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# Order Adjustment Approach Using Cayley Graphs for the Order/Degree Problem

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**SUMMARY** The order/degree problem consists of finding the smallest diameter graph for a given order and degree. Such a graph is beneficial for designing low-latency networks with high performance for massively parallel computers. The average shortest path length (ASPL) of a graph has an influence on latency. In this paper, we propose a novel order adjustment approach. In the proposed approach, we search for Cayley graphs of the given degree that are close to the given order. We then adjust the order of the best Cayley graph to meet the given order. For some order and degree pairs, we explain how to derive the smallest known graphs from the Graph Golf 2016 and 2017 competitions.
***key words:*** *order/degree problem, Cayley graph, diameter, average shortest path length (ASPL), vertex bisection, vertex injection, vertex removal*

## 1. Introduction

The order/degree problem (ODP) lies in the graph theory domain and consists of finding the smallest diameter graph for given order/degree constraints. ODP solutions are beneficial for designing low-latency network topologies. In recent years, the ODP has drawn significant attention based on the introduction of high performance computers using many cores [1]–[5].

A graph $G = (V, E)$ consists of $|V|$ vertices and $|E|$ edges. The degree of graph $G$ is $\max_{v\in V} |Adj(v)|$, where $Adj(v)$ is the set of vertices adjacent to vertex $v$. All edges have a constant weight. The diameter $k(G)$ is the maximum distance between any two vertices. For a given $n$ and $d$, the ODP is defined as follows:

$$\min_{G=(V,E)} k(G), \quad \text{s.t. } |V| = n, \quad \max_{v\in V} |Adj(v)| \le d.$$

A list of the smallest known ODP solutions for certain order/degree constraints is available on the Graph Golf website [6]. We found the five best solutions in the 2017 competition using our heuristics and received the general graph widest improvement award and grid graph deepest improvement award**. Table 1 contains our five graphs submitted to Graph Golf 2017 in the general graph category and one graph submitted to Graph Golf 2016. For each given order and degree pair, the diameter, average shortest path length (ASPL), and ASPL gap of our graphs are shown. Each row of (random) contains a random regular graph provided by the competition organizers as a baseline for diameter and ASPL. The rows of (lower bound) contain the lower bounds of the diameter and ASPL described by Cerf et al. [7] for each order and degree pair. The ASPL gap is defined as $(l - L)$, where $l$ is the ASPL of the graph and $L$ is the lower bound of the ASPL. In the Graph Golf competition, if two graphs have the same diameter, the graph with the smaller ASPL is superior. ASPL $l$ is defined as

$$l = \frac{1}{n(n-1)} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \delta_{ij},$$

where the distance between vertices $i$ and $j$ is denoted by $\delta_{ij}$ and the number of vertices is $n$. The diameter of the graph is defined as $\max_{0\le i,j<n} \delta_{ij}$.

The approach proposed in this paper was used to create five graphs in Table 1, but not the graph of order 1344 and degree 30***. The graph of order 512 and degree 8 was

**Table 1** Our best submissions for Graph Golf 2016 and 2017. Random graph and lower bounds are shown for comparison.

| Order | Degree | Diameter | ASPL | ASPL gap |
|---|---|---|---|---|
| 512 | 8 | 4 | 3.11138 | 0.14465 |
| (random) | | 5 | 3.26793 | 0.30120 |
| (lower bound) | | 4 | 2.96673 | — |
| 4,896 | 24 | 4 | 2.89823 | 0.02080 |
| (random) | | 4 | 2.94896 | 0.07153 |
| (lower bound) | | 3 | 2.87743 | — |
| 9,344 | 10 | 5 | 4.24654 | 0.23338 |
| (random) | | 6 | 4.31101 | 0.29784 |
| (lower bound) | | 5 | 4.01316 | — |
| 88,128 | 12 | 6 | 4.88278 | 0.10201 |
| (random) | | 7 | 4.91098 | 0.13021 |
| (lower bound) | | 5 | 4.78077 | — |
| 98,304 | 10 | 7 | 5.35521 | 0.19984 |
| (random) | | 7 | 5.38272 | 0.22735 |
| (lower bound) | | 6 | 5.15537 | — |
| 1,344 | 30 | 3 | 2.34582 | 0.03830 |
| (random) | | 3 | 2.48257 | 0.17505 |
| (lower bound) | | 3 | 2.30752 | — |

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[**]In this paper, we do not discuss grid graphs.
[***]The graph of order 1344 and degree 30 in Table 1 was created using another method. To create this graph, we concatenated two Brown graphs [11] of $q = 5^2$, which have order $q^2 + q + 1 = 651$ and degree $q + 1 = 26$. We combined them to get a graph of order 1302 and then add 42 vertices. We then added edges randomly.

submitted to Graph Golf 2016 and other five graphs were submitted to Graph Golf 2017.

