Radar Cross Section Estimation using the Variational Quantum Linear Solver Algorithm

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***Abstract* — Electromagnetic simulations are computationally intensive yet paramount in radar technologies. Finite Element Methods (FEM) require expensive linear algebra algorithms, with a polynomial complexity in the mesh size. Variational Quantum Linear Solver (VQLS) is a quantum algorithm that might be able to offer an exponential speed-up for such tasks. Despite this quantum algorithm not being relevant for every application, it might be a promising approach to tackle the problem of radar cross-section computation efficiently. In this paper, we explain how to apply such a quantum algorithm to this radar-related problem, with a focus on the statistical estimation part of VQLS.**

***Keywords* — radar, RCS, quantum, VQLS, FEM**

I. Introduction

The radar cross section or RCS is a key concept in radar technology. This is not only because its mean value can be plugged into the radar range equation to determine the SNR or detection range in a specific scenario. It also conveys signature information that can be used to classify or identify the object under surveillance. This is because the instantaneous value of the RCS is determined by the geometry and material of the target. As a result, the measured RCS is highly dependent on the measuring aspect angle and wavelength used, revealing those geometric properties. Swerling fluctuation models allow us to simplify that behavior into a small set of statistical models or 'cases' used in detection theory [1], but for other applications such as classification, the instantaneous RCS values are relevant. By performing Doppler analysis on a coherent set of complex RCS measurements on a target, so called micro-Doppler contributions of a target can be made visible, revealing moving parts on the geometry such as propellers on airplanes, or wheels and gait movements on vehicles and humans, respectively, further improving classification abilities. For objects that cannot be measured in an anechoic chamber, either because they are too large or they are simply physically not available, like enemy platforms, it is interesting to estimate the RCS based on CAD models. Fidelity is of prime concern here. High frequency methods such as Physical Optics (PO) and Geometrical Optics (GO) are fast methods to obtain a first order value for the RCS, but they take only a limited set of EM interaction types into account, in particular specular reflections. For object size in the order of the wavelength ($< 10\lambda$), other effects like diffraction become dominant and the estimated RCS values using optical techniques become inaccurate. In contrast, low frequency methods, or 'full wave methods' approach Maxwell's equations and give more accurate results. They suffer however from significant calculating time despite the fact that some full wave solvers allow parallel processing, using for instance GPUs. To alleviate the calculation time for RCS estimation, or indeed for any other applications where Maxwell solving is required, it is interesting to turn to quantum computing.

A common approach to this problem is to use Finite Element Method (FEM) to approximate the solution of the Helmholtz equation. This method boils down to solving a linear system of equations, whose dimension is the size of the mesh: a finer approximation requires a larger system. Granted the system is sparse, state-of-the-art algorithms can solve in $\mathcal{O}(N\kappa)$ time, where N is the dimension of the system and κ its condition number.

Quantum computing is a relatively new field of research, yet the unprecedented computational power it provides may have major consequences in a variety of fields, from cryptography to optimization. In 1994, a quantum algorithm developed by Shor [2] demonstrated the potential of quantum computing, with an algorithm solving integer factorization in logarithmic time. In 2008, Harrow, Hassidim, and Lloyd [3] described a quantum algorithm for linear systems of equations, named HHL, with complexity $\mathcal{O}(\log(N)\kappa^2)$. Unfortunately, both these algorithms require quantum subroutines with low fault-tolerance, notably quantum phase estimation, making them unfit for short-term applications. Indeed, today's quantum computing is referred to as the noisy intermediate-scale quantum (NISQ) era, as current quantum computers are not yet capable of efficient error correction, thus very sensitive to environmental noise. On the other hand, Variational Quantum Eigensolver (VQE) algorithms, introduced in [4] and including the Variational Quantum Linear Solver (VQLS), are considered NISQ compatible.

We will start by briefly introducing quantum algorithm framework and VQE. Then, we will consider a toy example of scattered field computation with FEM and explain why VQLS could speed-up the estimation of the radar cross-section. In particular, we will explain a sampling strategy allowing to preserve an exponential speed-up when estimating a mean energy, a crucial part of VQLS.

II. Quantum Computing Introduction

The basic component of a quantum computer is the quantum bit, or qubit. Mathematically, it is a linear combination

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \qquad \alpha, \beta \in \mathbb{C} \tag{1}$$

A qubit can be represented as a complex vector with two coefficients. Measuring a qubit will project it on $|0\rangle$ (resp. $|1\rangle$) with probability $|\alpha|^2$ (resp. $|\beta|^2$) normalized by $|\alpha|^2 + |\beta|^2$. For this reason, we usually normalize them.

