FINAL EXAM
Linear Algebra and Probability
Spring 2019

IT University of Copenhagen

May 27, 2019

Instructions

- This is a 4 hour written exam with all written and printed aids allowed (A2)
- There are 6 problems and 5 pages.
- Each problem is divided into questions
- The point value of each problem and the distribution of points among questions is given explicitly.
- There is a total of 100 points to be earned.
- The problems are formulated in english and should be answered in english.
- Read each question completely before trying to solve it.
- Please order and number the pages before handing in.
- Solutions should be hand-written
- Access to aid in the form of books, own notes, e-books, also on laptops and iPads is permitted.
- Use of internet including email and social media is not permitted. As an exception to this, downloading course material from learnit is permitted.
- Use of any other hardware or software such as MatLab or pocket calculators is not permitted
- Any form of communication between students or with the outside world is not permitted.

1 Systems of linear equations (17 points)

For this problem we consider the matrix A and vector b given as follows.

$$A = \begin{bmatrix} -1 & 3 & -2 & 10 \\ 2 & -1 & 4 & -5 \\ 4 & 2 & 8 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 9 \\ -8 \\ -8 \end{bmatrix}$$

a) [10 points] Compute the set of solutions to the system of linear equations given as the matrix equation

$$A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = b$$

b) [7 points] Compute the following data for the matrix A: The rank, dimensions of row space and column space and give a basis for the null space of A.

Solution

a) First reduce the augmented matrix to reduced echelon form as follows

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & 10 & 9 \\ 2 & -1 & 4 & -5 & -8 \\ 4 & 2 & 8 & 2 & -8 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -3 & 2 & -10 & -9 \\ 0 & 5 & 0 & 15 & 10 \\ 0 & 14 & 0 & 42 & 28 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -3 & 2 & -10 & -9 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & -3 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So z and w are free variables, which can range over real numbers s and t. This gives the solutions

$$\begin{aligned} x &= -3 - 2s + t \\ y &= 2 - 3t \\ z &= s \\ w &= t \end{aligned}$$

b) The reductions above give the reduced echelon form of A:

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which tells us that the rank is 2, and so the dimensions of the row and column spaces are also 2. The null space is spanned by $(-2, 0, 1, 0)$ and $(1, -3, 0, 1)$.

2 Eigenvalues and eigenvectors (10 points)

Let A be the matrix

$$\begin{bmatrix} 2 & 5 \\ 2 & -1 \end{bmatrix}$$

a) [10 points] Determine the eigenvalues and eigenvectors of A.

Solution

a) The characteristic polynomial is

$$(\lambda - 2)(\lambda + 1) - 10 = \lambda^2 - \lambda - 12$$

which has roots $4, -3$, so these are the eigenvalues of A. The eigenvectors for eigenvalue 4 are the elements in the null space of

$$4I - A = \begin{bmatrix} 2 & -5 \\ -2 & 5 \end{bmatrix}$$

these are all vectors of the form

$$t \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

for $t \in \mathbb{R}$. The eigenvectors for eigenvalue -3 are the elements of the null space of

$$-3I - A = \begin{bmatrix} -5 & -5 \\ -2 & -2 \end{bmatrix}$$

these are all vectors of the form

$$t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for $t \in \mathbb{R}$.

3 Bases and projections (18 points)

For this problem consider the following vectors in $\mathbb{R}^3$.

$$u = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad w = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} \qquad z = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Let V be the vector space spanned by u and v. In this exercise, you may take for granted that $B = \{u, v\}$ is a basis for V.

a) [8 points] Compute the coordinates of w relative to B.

b) [10 points] Compute the projection p of z onto V and the projection matrix from $\mathbb{R}^3$ down to V.

Solution

a) Since $w = 3u - 2v$, the coordinates are $(3, -2)$.

b) The projection matrix is

$$P = A(A^T A)^{-1} A^T$$

where A is the matrix whose columns are u and v. We compute

$$A^T A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix}$$

So

$$(A^T A)^{-1} = \frac{1}{21} \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix}$$

and so

$$\begin{aligned} P &= \frac{1}{21} \begin{bmatrix} 1 & 2 \\ -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{21} \begin{bmatrix} 1 & 2 \\ -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -10 & 2 \\ 9 & 6 & 3 \end{bmatrix} \\ &= \frac{1}{21} \begin{bmatrix} 17 & 2 & 8 \\ 2 & 20 & -4 \\ 8 & -4 & 5 \end{bmatrix} \end{aligned}$$

and so the projection of z onto V is

$$\frac{1}{21} \begin{bmatrix} 17 & 2 & 8 \\ 2 & 20 & -4 \\ 8 & -4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 37 \\ 34 \\ 10 \end{bmatrix}$$

