Intelligent Systems Programming

Lecture 11: **Linear Programming II**

Properties of simplex

IT University of Copenhagen

Pitfalls of Simplex

1) **Initialization**: What if the initial dictionary is infeasible?
2) **Iteration**: What if we cannot find an entering or leaving variable?
3) **Termination**: what if simplex never terminates?

Today's Program

- [10:00-10:50]
  - Initialization
    - Two phase simplex
  - Iteration
    - Unboundedness
- [11:00-11:50]
  - Iteration
    - Degeneracy
  - Termination
    - Cycling
  - Efficiency of simplex
  - Professional solvers

Initialization

Initialization

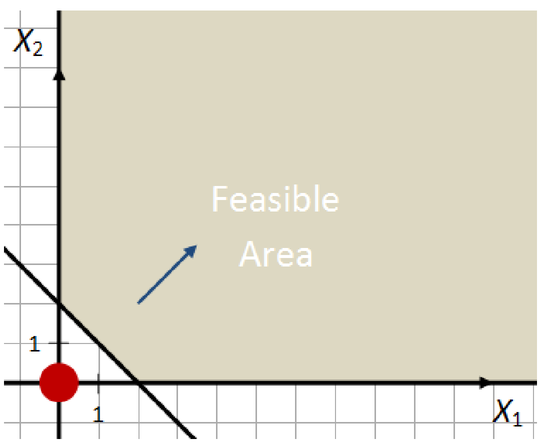
- Initial slack dictionary is feasible if and only if $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is feasible

  Why?

Initialization

- Initial slack dictionary is feasible if and only if $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is feasible
- Example of infeasible slack dictionary:

Geometric Form



Standard Form

Maximize

$x_1 + x_2$

Subject to

$x_1 + x_2 \geq 2 \Leftrightarrow$

$-x_1 - x_2 \leq -2$

$x_1, x_2 \geq 0$

Slack Form

Maximize

z

Subject to

$x_3 = -2 + x_1 + x_2$

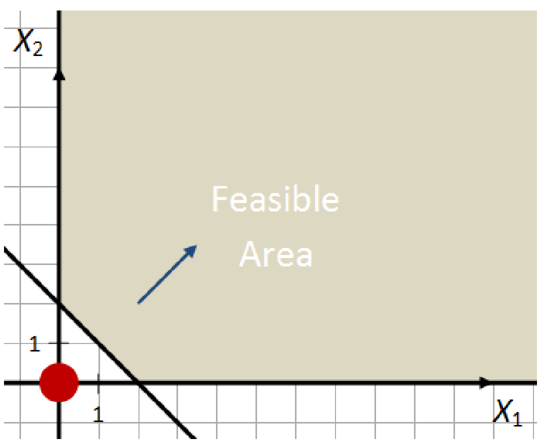
$z = x_1 + x_2$

$x_1, x_2, x_3 \geq 0$

Initialization

- Initial slack dictionary is feasible if and only if $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is feasible
- Example of infeasible slack dictionary:

Geometric Form



Standard Form

Maximize

$$x_1 + x_2$$

Subject to

$$x_1 + x_2 \geq 2 \Leftrightarrow$$

$$-x_1 - x_2 \leq -2$$

$$x_1, x_2 \geq 0$$

Slack Form

Maximize

$$z$$

Subject to

$$x_3 = -2 + x_1 + x_2$$

$$z = x_1 + x_2$$

$$x_1, x_2, x_3 \geq 0$$

Initialization

- Initial slack dictionary is feasible if and only if $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is feasible
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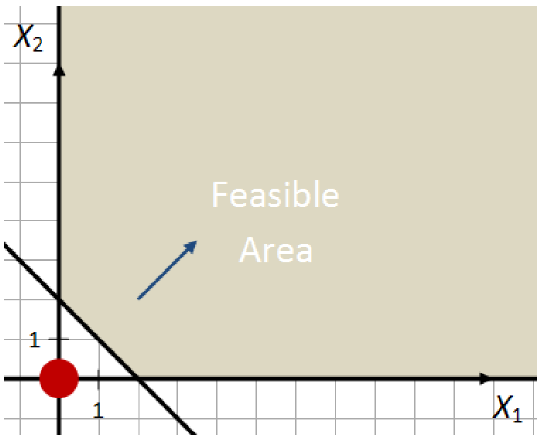
Geometric Form	Standard Form	Slack Form
	Maximize $x_1 + x_2$ Subject to $x_1 + x_2 \geq 2 \Leftrightarrow$ $-x_1 - x_2 \leq -2$ $x_1, x_2 \geq 0$	Maximize z Subject to $x_3 = -2 + x_1 + x_2$ $z = x_1 + x_2$ $x_1, x_2, x_3 \geq 0$

Initialization

- Two initialization challenges
  1) It may not be clear if our problem has any feasible solution at all.
  2) Even of a feasible solution is apparent, a feasible dictionary may not be.
- Solution: the two-phase simplex method
  - Solves both 1) and 2)

Two Phase Simplex

- **Idea**: In first phase, solve an auxiliary LP problem that minimizes an artificial slack variable x_0