A n-qubit system can be obtained by taking the tensor product of each qubit. It is described by the 2^n coefficients corresponding to the 2^n possible bitstrings spanning the total space:

$$|\psi\rangle = \sum_{k=0}^{2^n-1} \lambda_k\,|k\rangle, \qquad \lambda_k \in \mathbb{C} \tag{2}$$

with k written in binary, for instance $|1001\rangle + |0001\rangle$. n-qubit sequences thus live in a tensor space of dimension 2^n: an Hilbert space $\mathcal{H}$ isomorphic to $\mathbb{C}^{2^n}$, as we can see with the vector representation. This makes computations convenient: operations on qubits can be represented as unitary matrices. The Hermitian transpose of a vector or an operator is written with the symbol $\dagger$, and we write $|\psi\rangle^\dagger = \langle\psi|$.

A sequence of such operations is usually represented as a circuit with n registers corresponding to qubits. Measuring the output of a quantum circuit will yield only one bitstring, with a probability related to the modulus square of its complex coefficient. Writing the output as a vector allow one to see it as a probability distribution. As estimating each of the 2^n coefficients requires an exponential number of samples, this probability distribution should carry information that only a few samples allow to retrieve.

In VQE [4], one considers a circuit depending on a set of parameters θ, producing an output state $|\psi(\theta)\rangle$. The mean value of an Hamiltonian H for the state $|\psi\rangle$, denoted by $\langle\psi|H|\psi\rangle = \langle H\rangle$ is estimated by sampling the output state. This procedure and its efficiency will be discussed in section IV. Then, a classical optimizer provides a new θ, and those operations are repeated until the mean value (or energy of $|\psi(\theta)\rangle$) is small enough.

This algorithm is called hybrid, as a classical computer is needed to optimize the circuit parameters. The algorithm boils down to optimizing a real-valued function, which it computed by estimating the energy of the output state of a quantum circuit. The Hamiltonian giving this energy should reflect the specific problem we consider: its ground state (*i.e* lowest energy state) should be the solution to the problem. The architecture of the VQE circuit is an important parameter of the algorithm, and some choices will give better results than others depending on the specificity of the problem (that is, its Hamiltonian). However, we will not focus on this aspect here.

III. EM Simulation

The toy example we are considering in the paper is a FEM scheme for a 2D space discretized with a square mesh and

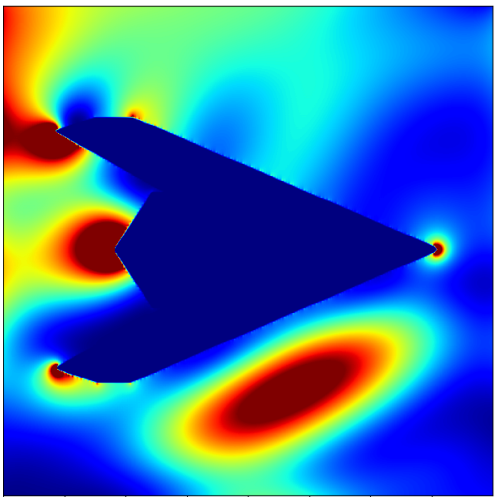

Fig. 1. A 2D scattered field computed with FEM. An electric field hits the object from a given angle (here 60°), and the colors indicate the scattered field intensity at any point within the box.

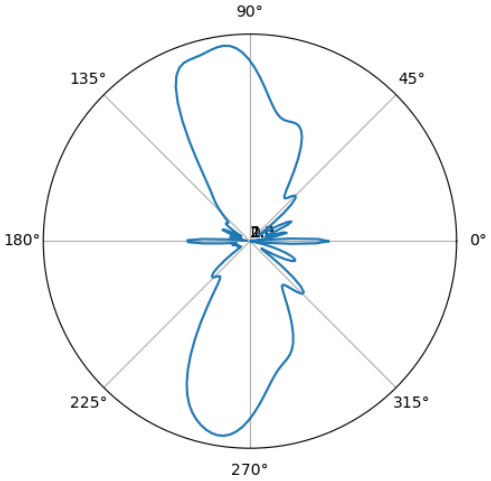

Fig. 2. The radar-cross section associated with the object. For every possible incident angle, we compute the far-field intensity going back in this direction.

Nédelec basis functions. A simple example with a plane-like object is shown in Fig. 1 and 2. The total electric field is the sum of the incoming and the scattered field, *i.e* the one reflected by the object, whose boundary is a perfectly conducting material in this example. This is a very simplistic example with a low-detail 2D shape: Fig 1 and 2 are not supposed to be realistic, but are included to illustrate the setting we consider in this work.

