Propositional and Predicate Logic

Course on Discrete Maths

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ITU,August, 2014

Why “Formal Concepts in Computer Science”?

An (incomplete) list of arguments:

- Concrete knowledge
  - Common knowledge for computer scientists,
  - Common frame of reference, and
  - Prerequisite/advantage in future courses and projects.
- Meta-level knowledge, aid in providing
  - Abstract thinking,
  - Cogent reasoning,
  - Precise argumentation, and
  - Precise descriptions.

Introduction (I)

- **Logic** is a branch of mathematics used for precise reasoning, e.g. helpful in doing proofs and also used in programming languages:

```
while ( x < 5 && (y != 8 || x <= 8) )
  if ( x + y > 13 && x > 2 && !(y < 10) ) ...
  else ...
```

- **Propositional logic** is a simple logic:

$$P,\, Q ::= p \mid \overline{P} \mid P \wedge Q \mid P \vee Q \mid P \underline{\vee} Q \mid P \rightarrow Q \mid P \leftrightarrow Q$$

where p is a basic (atomic) **proposition** e.g. *the rose is red*

- its extension with **variables** and **quantifiers**

$$P,\, Q ::= \ldots \mid x \mid (\forall x)P \mid (\exists x)P$$

is called **predicate logic**.

Introduction (II)

Logic is used to establish the truth value, false (F) or true (T), of statements. Some **propositions** are always true or false

- *A triangle has three sides*
- *London is in Denmark*

but the value of others may depend on the context

- *It's raining*
- *The sun is shining*
- *I'm getting wet*

Not all propositions are basic, they may also be ***compound*** by logical **connectives**

- ***If*** *it's raining* ***then*** *I'm getting wet*
- ***If*** *the sun is shining* ***then*** *it's* ***not*** *raining*

Introduction (III)

Propositional Logic can be considered as a mathematical model to investigate certain ***arguments*** i.e. find a proof.

Given the propositions

- ***If*** *it's* ***not*** *raining* ***then*** *I'm* ***not*** *getting wet*
- ***If*** *the sun is shining* ***then*** *it's* ***not*** *raining*
- *The sun is shining*

we can then conclude

- *I'm* ***not*** *getting wet*

Later we shall see how to prove formally this kind of things

Overview (Logic)

- *Propositional Logic*

  Logical Connectives

  Tautologies

  Logical Equivalence and Logical Implication

  Arguments and Inference Rules

- *Predicate Logic*

  Universal and Existential Quantification

  Arguments and Inference Rules

  Two-place/n-ary Predicates

  Negation of Quantified Predicates

Compound Proposition: Negation

- Denoted by: $\boxed{\overline{p}}$ (sometimes also by $\neg p$, $\tilde{p}$, $neg(p)$)

- Meaning: $\boxed{\overline{p}}$ ***is false whenever*** p ***is true, and vice versa***

- Example:

p	=	*It's raining*
$\overline{p}$	=	*it's* ***not*** *raining*

- We can formally define it using a **truth table**:

p	$\overline{p}$
T	F
F	T

Compound Proposition: Conjunction

- Denoted by: $\boxed{p \wedge q}$ (it is also called **and**, binary operator)
- Meaning: ***$p \wedge q$ is true whenever both p and q are true, otherwise it's false***
- Example: *It's raining and I'm getting wet.*
- **Truth table**:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Compound Proposition: Disjunction

- Denoted by: $\boxed{p \vee q}$ (it is also called **or**, binary operator)
- Meaning: ***$p \vee q$ is true whenever either p or q is true (or both), otherwise it's false***
- Example: *It's raining **or** I'm getting wet.*
- **Truth table**:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

What if two propositions, say *it's raining* and *the sun is shining*, should not be true simultaneously?

Inclusive disjunction is not appropriate.

Compound Proposition: Exclusive Disjunction

- Denoted by: $\boxed{p \veebar q}$ (also by $\oplus$, called **xor**, binary operator)
- Meaning: ***$p \veebar q$ is true whenever either p or q is true (exclusively), otherwise it's false***
- Example: *It's raining **xor** the sun is shining*
- **Truth table**:

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

Compound Proposition: Conditional

- Denoted by: $\boxed{p \rightarrow q}$ (also by $\Rightarrow$, $\supset$, aka **logical implication**)
- Meaning: ***if p holds then also q must be true, however if p is false no requirement on q is posed***
- Example: *If it's raining **then** I'm getting wet*
- **Truth table**:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Compound Proposition: Bidirectional

- Denoted by: $\boxed{p \leftrightarrow q}$ (also by *iff*, $\Leftrightarrow$, aka **logical equivalence**)
- Meaning: $\boxed{p \text{ and } q \text{ have the same value}}$
- Example: $\boxed{\text{it's raining if and only if I'm getting wet}}$
- **Truth table**:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example (I)