Original Problem

Maximize $$\sum_{j=1}^{n} c_j x_j$$

Subject to $$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

$$x_j \geq 0$$

Auxiliary Problem

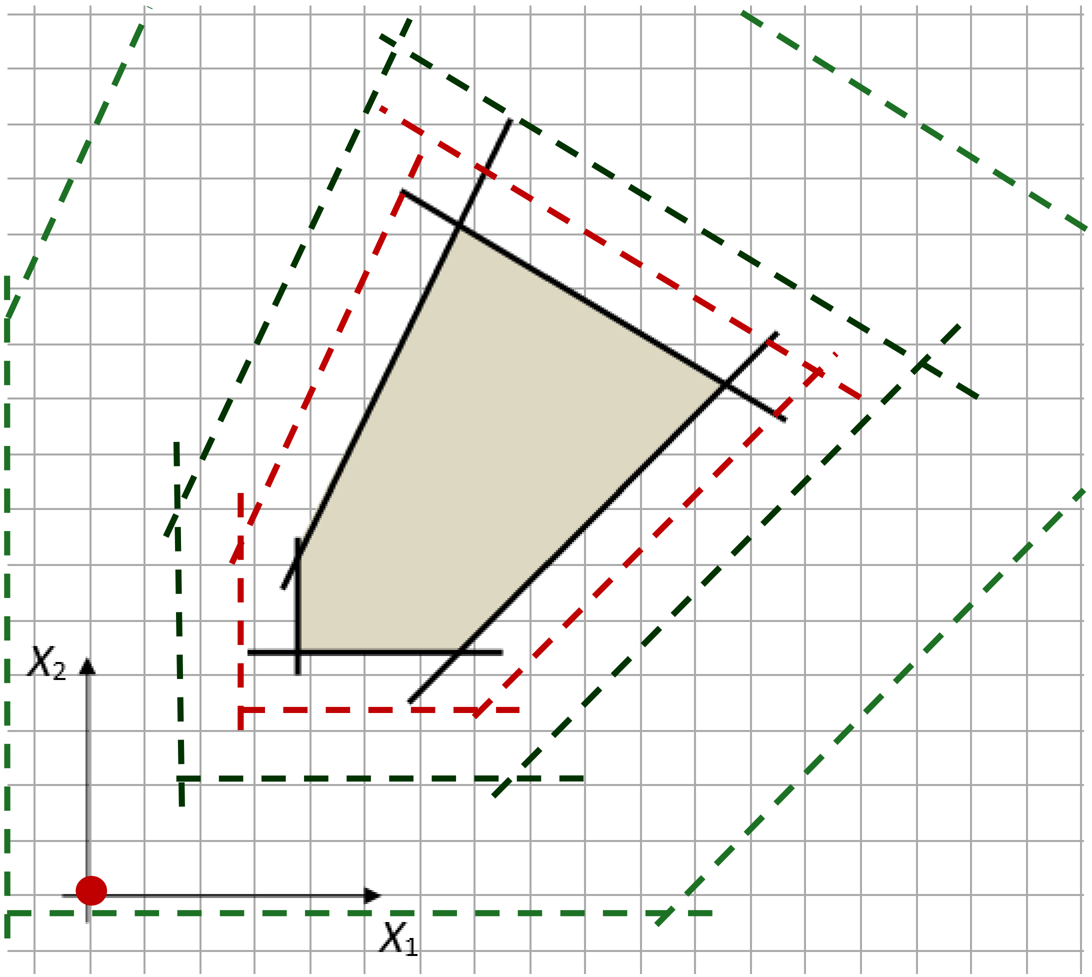
Maximize $-x_0$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i$$

$$x_j \geq 0$$

Geometric Interpretation of Auxiliary LP



First Phase on Example

- **Step 1**: Formulate auxiliary problem

Problem

Maximize

$$x_1 - x_2 + x_3$$

Subject to

$$\begin{aligned} 2x_1 - x_2 + 2x_3 &\leq 4 \\ 2x_1 - 3x_2 + x_3 &\leq -5 \\ -x_1 + x_2 - 2x_3 &\leq -1 \end{aligned}$$

$$x_1, x_2, x_3 \geq 0$$

Auxiliary Problem

Maximize

$$-x_0$$

Subject to

$$\begin{aligned} 2x_1 - x_2 + 2x_3 - x_0 &\leq 4 \\ 2x_1 - 3x_2 + x_3 - x_0 &\leq -5 \\ -x_1 + x_2 - 2x_3 - x_0 &\leq -1 \end{aligned}$$

$$x_0, x_1, x_2, x_3 \geq 0$$

First Phase on Example

- **Step 2**: Translate auxiliary problem to slack form

Auxiliary Problem

Maximize

$$-x_0$$

Subject to

$$\begin{aligned} 2x_1 - x_2 + 2x_3 - x_0 &\leq 4 \\ 2x_1 - 3x_2 + x_3 - x_0 &\leq -5 \\ -x_1 + x_2 - 2x_3 - x_0 &\leq -1 \end{aligned}$$

$$x_0, x_1, x_2, x_3 \geq 0$$

Slack Form

Maximize

$$w$$

Subject to

$$\begin{aligned} x_4 &= 4 - 2x_1 + x_2 - 2x_3 + x_0 \\ x_5 &= -5 - 2x_1 + 3x_2 - x_3 + x_0 \\ x_6 &= -1 + x_1 - x_2 - 2x_3 + x_0 \\ w &= -x_0 \end{aligned}$$

$$x_0, x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

First Phase on Example

- **Step 3**: Pivot most negative basis variable with x_0

Slack Form

Maximize

w

Subject to

$$\begin{aligned} x_4 &= 4 - 2x_1 + x_2 - 2x_3 + x_0 \\ x_5 &= -5 - 2x_1 + 3x_2 - x_3 + x_0 \\ x_6 &= -1 + x_1 - x_2 - 2x_3 + x_0 \\ w &= -x_0 \end{aligned}$$

Initial Dictionary

Maximize

w

Subject to

$$\begin{aligned} x_0 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\ x_4 &= 9 - 2x_2 - x_3 + x_5 \\ x_6 &= 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\ w &= -5 - 2x_1 + 3x_2 - x_3 - x_5 \end{aligned}$$

Resulting dictionary is feasible!

First Phase on Example

- **Step 4**: Do simplex. If $w < 0$ then problem infeasible, Otherwise Pivot x_0 out of basis. Why always possible if problem is feasible?

