Foundations of Computing - Discrete Mathematics

Graphs and Trees

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Today's lecture

1. Graphs and Graph Models
   - What is a graph?
   - Examples of Graphs
   - Vertex adjacency
   - Vertex degree in undirected graphs
   - Vertex degree in directed graphs
   - The Handshaking theorem
2. Families of graphs
   - Complete graphs
   - Cycles
   - Wheels
   - Star graph
   - Path graph
   - Bipartite graphs
3. Connectivity
   - Walk, Trail, Path, and Circuit
   - Subgraph
   - Connectedness in undirected graphs
4. Euler Circuits
   - Routing problem
   - Euler circuits
   - Finding Euler circuits
5. Hamiltonian Circuits
   - Hamiltonian circuits
   - Applications
6. Trees
   - Trees as models
   - Forests
   - Terminal and internal vertices
   - Rooted trees
   - Binary trees

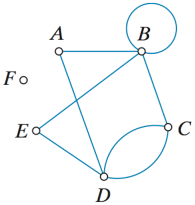
Outline

What is a graph? (I)

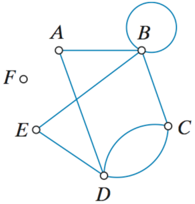
A graph $G = (V, E)$ is a structure consisting of a set of vertices (or nodes) V, and a set of edges E connecting some of these vertices.

In general, we use points to represent vertices, and line segments (possibly curved) to represent edges.

Example This graph has six vertices $V = \{A, B, C, D, E, F\}$ and eight edges: $\{\{A, B\}, \{B, C\}, \{C, D\}, \{C, D\}, \{E, D\}, \{B, E\}, \{A, D\}, \{B, B\}\}$



What is a graph? (II)



An edge can connect a vertex back to itself, as with $\{B, B\}$. These type of edges are called loops.

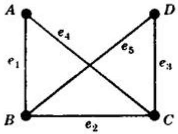
Two edges can connect the same pair of vertices, as with $\{C, D\}$. In general, we refer to such edges as multiple edges.

Edges can "cross" each other at incidental crossing points that are not themselves vertices of the graph, as is the case with the crossing point created by edges $\{A, D\}$ and $\{B, E\}$.

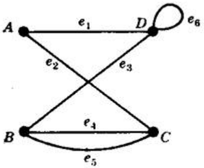
What is a graph? (III)

A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Graphs that have more than one edge connecting the same pair of vertices are called multigraphs.



(a) Graph



(b) Multigraph

Whats is a graph? (IV)

When edges are directed edges the graph is said to be a directed graph (or digraph). When depicting directed graphs, we use an arrow pointing from u to v to indicate the direction of an edge that starts at vertex u and ends at vertex v.

When a directed graph has no loops and has no multiple directed edges, it is called a simple directed graph.

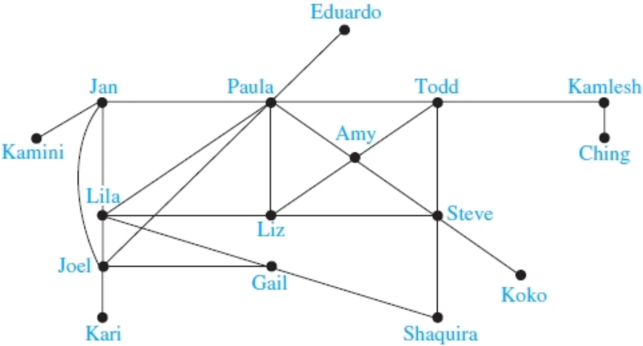
The term graph is used as a general term to denote graphs with directed or undirected edges, with or without loops, and with or without multiple edges.

Graph Models

Social networks

Graphs are used to model social structures based on different kinds of relationships between people or groups of people.

Individuals or organizations are represented by vertices, and relationships between them are represented by edges. No multiple edges nor loops are used.

