Discrete Mathematics

Counting

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Outline

1. Multiplication Rule
2. Addition Rule
3. The Pigeonhole Principle
4. Combinations
5. r-Combinations with Repetition Allowed

The Multiplication Rule

Definition (The Multiplication Rule)

If an operation consists of k steps and

- the **first** step can be performed in n_1 ways
- the **second** step can be performed in n_2 ways [regardless of how the first step was performed]
- ...
- the k-**th** step can be performed in n_k ways [regardless of how the preceding steps were performed]

then the entire operation can be performed in:

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k$$

different ways.

The Multiplication Rule – Example 1

Suppose a computer installation has 4 different input/output units (A, B, C, and D) and 3 different central processing units (X, Y, and Z).

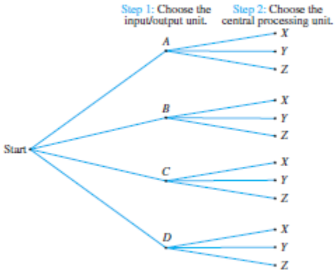
How many ways are there to pair any input/output unit with any central processing unit?

Imagine the pairing as 2-step operation:

Step 1 Choose the input/output unit

Step 2 Choose the central processing unit

The possible outcomes of this operation are illustrated by the following tree:



The total number of ways to pair the two types of units is the same as the number of branches of the tree, which is:

$$3 + 3 + 3 + 3 = 4 \cdot 3 = 12$$

The Multiplication Rule – Example 2

A typical PIN (personal identification number) is a sequence of any 4 symbols chosen from the 26 letters and 10 digits (36 different ways in total), with **repetition allowed**. How many different PINs are possible?

Forming a 4-symbol PIN is a 4 step operation:

Step 1 Choose the first symbol

Step 2 Choose the second symbol

Step 3 Choose the third symbol

Step 4 Choose the fourth symbol

There 36 different ways to perform each step, regardless of how preceding steps were performed. By the multiplication rule, we have:

$$36 \cdot 36 \cdot 36 \cdot 36 = 36^4 = 1,679,616$$

different PINs in total.

Suppose that **repetition is not allowed** in forming PINs. How many different PINs are there?

Permutations

A **permutation** of a set of objects is an ordering of the objects in a row.

For example, the set of elements a, b, and c has 6 permutations:

$$abc \quad acb \quad bac \quad bca \quad cab \quad cba$$

In general, given a set of n objects, forming a permutation can be seen as an n-step operation:

Step 1 Choose the first element

Step 2 Choose the second element

Step ... ...

Step n Choose the n-th element

There are n ways to perform Step 1, $n-1$ ways to perform Step 2, ..., and there is only 1 way to perform Step n. In general, the number of ways to perform each step is one less than the number of ways to perform the preceding step. Hence, by the multiplication rule, there are

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n!$$

ways to perform the entire operation.

Permutations

Theorem

For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$

Example1. How many ways can the letters in the word COMPUTER be arranged in a row?
First, find how many distinct letters are in the word! That is 8.
The the number of permutations of a set of 8 elements is: $8! = 40,320$.

Example2. How many ways can the letters in the word COMPUTER be arranged in a row if the letters CO must remain next to each other as a unit?

Permutations of Selected Elements

There are 6 ways to select 2 letters from the set $\{a, b, c\}$ and write them in order:

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb$$

Each such ordering is called 2-permutation of $\{a, b, c\}$.

Definition (r-Permutation)

An r-permutation of a set of n elements is an ordered selection of r elements taken from the set of n elements.
The number of r-permutations of a set of n elements is denoted as $P(n, r)$, defined as:

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \ldots (n-r+1)$$

or, equivalently

$$P(n, r) = \frac{n!}{(n-r)!}$$

Permutations of Selected Elements – Example

Example1. How many ways can 3 of the letters in the word BYTES be arranged in a row?
The answer is the number of 3-permutations of a set of 5 elements, which is equal to

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 5 \cdot 4 \cdot 3 = 60$$

Example2. How many ways this can be done if the first letter must be B?

Outline

The Addition Rule

We now look at counting problems that can be solved by counting the number of elements in sets.

Definition (The Addition Rule)

Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then:

$$N(A) = N(A_1) + N(A_2) + \ldots + N(A_k)$$

Example. A password consists of from one to three letters chosen from the 26 in the alphabet with repetitions allowed. How many different passwords are possible?