Initial Dictionary

Maximize

w

Subject to

$$\begin{aligned}
x_0 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
x_4 &= 9 - 2x_2 - x_3 + x_5 \\
x_6 &= 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\
w &= -5 - 2x_1 + 3x_2 - x_3 - x_5
\end{aligned}$$

Final Dictionary

Maximize

w

Subject to

$$\begin{aligned}
x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 - 0.8x_0 \\
x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 - 0.6x_0 \\
x_4 &= 3 - x_1 - x_6 + 2x_0 \\
w &= -x_0
\end{aligned}$$

Two Phase Simplex on Example

- **Step 5**: 1) Remove x_0, 2) Express z in terms of non-basic variables of final dictionary, 3) run simplex on resulting dictionary (2nd phase)

Final Dictionary

Maximize

w

Subject to

$$
\begin{aligned}
x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 - 0.8x_0 \\
x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 - 0.6x_0 \\
x_4 &= 3 - x_1 - x_6 + 2x_0 \\
w &= -x_0
\end{aligned}
$$

First Dictionary 2nd Phase

Maximize

z

Subject to

$$
\begin{aligned}
x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 \\
x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 \\
x_4 &= 3 - x_1 - x_6 \\
z &= x_1 - x_2 + x_3 \\
&= x_1 - (2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6) + (1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6) \\
&= -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6
\end{aligned}
$$

Iteration

Iteration

- **No entering variable** $\Rightarrow$
  No positive coefficients in z expression $\Rightarrow$
  Optimal solution found!
- Example:

$$z = 22 - \frac{1}{2}x_3 - 3x_5 - x_7$$

Iteration

- **No entering variable** $\Rightarrow$
  No positive coefficients in z expression $\Rightarrow$
  Optimal solution found!
- Example:

$$z = 22 - \frac{1}{2}x_3 - 3x_5 - x_7$$

- **No leaving variable** $\Rightarrow$ Unbounded problem!
- **Several leaving variables** $\Rightarrow$ Degenerate problem!

Iteration

- **No entering variable** $\Rightarrow$
  No positive coefficients in z expression $\Rightarrow$
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- Example:

$$z = 22 - \frac{1}{2}x_3 - 3x_5 - x_7$$

- **No leaving variable** $\Rightarrow$ Unbounded problem!
- **Several leaving variables** $\Rightarrow$ Degenerate problem!

Unbounded Example 1/2

Standard Form

Maximize $x_1 + x_2$

Subject to

$$x_2 \geq -3 + x_1 \Leftrightarrow x_1 - x_2 \leq 3$$
$$x_2 \leq 2 + 2x_1 \Leftrightarrow -2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

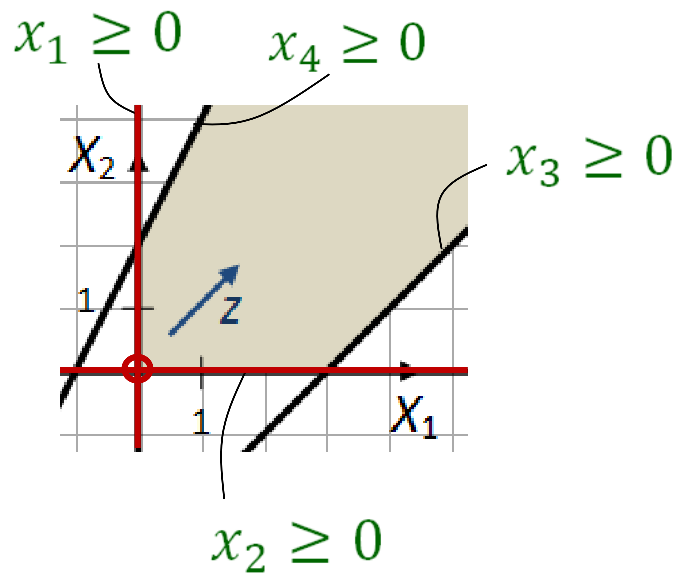
Initial Dictionary

Maximize z

Subject to

$$\begin{aligned} x_3 &= 3 - x_1 + x_2 \\ x_4 &= 2 + 2x_1 - x_2 \\ z &= x_1 + x_2 \end{aligned}$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Geometric Interpretation



Unbounded Example 1/2

Standard Form

Maximize $\quad x_1 + x_2$

Subject to

$$x_2 \geq -3 + x_1 \Leftrightarrow x_1 - x_2 \leq 3$$
$$x_2 \leq 2 + 2x_1 \Leftrightarrow -2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Initial Dictionary

Maximize $\quad z$

Subject to

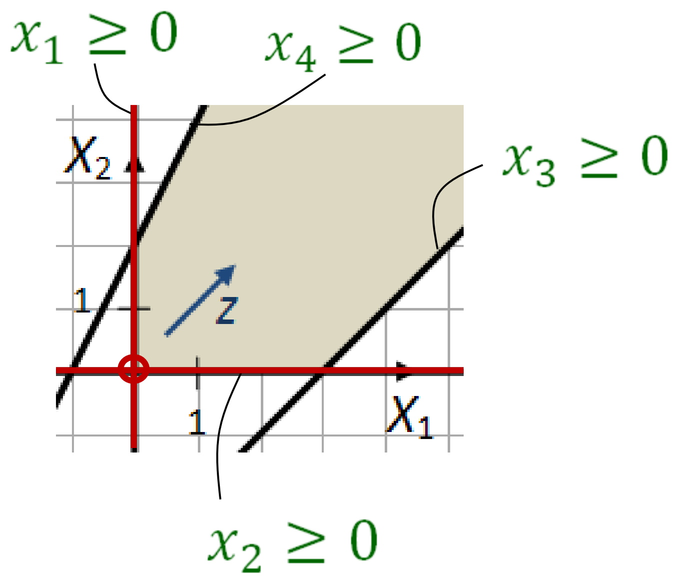
$$\begin{aligned} x_3 &= 3 - x_1 + x_2 \\ x_4 &= 2 + 2x_1 - x_2 \\ z &= x_1 + x_2 \end{aligned}$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Max x_1 increase