Finding superior solutions for the ODP is a time-consuming task [8], [9]. In the Graph Golf [6] workshop, Shimizu et al., Mizuno et al., Inoue, Kawamata, and others presented their techniques. Meanwhile, graph theoreticians have studied the degree/diameter problem (DDP) for decades [10]. For example, Brown [11] presented an approach for making graphs of diameter two, which is known as Brown's construction. Loz and Pineda-Villavicencio [12] produced the largest known graphs of maximum degree $17 \leq d \leq 20$ and diameter $2 \leq k \leq 10$. More than half of their largest known graphs are Cayley graphs. Miller and Širáň [13] continuously survey DDP methods.

However, DDP solutions will never be usable for the ODP problem unless they meet given order and degree constraints. Currently, there is no trivial approach to derive arbitrary ODP solutions from DDP solutions. Our objective is to create better ODP solutions than a random graph by using the graph construction method of the DDP.

A similar approach to ours was proposed by Koibuchi et al. [1]. They discussed not only generating graphs of small diameter, but also solution feasibility, such as parallel application performance and cabling complexity. They used the best known DDP solutions with orders close to a given order $n$ as a base graph for order adjustment. The major difference between our methods is that they use the best known DDP graphs as candidates for base graphs. In contrast, our approach employs a method that is frequently used for the DDP to construct new Cayley graphs. Therefore, our base graphs typically have closer orders to the given order $n$ compared to theirs.

The remainder of this paper is organized as follows. In Sect. 2, our order adjustment approach is proposed, where we generate Cayley graphs based on the DDP method. Cayley graphs cannot be constructed for arbitrary orders. Therefore, we search for Cayley graphs of not only the exact given order, but also of similar orders to the given order. We then adjust the order of the Cayley graph that is closest to the given order and has the smallest ASPL. Our experiments to verify the proposed approach are described in Sect. 3. An open question is described in Sect. 4. Finally, we conclude this paper in Sect. 5.

## 2. Algorithm

### 2.1 Creating Arbitrary Order Graphs from Cayley Graphs

For a given order $n$ and degree $d$, we wish to create a graph $G' = (V', E')$ from a base graph $G_0 = (V_0, E_0)$ to satisfy the constraints $|V'| = n$ and $\max_{v \in V'} |Adj(v)| \leq d$. Our approach consists of the following two steps.

First, we search for Cayley graphs with orders equal to $|V_0|$, where $|V_0| = n \pm \alpha$ ($\alpha \ll n$) and the given degree $d$. The Cayley graph with the smallest diameter and ASPL is treated as a base graph $G_0 = (V_0, E_0)$. We describe the details of Cayley graphs in Sect. 2.2.

Second, we adjust the order of the base graph $G_0$. The order $|V_0|$ is typically larger (or smaller) than the given order $n$. Therefore, we must add (or remove) $n - |V_0|$ vertices to (from) $G_0$. As described in Sects. 2.3 and 2.4, when the order is smaller than given order (i.e., $|V_0| < n$), either vertex bisection or vertex injection are applied to $G_0$. When the order is larger than the given order (i.e., $|V_0| > n$), the vertex removal process described in Sect. 2.5 is applied to $G_0$.

### 2.2 Cayley Graphs

We describe Cayley graphs, which are used as our base graphs, in the next several subsections. The method for generating Cayley graphs is described below. Cayley graphs are regular graphs, meaning all vertices of Cayley graphs have the same number of edges.

A Cayley graph is vertex-transitive. As explained below, one can calculate the ASPL of a vertex-transitive graph faster than that of a general graph. This fast ASPL computation aids us in finding small ASPL graphs with large orders quickly. Note that the graphs obtained by vertex bisection, injection, and removal, which are described in Sects. 2.3 to 2.5, are not vertex-transitive.

Vertex-transitive graphs satisfy the following property. In a vertex-transitive graph, every vertex has the same number of neighboring vertices at the same distance. In other words, when the number of vertices at distance $k$ from vertex $i$ is denoted by $n(i, k)$, $n(i, k) = n(i', k)$ is satisfied for any two vertices $i$ and $i'$, and any distance $k$. This property ensures that the ASPL $l$ of all pairs of vertices is equal to the ASPL $l'$ of a single vertex to all others. Therefore, to calculate the ASPL $l$ of a Cayley graph, we only need to compute the shortest path lengths from vertex 0 to all other vertices $j$ ($0 < j < n$). We never need to compute the distances between other vertex pairs, such as vertex pair $i$ and $j$ ($0 < i, j < n$). The ASPLs $l$ and $l'$ are defined as follows:

$$l' = \frac{1}{n-1} \sum_{j=0}^{n-1} \delta_{0j} \qquad \text{(for Cayley graph)},$$

$$l = \frac{1}{n(n-1)} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \delta_{ij} \qquad \text{(for general graph)},$$

where the distance between vertices $i$ and $j$ is denoted by $\delta_{ij}$. To compute the ASPL $l$ of a general graph, we must compute the distances $\delta_{ij}$ between all pairs of nodes ($0 \leq i, j < n$). The computation time of ASPL $l'$ for a Cayley graph is roughly $n$ times faster than that of ASPL $l$ for a general graph.

