Foundations of Computing - Discrete Mathematics

Relations

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Today's lecture

1. Relations and their properties
   - What is a relation?
   - Functions as relations
   - Relations on numbers
   - Directed graph of a relation
   - Properties of relations
   - Composition of relations
2. n-ary Relations and Their Applications
   - n-ary relations
3. Closures of Relations
4. Equivalence Relations
   - Equivalence relations
   - Equivalence classes
5. Order relations
   - Partial orderings
   - Total orderings
   - Hasse diagrams

Sets and Cartesian product (reminder)

Sets are unordered, so $\{1, 2, 3\} = \{1, 3, 2\}$. However, sometimes we need to establish an order.

$A = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25), \ldots\} = \{(x, y) \ : \ x, y \in \mathbb{N}$ and $y = x^2\}$

In this example, $(2, 4) \in A$ but $(4, 2) \notin A$.

An ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is a collection with an established order, where a_1 is the first element, a_2 is the second element, etc. If $n = 2$ then they are called ordered pairs.

The Cartesian product of two or more sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for $1 \leq i \leq n$.

In general, the Cartesian product is not commutative, so $A \times B \neq B \times A$ unless $A = \emptyset$, $B = \emptyset$ or $A = B$.

Cartesian product (reminder)

Example Consider the sets *Student_name*, *Course*, *Grade*, *Textbook*, and *Classroom*:

Student_name is a set of names of the students in ITU

Course is the set of courses in ITU

Grade is the set of all possible grades that can be given to a student

Classroom is the set of classrooms in ITU

We may have Cartesian products like:

Student_name $\times$ Course

Course $\times$ Classroom

Student_name $\times$ Course $\times$ Grade

What is a relation?

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

Notation:

$a\ R\ b$ denotes $(a, b) \in R$. Then a is said to be related to b by R.

$a \not R\ b$ denotes $(a, b) \notin R$.

Example Let the sets *Course* and *Classroom* be the ones previously defined, and let R be the relation that consists of those pairs (a, b), where a is a course given in classroom b.

$$R \subseteq \mathit{Course} \times \mathit{Classroom}$$

- (Discr. Math., Auditorium 1) $\in R$ (or Discr. Math. R Auditorium 1)
- (Discr. Math., Auditorium 2) $\notin R$ (or Discr. Math. $\not R$ Auditorium 2)
- If a certain course c is not offered by ITU ($c \notin$ *Course*), then there will be no pairs in R having this course as the first element.

Functions as relations

A function f from a set A (domain) to a set B (codomain) assigns exactly one element of B to each element of A.

Let us consider the set of ordered pairs

$$C = \{(a, b) \ : \ a \in A \text{ and } b \in B \text{ and } b = f(a)\}.$$

This points form the graph of f.

Since $C \subseteq A \times B$, we have that the graph of any function f from A to B is a relation from A to B.

Relations on numbers

Relations on numbers can be defined by using $\leq, <, =, \geq, >$.

Examples

$$R_1 = \{(a,b) \;:\; a \leq b\}$$
$$R_2 = \{(a,b) \;:\; a > b\}$$
$$R_3 = \{(a,b) \;:\; a = 2b + 3\}$$
$$R_4 = \{(a,b) \;:\; a - b \geq 1\}$$

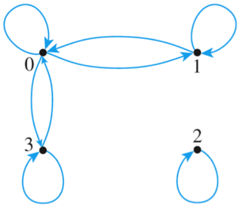
Directed graph of a relation

Any relation R defined on a set A can be represented with a directed graph.

How to draw the graph of a relation R:

- Draw an arrow from node x to node y if and only if $(x, y) \in R$.
- If an element is related to itself, a loop is drawn that extends out from the point and goes back to it.

Example Let $A = \{0, 1, 2, 3\}$ and define the relation
$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$.



Properties of relations (I)

A relation on a set A is a relation from A to A (i.e, a subset of $A \times A$).

A relation R on a set A is reflexive if

$$\forall a \in A, (a, a) \in R.$$

Exercise Are these relations reflexive?

a) The relation obtained from the graph of function $f(x) = x^2$, where $f : \mathbb{Z} \to \mathbb{Z}$.
No, since the relation $R = \{(0,0), (1,1), (2,4), (3,9), \ldots\}$ does not contain all possible pairs of the form (x, x), $\forall x \in \mathbb{Z}$.

b) The graph of a function $g(x) = x$, where $g : \mathbb{R} \to \mathbb{R}$.
Yes, by the definition itself.

c) The relation "is a subset of" (set inclusion $\subseteq$).
Yes, because it includes the equality.

d) The relation "is greater than" on the set of integers.
No, but the relation "is greater than or equal to" would be.

e) The relation "divides" (divisibility) on the set of all negative integers.
Yes, because any number is a divisor of itself.