3

∞

Geometric Interpretation



Unbounded Example 2/2

First Dictionary (in x_1,out x_3)

Maximize $\quad z$

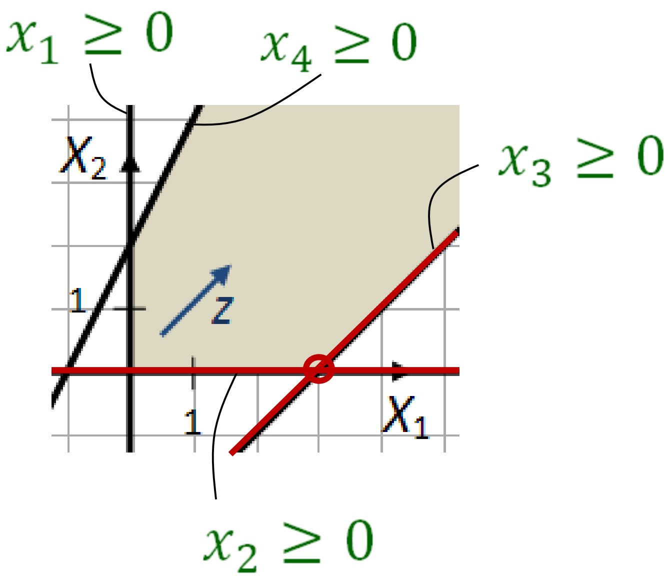
Subject to

$$x_1 = 3 + x_2 - x_3$$
$$x_4 = 2 + 2(3 + x_2 - x_3) - x_2$$
$$= 8 + x_2 - 2x_3$$

$$z = (3 + x_2 - x_3) + x_2$$
$$= 3 + 2x_2 - x_3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Geometric Interpretation



Unbounded Example 2/2

First Dictionary (in x_1,out x_3)

Maximize $\quad z$

Subject to

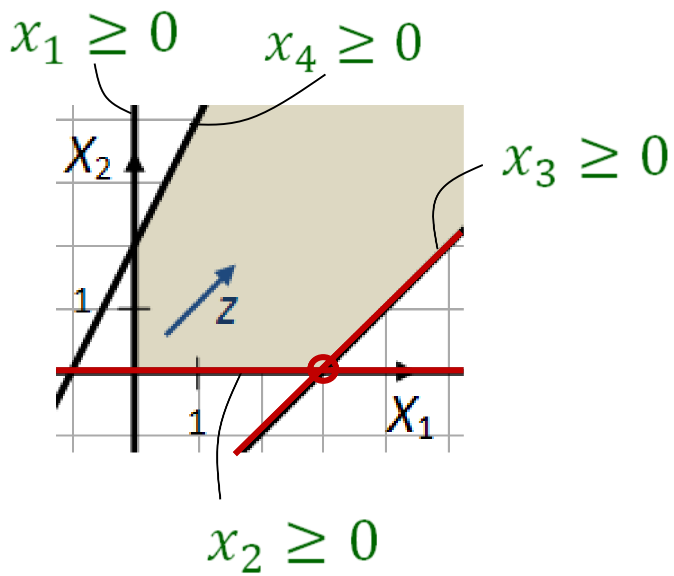
$$x_1 = 3 + x_2 - x_3$$

$$x_4 = 2 + 2(3 + x_2 - x_3) - x_2$$
$$= 8 + x_2 - 2x_3$$

$$z = (3 + x_2 - x_3) + x_2$$
$$= 3 + 2x_2 - x_3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Geometric Interpretation



Unbounded Example 2/2

First Dictionary (in x_1,out x_3)

Maximize $\quad z$

Subject to

$$x_1 = 3 + x_2 - x_3$$

$$x_4 = 2 + 2(3 + x_2 - x_3) - x_2$$

$$= 8 + x_2 - 2x_3$$

$$z = (3 + x_2 - x_3) + x_2$$

$$= 3 + 2x_2 - x_3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Max x_2 increase

∞ ($x_1 = 3 + x_2$)

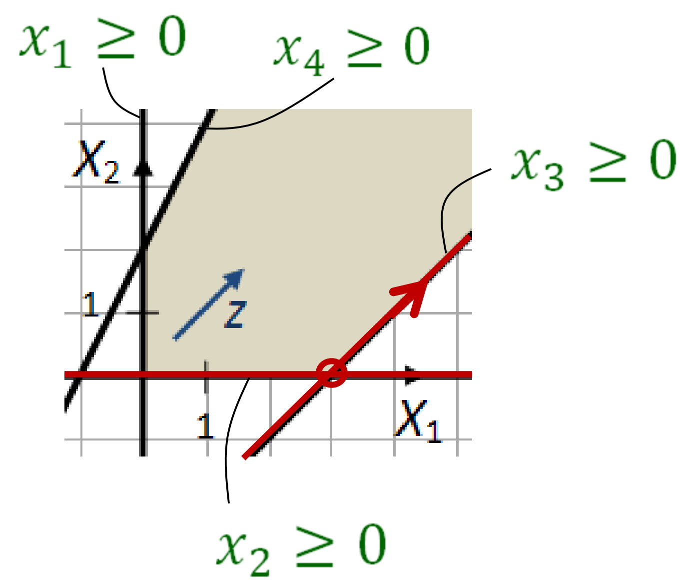
∞ ($x_4 = 8 + x_2$)

∞ ($z = 3 + 2x_2$)

Second Dictionary (in x_2,out ?)