Cayley graphs are a type of voltage graph. Here, we define a Cayley graph using a semi-direct product of the form $\mathbb{Z}_s \rtimes_r \mathbb{Z}_t$ and a set of voltages $X$. $s, t$, and $r$ are integers. $\mathbb{Z}_s$ and $\mathbb{Z}_t$ are cyclic groups such that $\gcd(\phi(s), t) > 1$. $\phi(s)$ is Euler's totient function. The parameter $r$ is selected to satisfy $r^t \equiv 1 \pmod{s}$. The order of a Cayley graph is $n = st$. Each vertex $i$ in a Cayley graph corresponds to an element $(s_i, t_i)$ in $\mathbb{Z}_s \rtimes_r \mathbb{Z}_t$, which satisfies $i = s_i + s \cdot t_i$,

$0 \le s_i < s$, and $0 \le t_i < t$. All edges in a Cayley graph are defined by the set $X$. Each voltage $x \in X$ is an element of $\mathbb{Z}_s \rtimes_r \mathbb{Z}_t$. Let $x = (s_x, t_x)$, $f_s(s_i, t_i, s_x) \equiv s_i + r^{t_i} \cdot s_x \pmod{s}$, and $f_t(t_i, t_x) \equiv t_i + t_x \pmod{t}$. For a voltage $x$, we add an edge between vertices $i$ and $j$, which correspond to elements $(s_i, t_i)$ and $(s_j, t_j)$ in $\mathbb{Z}_s \rtimes_r \mathbb{Z}_t$, respectively, such that

$$s_j = f_s(s_i, t_i, s_x) \;\text{ and }\; t_j = f_t(t_i, t_x).$$

The details of this definition are provided by E. Loz and G. Pineda-Villavicencio [12]†.

For each $(s, t, r)$ parameter, we locally search for a suitable set of voltages $X$ that yield a smaller diameter and ASPL. We first generate multiple sets of voltages. We then randomly replace a voltage $x$ in each set $X$ to find a better Cayley graph. The concrete graphs we found for given pairs of order and degree are presented in Sect. 3.

## 2.3 Vertex Bisection

Vertex bisection is a method to increase the order of a base graph. A new vertex $a$ is added to a base graph $G_0$. Half of the edges incident to vertex $u$ in $G_0$ are removed from $u$ and connected to $a$. Then, unless the degree constraint is already satisfied, new edges are added between vertices with small degrees.

Algorithm 1 is the process for a single vertex bisection. Let $G_0 = (V_0, E_0)$, $V_0 = \{1, 2, 3, 4, 5, 6\}$, $E_0 = \{(1,2), (1,3), (1,4), (1,6), \dots, (5,6)\}$, and $d = 4$, as shown in Fig. 1 (a). Figure 1 (b) is the output graph $G'$ from the following steps. The thick edges in Fig. 1 (b) are newly added edges. In Algorithm 1, we select a vertex $u = 3$ and its $d/2 = 2$ neighbors $v_1 = 2$ and $v_2 = 4$ (lines 1–2). These neighbors are selected randomly. We then add a new vertex $a = 7$ and edge $(a, u)$ (lines 3–4). We remove edges $(u, v_1)$ and $(u, v_2)$, and add edges $(a, v_1)$ and $(a, v_2)$ (lines 6–7). We then choose an additional edge $e$ randomly to affect the diameter and ASPL (line 9). In this example, the vertices whose degrees are less than $d$ are 3 and 7. However, vertices 3 and 7 already have an edge between them. Therefore, Algorithm 1 terminates without adding edges in line 10.

We can now check the distance between $u = 3$ (or $a = 7$) and the vertices $Adj(3) = \{1, 2, 4, 5\}$ in $G_0$. In line 2 of Algorithm 1, $Adj(3)$ is divided into $Adj_a = \{2, 4\}$ and $Adj_u = \{1, 5\}$. Recall that $\delta_{ij}$ is the distance between vertices $i$ and $j$. In $G_0$, for all $v \in Adj(3)$, $\delta_{3v} = 1$. In the output of the bisection of $G'$, for all $v \in Adj_a$, $\delta_{3v}$ is increased to 2 and $\delta_{7v} = 1$. For all $v' \in Adj_u$, $\delta_{3v'}$ is not changed from 1 and $\delta_{7v'} = 2$.

For multiple vertex bisection, we execute lines 1 to 7 repeatedly to reach the given order $n$. Once $|V'|$ is equal to the given order, we execute line 8.

