Notes on Characteristic Polytope

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1 Characteristic Polytope

1.1 Characteristic vectors

Fix a positive integer n, and let $[n] := \{1, 2, \cdots, n\}$. To any subset $S \subseteq [n]$, we associate the *characteristic vector*:

$$\chi_S = (\chi_S(1), \chi_S(2), \dots, \chi_S(n))$$

where

$$\chi_S(i) = \begin{cases} 1, & i \in S \\ 0, & i \notin S \end{cases}$$

Given a simplicial complex Δ over $[n]$, we associate to each face $\tau \in \Delta$ its characteristic vector χ_τ and get the characteristic polytope as follows:

Definition 1.1.1. The **characteristic polytope** $P_\chi(\Delta) \subset \mathbb{R}^n$ of Δ is the convex hull of its characteristic vectors:

$$P_\chi(\Delta) = \text{conv}\{\chi_\tau \mid \tau \in \Delta\}.$$

1.2 Characteristic polytope of a graph

Given a graph G with vertices $[n]$, and we view the graph as a simplicial complex. For any graph we can build a characteristic polytope as defined above. In what follows we will first do a few examples and then focus on two graphs, namely the complete graph K_n and the cycle graph C_n.

1.2.1 Example: Complete graph K_n

The complete graph K_n has all subsets of $[n]$ as its faces. The characteristic polytope of K_n is given by:

$$P_\chi(K_n) = \text{conv}\{\chi_S \mid S \subseteq [n]\}.$$

1.2.2 Example: Cycle graph C_n

The cycle graph C_n has faces corresponding to all subsets of vertices that induce a connected subgraph. The characteristic polytope of C_n is given by:

$$P_\chi(C_n) = \text{conv}\{\chi_S \mid S \subseteq [n], \text{ induces a connected subgraph}\}.$$

1.3 Volume and Ehrhart series

We are mainly interested in the volume and Ehrhart series of the characteristic polytopes. We list the volume and Ehrhart series of n-gon up to 5-gon as follows:

- 3-gon:

$$\mathrm{vol}\, P_\chi(Q_3) = 1,$$
$$\mathrm{Ehr}_{P_\chi(Q_3)}(z) = \frac{1+4z+z^2}{(1-z)^3}.$$