Unbounded!

Geometric Interpretation



Iteration

- **No entering variable** $\Rightarrow$
  No positive coefficients in z expression $\Rightarrow$
  Optimal solution found!
- Example:

$$z = 22 - \frac{1}{2}x_3 - 3x_5 - x_7$$

- **No leaving variable** $\Rightarrow$ Unbounded problem!
- **Several leaving variables** $\Rightarrow$ Degenerate problem!

Degenerate Example 1/3

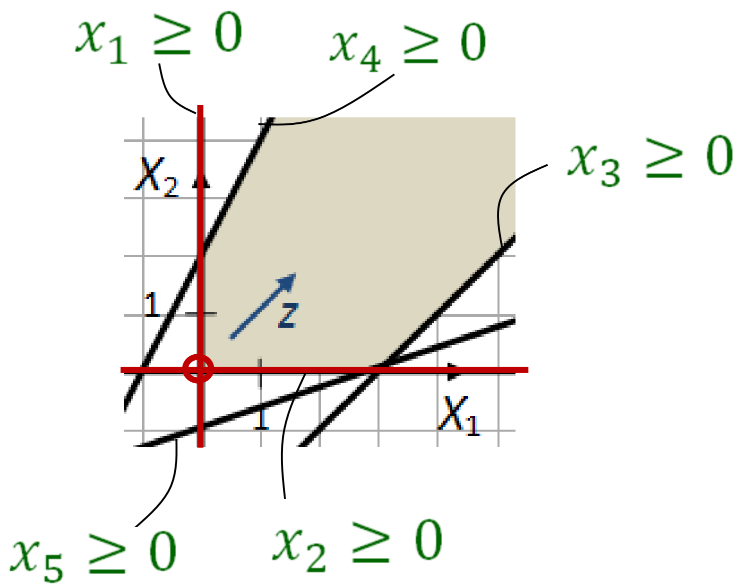
Initial Dictionary

Maximize $\quad z$

Subject to

$$\begin{aligned} x_3 &= 3 - x_1 + x_2 \\ x_4 &= 2 + 2x_1 - x_2 \\ \boxed{x_5 = 3 - x_1 + 3x_2} \\ z &= x_1 + x_2 \end{aligned}$$

Geometric Interpretation



Degenerate Example 1/3

Initial Dictionary

Maximize $\quad z$

Subject to

$$
\begin{aligned}
x_3 &= 3 - x_1 + x_2\\
x_4 &= 2 + 2x_1 - x_2\\
x_5 &= 3 - x_1 + 3x_2\\
z &= x_1 + x_2
\end{aligned}
$$

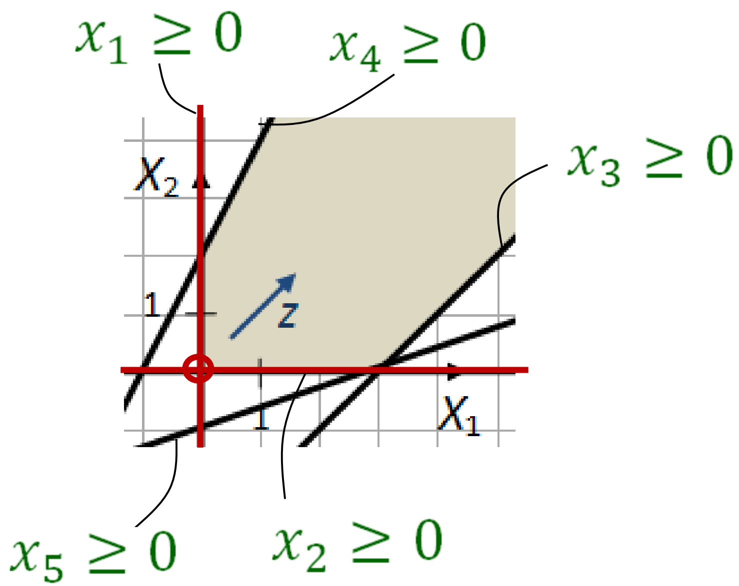
Max x_1 increase

3

∞

3

Geometric Interpretation



Degenerate Example 2/3

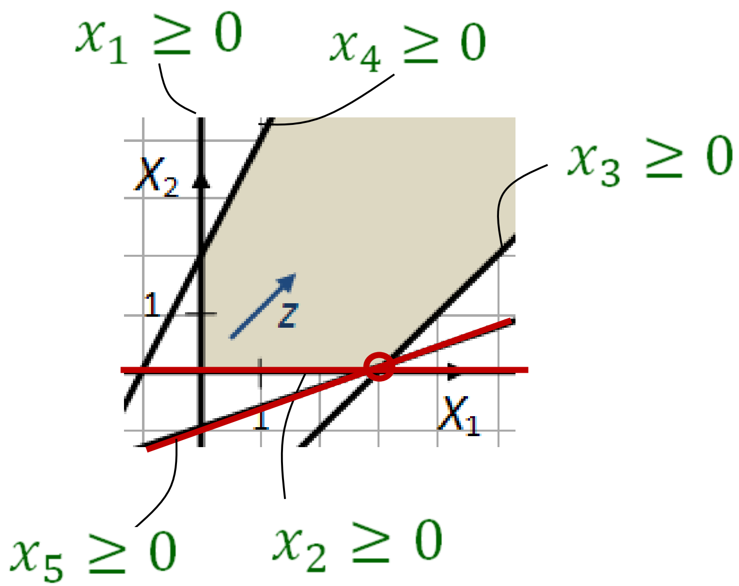
First Dictionary (in x_1,out x_5)

