Discrete Mathematics

Sets, Functions, and Sequences

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Today's lecture

1. Sets
   - Important sets in Discrete Mathematics
   - Intervals
   - Equality of Sets
   - Subsets
   - Venn Diagrams
   - Universal Sets
   - Operations on Sets
   - Partition of a Set
   - The Cartesian product
   - Properties of Sets
2. Functions
   - Types of Correspondences
   - Composition of Functions
   - Inverse of a Function
3. Sequences
   - Geometric progression
   - Arithmetic progression
4. Summation and Product Notations
   - Summations
   - Product Notation
   - Properties of Summations and Products

Outline

1. Sets
2. Functions
3. Sequences
4. Summation and Product Notations

What is a set?

A set A is an (unordered) collection of objects, called elements or members of A.

Examples of sets:

- different brands of bikes
- set of multiples of 2

Sets can be finite or infinite.

Describing sets (I)

We can list all members of a set between braces "{" and "}"

$$A = \{\text{red, blue, yellow}\}$$

$$B = \{1, 2, 3, 5, 6\}$$

When the set is infinite, we use ellipses (...) when the general pattern of the elements is obvious.

$$C = \{1, 2, 3, 5, 7, \ldots\}$$

$$E_n = \{0, 1, 4, 9, 16, 25, \ldots\}$$

Infinite sets are better defined by explicitly stating the membership law (set builder notation)

$$C = \{x \ : \ x \text{ is a prime}\}$$

$$E_n = \{x^2 \ : \ x \in \mathbb{N}\}$$

Describing sets (II)

The cardinality of a set A, denoted by $|A|$, is the number of (distinct) elements it contains.
In the case of infinite sets we have $|A| = \infty$.

The empty set (or null set) is denoted by $\emptyset$ or $\{\}$.

Obviously, $|\emptyset| = 0$.

Notation

$a \in A$ means that a is an element of A
$a \notin A$ means that a is not an element of A

Important:

- $\{a\} \neq a$

  $\{a\}$ is a set consisting of one element a
  a is the element itself

- $\{\emptyset\} \neq \emptyset$

  $\{\emptyset\}$ has one more element than $\emptyset$

- Any set can be a member of another set

$$B = \{1, 2, \{1\}, \{2\}, \{1, 2\}\}$$

Important sets in Discrete Mathematics

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the set of natural numbers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of integers

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, the set of positive integers

$\mathbb{Q} = \{\frac{p}{q} \ : \ p \in \mathbb{Z}, q \in \mathbb{Z},$ and $q \neq 0\}$, the set of rational numbers

$\mathbb{R}$, the set of real numbers

$\mathbb{R}^+$, the set of positive real numbers

Intervals

Recall the notation for intervals of real numbers.

When a and b are real numbers with $a < b$, we write

$$[a, b] = \{x \ : \ a \leq x \leq b\}$$
$$[a, b) = \{x \ : \ a \leq x < b\}$$
$$(a, b] = \{x \ : \ a < x \leq b\}$$
$$(a, b) = \{x \ : \ a < x < b\}$$

$[a, b]$ is the closed interval from a to b, and (a, b) is the open interval from a to b.

Equality of Sets

Two sets A and B are equal if and only if they have the same elements.

$$A = B \text{ if and only if } (\forall x)[x \in A \leftrightarrow x \in B]$$

Example Do $\{1, 2, 3\}$, $\{1, 3, 2\}$ and $\{1, 2, 3, 1\}$ represent the same set?

Yes, they represent the same set, which can also be written as

$$\{x \ : \ x \in \mathbb{Z}^+ \text{ and } x \leq 3\}.$$

Exercise Are the sets $\{x \ : x \in \mathbb{Z} \text{ and } 2x - 3 = 5\}$ and $\{1, 4\}$ equal?

No, the first set is equal to the set $\{4\}$.

Subsets

The set A is a subset of B if and only if every element of A is also in B.

$$A \subseteq B \text{ if and only if } (\forall x)[x \in A \to x \in B]$$

The set B is then said to be a superset of A.

For every set A, $\emptyset \subseteq A$ and $A \subseteq A$.

A set A is a proper subset of B, denoted by $A \subset B$, if A is a subset of B but $A \neq B$.

The power set of a set A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A.

Example What is the power set of $A = \{$ Paris, Rome, Barcelona $\}$?

