Mandatory 3

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Assignment 1

For the matrix A, compute the following:

- The rank
- The dimension of the row space
- The dimensions of the column space
- A basis for the nullspace (not just the dimension of the nullspace)

$$A = \begin{matrix} 1 & 2 & 3 & 3 & 2 \\ 4 & 6 & 4 & 8 & 9 \\ 3 & 4 & 1 & 4 & 8 \end{matrix}$$

Computing the rank

The rank of a matrix represents the maximum number of linearly independent vectors in a matrix which is equal to the number of non-zero rows in its row echelon form.

Thus to find the rank of A we can reduce A to row echelon form.
Thereafter the number of nonzero rows represents the rank of A

$$A = \begin{matrix} 1 & 2 & 3 & 3 & 2 \\ 4 & 6 & 4 & 8 & 9 \\ 3 & 4 & 1 & 4 & 8 \end{matrix}$$

$R_2 + (-4)R_1 \Rightarrow R_2$

$$\begin{matrix} 1 & 2 & 3 & 3 & 2 \\ 0 & -2 & -8 & -4 & 1 \\ 3 & 4 & 1 & 4 & 8 \end{matrix}$$

$(-\frac{1}{2})R_2 \Rightarrow R_2$

$$\begin{matrix} 1 & 2 & 3 & 3 & 2 \\ 0 & 1 & 4 & 2 & -\frac{1}{2} \\ 3 & 4 & 1 & 4 & 8 \end{matrix}$$

$R_3 + (-3)R_1 \Rightarrow R_3$

$$\begin{matrix} 1 & 2 & 3 & 3 & 2 \\ 0 & 1 & 4 & 2 & -\frac{1}{2} \\ 0 & -2 & -8 & -5 & 2 \end{matrix}$$

$R_3 - (-2)R_2 \Rightarrow R_3$

$$\begin{matrix} 1 & 2 & 3 & 3 & 2 \\ 0 & 1 & 4 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & -1 & 1 \end{matrix}$$

$(-1)R_3 \Rightarrow R_3$

$$\begin{matrix} 1 & 2 & 3 & 3 & 2 \\ 0 & 1 & 4 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -1 \end{matrix}$$

Since A in echelon form has 3 nonzero rows then
$rank(A) = 3$

Computing dimension of the row space and column space
We know that the dimension of the row/column space of Matrix A is equal to the rank of A.
Thus the dimension of the column/row space is 3 as we know that the rank of matrix A is 3

$$\mathrm{A} = \begin{matrix} 1 & 2 & 3 & 3 & 2 \\ 0 & 1 & 4 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -1 \end{matrix}$$

Computing the basis for the nullspace
The nullspace of A is the solution space of the homogenous space
$A\boldsymbol{x} = \mathbf{0}$

To solve this we can write the augmented matrix $[A\ \mathbf{0}]$ in reduced echelon row echelon form

We can reuse the row echelon form from the previous assignment

$$A = \begin{matrix} 1 & 2 & 3 & 3 & 2 \\ 0 & 1 & 4 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \end{matrix}$$

The system of equations corresponding to the reduced row-echelon form is
$x_1 + 2x_2 + 3x_3 + 3x_4 + 2x_5 = 0$
$x_2 + 4x_3 + 2x_4 - \frac{1}{2}x_5 = 0$
$x_4 - x_5 = 0$

$let\ x_3\ and\ x_5\ be\ the\ free\ variables\ to\ present\ the\ solutions\ in\ parametric\ form$
$x_1 = -2x_2 - 3s - 3x_4 - 2t$
$\quad = -2\left(-4s + \frac{3}{2}t\right) - 3s - 3x_4 - 2t$
$\quad = 5s + \frac{6}{2}t - \frac{10}{2}t$
$\quad = 5s - \frac{4}{2}t$
$\quad = 5s - 2t$
$x_2 = -4s - 2t + \frac{1}{2}x_5 = -4s - 2t + \frac{1}{2}t = -4s + \frac{3}{2}t$
$x_3 = s$
$x_4 = t$
$x_5 = t$