Properties of relations (II)

A relation R on a set A is called symmetric if

$$\forall a, b \in A, (a, b) \in R \Rightarrow (b, a) \in R.$$

Similarly, a relation R on a set A is antisymmetric if

$$\forall a, b \in A, \; ((a, b) \in R \wedge (b, a) \in R) \Rightarrow a = b.$$

Exercise Are these relations symmetric and/or antisymmetric?

a) The relation "divides" (divisibility) between any two natural numbers.
   Antisymmetric. The only way two numbers can be divisible by each other is if the two are the same number.

b) The relation "is married to" between any two persons.
   Symmetric. Whenever 'a' is married to 'b', then 'b' is also married to 'a'.

c) The relation $R = \{(0, 1), (1, 2), (2, 1)\}$.
   Not symmetric because $(1, 0)$ is missing. Not antisymmetric either because $(1, 2) \in R$, $(2, 1) \in R$ and $1 \neq 2$.

d) The relation $S = \{(1, 1), (2, 2), (3, 3)\}$.
   Symmetric and antisymmetric. In fact, this is the only way a relation can be both symmetric and antisymmetric.

Properties of relations (III)

A relation R on a set A is transitive if

$$\forall a, b, c \in A,\ ((a,b) \in R \wedge (b,c) \in R) \Rightarrow (a,c) \in R.$$

Example The following are transitive relations.

a) The relation "is greater than" on the set of integers: whenever $a > b$ and $b > c$, then also $a > c$.

b) The relation "is a subset of" (set inclusion).

c) The relation "divides" (divisibility) on the set of natural numbers.

d) The relation "implies" (implication).

Exercise Is the relation "is the mother of" transitive?

No. If Alice is the mother of Beth, and Beth is the mother of Charlie, that does not mean that Alice is the mother of Charlie.

$$(Alice, Beth) \in R \text{ and } (Beth, Charlie) \in R$$
$$\text{but } (Alice, Charlie) \notin R$$

Composition of relations

Given two relations $R \subseteq A \times B$ and $S \subseteq B \times C$, the composition of R and S, denoted by $S \circ R$, is the relation defined by

$$S \circ R = \{(a, c) \in A \times C \ : \ \exists b \in B \text{ such that } (a, b) \in R \land (b, c) \in S\}$$

Note that if R and S are functions, then $S \circ R$ is the usual function composition.

Exercise

a) What is the composition of the relations "is brother of" and "is the parent of"?
The relation "is uncle of": if Alex is the brother of Bob, and Bob is the parent (father) of Charlie, then Alex is the uncle of Charlie.

b) What is the composition of the relations
$R = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$ and
$S = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$?
The relation $S \circ R = \{(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)\}$.

n-ary relations (I)

Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and n is its degree.

Example Given the sets *Student_name*, *Course*, *Grade*, and *Classroom*, we said one of the possible Cartesian products could be (among others):

$$\text{Student\_name} \times \text{Course} \times \text{Grade}$$

Let us consider the relation $R \subseteq \text{Student\_name} \times \text{Course} \times \text{Grade}$. This is a ternary relation of degree 3.

n-ary relations (II)

Example (cont.)

$$R \subseteq \text{Student\_name} \times \text{Course} \times \text{Grade}$$

where

Student_name = { Daniel M., Maria S., Mads T., Sarah D., Tobias A.}

Course = { Discr. Math., Global SW dev., SW Eng., Alg. and Data Str., Syst. Arch. and Security }

Grade $= \{-3, 0, 2, 4, 7, 10, 12\}$

$$\begin{gathered} R = \{(\text{Daniel M.}, \text{SW Eng.}, 7), (\text{Daniel M.}, \text{Syst. Arch. and Security}, 4), \\ (\text{Maria S.}, \text{Disc. Math.}, 12), (\text{Mads T.}, \text{SW Eng.}, -3), \\ (\text{Sarah D.}, \text{Discr. Math.}, 2), (\text{Tobias A.}, \text{Global SW dev.}, 0), \ldots\} \end{gathered}$$

n-ary relations (III)

Example (cont.)

$$R = \{(\text{Daniel M.}, \text{SW Eng.}, 7), (\text{Daniel M.}, \text{Syst. Arch. and Security}, 4),$$
$$(\text{Maria S.}, \text{Disc. Math.}, 12), (\text{Mads T.}, \text{SW Eng.}, -3),$$
$$(\text{Sarah D.}, \text{Discr. Math.}, 2), (\text{Tobias A.}, \text{Global SW dev.}, 0), \ldots\}$$

$S = \{a : (a, b, c) \in R \wedge c \geq 7\}$ is a selection from R of all students registered to some course in ITU, having a grade greater than or equal 7.