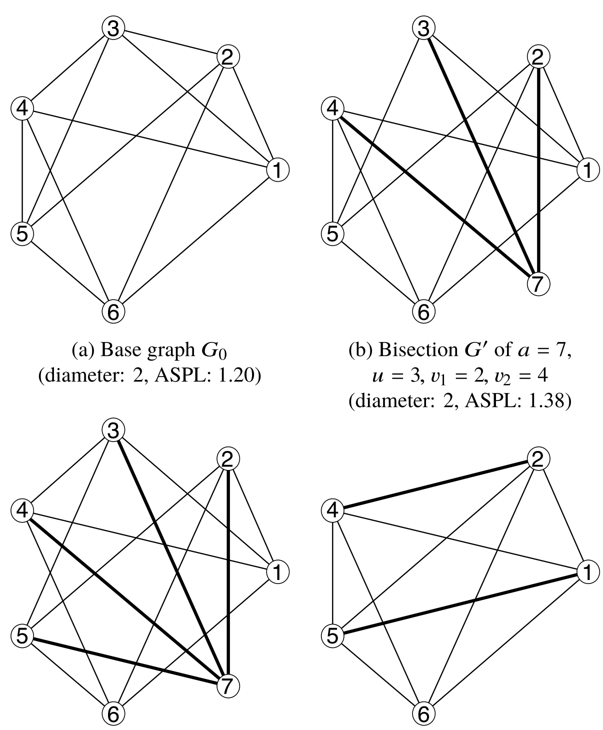
**Algorithm 1:** Vertex bisection

```
   Input: base graph G_0 = (V_0, E_0)
   Input: degree d
   Output: graph G' = (V', E')
1  Sample u ~ V_0
2  Sample v_1, ..., v_{d/2} ~ {v|(u, v) ∈ E_0}
3  V' ← {a} ∪ V_0
4  E' ← {(a, u)} ∪ E_0
5  for v ∈ {v_1, ..., v_{d/2}} do
6      E' ← E' \ {(u, v)}
7      E' ← {(a, v)} ∪ E'
8  edges ← {(u, v)|(u, v) ∉ E' ∀u, v ∈ V'}
9  for e = (u, v) ∈ edges do
10     if |δ(u)| < d and |δ(v)| < d then E' ← e ∪ E'
11 return G'
```

## 2.4 Vertex Injection

Vertex injection is another method to increase the order of a base graph. A new vertex $a$ is added. In order to avoid exceeding the degree constraint, several edges are removed from the graph. Both ends of the removed edges are then connected to vertex $a$.

Here, we explain steps of Algorithm 2 using an example. Figures 1 (a) and 1 (c) present example input and output graphs of Algorithm 2. We add a vertex $a = 7$ (line 2). We choose $\lfloor d/2 \rfloor = 2$ edges $(2, 3)$ and $(4, 5)$ (lines 5–8). We remove edges $(2, 3)$ and $(4, 5)$ (line 10), and add edges $(7, 2)$, $(7, 3)$, $(7, 4)$, and $(7, 5)$ (line 11). The added edges are represented by thick lines in Fig. 1 (c).




(a) Base graph $G_0$ (diameter: 2, ASPL: 1.20)

(b) Bisection $G'$ of $a = 7$, $u = 3$, $v_1 = 2$, $v_2 = 4$ (diameter: 2, ASPL: 1.38)

(c) Injection $G'$ of $a = 7$, $edges = \{(2, 3), (4, 5)\}$ (diameter: 2, ASPL: 1.33)

(d) Removal $G'$ of vertex 3 (diameter: 1, ASPL: 1.00)

**Fig. 1** Examples of vertex bisection, injection, and removal.

†Instead of $s$ and $t$, $m$ and $n$ are used in [12].

**Algorithm 2:** Vertex injection

**Input**: base graph $G_0 = (V_0, E_0)$
**Input**: degree $d$
**Output**: graph $G' = (V', E')$

1 $edges \leftarrow \emptyset$
2 $V' \leftarrow \{a\} \cup V_0$
3 $E' \leftarrow E_0$
4 $E_{rest} \leftarrow E_0$
5 **for** $i \in \{1, \ldots, \lfloor d/2 \rfloor\}$ **do**
6 $\quad$ Sample $(u, v) \sim E_{rest}$
7 $\quad$ $E_{rest} \leftarrow E_{rest} \setminus \{(u, w), (v, w) | \forall w \in V_0\}$
8 $\quad$ $edges \leftarrow \{(u, v)\} \cup edges$
9 **for** $e = (u, v) \in edges$ **do**
10 $\quad$ $E' \leftarrow E' \setminus \{e\}$
11 $\quad$ $E' \leftarrow \{(a, u), (a, v)\} \cup E'$
12 **return** $G'$

When we sample $\lfloor d/2 \rfloor$ edges in lines 5 to 8 of Algorithm 2, any two sampled edges should not be adjacent to each other. This is ensured by using $E_{rest}$ instead of $E_0$. If a pair of sampled edges are adjacent to each other, the nodes between these edges decrease by one degree at the end of Algorithm 2.

### 2.5 Vertex Removal

If $|V_0|$ is greater than the given order $n$, we remove $|V_0| - n$ vertices from $G_0$. These vertices are randomly chosen. By removing vertices, some remaining vertices are left with a smaller degree than the given degree $d$. Therefore, we add edges between vertex pairs with smaller degrees randomly, unless the degrees of all vertices are less than or equal to the given degree $d$.

For example, vertex 3 is removed from graph $G_0$, as shown in Fig. 1 (a). Four edges $(3, v)$, $v \in Adj(3)$, are removed. Then, four vertices (1, 2, 4, and 5) have a degree of three. We can add two edges $(1, 5)$ and $(2, 4)$. Figure 1 (d) presents the result of vertex removal and the complete graph of order five.

