Intelligent Systems Programming

Lecture 10: **Linear Programming I**

History of Linear Programming (LP)

- Simplex invented 1947 by Georg B. Danzig
- Expected to be the answer to ***everything***
  - Oil blending
  - Crew assignment
  - Production planning
  - Games
- LP proven to be in **P** in 1979
- Important tool in algorithms

Today's Program

- [10:00-10:50]
  - LP problem examples
  - Definition of LP problems
  - The standard form
  - Geometric interpretation
- [11:00-12:00]
  - Slack form
  - The simplex algorithm
  - Dictionaries
  - Geometric interpretation of the simplex algorithm

Problem Example: Diet Problem

- Choose number of servings of six foods such that:
  - 2000 kcal, 55g protein, 800 mg calcium, and min cost

Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Cost (cents)	Max servings
Oatmeal	28 g	110	4	2	3	4
Chicken	100 g	205	32	12	24	3
Eggs	2 large	160	13	54	13	2
Milk	237 cc	160	8	285	9	8
Cherry pie	170 g	420	4	22	20	2
Pork w/ beans	260 g	260	14	80	19	2

Decision Variables

- x_1 : number of oatmeal servings
- x_2 : number of chicken servings
- x_3 : number of eggs servings
- x_4 : number of milk servings
- x_5 : number of cherry pie servings
- x_6 : number of pork w/ beans servings

Objective and Constraints

Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Cost (cents)	Max servings
Oatmeal	28 g	110	4	2	3	4
Chicken	100 g	205	32	12	24	3
Eggs	2 large	160	13	54	13	2
Milk	237 cc	160	8	285	9	8
Cherry pie	170 g	420	4	22	20	2
Pork w/ beans	260 g	260	14	80	19	2

- Objective

$$\min 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6$$

- Constraints

$$110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \geq 2000$$

$$4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \geq 55$$

$$2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \geq 800$$

$$0 \leq x_1 \leq 4 \quad 0 \leq x_2 \leq 3 \quad 0 \leq x_3 \leq 2$$

$$0 \leq x_4 \leq 8 \quad 0 \leq x_5 \leq 2 \quad 0 \leq x_6 \leq 2$$

Definition of LP Problems

Linear function

$f(x_1,x_2,\ldots,x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$

1. **Objective: maximize/minimize linear function**

$$\max f(x_1,x_2,\ldots,x_n),$$
$$\min f(x_1,x_2,\ldots,x_n)$$

2. **Constraints**
   a) Linear equations: $f(x_1,x_2,\ldots,x_n) = b$
   b) Linear inequalities: $f(x_1,x_2,\ldots,x_n) \geq b$
   $f(x_1,x_2,\ldots,x_n) \leq b$

3. **Continuous decision variables:** $x_1,x_2,\ldots,x_n \in \mathbf{R}$

LP Problems in Standard Form

- Maximize

$$\sum_{j=1}^{n} c_j x_j$$

- Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1,2,\dots,m)$$

$$x_j \geq 0 \qquad (j = 1,2,\dots,n)$$

$$a_{ij}, b_i, c_j, x_j \in \mathbb{R}$$

Conversion to Standard Form

Standard Form

Maximize

$$\sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

$$x_j \geq 0$$

- How do we convert the following to standard form:
  - Minimization rather than maximization ?
  - Larger-than-or-equal constraints (≥) ?
  - Equality constraints (=) ?
  - $x \leq 0$ variables ?
  - Free variables (i.e., domain is all reals) ?

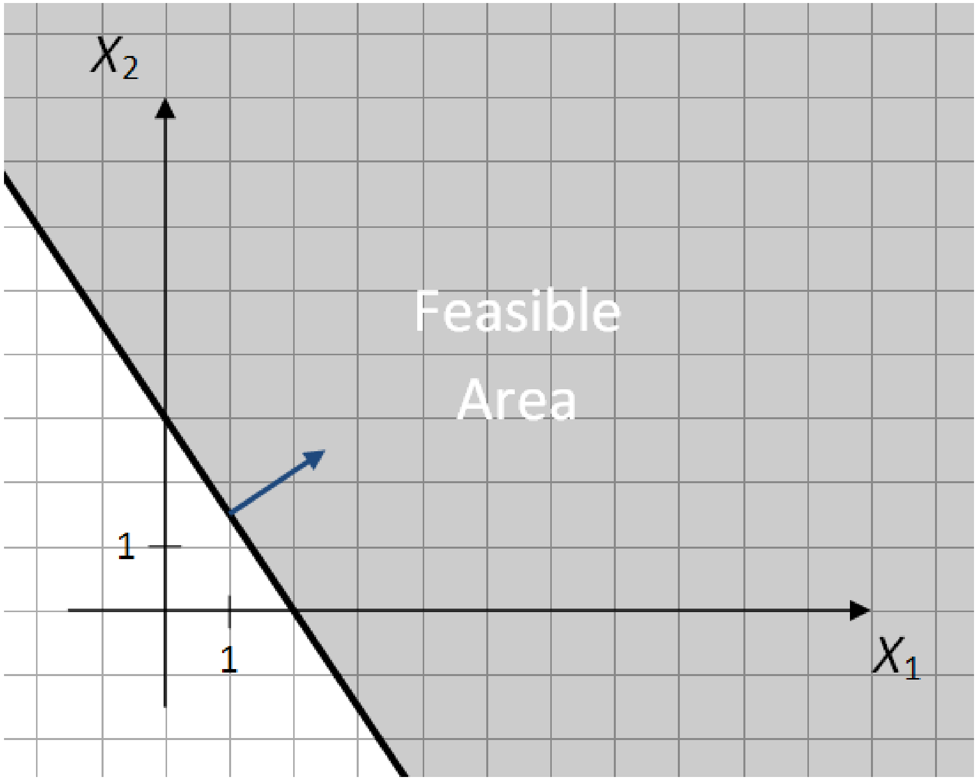
What constraints are linear

- $x_1 = x_2 x_4$? No, unfortunately
- $AllDiff(x_1, x_2, \ldots, x_k)$? No + makes little sense
- $12.3 = \frac{x_1}{x_2} + 9$? Ok, $12.3x_2 = x_1 + 9x_2$
- $x_1 \wedge x_4 \Rightarrow x_2$? No + makes little sense
- $x_1 = max(x_2, x_3)$? Ok, $x_1 \geq x_2$, $x_1 \geq x_3$, $min(x_1)$

