Mandatory 4

Monday, March 25, 2019 6:11 PM

Exercise 1

1. Compute eigenvalues and eigenvectors for the following matrix

$$B = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Diagonalise the matrix using a orthogonal matrix, i.e., find orthogonal Q, diagonal Λ such that $B = Q\Lambda Q^T$. Write $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a linear combination of eigenvectors and use it to compute a closed formula for

$$B^n \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Let B be a 2x2 square matrix, v a vector and λ a scalar that satisfy $Bv = \lambda v$, then v is an eigenvector of B and λ is the eigenvalue of B

Eigenvalues

First we form the characteristic equation to find the eigenvalues

$\det(\lambda I - B) = 0$

$\lambda I = \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix}$

$\lambda I - B = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$

$\lambda I - B = \begin{vmatrix} \lambda - 2 & 3 \\ 3 & \lambda - 2 \end{vmatrix}$

$\det(\lambda I - B) = \begin{vmatrix} \lambda - 2 & 3 \\ 3 & \lambda - 2 \end{vmatrix} = (\lambda - 2)(\lambda - 2) - 3 * 3$

$\det(\lambda I - B) = \lambda^2 - 2\lambda - 2\lambda + 4 - 9$

$\det(\lambda I - B) = \lambda^2 - 4\lambda - 5$

We can factorize the characteristic equation and use the zero-rule to determine the roots

$\lambda^2 - 4\lambda - 5 = 0$

$(x + 1)(x - 5) = 0$

The eigenvalues are thus

$\lambda_1 = -1$

$\lambda_2 = 5$

Eigenvectors

We can then use the two eigenvalues to calculate the eigenvectors

The first eigenvector can be found:

$\lambda_1 I - B = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} -1 - 2 & 3 \\ 3 & -1 - 2 \end{vmatrix} = \begin{vmatrix} -3 & 3 \\ 3 & -3 \end{vmatrix}$

Reducing the matrix with row operations produces a matrix in reduced row echelon form

$\begin{vmatrix} -3 & 3 \\ 3 & -3 \end{vmatrix} \Rightarrow \begin{vmatrix} -3 & 3 \\ 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix}$

The system associated with the eigenvalue $\lambda_1 = -1$ can be written as

$(\lambda I - B) \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

The system above can be reduced to
$-x_1 + x_2 = 0$
$let\ x_2 = t$ we can conclude every eigenvector of λ_1 is form
$v_1 = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} -t \\ t \end{vmatrix} = t \begin{vmatrix} -1 \\ 1 \end{vmatrix}, \qquad t \neq 0$

One eigenvector of λ_1 is
Let t = 1
$v_1 = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$

The second eigenvector can be found
$\lambda_2 I - B = \begin{vmatrix} 5 & 0 \\ 0 & 5 \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} 5-2 & 3 \\ 3 & 5-2 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 3 & 3 \end{vmatrix}$

Reducing the matrix with row operations produces a matrix in reduced row echelon form
$\begin{vmatrix} 3 & 3 \\ 3 & 3 \end{vmatrix} \Rightarrow \begin{vmatrix} 3 & 3 \\ 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$

The system associated with the eigenvalue $\lambda_2 = 5$ can be written as
$(\lambda I - B) \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$
$\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

The system above can be reduced to
$x_1 + x_2 = 0$
$let\ x_2 = t$ we can conclude every eigenvector of λ_2 is form
$v_2 = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} t \\ t \end{vmatrix} = t \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \qquad t \neq 0$

One eigenvector of λ_2 is
Let t = 1
$v_2 = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

And so the eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of B is
$\lambda_1 = -1, \qquad \lambda_2 = 5$
$v_1 = \begin{vmatrix} -1 \\ 1 \end{vmatrix}, \qquad v_2 = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

Diagonalizing the Matrix using an Orthogonal matrix
Recall the matrix B is a 2x2 matrix
$B = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$