The details of the implementation can be found in [5], where the performance of the HHL algorithm for this specific problem is evaluated. Here, we are considering VQLS instead of HHL but the formulation of the FEM is the same. To recover an approximation of the field scattered by an object for an incoming field whose components in the Nédelec basis is given by vector b, one must solve

$$Ax = b \tag{3}$$

where x is the vector of the scattered field in the Nédelec basis and A is a sparse matrix with at most 7 non-zero entries per

row and dimension $N = 2k(k+1)$, the number of edges in the mesh (k being the number of cells). The matrix is made of 9 differents bands (see Fig. 3)).

To solve (3) with VQE, the idea is to consider x and b as quantum states $|x\rangle$ and $|b\rangle$, using the representation of a n qubits system as a vector with 2^n complex coefficients. Vectors of size N thus only require about $\log(N)$ qubits, hence the exponential speed-up. Matrix A must then be encoded as an operator acting on $n = \log(N)$ qubits. The paper introducing VQLS [6] proposed the Hamiltonian

$$H = A^\dagger(I - |b\rangle\langle b|)A \tag{4}$$

whose ground state is precisely $|x\rangle$, the quantum state encoding normalized solution vector x. To be more precise, $|x\rangle$ is the only state reaching the lowest energy level, in this case 0. As explained in [7], if $|\psi\rangle$ is the circuit output state,

$$|\langle x|\psi\rangle|^2 \geq 1 - \kappa^2 \langle\psi| H |\psi\rangle \tag{5}$$

meaning a state with near zero energy will be close to the solution. This inequality also shows how the condition number of A, κ, hinders the precision of the algorithm, similarly to classical sparse linear system solvers and HHL algorithm.

However, as explained in section II, to recover $|\psi\rangle$ would require number of samples exponential in the number of qubits, destroying the speed-up. This is why radar cross-section computation is one of the few examples where quantum algorithms for linear systems of equations, be it HHL or VQLS, is relevant. This application example was first introduced in [8]. Even though we cannot retrieve the full solution vector $|x\rangle$, we can compute its overlap with another state with a SWAP test, and only then measure the circuit to get this scalar value. In this case, the inner product of x with a vector giving the far-field extrapolation of the Nédelec basis functions gives the radar cross-section for a chosen direction [5].

IV. Matrix Decomposition

To estimate the mean value $\langle\psi| H |\psi\rangle$ of (4), it is necessary to write the matrix A as a sum of unitaries, as explained for instance in [7]. Indeed, one needs both

$$|\langle b|A|\psi\rangle|^2 \text{ and } |\langle\psi|A^\dagger A|\psi\rangle| \tag{6}$$

The first term can be evaluated with indirect measurements called *Hadamard test*, which require controlled unitary operators for both the VQE circuit and the state $|b\rangle$. The second correspond to the mean value of $A^\dagger A$, so it can be evaluated directly as the mean value of this operator for the output state. In both cases, we need to write A as a sum of unitaries.

Those unitaries should correspond to gates that can be applied within a circuit, and a common choice are the Pauli operators acting on n qubits, that is, tensor products of the 2-qubit Pauli matrices X, Y and Z along with identity I.

Thus, we need to write

$$A = \sum \lambda_i \sigma_i \tag{7}$$

where λ_i are complex numbers and σ_i Pauli operators on n qubits. Computing this decomposition amounts to writing A in a basis with 4^n elements, hence 4^n matrix products. In order to preserve the exponential speed-up, it is thus necessary to work out a formula giving this decomposition for any n.

This question is addressed in [9], where VQLS for a 1D Poisson equation is considered. With this equation, matrix A is the 1D Laplacian operator with Dirichlet boundary conditions, the sum of the identity matrix and a matrix with two bands (8)

$$\begin{pmatrix} 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{8}$$

The author of [9] notices such a matrix can be written as a sum of Paulis with 2^n terms, and proposes a recursive expression for the decomposition. We provide a simple formula for this decomposition:

$$\sum_{k=1}^{n} I^{\otimes n-k} \otimes \frac{X - iY}{2} \otimes (\frac{X + iY}{2})^{\otimes k-1} + cc \tag{9}$$

with cc the complex conjugate. This expression is found by writing the bitwise addition operator $\sum |k\rangle\langle k+1|$. Expending this expression tells us every Pauli of the form $I...I$ followed by some X and Y with an even number of Y appears. The number $1 \leq k \leq n$ of X, Y gives a magnitude of 2^{1-k}, with a sign depending on the first X, Y and the number of $Y \mod 4$, and there are 2^{k-1} of those.

Furthermore, [7] and [10] explain how sampling m of those Paulis according to their magnitude yields a variance with a dependence on $C = \sum |\lambda_i|$:

$$\varepsilon^2 \approx \frac{C^2}{m} \tag{10}$$

for the estimation of $\langle A\rangle$. In other words, granted C scales polynomially with n, an estimation with precision ε requires a polynomial number of samples.

