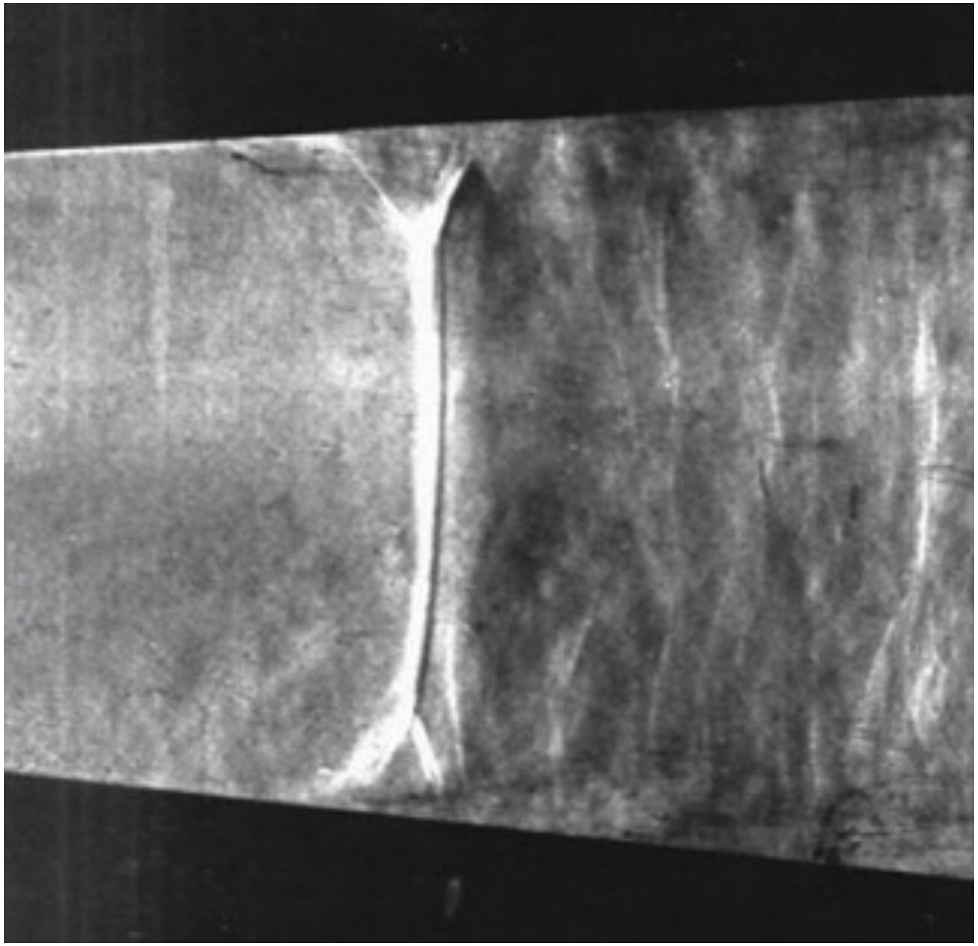
$$\frac{p_{i,0}A_0^\star}{\sqrt{T_{i,0}}} = \frac{p_{i,e}A_e^\star}{\sqrt{T_{i,e}}}$$