Geometric Interpretation of LP Problems

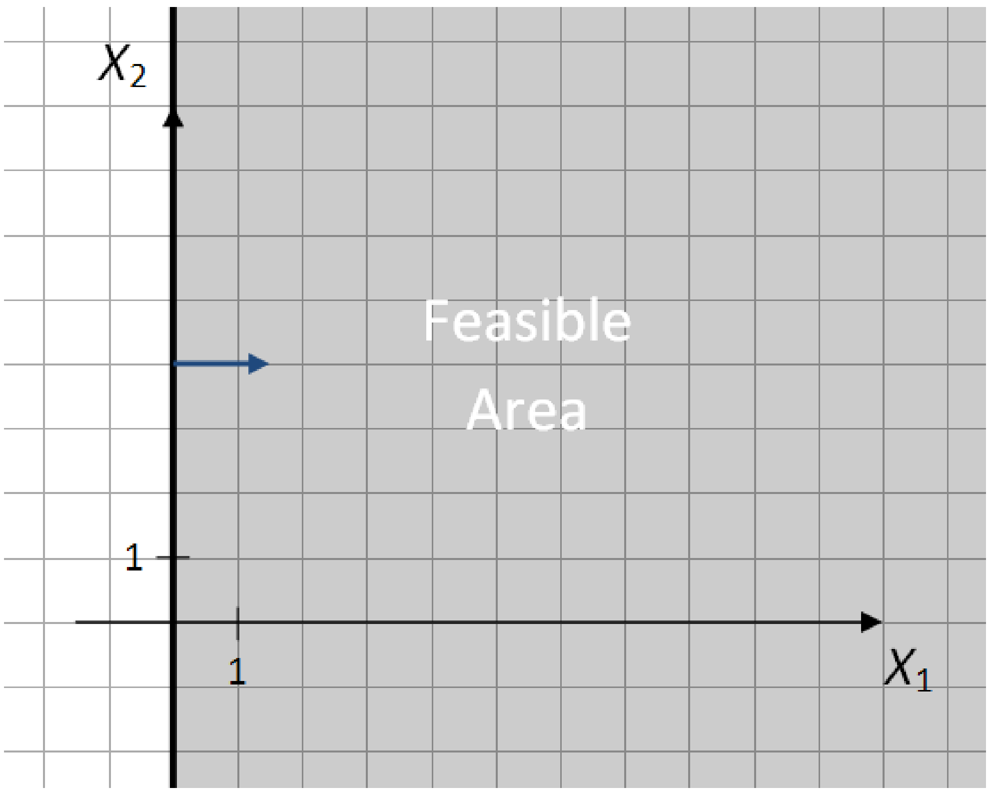
Half-Spaces

- Each LP constraint forms a closed halfspace in the coordinate system of the decision variables
- Example 1:
  $3x_1 + 2x_2 \geq 6$



Half-Spaces

- Each LP constraint forms a closed halfspace in the coordinate system of the decision variables
- Example 2:
  $x_1 \geq 0$

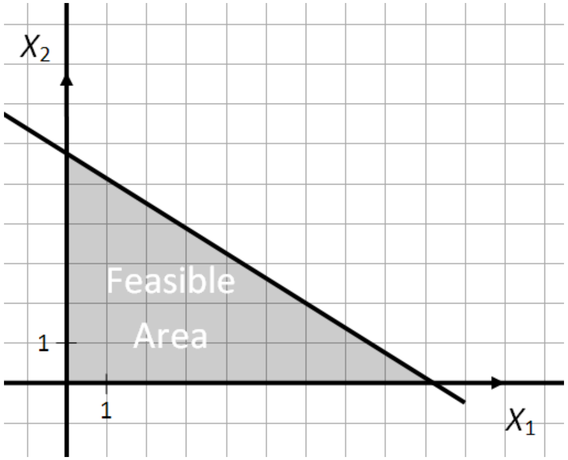


Constraints form a Polyhedron

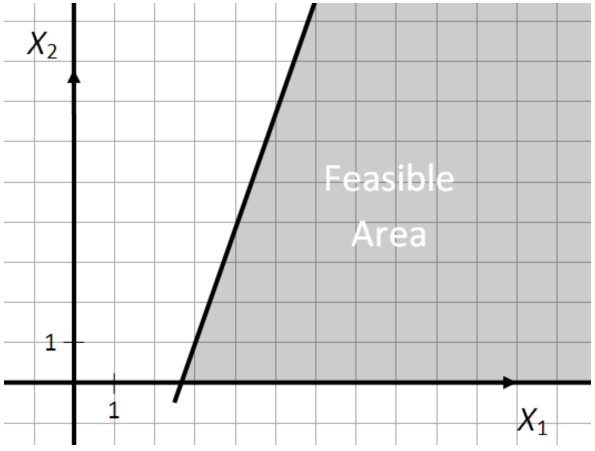
- A Polyhedron P is the intersection of finitely many closed halfspaces in some $\mathbf{R}^n$

$$P = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for} \quad i = 1,2,\dots,m\}$$

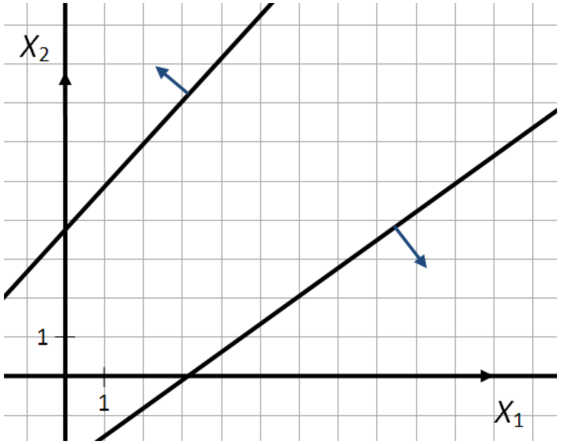
- 2D Examples



Bounded



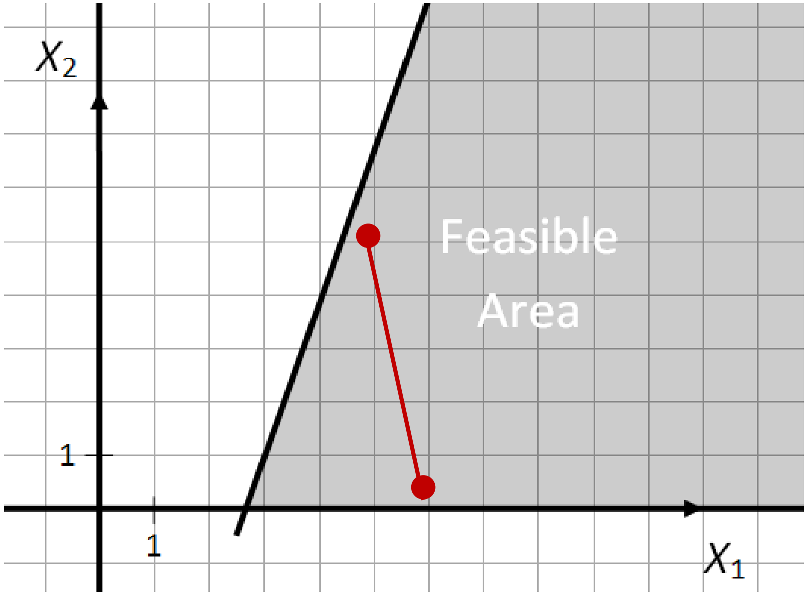
Unbounded



Infeasible

Convexity of Polyhedron

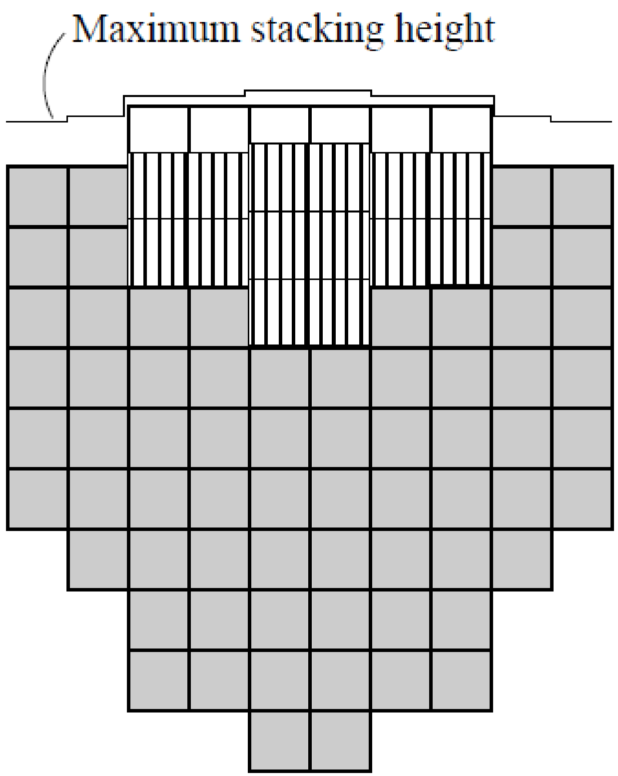
- A polyhedron is a convex set: a line between two points does not leave the set

