Characteristic Polytope of Graphs

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1 Characteristic Polytope

1.1 Characteristic vectors of subsets

Fix a positive integer n, and let $[n] := \{1, 2, \cdots, n\}$. To any subset $S \subseteq [n]$, we associate the *characteristic vector*:

$$\chi_S = (\chi_S(1), \chi_S(2), \ldots, \chi_S(n))$$

where

$$\chi_S(i) = \begin{cases} 1, & i \in S \\ 0, & i \notin S \end{cases}$$

Given a simplicial complex Δ over $[n]$, we associate to each face $\tau \in \Delta$ its characteristic vector χ_τ and get the characteristic polytope as follows:

Definition 1.1.1. The **characteristic polytope** $P_\chi(\Delta) \subset \mathbb{R}^n$ of Δ is the convex hull of its characteristic vectors:

$$P_\chi(\Delta) = \text{conv}\{\chi_\tau \mid \tau \in \Delta\}.$$

1.2 Characteristic polytope of a graph

We view a graph G with vertices $[n]$ as a simplicial complex. For any graph we can build a characteristic polytope as defined above. In what follows we will first do a few examples and then focus on two graphs, namely the complete graph K_n and the cycle graph C_n.

1.2.1 Example: Empty graph E_n

The empty graph E_n has no edges, and thus its faces are only the vertices and empty set. The characteristic polytope of E_n is given by:

$$P_\chi(E_n) = \text{conv}\{0, \chi_i \mid i \in [n]\}.$$

Clearly, this is just the convex hull of the unit vectors together with the origin $\mathbf{0}$ in $\mathbb{R}^n$, which is the standard simplex Δ_n.

1.2.2 Example: Complete graph K_n

The complete graph K_n has all subsets of $[n]$ with cardinality at most 2 as its faces. The characteristic polytope of K_n is given by:

$$P_\chi(K_n) = \text{conv}\{\chi_S \mid S \subseteq [n], |S| \leqslant 2\}.$$

1.2.3 Example: Cycle graph C_n

The cycle graph C_n has faces corresponding to all subsets of vertices that induce a connected subgraph. The characteristic polytope of C_n is given by:

$$P_\chi(C_n) = \text{conv}\{\chi_S \mid S \subseteq [n], \text{ induces a connected subgraph}\}.$$

1.3 Volume and Ehrhart series

We are mainly interested in the volume and Ehrhart series of the characteristic polytopes. We list the volume and Ehrhart series of n-gon up to 5-gon as follows:

- 3-gon:

$$\begin{aligned} \text{vol } P_\chi(Q_3) &= 1, \\ \text{Ehr}_{P_\chi(Q_3)}(z) &= \frac{1+4z+z^2}{(1-z)^3}. \end{aligned}$$