Notes on Unimodality of Hypersimplex

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1 Introduction

1.1 Hypersimplex

Fix a positive integer n, and let $[n] := \{1, 2, \cdots, n\}$. To any subset $S \subseteq [n]$, we associate the indicator vector:

$$\chi_S = (\chi_S(1), \chi_S(2), \ldots, \chi_S(n))$$

where

$$\chi_S(i) = \begin{cases} 1, & i \in S \\ 0, & i \notin S \end{cases}$$

For $0 < k < n$, let $\binom{[n]}{k}$ be the family of all k-subsets of $[n]$. The *hypersimplex* $\Delta_{k,n} \subseteq \mathbb{R}^n$ is the convex hull of the indicator vectors χ_I for $I \in \binom{[n]}{k}$. Equivalently,

$$\Delta_{k,n} = \left\{ (x_1, \ldots, x_n) \mid 0 \leqslant x_i \leqslant 1, \sum_{i=1}^{n} x_i = k \right\}$$

1.2 Stanley's Triangulations

Why is the set defined by a permutation

$$\nabla_w = \{(y_1, \ldots, y_{n-1}) \in [0,1]^{n-1} \mid 0 < y_{w(1)} < \cdots < y_{w(n-1)} < 1\}$$

a simplex?

Step-by-Step Explanation

1. **What is a simplex?**
   A simplex is the simplest generalization of a triangle or tetrahedron to higher dimensions:

   - 1-simplex: line segment (2 vertices)
   - 2-simplex: triangle (3 vertices)
   - 3-simplex: tetrahedron (4 vertices)

- In general, an $(n-1)$-simplex in $\mathbb{R}^{n-1}$ is the convex hull of n affinely independent points.

Formally,

$$\operatorname{conv}(v_0, \dots, v_d) = \left\{ \sum_{i=0}^{d} \lambda_i v_i \mid \lambda_i \geqslant 0, \sum_{i=0}^{d} \lambda_i = 1 \right\}$$

2. **Connecting our definition to a simplex.**
   The set is defined by strict inequalities:

$$0 < y_{w(1)} < \cdots < y_{w(n-1)} < 1$$

   The closure allows equality:

$$0 \leqslant y_{w(1)} \leqslant \cdots \leqslant y_{w(n-1)} \leqslant 1$$

3. **Identifying the vertices.**
   The vertices are:

$$(0, 0, \dots, 0), \quad e_{w(1)}, \quad e_{w(1)} + e_{w(2)}, \quad \dots, \quad (1, 1, \dots, 1)$$

   where e_i is the i-th unit vector.

4. **Affine independence.**
   Each vertex introduces a new coordinate direction, so they are affinely independent.

5. **Geometric intuition.**
   The region is a "slice" of the cube, forming a staircase-like path, which is a simplex.

Summary

1. The set identifies n vertices at cube corners.

2. There are n vertices, forming an $(n-1)$-simplex.

3. Vertices are affinely independent.

Thus, ∇_w is a simplex.

1.3 Volume and Ehrhart series

We are mainly interested in the volume and Ehrhart series of the characteristic polytopes. For n-gons up to 5-gon:

- 3-gon:

$$\mathrm{vol}\, P_\chi(Q_3) = 1,$$
$$\mathrm{Ehr}_{P_\chi(Q_3)}(z) = \frac{1+4z+z^2}{(1-z)^3}.$$

- 4-gon:

$$\mathrm{vol}\, P_\chi(Q_4) = \frac{1}{2},$$
$$\mathrm{Ehr}_{P_\chi(Q_4)}(z) = \frac{1+5z+5z^2+z^3}{(1-z)^4}.$$

- 5-gon:

$$\mathrm{vol}\, P_\chi(Q_5) = \frac{5}{24},$$
$$\mathrm{Ehr}_{P_\chi(Q_5)}(z) = \frac{1+6z+11z^2+6z^3+z^4}{(1-z)^5}.$$

From these examples, we see unimodality in the h^*-polynomial. Thus, we conjecture:

Conjecture 1.3.1. The coefficients h_i^* in the h^*-polynomial of the Ehrhart series of characteristic polytopes of n-gon form a unimodal sequence.

2 Characteristic Polytope of d-cube $\square_d$

References

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