$$T_{i,0} = T_{i,e}$$

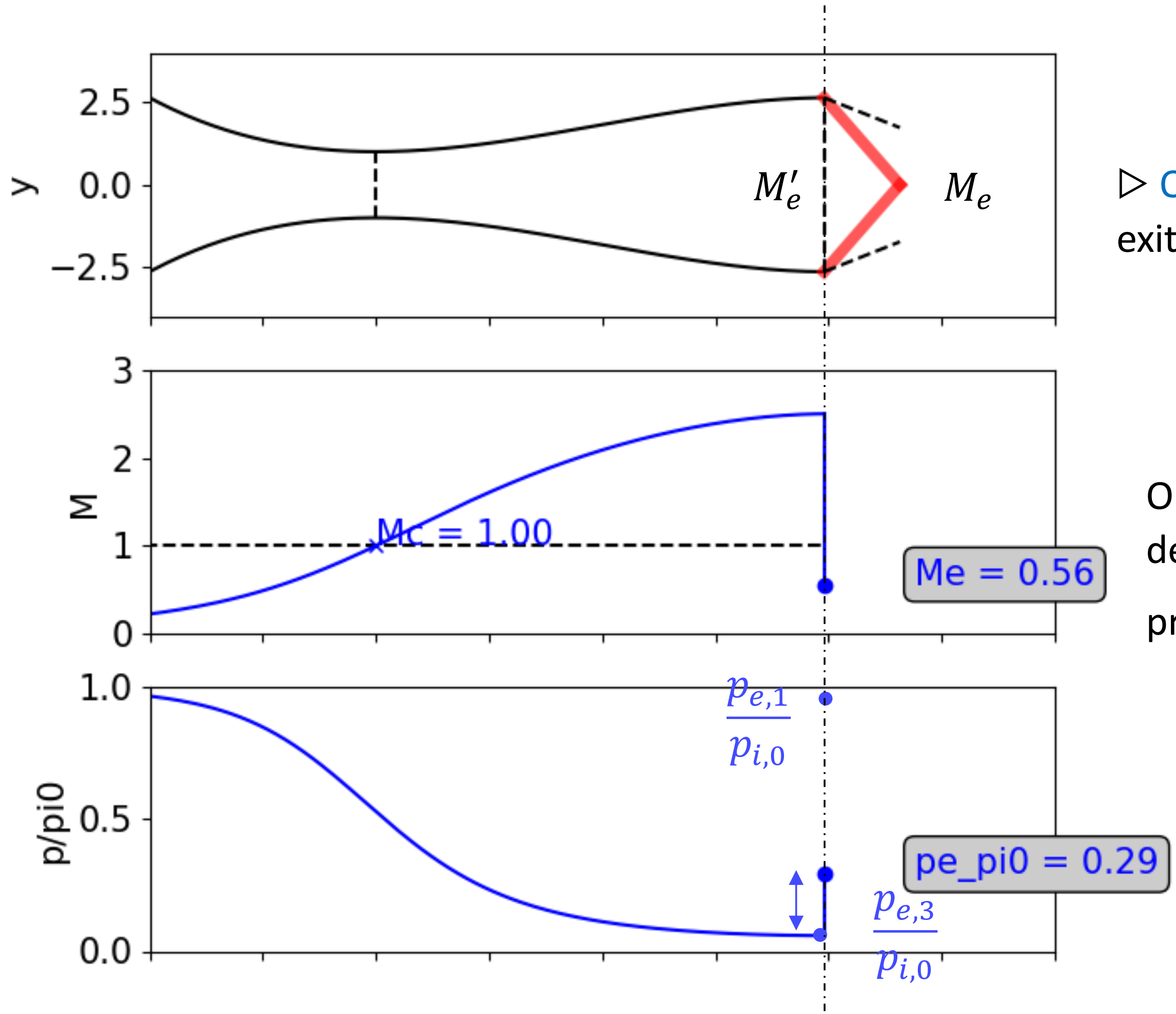
$$\rightarrow \frac{p_{i,e}}{p_{i,0}} = \frac{A_0^\star}{A_e^\star} = \frac{A_c}{A_e^\star} = \frac{A_c}{A_e}\Sigma(M_e)$$

$$\frac{A_e}{A_e^\star} = \Sigma(M_e)$$

Over-expanded nozzle: normal shock (2)



Over-expanded nozzle: oblique shocks (1)