Set of all passwords of Length≤ 3 = Set of pass. of Length 1 $\cup$ Set of pass. of Length 2 $\cup$ Set of pass. of Length 3
We have:
number of pass. of Length 1 = 26
number of pass. of Length 2 = $26 \cdot 26$
number of pass. of Length 3 = $26 \cdot 26 \cdot 26$
The total number of password is: $26 + 26^2 + 26^3 = 18,278$

The Inclusion/Exclusion Rule

We now look at the problem of counting the number of elements in a union of sets when some of the sets overlap.

Definition (The Inclusion/Exclusion Rule for Two or Three Sets)

If A, B, C are any finite sets, then:

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$$

The Inclusion/Exclusion Rule – Example

How many integers from 1 through 1000 are multiples of 3 or multiples of 5?

Let A be the set of integers from 1 to 1000 that are multiples of 3
Let B be the set of integers from 1 to 1000 that are multiples of 5
Then
$A \cup B$ is the set of int. from 1 to 1000 that are multiples of 3 or 5
$A \cap B$ is the set of int. from 1 to 1000 that are multiples of both 3 and 5 (=15)

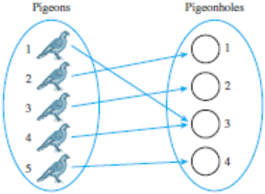
Every third integer from 3 to 999 is a multiple of 3, and it can be represented as $3 \cdot k$ for k from 1 to 333. Thus, $N(A) = 333$.
Every fifth integer from 5 to 1000 is a multiple of 5, and it can be represented as $5 \cdot k$ for k from 1 to 200. Thus, $N(B) = 200$.
Every fifteenth integer from 15 to 990 is a multiple of 15, and it can be represented as $15 \cdot k$ for k from 1 to 66. Thus, $N(A \cap B) = 66$.

Thus, $N(A \cup B) = 333 + 200 - 66 = 467$.

Outline

The Pigeonhole Principle

It states that if n pigeons fly into m pigeonholes and $n > m$, then at least one hole must contain two or more pigeons.



Definition (Pigeonhole Principle)

A function from one finite set to a smaller finite set cannot be one-to-one. There must be at least two elements in the domain that have the same image in the co-domain.

The Pigeonhole Principle – Example

Example1. In a group of 13 people, must there be at least 2 who were born in the same month? Why?

Think of 13 people as pigeons and 12 months as the pigeonholes. Define a function from the set of people to the 12 months. By the pigeonhole principle, there must be at least two people pointing to the same month.

Example2. Among the residents of New York City, there must be at least two people with the same number of hairs on their head? Why?

It is known that the population of New York City P is at least 5,000,000.
Also the maximum number of hairs on any person's head is known to be no more than 300,000.
Define a function H from the set of people of New York to the set $\{0, 1, \ldots, 300000\}$.
By the pigeonhole principle, H is not one-to-one.

Outline

Counting Subsets of a Set: Combinations

Consider the following question:
Given a set S of n elements, how many different subsets of size r can be chosen from S?
Each individual subset of size r is called an r-combination of S.

Definition

An r-**combination** of a set of n elements ($r \leq n$) is a subset of size r of the n elements.
The number of r-combinations is

$$\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$$

which is read "n choose r".

Combinations – Example

Let $S = \{Ann, Bob, Cyd, Dan\}$. Each committee consisting of 3 of the people in S is a 3-combination of S.
Find the number of all such 3-combinations of S, and list them.

The number is

$$\binom{4}{3} = \frac{4!}{3! \cdot (4-3)!} = 4$$

The 3-combinations are:
$\{Bob, Cyd, Dan\}$
$\{Ann, Cyd, Dan\}$
$\{Ann, Bob, Dan\}$
$\{Ann, Bob, Cyd\}$

Relation between Permutations and Combinations

There are 2 distinct methods that can be used to select r objects from a set of n elements.

In an **ordered selection**, it is not only what elements are chosen but also the order in which they are chosen that matters. An ordered selection of r elements from a set of n elements is an **r-permutation** of the set.

In an **unordered selection**, it is only the identity of the chosen elements that matters. An unordered selection of r elements from a set of n elements is an **r-combination** of the set.

Relation between Permutations and Combinations

Write all 2-permutations and 2-combinations of the set $\{0, 1, 2, 3\}$.

The number of 2-permutations of $\{0, 1, 2, 3\}$ is:

$$P(4,2) = \frac{4!}{(4-2)!} = 12$$

The number of 2-combinations of $\{0, 1, 2, 3\}$ is:

$$\binom{4}{2} = \frac{4!}{2! \cdot (4-2)!} = 6$$

Combinations – Example: Number of Teams

1. How many distinct 5-person teams can be chosen from a set of 12 people?

That is the number of 5-combinations of 12-sized set:

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = 11 \cdot 9 \cdot 8 = 792$$

2. Suppose that 2 members of the set insist on working together – any team must contain either both or neither. How many 5-person teams can be formed?