We know that B is orthogonally diagonalizable if and only if B is symmetric
Recall that a symmetric matrix is a square matrix that is equal to its transpose:
$B = B^T$
$\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$

Since we know that matrix B is symmetric we can show that B can be orthogonal diagonalized

Recall the characteristic equation of B was
$p(x) = (x+1)(x-5)$
Which produced two distinct eigenvalues ($\lambda_1 = -1,\ \lambda_2 = 5$) and each occurs only once and thus the multiplicity of each eigenvalue is 1. We know that by theorem 7.7 from the book that an eigenvalue λ of a symmetric matrix A with multiplicity k has k linearly independent eigenvectors.
Thus both eigenvalues of B has 1 linearly independent eigenvector

Recall that the eigenvectors v_1, v_2 of B is
$v_1 = (-1,1), \qquad v_2 = (1,1)$

First we can check if the set of vectors {v1,v2} is orthogonal
Recall that two vectors are orthogonal if their dot product equals to zero
$v_1 \cdot v_2 = (-1 * 1) + (1 * 1) = 0$

And so the eigenvectors v1 and v2 form an orthogonal basis for R^2. We can normalize these eigenvectors to produce an orthonormal basis:

$$p_1 = \frac{v_1}{||v_1||} = \frac{(-1,1)}{\sqrt{-1^2+1^2}} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$p_2 = \frac{v_2}{||v_2||} = \frac{(1,1)}{\sqrt{1^2+1^2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

We can then form a matrix Q where p1 and p2 is its columns:

$$Q = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

And verify that Q orthogonally diagonalizes B by finding
$Q^{-1}BQ = Q^TBQ$

$$Q^TBQ = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$Q^TBQ = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix}$$

The product yields a 2x2 matrix where the main diagonal consists of the eigenvalues which implies that B was orthogonally diagonalized correctly.

And finally we can show that $B = Q^T\mathrm{A}Q$

$$let\ \mathrm{A} = \begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix}$$

$$B = Q^T\mathrm{A}Q = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} \begin{vmatrix} -1 & 0 \\ 0 & 5 \end{vmatrix} \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$$

Conclusion
Thus, the eigenvalues λ_1, λ_2 and eigenvectors $\mathrm{v}_1, \mathrm{v}_2$ of B is
$\lambda_1 = -1, \qquad \lambda_2 = 5$
$v_1 = \begin{vmatrix} -1 \\ 1 \end{vmatrix}, \qquad v_2 = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

And the symmetric matrix B is orthogonally diagonalizable, where the orthogonal matrix Q

$$Q = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

Orthogonally diagonalizes B such that
$Q^T\mathrm{B}Q = \mathrm{A} \Rightarrow Q^T\mathrm{A}Q = B$

Exercise 1.1

Write $\begin{vmatrix} 2 \\ 0 \end{vmatrix}$ as a linear combination of eigenvectors and use it to compute a closed formula for
$B^n \begin{vmatrix} 2 \\ 0 \end{vmatrix}$

Recall that the every eigenvectors of $\lambda_1$and λ_2 can be written in form of
$v_1 = t_1 \begin{vmatrix} -1 \\ 1 \end{vmatrix}, \qquad t_1 \neq 0$

$v_2 = t_2 \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \qquad t_2 \neq 0$

Thus a linear combination of the eigenvectors can be found to form the vector $\begin{vmatrix}2\\0\end{vmatrix}$

$let\ t_1 = -1\,, \qquad t_2 = 1$

$\begin{vmatrix}2\\0\end{vmatrix} = -1\mathrm{v}_1 + 1v_2$

$\begin{vmatrix}2\\0\end{vmatrix} = -1\begin{vmatrix}-1\\1\end{vmatrix} + 1\begin{vmatrix}1\\1\end{vmatrix}$

$\begin{vmatrix}2\\0\end{vmatrix} = \begin{vmatrix}1\\-1\end{vmatrix} + \begin{vmatrix}1\\1\end{vmatrix}$