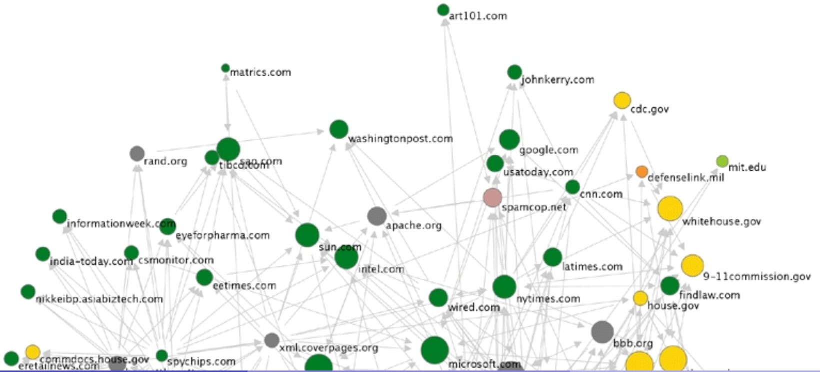


Graph Models (II)

Communication networks

Vertices represent devices and edges represent the particular type of communications links between the devices.

For example, in a web graph, each webpage is represented by a vertex and an edge starts at the webpage a and ends at the webpage b if there is a link on a pointing to b.



Graph Models (III)

Transportation networks

Vertices represent intersections in a road (or in the metro line), and edges represent roads (metro lines). If all roads are two-way, then we can use a simple undirected graph to model the road network.

Vertex adjacency

- Adjacency in undirected graphs:

  Two vertices u and v in an undirected graph G are called adjacent (or neighbours) in G if they are directly connected through an edge.

  Notation:

  > $\{u, v\}$ denotes the edge between vertices u and v. This edge is said to be incident with the vertices u and v.

- Adjacency in directed graphs:

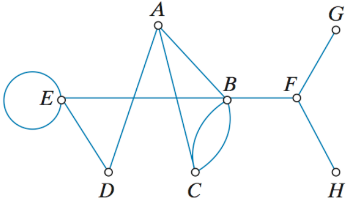
  A vertex u is said to be adjacent to v if there is a directed edge from u to v. Then v is said to be adjacent from u.

  Notation:

  > (u, v) denotes the directed edge from u to v. The vertex u is called the initial vertex of (u, v), and v is called the terminal (or end vertex) of (u, v).

Vertex adjacency (II)

Example Which vertices are adjacent to other vertices in the following undirected graph?



Vertices A and B are adjacent, and so are vertices B and C.

Vertices C and D are not adjacent, and neither are vertices A and E.

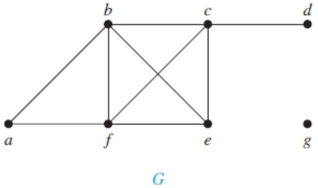
Because of the loop $\{E, E\}$, we say that vertex E is adjacent to itself.

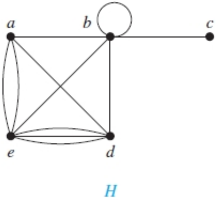
Vertex degree in undirected graphs

The degree of a vertex v in an undirected graph, denoted by $\deg(v)$, is the number of edges incident with it, except for loops, that contribute twice to the degree of that vertex.

A vertex is pendant if and only if it has degree one, while a vertex of degree zero is called isolated.

Example



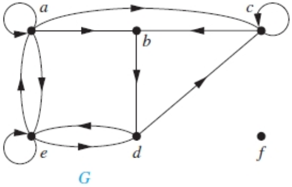


Degree of vertices in graph G: $deg(a) = 2, deg(b) = 4, deg(c) = 4,$
$deg(d) = 1, deg(e) = 3, deg(f) = 4, deg(g) = 0.$
Degree of vertices in graph H: $deg(a) = 4, deg(b) = 6, deg(c) = 1,$
$deg(d) = 5, deg(e) = 6.$

Vertex degree in directed graphs

The in-degree of a vertex v, denoted by $\deg^{-}(v)$, is the number of edges with v as their terminal vertex.
The out-degree of v, denoted by $\deg^{+}(v)$, is the number of edges with v as their initial vertex.
A loop contributes 1 to both the in-degree and the out-degree of a vertex.