With the matrix defined in (8), it follows from the decomposition that $C = n$. Despite a number of Paulis scaling as 2^n, efficient sampling is thus still possible. The strategy is to choose a number k of X, Y uniformly at random, and then consider a random Pauli having the form prescribed above, whose mean value is estimated by running the circuit. Equation (10) tells us that repeating this operation $(n/\varepsilon)^2$ times yield an estimation of $\langle A\rangle$ with precision ε.

In our case, the quantity of interest is not directly $\langle A\rangle$, but rather (6). Weighted sampling strategy for the Hadamard tests gives an error similar to (10) for $|\langle b|A|\psi\rangle|$, and assuming the estimator is Gaussian, the error for $|\langle b|A|\psi\rangle|^2$ is of order C^4/m. For the second term, the above reasoning applied to $\langle A^\dagger A\rangle$ gives a similar result. The consequence is a variance scaling as C^4/m, hence n^4/ε^2 samples to achieve ε precision.

As mentioned earlier, for the scattered field problem, the matrix is made of 9 bands. Furthermore, the alternation

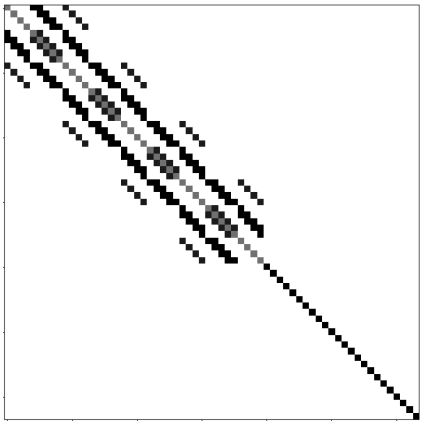

Fig. 3. Without removing edges, we get this matrix by following [5]. The discontinued bands reflect the succession of k horizontal edges followed by $k+1$ verticals ones. Here $k=4$ and the matrix is 64×64.

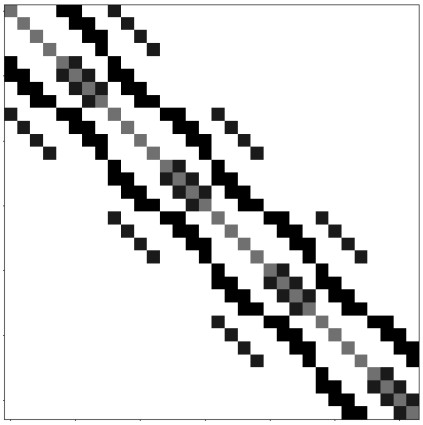

Fig. 4. By removing edges, we get a similar matrix but with a simple Pauli decomposition. k is still equal to 4 but now the matrix is only 32×32, as no padding is required.

between k horizontal edges and $k+1$ vertical ones (with the numbering scheme explained in [5]) leads to irregularities inside those bands one can observe on Fig. 3. That and the need to pad the matrix with 1 on the diagonal to fit a 2^n dimension (since $N = 2k(k+1)$ is no power of 2) are probably the reasons why decomposing this matrix as a sum of Paulis exhibits no pattern, unlike matrix (8). We suggest to ignore the rightmost vertical edges, and the upmost horizontal edges of the mesh. This fixes the two reasons behind the difficult Pauli decomposition. See the slightly modified matrix in Fig. 4.

V. Conclusion

Electromagnetic simulations may benefit from quantum computing, even in the NISQ era, as VQLS can be applied to FEM. We have seen a simple implementation example, and described a way to preserve the exponential speed-up during the sampling part of VQLS. This work is of theoretical nature, as simulating a quantum algorithm on a classical computer is, by design, very computationally expensive. Without a quantum computer, simulating the VQLS algorithm for a realistic RCS problem was thus not in the scope of this work.

There are, however, some aspects of VQLS that still need to be addressed. First, the energy we need to estimate could get closer to zero, on average, when n increases. Even if keeping a constant ε is possible, the relative precision will plummet if VQLS gets stuck in a region with vanishing gradient. This vanishing is one of the main issues with VQLS, but strategies like Hamiltonian morphing [7] and local cost function [6] were designed to prevent this problem. We also need to input $|b\rangle$ in the circuit (quantum oracle), to estimate the mean energy. Finally, the Pauli decomposition we introduced did not take into account the object whose scattered field is of interest. In order to preserve the advantage of the Pauli decomposition, the shape of the object, or rather the modifications it induces on the matrix should be easy to decompose as a sum of Paulis as well.

Despite those limitations, RCS computation is still a promising application of quantum computing due to the scalar nature of the quantity one needs to extract from the scattered field vector. Furthermore, this work suggests that using VQLS for this problem might be relevant, with an example of simple FEM simulation where the matrix of interest seems adapted to the energy measurement part of VQLS.

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