Maximize $\quad z$

Subject to

$$x_3 = 3 - (3 + 3x_2 - x_5) + x_2$$
$$= 0 - 2x_2 + x_5$$
$$x_4 = 2 + 2(3 + 3x_2 - x_5) - x_2$$
$$= 8 + 5x_2 - 2x_5$$
$$x_1 = 3 + 3x_2 - x_5$$
$$z = (3 + 3x_2 - x_5) + x_2$$
$$= 3 + 4x_2 - x_5$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Geometric Interpretation



Degenerate Example 2/3

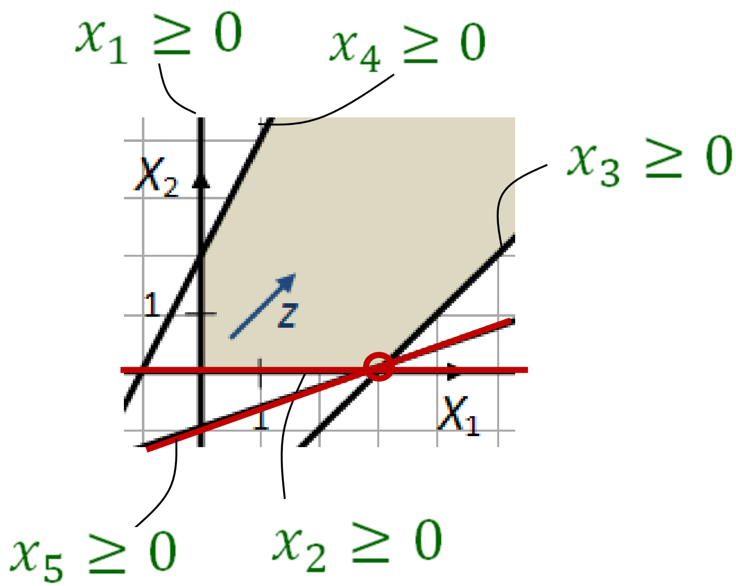
First Dictionary (in x_1,out x_5)

Maximize $\quad z$

Subject to

$$x_3 = 3 - (3 + 3x_2 - x_5) + x_2$$
$$= 0 - 2x_2 + x_5$$
$$x_4 = 2 + 2(3 + 3x_2 - x_5) - x_2$$
$$= 8 + 5x_2 - 2x_5$$
$$x_1 = 3 + 3x_2 - x_5$$
$$z = (3 + 3x_2 - x_5) + x_2$$
$$= 3 + 4x_2 - x_5$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Geometric Interpretation



Degenerate Example 2/3

First Dictionary (in x_1,out x_5)

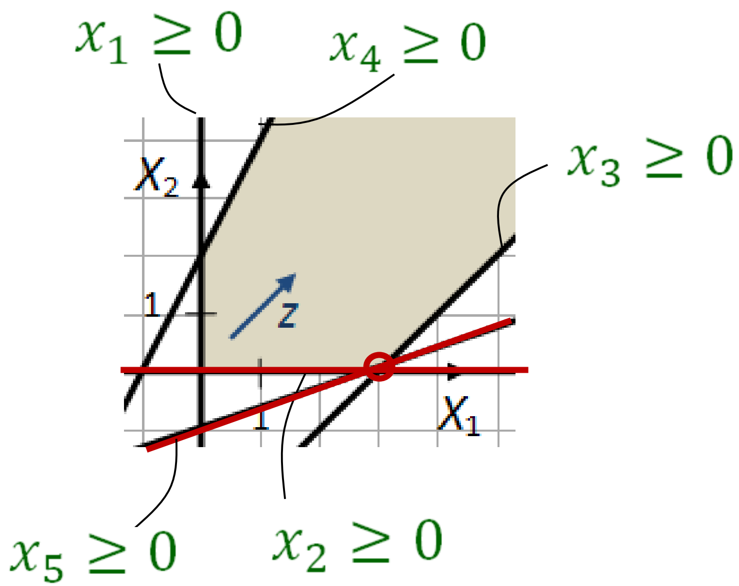
Maximize $\quad z$

Subject to

$$x_3 = 3 - (3 + 3x_2 - x_5) + x_2$$
$$= 0 - 2x_2 + x_5$$
$$x_4 = 2 + 2(3 + 3x_2 - x_5) - x_2$$
$$= 8 + 5x_2 - 2x_5$$
$$x_1 = 3 + 3x_2 - x_5$$
$$z = (3 + 3x_2 - x_5) + x_2$$
$$= 3 + 4x_2 - x_5$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Max x_2 increase
0
∞
∞

Geometric Interpretation



Degenerate Example 3/3

Second Dictionary (in x_2,out x_3)

Maximize z

Subject to

$$x_2 = -\frac{1}{2}x_3 + \frac{1}{2}x_5$$

$$x_4 = 8 + 5\left(-\frac{1}{2}x_3 + \frac{1}{2}x_5\right) - 2x_5$$

$$= 8 - \frac{5}{2}x_3 + \frac{1}{2}x_5$$

$$x_1 = 3 + 3\left(-\frac{1}{2}x_3 + \frac{1}{2}x_5\right) - x_5$$

$$= 3 - \frac{3}{2}x_3 + \frac{1}{2}x_5$$

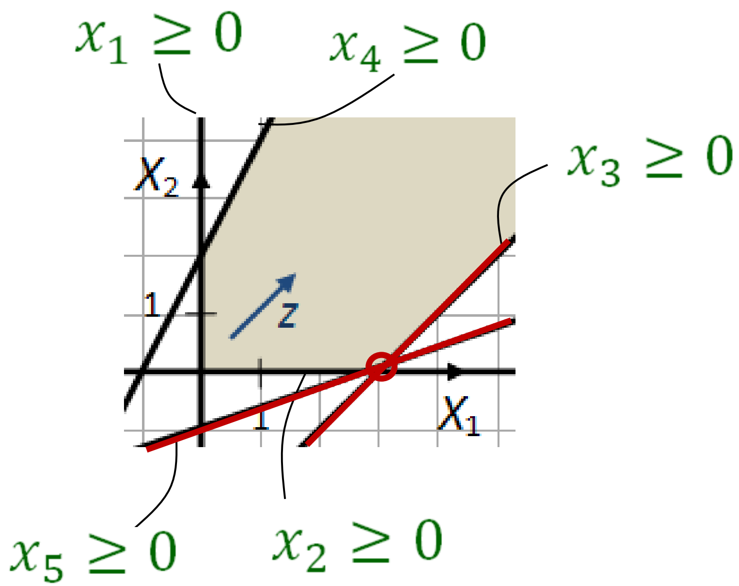
$$z = 3 + 4\left(-\frac{1}{2}x_3 + \frac{1}{2}x_5\right) - x_5$$