Example



$deg^{-}(c) = 3, deg^{+}(c) = 2.$
$deg^{-}(f) = deg^{+}(f) = 0.$

The Handshake theorem (I)

The Handshake theorem

Let $G = (V, E)$ be an undirected graph with $|V|$ vertices $V = \{v_1, v_2, \ldots, v_{|V|}\}$, and $|E|$ edges in total. Then

$$\sum_{i=1}^{|V|} \deg(v_i) = 2|E|.$$

The above theorem applies also when there are multiple edges and loops.

Corollary An undirected graph has an even number of vertices of odd degree.

The Handshake theorem (II)

Exercise Suppose there are 5 persons in a meeting. Can every person have shaken hands with only 3 other people? And what about every person shaking hands with 4 people?

Two persons are involved in each handshake. So every person is represented as a vertex in a graph, and a handshake is represented by edges.
By the Handshake theorem, the sum of the degrees is $3 + 3 + 3 + 3 + 3 = 3 \cdot 5$, which is an odd number, so this situation is not possible.

On the other hand, it is indeed possible that each vertex had degree 4.
Sum of the degrees: $4 \cdot 5 = 20$. This is called the total degree of the graph.

Exercise A graph has seven vertices of degree 1, three of degree 2, seven of degree 3, and two of degree 4. How many edges does it have?

By the Handshake theorem,

$$7 \cdot 1 + 3 \cdot 2 + 7 \cdot 3 + 2 \cdot 4 = 2|E|,$$

so $|E| = 21$.

The Handshake theorem (III)

When considering directed graphs we differentiate between the in-degree and the out-degree of a vertex.

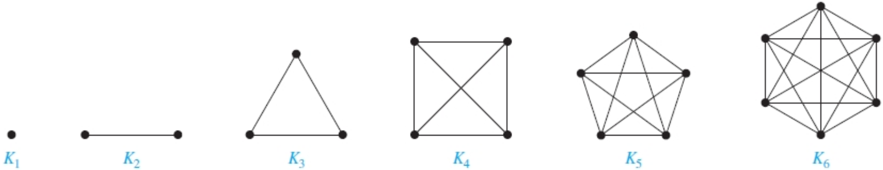
Theorem Let $G = (V, E)$ be a digraph with $|V|$ vertices $V = \{v_1, v_2, \ldots, v_{|V|}\}$, and $|E|$ edges in total. Then,

$$\sum_{i=1}^{|V|} \mathsf{deg}^-(v_i) = \sum_{i=1}^{|V|} \mathsf{deg}^+(v_i) = |E|.$$

Outline

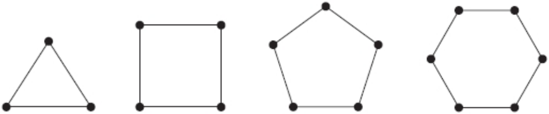
Complete graphs

A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.



Cycles

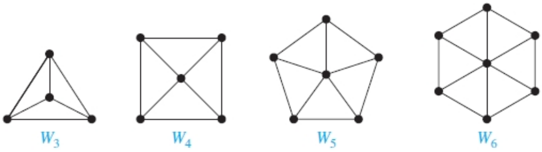
A cycle C_n, for $n \geq 3$, consists of n vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_n, v_1\}$.



C_3 C_4 C_5 C_6

Wheels

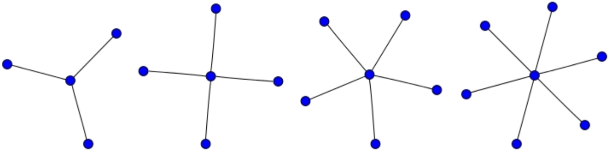
A wheel W_n is a graph obtained by adding an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.