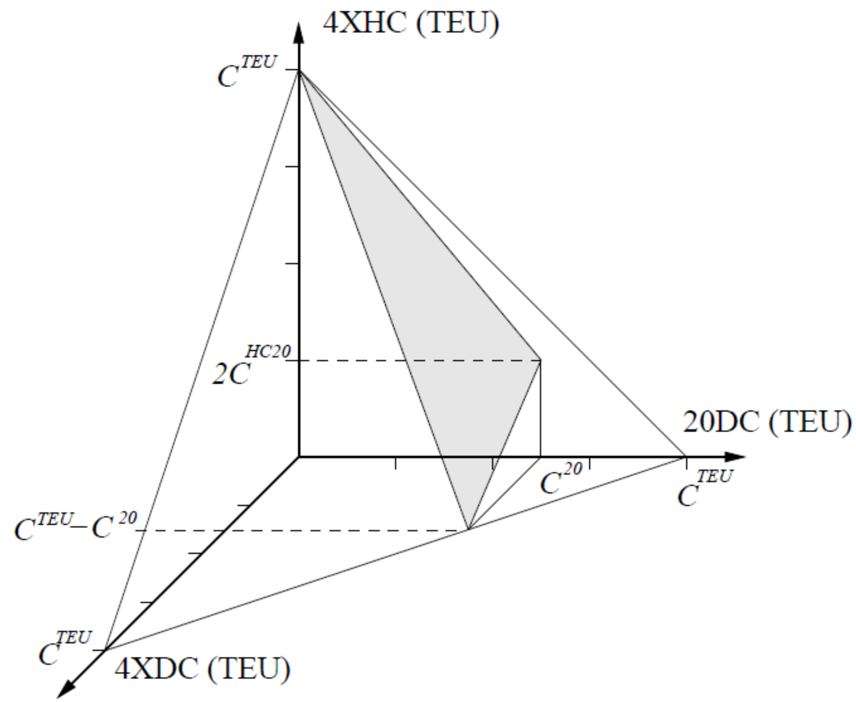


- How do we define a concave set using linear constraints?