$\mathcal{P}(A) = \{\emptyset$, {Paris}, {Rome}, {Barcelona}, {Paris, Rome}, {Paris, Barcelona}, {Rome, Barcelona}, {Paris, Rome, Barcelona}}

Venn Diagrams

Venn Diagrams are a way of representing sets.

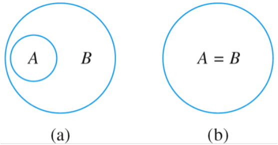


Figure : $A \subseteq B$

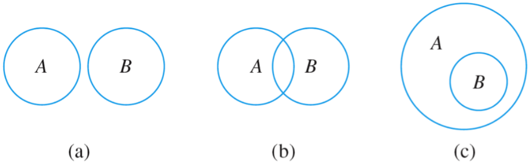


Figure : $A \not\subseteq B$

Universal Sets

The universal set $\mathcal{U}$, contains all the elements currently under consideration. It may be finite or infinite.

- Content depends on the context.
- Sometimes explicitly stated, sometimes implicit.

Examples

a) Writing $\{x : P(x)\}$ actually means $\{x : x \in \mathcal{U}$ and $P(x)\}$.

b) In the set $E_n = \{x^2 : x \in \mathbb{N}$ and $x \leq 10\}$, $\mathbb{N}$ is the universal set.

Operations on Sets (I)

The union of two sets A and B is the set

$$A \cup B = \{x \ : \ x \in A \vee x \in B\}.$$

The intersection of A and B is the set

$$A \cap B = \{x \ : \ x \in A \wedge x \in B\}.$$

Given n sets $A_1, A_2, \ldots, A_n$,

$$\bigcup_{i=1}^{n} A_i = A_1 \cup \ldots \cup A_n \qquad \bigcap_{i=1}^{n} A_i = A_1 \cap \ldots \cap A_n.$$

Exercise Given the sets $A = \{a, c, e\}$ and $B = \{a, b, c\}$,

a) What is their union?

$$A \cup B = \{a, b, c, e\}.$$

b) What is their intersection?

$$A \cap B = \{a, c\}.$$

Operations on Sets (II)

Two sets A and B are disjoint if $A \cap B = \emptyset$.

The principle of inclusion-exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

The difference of two sets A and B, denoted by $A - B$ (or by $A \setminus B$), is the set containing those elements in A but not in B. It is sometimes called the complement of B with respect to A.

Example Given the sets $A = \{a, c, e\}$ and $B = \{a, b, c\}$, we have that

$$A - B = \{e\}.$$

However,

$$B - A = \{b\}.$$

Operations on Sets (III)

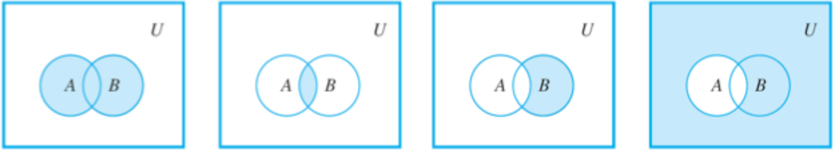
The complement of the set A, denoted by $\overline{A}$ or A^c, is the complement of A with respect to $\mathcal{U}$.

$$\overline{A} = \mathcal{U} - A = \{x \in \mathcal{U} \; : \; x \notin A\}$$

Example Given the set $A = \{a, c, e\}$, we have that

$$\overline{A} = \{b, d, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\},$$

where $\mathcal{U}$ is the set of letters of the English alphabet.



Shaded region represents $A \cup B$.

Shaded region represents $A \cap B$.

Shaded region represents $B - A$.

Shaded region represents A^c.

Subsets and Equality

Theorem $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

How to prove $A = B$

1. Prove that $A \subseteq B$.
2. Prove that $B \subseteq A$.
3. Therefore, since $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$.

Partition of a Set

A partition of a set A is a family of sets P of non-empty subsets of A, such that every element in A is in exactly one of these subsets.

Equivalently, a family of sets P is a partition of A if and only if:

1. $\emptyset \notin P$,
2. $\bigcup_{B \in P} B = A$
3. if $B, C \in P$ and $B \neq C$, then $B \cap C = \emptyset$ (the elements of P are pairwise disjoint).

Example The set $\{a, c, e\}$ has five partitions:

$$\{\ \{a\}, \{c\}, \{e\}\ \}$$

$$\{\ \{a, c\}, \{e\}\ \} \qquad \{\ \{a\}, \{c, e\}\ \} \qquad \{\ \{a, e\}, \{c\}\ \}$$

$$\{\ \{a, c, e\}\ \}$$

The Cartesian Product

An ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is a collection with an established order, where a_1 is the first element, a_2 is the second element, etc.