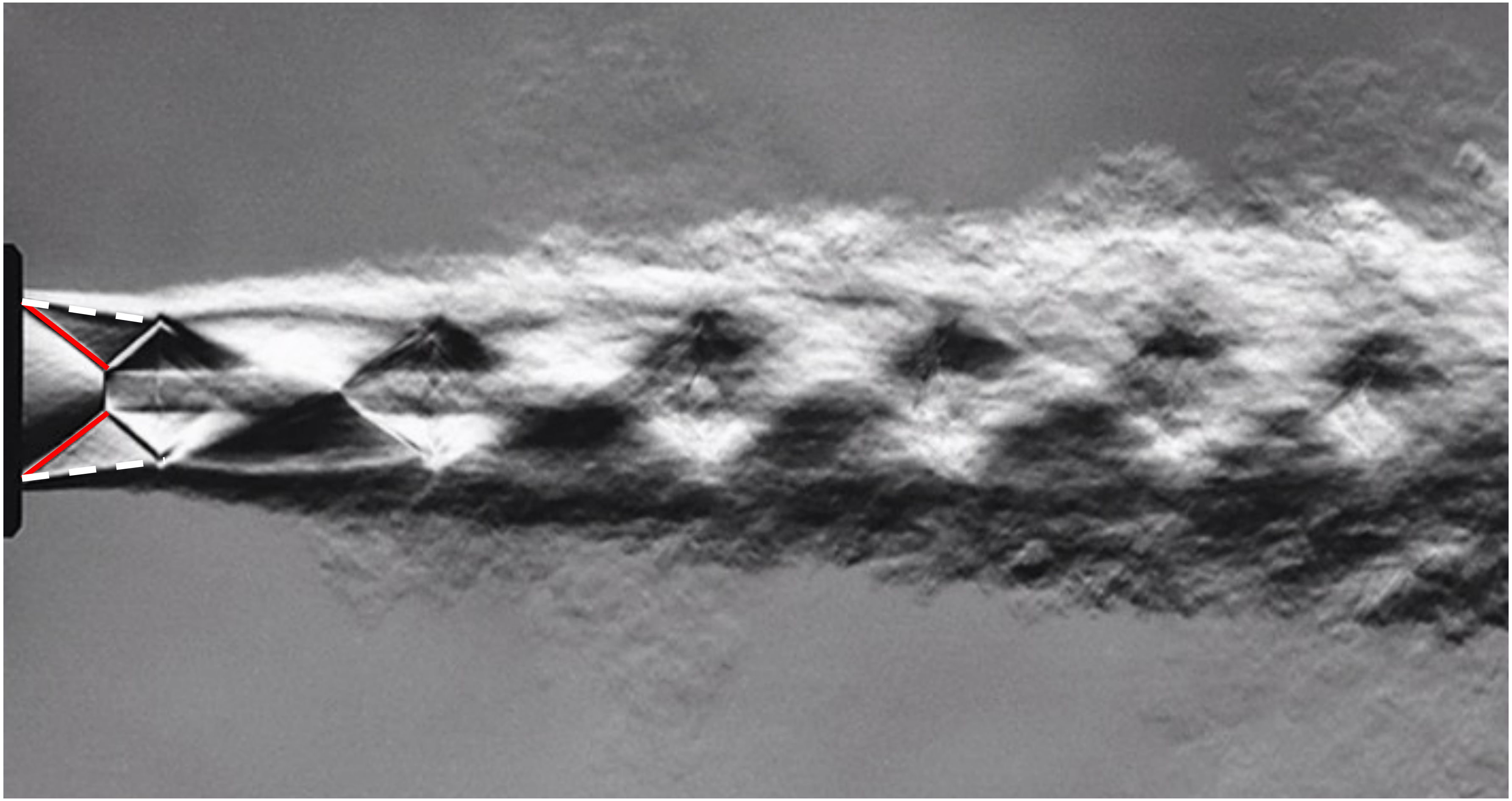
$$\frac{p_{e,1}}{p_{i,0}} > \frac{p_e}{p_{i,0}} > \frac{p_{e,3}}{p_{i,0}}$$