Linear Constraints as Polyhedron

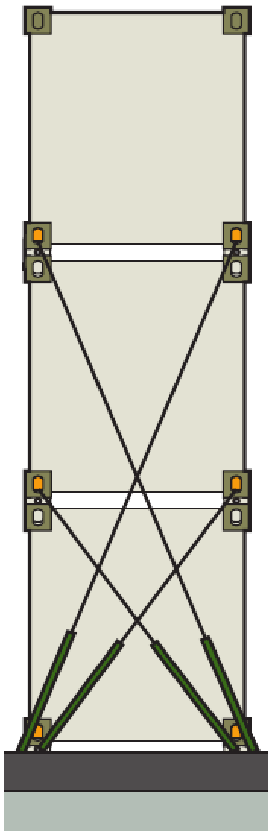
- Container mix below deck

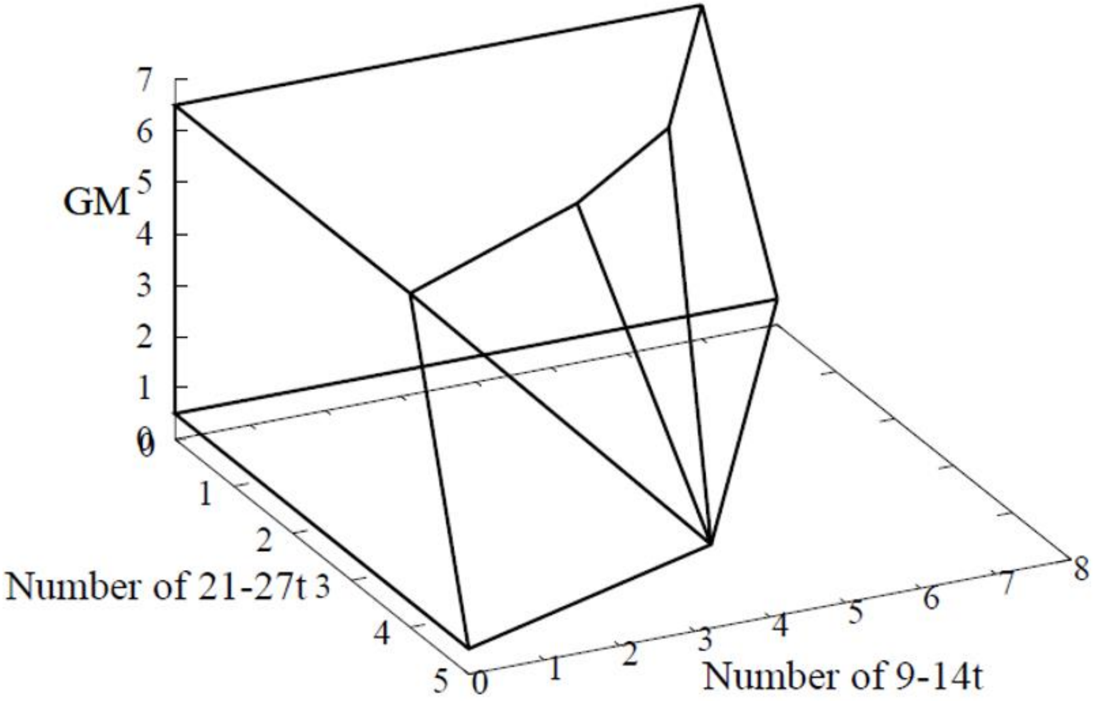




Linear Constraints as Polyhedron

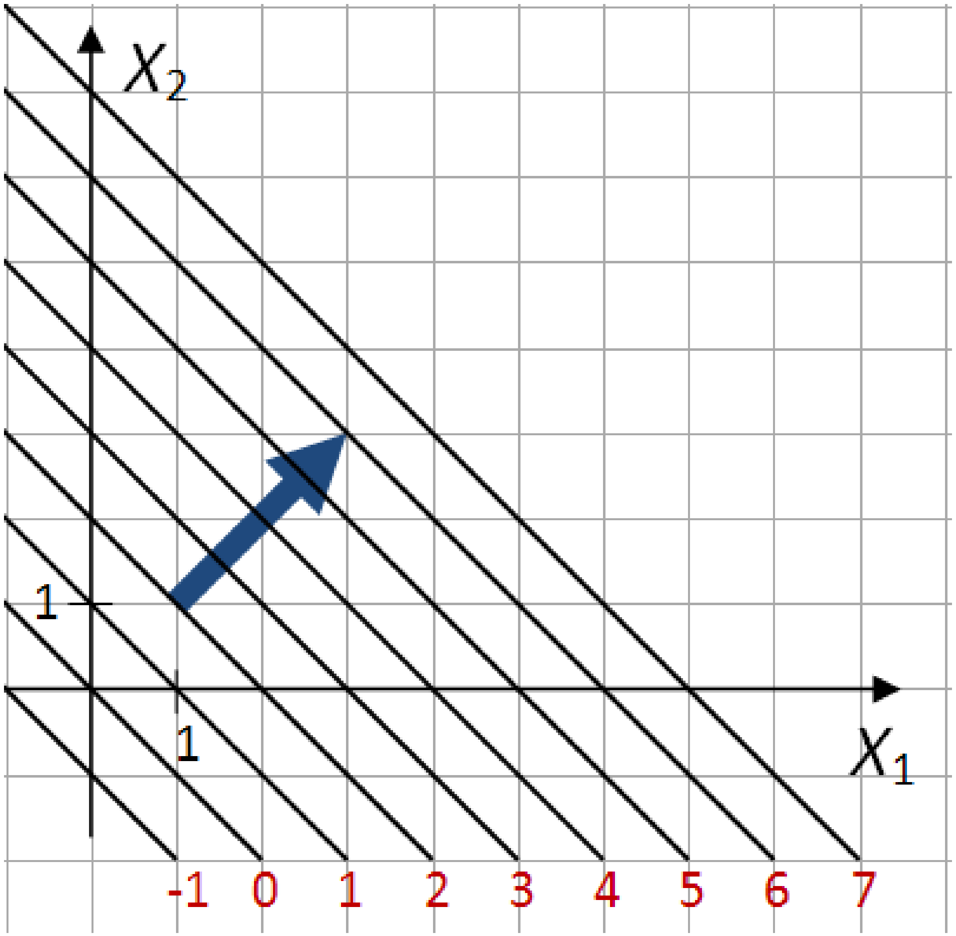
- Forces in securing equipment





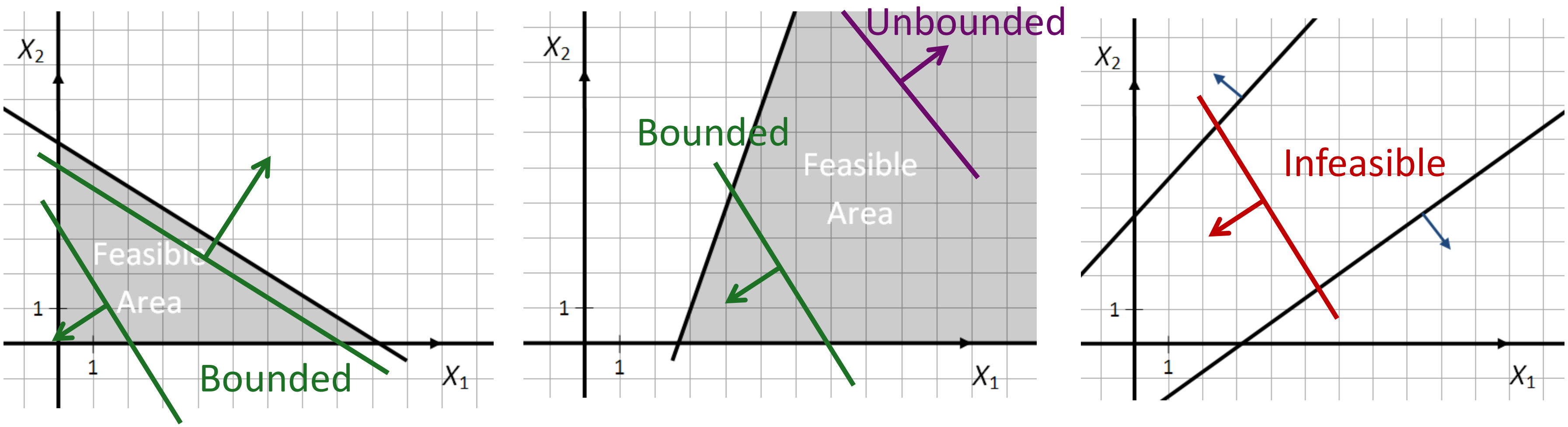
Objective Contours Are Lines

Example: *maximize* $x_1 + x_2$



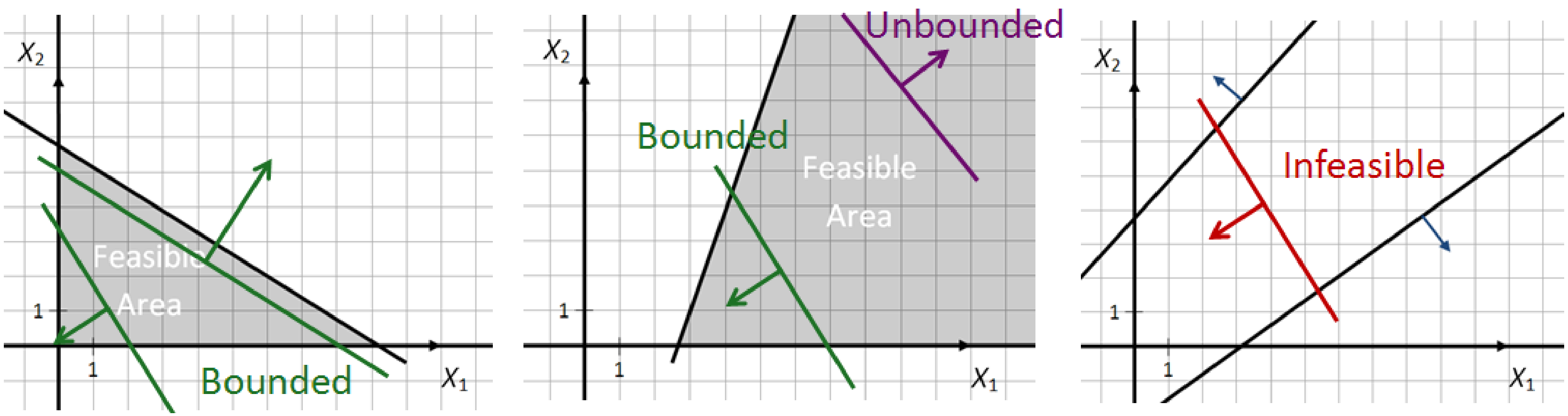
Geometric Interpretation of LP Problems in 2D