If $n = 2$ then they are called ordered pairs.

The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \ : \ a \in A \cap b \in B\}$$

In general, $A \times B \neq B \times A$ unless $A = \emptyset$, $B = \emptyset$ or $A = B$.

The concept of Cartesian product can be extended to that of more than two sets.

Example Let $A = \{1, 2\}$, $B = \{a, b\}$ and $C = \{5, 6\}$.

Then

$$A \times B \times C = \{\ (1, a, 5), (1, a, 6), (1, b, 5), (1, b, 6), (2, a, 5), (2, a, 6), (2, b, 5), (2, b, 6)\ \}.$$

Properties of Sets

These are some set properties that involve subset relations.

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

Set Identities (I)

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An identity is an equation that is universally true for all elements in some set.

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

Set Identities (II)

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

Outline

1. Sets
2. Functions
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4. Summation and Product Notations

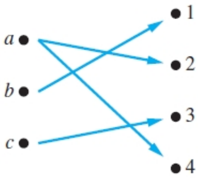
Functions (I)

Given two non-empty sets A and B, a function (or mapping) f from A to B is an assignment of exactly one element of B to each element of A.

If f is a function from A to B we write

$$f : A \to B.$$

The following is NOT a function.



Functions (II)

The set A is the domain of f and B is the codomain of f.

If a function f from A to B assigns the element $b \in B$ to the element $a \in A$, then we write

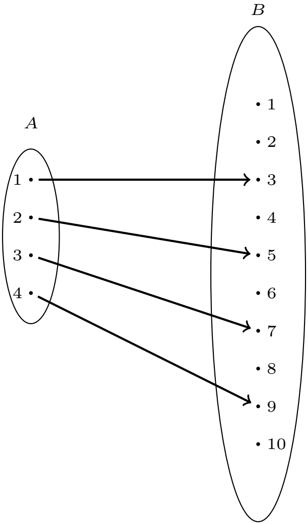
$$f(a) = b.$$

We say that b is the image of a, and a is the preimage of b. The range (or image) of f is the set of all images of elements of A.

Two functions are equal if they have the same domain, codomain, and they map each element of their domain to the same element in the codomain.

Functions (III)

Example Let $f : A \to B$, where $f(x) = 2x + 1$ for any $x \in A$.



Domain: $A = \{1, 2, 3, 4\}$
Codomain: $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
Range or image: $\{3, 5, 7, 9\}$

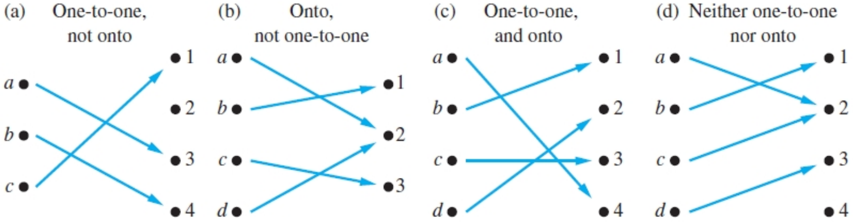
f(3) = 7

Types of Correspondences (I)

A function $f : A \to B$ is onto (or a surjection) if and only if for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

A function $f : A \to B$ is one-to-one (or an injunction) if $f(a) = f(b)$ implies $a = b$ for all a and b in the domain of f.

A function f is a bijection if it is both one-to-one and onto.



Types of Correspondences (II)

Examples

a) Let $g : \mathbb{R} \to \mathbb{R}$ be the function $g(x) = x^2 - 2$. Is this function onto? Is it one-to-one?

   It is NOT onto, since not all possible values in the codomain are "used": the values below -2 are not the image of any of the values in the domain.

   It is NOT one-to-one either, since for example $g(3) = g(-3) = 7$, so there are at least two different values in the domain, having the same image.

b) Is the function $f(x) = |x - 2|$ one-to-one, where $f : \mathbb{R} \to [0, +\infty)$?