$$= 3 - 2x_3 + x_5$$

Degenerate pivot, no increase in *z*!!

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Geometric Interpretation



Degenerate Example 3/3

Second Dictionary (in x_2,out x_3)

Maximize $\quad z$

Subject to

$$x_2 = -\frac{1}{2}x_3 + \frac{1}{2}x_5$$

$$x_4 = 8 + 5\left(-\frac{1}{2}x_3 + \frac{1}{2}x_5\right) - 2x_5$$

$$= 8 - \frac{5}{2}x_3 + \frac{1}{2}x_5$$

$$x_1 = 3 + 3\left(-\frac{1}{2}x_3 + \frac{1}{2}x_5\right) - x_5$$

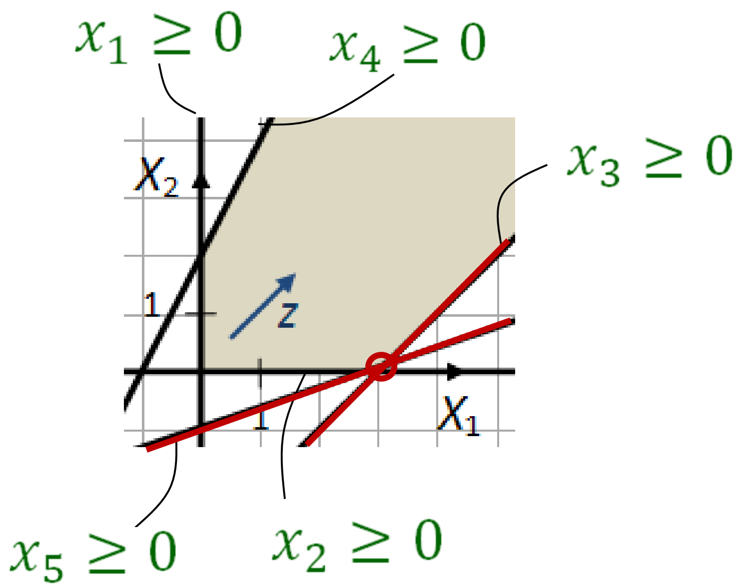
$$= 3 - \frac{3}{2}x_3 + \frac{1}{2}x_5$$

$$z = 3 + 4\left(-\frac{1}{2}x_3 + \frac{1}{2}x_5\right) - x_5$$

$$= 3 - 2x_3 + x_5$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Geometric Interpretation



Degenerate Example 3/3

Second Dictionary (in x_2,out x_3)

Maximize $\quad z$

Subject to

$$x_2 = -\frac{1}{2}x_3 + \frac{1}{2}x_5$$

$$x_4 = 8 + 5\left(-\frac{1}{2}x_3 + \frac{1}{2}x_5\right) - 2x_5$$

$$= 8 - \frac{5}{2}x_3 + \frac{1}{2}x_5$$

$$x_1 = 3 + 3\left(-\frac{1}{2}x_3 + \frac{1}{2}x_5\right) - x_5$$

$$= 3 - \frac{3}{2}x_3 + \frac{1}{2}x_5$$

$$z = 3 + 4\left(-\frac{1}{2}x_3 + \frac{1}{2}x_5\right) - x_5$$

$$= 3 - 2x_3 + x_5$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Max x_5 increase

∞ ($x_2 = 0.5x_5$)

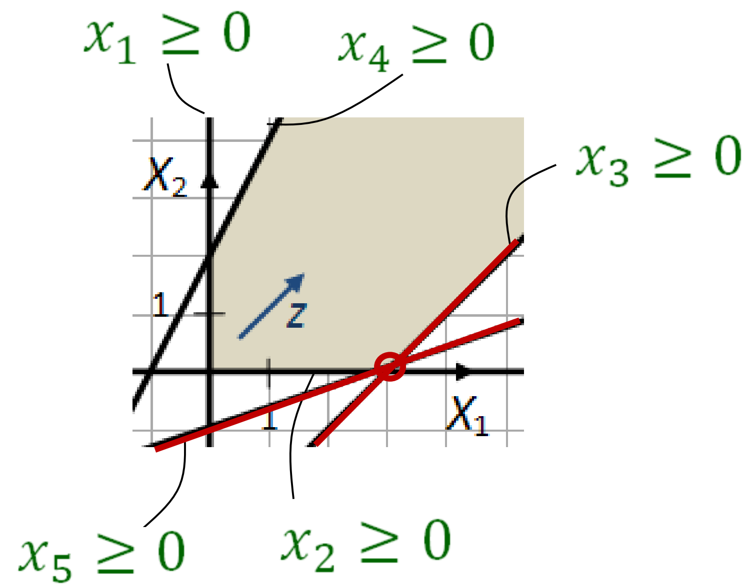
∞ ($x_4 = 8 + 0.5x_5$)

∞ ($x_1 = 3 + 0.5x_5$)

∞ ($z = 3 + x_5$)

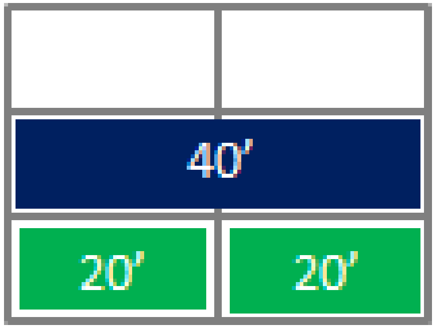
Unbounded as expected !!