$\begin{vmatrix}2\\0\end{vmatrix} = \begin{vmatrix}2\\0\end{vmatrix}$

And so a linear combination that can be used to compute a closed formula is

$\begin{vmatrix}2\\0\end{vmatrix} = -1\begin{vmatrix}-1\\1\end{vmatrix} + 1\begin{vmatrix}1\\1\end{vmatrix}$

Recall the closed formula

$B^n\begin{vmatrix}2\\0\end{vmatrix}$

We can replace the formula with

$B^n\begin{vmatrix}2\\0\end{vmatrix} = B^n * \left(-1\begin{vmatrix}-1\\1\end{vmatrix} + 1\begin{vmatrix}1\\1\end{vmatrix}\right)$

Since we we know that

$B^n * \vec{v} = \lambda^n * \vec{v}$ | where $\vec{v}$ is an eigenvector.

we can write

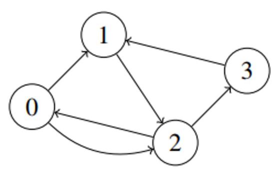
$B^n * \left(-1\begin{vmatrix}-1\\1\end{vmatrix} + 1\begin{vmatrix}1\\1\end{vmatrix}\right) = \lambda^n * \left(-1\begin{vmatrix}-1\\1\end{vmatrix} + 1\begin{vmatrix}1\\1\end{vmatrix}\right) = \left(-1 * \lambda^n\begin{vmatrix}-1\\1\end{vmatrix} + 1 * \lambda^n\begin{vmatrix}1\\1\end{vmatrix}\right) = \left(\lambda^n\begin{vmatrix}1\\-1\end{vmatrix} + \lambda^n\begin{vmatrix}1\\1\end{vmatrix}\right) = \left(\begin{vmatrix}\lambda^n\\-\lambda^n\end{vmatrix} + \begin{vmatrix}\lambda^n\\\lambda^n\end{vmatrix}\right)$

And so we have

$B^n\begin{vmatrix}2\\0\end{vmatrix} = \left(\begin{vmatrix}\lambda^n\\-\lambda^n\end{vmatrix} + \begin{vmatrix}\lambda^n\\\lambda^n\end{vmatrix}\right)$

Exercise 2

2. Consider the web corresponding to the graph below. The web consists of 4 web pages with links as indicated, i.e., page 0 links to pages 1 and 2 etc.



Construct the matrix M for which the importance score vector x should be an eigenvector. The matrix M should depend on an unknown damping factor m. For the specific case of $m = 0$ **write out** the 4 linear equations defining the importance scores x_0, x_1, x_2, x_3, and **solve these** to derive a ranking of the pages. Of the two highest ranking pages, **explain in words** why one is considered more important than the other. See the lecture notes for the formula for the matrix M.

The PageRank algorithm ranks web pages by their importance by counting the number and quality of links to a page to determine a rough estimate. The idea is that more important websites are likely to receive more clicks from others.

The PageRank algorithm outputs a probability distribution used to represent the probability
that a person randomly clicking on links will arrive at any particular page. One may then sort this output by probability of each page in descending order and get a ranked list.

Given graph above we can create a system of linear equations that determines the importance score x_i of page i

$x_0 = \frac{1}{2}x_2$

$x_1 = \frac{1}{2}x_0 + x_3$
$x_2 = \frac{1}{2}x_0 + 1x_1$
$x_3 = \frac{1}{2}x_2$

We can then form a 4x4 square coefficient matrix M to represent the system of equations above:

$$M\boldsymbol{x} = \begin{vmatrix} 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{vmatrix} * \begin{vmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{vmatrix}$$

Existence
We need to ensure that for M, an eigenvector of importance is an eigenvector for eigenvalue 1 exists.