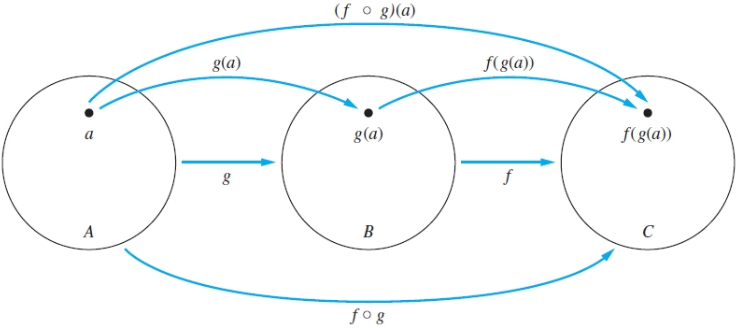
   It is NOT one-to-one (note that $f(3) = f(1) = 1$), but it IS onto.

Composition of Functions (I)

Given two functions $g : A \to B$ and $f : B \to C$, the composition of f and g is defined by

$$(f \circ g)(a) = f(g(a)).$$

Note that the composition $(f \circ g)$ cannot be defined unless the range of g is a subset of the domain of f.



Composition of Functions (II)

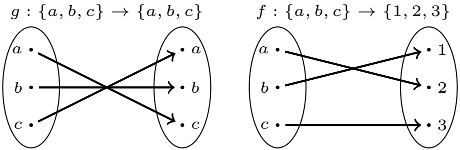
Examples

a) Given the functions $f(x) = 5x$ and $g(x) = x^2 + 1$, where $f, g : \mathbb{R} \rightarrow \mathbb{R}$,

- $(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = 5(x^2 + 1) = 5x^2 + 5$
- $(g \circ f)(x) = g(f(x)) = g(5x) = (5x)^2 + 1 = 25x^2 + 1$

Note: $(f \circ g)(x)$ and $(g \circ f)(x)$ do not necessarily yield the same answer. Composition of functions is not commutative.

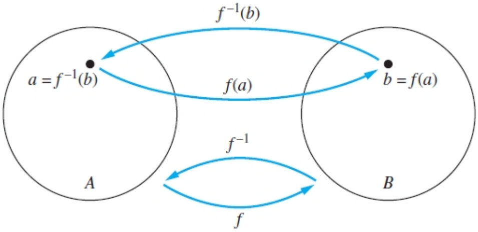
b) Given these two functions



- $(f \circ g)(a) = f(g(a)) = f(c) = 3$
- $(f \circ g)(b) = f(g(b)) = f(b) = 1$
- $(f \circ g)(c) = f(g(c)) = f(a) = 2$

Inverse of a Function (I)

Let f be a one-to-one function from A to B. The inverse function of f, denoted by f^{-1}, is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.
Hence, $f^{-1}(b) = a$ when $f(a) = b$.



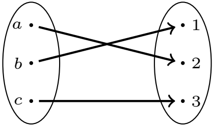
If a function f is not one-to-one, we cannot define an inverse function. In this case f is said to be not invertible.

Inverse of a function (II)

Examples

a) Let f be the following function

$$f : \{a, b, c\} \to \{1, 2, 3\}$$



Is f invertible? If so, what is its inverse?

The function f is invertible, because it is a one-to-one correspondence. The inverse function is

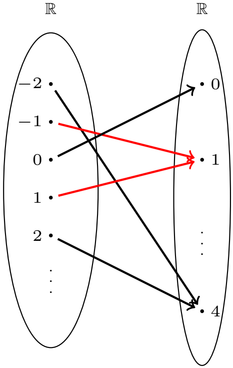
$$f^{-1} : \{1, 2, 3\} \to \{a, b, c\}$$

such that $f^{-1}(1) = b$, $f^{-1}(2) = a$, and $f^{-1}(3) = c$.

Inverse of a Function (III)

Examples (cont.)

b) Let $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = x^2$. Is g invertible?



We can see f is not one-to-one.
If there were an inverse function, it would have $g^{-1}(1) = 1$ and $g^{-1}(1) = -1$ at the same time, which cannot happen!
Then, g is not invertible.

However, if g were $g : \mathbb{R}^+ \to \mathbb{R}^+$, then it would be invertible.

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1 Sets

2 Functions

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4 Summation and Product Notations

Sequences (I)

A sequence is a function from a set of indexes (usually integers) to some set S.
The image of the integer n is denoted by a_n, and $\{a_n\}$ denotes the whole sequence.

Sometimes we just write "the sequence $a_0, a_1, a_2, \ldots$" instead of $\{a_n\}$ to make sure that the set of indexes is clear.