This means that the solution space of $A\boldsymbol{x} = \mathbf{0}$ $consists\ of\ all\ solution\ vectors\ of\ the\ form$

$$\boldsymbol{x} = \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} = \begin{matrix} 5s - 2t \\ -4s + \frac{3}{2}t \\ s \\ t \\ t \end{matrix} = s \begin{matrix} 5 \\ -4 \\ 1 \\ 0 \\ 0 \end{matrix} + t \begin{matrix} -2 \\ \frac{3}{2} \\ 0 \\ 1 \\ 1 \end{matrix}$$

So a basis for the null space of A consists of the vectors:

$$\begin{matrix} 5 \\ -4 \\ 1 \\ 0 \\ 0 \end{matrix} \text{ and } \begin{matrix} -2 \\ \frac{3}{2} \\ 0 \\ 1 \\ 1 \end{matrix}$$

In other words these two vectors are solutions for $A\boldsymbol{x} = \boldsymbol{0}$ and all linear combinations of these two vectors are also solutions.

Assignment 2

Consider the following vectors in $\mathbb{R}^3$

$$v = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad u = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$$

Compute the projection of **u** onto the subspace spanned by **v** and **w.**
Find the coordinates *a, b* such that the projection can be written as
*a***v** + *b***w**

Computing the projection of u onto subspace spanned by v and w

Recall that a set of vectors $S = \{v_1, v_2, \dots, v_n\}$ in a vector space V is a basis for V when

- S spans V
- S is linearly independent

In our case we know that
$S = \{v, w\}$
Given in the assignment we know that S spans $\mathbb{R}^3$
And we know S is linearly independent because the vector equation

$$c_1 v + c_2 w = \boldsymbol{0}$$

Where $c_1 = 0, and\ c_2 = 0$ is the only trivial solution to the vector equation above

Hence the basis A can be found by adjoining v and w => $[v \quad w]$

$$A = \begin{matrix} 1 & 0 \\ 0 & 1 \\ 3 & 2 \end{matrix}$$

By theorem 1 from the lecture notes we know that the projection of a vector u in $\mathbb{R}^3$ to S can be computed as

$$P\boldsymbol{u} \; where \; P = A(A^T A)^{-1} A^T$$

And so

$$A^T A = \begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{matrix} * \begin{matrix} 1 & 0 \\ 0 & 1 \\ 3 & 2 \end{matrix}$$

$$= \begin{matrix} 1*1+0*0+3*3 & 1*0+0*1+3*2 \\ 0*1+1*0+2*3 & 0*0+1*1+2*2 \end{matrix}$$

$$= \begin{matrix} 10 & 6 \\ 6 & 5 \end{matrix}$$

Since ATA is a 2x2 matrix recall that we can find the inverse of a 2x2 matrix with the following formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ c & a \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{10*5-6*6}\begin{bmatrix} 5 & -6 \\ 6 & 10 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{50-36}\begin{bmatrix} 5 & -6 \\ 6 & 10 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{14}\begin{bmatrix} 5 & -6 \\ 6 & 10 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} \frac{5}{14} & -\frac{6}{14} \\ \frac{6}{14} & \frac{10}{14} \end{bmatrix}$$

Then we can calculate

$$A(A^T A)^{-1} = \frac{1}{14} * \left(\begin{matrix} 1 & 0 \\ 0 & 1 \\ 3 & 2 \end{matrix} * \begin{matrix} 5 & -6 \\ 6 & 10 \end{matrix} \right)$$

$$A(A^T A)^{-1} = \frac{1}{14} * \begin{bmatrix} 5 & -6 \\ 6 & 10 \\ 27 & 2 \end{bmatrix}$$

Finally we can combine our previous calculations to compute the projection matrix P

$$P = A(A^T A)^{-1}\, A^T$$

$$P = \frac{1}{14} * \left(\begin{bmatrix} 5 & -6 \\ 6 & 10 \\ 27 & 2 \end{bmatrix} * \begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{matrix} \right)$$

$$P = \frac{1}{14} * \begin{bmatrix} 5 & -6 & 3 \\ 6 & 10 & 38 \\ 27 & 2 & 85 \end{bmatrix}$$

