MOCK EXAM
Linear Algebra and Probability
Spring 2019

IT University of Copenhagen

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Instructions

- This is a 4 hour written exam with all written and printed aids allowed (A2)
- There are 5 problems and 4 pages.
- Each problem is divided into questions
- The point value of each problem and the distribution of points among questions is given explicitly.
- There is a total of 100 points to be earned.
- The problems are formulated in english and should be answered in english.
- Read each question completely before trying to solve it.
- **Make sure** you do all actions marked with bold font
- Please **order** and **number** the pages before handing in.
- Solutions should be hand-written
- Access to aid in the form of books, own notes, e-books, also on laptops and iPads is permitted.
- Use of internet including email and social media is not permitted.
- Use of any other hardware or software such as MatLab or pocket calculators is not permitted
- Any form of communication between students or with the outside world is not permitted.

1 Systems of linear equations (16 points)

a) [10 points] Compute the set of solutions to the following system of linear equations.

$$\begin{aligned} w + \ \ x + y - z &= 3 \\ 2x + y \qquad &= 4 \\ -2w + \ \ x \qquad + z &= 1 \end{aligned}$$

For the next question, consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{bmatrix}$$

b) [6 points] Compute the rank of A and the dimensions of the row space, column space and null space of A.

Solution

a) First reduce the augmented matrix to reduced echelon form

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 2 & 1 & 0 & 4 \\ -2 & 1 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 2 & 1 & 0 & 4 \\ 0 & 3 & 2 & -1 & 7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 1 & 0.5 & 0 & 2 \\ 0 & 3 & 2 & -1 & 7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 1 & 0.5 & 0 & 2 \\ 0 & 0 & 0.5 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0.5 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & 2 \end{array}\right]$$

The last variable column is free, which means that for any value of z there is a solution to the system. Thus, for any t, the following is a solution

$$w = 0 \qquad x = 1 - t \qquad y = 2 + 2t \qquad z = t$$

b) The matrix A is the coefficient matrix for the system of question a. The reductions above show that A can be reduced to the echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

which has rank (number of leading 1s) 3. The dimensions of the row and column space are therefore also 3, and the dimension of the null space is the number of rows minus the rank, i.e., 1.

2 Eigenvectors and eigenvalues (12 points)

For this problem, consider the following matrix

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$$

a) [12 points] Compute the eigenvectors and eigenvalues of the matrix A.

Solution

a) The characteristic polynomial for A is

$$(\lambda - 2)(\lambda - 5) + 2 = \lambda^2 - 7\lambda + 12$$

The roots of this are 4 and 3. The eigenvectors for eigenvalue 4 are the non trivial elements in the null space for $4I - A$. This is computed by reducing to echelon form

$$4I - A = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The eigenvectors for eigenvalue 4 are therefore any vector of the form

$$t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for non-zero t. For eigenvalue 3 we compute likewise

$$3I - A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

The eigenvectors for eigenvalue 3 are therefore any vector of the form

$$t \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

for non-zero t.

3 Vector spaces (17 points)

For the first question, consider the following three vectors

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad v = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

a) [10 points] Are the vectors u, v, w linearly independent? Argue for your answer.

For the next question, consider the following two subsets of $\mathbb{R}^2$:

$$V = \{(x, y) \mid xy + y = 4\}$$
$$W = \{(x, y) \mid 2x = 5y\}$$

b) [7 points] Which of the two sets V, W are subspaces of $\mathbb{R}^2$? Argue for your answer.

Solution

a) By definition, the three vectors are independent if

$$x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies $x = y = z = 0$. This equation can be written as the following system of linear equations

$$\begin{aligned} x + 2y \quad\quad &= 0 \\ 2x + 3y + \quad z &= 0 \\ 3x + 5y + 2z &= 0 \end{aligned}$$

This set of homogeneous equations has a unique solution if and only if the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}$$

has non-zero determinant (Equivalent conditions listed page 129). The determinant is -1, so the vectors are linearly independent.

b) V is not a subspace because it is not closed under addition: $(1, 2) \in V$, but $(1, 2) + (1, 2) = (2, 4)$ is not because $2 \cdot 4 + 4 = 12 \neq 4$. The set W is a subspace because it is the null space of the matrix

$$\begin{bmatrix} 2 & -5 \end{bmatrix}$$

Alternatively, one can show directly that it is closed under sum and scalar multiplication and contains $(0, 0)$: $(0, 0) \in W$ because $2 \cdot 0 = 5 \cdot 0$. If $(x_0, y_0) \in W$ and $(x_1, y_1) \in W$ then $(x_0 + x_1, y_0 + y_1) \in W$ because

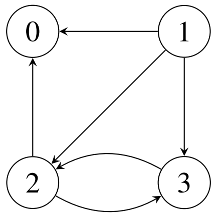
$$2(x_0 + x_1) = 2x_0 + 2x_1 = 5y_0 + 5y_1 = 5(y_0 + y_1)$$

If $(x, y) \in W$ and $c \in \mathbb{R}$, also $c(x, y) = (cx, cy) \in W$ because

$$2cx = c2x = c5y = 5cy$$

4 PageRank (10 points)

Consider the web consisting of 4 pages with links as indicated in the below diagram.



a) [10 points] Construct the matrix M for which the page ranking for the above web is an eigenvector for eigenvalue 1. The matrix should take dangling nodes into account and should use damping factor 0. Note that you are *not* asked to compute the eigenvector of M.