Examples

a) The sequence $\{a_n\}$ where $a_n = (-4)^n$ for all integers $n \geq 0$ is

$$1, -4, 16, -64, 256, \ldots$$

b) The sequence $\{b_n\}$ where $b_n = \frac{1}{n^2}$ for all integers $n \geq 1$ is

$$1, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \ldots$$

Sequences (II)

A geometric progression is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where r is called the common ratio, a is the initial term, and they are both real numbers. Equivalently, every term is defined in general as $a_n = ar^n$.

Examples

a) The sequence $\{c_n\}$ with $c_n = 2 \cdot 10^n$ for all integers $n \geq 0$ is a geometric progression with initial term equal to 2, and common ratio $r = 10$:

$$2, 20, 200, 2000, 20000, \ldots$$

b) The sequence $\{d_n\}$ with $d_n = 3 \cdot (-2)^n$ for all integers $n \geq 0$ is a geometric progression with initial term equal to 3, and common ratio $r = -2$. The list of terms $d_0, d_1, d_2, d_3, d_4, \ldots$ begins with

$$3, -6, 12, -24, 48, \ldots$$

Sequences (III)

An arithmetic progression is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where d is called the common difference, a is the initial term, and they are both real numbers.
Equivalently, every term is defined in general as $a_n = a + nd$.

Examples

a) The sequence $\{e_n\}$ with $e_n = 4 + 3n$ for all integers $n \geq 0$ is an arithmetic progression with initial term equal to 4, and common difference $d = 3$. The list of terms $a_0, a_1, a_2, a_3, a_4, \ldots$ begins with

$$4, 7, 10, 13, 16, \ldots$$

b) The sequence $\{c_n\}$ with $c_n = 7 - 3n$ for all integers $n \geq 0$ is an arithmetic progression with initial term equal to 7, and common difference $d = -3$:

$$7, 4, 1, -2, -5, \ldots$$

Sequences (IV)

What's the next number?

$$1, 2, 3, 4, \ldots$$
$$1, 3, 5, 7, 9, \ldots$$
$$2, 3, 5, 7, 11, \ldots$$

Sometimes, given the first terms of a sequence, we are asked to find the function generating that sequence, or a procedure to enumerate the sequence.

The problem of finding the generating function given just an initial subsequence is not easy. There are infinitely many computable functions that will generate any given initial subsequence.

Sequences will be used in summations and product notation.

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Summations

A summation

$$\sum_{j=m}^{n} a_j$$

denotes the sum $a_m + a_{m+1} + \ldots + a_{n-1} + a_n$ from a sequence $\{a_n\}$. The variable j is the index of summation and n is the upper limit of the summation.

Examples

a) $\displaystyle\sum_{i=2}^{4}(i^2+1) = \; (2^2+1)+(3^2+1)+(4^2+1) = \; 5+10+17=32.$

b) The sum of the first 50 elements of the sequence $\{a_n\}$ for $a_n = 7-3n$ and $n = 1, 2, 3, \ldots$ is

$$\sum_{i=1}^{50}(7-3i).$$

c) $\displaystyle\sum_{i=1}^{2}\sum_{j=4}^{6}(3ij) = \sum_{i=1}^{2}(3i\cdot 4+3i\cdot 5+3i\cdot 6) =$
$(3\cdot 1\cdot 4+3\cdot 1\cdot 5+3\cdot 1\cdot 6)+(3\cdot 2\cdot 4+3\cdot 2\cdot 5+3\cdot 2\cdot 6).$

Product notation

The notation

$$\prod_{j=m}^{n} a_j$$

denotes the multiplication $a_m \cdot a_{m+1} \cdot \ldots \cdot a_{n-1} \cdot a_n$ from a sequence $\{a_n\}$.

Example

a) $\prod_{i=1}^{6} i^2 = 1 \cdot 2^2 \cdot 3^2 \cdot \ldots \cdot 6^2 = \; 1 \cdot 4 \cdot 9 \cdot 16 \cdot 25 \cdot 36.$

b) $\prod_{i=1}^{2} \prod_{j=4}^{6} (3ij) = \; \prod_{i=1}^{2} ((3i \cdot 4)(3i \cdot 5)(3i \cdot 6)) =$
$((3 \cdot 1 \cdot 4)(3 \cdot 1 \cdot 5)(3 \cdot 1 \cdot 6))((3 \cdot 2 \cdot 4)(3 \cdot 2 \cdot 5)(3 \cdot 2 \cdot 6))$

Properties of Summations and Products

The following theorem states general properties of summations and products.

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$
2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law
3. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k).$