Let's construct the truth table for the compound proposition

$$(\overline{p} \vee q) \leftrightarrow (p \to q)$$

p	q	$\overline{p}$	$\overline{p} \vee q$	$p \to q$	$(\overline{p} \vee q) \leftrightarrow (p \to q)$
T	T				
T	F				
F	T				
F	F				

Example (II)

Let's construct the truth table for the compound proposition

$$(\overline{p} \vee q) \leftrightarrow (p \to q)$$

p	q	$\overline{p}$	$\overline{p} \vee q$	$p \to q$	$(\overline{p} \vee q) \leftrightarrow (p \to q)$
T	T	F			
T	F	F			
F	T	T			
F	F	T			

Example (III)

Let's construct the truth table for the following compound proposition

$$(\overline{p} \vee q) \leftrightarrow (p \rightarrow q)$$

p	q	$\overline{p}$	$\overline{p} \vee q$	$p \rightarrow q$	$(\overline{p} \vee q) \leftrightarrow (p \rightarrow q)$
T	T	F	T		
T	F	F	F		
F	T	T	T		
F	F	T	T		

Example (IV)

Let's construct the truth table for the following compound proposition

$$(\overline{p} \vee q) \leftrightarrow (p \rightarrow q)$$

p	q	$\overline{p}$	$\overline{p} \vee q$	$p \rightarrow q$	$(\overline{p} \vee q) \leftrightarrow (p \rightarrow q)$
T	T	F	T	T	
T	F	F	F	F	
F	T	T	T	T	
F	F	T	T	T	

Example (V)

Let's construct the truth table for the following compound proposition

$$(\overline{p} \vee q) \leftrightarrow (p \rightarrow q)$$

p	q	$\overline{p}$	$\overline{p} \vee q$	$p \rightarrow q$	$(\overline{p} \vee q) \leftrightarrow (p \rightarrow q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Tautologies

...some compound propositions are always true!

- A **tautology** is a proposition which is true no matter what the truth values of its simple components are.

- For example: $\boxed{p \vee \overline{p}}$ and $\boxed{(\overline{p} \vee q) \leftrightarrow (p \rightarrow q)}$

p	$\overline{p}$	$p \vee \overline{p}$
T	F	T
T	F	T
F	T	T
F	T	T

p	q	$\overline{p}$	$\overline{p} \vee q$	$p \rightarrow q$	$(\overline{p} \vee q) \leftrightarrow (p \rightarrow q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Tautologies: Exercises

- **Exercise:** Show $\boxed{\overline{\overline{p}} \leftrightarrow p}$, $\boxed{p \wedge p \leftrightarrow p}$, $\boxed{p \vee p \leftrightarrow p}$, $\boxed{p \wedge T \leftrightarrow p}$ and $\boxed{p \vee F \leftrightarrow p}$ are tautologies.

- **Exercise:** Show $\boxed{\overline{p \wedge q} \leftrightarrow \overline{p} \vee \overline{q}}$ and $\boxed{\overline{p \vee q} \leftrightarrow \overline{p} \wedge \overline{q}}$ are tautologies (De Morgan's Laws).

- **Exercise:** Show $\boxed{(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)}$ is a tautology.

- **Exercise:** Show $\boxed{(p \rightarrow q) \leftrightarrow (\overline{q} \rightarrow \overline{p})}$ is a tautology.

Logical Equivalence

Two propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology

- *If p and q are logically equivalent we write $p \equiv q$.*

- **Example:** $(\overline{p} \vee q) \equiv (p \rightarrow q)$ as $(\overline{p} \vee q) \leftrightarrow (p \rightarrow q)$ is a tautology
- **Example:** $(p \rightarrow q) \equiv (\overline{q} \rightarrow \overline{p})$ as $(p \rightarrow q) \leftrightarrow (\overline{q} \rightarrow \overline{p})$ is a tautology

Logically equivalent propositions can be interchanged in a compound proposition without altering its truth value.

- **Example:** $p \wedge (\overline{p} \vee q) \equiv p \wedge (p \rightarrow q)$ as $(\overline{p} \vee q) \equiv (p \rightarrow q)$

Logical Implication

A proposition p logically implies proposition q if $p \to q$ is a tautology.

- ***If p logically implies q we write $p \vdash q$.***

Example $p \wedge (p \to q) \vdash q$ because $(p \wedge (p \to q)) \to q$ is a tautology.

Example $(p \to q) \vdash (\overline{q} \to \overline{p})$ because $(p \to q) \to (\overline{q} \to \overline{p})$ is a tautology.

