MATH 3100 – Learning objectives to meet for the final exam

The final exam is <u>cumulative</u> and all of the material discussed during the semester is fair game. **The following review sheet covers only material after Exam #3, from §31 of the Ross text.** You are strongly encouraged to consult earlier review sheets and past exams in your preparation for the final.

What to be able to state (since last exam)

Basic definitions

You should be able to give complete and precise definitions of each of the following items:

- Taylor series of f
- nth Taylor polynomial $P_n^f(x)$
- Taylor series centered at $x = x_0$

Big theorems

Be familiar with all of the following theorems, making sure that you can give full statements for those not already spelled out below. For results proved in class, describe the components and main ideas of the proof.

- If f is represented by a power series (centered at 0) on some interval $(-r, r)$, where $r > 0$, that power series is the Taylor series of f.
- Taylor's theorem: Let n be a nonnegative integer. Suppose f is $(n+1)$-times differentiable on an open interval I containing 0. For every nonzero $x \in I$,

$$f(x) - P_n^f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \quad \text{for some } c \text{ strictly between 0 and } x.$$

- Analogues of the above results for power series centered at $x = x_0$.

What to expect on the final exam

The format of the final exam will be similar to your three midterms but the length will be approximately double: You should expect the number of questions to land in the closed interval $[8, 10]$. **Among other things**, there will be ...

- A problem asking you to establish the value of a limit directly from the definition of a limit.
- A problem asking you to determine convergence/divergence of given series (possibly also absolute or conditional convergence).
- A problem asking you to apply Taylor's theorem.

Practice problems for §31

1. Prove that if $f(x)$ is a polynomial, say $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ (with all the a_i real numbers), then the Taylor series for $f(x)$ is

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + 0x^{n+1} + 0x^{n+2} + \ldots.$$

2. Let $f(x) = x^2/(1-2x^3)$.

   (a) Find the Taylor series of $f(x)$. (Hint: You can do this without computing derivatives of f !)

   (b) Find the exact value of $f^{(5)}(0)$. Justify your answer.

3. Use Taylor's theorem with $f(x) = \cos x$ to show that $\cos x > 1 - x^2/2$ for every x with $0 < x < \pi/2$. Then argue that the same inequality holds for all nonzero numbers x. (For the latter half of the problem, you may take as known that $\pi > 3$.)

4. (a) Using Taylor's theorem, prove that for all $x \in [0,1]$, and all positive integers n, we have

$$\left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| \le e\frac{x^{n+1}}{(n+1)!}.$$

   (b) Write down the inequality obtained by taking $x = 1$ and $n = 2$ in part (a). Use this to show that $e \le 3$.

   (c) Using the results of (a) and (b), show that e is within 10^{-4} of $\sum_{k=0}^{8} 1/k!$. Is e larger or smaller than this sum?

5. (a) Let $f(x) = \sqrt{x}$. Find $f(4), f'(4)$, and $f''(4)$.

   (b) Let $P(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2$. Show that

$$|\sqrt{5} - P(5)| < 2^{-9}.$$

   (c) Is $P(5)$ larger or smaller than $\sqrt{5}$?

   Justify your answers in (b) and (c) using Taylor's theorem.