4 Vector spaces (10 points)

Let $M_{3,3}$ be the vector space of 3×3 matrices, and let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

Let V be the subset of $M_{3,3}$ given as

$$V = \{B \in M_{3,3} \mid BA = 0\}$$

where 0 is the matrix of all 0's.

a) [10 points] Is V a subspace of $M_{3,3}$? Argue for your answer.

Solution

a) This is a subspace: It is non-empty since $0A = 0$, so $0 \in V$, if $B, C \in V$ then

$$(B + C)A = BA + BC = 0 + 0 = 0$$

so $B + C$ is in V. If B in V and c is a real number (scalar), then $(cB)A = c(BA) = c0 = 0$ so also $cB \in V$.

5 Basic probability (22 points)

The betting company Safe Bets offers a football pool, i.e., a type of bet where the customer bets simultaneously on the outcome of 13 football games. The outcome of a game is marked as either 1 (the home team wins), $\times$ (a draw), or 2 (the away team wins). In the following problems, suppose that for all games, all three possible outcomes are equally likely.

a) [5 points] Donald Duck turns in a coupon consisting of 13 $\times$'s, i.e. betting on all 13 games to end in a draw. Let X be a random variable describing the number of correct guesses on Donald Duck's coupon. Give a formula describing the probability mass function for X.

The following week Safe Bets introduces the option of *double chance*, i.e., the possibility of selecting two possible outcomes of a single game, increasing the chance of winning. The outcome of the bet is still based on the number of correct guesses of game outcomes on a coupon. In the case of a double chance, a guess is correct if the outcome of the game is one of the two selected on the coupon.

b) [9 points] Daisy Duck turns in a new coupon with all $\times$'s, except for the first game, where she uses the double chance to bet on 1 and $\times$. Let Y be a random variable describing the number of correct guesses on this coupon. Compute the probability $P(Y = 10)$. Your solution may contain exponentials of fractions and binomial coefficients that you are not required to reduce to numbers.

Safe Bets also has a long football pools coupon where customers can bet on the outcome of 100 games. As in the questions above, we assume that all outcomes of all games are equally likely.

c) [8 points] Goofy turns in a long coupon consisting of 100 $\times$'s. Use the central limit theorem to estimate the probability that Goofy gets at least 41 correct guesses. For this problem, you may approximate $\sqrt{2}$ by 1.41.

Solution

a) $P(X = k) = \binom{13}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{13-k}$

b) We apply the total probability theorem. Let A be the event that the outcome of the first game is 1 or $\times$. Given A, the probability of getting 10 correct is the same as getting 9 out of the remaining 12 games correct. Likewise, given A^c there will be 10 correct answers on the coupon precisely when 10 out of the remaining 12 are correct.

$$\begin{aligned}
P(Y = 10) &= P(A)P(Y = 10 \mid A) + P(A^c)P(Y = 10 \mid A^c) \\
&= \frac{2}{3}\left(\binom{12}{9}\left(\frac{1}{3}\right)^9\left(\frac{2}{3}\right)^3\right) + \frac{1}{3}\left(\binom{12}{10}\left(\frac{1}{3}\right)^{10}\left(\frac{2}{3}\right)^2\right) \\
&= \binom{12}{9}\frac{16}{3^{13}} + \binom{12}{10}\frac{4}{3^{13}}
\end{aligned}$$

c) Let X_i be a random variable which is 1 if the guess for the ith game is correct, and 0 otherwise. This is a Bernoulli variable with probability $\frac{1}{3}$, and so has mean $\mu = \frac{1}{3}$ and variance $\sigma^2 = \frac{1}{3}(1 - \frac{1}{3}) = \frac{2}{9}$. Let

