Notes on Characteristic Polytope

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1 Characteristic Polytope

1.1 Characteristic Polytope of a Simplicial Complex Δ

Given a simplicial complex Δ over $[n]$, we associate to each face $\tau \in \Delta$ its characteristic vector as follows:

$$\chi_\tau = (\delta_\tau(1), \delta_\tau(2), \dots, \delta_\tau(n))$$

where $\delta_\tau(i) = \begin{cases} 1, & i \in \tau \\ 0, & i \notin \tau \end{cases}$.

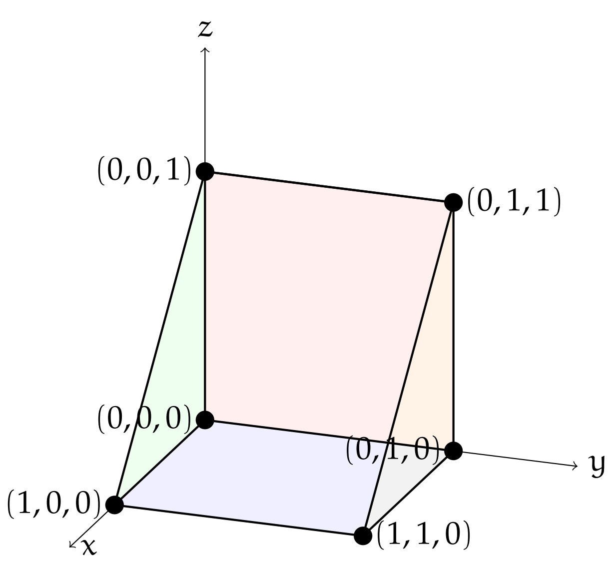
Definition 1.1.1. The **characteristic polytope** $P_\chi(\Delta) \subset \mathbb{R}^n$ of Δ is the convex hull of its characteristic vectors:

$$P_\chi(\Delta) = \text{conv}\{\chi_\tau \mid \tau \in \Delta\}.$$

Example 1.1.2. $\Delta = \{\emptyset, 1, 2, 3, 12, 23\}$, and we have the characteristic vectors:

$$\begin{aligned} \chi_\emptyset &= (0,0,0) & \chi_1 &= (1,0,0) \\ \chi_2 &= (0,1,0) & \chi_3 &= (0,0,1) \\ \chi_{12} &= (1,1,0) & \chi_{23} &= (0,1,1) \end{aligned}$$

Then $P_\chi(\Delta) = \text{conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1)\}$, which is depicted as follows.



Remark 1.1.3. It is direct to see that the characteristic vectors of 0-dimensional faces are the standard basis vectors e_i's. In general, $\chi_\emptyset = \mathbf{0}$ and $\chi_\tau = \sum_{j \in \tau} e_j$ for $\emptyset \neq \tau \in \Delta$.

1.2 Characteristic Polytope of a Polytope Q

Given a polytope $Q \subset \mathbb{R}^d$ with vertices $[n]$, we associate to each flat τ in its lattice of flats L_Q a characteristic vector in the same way:

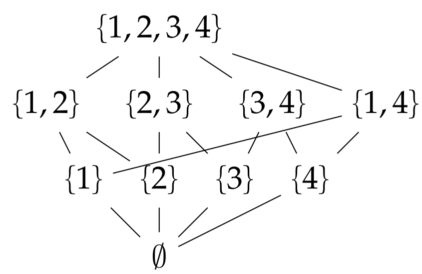
$$\chi_\tau = (\delta_\tau(1), \delta_\tau(2), \dots, \delta_\tau(n))$$

where $\delta_\tau(i) = \begin{cases} 1, & i \in \tau \\ 0, & i \notin \tau \end{cases}$.

