2 Model Settings

In this section, we present the mathematical framework for a portfolio optimization problem over a finite investment horizon divided into multiple time periods. The model includes n risky assets alongside one risk-free asset (bank deposit). At each period, the investor has the opportunity to reallocate assets within the portfolio, which incurs transaction costs. Short selling is permitted. The objective of the investor is to maximize total wealth and minimize risk across the investment horizon, while satisfying several constraints: the availability of capital throughout the period, a ceiling on the number of distinct assets that can be held, and limits on the quantity of each asset. The formulation aims to develop a discrete-time dynamic portfolio allocation strategy that conforms to the Markowitz mean-variance objective function.

2.1 Classic Mean-Variance Portfolio Optimization Problem

In the classic Markowitz's portfolio optimization problem, the simplest form requires choosing exactly B assets out of n, which can be formulated as finding the binary decision variables $x_i \in \{0, 1\}$ for each asset i that maximize the quadratic utility over one single period:

$$\max_{\mathbf{x}\in\{0,1\}^n} \boldsymbol{\mu}^\mathsf{T}\mathbf{x} - q\mathbf{x}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{x} \quad \text{subject to} \quad \mathbf{1}^\mathsf{T}\mathbf{x} = B,$$

where $\boldsymbol{\mu}$ represents the vector of expected returns, $\boldsymbol{\Sigma}$ is the covariance matrix of asset prices, and $q > 0$ is the so-called risk tolerance factor. This factor balances the risk against expected returns in the portfolio; a higher value of q emphasizes risk aversion, leading to a more conservative investment strategy, while a value of $q = 0$ completely ignores risks.

Using a sufficiently large penalty P and reversing the objective, the problem can be reformulated as an unconstrained one:

$$\min_{\mathbf{x}\in\{0,1\}^n} -\boldsymbol{\mu}^\mathsf{T}\mathbf{x} + q\mathbf{x}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{x} + P(B - \mathbf{1}^\mathsf{T}\mathbf{x})^2,$$

or equivalently in element-wise form:

$$\min_{x_i\in\{0,1\}} -\sum_i \mu_i x_i + q\sum_i\sum_j x_i\sigma_{ij}x_j + P\left(B - \sum_i x_i\right)^2.$$

Note that this single-period problem takes the form of Quadratic Unconstrained Binary Optimization (QUBO), where the objective is to minimize a quadratic polynomial

$$\min_{\mathbf{x}\in\{0,1\}^n} \mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij}x_i x_j,$$

where $\mathbf{Q}$ is a matrix that is dense and grows quadratically with the number of assets n.

2.2 Multiple Time Periods with Transaction Costs and Short Selling

Consider an investment horizon comprising multiple time periods $t \in \{1, \ldots, T\}$, each separated by a unit length of time. Introduce an initial period 0 with the asset selection $x_{i0} = 0$. The investor is to allocate the available (normalized) units of capital C at each period, by either reallocating up to B assets or holding cash, subject to the constraint $C \leq B$. Let $p_{i,t}$ represent the price of one unit of asset i at time t, and let $\sigma_{ij,t}$ denote the covariance between stocks i and j at time t. The portfolio risk for this model is defined by:

$$\sum_i \sum_j p_{it} x_{it} \sigma_{ijt} x_{jt} p_{jt}, \tag{1}$$

The task is to determine the trajectory $x_{it} \in \{0, 1\}$ that minimizes the overall portfolio risk while maximizing profit, expressed as:

$$\sum_{t=1}^{T} \left(q \sum_i \sum_j p_{it} x_{it} \sigma_{ijt} x_{jt} p_{jt} - \sum_i (p_{i,t+1} - p_{it}) x_{it} \right), \tag{2}$$

We allow that each asset i can be selected up to k times and define a $2kn$-dimensional binary vector $\mathbf{x}$ for the n assets at each time t, where each block of k binary variables represents a single asset type. The inclusion of a factor of 2 accounts for the allowance of short selling, with an equal number of vector elements allocated to represent both long and short positions. To mirror realistic trading scenarios, constraints on transaction costs and short selling are imposed at each time step. Both impose limits on trading activities, and affect the path-dependent cash.

Transaction Costs: We denote δ as the transaction cost rate applied to both buying and selling. Since x_{it} is binary, its absolute value change is given by $|x_{it-1} - x_{it}| = x_{it-1} + x_{it} - 2x_{it-1}x_{it}$. The total transaction cost is represented by:

$$\delta \sum_i \sum_{t=1}^{T} p_{it}(x_{it-1} + x_{it} - 2x_{it-1}x_{it}). \tag{3}$$

We also account for the transaction cost associated with the liquidation of all assets at the terminal time T:

$$\delta \sum_i p_{iT} x_{iT}. \tag{4}$$

Short Selling Cost and Restriction: We introduce a short-selling indicator, $\tau \in \{-1, +1\}$, where -1 indicates a short position and $+1$ a long position. Our model includes a borrowing cost rate for short sales, denoted by ρ_s, to enforce prudence in short-selling activities:

$$\rho_s \sum_{i \in S} p_{it} x_{it}, \tag{5}$$

where S denotes the set of assets selected for short selling. Each short position allows the acquisition of additional capital for investment. We introduce slack variables $s_{bt} \in \{0, 1\}, b \in$