Star graph

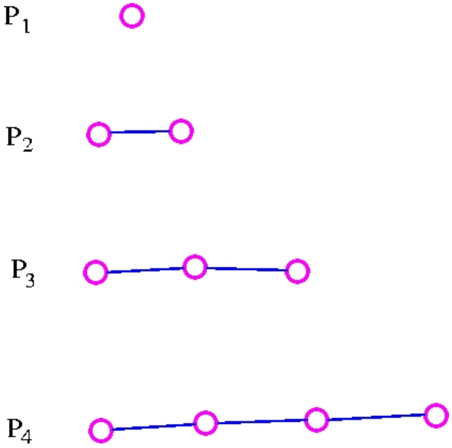
The star graph, denoted by St_n, consists of $n+1$ vertices $V = \{v_1, v_2, \ldots, v_{n+1}\}$ and n edges $\{v_1, v_2\}$, $\{v_1, v_3\}$, ..., $\{v_1, v_n\}$.

Applications
Computers, peripheral devices (printers, plotters, etc.), can be connected using a local area network. Some of these network are based on a star topology, where all devices are connected to a central control device.



Path graph

A path graph, denoted by P_n, consists of $n+1$ vertices $v_1, v_2, \ldots, v_{n+1}$, and n edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_n, v_{n+1}\}$.
The length of the path is the number of edges in the path.

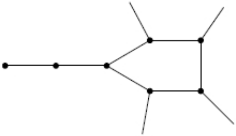


Bipartite graphs (I)

A simple graph G is bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

Exercise Are these two graphs bipartite?

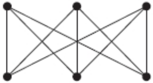
C_6



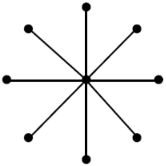
Bipartite graphs (II)

A complete bipartite graph, denoted by $K_{m,n}$, is a bipartite graph whose vertices can be partitioned into two subsets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$, which has an edge connecting every vertex in V_1 to every vertex in V_2.

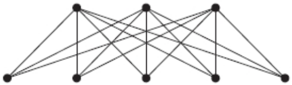
$K_{2,3}$



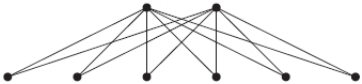
$K_{3,3}$



$K_{1,8}$



$K_{3,5}$



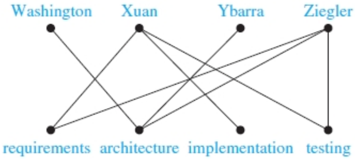
$K_{2,6}$

Bipartite graphs (III)

Applications
Bipartite graphs can be used to model applications involving matching the elements of one set to elements of another set.

For example, in **job assignments**.
We want to assign an employee to each job.

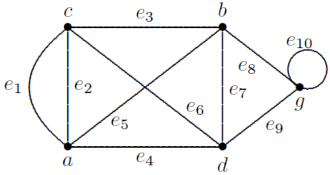
This problem is modeled by a bipartite graph $G = (\{Employees, Jobs\}, E)$.



Outline

Walk, Trail, Path, and Circuit

Let n be a nonnegative integer, and v, w two vertices in an undirected graph G.



A walk from v to w is an alternating sequence of vertices and edges

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$$

going from $v = v_0$ to $w = v_n$. We can repeat edges and vertices.

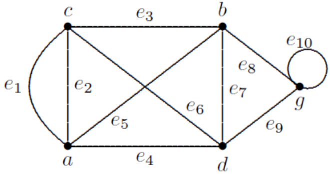
Example: A possible walk from b to a: $b, e_7, d, e_9, g, e_8, b, e_7, d, e_4, a$.

A trail from v to w is a walk from v to w with no repeated edges.

Example: A possible trail from b to a: $b, e_7, d, e_9, g, e_8, b, e_5, a$.

Walk, Trail, Path, and Circuit (II)

Let n be a nonnegative integer, and v, w two vertices in an undirected graph G.



