# Notes – From Face Ring to Partition Complex

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# 0 Background

This note is constantly evolving. The ultimate goal is to develop an understanding of the field **combinatorial commutative algebra**.

This note is very personal. The use of diagrams, analogies, stories, and other means is only a shadow of how the author thinks or likes to think/understand.

# 1 Sets and Multisets

## 1.1 Sets and subsets

Let

$$E = \{x_1, x_2, \dots, x_n\}$$

We will abbreviate or identify it with

$$[n] := \{1, 2, \dots, n\}.$$

**Definition 1.1.1.** For a subset $I \subseteq E$, the **characteristic function** of $I$ is defined as:

$$\delta_I : E \to \{0, 1\}$$

where

$$\delta_I(i) = \begin{cases} 1, & \text{if} \quad i \in I, \\ 0, & \text{if} \quad i \notin I. \end{cases}$$

Identify $I$ with the monomial $\prod\limits_{i \in I} x_i$ or its characteristic vector $\nu_I = (\delta_I(1), \delta_I(2), \dots, \delta_I(n))$.

**Definition 1.1.2.** A $k$**-subset** of $E$ is a subset $I \subseteq E$ whose cardinality is $k$.

## 1.2 Multisets

Fix $E = [n]$.

**Definition 1.2.1.** A **multiset** $M$ is a pair $(E, \delta_E)$, where

$$\delta_E(i) = \begin{cases} k_i & \text{if } i \in E, \\ 0 & \text{if } i \notin E, \end{cases}$$

where $k_i \in \mathbb{Z}_{>0}$ is the **multiplicity** of $i$.

We call $S = \{i \in E \mid \delta_E(i) \neq 0\}$ the **support** of the multiset, written as $\mathrm{supp}(M) = S$.

**Example 1.2.2.** Let $E = \{1, 2, 3\}$, and

$$\delta_E(1) = 4, \quad \delta_E(2) = 0, \quad \delta_E(3) = 1.$$

The multiset is represented as: $M = \{1, 1, 1, 1, 3\}$, and supp(M) = (1,3).

**Remark 1.2.3.** A submultiset is subset of a multiset, and a $k$**-submultiset** is a submultiset whose cardinality is $k$.

Given $I \subseteq E$, we want to characterize the $k$-submultisets with support $I$.

Take $I = \{1, 3\} \subseteq [4]$ as an example. What are the 4-submulsets with support I? Let's list them all: {1,1,1,3}, {1,1,3,3}, {1,3,3,3}. We have 3 4-submultisets with support $\{1, 3\}$.

**Lemma 1.2.4.** Let $I \subseteq E = [n]$ and $m = |I|$. The set of $k$-submultisets with support I are in bijection with the set $\{(z_1, z_2, \ldots, z_m) \in \mathbb{Z}^m_{\geqslant 1} \mid \sum_{i=1}^m z_i = k\}$.

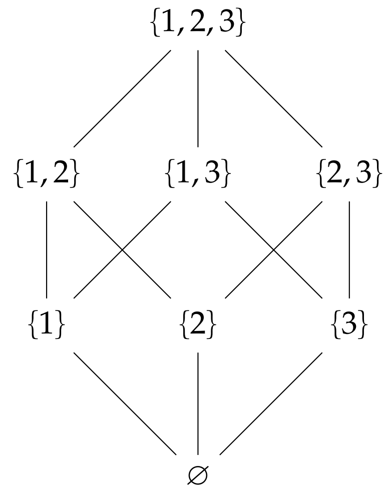
**Corollary 1.2.5.** Let $I \subseteq E = [n]$ and $m = |I|$. The number of $k$-submultisets with support I is $\binom{k-1}{m-1}$.

## 1.3 Graded structures and reciprocity

The set of subsets of $E$ forms a finite **graded Boolean lattice** $B_n$ with the rank function:

$$\mathrm{rk} : 2^E \to \mathbb{Z}_{\geqslant 0}, \quad I \mapsto |I| \quad \text{(cardinality)}.$$

**Example 1.3.1.** Let $E = [3]$ and we have the Hasse diagram of $B_3$:




The generating function of the Boolean lattice $B_n$ is

$$F(x_1, x_2, \ldots, x_n) = 1 + \sum_{i=1}^{n} x_i + \sum_{i<j} x_i x_j + \cdots + x_1 x_2 \cdots x_n = \prod_{i=1}^{n} (1 + x_i)$$

Let $x_1 = x_2 = \cdots = x_n = x$. The generating function becomes:

$$F(x, x, \ldots, x) = (1+x)^n := \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n.$$

and we get the definition of **binomial coefficients**, the most familiar counting function.

**Definition 1.3.2.** $\binom{n}{k}$ is the number of k-subsets in $[n]$.

**Remark 1.3.3.** The vector $\left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}\right)$ is the rank vector (or face vector) of the Boolean lattice (or simplicial complex) $B_n$.

All submultisets of a multiset form a graded lattice $\mathcal{L}^\infty$. The generating function of the Boolean lattice $\mathcal{L}^\infty$ is

$$F(x_1, x_2, \ldots, x_n) = 1 + \sum_{i=1}^{\infty} x_i + \sum_{i<j} x_i x_j + \cdots = \prod_{i=1}^{n} \frac{1}{1 - x_i}$$

Let $x_1 = x_2 = \cdots = x_n = x$. The generating function becomes:

$$F(x, x, \ldots, x) = \frac{1}{(1-x)^n} := \left(\!\!\binom{n}{0}\!\!\right) + \left(\!\!\binom{n}{1}\!\!\right) x + \left(\!\!\binom{n}{2}\!\!\right) x^2 + \cdots + \left(\!\!\binom{n}{n}\!\!\right) x^n + \ldots.$$

In what follows we assume that each element in S has multiplicity infinity. Denote this multiset as $S^\infty$.

### 1.4 Down-closed property

For a family $\mathcal{F}$ of subsets of $[n]$, we define the **down-closed property** as follows:

$$\text{For any } \sigma \subseteq \tau \in \mathcal{F}, \text{ then } \sigma \in \mathcal{F}.$$

### 1.5 Euler characteristic

## 2 Stanley-Reisner Ring

### 2.1 Simplicial complex

### 2.2 Hilbert function

### 2.3 Cohen-Macaulay property

### 2.4 Upper Bound Theorem

### 2.5 Lefschetz property

### 2.6 G-theorem

## References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