$\{0, \ldots, \lfloor \log_2 B \rfloor\}$ to cap the total assets:

$$\sum_i x_{it} + \sum_b 2^b s_{bt} = B \quad \text{for all } t \in \{1, \ldots, T\}. \tag{6}$$

Path-dependent Cash: In each time period, the model restricts the total available cash to not exceed C units. We introduce another slack variables $s_{ct} \in \{0, 1\}$, $c \in \{0, \ldots, \lfloor \log_2 C \rfloor\}$. The constraint for cash, given short-selling is permitted, is formulated as:

$$\sum_i \tau_i x_{it} + \sum_c 2^c y_{ct} = C \quad \text{for all } t \in \{1, \ldots, T\}. \tag{7}$$

Note that the slack variables s_{ct} indicate the amount of the budget that remains unspent and incur interest:

$$-\rho_c u \sum_t \sum_c 2^c y_{ct},$$

where ρ_c denotes the risk-free interest rate, and u is the value of one normalized cash unit. The constraints (6) and (7) can be enforced in the objective function through a quadratic penalty term:

$$P \sum_{t=1}^{T} \left\{ (B - \sum_i x_{it} - \sum_b 2^b s_{bt})^2 + (C - \sum_i \tau_i x_{it} - \sum_c 2^c y_{ct})^2 \right\}. \tag{8}$$

2.3 QUBO

For multi-period portfolio optimization under market frictions, it is not only challenging to obtain a solution when handling large dimensional assets, but also difficult to prove exact global optimal solutions. Quadratic Unconstrained Binary Optimization (QUBO) provides a framework suitable for a broad range of combinatorial optimization problems. The adoption of QUBO in benchmarking this optimization problem is justified for several reasons: Above all, QUBO is a vibrant area in optimization where digital algorithms have been developed to reliable standards, allowing us to not only obtain solutions but also measure their optimality with a precise bound. More importantly, QUBO is quantum-ready. Any QUBO problem can be transformed into Ising Hamiltonians, allowing direct implementation of quantum algorithms using QUBO inputs on quantum computing platforms like IBM Qiskit, D-Wave annealing, or hybrid systems.

The portfolio optimization problem can be formulated in QUBO form as follows:

$$\min_{\substack{x\in\{0,1\}^{n\times t}\\ y\in\{0,1\}^{c\times t}\\ s\in\{0,1\}^{b\times t}}} \sum_{t=1}^{T} \left(\underbrace{q\sum_i\sum_j p_{it}x_{it}\sigma_{ijt}x_{jt}p_{jt}}_{\text{risk}} - \sum_i \left(\underbrace{(p_{it+1}-p_{it})x_{it}}_{\text{profit}} - \underbrace{\delta p_{it}(x_{it-1}+x_{it}-2x_{it-1}x_{it})}_{\text{transaction cost}} \right) \right. \tag{9}$$

$$\left. - \underbrace{\rho_c u\sum_c 2^c y_{ct}}_{\text{cash interest}} + \underbrace{\rho_s\sum_{i\in S} p_{it}x_{it}}_{\text{short selling cost}} + \underbrace{\delta\sum_i p_{iT}x_{iT}}_{\text{liquidation cost}} \right)$$

$$+P\sum_{t=1}^{T}\left(\underbrace{\left(C-\sum_i \tau_i x_{it}-\sum_c 2^c y_{ct}\right)^2}_{\text{capital limit}} + \underbrace{\left(B-\sum_i x_{it}-\sum_b 2^b s_{bt}\right)^2}_{\text{number of assets limit}} \right)$$

The problem can be formulated as an equivalent Binary Quadratic Program (BQP) where the objective is a quadratic function subject to linear constraints, and the decision variables are constrained to be binary.

$$\min_{\substack{x\in\{0,1\}^{n\times t}\\ y\in\{0,1\}^{c\times t}\\ s\in\{0,1\}^{b\times t}}} \sum_{t=1}^{T} \left(\underbrace{q\sum_i\sum_j p_{it}x_{it}\sigma_{ijt}x_{jt}p_{jt}}_{\text{risk}} - \sum_i \left(\underbrace{(p_{it+1}-p_{it})x_{it}}_{\text{profit}} - \underbrace{\delta p_{it}(x_{it-1}+x_{it}-2x_{it-1}x_{it})}_{\text{transaction cost}} \right) \right. \tag{10}$$

$$\left. - \underbrace{\rho_c u\sum_c 2^c y_{ct}}_{\text{cash interest}} + \underbrace{\rho_s\sum_{i\in S} p_{it}x_{it}}_{\text{short selling cost}} + \underbrace{\delta\sum_i p_{iT}x_{iT}}_{\text{liquidation cost}} \right)$$

subject to

$$\sum_i \tau_i x_{it} + \sum_c 2^c y_{ct} = C \quad \text{for all } t\in\{1,\dots,T\} \qquad \text{capital limit}$$

$$\sum_i x_{it} + \sum_b 2^b s_{bt} = B \quad \text{for all } t\in\{1,\dots,T\} \qquad \text{number of assets limit}$$

The BQP can only be solved by classical digital computing, but with precise bound and reliable performance. It can serve as a baseline to evaluate the optimal solutios of QUBO via digital and quantum computing.