Definition 1.2.1. The **characteristic polytope** $P_\chi(Q) \subset \mathbb{R}^n$ of Q is the convex hull of its characteristic vectors:

$$P_\chi(Q) = \text{conv}\{\chi_\tau \mid \tau \in L_Q\}.$$

Example 1.2.2. Let Q be a 4-gon, and the lattice of flats L_Q is as follows:



We have the corresponding characteristic vectors:

$$\begin{aligned} \chi_\emptyset &= (0,0,0,0) & \chi_{12} &= (1,1,0,0) \\ \chi_1 &= (1,0,0,0) & \chi_{23} &= (0,1,1,0) \\ \chi_2 &= (0,1,0,0) & \chi_{34} &= (0,0,1,1) \\ \chi_3 &= (0,0,1,0) & \chi_{14} &= (1,0,0,1) \\ \chi_4 &= (0,0,0,1) & \chi_{1234} &= (1,1,1,1) \end{aligned}$$

$P_\chi(Q)$ is the convex hull of the above vectors and it is 4-dimensional.

1.3 Volume and Ehrhart series

We are mainly interested in the volume and Ehrhart seires of the characteristic polytopes. We list the volume and Ehrhart series of n-gon up to 5-gon as follows:

- 3-gon:

$$\text{vol } P_\chi(Q_3) = 1,$$
$$\text{Ehr}_{P_\chi(Q_3)}(z) = \frac{1 + 4z + z^2}{(1-z)^3}.$$

- 4-gon:

$$\begin{aligned} \text{vol } P_\chi(Q_4) &= \frac{1}{2}, \\ \text{Ehr}_{P_\chi(Q_4)}(z) &= \frac{1+5z+5z^2+z^3}{(1-z)^4}. \end{aligned}$$

- 5-gon:

$$\begin{aligned} \text{vol } P_\chi(Q_5) &= \frac{5}{24}, \\ \text{Ehr}_{P_\chi(Q_5)}(z) &= \frac{1+6z+11z^2+6z^3+z^4}{(1-z)^5}. \end{aligned}$$

From the examples we see unimodality in the h^*-polynomial, and it is natural to conjecture that:

Conjecture 1.3.1. The coefficients h_i^* in the h^*-polynomial of the Ehrhart series of characteristic polytopes of n-gon form a unimodal sequence.

2 Characteristic Polytope of d-cube $\square_d$

2.1 Constructing by tensoring

The face lattice in example 1.2.2 is also the face lattice for 2-cube, and we could already see that it is hard to visualize the characteristic polytope. An algorithm to construct the characteristic polytope of d-cube is provided as follows.

- Step 1: Start with the $(d-1)$-dim simplicial complex Δ over $[d]$. Associate each face $\tau \in \Delta$ with the corresponding standard basis vector $e_\tau \in \mathbb{R}^{2^d}$.
- Step 2: For $\sigma_1, \sigma_2, \rho \in \Delta$, define

$$\begin{aligned} e_{[\sigma_1,\sigma_2]} &:= \sum_{\sigma_1 \subset \tau \subset \sigma_2} e_\tau, \\ e_{[\sigma_1,\sigma_2]} \otimes e_\rho &:= e_{[\sigma_1 \cup \rho, \sigma_2 \cup \rho]}. \end{aligned}$$

  Specifically, $e_\emptyset \otimes e_\rho = e_\rho$.

- Step 3: Let $P_\chi(\square_d) := \text{conv}\{\mathbf{0}, \bigsqcup_{\substack{\tau \in \Delta \\ \rho \in \text{lk}_\tau(\Delta)}} e_{[\emptyset,\tau]} \otimes e_\rho\}$, and we claim that:

Lemma 2.1.1. $P_\chi(\square_d)$ is the characteristic polytope of the d-cube $\square_d$.

Proof. It is direct from the construction above.

We illustrate the construction with the following example.

