

# Solutions Lecture 12

## Intelligent System Programming (ISP)

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### Exercise 1 (adapted from C83 3.9)

#### First Phase

Aux. problem

Maximize  $-x_0$

Subject to

$$x_1 - x_2 - x_0 \leq -1$$

$$-x_1 - x_2 - x_0 \leq -3$$

$$2x_1 + x_2 - x_0 \leq 4$$

#### Pre-Initial dictionary

$$x_3 = -1 - x_1 + x_2 + x_0$$

$$x_4 = -3 + x_1 + x_2 + x_0 \quad \text{Pivot out most negative variable to get feasible dictionary}$$

$$x_5 = 4 - 2x_1 - x_2 + x_0$$

$$W = \quad \quad \quad -x_0$$

#### Initial dictionary

Max increase of  $x_2$

$$x_3 = -1 - x_1 + x_2 + (3 - x_1 - x_2 + x_4)$$

$$= 2 - 2x_1 + x_4 \quad 1$$

$$x_0 = 3 - x_1 - x_2 + x_4 \quad 3$$

$$x_5 = 4 - 2x_1 - x_2 + (3 - x_1 - x_2 + x_4)$$

$$= 7 - 3x_1 - 2x_2 + x_4 \quad 3.5$$

$$W = -(3 - x_1 - x_2 + x_4)$$

$$= -3 + x_1 + x_2 - x_4$$

**First dictionary: increase  $x_2$  to 3 (we choose  $x_2$  rather than  $x_1$  since this makes  $x_0$  non-basic)**

$$x_3 = 2 - 2x_1 + x_4$$

$$x_2 = 3 - x_1 + x_4 - x_0$$

$$x_5 = 7 - 3x_1 - 2(3 - x_1 + x_4 - x_0) + x_4$$

$$= 1 - x_1 - x_4 + 2x_0$$

$$W = -3 + x_1 + (3 - x_1 + x_4 - x_0) - x_4$$

$$= -x_0 \quad (\text{not surprising})$$

First phase over, remove  $x_0$  and express  $z$  in terms the other non-basic variables  $x_1$  and  $x_4$

## Second Phase

### Initial dictionary

Max increase of  $x_4$

$$\begin{array}{ll} x_3 = 2 - 2x_1 + x_4 & \text{inf} \\ x_2 = 3 - x_1 + x_4 & \text{inf} \\ x_5 = 1 - x_1 - x_4 & 1 \\ z = 3x_1 + x_2 & \\ = 3x_1 + (3 - x_1 + x_4) & \\ = 3 + 2x_1 + x_4 & \end{array}$$

Obs.: The text-book often chooses the variable in the  $z$  expression with largest coefficient. This would be  $x_1$ . But you are allowed to choose any variable with positive coefficient (in particular if this leads to simpler computations for you). In this case, we choose  $x_4$ .

### First dictionary

Max increase of  $x_1$

$$\begin{array}{ll} x_3 = 2 - 2x_1 + (1 - x_1 - x_5) & \\ = 3 - 3x_1 - x_5 & 1 \\ x_2 = 3 - x_1 + (1 - x_1 - x_5) & \\ = 4 - 2x_1 - x_5 & 2 \\ x_4 = 1 - x_1 - x_5 & 1 \\ z = 3 + 2x_1 + (1 - x_1 - x_5) & \\ = 4 + x_1 - x_5 & \end{array}$$

### Second dictionary (optimal)

$$\begin{array}{ll} x_3 = 3 - 3(1 - x_4 - x_5) - x_5 & \\ = 3x_4 + 2x_5 & \\ x_2 = 4 - 2(1 - x_4 - x_5) - x_5 & \\ = 2 + 2x_4 + x_5 & \\ x_1 = 1 - x_4 - x_5 & \\ z = 4 + (1 - x_4 - x_5) - x_5 & \\ = 5 - x_4 - 2x_5 & \text{(error in solutions in C83!)} \end{array}$$