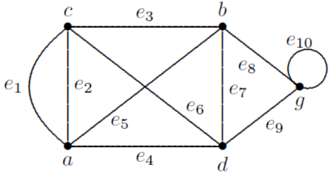
A path from v to w is a trail with no repeated vertices. Thus it is a sequence of vertices and edges with no repeated edges nor vertices.

When the graph is simple, we can denote a path by its vertex sequence $v_0, v_1, \ldots, v_n$. Listing these vertices uniquely determines the path.

Example: A possible path from b to a: b, e_7, d, e_4, a.

Walk, Trail, Path, and Circuit (III)

Let n be a nonnegative integer, and v, w two vertices in an undirected graph G.



A circuit is a trail that starts and ends at the same vertex, and has length greater than zero.

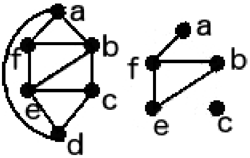
A circuit is simple if it does not contain repeat vertices (except the first and last).

Example: A possible simple circuit: $b, e_7, d, e_4, a, e_2, c, e_3, b$.

Subgraph

A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. The graph H is a proper subgraph of G if $H \neq G$.

Example



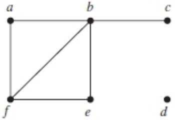
The graph on the right is a subgraph of the graph on the left.

Connectedness in undirected graphs

An undirected graph is called connected if there is a walk between every pair of distinct vertices of a graph. Otherwise, it is called disconnected.

A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.

Example How many connected components has the following graph G?



It has two connected components. They are disjoint and their union is the whole graph G.

Outline

Routing Problem

Routing problems are concerned with finding ways to route the delivery of goods and/or services to an assortment of destinations.

Goods or services: packages, mail, newspapers, pizzas, garbage collection, bus service, ...
Delivery destinations: homes, warehouses, distribution centers, terminals, ...

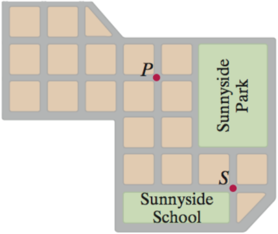
Existence question: **Is an actual route possible?**

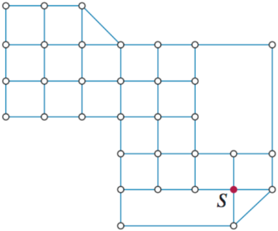
For most routing problems, the existence question is easy to answer, either yes or no.

Routing Problem (II)

Example Walking the neighbourhood.
A private security guard is hired to patrol the streets of a neighborhood.

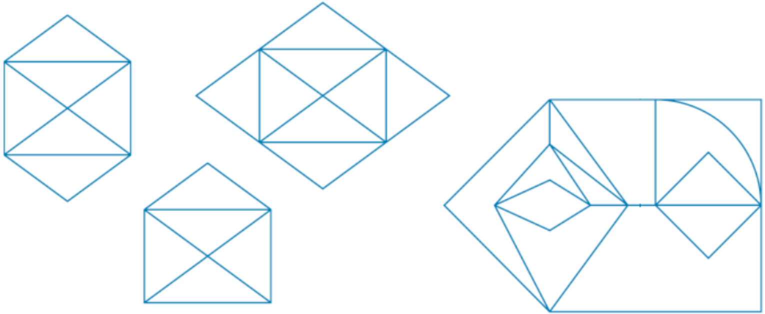
His task is to make an exhaustive patrol, on foot, through the entire neighborhood.
He doesn't want to walk more than what is necessary. His starting point is the school (S), that's where he parks his car.





Routing Problem (III)

Example Can you trace each drawing without lifting the pencil or retracing any of the lines?



Euler Circuits

Given a graph G, an Euler circuit for that graph is a circuit that travels through **every** edge of a connected graph without repeating edges.

An Euler trail in a graph G is a trail containing every edge of G.