$SELECT\ student\_name$
$FROM\ R$
$WHERE\ grade >= 7$

The result of the selection would be

$$S = \{\text{Daniel M.}, \text{Maria S.}\}$$

Closures of relations (I)

The closure of a relation R with respect to a certain property p is the relation obtained by adding the minimum number of ordered pairs to R to obtain property p.

- To find the reflexive closure of R: add all the pairs of the form (a, a) (if not already included in R).
- To find the symmetric closure of R: for every pair (a, b), add the pair (b, a) (if not already included in R).
- To find the transitive closure of R: if there is a pair (a, b) and also (b, c), add the pair (a, c) (if not already included in R).

Reflexive and symmetric closures are easy. Transitive closures can be very complicated.

Closures of relations (II)

Exercise Let R be the relation

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}.$$

a) Is the relation R reflexive? If not, what is the reflexive closure of R?
No. We just need to add the pair $(3,3)$ for R to be reflexive.

b) Is it symmetric? If not, what is the symmetric closure of R?
No. We need to add the pairs $(4,3)$ and $(1,4)$ for R to be symmetric.

c) Is it transitive? If not, what is the transitive closure of R?
No. We need to add the following three pairs:

$(3,1)$ (from $(3,4)$ and $(4,1)$)
$(4,2)$ (from $(4,1)$ and $(1,2)$)
$(3,2)$ (from $(3,4)$ and $(4,2)$).

Equivalence relations

A relation R is an equivalence relation if R is reflexive, symmetric, and transitive.

Two elements a and b that are related by an equivalence relation are called equivalent. Often, the notation $a \sim b$ is used.

Example The relation $=$ is an equivalence relation on $\mathbb{N}$.

Exercise The relation $<$ on $\mathbb{N}$ is **not** an equivalence relation. Why not? And what about the relation $\leq$?
The relation $<$ on $\mathbb{N}$ is not an equivalence relation because it is not reflexive nor symmetric.

The relation $\leq$ is not an equivalence relation either, because it is reflexive and transitive, but not symmetric. It is, however, a partial order, as we will later see.

Equivalence classes (I)

Let R be an equivalence relation on a set A. The set of elements that are related to an element a of A is called the equivalence class of a.

$$[a]_R = \{x \in A \ : \ (a, x) \in R\}.$$

It can also be denoted by $[a]$, if there is only one relation in consideration.

If $b \in [a]_R$, then b is called a representative of the equivalence class $[a]_R$. Any element in a class can be used as a representative of this class.

Equivalence classes (II)

Example Let S be the set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. We can define many equivalence relations on this set.

$\forall x, y \in S, (x, y) \in R$ if and only if $x \equiv y$ mod 3 (i.e. $x - y$ is divisible by 3).

$$\begin{aligned} R = \{&(0,3),(3,6),(0,6),(1,4),(4,7),(1,7),(2,5),(5,8),(2,8),\\ &(3,0),(6,3),(6,0),(4,1),(7,4),(7,1),(5,2),(8,5),(8,2)\\ &(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8)\} \end{aligned}$$

Then our equivalence classes are:

$$[0] = \{0, 3, 6\} \text{ remainder } 0 \text{ when divided by } 3$$

$$[1] = \{1, 4, 7\} \text{ remainder } 1 \text{ when divided by } 3$$

$$[2] = \{2, 5, 8\} \text{ remainder } 2 \text{ when divided by } 3$$

These classes are called congruence classes.

Equivalence classes (III)

Theorem Let R be an equivalence relation on a set A. Then, given any two elements $a, b \in A$, the following statements are equivalent:

i) $(a, b) \in R$

ii) $[a] = [b]$

iii) $[a] \cap [b] \neq \emptyset$

Equivalence classes (IV)

Recall that a partition of a set is a collection of mutually disjoint subsets whose union is the original set.

Theorem Let R be an equivalence relation on a set A. Then $\{[a]_R \ : \ a \in A\}$ is a partition of A.

Example Given the relation of congruence mod 3 on the integers $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, we obtained the partition:

$$\mathbb{Z} = [0] \cup [1] \cup [2]$$

It is easy to check that these congruence classes are disjoint, and every integer is in exactly one of them.