## 3. Graph Instances

We now describe how to find the graphs contained in Table 1 and present some additional experimental results of vertex bisection, injection, and removal.

We adopt the Cayley graph contained in Table 2 as a base graph for each given order and degree. Cayley graphs were discussed in Sect. 2.2.

Vertex bisection is used to create graphs of (order, degree) = (4 896, 24) and (88 128, 12). To obtain a graph of order 4,896, 24 vertices are added to the base graph of order 4,872 via vertex bisection. The increase in ASPL from the vertex bisection is only $2.73 \times 10^{-3}$. Note that ASPL of the random graph is 2.94896, which is $5.07 \times 10^{-2}$ larger than that of our submitted graph. A similar process is applied for the graph of (88 128, 12). The graph of order 88,128 is obtained by adding 18 vertices to the base graph of order 88,110 via vertex bisection. This increases the ASPL by $1.6 \times 10^{-4}$.

In Fig. 2, we present multiple trials (+) of vertex bisection for the graphs of order 4,873 and 4,896 to confirm that vertex bisection can be applied for these orders. For each order, we attempt vertex bisection approximately 10 times. For all trials, we used the graph of order 4,872 ($\ast$) in Table 2 as the base graph. All graphs for these trials have a diameter of four. It should be noted that for an arbitrary base graph, there is no guarantee that a graph obtained by vertex bisection has the same diameter as the base graph. In Fig. 2, one can see that graphs of the same order have similar ASPLs. We calculated the difference between the minimum and the maximum ASPLs of the multiple trials for each order. The differences between them are no greater than $7.65 \times 10^{-5}$ for every order.

The other points ($\times$) in Fig. 2 show the ASPLs of the Cayley graphs other than the base Cayley graph ($\ast$) in Table 2. To find the base Cayley graph, we searched for Cayley graphs of orders between 4,848 and 4,905. There are multiple points for this order because each point corresponds to the smallest-ASPL graph of a group in that order. For example, for the order 4,872, graphs of ASPL 2.8975, 2.8983, 2.8991, and 2.8996 are found in the groups of $(s, t, r) = (203, 24, 17)$, $(87, 56, 11)$, $(203, 24, 12)$, and $(348, 14, 7)$, respectively. The best Cayley graphs of orders 4,895, 4,896, and 4,905 in Fig. 2 are in the groups of

**Table 2** Properties and parameters of Cayley graphs used as base graphs.

| | | | | Group | | |
|---|---|---|---|---|---|---|
| Order | Degree | Diameter | ASPL | $s$ | $t$ | $r$ |
| 506 | 8 | 4 | 3.10099 | 46 | 11 | 9 |
| Voltages | [(8,3)(11,7)(7,1)(7,0)] | | | | | |
| 4,872 | 24 | 4 | 2.89550 | 87 | 56 | 5 |
| Voltages | [(6,14)(7,17)(12,12)(16,4)(27,31)(29,15)(31,30)(31,47)(39,16)(46,53)(65,16)(73,18)] | | | | | |
| 9,360 | 10 | 5 | 4.24703 | 390 | 24 | 23 |
| Voltages | [(56,20)(89,11)(101,18)(249,5)(293,17)] | | | | | |
| 88,110 | 12 | 6 | 4.88262 | 1335 | 66 | 64 |
| Voltages | [(71,61)(198,40)(202,2)(770,51)(1034,13)(1081,8)] | | | | | |
| 98,350 | 10 | 7 | 5.35534 | 1405 | 70 | 64 |
| Voltages | [(81,13)(190,53)(853,66)(1099,59)(1196,10)] | | | | | |

