Solutions to Exercises - Week 6
Intelligent Systems Programming

Exercise 1

a) We define the proposition symbols ss = "'The sun shines"', r = "'It rains"', w = "'I get wet"', su = "'It is summer"'.

b) 1. $ss \Rightarrow (r \Rightarrow w)$.
   2. $ss \Rightarrow su$.
   3. $\neg su$.
   4. $ss \Rightarrow w$.

c) We will show that 1., 2. and 3. entail 4 by showing that $1. \wedge 2. \wedge 3 \Rightarrow 4$ is valid:

$$
\begin{aligned}
&\big((ss \Rightarrow (r \Rightarrow w)) \wedge (ss \Rightarrow su) \wedge (\neg su)\big) \Rightarrow (ss \Rightarrow w)\\
&\qquad \equiv \big((\neg ss \vee \neg r \vee w) \wedge (\neg ss \vee su) \wedge \neg su\big) \Rightarrow (\neg ss \vee w)\\
&\qquad \equiv \neg\big((\neg ss \vee \neg r \vee w) \wedge (\neg ss \vee su) \wedge \neg su\big) \vee (\neg ss \vee w)\\
&\qquad \equiv (ss \wedge r \wedge \neg w) \vee (ss \wedge \neg su) \vee su \vee (\neg ss \vee w)\\
&\qquad \equiv (ss \wedge r \wedge \neg w) \vee \big((ss \vee su) \wedge (\neg su \vee su)\big) \vee (\neg ss \vee w)\\
&\qquad \equiv (ss \wedge r \wedge \neg w) \vee \big((ss \vee su) \wedge \textsc{True}\big) \vee (\neg ss \vee w)\\
&\qquad \equiv (ss \wedge r \wedge \neg w) \vee (ss \vee su) \vee (\neg ss \vee w)\\
&\qquad \equiv (ss \wedge r \wedge \neg w) \vee (ss \vee \neg ss) \vee su \vee w\\
&\qquad \equiv (ss \wedge r \wedge \neg w) \vee \textsc{True} \vee su \vee w\\
&\qquad \equiv \textsc{True}
\end{aligned}
$$

Exercise 2

For space and time saving we use s for "'Smoke"', f for "'Fire"', h for "'Heat"' and so on.

a)

$$s \Rightarrow s \equiv \neg s \vee s \equiv \textsc{True}.$$

Since True is true in all models, $s \Rightarrow s$ is valid.

b)

$$s \Rightarrow f \equiv \neg s \vee f.$$

This expression evaluates to False for $s = \textsc{True}$ and $f = \textsc{False}$, while it evaluates to True for e.g. $s = f = \textsc{True}$. I.e. $s \Rightarrow f$ is neither valid nor unsatisfiabe.

c)

$$
\begin{aligned}
(s \Rightarrow f) \Rightarrow (\neg s \Rightarrow \neg f) &\equiv (\neg s \vee f) \Rightarrow (\neg f \vee s) \\
&\equiv \neg(\neg s \vee f) \vee (\neg f \vee s) \\
&\equiv (s \wedge \neg f) \vee (\neg f \vee s).
\end{aligned}
$$

For $s = \text{FALSE}$ and $f = \text{TRUE}$, the expresion is false, while for $s = \text{TRUE}$ and $f = \text{FALSE}$ it is true. That is, $(s \Rightarrow f) \Rightarrow (\neg s \Rightarrow \neg f)$ is neither valid nor unsatisfiable.

d)

$$
s \vee f \vee \neg f \equiv s \vee \text{TRUE} \equiv \text{TRUE},
$$

i.e. the expression is valid.

e)

$$
\begin{aligned}
((s \vee h) \Rightarrow f) \Leftrightarrow ((s \Rightarrow f) \vee (h \Rightarrow f)) &\equiv (\neg(s \wedge h) \vee f) \Leftrightarrow (\neg s \vee f \vee \neg h \vee f) \\
&\equiv (\neg s \vee \neg h \vee f) \Leftrightarrow (\neg s \vee \neg h \vee f).
\end{aligned}
$$

Since the left hand side in the last expression is the exactly same as he right hand side of the last expression, the truth value of the left hand side will allways equal the truth value of the right hand side, so (compare with truth table of $\Leftrightarrow$) the expression is always true, i.e. it is valid.