Order relations (I)

Sometimes relations are used to order some or all the elements of a set.

A relation R on a set S is a partial ordering if it is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a partially ordered set (or poset), denoted by (S, R).

Example Let us consider the relation "$\geq$" on the set $\mathbb{Z}$. For any three integers $a, b, c \in \mathbb{Z}$, the relation "$\geq$" is

- Reflexive: $a \geq a$.
- Antisymmetric: if $a \geq b$ and $b \geq a$, then $a = b$.
- Transitive: $a \geq b$ and $b \geq c$ imply that $a \geq c$.

Therefore, the relation "$\geq$" is a partial ordering on the set $\mathbb{Z}$, and $(\mathbb{Z}, \geq)$ is a poset.

Order relations (II)

Notation:

Given a partial order R,

$a \preccurlyeq b$ denotes $(a, b) \in R$.

$a \prec b$ denotes $(a, b) \in R$ but $a \neq b$.

Note: The notation $\preccurlyeq$ is not the same as $\leq$ ("less than or equal to").

Two elements a and b of a poset $(S, \preccurlyeq)$ are comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise, they are called noncomparable.

Examples

- Given a poset $(\mathbb{Z}, \leq)$ and any two integers $a, b \in \mathbb{Z}$, either $a \leq b$ or $b \leq a$. Then a and b are comparable.
- Given a poset $(S, \subseteq)$ and two subsets A, B of S, where $A = \{0\}$ and $B = \{1\}$, we have that $A \nsubseteq B$ and $B \nsubseteq A$, so A and B are noncomparable.

Order relations (III)

If **every pair** of elements in a poset $(S, \preccurlyeq)$ is comparable, then the set S is called a total order set (or a chain), while $\preccurlyeq$ is called a total order.

Example The poset $(\mathbb{Z}, \geq)$ is a total order because for every $a, b \in \mathbb{Z}$ we have that either $a \geq b$ or $b \geq a$.

Example Take the power set $\mathcal{P}(S) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ of the set $S = \{0, 1\}$.

We can see that, for any three elements $a, b, c \in \mathcal{P}(S)$, the relation $\subseteq$ satisfies the axioms for a partial order:

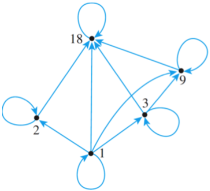
- $a \subseteq a$
- if $a \subseteq b$ and $b \subseteq a$, then $a = b$.
- $a \subseteq b$ and $b \subseteq c$ imply that $a \subseteq c$.

So $(\mathcal{P}(S), \subseteq)$ is a poset. However, there are pairs of elements in this poset which are not comparable: neither $\{0\} \subseteq \{1\}$ nor $\{1\} \subseteq \{0\}$, then $(\mathcal{P}(S), \subseteq)$ is not a total order.

Hasse diagrams

Whenever we have a partial order, we have a directed graph where:

- There is a loop at every vertex.
- All other arrows point in the same direction (upward).
- Any time there is an arrow from one point to a second, and from the second point to a third point, there is also an arrow from the first point to the third.



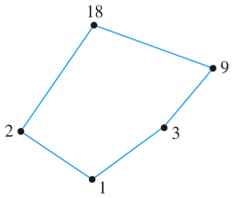
Hasse diagrams (II)

We can associate a simpler graph to partial order relations, called Hasse diagram.

Start with a directed graph of the relation, placing vertices so that all arrows point upward. Then eliminate:

1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property,
3. the direction indicators on the arrows.

For the relation given previously, the Hasse diagram is as follows:



Topological Sorting

An element a in a poset $(S, \preccurlyeq)$ is maximal if it is not less than any other element in the poset. If there is only one maximal element, we call it the greatest element.

An element a in a poset $(S, \preccurlyeq)$ is minimal if it is not greater than any other element in the poset. If there is only one minimal element, we call it the least element.

Example Determine whether the following posets have a greatest element and a least element. (For simplicity, posets have been written omitting the pairs coming from reflexivity, transitivity and antisymmetry).

a) $(S, \preccurlyeq) = \{(a, b), (a, c), (b, d), (c, d)\}$.
Least element: a. Greatest element: d.

b) $(S, \preccurlyeq) = \{(a, c), (b, c), (c, d)\}$.
It has no least element (min. elements: a, b). Greatest element: d.

c) $(S, \preccurlyeq) = \{(a, b), (b, c), (b, d), (b, e)\}$.
Least element: a. It has no greatest element (max. elements: c, d, e).