▷ Oblique shock at the nozzle exit

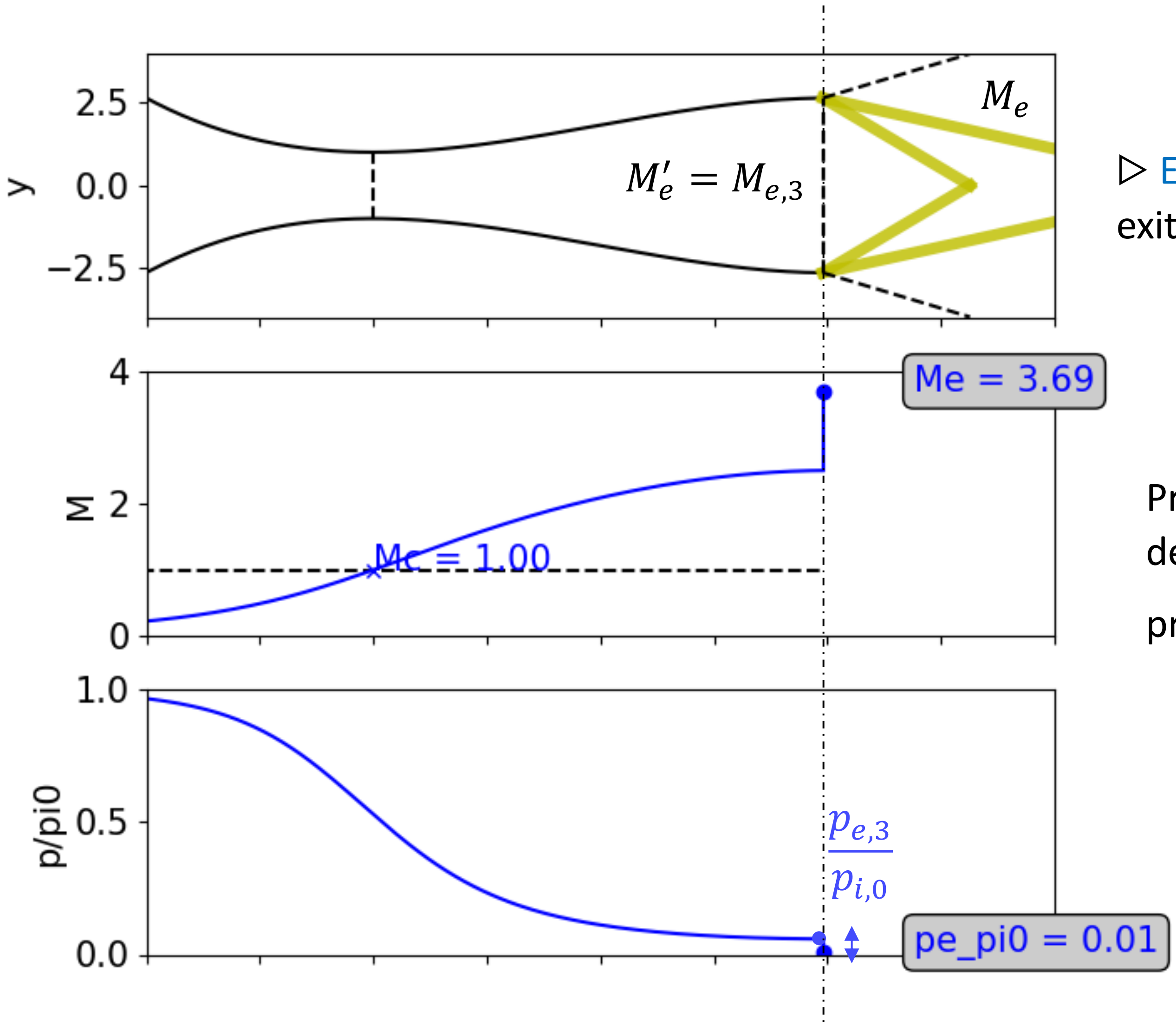
$$\frac{p_e}{p_{i,0}} = \frac{p_e}{p_e{}'}\frac{p_e{}'}{p_{i,0}} = \frac{p_e}{p_e{}'}\omega(M_e')$$

Oblique shock angle β is determined so as to give the proper static pressure ratio $\frac{p_e}{p_e{}'}$.