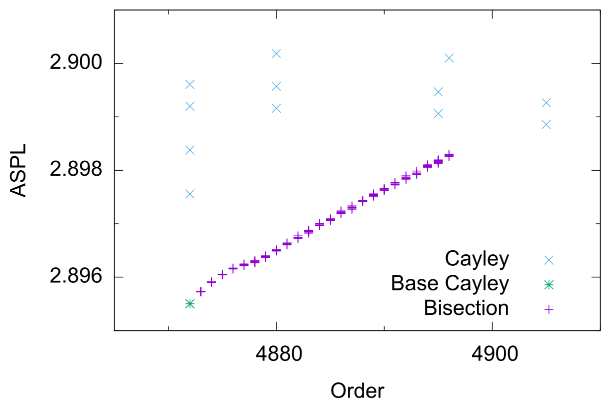



**Fig. 2** Results of Cayley graph searches and vertex bisection (order of approximately 4,896 and degree 24).

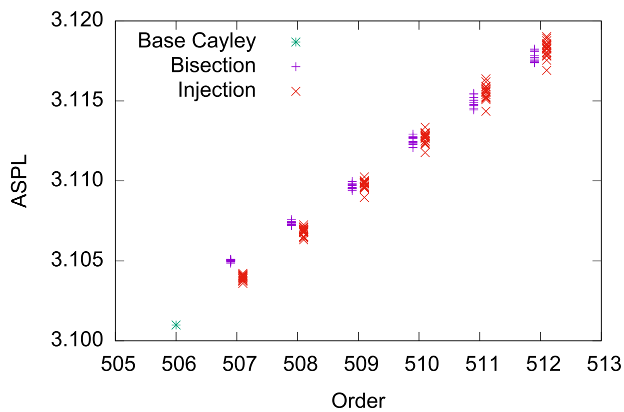



**Fig. 3** Results of vertex injection compared to vertex bisection (order of approximately 512 and degree 8).

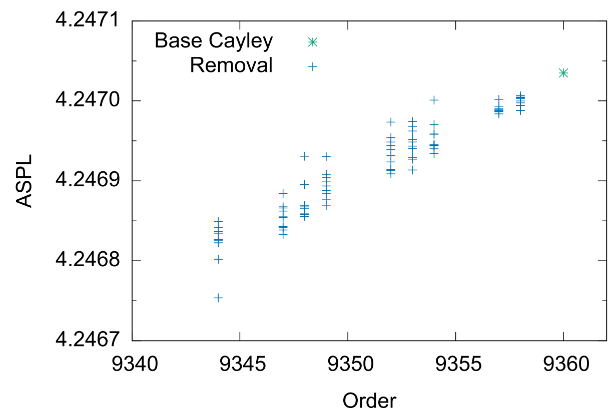



**Fig. 4** Results of vertex removal (order of approximately 9344 and degree 10).

$(s, t, r) = (89, 55, 32)$, $(153, 32, 8)$, and $(109, 45, 16)$, respectively[†].

We utilized vertex injection for obtaining a graph of $(\text{order}, \text{degree}) = (512, 8)$ from the Cayley graph of $(506, 8)$ in Table 2. We present the ASPLs of the graphs obtained by vertex injection in Fig. 3. We also increase the order to 512. In this figure, we also present the results of vertex bisection for comparison. We plot the ASPLs of 10 to 30 trials for each order. Most graphs have a diameter of five. Comparing the smallest ASPLs, vertex injection is superior for all orders. Even in terms average values, vertex injection is superior to bisection when several vertices are added. When the order is greater than 509, the average values of the two methods seem to be comparable. In Graph Golf 2016, we submitted a graph of diameter four and ASPL 3.11138 (ASPL gap 0.14465) by applying an additional local search to the best graph from vertex injection. For the additional local search, we utilized a 2-opt local search with an edge-importance function [14]. This graph is currently the smallest known graph. Note that the diameter and ASPL of the random graph are 5 and 3.26793, respectively. The lower bound has a diameter of four and ASPL of 2.96673.

Vertex injection was also used to construct the smallest known graphs of $(\text{order}, \text{degree}) = (10\,000, 7)$, $(10\,000, 11)$, $(10\,000, 20)$, and $(100\,000, 20)$ in Graph Golf 2016. All four of these graphs are based on Cayley graphs of $(s, t, r) = (555, 18, 4)$ and $(4165, 24, 19)$. These graphs were rendered obsolete by the voltage graphs of Kawamata [15] in 2017. He decreased the diameter to six for $(10\,000, 7)$. For the other three graphs, he submitted graphs with the same diameter but smaller ASPLs compared to ours ($1.5$–$7.8 \times 10^{-4}$ reduction in ASPL).

Vertex removal was used to create graphs of $(\text{order}, \text{degree}) = (9\,344, 10)$ and $(98\,304, 10)$ in Graph Golf 2017. The two graphs of order 9,360 and 98,350 in Table 2 are their base graphs. In our experience, vertex-removed graphs have smaller ASPLs than their base graphs. The best graph of order 9,344 reduced the ASPL by approximately $2.8{\times}10^{-4}$ compared to the base graph. For Graph Golf 2017, we continued to locally search for smaller-ASPL graphs [14] based on the smallest-ASPL graph from vertex removal. We eventually found a graph with a $2.3 \times 10^{-4}$ smaller ASPL, as shown in Table 1. For the graph of $(98\,304, 10)$, we removed 46 vertices from the base graph. Vertex removal kept the diameter at six and reduced the ASPL by $1.3 \times 10^{-4}$ compared to the base Cayley graph.

In Fig. 4, we present ASPL changes based on vertex removal from graphs of order 9,344 to 9,359. We attempted vertex removal approximately 10 times for each order. Although we present only nine orders, vertex removal can be applied for arbitrary order graphs between order 9,344 and 9,359.

## 4. Open Question

We discovered an interesting case when creating the graph of order 9,344, degree 10, and diameter 5. We attempted to use two Cayley graphs $G_1$ and $G_2$ as base graphs for vertex removal. Both graphs had the same order of 9,360, degree of 10, and diameter of 5. The ASPLs of $G_1$ and $G_2$ were 4.24554 and 4.24703, respectively.

Although the ASPL of graph $G_2$ was larger than that of $G_1$, when we applied vertex removal of 16 vertices from $G_1$ and $G_2$ many times, some graphs derived from $G_2$ had a diameter five, whereas $G_1$ never became a base graph of diameter five in our trials. Most graphs derived from $G_1$ and $G_2$ had a diameter of six. We attempted vertex removal 1,358 times with base graph $G_1$. Additionally, we found 10 graphs of diameter five and 169 graphs of diameter six by using $G_2$ as a base graph.