Geometric Interpretation



Over Specified Problems

- **Challenge**: Degeneracy is often caused by over specified problems and these are natural!
- **Example**: A bay can hold 6 TEU. A 20’ container takes 1 TEU a 40’ container takes 2 TEU



Over Specified Problems

- **Example**: A bay can hold 6 TEU. A 20' container takes 1 TEU a 40' container takes 2 TEU
- Derived rules:

  x^{20} : number of 20' containers

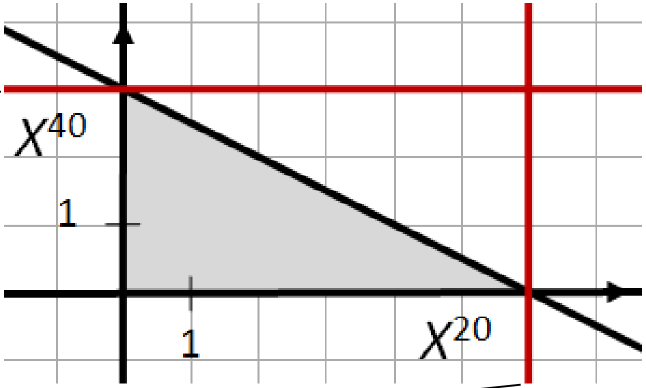
  x^{40} : number of 40' containers

$x^{20} + 2x^{40} \leq 6$

$x^{40} \leq 3$

$x^{20} \leq 6$

Over specification!



Termination

Termination

- Can simplex cycle in degenerate iterations and never terminate?
- Yes:

$$
\begin{array}{rcl}
x_5 & = & -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 & = & -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 & = & 1 - x_1 \\
\hline
z & = & 10x_1 - 57x_2 - 9x_3 - 24x_4
\end{array}
$$

$$
\begin{array}{rcl}
x_1 & = & 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 & = & -4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 & = & 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5 \\
\hline
z & = & 53x_2 + 41x_3 - 204x_4 - 20x_5.
\end{array}
$$

$$
\begin{array}{rcl}
x_2 & = & -0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\
x_1 & = & -0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\
x_7 & = & 1 + 0.5x_3 - 4x_4 - 0.75x_5 - 13.25x_6 \\
\hline
z & = & 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6.
\end{array}
$$

$$
\begin{array}{rcl}
x_3 & = & 8x_4 + 1.5x_5 - 5.5x_6 - 2x_1 \\
x_2 & = & -2x_4 - 0.5x_5 + 2.5x_6 + x_1 \\
x_7 & = & 1 - x_1 \\
\hline
z & = & 18x_4 + 15x_5 - 93x_6 - 29x_1.
\end{array}
$$

$$
\begin{array}{rcl}
x_4 & = & -0.25x_5 + 1.25x_6 + 0.5x_1 - 0.5x_2 \\
x_3 & = & -0.5x_5 + 4.5x_6 + 2x_1 - 4x_2 \\
x_7 & = & 1 - x_1 \\
\hline
z & = & 10.5x_5 - 70.5x_6 - 20x_1 - 9x_2.
\end{array}
$$

$$
\begin{array}{rcl}
x_5 & = & 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
x_4 & = & -x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\
x_7 & = & 1 - x_1 \\
\hline
z & = & 24x_6 + 22x_1 - 93x_2 - 21x_3.
\end{array}
$$

Cycle Breaking

- **Smallest Subscript-Rule:** Resolve ties between choices of entering and leaving variables by always taking the one with smallest subscript index
- Cycling very rare, so above rule seldom implemented

Can you think of any advantages of not implementing a cycle breaking rule?

Efficiency of Simplex

Efficiency of simplex

- LP in **P** in 1979, but simplex is exponential!
- Pathological Kleen-Minty LP ($2^n - 1$ iterations)

Maximize $$\sum_{j=1}^{n} 10^{n-j} x_j$$

Subject to $$\left(2 \sum_{j=1}^{i-1} 10^{i-j} x_j\right) + x_i \leq 100^{i-1} \qquad (i = 1, \dots, n)$$

- Average number of simplex iterations:
  - Grows linear in number of rows ($< 3m$)
  - Grows logarithmically in number of variables

Pivot Rules

- **Largest coefficient rule** (Kleen-Minty exponential)
- **Largest *z* increase rule** (Kleen-Minty constant)
- Trade-off
  - Largest increase rule gives fewer iterations
  - But simpler rules often overall makes simplex faster!

Making Simplex Efficient

- Revised simplex:
  - Only compute non-basic columns as needed
  - Take first produced non-basic variable with positive coefficient
- Computational tricks:
  - Generalize simplex to lower and upper bounds on variables and constraints (no slack variables)
  - Keep coefficient in sparse matrices
  - Keep computations in decomposed inverse matrix form
  - Make special versions of simplex whenever possible

Tools

- Open source: COIN OR (big), LP Solve (easy to install)
- Commercial: IBM ILOG CPLEX, Gurobi, MOSEK
  - Optimization Programming Language (OPL/CPLEX)

```
dvar float+ Gas;
dvar float+ Chloride;
maximize
  40 * Gas + 50 * Chloride;
subject to {
    Gas + Chloride <= 50;
    3 * Gas + 4 * Chloride <= 180;
    Chloride <= 40;
}
```