Call the two members that want to work together as A and B.
Set of teams with both A and B is: $\binom{10}{3} = 120$.
Set of teams with neither A nor B is: $\binom{10}{5} = 252$.
The total number of teams that contain either both A and B, or neither A nor B is: $120 + 252 = 372$.

3. Suppose that 2 members of the set refuse on working together. How many 5-person teams can be formed?

Call the two members that refuse to work together as C and D.
Set of teams that contain C but not D is: ?.
Set of teams that contain D but not C is: ?.
Set of teams that contain neither C nor D is: ?.

Outline

r-Combinations with Repetition Allowed

We showed that there are $\binom{n}{r}$ r-combinations of a set of n elements.

For example, there are $\binom{4}{3} = 4$ ways to choose 3 elements out of a set of 4: $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$.

How many ways are there to choose r elements without regard to order from a set of n elements **if repetition is allowed**?

For example, out of a set of size 4, it is possible to choose two elements of type 3 and one of type 1, or all three of type 2. We denote such choices by $[3, 3, 1]$ and $[2, 2, 2]$, respectively. Note that since the order does not matter: $[3, 3, 1] = [1, 3, 3] = [3, 1, 3]$.

List all 3-combinations with repetition allowed from $\{1, 2, 3, 4\}$:

$$[1, 1, 1]; [1, 1, 2]; [1, 1, 3]; [1, 1, 4]$$
$$[1, 2, 2]; [1, 2, 3]; [1, 2, 4]$$
$$[1, 3, 3]; [1, 3, 4]; [1, 4, 4]$$
$$[2, 2, 2]; [2, 2, 3]; [2, 2, 4]$$
$$[2, 3, 3]; [2, 3, 4]; [2, 4, 4]$$
$$[3, 3, 3]; [3, 3, 4]; [3, 4, 4]$$
$$[4, 4, 4]$$

In total: 20

r-Combinations with Repetition Allowed

Theorem

The number of r-combinations with repetition allowed, or **multisets of size r**, *that can be selected from a set of n elements is:*

$$\binom{r+n-1}{r}$$

r-Combinations with Repetition Allowed – Example

How many possible ways of placing an order are there, if we want to buy 15 cans of soft drinks from a store that sells 5 different types of soft drinks.

Here order does not matter, and repetitions are possible. We can think about 5 different types of soft drinks as $n = 5$, and 15 cans to be chosen as $r = 15$. Then, we have:

$$\binom{15+5-1}{15} = \frac{19!}{15! \cdot 4!} = 3,876$$

r-Permutations with Repetition Allowed

We count different ways to choose r elements with regard to order from a set of n elements if repetition is allowed.

Theorem

The number of r-permutations with repetition allowed that can be selected from a set of n elements is:

$$n^r$$

Example. How many 3 digit numbers can be formed with the digits: 1, 2, 3, 4, 5?
The order of the elements matters, and we allow repetitions.
Therefore, there are

$$5^3 = 125$$

different numbers that can be formed.

Summary

Choose r objects from n	Order matters	Order does not matter
Repetition is allowed	n^r	$\binom{r+n-1}{r}$
Repetition is not allowed	$P(n,r) = \frac{n!}{(n-r)!}$	$\binom{n}{r}$

Pascal's Triangle for Values of $\binom{n}{r}$

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r / n	0	1	2	3	4	5	$\dots$	$r-1$	r	$\dots$
0	1							.	.	$\dots$
1	1	1						.	.	$\dots$
2	1	2	1					.	.	$\dots$
3	1	3	3	1				.	.	$\dots$
4	1	4	6 +	4	1			.	.	$\dots$
5	1	5	10 =	10	5	1		.	.	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\dots$	$\binom{n}{r-1}$ +	$\binom{n}{r}$	$\dots$
$n+1$	$\binom{n+1}{0}$	$\binom{n+1}{1}$	$\binom{n+1}{2}$	$\binom{n+1}{3}$	$\binom{n+1}{4}$	$\binom{n+1}{5}$	$\dots$		= $\binom{n+1}{r}$	$\dots$
.		.	.	.	.	.		.	.	$\dots$
.		.	.	.	.	.		.	.	$\dots$
.		.	.	.	.	.		.	.	$\dots$