And so we can compute the projection of vector u onto to S with the computed Projection matrix P

$$P\boldsymbol{u} = \frac{1}{14} * \left(\begin{bmatrix} 5 & -6 & 3 \\ 6 & 10 & 38 \\ 27 & 2 & 85 \end{bmatrix} * \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} \right)$$

$$P\boldsymbol{u} = \frac{1}{14} * \begin{bmatrix} -7 \\ 198 \\ 375 \end{bmatrix}$$

$$P\boldsymbol{u} = \begin{bmatrix} -\frac{7}{14} \\ \frac{198}{14} \\ \frac{375}{14} \end{bmatrix}$$

$$P\boldsymbol{u} = \begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{bmatrix}$$

This implies that the projection of vector u onto S results in the following vector

$$\begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{bmatrix}$$

Finally we can compute the coordinates a and b to show that
*a***v** + *b***w = Pu**

$$a * \begin{matrix} 1 \\ 0 \\ 3 \end{matrix} + b * \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} = \begin{matrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{matrix}$$

Which corresponds to the following system of linear equations

$$a = -\frac{1}{2}$$

$$b = \frac{99}{7}$$

$$3a + 2b = \frac{375}{14}$$

We can solve the system with Gauss Jordan elimination on the augmented matrix

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{99}{7} \\ 3 & 2 & \frac{375}{14} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{99}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

Hence the coordinate a and b are

$$a = -\frac{1}{2}, \qquad b = \frac{99}{7}$$

We can test if this solution is correct by plotting it in our equation :

$$a\boldsymbol{v} + b\boldsymbol{w} = P\boldsymbol{u}$$

$$-\frac{1}{2} * \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \frac{99}{7} * \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{matrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{matrix}$$

And solve the left side of the equation

$$\begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{3}{2} \end{bmatrix} + \frac{99}{7} * \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{3}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{99}{7} \\ \frac{198}{7} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{bmatrix}$$

$$\begin{bmatrix} 0 - \frac{1}{2} \\ \frac{99}{7} + 0 \\ \frac{198}{7} + -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{396}{14} - \frac{21}{14} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{bmatrix}$$

Which implies
$a\boldsymbol{v} + b\boldsymbol{w} = P\boldsymbol{u}$

And so we can conclude that the projection of **u** onto the subspace spanned by **v** and **w** results in the vector

$$P\boldsymbol{u} = \begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{bmatrix}$$

Where P is the projection matrix

And also that the same projection can be written with the coordinates a and b in the equation
$a\boldsymbol{v} + b\boldsymbol{w}$

Since we have shown that

$$P\mathbf{u} = a\boldsymbol{v} + b\boldsymbol{w} = \begin{bmatrix} -\frac{1}{2} \\ \frac{99}{7} \\ \frac{375}{14} \end{bmatrix}$$

Assignment 3

Find linear combination of the functions
$f(x) = 2x^2 + x$
$g(x) = x + 1$

That best fits the observation points
$\{(-1,-2), (0,2), (1,5)\}$

In other words find numbers a,b such that
$h(x) = a\, f(x) + b\, g(x)$ minimises the sum
$(h(-1) + 2)^2 + (h(0) - 2)^2 + (h(1) - 5)^2$

We can substitute the points into the function
$h(x) = a\, f(x) + b\, g(x)$

Such that we get a system of 3 linear equations
$-2 = a\,((2(-1)^2 - 1) + b * (-1 + 1)$
$2 = a\,(2(0)^2 + 0) + b\,(0 + 1)$
$5 = a(\,2(1)^2 + 1) + b\,(1 + 1)$

We can expand the linear equations
$-2 = a\,(1) + b(0)$
$2 = a\,(0) + b\,(1)$
$5 = a(\,3) + b\,(2)$

$-2 = a$
$2 = b$
$5 = 3a + 2b$

Plotting the equations into a coefficient matrix we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 2 \end{bmatrix} * \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$$

And so we can define the following vectors v', w' and u' based on matrix above such that
$v', w', u' \in \mathbb{R}^3$

$$v' = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad w' = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad u' = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$$

Recall that in exercise two we have the following vectors we used to solve the least square problem

$$v = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad u = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$$

As it can be seen, this least square problem is the same as in exercise 2 and therefore
the calculations and solution to this least square problem is exactly the same as
In exercise 2