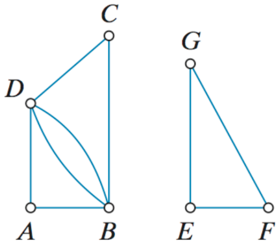
Euler's circuit and trail theorem

Given an undirected graph G,

- it has an Euler circuit if, and only if, G is connected and **every** vertex has even degree.
- it has an Euler trail if, and only if, G is connected and it has exactly 0 or 2 vertices of odd degree.

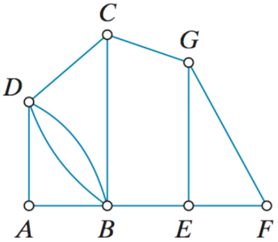
Euler Circuits (II)

Example This graph cannot have an Euler circuit nor Euler trail, for the simple reason that it is disconnected.



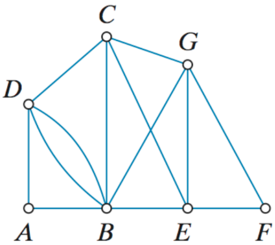
Euler Circuits (III)

Example This graph is connected but we can quickly spot odd vertices, so this graph has no Euler circuits. What about Euler trail?



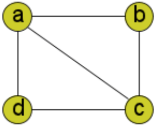
Euler Circuits (IV)

Example This graph is connected and all vertices are even. Thus this graph does have Euler circuits.



Euler Circuits (V)

Exercise Does the following graph have an Euler circuit? And an Euler trail?



Euler trail: a,b,c,d,a,c.

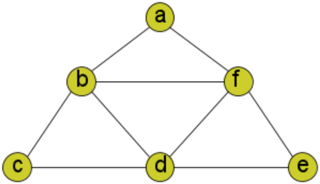
Finding Euler Circuits

Given a graph G connected and with all vertices of even degree, we can obtain an Euler circuit as follows.

1. Pick any vertex v of G at which to start. Initially C_{aux} and C are empty.
2. Pick any sequence of adjacent vertices, starting and ending at v and never repeating an edge. Write the resulting circuit in C_{aux} and C.
3. Check whether C contains every edge of G. If so, C is already an Euler circuit, and we are done. If not, perform the following steps:
    1. Remove all edges of C_{aux} from the graph G and also any vertices that become isolated when these edges are removed. Call the resulting subgraph G'.
    2. Pick any vertex w common to both C_{aux} and G'.
    3. Pick any sequence of adjacent vertices and edges of G', starting and ending at w and never repeating an edge. Write the resulting circuit in C_{aux}.
    4. Extend C with the circuit C_{aux}.
    5. Go back to step 3.

Finding Euler Circuits (II)

Exercise Find an Euler circuit in the following graph.



Iteration	v	C_{aux}	C
0	a		a
1	a	a, b, f, a	a, b, f, a
2	b	b, c, d, b	a,**b,c,d,b**,f,a
3	d	d, e, f, d	a,b,c,**d,e,f,d**,b,f,a

Outline

Hamiltonian Circuits

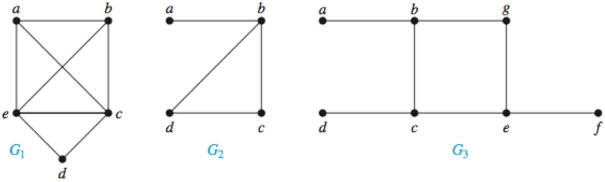
Given a graph G, a Hamiltonian circuit for that graph is a circuit that travels through **every** vertex of a connected graph without repeating vertices.

Note on Euler and Hamiltonian circuits:

- An Euler circuit includes every vertex of G, but it may visit some vertices more than once and thus may not be a Hamiltonian circuit.
- A Hamiltonian circuit does not need to include all the edges of G.

Hamiltonian Circuits (II)

Example Which of the following simple graphs have a Hamiltonian circuit?



G_1 has a Hamiltonian circuit: a, b, c, d, e, a.

There is no Hamiltonian circuit in G_2. This can be seen by noting that any circuit containing every vertex must contain the edge $\{a, b\}$ twice.