f)

$$
\begin{aligned}
(s \Rightarrow f) \Rightarrow ((s \wedge h) \Rightarrow f) &\equiv (\neg s \vee f) \Rightarrow (\neg(s \wedge h) \vee f) \\
&\equiv (\neg s \vee f) \Rightarrow (\neg s \vee \neg h \vee f) \\
&\equiv \neg(\neg s \vee f) \vee (\neg s \vee \neg h \vee f) \\
&\equiv \neg(\neg s \vee f) \vee (\neg s \vee f) \vee \neg h \\
&\equiv \text{TRUE} \vee \neg h \\
&\equiv \text{TRUE},
\end{aligned}
$$

i.e. the expression is valid.

g)

$$
(b \wedge d) \vee \neg d.
$$

For $b = \text{FALSE}$ and $d = \text{TRUE}$ the expression is false, while it is true for $b = d = \text{TRUE}$. Therefore it is neither valid nor unsatisfiable.

Exercise 3

a) $\text{TRUE} \models \alpha$ is by definition the case if and only if α is true in every model where TRUE is true. Since TRUE is always true, we therefore have that $\text{TRUE} \models \alpha$ if and only if α is true in every model. This is by definition the case if and ony is α is valid.

b) $\text{FALSE} \models \alpha$ is by definition the case if and only if α is true in every model where FALSE is true. Since FALSE never is true (i.e. it is true in no model), this gives us no constraints on the truth value on α, and therefore $\text{FALSE} \models \alpha$ is true for any α.

c) $\alpha \models \beta$ if and only if in all models where α is true, β is also true (by definition). On the other hand $\alpha \Rightarrow \beta$ is valid if and only if $\alpha \Rightarrow \beta$ is true in all models, i.e. if and only if $\neg\alpha \vee \beta$ is true in all models. This is the case if and only if either α is false or β is true, so in order for $\alpha \Rightarrow \beta$ to be valid we see that in all model where α is true, β has to be true too.
From this we conclude, that $\alpha \models \beta$ if and only if $\alpha \Rightarrow \beta$ is valid.

d) $\alpha \equiv \beta$ if and only if $\alpha \models \beta$ and $\beta \models \alpha$ (by definiton). So according to c), $\alpha \equiv \beta$ if and only if $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. According to the definitin of "‘$\Leftrightarrow$"’, we hereby get that $\alpha \equiv \beta$ if and ony if $\alpha \Leftrightarrow \beta$.

e) Due to c) we have that $\alpha \models \beta$ if and only if $(\alpha \Rightarrow \beta)$ is valid. It is therefore enough to show that $\alpha \Rightarrow \beta$ is valid if and only if $(\alpha \wedge \neg\beta)$ is unsatisfiable. That $\alpha \wedge \neg\beta$ is unsatisfiable means that it is false in all models, i.e. that the negation is true in all models. I.e. we have to show that $\neg(\alpha \wedge \neg\beta)$ is valid if and only if $\alpha \Rightarrow \beta$ is valid. But this follows easily since

$$\neg(\alpha \wedge \neg\beta) \equiv (\neg\alpha \vee \beta) \equiv \alpha \Rightarrow \beta.$$

Exercise 4

a)

$$\begin{aligned}
&\neg(a \Rightarrow b) \wedge \big(a \vee b \Leftrightarrow (c \Rightarrow b)\big) \\
&\qquad \equiv \neg(\neg a \vee b) \wedge \big((a \vee b) \Rightarrow (\neg c \vee b)\big) \wedge \big((\neg c \vee b) \Rightarrow (a \vee b)\big) \\
&\qquad \equiv (a \wedge \neg b) \wedge \big(\neg(a \vee b) \vee (\neg c \vee b)\big) \wedge \big(\neg(\neg c \vee b) \vee (a \vee b)\big) \\
&\qquad \equiv a \wedge \neg b \wedge \big((\neg a \wedge \neg b) \vee (\neg c \vee b)\big) \wedge \big((c \wedge \neg b) \vee (a \vee b)\big) \\
&\qquad \equiv a \wedge \neg b \wedge (\neg a \vee \neg c \vee b) \wedge (\neg b \vee \neg c \vee b) \wedge (c \vee a \vee b) \wedge (\neg b \vee a \vee b)
\end{aligned}$$

b) By unit resolution on the literals a and $\neg b$ we get:

$$\frac{\neg b, \quad \dfrac{a, (\neg a \vee \neg c \vee b)}{(\neg c \vee b)}}{\neg c}$$

that is $\neg(a \Rightarrow b) \wedge (a \vee b \Leftrightarrow (c \Rightarrow b)) \models \neg c$.