Prior to this experience, we believed that diameter and ASPL are the most important criteria and applied vertex removal to the Cayley graph with the smallest ASPL. Even

[†]To obtain the graphs, the following voltages were assigned for groups $(89, 55, 32)$, $(153, 32, 8)$, and $(109, 45, 16)$, respectively. Voltages: [(4,53)(7,9)(12,45)(16,17)(23,16)(24,19)(31,27)(43,32)(47,8)(48,14)(48,35)(71,15)], [(15,11)(29,31)(32,15)(41,22)(43,12)(62,23)(65,24)(65,27)(67,7)(87,6)(127,6)(142,26)], and [(6,38)(8,1)(9,1)(13,43)(14,19)(21,21)(30,11)(53,23)(54,9)(62,8)(62,32)(92,43)].

after this experience, these criteria are still valuable. However, in this case, ASPL alone was not sufficient to select a base Cayley graph for vertex removal.

The question is which type of criteria can give larger priority to $G_2$ than $G_1$ when selecting a base graph for vertex removal. We expect that similar criteria will be needed for vertex bisection and injection. Further investigation regarding this matter is required.

We describe the detail of $G_1$ and $G_2$ below. Both Cayley graphs $G_1$ and $G_2$ belong to the same group $(s, t, r) = (390, 24, 23)$. The set of voltages of $G_1$ is [(73,20)(86,5)(134,6)(138,13)(174,7)] and that of $G_2$ is shown in Table 2. We also examine another property of these graphs. Let $distance(G) = (n(0,0), n(0,1), \ldots, n(0,k))$, where $k$ is the diameter of the Cayley graph $G = (V, E)$ and $n(0,k') = |\{v \in V | \delta_{0v} = k'\}|$ is the number of vertices at a distance $k'$ from vertex 0. $distance(G_1) = (1, 10, 90, 810, 5131, 3318)$ and $distance(G_2) = (1, 10, 90, 810, 5117, 3332)$. There are differences at distances four and five. Both graphs contain no six-vertex cycles or shorter cycles. Approximately 35% of nodes are at distance five. Therefore, small changes in these base graphs are likely to increase their diameters to six.

## 5. Conclusion

In this paper, we proposed an order-adjustment approach to construct superior graphs for the order/degree problem. We adopted Cayley graphs, which are frequently used to create the best known graphs for degree/diameter problem, as our base. Because Cayley graphs cannot be constructed for arbitrary orders, we proposed vertex bisection, injection, and removal to adjust the order of a graph. These three methods allow one to create graphs of an arbitrary order. We applied this approach in the Graph Golf 2017 and 2016 competitions [6] and received the widest improvement award for the general graph category, which is given to the authors who find the largest number of superior solutions.

We also described an open question that we encountered for a graph of order 9,344 and degree 10. We explained that the smallest-ASPL graph is not always the best graph for vertex removal. We do not have any more suitable criteria to determine the most suitable base graph for vertex removal at this time. This question should be investigated in future work.

Considering a network of high-performance computers, a small-diameter graph is a good candidate for a low-latency network topology. To increase feasibility, routing mechanisms and fault tolerances should be investigated in future work.

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## Appendix A: Parameters $(s, t, r)$ of the Semi-Direct Product $\mathbb{Z}_s \rtimes_r \mathbb{Z}_t$ and Voltages $X$ of a Cayley Graph

We briefly introduced Cayley graphs in Sect. 2.2. For readers unfamiliar with Cayley graphs, we explain additional details in this section. We use an order of 84 as an example. The prime factors of 84 are $2^2 \times 3 \times 7$. In Table A·1, we list the $(s, t, r)$ parameters of the semi-direct product $\mathbb{Z}_s \rtimes_r \mathbb{Z}_t$ such that $st = 84$. All parameters satisfy $\gcd(\phi(s), t) > 1$, where $\phi(s)$ is Euler's totient function. $\phi(s)$ is defined as the number of positive integers smaller than $s$ and relatively prime to $s$. We only show $r$ values that satisfy $r^t \equiv 1 \pmod{s}$ and $r < t$. The parameters $(s, t) = (12, 7)$ and $(4, 21)$ are not listed in Table A·1 because they do not sat-

**Table A· 1** Groups $(s, t, r)$ of order 84 with the value of Euler's totient function $\phi(s)$ and greatest common denominator $(\phi(s), t)$.

| $s$ | $t$ | $r$-list | $\phi(s)$ | $\gcd(\phi(s), t)$ |
|---|---|---|---|---|
| 14 | 6 | {3, 5} | 6 | 6 |
| 7 | 12 | {2, 3, 4, 5, 6, 8, 9, 10, 11} | 6 | 6 |
| 6 | 14 | {5, 7, 11, 13} | 2 | 2 |
| 3 | 28 | {2, 4, 5, ..., 26} | 2 | 2 |

isfy $\gcd(\phi(s), t) > 1$. The parameters $(s, t) = (42, 2)$, $(28, 3)$, and $(21, 4)$ do satisfy $\gcd(\phi(s), t) > 1$. However, they are not listed in Table A· 1 because there are no $r$ values that satisfy $r^t \equiv 1 \pmod{s}$ and $r < t$. The condition $r < t$ is not required to generate a Cayley graph, but we use it because we wish to reduce the number of combinations. To save rows in Table A· 1, we merge multiple $r$ values into a single row. For example, groups $(14, 6, 3)$ and $(14, 6, 5)$ are listed in a row of $(s, t) = (14, 6)$.