G_3 has not a Hamiltonian circuit either, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$ and $\{c, d\}$ more than once.

Hamiltonian Circuits (III)

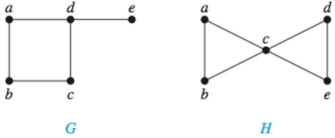
There is no efficient algorithm to find a Hamiltonian circuit.

If a graph G has a Hamiltonian circuit, then G contains a subgraph H with the following properties:

1. H has the same vertices as G
2. H is connected
3. H has the same number of edges as vertices
4. Every vertex of H has degree 2.

Thus if a graph G does not have a subgraph H with these properties, then G does not have a Hamiltonian circuit.

Example These two graphs do not have a Hamiltonian circuit.



Applications of Hamiltonian Circuits

Example The Traveling Salesman Problem. The salesman does not want to visit a city more than once!

Outline

Trees

A tree is a connected undirected graph with no circuits.
Trees cannot have multiple edges nor loops.

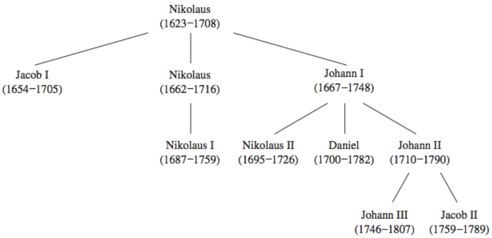
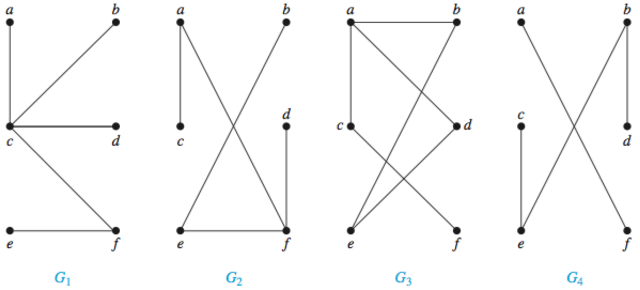


Figure : The family tree of the male members of the Bernoulli family of Swiss mathematicians

Alternative definition:
An undirected graph is a tree if and only if there is a unique trail between any two of its vertices.

Trees (II)

Example Which of the following graphs are trees?



G_1 and G_2 are trees, because both are connected graphs with no circuits.

G_3 is not a tree because e, b, a, d, e is a circuit in this graph. G_4 is not a tree because it is not connected.

Trees as models

Trees are used in computer science, chemistry, geology, botany, pyschology, and so on.

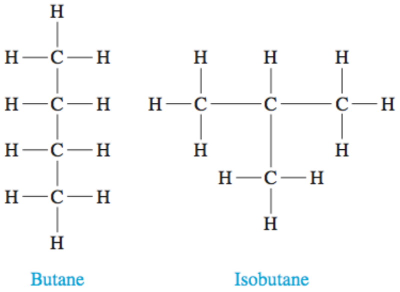
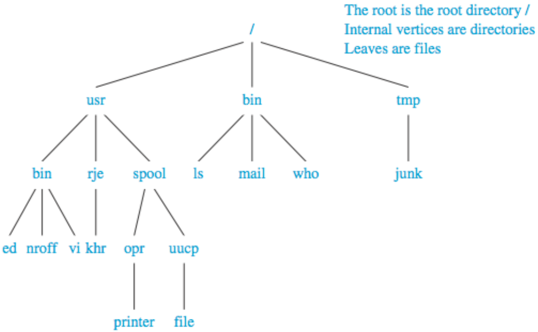


Figure : Trees can be used to represent molecules, where atoms are represented by vertices and bonds between them by edges.

