Notes on Unimodality of Hypersimplex

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1 Introduction

1.1 Hypersimplex

Fix a positive integer n, and let $[n] := \{1, 2, \cdots, n\}$. To any subset $S \subseteq [n]$, we associate the indicator vector:

$$\chi_S = (\chi_S(1), \chi_S(2), \dots, \chi_S(n))$$

where

$$\chi_S(i) = \begin{cases} 1, & i \in S \\ 0, & i \notin S \end{cases}$$

For $0 < k < n$, let $\binom{[n]}{k}$ be the family of all k-subsets of $[n]$. The *hypersimplex* $\Delta_{k,n} \subseteq \mathbb{R}^n$ is the convex hull of the indicator vectors χ_I for $I \in \binom{[n]}{k}$. Equivalently,

$$\Delta_{k,n} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \leqslant x_i \leqslant 1, \sum_{i=1}^{n} x_i = k \right\}$$

1.2 Stanley's Triangulations

1.2.1 The Original Triangulation

Let's break down the original idea that Stanley used to bridge the triangulations of polytopes $R_{n,k}$ and $S_{n,k}$ whose definitions will be given below.

Definition of $R_{n,k}$ and $S_{n,k}$.

- $R_{n,k}$ is the *regular* hypersimplex, defined as:

$$R_{n,k} = \left\{ (x_1, \dots, x_n) \in [0,1]^n \mid 0 \leqslant x_i \leqslant 1, \sum_{i=1}^{n} x_i = k \right\}$$

- $S_{n,k}$ is the hypersimplex with k rises (with the convention that $y_0 = 0$):

$$S_{n,k} = \{(y_1, \dots, y_n) \in [0,1]^n \mid y_0 = 0, \ |\{i : y_{i-1} < y_i\}| = k\}$$

Clearly, $R_{n,k}$ is the hypersimplex $\Delta_{k,n}$ we defined above, while $S_{n,k}$ is a polytope where we count the number of rises in the coordinates. Both have the same volume which is equal to $\frac{A_{n,k}}{n!}$, where $A_{n,k}$ is the Eulerian number counting the number of permutations of $[n]$ with k rises.

Before we proceed to the triangulation, let's clarify with an example what the polytope $S_{n,k}$ looks like. For $n = 3$ and $k = 2$, interior of the polytope $S_{3,2}$ consists of points (y_1, y_2, y_3) such that:

$$\begin{aligned} &0 = y_0 < y_1 < y_3 < y_2 < 1, \quad 0 = y_0 < y_3 < y_1 < y_2 < 1, \\ &0 = y_0 < y_2 < y_1 < y_3 < 1, \quad 0 = y_0 < y_2 < y_3 < y_1 < 1. \end{aligned}$$

What we get is actually a triangulation of the polytope $S_{3,2}$. It is natural to index each of these simplices by the permutation of [3] that gives the order of the coordinates, i.e., the first one corresponds to the identity permutation $(1, 2, 3)$, the second one corresponds to the permutation $(3, 1, 2)$, and so on. We will come back to this idea later in Lam and Postnikov's reformulation of the triangulation.

1.2.2 The Volume-Preserving Map

The question Stanley tried to answer is whether there is a map from $S_{n,k}$ to $R_{n,k}$ that preserves the volume. The answer is yes, and the map is given by:

$$\begin{aligned} \phi : \quad & S_{n,k} \longrightarrow R_{n,k} \\ & (y_1, \ldots, y_n) \longmapsto (x_1, \ldots, x_n) \end{aligned}$$

where

$$x_i = \begin{cases} y_{i-1} - y_i, & y_i < y_{i-1} \\ 1 + y_{i-1} - y_i, & y_i > y_{i-1}. \end{cases}$$

This map is piecewise linear, and we can see from the map that the x_i coordinates record the rises (ascents) or falls (descents) of the y_i coordinates. By two basic facts in permutation theory: (1) the number of rises in a permutation (with the convention that the first element is always a rise) plus the number of its falls is always n, and (2) the number of rises in a permutation equals the number of falls of its reverse, we can infer that the coordinates in the polytope $S_{n,k}$ have $n - 1 - (n - k) = k - 1$ descents.

1.2.3 Lam and Postnikov's Reformulation

Lam and Postnikov [1] reformulated the triangulation of $S_{n,k}$ in terms of permutations. Consider the polytopes

$$\nabla_w = \{(y_1, \ldots, y_{n-1}) \in [0, 1]^{n-1} \mid 0 < y_{w(1)} < \cdots < y_{w(n-1)} < 1\}$$

where w is a permutation of $[n - 1]$. The question is: Is ∇_w a simplex?

Step-by-Step Explanation

1. **What is a simplex?** A simplex is the simplest generalization of a triangle or tetrahedron to higher dimensions:

   - 1-simplex: line segment (2 vertices)
   - 2-simplex: triangle (3 vertices)
   - 3-simplex: tetrahedron (4 vertices)
   - In general, an $(n-1)$-simplex in $\mathbb{R}^{n-1}$ is the convex hull of n affinely independent points.

   Formally,

   $$\operatorname{conv}(v_0, \dots, v_d) = \left\{ \sum_{i=0}^{d} \lambda_i v_i \mid \lambda_i \geqslant 0, \sum_{i=0}^{d} \lambda_i = 1 \right\}$$

2. **Connecting our definition to a simplex.**
   The set is defined by strict inequalities:

   $$0 < y_{w(1)} < \cdots < y_{w(n-1)} < 1$$

   The closure allows equality:

   $$0 \leqslant y_{w(1)} \leqslant \cdots \leqslant y_{w(n-1)} \leqslant 1$$

3. **Identifying the vertices.**
   The vertices are:

   $$(0, 0, \dots, 0), \quad e_{w(1)}, \quad e_{w(1)} + e_{w(2)}, \quad \dots, \quad (1, 1, \dots, 1)$$

   where e_i is the i-th unit vector.

4. **Affine independence.**
   Each vertex introduces a new coordinate direction, so they are affinely independent.

5. **Geometric intuition.**
   The region is a "slice" of the cube, forming a staircase-like path, which is a simplex.

Summary

1. The set identifies n vertices at cube corners.

2. There are n vertices, forming an $(n-1)$-simplex.

3. Vertices are affinely independent.

Thus, ∇_w is a simplex.

1.3 Volume and Ehrhart series

The volume of the hypersimplex $\Delta_{k,n}$ is given by the Eulerian number $A_{n-1,k-1}$ divided by $(n-1)!$. The Ehrhart series of $\Delta_{k,n}$ is given by:

$$\mathrm{Ehr}(\Delta_{k,n}, t) = \frac{\sum_{j=0}^{k-1} A_{n,j} t^j}{(1-t)^n}$$

where $A_{n,j}$ are the Eulerian numbers.

References

[1] T. Lam and A. Postnikov, "Alcoved polytopes I," Discrete and Computational Geometry, vol. 38, no. 3, pp. 453–478, 2007.