Arguments (1)

An **argument** consists of **premises** (propositions $P_1, \ldots, P_n$) and a **conclusion** (a proposition Q) supposed to follow logically from the premises.

The argument with premises $P_1, \ldots, P_n$ and conclusion Q is **valid** if

$$P_1 \wedge \ldots \wedge P_n \vdash Q$$

i.e. if the premises logically implies the conclusion.

Arguments (2)

Examples.

$$\text{it's raining} \wedge (\text{It's raining} \rightarrow \text{I'm getting wet}) \quad \vdash \quad \text{I'm getting wet}$$

and

$$(\text{I'm getting wet} \rightarrow \text{it's raining}) \wedge$$
$$(\text{the sun is shining} \rightarrow \text{it's NOT raining}) \wedge \text{the sun is shining}$$
$$\vdash$$
$$\text{I'm NOT getting wet}$$

We may show an argument to be correct using truth tables, or we can reason at a higher level using generally valid rules (inference rules)

Inference Rules (I)

To help showing that a conclusion follows logically from a set of premises we may apply **inference rules** on the form:

$$\begin{array}{l} p_1 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

The validity of the rule is ensured if $(p_1 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.

Inference Rules (II)

(Modus Ponens) $\dfrac{\begin{array}{c} p \\ p \rightarrow q \end{array}}{\therefore q}$ (based on $[p \wedge (p \rightarrow q)] \rightarrow q$)

(Addition) $\dfrac{p}{\therefore p \vee q}$ (Simplification) $\dfrac{p \wedge q}{\therefore p}$ (Conjuntion) $\dfrac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q}$

(Modus Tollens) $\dfrac{\begin{array}{c} \overline{q} \\ p \rightarrow q \end{array}}{\therefore \overline{p}}$ (based on $[\overline{q} \wedge (p \rightarrow q)] \rightarrow \overline{p}$)

Example

Example Let p be *it's raining*, let q be *I'm getting wet*, and let r be *the sun is shining*. Let's argue that $(q \rightarrow p) \wedge (r \rightarrow \overline{p}) \wedge r \vdash \overline{q}$.

Given premises $q \rightarrow p$, $r \rightarrow \overline{p}$, and r the following steps show the conclusion $\overline{q}$ follows logically from the premises:

1	r	(premise)
2	$r \rightarrow \overline{p}$	(premise)
3	$\overline{p}$	(from 1 and 2 using Modus Ponens)
4	$q \rightarrow p$	(premise)
5	$\overline{q}$	(from 3 and 4 using Modus Tollens)

Notice the importance of the order of the steps.

Exercise Let p, q, and r be as above. What about the correctness of the argument $(p \rightarrow q) \wedge (r \rightarrow \overline{p}) \wedge r \vdash \overline{q}$?

Predicate Logic (I)

In propositional logic we may state

Peter is a human being and *Ann is a human being*.

However, we have no means to express that the two propositions are about the same general property: *is a human being.*

A **predicate** $P(x)$ describes a property, say

x is a human being.

where x is a free **variable** that may be substituted by values in the **universe of discourse** (UoD) of the predicate.

We can write the two propositions above as $P(Peter)$ and $P(Ann)$.

Predicate Logic (II)

Predicate Logic can be considered a model to investigate certain arguments, arguments that can't be expressed in propositional logic.

Given the following propositions as premises

> ***All human beings are mammals***
>
> ***Peter is a human being***

We should be able to conclude that

> ***Peter is a mammal.***

This immediate raises the two questions:

- How to express propositions like the first premise?
- How to provide rules to help judge the validity of the argument?

Universal Quantification

The proposition

All human beings are intelligent.

can be expressed in predicate logic using **universal quantification**.

If $I(x)$ describes

x is intelligent

and $H(x)$ describes

x is a human being.

then the proposition above can be defined by $\boxed{(\forall x)[H(x) \rightarrow I(x)]}$

If it's understood that the UoD is human beings, we may write $(\forall x)I(x)$ instead.

Existential Quantification

If $M(x)$ describes

x is a mammal.

Then the proposition

Not all mammals are human beings.

can be expressed by ($\neg$ is negation) $\boxed{\neg(\forall x)[M(x) \rightarrow H(x)]}$

Alternatively, using **existential quantification** we may write

$$\boxed{(\exists x)[M(x) \wedge \neg H(x)]}$$

or alternatively, if the UoD is mammals, $(\exists x)[\neg H(x)]$ saying that (at least) one mammal is not a human being.

Arguments in Predicate Logic

- What about the argument on slide 30, is it sound?

- We can represent the premises by

$$\boxed{(\forall x)[H(x) \to M(x)]} \qquad \text{and} \qquad \boxed{H(\textit{Peter})}$$

- But how to conclude $\boxed{M(\textit{Peter})}$?