Before we show the particular voltage set $X$ of order 84, we introduce the quotient $B(s, l)$. The quotient $B(s, l)$ is utilized for a brief explanation of both the size of the voltage set $X$ and degree $d$ of the derived Cayley graph. The following equations are satisfied by these two values.

$$|X| = s + l$$
$$d = s + 2l$$

A voltage $x \in X$ is classified as symmetric or asymmetric. The numbers of symmetric and asymmetric voltages in $X$ are $s$ and $l$, respectively. Here, we denote the elements in $\mathbb{Z}_s \rtimes_r \mathbb{Z}_t$ using $(s_i, t_i)$ and $(s_j, t_j)$. As explained in Sect. 2.2, for a voltage $x \in X$ and each $(s_i, t_i)$, we calculate $(s_j, t_j)$ using the following equations:

$$s_j = f_s(s_i, t_i, s_x) \text{ and } t_j = f_t(t_i, t_x) \quad \text{(A· 1)}$$

where $x = (s_x, t_x)$. Then, because an element $(s', t') \in \mathbb{Z}_s \rtimes_r \mathbb{Z}_t$ corresponds to a vertex $i = s' + st'$, we add an edge between vertices $i = s_i + st_i$ and $j = s_j + st_j$ in the derived Cayley graph. For any $(s_i, t_i)$ and $(s_j, t_j)$ that satisfy Eq. (A· 1), if the following equations are satisfied, the voltage $x$ is "symmetric." Otherwise, the voltage $x$ is "asymmetric."

$$s_i = f_s(s_j, t_j, s_x) \text{ and } t_i = f_t(t_j, t_x)$$

In simpler terms, we define a function $j = f(i, x)$ that creates an edge between the vertices $i$ and $j$ based on the voltage $x$. A voltage $x$ is symmetric if and only if $i = f(j, x)$ for any $i$ and $j = f(i, x)$.

We consider not only symmetry, but also the loop length for each voltage $x$. From any vertex $i_0$, we can create the sequence $(i_0, i_1, i_2, \ldots)$ by using the function $i_{l+1} = f(i_l, x)$ for a particular $x$ repeatedly. If $x$ is symmetric, the sequence satisfies $i_0 = i_2$. For an asymmetric voltage, it satisfies $i_0 \neq i_1$ and $i_0 \neq i_2$. Additionally, for an asymmetric voltage, there exists a loop length $l > 2$ in the sequence such that $i_0 = i_l$. An inverse voltage $x^{-1} (\neq x)$ for an asymmetric voltage $x$ also exists and satisfies $i = f(j, x^{-1})$, where $j = f(i, x)$.

We use the following notation for a voltage set. For a quotient $B(1, l)$, we use the form $[(s_0, t_0)|(s_1, t_1) \ldots (s_l, t_l)]$, where $(s_0, t_0)$ is a symmetric voltage and the others are asymmetric voltages. For a quotient $B(0, l)$, we use the form $[(s_1, t_1) \ldots (s_l, t_l)]$, where all voltages are asymmetric. All voltages described in Table 2 are asymmetric. The quotients of the Cayley graphs in Table 2 are $B(0, 12)$, $B(0, 5)$, $B(0, 6)$, $B(0, 5)$, $B(0, 4)$, from top to bottom.

We now return to an example of order 84. We pick a semi-direct product $\mathbb{Z}_7 \rtimes_2 \mathbb{Z}_{12}$ (i.e., $(s, t, r) = (7, 12, 2)$) from Table A· 1. We consider the quotient $B(0, 2)$ for the semi-direct product. The voltage set $X = [(1, 2)(1, 7)]$ derives a Cayley graph of order 84, degree 4, diameter 4 and ASPL 3.14458. We will now discuss two more examples of the same semi-direct product. When we assign the voltage set $X$ to $[(0, 6)|(1, 7)(2, 1)]$ as the quotient $B(1, 2)$, the derived Cayley graph has order 84, degree 5, diameter 4, and ASPL 2.95181. When we consider the quotient $B(0, 3)$ and assign the voltage set $X$ to $[(1, 2)(2, 2)(2, 7)]$, the derived Cayley graph has order 84, degree 6, diameter 3, and ASPL 2.49398. In these examples, all voltages other than $(0, 6)$ are asymmetric. The loop lengths of voltages $(1, 2)$, $(1, 7)$, $(2, 1)$, $(2, 2)$, and $(2, 7)$ are 6, 12, 12, 6, and 12, respectively. Their inverse voltages are $(5, 10)$, $(3, 5)$, $(6, 11)$, $(3, 10)$, and $(6, 5)$, respectively.

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