Over-expanded nozzle: oblique shocks (2)



Under-expanded nozzle: expansion wave (1)



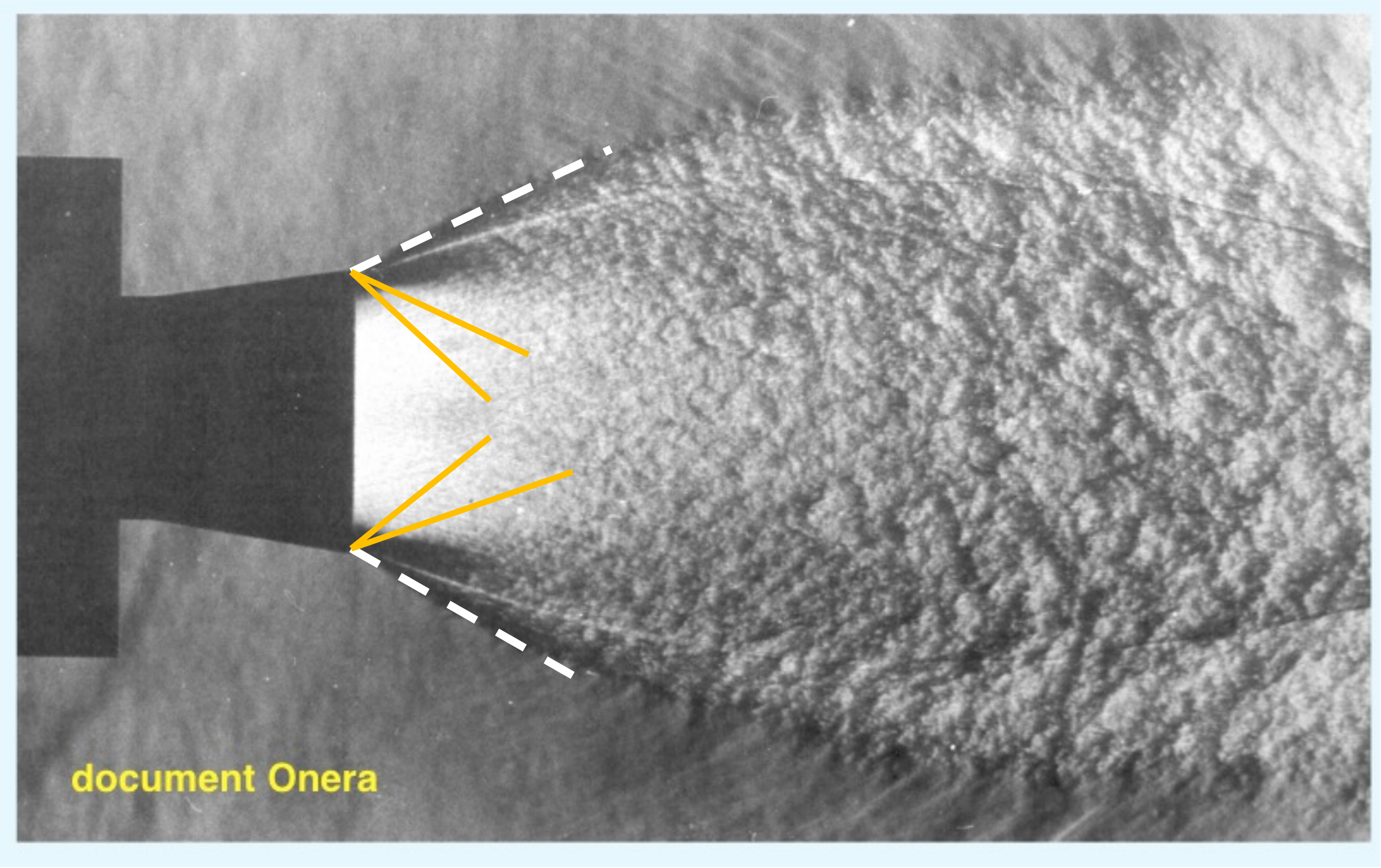
$$\frac{p_e}{p_{i,0}} < \frac{p_{e,3}}{p_{i,0}}$$

▷ Expansion wave at the nozzle exit .

$$\frac{p_e}{p_{i,0}} = \frac{p_e}{p_e{}'}\frac{p_e{}'}{p_{i,0}} = \frac{p_e}{p_e{}'}\omega(M_e')$$

Prandtl-Meyer angles ν are determined so as to give the proper static pressure ratio $\frac{p_e}{p_e{}'}$.

Under-expanded nozzle: expansion wave (2)



Unadapted supersonic nozzle