MATH 4000/6000

Practice problems for Exam 2 on Chapter 3 material

Throughout, F denotes a field.

Getting comfortable with the definitions

1. Show that if $p(x)$ is irreducible in $F[x]$, then so is $c \cdot p(x)$ for any nonzero $c \in F$.
2. Show that if $a(x), b(x) \in F[x]$ and $d(x)$ is a gcd of $a(x)$ and $b(x)$, then so is $c \cdot d(x)$ for any nonzero $c \in F$.
3. Suppose that $F \subseteq K$, where F and K are both fields. In class, we defined $F[\alpha]$ for every $\alpha \in K$.

   Show that if in fact $\alpha \in F$, then $F[\alpha] = F$. (For example, $\mathbb{Q}[\frac{2}{3}] = \mathbb{Q}$, and $\mathbb{R}[\pi] = \mathbb{R}$.)
4. We know $F[x]$ is a domain. Is $F[x]$ a field? *Hint:* Does x have an inverse?

Further practice problems

1. Improving on the result of Problem 4 above, show that the units in $F[x]$ are precisely the nonzero constant polynomials.
2. Recall that every nonconstant polynomial in $\mathbb{C}[x]$ has a root in $\mathbb{C}$. (This is the **fundamental theorem of algebra**.)

   Suppose that $f(x) \in \mathbb{C}[x]$ is a polynomial with degree $n \geq 1$.

   (a) Show that there are complex numbers $r_1, \ldots, r_n$ with
   $$f(x) = (x - r_1) \cdots (x - r_n).$$
   Hint: Apply the root factor theorem n times.

   (b) Show that the numbers $r_1, \ldots, r_n$ in part (a) are precisely the complex roots of $f(x)$.

   (c) Now write $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Show that $r_1 + r_2 + \cdots + r_n = -a_{n-1}$.
3. Let p be a prime number. Show that every element of $\mathbb{Z}_p$ is a root of the polynomial $f(x) = x^p - x \in \mathbb{Z}_p[x]$. Explain carefully why this implies that $f(x) = x(x - \overline{1})(x - \overline{2})(x - \overline{(p-1)})$.
4. Find polynomials $X(x), Y(x) \in \mathbb{Q}[x]$ with
   $$(x^3 - 2)X(x) + (x^2 + 3x + 1)Y(x) = 1.$$
5. Problem 3.1.10(a,c,e,f) from the textbook.
6. Problem 3.1.18 from the textbook.
7. (a) Suppose $d(x), e(x) \in F[x]$ and $d(x) \mid e(x)$, and $e(x) \mid d(x)$. Show that $d(x) = c \cdot e(x)$ for some nonzero constant $c \in F$.

(b) Suppose that $d(x)$ and $e(x)$ are both gcds of the same pair of polynomials in $F[x]$. Show that $d(x) = c \cdot e(x)$ for some nonzero $c \in F$. (This is a converse to Problem 2 from the first section.)

8. Let $\omega = \cos(2\pi/5) + i\sin(2\pi/5)$. Show that $\mathbb{Q}[\omega]$ is a splitting field for $x^5 - 1$ over $\mathbb{Q}$.

9. Keeping the same definition of ω as in the last problem, show that $\mathbb{Q}[\omega, \sqrt[5]{2}]$ is a splitting field for $x^5 - 2$ over $\mathbb{Q}$.