Mandatory 6

Sunday, May 05, 2019 12:02 PM

Tree exercise

Tree Exercise We model the diameter of a tree with a random variable X and its height with a random variable Y. Their joint probability density function is given by:

$$f_{X,Y} = \begin{cases} \frac{1}{4}(x+y)e^{-y} & \text{when } y \geq 0, 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

NB. In this exercise, you can use the values $\int_0^\infty ye^{-y}dy = 1$ and $\int_0^\infty y^2e^{-y}dy = 2$ from the book with no justification.

$X = Diameter\ of\ tree$
$Y = Height\ of\ tree$

1) Compute the marginal PDF of X and the expected value $\mathbb{E}[X]$

We can compute the marginal PDF of X by integrating the joint PDF in regards to Y

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)\,dy$$

$$f_X(x) = \int_0^\infty \frac{1}{4}(x+y)e^{-y}\,dy$$

$$f_X(x) = \frac{1}{4}\int_0^\infty (x+y)e^{-y}\,dy$$

$$f_X(x) = \frac{1}{4}x\int_0^\infty ye^{-y}\,dy$$

$$f_X(x) = \frac{1}{4}x * 1$$

$$f_X(x) = \begin{Bmatrix} \frac{1}{4}x & 0 \leq x \leq 2 \\ 0 & otherwise \end{Bmatrix}$$

We can then use the marginal PDF of X to compute the expected value of X

$$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx$$

$$E[X] = \int_0^2 x\frac{1}{4}x\,dx$$

$$E[X] = \frac{1}{4}\int_0^2 x^2\,dx$$

$$E[X] = \frac{1}{4} * \left[\frac{1}{3}x^3\Big|_0^2\right]$$

$$E[X] = \frac{1}{4} * \left[\frac{1}{3}2^3 - \frac{1}{3}0^3\right]$$

$$E[X] = \frac{1}{4} * \frac{1}{3}2^3$$

$$E[X] = \frac{1}{4} * \frac{8}{3}$$
$$E[X] = \frac{8}{12} = \frac{2}{3}$$

2) Compute the conditional PDF $f_{Y|X}(y \mid x)$, are X and Y indepndent?
The conditional PDF can be computed
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{Y|X}(y|x) = \frac{\frac{1}{4}(x+y)e^{-y}}{\frac{1}{4}x}$$

$$f_{Y|X}(y|x) = \frac{(x+y)e^{-y}}{x}$$
Note that we cannot remove 1/4x but only 1/4 because it is not multiplication

And so we have the conditional PDF Y given X
$$f_{Y|X}(y|x) = \begin{Bmatrix} \frac{(x+y)e^{-y}}{x} & y \geq 0, 0 \leq x \leq 2 \\ 0 & otherwise \end{Bmatrix}$$

Are X and Y independent?
For X and Y to be independent it must be shown that
$$f_{X,Y}(y,x) = f_X(x) f_Y(y)$$

So we can find the marginal PDF of y
$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)\, dy$$
$$f_Y(y) = \int_0^2 \frac{1}{4}(x+y)e^{-y}\, dx$$
Applying the sum rule
$$f_Y(y) = \frac{1}{4}e^{-y}\left(\int_0^2 x\, dx + \int_0^2 y\, dx\right)$$
Applying integral of a constant
$$f_Y(y) = \frac{1}{4}e^{-y} * \left(\left[\frac{x^2}{2}\Big|_0^2\right] + \left[yx\Big|_0^2\right]\right)$$

$$f_Y(y) = \frac{1}{4}e^{-y} * \left(\left[\frac{2^2}{2} - 0\right] + [y2 - 0]\right)$$

$$f_Y(y) = \frac{1}{4}e^{-y} * \left(\frac{2^2}{2} + 2y\right)$$
$$f_Y(y) = \frac{1}{4}e^{-y} * (2 + y2)$$
$$f_Y(y) = \frac{1}{4}e^{-y} * (2 + y2)$$

$f_Y(y) = \frac{1}{2}e^{-y} * (1 + y)$

Recall

$f_{X,Y}(y, x) = f_X(x)f_Y(y)$

$f_{X,Y}(y, x) = \left(\frac{1}{4}x\right) * \left(\frac{1}{2}e^{-y} * (1 + y)\right)$

$f_{X,Y}(y, x) = \frac{1}{8}xe^{-y}(y + 1)$

But

$\frac{1}{4}(x + y)e^{-y} \neq \frac{1}{8}xe^{-y}(y + 1)$

Which implies

$f_{X,Y}(y, x) \neq f_X(x)f_Y(y)$

And so X and Y are not independent

3) Age of a tree is obtained by $W = 12XY$, Compute expected age of tree and diameter *x*