Solution

The matrix M is a sum of matrices

$$M = (1 - m)(A + D) + mS$$

where m is the damping factor and

$$A = \begin{bmatrix} 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1 \\ 0 & 1/3 & 1/2 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \end{bmatrix}$$

and all entries of S are $1/4$. In total

$$M = \begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 \\ 1/4 & 0 & 0 & 0 \\ 1/4 & 1/3 & 0 & 1 \\ 1/4 & 1/3 & 1/2 & 0 \end{bmatrix}$$

5 Discrete random variables (15 points)

The intersection of Danger Road and Full Throttle Street is the cite of an accident on one of 10 working days on average. In this exercise we will assume that there are never two accidents in the same day. The police decide to set up a hidden camera to monitor the intersection.

a) [5 points] What is the probability that the there will be at least one accident in the intersection within the first work week (5 days) after the camera is installed?

b) [5 points] Give a formula for computing the probability that 4 accidents will happen within a year (50 weeks).

c) [5 points] What is the expected number of days before 3 accidents have happened?

Solution

a) The probability is 1 minus the probability that there are no accidents

$$1 - \left(\frac{9}{10}\right)^5$$

b)

$$P(X = 4) = \binom{250}{4} \left(\frac{1}{10}\right)^4 \left(\frac{9}{10}\right)^{246}$$

c) Let Z be a random variable describing the number of days before 3 accidents happen. This is the same as the sum of three independent random variables Y_1, Y_2, Y_3 each giving the number of days before one accident. Each of these follow a geometric distribution with parameter $p = .1$ and thus mean 10. So,

$$\mathsf{E}[Z] = \mathsf{E}[Y_1 + Y_2 + Y_3] = \mathsf{E}[Y_1] + \mathsf{E}[Y_2] + \mathsf{E}[Y_2] = 30$$

6 Exponential distributions and conditional probability (12 points)

A train arrives at the station at 920 with a delay given by a random variable X following an exponential distribution with a mean of 10 minutes.

a) [5 points] Give an expression to compute the probability that the train is more than 5 minutes delayed. You should give a formula for your answer, and you are not required to reduce exponentials of the form e^y to decimal numbers.

b) [7 points] Mrs Patience is on board the train, hoping to catch the 925 train leaving from the same platform. If she does, she will be home at 945. If not, she will have to walk home, which takes 30 minutes. What is the expected time that Mrs Patience will arrive home? As in the first subquestion, you should give a formula for your answer.

Solution

a)

$$1 - \int_0^5 \frac{1}{10} e^{(\frac{-x}{10})} dx = 1 - \left[-e^{(\frac{-x}{10})} \right]_0^5 = 1 - (1 - e^{-.5}) = e^{-.5}$$

b) Let A be the event that she catches the train, and let T be the length of the time interval from 925 until she arrives at home. Then

$$\mathsf{E}[T|A] = 20$$

and

$$\mathsf{E}[T|A^c] = 30 + \mathsf{E}[X|X \geq 5] - 5 = 25 + \mathsf{E}[X] = 35$$

using the fact that exponential random variables are timeless in the second equality. So by the total expectation theorem

$$\mathsf{E}[T] = P(A)\mathsf{E}[T|A] + P(A^c)\mathsf{E}[T|A^c] = e^{-.5} \cdot 20 + (1 - e^{-.5}) \cdot 35$$

which gives the answer to the question. If pocket calculators were allowed, then you could reduce this to

$$0.61 \cdot 20 + 0.39 \cdot 35 = 12.1 + 13.7 = 25.8$$

7 Joint discrete random variables (18 points)

Suppose we are given n boxes numbered 1 to n, and that box k contains k balls numbered 1 to k. We choose a box at random, and from this box, we draw a ball at random. Let X be the number of the box and Y the number of the ball.

a) [6 points] Determine the joint PMF of (X, Y).

b) [6 points] Determine the marginal distribution for Y.

c) [6 points] Compute the mean of Y.