First we check that that Matrix M is column stochastic. Recall that a matrix is column stochastic if and only if all entries are nonnegative (≥ 0) and the columns sum to 1

In this regard this is true for M

$$M = \begin{vmatrix} 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{vmatrix}$$
$$\begin{matrix} 1 & 1 & 1 & 1 \end{matrix}$$

In case that M was not column stochastic which means that the network has at least one dangling node then one may replace M with

$(M + D)$ Where $\mathrm{D_{ij}} = \begin{Bmatrix} 0\ if\ there\ is\ link\ out\ of\ j \\ otherwise\ \frac{1}{n} \end{Bmatrix} \mid n = number\ of\ nodes$

which produces a column stochastic matrix.
However, since there exists no dangling nodes in M then matrix D would be a zero matrix and therefore

$M + D = M$

Thus, M is column stochastic and there exists an eigenvector for eigenvalue 1

Uniqueness
Secondly, we need to prove uniqueness. That is we need to ensure that there is a unique importance vector such that each page has a unique ranking. Such importance vector has to ensure that all ranking scores are positive and sum to 1

Let $m = damping\ factor : 0 < m < 1$
Let $S = \left|\frac{1}{4}\right| : 4x4\ matrix\ where\ all\ entries\ are\ \frac{1}{4}$

We can then use the equation to produce a column stochastic matrix

$W = (1 - m)(M + D) + mS$

In this assignment we set damping factor to zero regardless of the condition above
Let $m = 0$

$$W = (1 - 0)\begin{vmatrix} 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{vmatrix} 0 * S$$

$$W = \begin{vmatrix} 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{vmatrix}$$

Compute eigenvector of webpage rankings
And so we can compute the importance eigenvector from the matrix W

$$W = \begin{vmatrix} 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{vmatrix}$$

First we write out a system of linear equation that represents the importance scores

$x_0 = \frac{1}{2}x_2$

$x_1 = \frac{1}{2}x_0 + x_3$

$x_2 = \frac{1}{2}x_0 + 1x_1$

$x_3 = \frac{1}{2}x_2$

We can then choose an arbitrary value for one of the variables above so
$let\ x_2 = 4$

We can then solve the rest of the system of equations above

$x_0 = \frac{1}{2}x_2 \Rightarrow \frac{1}{2} * 4 = 2$

$x_1 = \frac{1}{2}x_0 + x_3 \Rightarrow \frac{1}{2} * 2 + 2 = 3$

$x_2 = 4$

$x_3 = \frac{1}{2}x_2 \Rightarrow \frac{1}{2} * 4 = 2$

Verify that $x_2 = \frac{1}{2}x_0 + 1x_1 = 4$

$x_2 = \frac{1}{2} * 2 + 1 * 3 = 4$

And so we can produce a coefficient matrix representing the importance score such that the higher the value the higher the importance

$$R\boldsymbol{x} = \begin{vmatrix} 2 \\ 3 \\ 4 \\ 2 \end{vmatrix} * \begin{vmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{vmatrix}$$

We can then sort the matrix by importance score in descending order

$$R\boldsymbol{x} = \begin{vmatrix} 4 \\ 3 \\ 2 \\ 2 \end{vmatrix} * \begin{vmatrix} x_2 \\ x_1 \\ x_0 \\ x_3 \end{vmatrix}$$

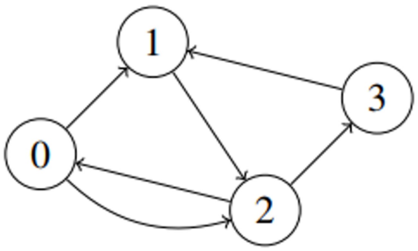
Thus we know that the importance of the pages is as following