Example 2.1.2. Let $d = 2$, and we have the simplicial complex $\Delta = \{\emptyset, 1, 2, 12\}$. The corresponding basis vectors are: $e_\emptyset = (1,0,0,0), e_1 = (0,1,0,0), e_2 = (0,0,1,0), e_{12} = (0,0,0,1)$. By tensoring we have:

$$
\begin{aligned}
&e_\emptyset \otimes e_\emptyset = e_\emptyset = (1,0,0,0) := v_1\\
&e_\emptyset \otimes e_1 = e_1 = (0,1,0,0) := v_2\\
&e_\emptyset \otimes e_2 = e_2 = (0,0,1,0) := v_3\\
&e_\emptyset \otimes e_{12} = e_{12} = (0,0,0,1) := v_4\\
&e_{[\emptyset,1]} \otimes e_\emptyset = e_{[\emptyset,1]} = e_\emptyset + e_1 = (1,1,0,0) := v_5\\
&e_{[\emptyset,1]} \otimes e_2 = e_{[2,12]} = e_2 + e_{12} = (0,0,1,1) := v_6\\
&e_{[\emptyset,2]} \otimes e_\emptyset = e_{[\emptyset,2]} = e_\emptyset + e_2 = (1,0,1,0) := v_7\\
&e_{[\emptyset,2]} \otimes e_1 = e_{[1,12]} = e_1 + e_{12} = (0,1,0,1) := v_8\\
&e_{[\emptyset,12]} \otimes e_\emptyset = e_{[\emptyset,12]} = e_\emptyset + e_1 + e_2 + e_{12} = (1,1,1,1) := v_9
\end{aligned}
$$

We then get the characteristic polytope $P_\chi(\Box_2) = \text{conv}\{\mathbf{0}, v_i \mid i \in [9]\}$, which is the same as in example 1.2.2.

2.2 Face vector of the characteristic polytope

Based on the construction, we now analyze the face vector of the characteristic polytope of the d-cube.

Recall that we get the characteristic polytope $P_\chi(\Box_d)$ from the simplicial complex Δ_d. By tensoring the empty set with its link in Δ_d we get the vertices of $P_\chi(\Box_d)$, and thus we have

$$f_0(P_\chi(\Box_d)) = \binom{d}{0} 2^d = 2^d.$$

Similarly, number of i-faces of $P_\chi(\Box_d)$ can be expressed as

$$f_i(P_\chi(\Box_d)) = \binom{d}{i} 2^{d-i},$$

and we have the face vector $f = (1, 2^d, \dots, \binom{d}{i} 2^{d-i}, 2d, 1)$.

Remark 2.2.1. Note that $\sum_{i=0}^{d} f_i = 3^d$, which means that $P_\chi(\Box_d)$ has $3^d + 1$ vertices.

2.3 Volume and Ehrhart series

We have calculated the volume and Ehrhart series for $P_\chi(\Box_2)$, which is

$$\text{vol } P_\chi(\Box_2) = \frac{1}{2},$$

$$\mathrm{Ehr}_{P_\chi(\square_2)}(z) = \frac{1+5z+5z^2+z^3}{(1-z)^4}.$$

When $d = 3$, $\dim P_\chi(\square_3) = 2^3 = 8$. Again using Mathematica and Macaulay2 we get the results as follows

$$\mathrm{vol}\ P_\chi(\square_3) = \frac{59}{2520},$$
$$\mathrm{Ehr}_{P_\chi(\square_3)}(z) = \frac{1+19z+127z^2+321z^3+329z^4+127z^5+19z^6+z^7}{(1-z)^8}.$$

For $d > 3$, the computation is beyond our resource. But we notice the pattern again: the coefficients of the h^*-polynomial are unimodal/top-heavy.

Conjecture 2.3.1. The coefficients in the h^*-polynomial of the characteristic polytope of d-cube are unimodal/top-heavy.

Another question is to find a formula for the volume. The characteristic polytope is similar to the *independent set polytope* I_M [ABD10] of a matroid M, but it differs from I_M as the latter has vertices in each slice of the hyperplane while the vertices of the former are only on certain slices.

References

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