Solution

a)

$$p_{X,Y}(x, y) = P(X = x, Y = y) = \begin{cases} 0 & \text{if } x < y \\ \frac{1}{nx} & \text{if } x \geq y \end{cases}$$

b)

For $1 \leq y \leq n$: $p_Y(y) = P(Y = y) = \sum_{x=1}^{n} p_{X,Y}(x, y) = \sum_{x=y}^{n} \frac{1}{x}$

c) $\mathbb{E}[Y] = \sum_{y=1}^{n} y p_Y(y) = \sum_{y=1}^{n} y \sum_{x=y}^{n} \frac{1}{x} = \sum_{x=1}^{n} \frac{1}{x} \sum_{y=1}^{x} y = \sum_{1}^{n} \frac{x+1}{2} = \frac{n(n+2)}{4}$

8 Probability density functions (25 points)

I add this question because it is a typical exam question (except that the numbers get a bit larger than I would usually aim for). It makes this mock exam sheet longer than a realistic one, which would be 100 points.

The probability density function for the random variable X is given as

$$f_X(x) = c\left(13x^3 - 6x\right)$$

for $1 \leq x \leq 2$ and $f_X(x) = 0$ for x outside this interval. Here c is a real number.

a) [5 points] Which real value c makes f_X a probability density function?

Suppose now the random variable Y has conditional density function given as

$$f_{Y|X}(y \mid x) = \frac{3(x^2y^2 - y)}{26x^2 - 12}$$

for $1 \leq y \leq 3$ and $1 \leq x \leq 2$, and $f_{Y|X}(y \mid x) = 0$ for x, y outside these intervals.

b) [7 points] Give a formula for the joint probability distribution function of X and Y, and for the marginal of Y.

c) [7 points] Compute a formula for the conditional density function $f_{X\|Y}$

For the final question we consider X the parameters of a distribution of Y, as in the setup of Bayesian inference.

d) [6 points] Compute the MAP and LMS estimates of Y given $X = x$.

Solution

a) The requirement of a probability density function is that its integral over all of $\mathbb{R}$ is 1. Since

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)dx = \int_1^2 c\left(13x^3 - 6x\right)dx = c\left[\frac{13}{4}x^4 - 3x^2\right]_1^2 &= c\left(4 \cdot 13 - 12 - \frac{13}{4} + 3\right) \\ &= c\left(43 - \frac{13}{4}\right) \\ &= \frac{159}{4}c\end{aligned}$$

c must be $\frac{4}{159}$.

b) The joint PDF of X and Y is

$$\begin{aligned} f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x) &= \frac{3(x^2y^2 - y)}{26x^2 - 12} \left(13x^3 - 6x\right) \frac{4}{159} \\ &= \frac{3(x^2y^2 - y)x}{2} \frac{4}{159} \\ &= \frac{2}{53}(x^3y^2 - xy) \end{aligned}$$

for $1 \leq y \leq 3$ and $1 \leq x \leq 2$ and 0 else.

The marginal is

$$\begin{aligned} f_Y(y) &= \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_{x=1}^{2} \frac{2}{53}(x^3y^2 - xy)dx \\ &= \frac{2}{53}\left[\frac{1}{4}x^4y^2 - \frac{1}{2}x^2y\right]_{x=1}^{2} \\ &= \frac{2}{53}\left(4y^2 - 2y - \frac{1}{4}y^2 + \frac{1}{2}y\right) \\ &= \frac{2}{53}\left(\frac{15}{4}y^2 - \frac{3}{2}y\right) \\ &= \frac{15}{106}y^2 - \frac{3}{53}y \end{aligned}$$

for $1 \leq y \leq 3$ and 0 else.

c) By Bayes rule, the conditional is

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{4(x^3y^2 - xy)}{15y^2 - 6y} = \frac{4(x^3y - x)}{15y - 6}$$

with the usual bounds.

d) The MAP estimates gives the y maximising $f_{Y|X}$ for a given x. This corresponds to maximising $x^2y^2 - y$ on the interval from 1 to 3, for a given x between 1 and 2. For all values of x this is a strictly increasing function of y, so maximal at $y = 3$, which is also the MAP estimate.

The LMS estimate is defined and computed as follows

$$
\begin{aligned}
E[Y \mid X = x] &= \int_{y=1}^{3} y f_{Y|X}(y \mid x) dy \\
&= \int_{y=1}^{3} \frac{3(x^2y^3 - y^2)}{26x^2 - 12} dy \\
&= \frac{3}{26x^2 - 12} \int_{1}^{3} (x^2y^3 - y^2) dy \\
&= \frac{3}{26x^2 - 12} \left[\frac{1}{4} x^2 y^4 - \frac{1}{3} y^3 \right]_{y=1}^{3} \\
&= \frac{3}{26x^2 - 12} \left(\frac{81}{4} x^2 - 9 - \frac{1}{4} x^2 + \frac{1}{3} \right) \\
&= \frac{3}{26x^2 - 12} \left(20x^2 - \frac{26}{3} \right) \\
&= \frac{60x^2 - 26}{26x^2 - 12} \\
&= \frac{30x^2 - 13}{13x^2 - 6}
\end{aligned}
$$