APPENDIX TO 'A POLYNOMIAL ANALOGUE OF THE TWIN PRIME CONJECTURE'

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Appendix: An Upper Bound for Twin Prime Pairs in $\mathbf{F}_q[T]$

In this section we establish the following estimate:

Lemma 1. *Let $n \geq 2$ be an integer, and let $M \neq 0$ be a polynomial of degree $< n$ over the finite field $\mathbf{F}_q$. Then*

$$\#\{P : P, P+M \text{ are both monic irreducibles of degree } n\} \leq 8 \frac{|M|}{\phi(M)} \frac{q^n}{n^2}.$$

As a corollary, we have

$$R(n; M, q) \leq 8 \frac{|M|}{\phi(M)} \frac{q^{n+1}}{n^2},$$

whenever $0 \leq \deg M < n$. For the proof, we begin with a statement of Selberg's upper-bound sieve in the polynomial setting (cf. [2, Theorem 1]).

Lemma 2 (Selberg's Λ^2-sieve for $\mathbf{F}_q[T]$)**.** *Let $\mathcal{A}$ be a multiset of polynomials over $\mathbf{F}_q$, and let $\mathcal{P}$ be a finite set of monic irreducibles over $\mathbf{F}_q$. Suppose that f is a multiplicative function defined on the squarefree divisors of $\prod_{P \in \mathcal{P}} P$ with $1 < f(P) \leq |P|$ for each $P \in \mathcal{P}$, and write*

$$\sum_{\substack{A \in \mathcal{A} \\ D \mid A}} 1 = \frac{\#\mathcal{A}}{f(D)} + R_D. \tag{1}$$

Let $\mathcal{D}$ be any subset of the monic divisors of $\prod_{P \in \mathcal{P}} P$ which is divisor closed (i.e., every monic divisor of an element of $\mathcal{D}$ belongs to $\mathcal{D}$). Then

$$\sum_{\substack{A \in \mathcal{A} \\ \gcd(A, \prod_{P \in \mathcal{P}} P) = 1}} 1 \leq \frac{\#\mathcal{A}}{\sum_{D \in \mathcal{D}} f(D)^{-1} \prod_{P \mid D} \left(1 - \frac{1}{f(P)}\right)^{-1}} + \sum_{D_1, D_2 \in \mathcal{D}} |X_{D_1} X_{D_2} R_{[D_1, D_2]}|,$$

where

$$X_D = \mu(D) f(D) \frac{\sum_{C \in \mathcal{D}, D \mid C} f(C)^{-1} \prod_{P \mid C} \left(1 - \frac{1}{f(P)}\right)^{-1}}{\sum_{C \in \mathcal{D}} f(C)^{-1} \prod_{P \mid C} \left(1 - \frac{1}{f(P)}\right)^{-1}}.$$

Proof of Lemma 1. In the case when $q = 2$, we may assume that $T(T+1)$ divides M, since since otherwise there are no prime pairs $P, P+M$ of degree n. Define the multiset

$$\mathcal{A} := \{A(A+M) : A \text{ monic}, \deg A = n\}.$$

Let $\mathcal{P}$ be the set of monic primes of degree $< n/2$ not dividing M. Then the number of monic, degree n prime pairs $P, P+M$ is at most the number of elements of $\mathcal{A}$ coprime to $\prod_{P\in\mathcal{P}} P$, a quantity which may be estimated with Lemma 2.

We take $\mathcal{D}$ to be the (divisor-closed) set of squarefree, monic polynomials of degree $< n/2$ supported on $\mathcal{P}$. We define the multiplicative function f appearing in Lemma 2 by setting $f(P) = |P|/2$ for $P \in \mathcal{P}$ and extending by multiplicativity. It is easy to check that if the squarefree polynomial D has degree $< n$ and is supported on $\mathcal{P}$, then (1) holds without any error term, i.e., with $R_D = 0$. Since the least common multiple of any pair $D_1, D_2 \in \mathcal{D}$ has degree $< n$, we obtain from Lemma 2 the following clean inequality:

$$\sum_{\substack{A\in\mathcal{A}\\ \gcd(A,\prod_{P\in\mathcal{P}} P)=1}} 1 \le \frac{\#\mathcal{A}}{\sum_{D\in\mathcal{D}} f(D)^{-1}\prod_{P|D}\left(1-\frac{1}{f(P)}\right)^{-1}}. \tag{2}$$

To proceed we need a lower bound on the denominator in this expression. For each $D \in \mathcal{D}$, we have

$$f(D)^{-1}\prod_{P|D}\left(1-\frac{1}{f(P)}\right)^{-1} = \prod_{P|D}\frac{2}{|P|-2},$$

and so we have reduced the problem to obtaining a lower bound on

$$\begin{aligned}\sum_{D\in\mathcal{D}}\prod_{P|D}\frac{2}{|P|-2} &= \sum_{D\in\mathcal{D}}\prod_{P|D}\left(\frac{2}{|P|}+\frac{4}{|P|^2}+\frac{8}{|P|^3}+\dots\right)\\ &= \sum_{\substack{A \text{ monic,}\\ \text{supported on } \mathcal{P}}}\frac{2^{\Omega(A)}}{|A|}\sum_{\substack{D\in\mathcal{D}\\ \mathrm{rad}(A)=D}} 1.\end{aligned}$$

The inner sum is positive whenever $\deg A < n/2$, and so we have a lower bound of

$$\sum_{\substack{A \text{ monic}, \deg A<n/2\\ \gcd(A,M)=1}}\frac{2^{\Omega(A)}}{|A|} \ge \sum_{\substack{A \text{ monic}, \deg A<n/2\\ \gcd(A,M)=1}}\frac{d(A)}{|A|}.$$

Now

$$\sum_{\substack{A \text{ monic}, \deg A<n/2\\ \gcd(A,M)=1}}\frac{d(A)}{|A|}\prod_{P|M}\left(1+\frac{d(P)}{|P|}+\frac{d(P^2)}{|P|^3}+\dots\right) \ge \sum_{A \text{ monic}, \deg A<n/2}\frac{d(A)}{|A|}.$$

Since

$$1+\frac{d(P)}{|P|}+\frac{d(P^2)}{|P|^3}+\dots = 1+\frac{2}{|P|}+\frac{3}{|P|^2}+\dots = \frac{1}{(1-1/|P|)^2},$$

we find that

$$\sum_{\substack{A \text{ monic}, \deg A<n/2\\ \gcd(A,M)=1}}\frac{d(A)}{|A|} \ge \left(\prod_{P|M}\left(1-\frac{1}{|P|}\right)\right)^2\sum_{A \text{ monic}, \deg A<n/2}\frac{d(A)}{|A|}.$$

The product on the right-hand side is just $\phi(M)/|M|$. Combining this with Carlitz's result that $\sum_{\deg A=k} d(A) = (k+1)q^k$ (see [1]), we find a lower bound on the denominator in (2) of

$$\frac{\phi(M)^2}{|M|^2} \sum_{k<n/2} (k+1) \geq \frac{\phi(M)^2}{|M|^2} \frac{n^2}{8},$$

say. Since the numerator in (2) is

$$\#\mathcal{A} = \#\{A \text{ monic, degree } n, \gcd(A, M) = 1\} = q^n \frac{\phi(M)}{|M|},$$

we obtain the stated result. □

References

1. L. Carlitz, *The arithmetic of polynomials in a Galois field*, Proc. Nat. Acad. Sci. U. S. A. **17** (1931), 120–122.
2. W. A. Webb, *Sieve methods for polynomial rings over finite fields*, J. Number Theory **16** (1983), no. 3, 343–355. MR 84j:12021

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