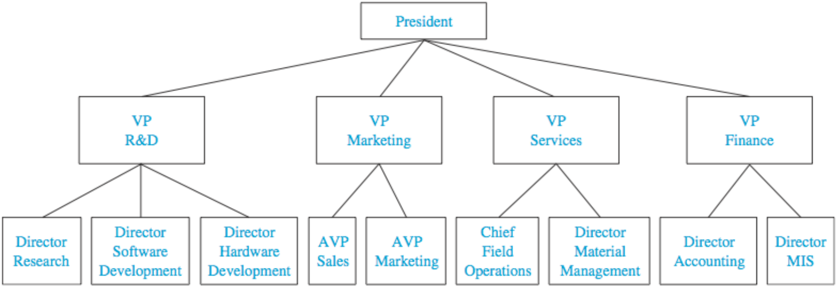
Trees as models (II)

Files in computer memory can be organized into directories. A directory can contain both files and subdirectories. The root directory contains the entire file system. Thus, a file system may be represented by a rooted tree, where the root represents the root directory, internal vertices represent subdirectories, and leaves represent ordinary files or empty directories.



Trees as models (III)

The structure of a large organization can be modeled using a tree. Each vertex in this tree represents a position in the organization.



Forests

A graph consisting of only one vertex is a trivial tree, while a graph consisting of more than one tree is called a forest.

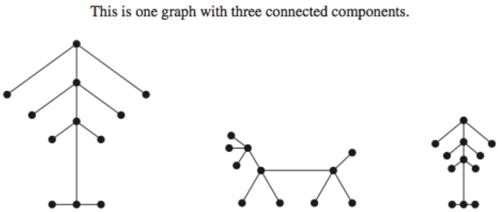


Figure : Example of a forest

Terminal and internal vertices

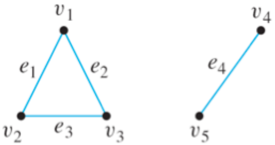
Let T be a tree. If T has $|V| \geq 3$, then a vertex of degree 1 in T is called a terminal vertex (or a leaf), and a vertex of degree greater than 1 is called an internal vertex (or a branch vertex).

If T has only $|V| = 2$, then each vertex is called a terminal vertex.

Theorem Any tree with $|V| = n$ vertices has $n - 1$ edges.

Theorem If G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.

Example The following is an example of a graph with five vertices and four edges that is not a tree.



Rooted Trees

A rooted tree is a tree in which one vertex is distinguished from the others and is called the root.

We can change an unrooted tree into a rooted tree by choosing any vertex as the root. Note that different choices of the root produce different rooted trees.

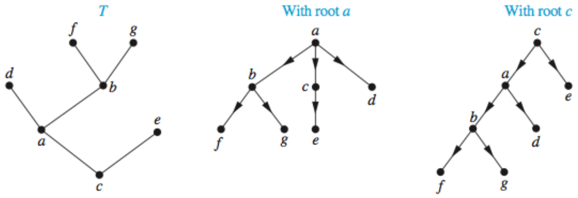
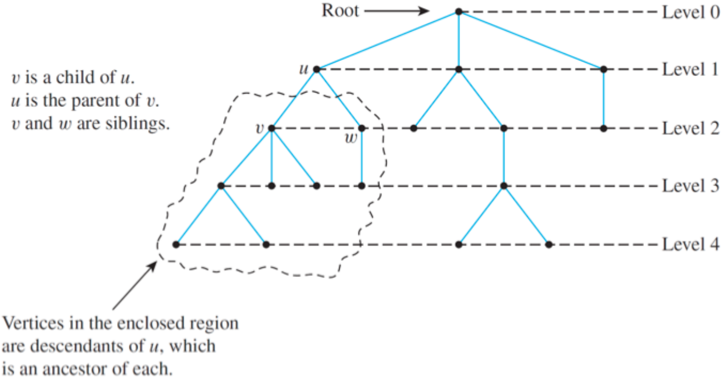


Figure : A tree and three rooted trees formed by designating two different roots.

Rooted Trees (II)

We can then refer to the parent of a vertex, the child, siblings, ancestors of a vertex, descendants of a vertex, etc.



Binary Trees

A binary tree is a rooted tree in which every parent has at most two children: a left child and a right child.

A full binary tree is a binary tree in which each parent has exactly two children.