Mandatory 2

Monday, February 18, 2019 5:00 PM

The deadline is Tuesday February 26 at 23:55. Prove that the set of polynomials of degree at most 2 for which 3 is a root is a subspace of P_2 (the vector space of all polynomials of degree at most 2). From the book do 4.4.26. **Please write clearly and explain each step you take in these exercises.** Writing formulas is not enough, text is needed as well. Part of the purpose of the exercise is to practice presenting mathematics in writing.

Here is a hint for 4.4.26: Consider an arbitrary vector $c_3x^3 + c_2x^2 + c_1x + c_0$ in P_3. Write u_0, u_1, u_2, u_3, for the vectors mentioned in the exercise. Write out the condition that $c_3x^3+c_2x^2+c_1x+c_0 = au_0+bu_1+cu_2+du_3$. Comparing the coefficients for x^3, x^2, x and the constant parts of the polynomials in the equation gives 4 equations in the unknowns a, b, c, d. Write out these equations, and show that they can be solved for every c_3, c_2, c_1, c_0.

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Prove that set of polynomials of degree at most 2 and root 3 is subspace of P2

Assume root of 3 is $x = 3$ *in our polynomials*

$let\ P_2 = set\ of\ polynomials\ of\ degree\ 2$

$W = \{(a, b, c) |\ a + b3 + c3^2 = 0\} \subset P_2$

For the subset W to be a subspace of vector space P_2 W must satisfy the following 3 conditions

- W *is nonempty*
- W *is closed under addition*
- W *is closed under scalar multiplication*

Prove if W is nonempty

let the zero vector $w = (0,0,0) \in W$

Such that

$0 + 0 * 3 + 0 * 3^2 = \mathbf{0}$

$0 = 0$

Which implies that W is nonempty

Prove that W is closed under addition

If $p \in W\ then\ p + q \in W$

assume $p = a + b3 + c3^2$, $q = a' + b'3 + c'3^2$ *where* $p, q \in W$

$p + q = (a + b3 + c3^2) + (a' + b'3 + c'3^2)$

By associativity of addition we get

$p + q = ((a + a') + (b + b')3 + (c + c')3^2)$

And so W is closed under addition

Prove that W is closed under scalar multiplication

if polynomial $w \in W$ *and the scalar* $k \in \mathbb{R}$ *then the product* $kw \in W$

$w = a + b3 + c3^2$

$kw = k(a + b3 + c3^2)$

$kw = ka + kb3 + kc3^2$

where $a = ka$, $b3 = kb3$, $c = kc3^2$

And so this implies that $kw \in W$ since kw produces a polynomial of degree 2 and hence $the\ subset\ W$ is also closed under scalar multiplication

Conclusion
Since W is nonempty, closed under addition and also closed under scalar multiplication it can be concluded that W is a subspace of P_2

4.4.26

Determine whether the set
$\mathbb{S} = \{-2x + x^2, 8 + x^3, -x^2 + x^3, -4 + x^2\}$

Spans P_3 $where\ P_3\ is\ the\ vector\ space\ of\ all\ polynomium\ \ where\ the\ degree\ is\ at\ most\ 3$

$That\ is, the\ set\ \mathbb{S}\ Spans\ P_3$
By definition of a spanning set of a vector space, the set $\mathbb{S}$ is a spanning set of P_3 when every vector in P_3 can be written as a linear combination of vectors in $\mathbb{S}$

Let u = (u_1, u_2, u_3, u_4) *be any vectors in* P_3
$We\ can\ produce\ \ a\ linear\ combination\ by\ finding\ scalars\ c_1, c_2\ , c_3 and\ c_4\ such\ that$
$(u_1, u_2, u_3, u_4)\ = c_1(-2x + x^2) + c_2(8 + x^3) + c_3(-x^2 + x^3) + c_4(-4 + x^2)$
$(u_1, u_2, u_3, u_4)\ = (-2c_1x + c_1x^2, c_28 + c_2x^3, -c_3x^2 + c_3x^3, -4c_4 + c_4x^2)$

We can rewrite linear combination such that we have a system of 4 polynomals of degree 3
$u_1 = 0 - 2x + x^2 + 0$
$u_2 = 8 + 0 + 0 + x^3$
$u_3 = 0 + 0 - x^2 + x^3$
$u_4 = -4 + 0 + x^2 + 0$

We can define a coefficient matrix where each row represent $c_1, c_2 \ldots c_4$ and each column represent a polynomial equation

$$\begin{matrix} 0 & 8 & 0 & -4 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{matrix} * \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix} = \begin{matrix} a \\ x \\ x^2 \\ x^3 \end{matrix}$$

We can calculate the determinant of the matrix to see if the system as a unique solution

$$A = \begin{matrix} 0 & 8 & 0 & -4 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

$R_2 + (2)R_3 = R^2$

$$\begin{matrix} 0 & 8 & 0 & -4 \\ 0 & 0 & -2 & 2 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

Column Cofactor expansion on a_{31}

$$1(-1)^{3+1} \begin{matrix} 8 & 0 & -4 \\ 0 & -2 & 2 \\ 1 & 1 & 0 \end{matrix}$$

Using 3x3 diagonal method by joining column 1 and 2 of A to the end of the matrix

$$\begin{matrix} 8 & 0 & -4 \\ 0 & -2 & 2 \\ 1 & 1 & 0 \end{matrix}$$

$$\begin{matrix} 8 & 0 & -4 & 8 & 0 \\ 0 & -2 & 2 & 0 & -2 \\ 1 & 1 & 0 & 1 & 1 \end{matrix}$$

$|A| = 8 * -2 * 0 + 0 * 2 * 1 + (-4) * 0 * 1 - 1 * -2 * -4 - 1 * 2 * 8 - 0 * 0 * 0$
$|A| = -24$

Since the determinant is nonzero for A the system has a unique solution.
And so the set $\mathbb{S}$ spans P_3, that is the set $\mathbb{S}$ spans the vector space where all polynomium is at most 3 degrees.