1. The constraints are a polyhedron in quadrant I
2. Objective contours are lines

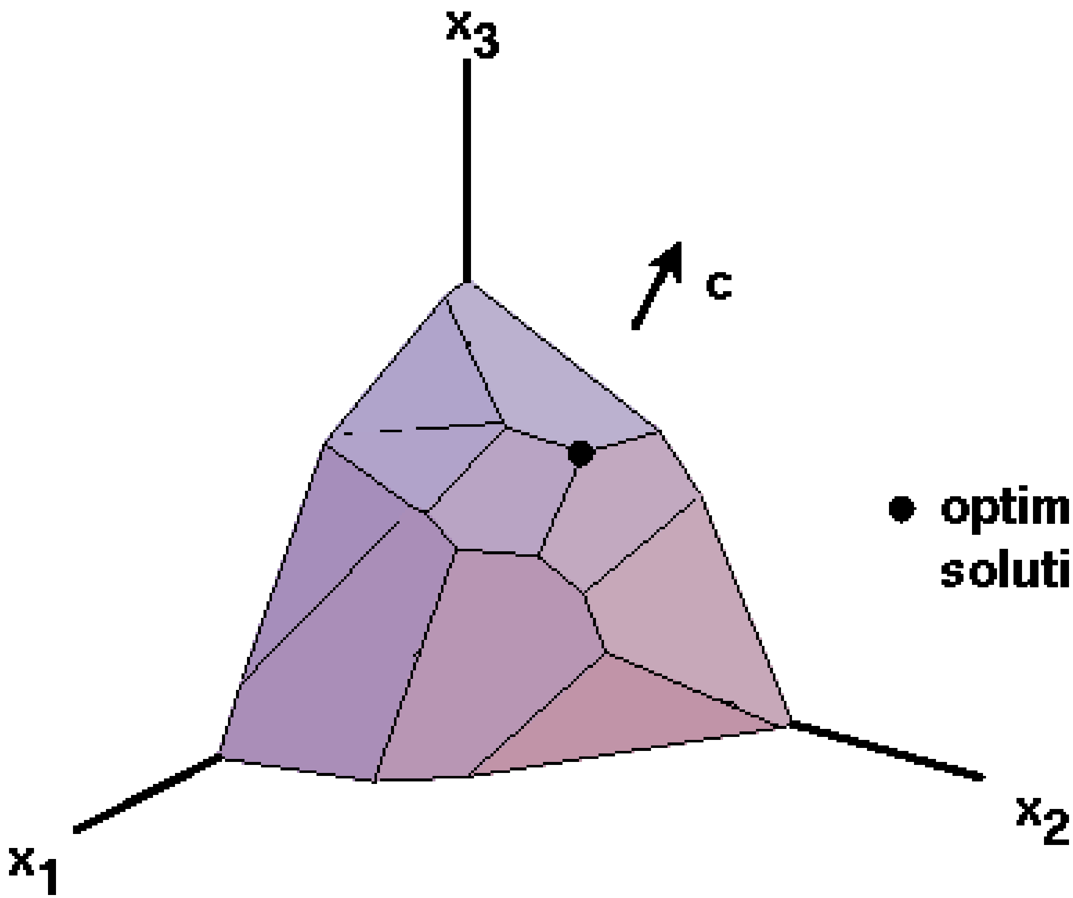


The Fundamental Theorem of LP

- **Theorem** Every LP problem in the standard form has the following three properties:
  1) If it has no optimal solution, then it is either infeasible or unbounded
  2) If it has a feasible solution, then it has a corner point solution
  3) If it has an optimal solution, then it has a corner point optimal solution



3D Example



● optimal solution

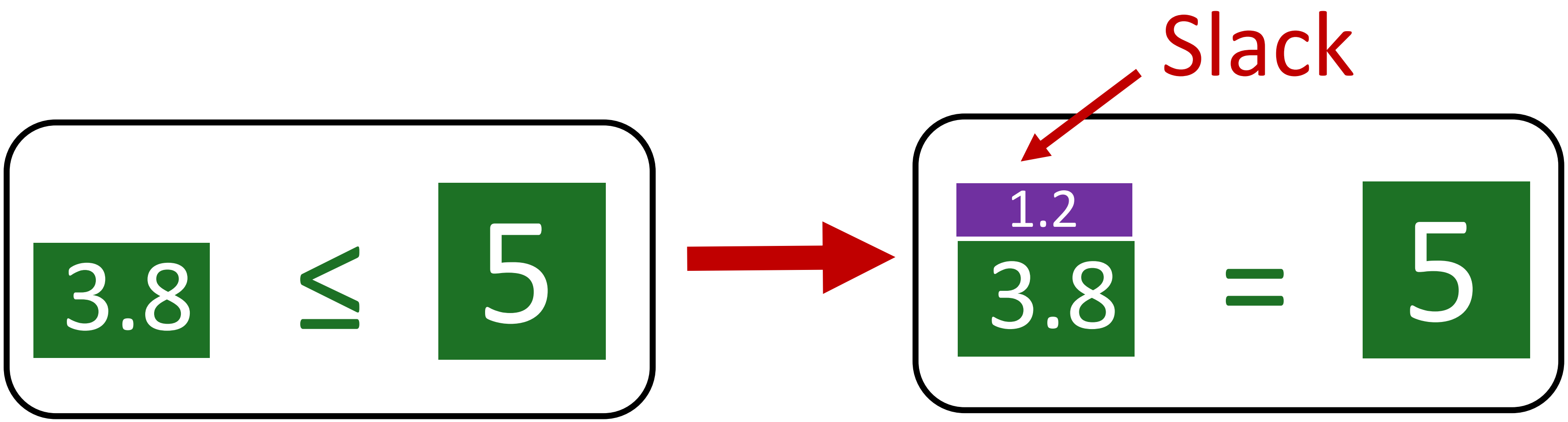
The Simplex Algorithm

Simplex Example

Maximize $$5x_1 + 4x_2 + 3x_3$$

Subject to
$$\begin{aligned} 2x_1 + 3x_2 + x_3 &\leq 5 \\ 4x_1 + x_2 + 2x_3 &\leq 11 \\ 3x_1 + 4x_2 + 2x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Slack Variables



- **Idea**: define slack variables and represent inequalities as equalities with non-negative slack requirements

Slack Variables

- Example first constraint:

Original form:

$$\boxed{2x_1 + 3x_2 + x_3 \leq 5}$$

Slack form:

$$\boxed{\begin{array}{c} x_4 = 5 - 2x_1 - 3x_2 - x_3 \\ x_4 \geq 0 \end{array}}$$

Slack variable

Standard Form and Slack Form

Standard Form

Maximize

$5x_1 + 4x_2 + 3x_3$

Subject to

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 &\le 5 \\
4x_1 + x_2 + 2x_3 &\le 11 \\
3x_1 + 4x_2 + 2x_3 &\le 8
\end{aligned}
$$

$$\underbrace{x_1,\ x_2,\ x_3}_{\text{decision variables}} \ge 0$$

Slack Form

Maximize

z

Subject to

$$
\begin{aligned}
x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\
x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\
x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
z &= 5x_1 + 4x_2 + 3x_3
\end{aligned}
$$

$$x_1, x_2, x_3, \underbrace{x_4, x_5, x_6}_{\text{slack variables}} \ge 0$$

Grand Strategy of Simplex

Successive improvement

- In each step:
  - Given current feasible solution:

    $$x_1, x_2, x_3, x_4, x_5, x_6$$

  - Find another feasible solution:

    $$\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}, \overline{x_6}$$

  - Such that:

    $$\bar{z} > z \Leftrightarrow$$

    $$5\overline{x_1} + 4\overline{x_2} + 3\overline{x_3} > 5x_1 + 4x_2 + 3x_3$$

- Repeat this a finite number of times to reach an optimal solution

Initial feasible solution

$$\begin{aligned}
x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\
x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\
x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
z &= \phantom{11 -{}} 5x_1 + 4x_2 + 3x_3
\end{aligned}$$

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 5, x_5 = 11, x_6 = 8$$

$$z = 0$$

How much can x_1 increase?

$$x_4 = 5 - 2x_1 - 3x_2 - x_3 \Rightarrow x_1 \leq \frac{5}{2}$$

$$x_5 = 11 - 4x_1 - x_2 - 2x_3 \Rightarrow x_1 \leq \frac{11}{4}$$

$$x_6 = 8 - 3x_1 - 4x_2 - 2x_3 \Rightarrow x_1 \leq \frac{8}{3}$$

Answer: $\frac{5}{2}$

Second Iteration

- New feasible solution:

$$x_1 = \frac{5}{2}, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 1, x_6 = \frac{1}{2}$$

$$z = \frac{25}{2}$$

- Idea: express variables with positive values in terms of variables with zero values

Second Iteration Cont.