$X = Diameter\ of\ tree$

$Y = Height\ of\ tree$

$E[W|X = x] = Expected\ age$

$$E[W|X = x] = E[g(X, Y)|X = x] = \int_0^\infty g(x, y) * f_{Y|X}(y|x)\, dy$$

$$E[g(X, Y)|X = x] = \int_0^\infty 12XY * \frac{(x + y)e^{-y}}{x} dy$$

$$E[g(X, Y)|X = x] = 12\int_0^\infty XY * \frac{(x + y)e^{-y}}{x} dy$$

$$E[g(X, Y)|X = x] = 12\int_0^\infty \frac{xy * (x + y)e^{-y}}{x} dy$$

$$E[g(X, Y)|X = x] = 12\int_0^\infty y * (x + y)e^{-y}\, dy$$

$$E[g(X, Y)|X = x] = 12\int_0^\infty (xy + y^2)e^{-y}\, dy$$

$$E[g(X, Y)|X = x] = 12\int_0^\infty (xye^{-y} + y^2e^{-y})\, dy$$

Applying the sum rule

$$E[g(X, Y)|X = x] = 12\left(\int_0^\infty xye^{-y}\, dy + \int_0^\infty y\, dy\right)$$

$$E[g(X, Y)|X = x] = 12\left(x\int_0^\infty ye^{-y}\, dy + \int_0^\infty y\, dy\right)$$

$E[g(X, Y)|X = x] = 12(x + 2)$

$E[g(X, Y)|X = x] = 12x * 24$

$E[g(X, Y)|X = x] = 12x + 24$

Which implies

$E[W|X = x] = 12x + 24$

4) Apply the total expectation theorem on the result of the previous exercise to compute expected age of a tree $E[W]$

$E[W|X = x] = 12x + 24$

Using the total expectation theorem we can compute E[W]

$$E[W] = \int_{-\infty}^{\infty} E[W|X = x] * f_X(x)\, dx$$

$$E[W] = \int_0^2 (12x + 24) * \frac{1}{4}x\, dx$$

$$E[W] = \frac{1}{4}\int_0^2 (12x + 24) * x\, dx$$

$$E[W] = \frac{1}{4}\int_0^2 12x^2 + 24x\, dx$$

Using sum rule

$$E[W] = \frac{1}{4}\left(\int_0^2 12x^2\, dx + \int_0^2 24x\, dx\right)$$

$$E[W] = \frac{1}{4}\left(12\int_0^2 x^2\, dx + 24\int_0^2 x\, dx\right)$$

$$E[W] = \frac{1}{4}\left(\left[4x^3\Big|_0^2\right] + \left[12x^2\Big|_0^2\right]\right)$$

$$E[W] = \frac{1}{4}(4 * 2^3 + 12 * 2^2)$$

$$E[W] = \frac{1}{4}(32 + 48)$$

$$E[W] = \frac{80}{4} = 20$$

SAP Exercise

Exercise 1

Sap office front desk is run by a single employee

Time to serve single student is an **exponential variable** with **mean 2 minutes**

Use Central limit theorem to estimate probability that **at least 36** students can be served within **first hour** after opening

Hint : Use this with 36 random variables each corresponding to the interarrival times

Let

$X_i = time\ to\ service\ single\ student\ in\ minutes$

$$E[X] = \frac{1}{\lambda} = 2 \quad \Rightarrow \quad \lambda = \frac{1}{2}$$

$$X_i \sim \exp\left(\lambda = \frac{1}{2}\right) = \mathrm{f(x)} = \begin{Bmatrix} \lambda e^{-\frac{1}{2}x} & x \geq 0 \\ 0 & otherwise \end{Bmatrix}$$

$$var(X_i) = \frac{1}{\lambda^2} = \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{1}{4}} = 4$$

We assume that the 36 students being served within the hour and are independent because they are served by a single employee and he can only serve one student at a time and that the serving time of each student has identical distribution

$P(S_{36} \leq 60) = 1 - P(S_{36} \geq 60)$

And so we have
$n = 36$
$S_{36} = X_1 + \cdots + X_{36}$
$var(S_{36}) = n * var(X_i) = 36 * 4 = 144$
$E[S_{36}] = n * E[X_i] = 36 * 2 = 72$

Using the central limit theorem we can calulate the z value to standadize S_n

$$z = \frac{60 - E[S_{36}]}{\sqrt{var(S_{36})}} = \frac{60 - 72}{\sqrt{144}} = \frac{-12}{12} = -1$$

We can then use the standard table to calculate the probability
$P(S_{36} \leq 60) \approx \Phi(-1) = 1 - \Phi(1) = 1 - 0.8413 = 0{,}1587$

And so the probability that SAP can service 36 students within 60 minutes is approximately 16%

Exercise 2
Management expects 100 students at sap front office each day in average
They would like to know with **at least 80 %** certainty that 100 students can be served within a day

Use **Central limit theorem** to estimate **minimum number of minutes** office should be open each day to meet this requirement

Let c be the minimum number of minutes that are required to serve 100 students, we have that
$n = 100$
$p = 80$

$p(S_{100} \leq c) \geq 0.80$

$E[S_{100}] = 100 * E[X_i] = 100 * 2 = 200$
$var(S_{100}) = 100 * 4 = 400$

Since we already know the probability we are looking for, we can get the z value that are at least 80 percent in the standard table and use that to find c
$\phi(0{,}85) = 0.8023$

$$0{,}85 = \frac{c - E[S_{100}]}{\sqrt{var(S_{100})}}$$

$$0{,}85 = \frac{c - 200}{20}$$

$0{,}85 * 20 = c - 200$

$c = 0{,}85 * 20 + 200$
$c = 217$

Which corresponds to
$217\ minutes$

Thus SAP requires at least 217 minutes to serve 100 students