Notes – From Face Ring to Partition Complex

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0 Background

This note is constantly evolving. The ultimate goal is to develop an understanding of the field **combinatorial commutative algebra**.

This note is very personal. The use of diagrams, analogies, stories, and other means is only a shadow of how the author thinks or likes to think/understand.

1 Sets and Multisets

1.1 Sets and subsets

Let

$$E = \{x_1, x_2, \ldots, x_n\}$$

We will abbreviate or identify it with

$$[n] := \{1, 2, \ldots, n\}.$$

Definition 1.1.1. For a subset $I \subseteq E$, the **characteristic function** of I is defined as:

$$\delta_I : E \to \{0, 1\}$$

where

$$\delta_I(i) = \begin{cases} 1, & \text{if} \quad i \in I, \\ 0, & \text{if} \quad i \notin I. \end{cases}$$

Identify I with the monomial $\prod\limits_{i \in I} x_i$ or its characteristic vector $\nu_I = (\delta_I(1), \delta_I(2), \ldots, \delta_I(n))$.

Definition 1.1.2. A k**-subset** of E is a subset $I \subseteq E$ whose cardinality is k.

1.2 Multisets

Fix $E = [n]$.

Definition 1.2.1. A **multiset** M is a pair (E, δ_E), where

$$\delta_E(i) = \begin{cases} k_i & \text{if } i \in E, \\ 0 & \text{if } i \notin E, \end{cases}$$

where $k_i \in \mathbb{Z}_{>0}$ is the **multiplicity** of i.

We call $S = \{i \in E \mid \delta_E(i) \neq 0\}$ the **support** of the multiset, written as $\operatorname{supp}(M) = S$.

Example 1.2.2. Let $E = \{1, 2, 3\}$, and

$$\delta_E(1) = 4, \quad \delta_E(2) = 0, \quad \delta_E(3) = 1.$$

The multiset is represented as: $M = \{1, 1, 1, 1, 3\}$, and supp(M) = (1,3).

Remark 1.2.3. A submultiset is subset of a multiset, and a k**-submultiset** is a submultiset whose cardinality is k.

Given $I \subseteq E$, we want to characterize the k-submultisets with support I.

Take $I = \{1, 3\} \subseteq [4]$ as an example. What are the 4-submulsets with support I? Let's list them all: {1,1,1,3}, {1,1,3,3}, {1,3,3,3}. We have 3 4-submultisets with support $\{1, 3\}$.

Lemma 1.2.4. Let $I \subseteq E = [n]$ and $m = |I|$. The set of k-submultisets with support I are in bijection with the set $\{(z_1, z_2, \ldots, z_m) \in \mathbb{Z}^m_{\geqslant 1} \mid \sum_{i=1}^m z_i = k\}$.

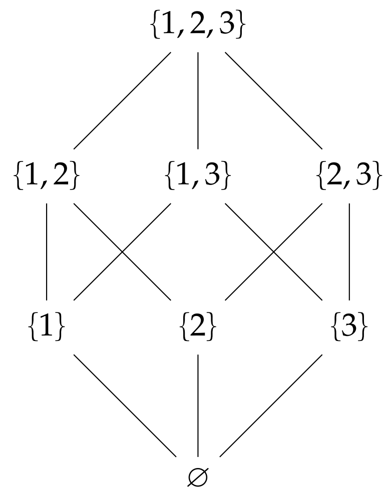
Corollary 1.2.5. Let $I \subseteq E = [n]$ and $m = |I|$. The number of k-submultisets with support I is $\binom{k-1}{m-1}$.