- x_1 can be expressed by rewriting the first equation

$$x_4 = 5 - 2x_1 - 3x_2 - x_3$$

$$\Leftrightarrow \quad x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$$

Second Iteration Cont.

- Substitute this expression into the remaining equations:

$$x_5 = 11 - 4\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) - \quad x_2 - 2x_3$$

$$x_6 = \quad 8 - 3\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) - 4x_2 - 2x_3$$

$$z = \qquad 5\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) + 4x_2 + 3x_3$$

New System

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$$

$$x_5 = 1 + 5x_2 + 2x_4$$

$$x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4$$

$$z = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4$$

How much can x_3 increase?

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \quad \Rightarrow \quad x_3 \le 5$$

$$x_5 = 1 + 5x_2 + 2x_4 \quad \Rightarrow \quad x_3 \le \infty$$

$$x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{3}{2}x_4 \quad \Rightarrow \quad x_3 \le 1$$

Answer: 1

Third Iteration

- New solution

$$x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0$$

$$z = 13$$

- New system

$$\begin{aligned} x_3 &= \ \ 1 + \ \ x_2 - 3x_4 - 2x_6 \\ x_1 &= \ \ 2 - 2x_2 - 2x_4 + \ \ x_6 \\ x_5 &= \ \ 1 + 5x_2 + 2x_4 \\ z &= 13 - 3x_2 - \ \ x_4 - \ \ x_6 \end{aligned}$$

Optimal!

Terminology

$$
\begin{aligned}
x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\
x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\
x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
z &= \phantom{11 - {}} 5x_1 + 4x_2 + 3x_3
\end{aligned}
$$

Dictionary

Terminology Cont.

Basic variables (m)

"The Basis"

$$
\begin{aligned}
\boxed{\begin{matrix} x_4 \\ x_5 \\ x_6 \end{matrix}} &\begin{matrix} = 5 - \\ = 11 - \\ = 8 - \end{matrix}
\boxed{\begin{matrix} 2x_1 - 3x_2 - x_3 \\ 4x_1 - x_2 - 2x_3 \\ 3x_1 - 4x_2 - 2x_3 \\ 5x_1 + 4x_2 + 3x_3 \end{matrix}} \\
z &=
\end{aligned}
$$

Non-basic variables (n)

Terminology Cont.

Pivot row

x_4: leaving variable

$$\begin{aligned} x_4 &= 5 - \boxed{2x_1} - 3x_2 - x_3 \\ x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\ x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \\ z &= 5x_1 + 4x_2 + 3x_3 \end{aligned}$$

Pivot column

x_1: entering variable

Terminology Cont.

$$
\begin{aligned}
x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\
x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\
x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
z &= 5x_1 + 4x_2 + 3x_3
\end{aligned}
$$

Pivoting

$$
\begin{aligned}
x_1 &= \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\
x_5 &= 1 + 5x_2 + 2x_4 \\
x_6 &= \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{3}{2}x_4 \\
z &= \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4
\end{aligned}
$$

Simplex Algorithm

1. Compute the dictionary of an initial solution
2. Choose a variable x_i with positive coefficient in objective expression (entering variable or pivot column)
3. Calculate its maximum increase given that all basic variables must remain non-negative
4. Choose an equation for a basic variable x_j that becomes zero when x_i is increased (leaving variable or pivot row)
5. Solve it with respect to x_i
6. Substitute this new expression for x_i in remaining basic variable expressions in dictionary and in z expression
7. Stop if resulting dictionary is optimal (no variable coefficients in z are positive), otherwise goto step 2