## Exercise 2

There are many possible solutions. Below we show one of them

### Standard Form

Maximize  $x_1 + x_2$

Subject to

$$x_1 + x_2 \leq 0$$

$$x_1, x_2 \geq 0$$

### Slack Form (initial dictionary)

Max increase of  $x_1$

$$x_3 = 0 - x_1 - x_2$$

0

$$z = x_1 + x_2$$

First dictionary (degenerate pivot since no change of basic solution)

$$x_1 = 0 - x_2 - x_3$$

$$z = (0 - x_2 - x_3) + x_2$$

$$= 0 - x_3$$

## Exercise 3 (adopted from C83 3.2)

### Initial dictionary

Maximum increase of  $x_2$

$$x_5 = 0 + 2x_1 + 9x_2 - x_3 - 9x_4$$

inf

$$x_6 = 0 - 1/3x_1 - x_2 + 1/3x_3 + 2x_4$$

0

$$z = 2x_1 + 3x_2 - x_3 - 12x_4$$

### 1. dictionary

Maximum increase of  $x_1$

$$x_5 = 0 + 2x_1 + 9(-1/3x_1 + 1/3x_3 + 2x_4 - x_6) - x_3 - 9x_4$$

0 (preferred due to tie breaking rule)

$$= 0 - 1/3x_1 + 1/3x_3 + 2x_4 - x_6$$

$$x_2 = 0 - 1/3x_1 + 1/3x_3 + 2x_4 - x_6$$

0

$$z = 2x_1 + 3(-1/3x_1 + 1/3x_3 + 2x_4 - x_6) - x_3 - 12x_4$$

$$= x_1 - 6x_4 - 3x_6$$

### 2. dictionary

$$\begin{aligned}
x_1 &= 0 + 2x_3 + 9x_4 - x_5 - 9x_6 \\
x_2 &= 0 - 1/3(2x_3 + 9x_4 - x_5 - 9x_6) + 1/3x_3 + 2x_4 - x_6 \\
&\quad 0 - 1/3x_3 - x_4 + 1/3x_5 + 2x_6 \\
z &= (2x_3 + 9x_4 - x_5 - 9x_6) - 6x_4 - 3x_6 \\
&= 2x_3 + 3x_4 - x_5 - 12x_6
\end{aligned}$$

Maximum increase of  $x_4$

inf

0

### 3. dictionary

$$\begin{aligned}
x_1 &= 0 + 2x_3 + 9(-x_2 - 1/3x_3 + 1/3x_5 + 2x_6) - x_5 - 9x_6 \\
&= 0 - 9x_2 - x_3 + 2x_5 + 9x_6 \\
x_4 &= 0 - x_2 - 1/3x_3 + 1/3x_5 + 2x_6 \\
z &= 2x_3 + 3(-x_2 - 1/3x_3 + 1/3x_5 + 2x_6) - x_5 - 12x_6 \\
&= -3x_2 + x_3 - 6x_6
\end{aligned}$$

Maximum increase of  $x_3$

0 (preferred due to tie breaking rule)

0

### 4. dictionary

$$\begin{aligned}
x_3 &= 0 - x_1 + 9x_2 + 2x_5 + 9x_6 \\
x_4 &= 0 - x_2 - 1/3(-x_1 + 9x_2 + 2x_5 + 9x_6) + 1/3x_5 + 2x_6 \\
&= 0 + 1/3x_1 + 2x_2 - 1/3x_5 - x_6 \\
z &= -3x_2 + (-x_1 + 9x_2 + 2x_5 + 9x_6) - 6x_6 \\
&= -x_1 - 12x_2 + 2x_5 + 3x_6
\end{aligned}$$

Maximum increase of  $x_6$

inf

0

### 5. dictionary

$$\begin{aligned}
x_3 &= 0 - x_1 + 9x_2 + 2x_5 + 9(1/3x_1 + 2x_2 - x_4 - 1/3x_5) \\
&= 0 + 2x_1 + 9x_2 - 9x_4 - x_5 \\
x_6 &= 0 + 1/3x_1 + 2x_2 - x_4 - 1/3x_5 \\
z &= -x_1 - 12x_2 + 2x_5 + 3(1/3x_1 + 2x_2 - x_4 - 1/3x_5) \\
&= -6x_2 - 3x_4 + x_5
\end{aligned}$$

Maximum increase of  $x_5$

0 (preferred due to tie breaking rule)

0

### 6. dictionary (identical to initial dictionary, thus cycle found)

$$x_5 = 0 + 2x_1 + 9x_2 - x_3 - 9x_4$$

$$x_6 = 0 + 1/3x_1 + 2x_2 - x_4 - 1/3(2x_1 + 9x_2 - x_3 - 9x_4)$$

$$= 0 - 1/3x_1 - x_2 + 1/3x_3 + 2x_4$$

$$z = -6x_2 - 3x_4 + (2x_1 + 9x_2 - x_3 - 9x_4)$$

$$= 2x_1 + 3x_2 - x_3 - 12x_4$$

## Exercise 4

No, the simplex algorithm cannot in general find optimal solutions to such problems. The problem is that an optimal solution may only exist in the interior of the polyhedron defined by the constraints. For a linear objective this is never the case which simplex exploits by only exploring the corner points.