$S_n = X_1 + \cdots + X_n$. The desired probability is

$$P(S_{100} \geq 41) = 1 - P(S_{100} \leq 40)$$

By the central limit theorem, $P(S_{100} \leq 40)$ can be approximated by $\Phi(z)$ where

$$\begin{aligned} z &= \frac{40 - 100 \cdot \frac{1}{3}}{\sqrt{\frac{2}{9}}\sqrt{100}} \\ &= \frac{40 - 100 \cdot \frac{1}{3}}{\frac{\sqrt{2}}{3}10} \\ &= \frac{12 - 10}{\sqrt{2}} \\ &= \sqrt{2} \\ &\approx 1.41 \end{aligned}$$

and $\Phi(1.41) = 0.9207$ (this can be found in a table in the book) so

$$P(S_{100} \geq 41) = 1 - P(S_{100} \leq 40) \approx 1 - 0.9207 = 0.0703$$

6 Conditional probabilities (23 points)

Suppose we are given two 3-sided dice with probabilities as given in the table below.

	Side 1	Side 2	Side 3
Die 1	1/3	1/2	1/6
Die 2	1/2	1/3	1/6

Suppose we pick a die at random (each die has equal probability of being picked) and roll it 3 times to get the sequence $(1, 2, 1)$.

a) [10 points] What is the probability that we picked die number 1?

A third three sided die is introduced. Unlike the other dice, the sides of this are labelled with colours rather than numbers. The probabilities are as follows

	Blue	Red	Green
Die 3	θ	2θ	1-3θ

for some unknown $\theta \in [0, \frac{1}{3}]$. We decide to determine θ by conducting an experiment, rolling the die twice. For the experiment we will model the value of θ by a random variable Θ. Let Z be a random variable which is 1 if rolling the die twice gives the sequence $(\text{Blue}, \text{Red})$, and 0 otherwise.

b) [5 points] Give a formula for the conditional probability mass function $p_{Z|\Theta}(z \mid \theta)$.

Suppose now we assume as our prior distribution on Θ a uniform distribution on $[0, \frac{1}{3}]$.

c) [8 points] Compute the posterior distribution PDF for Θ given that the 2 rolls of the die resulted in the sequence $(\text{Blue}, \text{Red})$, i.e. determine a formula for the probability density function $f_{\Theta|Z}(\theta \mid 1)$. Compute the maximum a posteriori probability (MAP) estimate of θ.

Solution

a) Let D be a random variable indicating the number of the die picked. Let A be the event that we observe the sequence $(1, 2, 1)$. Then

$$P(A \mid D = 1) = \frac{1}{3}\frac{1}{2}\frac{1}{3} = \frac{1}{18} \qquad\qquad P(A \mid D = 2) = \frac{1}{2}\frac{1}{3}\frac{1}{2} = \frac{1}{12}$$

and so

$$P(A \mid D = 1)P(D = 1) + P(A \mid D = 2)P(D = 2) = \frac{1}{18}\frac{1}{2} + \frac{1}{12}\frac{1}{2} = \frac{1}{36} + \frac{1}{24} = \frac{5}{72}$$

So by Bayes Theorem

$$P(D = 1 \mid A) = \frac{P(A \mid D = 1)P(D = 1)}{P(A \mid D = 1)P(D = 1) + P(A \mid D = 2)P(D = 2)} = \frac{72}{5 \cdot 36} = \frac{2}{5}$$

b) The conditional probability is

$$p_{Z|\Theta}(1 \mid \theta) = \theta(2\theta) = 2\theta^2$$

and $p_{Z|\Theta}(0 \mid \theta) = 1 - 2\theta^2$

c) The posterior probability function is given by Bayes rule

$$f_{\Theta|Z}(\theta \mid 1) = \frac{f_\Theta(\theta) p_{Z|\Theta}(1 \mid \theta)}{\int_0^{\frac{1}{3}} f_\Theta(\theta') p_{Z|\Theta}(1 \mid \theta') d\theta'}$$

The denominator of this is

$$\int_0^{\frac{1}{3}} 6\theta'^2 d\theta' = 6 \left[\frac{1}{3} \theta'^3 \right]_0^{\frac{1}{3}} = \frac{2}{3^3}$$

and the enumerator is

$$6\theta^2$$

Thus

$$f_{\Theta|Z}(\theta \mid 1) = 3^4 \theta^2$$

for $\theta \in [0, \frac{1}{3}]$ and 0 otherwise. To compute the MAP, we must find the θ that maximises this, i.e. we must maximise θ^2 on the interval $[0, \frac{1}{3}]$. The max is at $\frac{1}{3}$, and so the MAP is $\frac{1}{3}$.