Geometric Interpretation of Simplex

Example on Standard Form

Standard Form

Maximize

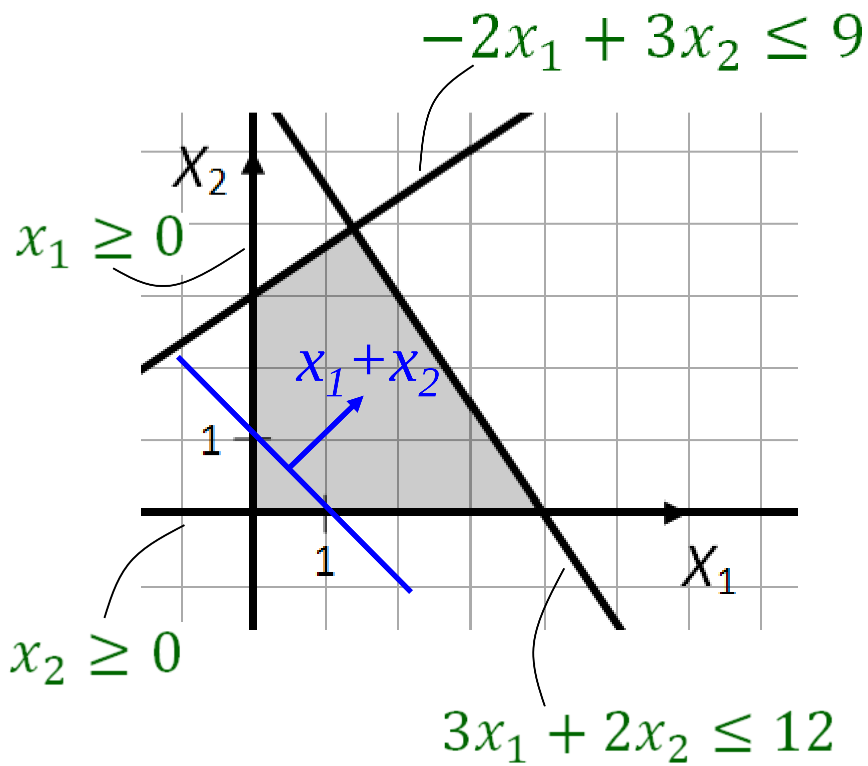
$$x_1 + \quad x_2$$

Subject to

$$-2x_1 + 3x_2 \leq \quad 9$$
$$3x_1 + 2x_2 \leq 12$$

$$x_1, x_2 \geq \quad 0$$

Geometric Interpretation



Example on Slack Form

Slack Form

Maximize

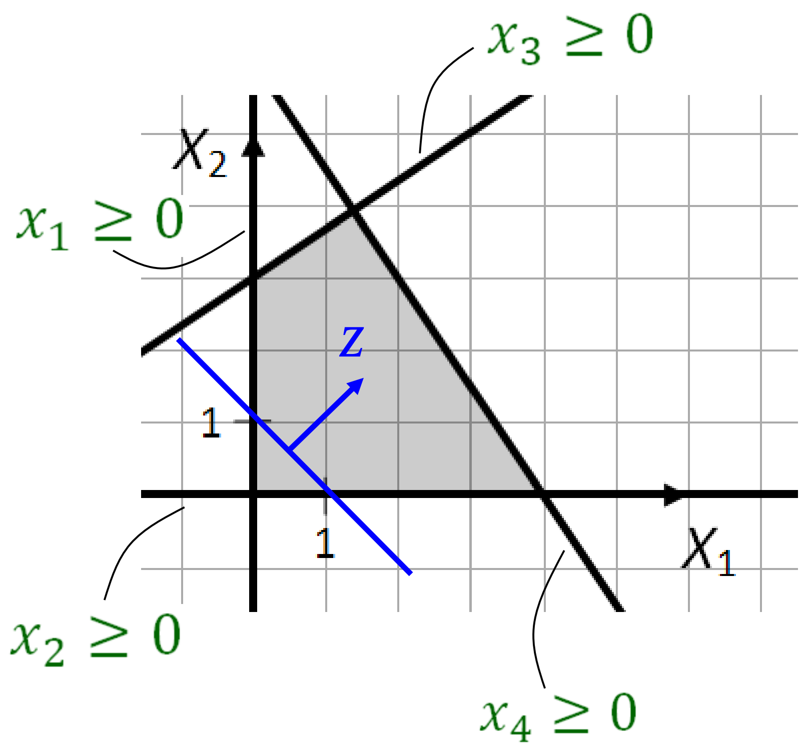
$$z$$

Subject to

$$\begin{aligned} x_3 &= \ \ 9 + 2x_1 - 3x_2 \\ x_4 &= 12 - 3x_1 - 2x_2 \\ z &= \qquad\ \ x_1 + \ \ x_2 \end{aligned}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Geometric Interpretation



Immediate Insights

- In a dictionary n non-basic variables are zero

  $\Rightarrow$ n constraints are binding!
  $\Rightarrow$ a dictionary corresponds to a corner point of the polyhedron

  $\Rightarrow$ Simplex searches in the space of corner points (called basic solutions)

- In a pivot one variables changes from non-basic to basic (entering variable) and one variable changes from basic to non-basic (leaving variable)

  $\Rightarrow$ Simplex pivots to an adjacent corner point

Simplex Step 0

Initial Dictionary

$$\begin{aligned} x_3 &= 9 + 2x_1 - 3x_2 \\ x_4 &= 12 - 3x_1 - 2x_2 \\ z &= x_1 + x_2 \end{aligned}$$

Basis $x_3 = 9$

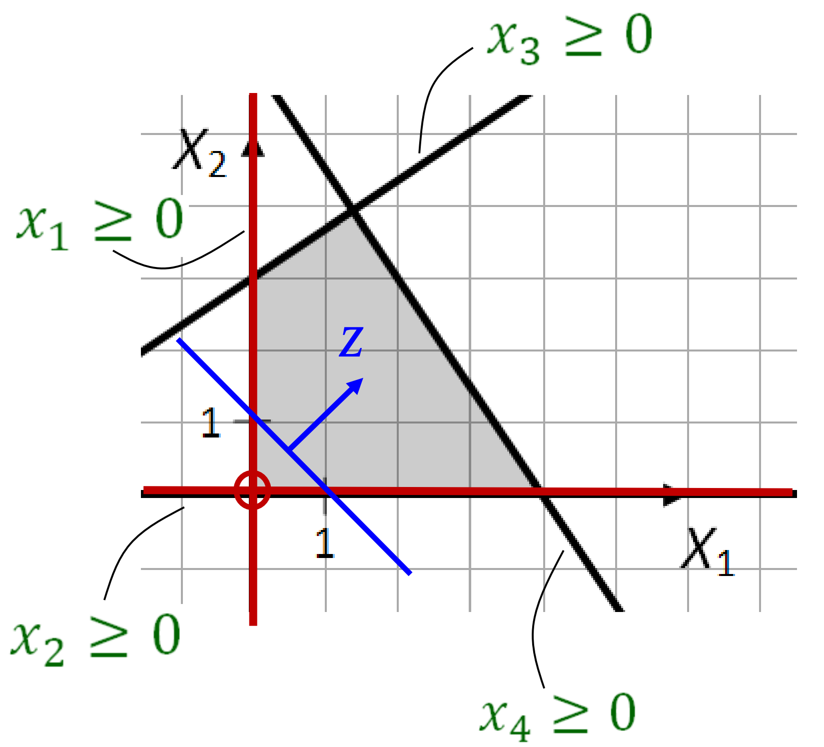
$x_4 = 12$

Non-Basis $x_1 = 0$

$x_2 = 0$

Objective $z = 0$

Geometric Interpretation



Entering variable x_1

$$x_3 = 9 + 2x_1 - 3x_2 \quad \Rightarrow \quad x_1 \le \infty$$

$$x_4 = 12 - 3x_1 - 2x_2 \quad \Rightarrow \quad x_1 \le 4$$

1) Solve wrt. x_1

$$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4$$

2) Substitute expression for x_1 into x_3 and z expression

$$x_3 = 9 + 2\left(4 - \frac{2}{3}x_2 - \frac{1}{3}x_4\right) - 3x_2 = 17 - \frac{13}{3}x_2 - \frac{2}{3}x_4$$

$$z = x_1 + x_2 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4 + x_2 = 4 + \frac{1}{3}x_2 - \frac{1}{3}x_4$$

Simplex Step 1

New Dictionary

$$x_3 = 17 - \frac{13}{3}x_2 - \frac{2}{3}x_4$$

$$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4$$

$$z = 4 + \frac{1}{3}x_2 - \frac{1}{3}x_4$$

Basis $x_1 = 4$

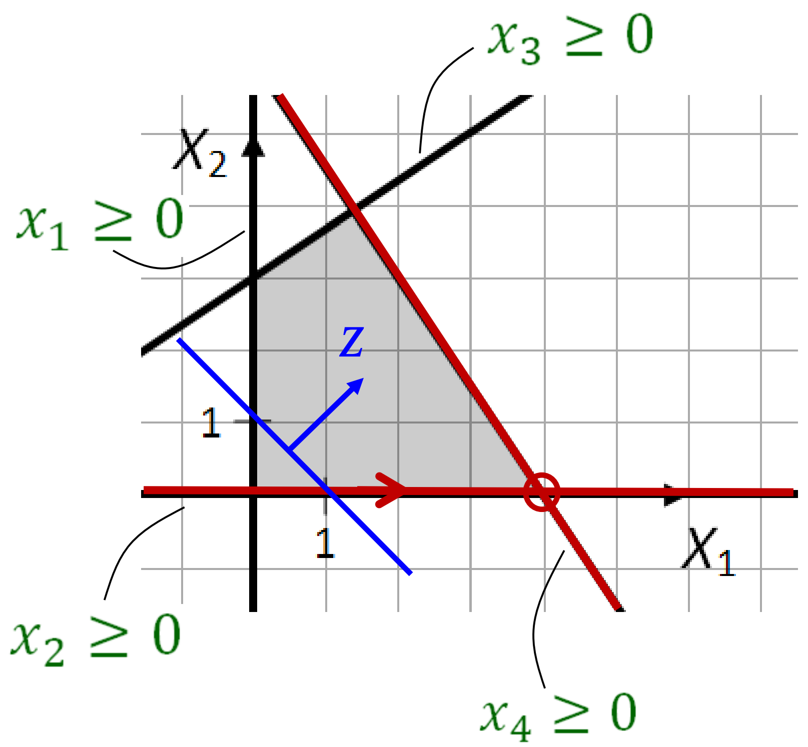
$x_3 = 17$

Non-Basis $x_2 = 0$

$x_4 = 0$

Objective $z = 4$

Geometric Interpretation



Entering variable x_2

$$x_3 = 17 - \frac{13}{3}x_2 - \frac{2}{3}x_4 \quad \Rightarrow \quad x_2 \leq \frac{51}{13}$$

$$x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4 \quad \Rightarrow \quad x_2 \leq 6$$