1.3 Graded structures and reciprocity

The set of subsets of E forms a finite **graded Boolean lattice** B_n with the rank function:

$$\mathrm{rk} : 2^E \to \mathbb{Z}_{\geqslant 0}, \quad I \mapsto |I| \quad \text{(cardinality)}.$$

Example 1.3.1. Let $E = [3]$ and we have the Hasse diagram of B_3:



The generating function of the Boolean lattice B_n is

$$F(x_1, x_2, \ldots, x_n) = 1 + \sum_{i=1}^{n} x_i + \sum_{i<j} x_i x_j + \cdots + x_1 x_2 \cdots x_n = \prod_{i=1}^{n}(1 + x_i)$$

Let $x_1 = x_2 = \cdots = x_n = x$. The generating function becomes:

$$F(x, x, \ldots, x) = (1+x)^n := \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

and we get the definition of **binomial coefficients**, the most familiar counting function.

Definition 1.3.2. $\binom{n}{k}$ is the number of k-subsets in $[n]$.

Remark 1.3.3. The vector $\left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}\right)$ is the rank vector (or face vector) of the Boolean lattice (or simplicial complex) B_n.

All submultisets of a multiset form a graded lattice $\mathcal{L}^\infty$. The generating function of the Boolean lattice $\mathcal{L}^\infty$ is

$$F(x_1, x_2, \ldots, x_n) = 1 + \sum_{i=1}^{\infty} x_i + \sum_{i<j} x_i x_j + \cdots = \prod_{i=1}^{n} \frac{1}{1 - x_i}$$

Let $x_1 = x_2 = \cdots = x_n = x$. The generating function becomes:

$$F(x, x, \ldots, x) = \frac{1}{(1-x)^n} := \left(\!\!\binom{n}{0}\!\!\right) + \left(\!\!\binom{n}{1}\!\!\right)x + \left(\!\!\binom{n}{2}\!\!\right)x^2 + \cdots + \left(\!\!\binom{n}{n}\!\!\right)x^n + \ldots.$$

In what follows we assume that each element in S has multiplicity infinity. Denote this multiset as S^∞.

1.4 Down-closed property

For a family $\mathcal{F}$ of subsets of $[n]$, we define the **down-closed property** as follows:

For any $\sigma \subseteq \tau \in \mathcal{F}$, then $\sigma \in \mathcal{F}$.

1.5 Euler characteristic

In combinatorial topology, the **Euler characteristic** is a topological invariant, a number that describes a topological space's structure regardless of how it is bent or stretched. For a finite simplicial complex K, the Euler characteristic $\chi(K)$ is defined as the alternating sum of the number of faces of each dimension:

$$\chi(K) = f_0 - f_1 + f_2 - f_3 + \cdots + (-1)^d f_d,$$

where f_i is the number of i-dimensional faces in K, and d is the highest dimension of a face in K.

Example 1.5.1.

1. For a single point, $f_0 = 1$, $f_i = 0$ for $i > 0$. So $\chi(\text{point}) = 1$.
2. For a line segment, $f_0 = 2$ (endpoints), $f_1 = 1$ (the segment itself). So $\chi(\text{segment}) = 2 - 1 = 1$.
3. For a circle (a 1-dimensional complex, e.g., a triangle's boundary), $f_0 = 3$, $f_1 = 3$. So $\chi(\text{circle}) = 3 - 3 = 0$.
4. For the boundary of a tetrahedron (a sphere-like surface), $f_0 = 4$ (vertices), $f_1 = 6$ (edges), $f_2 = 4$ (triangular faces). So $\chi(\text{sphere}) = 4 - 6 + 4 = 2$.

The Euler characteristic is a fundamental invariant that connects combinatorial properties (number of faces) with topological properties of the space. It plays a crucial role in various theorems, such as Euler's formula for polyhedra ($V - E + F = 2$) and the Gauss-Bonnet theorem in differential geometry. In the context of graded structures like the Boolean lattice, the Euler characteristic can often be related to properties of their generating functions or rank vectors.

2 Stanley-Reisner Ring

2.1 Simplicial complex

2.2 Hilbert function

2.3 Cohen-Macaulay property

2.4 Upper Bound Theorem

2.5 Lefschetz property

2.6 G-theorem

References

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