- to formally do so we need inference rules about how to interpret quantifications.

Inference Rules for Universal Quantification

Universal Specification

If $(\forall x)P(x)$ is true then $P(a)$ is true for every element a in UoD.

$$\boxed{\frac{(\forall x)P(x)}{\therefore\ P(a)\ \textbf{\textit{if } } a \textbf{\textit{ in UoD}}}}$$

Universal Generalization

If $P(a)$ is true for every element a in UoD then $(\forall x)P(x)$ is true.

$$\boxed{\frac{P(a)\ \textbf{\textit{for any }} a \textbf{\textit{ in UoD}}}{\therefore\ (\forall x)P(x)}}$$

Inference Rules for Quantification

Existential Specification

If $(\exists x)P(x)$ is true then $P(a)$ is true for (at least) one element a in UoD.

$$\boxed{\begin{array}{l} (\exists x)P(x) \\ \hline \therefore P(a) \textbf{\textit{ for some }} a \textbf{\textit{ in UoD}} \end{array}}$$

Existential Generalization

If $P(a)$ is true for some element a in UoD then $(\exists x)P(x)$ is true.

$$\boxed{\begin{array}{l} P(a) \textbf{\textit{ for some }} a \textbf{\textit{ in UoD}} \\ \hline \therefore (\exists x)P(x) \end{array}}$$

Example

Given the premises

$$(\forall x)[H(x) \to M(x)]$$

and

$$H(a)$$

the following steps show that the the conclusion $M(a)$ follows logically from the premises:

1	$(\forall x)[H(x) \to M(x)]$	(premise)
2	$H(a) \to M(a)$	(from 1 using Universal Specification)
3	$H(a)$	(premise)
4	$M(a)$	(from 3 and 2 using Modus Ponens)

Exercises

Exercise Translate the following into symbolic form:

$i)$ Everybody likes him.
$ii)$ Somebody cried out for help and called the police.
$iii)$ Nobody can ignore her.

Define your predicates, and consider the UoD.

Exercise Find an UoD and two unary predicates $P(x)$ and $Q(x)$ such that $(\forall x)[P(x) \to Q(x)]$ is true.

Exercise Find an UoD and two unary predicates $P(x)$ and $Q(x)$ such that $(\exists x)[P(x) \wedge Q(x)]$ is false but $(\exists x)P(x) \wedge (\exists x)Q(x)$ is true.

Exercise Show that $(\forall x)P(x) \vdash (\exists x)P(x)$.

Exercise Given the premises $(\exists x)P(x)$ and $(\forall x)[P(x) \to Q(x)]$ give a series of steps concluding that $(\exists x)Q(x)$.

Two-place Predicates (I)

Consider properties like

is larger than and *lives in the same city as*

We need two variables in order to define predicates describing such properties

Let $L(x, y)$ be the predicate

x *lives in the same city as* y

A single quantification, e.g. $(\forall x)L(x, y)$ leaves one variable free (and defines a predicate), so we must quantify over both x and y to make a proposition:

$$(\forall x)(\forall y)L(x,y),\ \ (\forall x)(\exists y)L(x,y),\ \ (\exists x)(\forall y)L(x,y),\ \ (\exists x)(\exists y)L(x,y)$$
$$(\forall y)(\forall x)L(x,y),\ \ (\forall y)(\exists x)L(x,y),\ \ (\exists y)(\forall x)L(x,y),\ \ (\exists y)(\exists x)L(x,y)$$

Two-place Predicates (II)

The order of the quantifications may be important, for instance letting UoD be Danish citizens then

$$(\exists y)(\forall x)L(x,y)$$

means that (at least) one Danish citizen lives in the same city as all other Danish citizens which is (hard to believe and) not the same as

$$(\forall x)(\exists y)L(x,y)$$

On the other hand

$$(\forall x)(\forall y)L(x,y) \equiv (\forall y)(\forall x)L(x,y)$$

and

$$(\exists x)(\exists y)L(x,y) \equiv (\exists y)(\exists x)L(x,y)$$

Exercise Give an intuitive argument for the validity of the logical equivalences above.

Negation of Quantified Predicates

The proposition

$$\neg(\forall x)P(x)$$

states that it's not the case that all a (in the UoD) satisfies $P(x)$, this is that same as

$$(\exists x)[\neg P(x)]$$

so

$$\neg(\forall x)P(x) \equiv (\exists x)[\neg P(x)]$$

likewise

$$\neg(\exists x)P(x) \equiv (\forall x)[\neg P(x)]$$

Exercise (Difficult) Show $\neg(\forall x)(\exists y)P(x,y) \vdash (\exists x)(\forall y)[\neg P(x,y)]$. Hint: $(\exists y)P(x,y)$ is a predicate over x.