1) Solve wrt. x_2

$$x_2 = \frac{51}{13} - \frac{3}{13}x_3 - \frac{2}{13}x_4$$

2) Substitute expression for x_2 into x_1 and z expression

$$x_1 = 4 - \frac{2}{3}\left(\frac{51}{13} - \frac{3}{13}x_3 - \frac{2}{13}x_4\right) - \frac{1}{3}x_4 = \frac{18}{13} + \frac{4}{39}x_3 - \frac{3}{13}x_4$$

$$z = 4 + \frac{1}{3}\left(\frac{51}{13} - \frac{3}{13}x_3 - \frac{2}{13}x_4\right) - \frac{1}{3}x_4 = \frac{69}{13} - \frac{1}{13}x_3 - \frac{15}{39}x_4$$

Simplex Step 2

New Dictionary

$$x_2 = \frac{51}{13} - \frac{3}{13}x_3 - \frac{2}{13}x_4$$
$$x_1 = \frac{18}{13} + \frac{4}{39}x_3 - \frac{3}{13}x_4$$
$$z = \frac{69}{13} - \frac{1}{13}x_3 - \frac{15}{39}x_4$$

Basis $x_1 = \frac{18}{13} \cong 1.38$

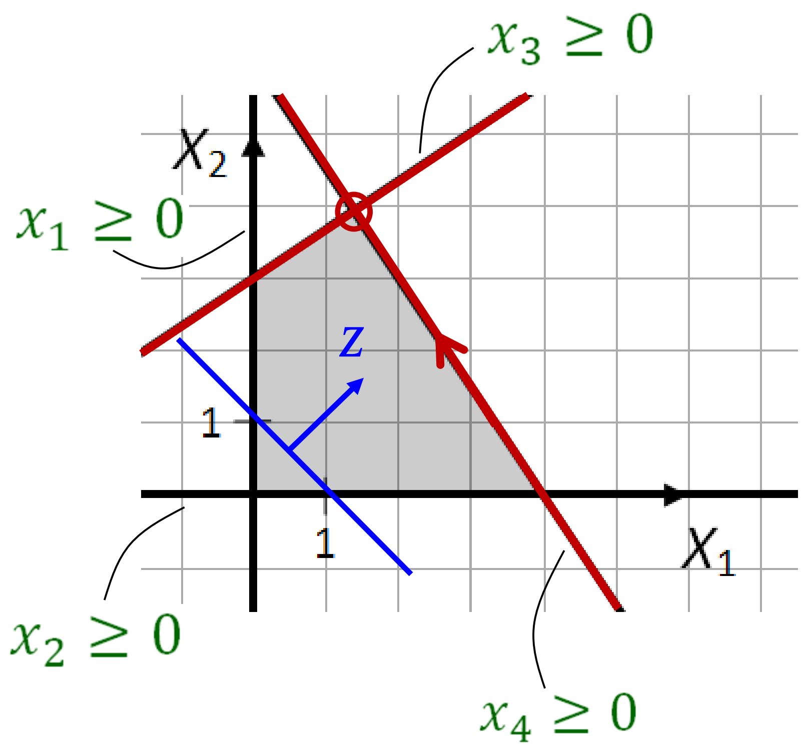
$x_2 = \frac{51}{13} \cong 3.92$

Non-Basis $x_3 = 0$

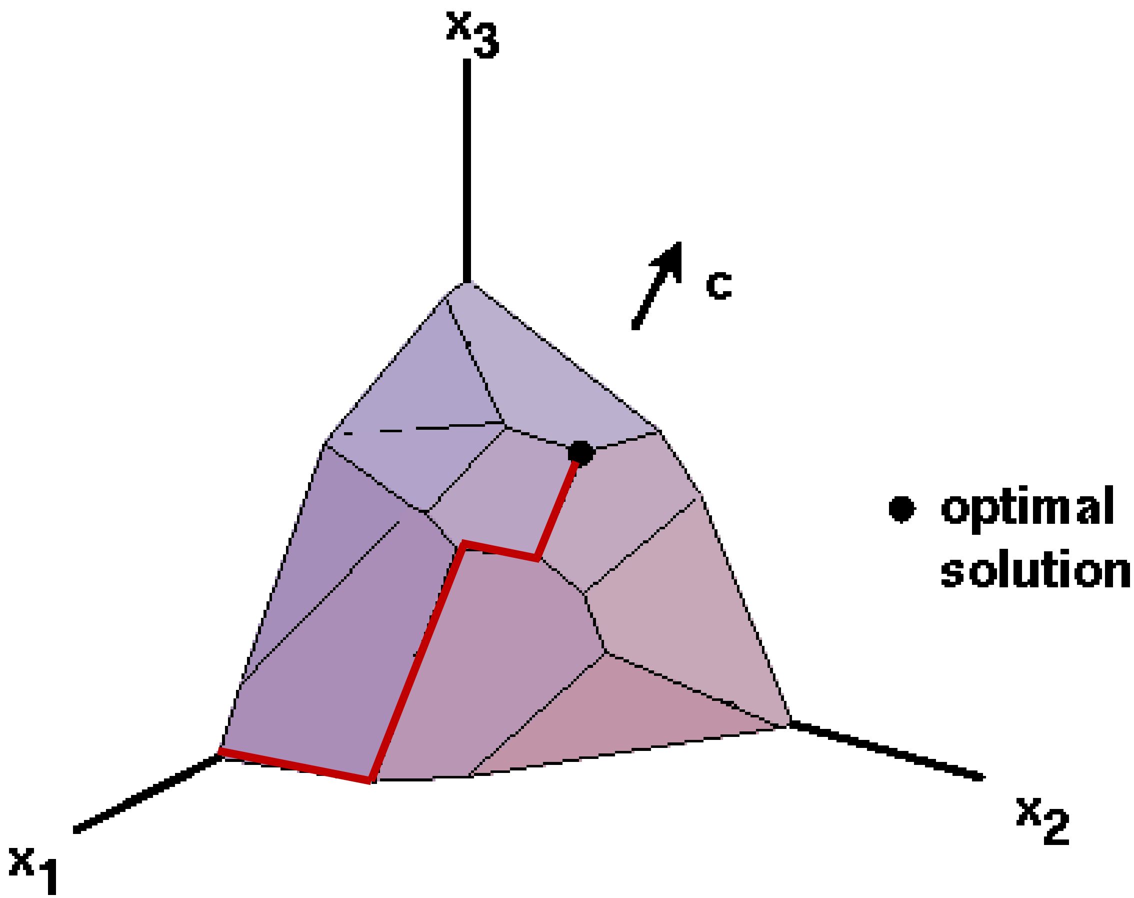
$x_4 = 0$

Objective $z = \frac{69}{13} \cong 5.31$

Geometric Interpretation



3D Example



Something to think about for next week

- Is the slack form always a feasible initial dictionary?
- How will we know if the problem is unbounded?
- Does each pivoting always improve z?
- Does Simplex terminate?