$x_2 > x_1 > x_0 = x_3$

An interesting fact is that $x_2 > x_1$ even though that both pages receive the same amount of votes
Recall that:

$$x_1 = \frac{1}{2}x_0 + 1x_3$$
$$x_2 = \frac{1}{2}x_0 + 1x_1$$

That is x_1 and x_2 both receive 1 and 1/2 votes. However the defining factor that make $x_2 > x_1$
is the fact that x_1 receives a full votes from x_3 which is $x_3 < x_1$ wheres x_2 receives a full vote from x_1.
This implies that the full vote that x_2 receives is of higer importance than the full vote that x_1 receives.
And so, we have that $x_2 > x_1$



Compute eigenvector of web page rankings using iterative method (*Optional where I used my python implementation* **)**
Alternatively assume that $m \neq 0$ we can calculate the page ranking with the iterative method
Let $m = 0.15$
To make my life a bit easier we can replace the matrix S with $\frac{1}{4}$ since all values are the same.

$$W' = (1 - 0.15)\begin{vmatrix} 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{vmatrix} 0.15 * \frac{1}{4}$$

$$W' = (0.85)\begin{vmatrix} 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{vmatrix} * 0.0375$$

$$W' = \begin{vmatrix} 0 & 0 & 0.425 & 0 \\ 0.425 & 0 & 0 & 0.85 \\ 0.425 & 0.85 & 0 & 0 \\ 0 & 0 & 0.425 & 0 \end{vmatrix} * 0.0375$$

$$W' = \begin{vmatrix} 0 & 0 & 0.0159375 & 0 \\ 0.0159375 & 0 & 0 & 0.031875 \\ 0.0159375 & 0.031875 & 0 & 0 \\ 0 & 0 & 0.0159375 & 0 \end{vmatrix}$$

Finally, we can then compute an approximation of the eigenvector by using the formula iteratively until the results converges
$\mathbf{x}_{k+1} = W'\boldsymbol{x}_{\boldsymbol{k}}$

Where the initial vector x_0 has $\frac{1}{n}$ on all entries and n is the number of nodes in graph

$$\mathbf{x}_0 = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{vmatrix}$$

$$\mathbf{x_1} = \boldsymbol{W'x_0} = \begin{vmatrix} 0 & 0 & 0.0159375 & 0 \\ 0.0159375 & 0 & 0 & 0.031875 \\ 0.0159375 & 0.031875 & 0 & 0 \\ 0 & 0 & 0.0159375 & 0 \end{vmatrix} * \begin{vmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{vmatrix}$$

…

I am using the PageRank implementation I made to do the rest of the calculations and it converges on k = 57 | [Link to PageRank.py]

$$\mathbf{x}_{57} = W\boldsymbol{x_{56}} = \begin{vmatrix} 0.1866997662580265 & 0.1866997662580265 & 0.1866997662580265 & 0.1866997662580265 \\ 0.27554219720119133 & 0.27554219720119133 & 0.27554219720119133 & 0.27554219720119133 \\ 0.27554219720119133 & 0.27554219720119133 & 0.27554219720119133 & 0.27554219720119133 \\ 0.1866997662580265 & 0.1866997662580265 & 0.1866997662580265 & 0.1866997662580265 \end{vmatrix}$$

$$\mathbf{x}_{57} = W\boldsymbol{x_{56}} \approx \begin{vmatrix} 0.187 & 0.187 & 0.187 & 0.187 \\ 0.276 & 0.276 & 0.276 & 0.276 \\ 0.276 & 0.276 & 0.276 & 0.276 \\ 0.187 & 0.187 & 0.187 & 0.187 \end{vmatrix}$$

One may then sort the matrix which produces an ordered sequence by probability that also represents the ranking of the pages
$ranks = \{x_2, x_1, x_0, x_3\}$

Note that x_2, x_1 and x_0,x_3 are equally ranked since their probailities are equal.

Recall that the PageRank algorithm outputs a probability distribution used to represent the probability that a person randomly clicks on links will arrive at any particular page.

Assume that a page x has a higher probability than a page y then the likelihood that a person, that randomly clicks on links in the web, will land on page x over page y. Hence page x will then have a higher rank than page y.