June 17, 2009

Hi Kevin,

I mentioned before that I have what seems like a proof that almost none of the numbers $\phi(pq)$ are in the range of σ. I now believe I have proofs of the following two additional theorems:

Theorem A. *For each fixed $k \geq 1$, a positive proportion of the totients of the form $\phi(p_1 \cdots p_k)$ are not in the range of σ.*

Theorem B. *The number of totients $\leq x$ not in the range of σ is at least*

$$\frac{x}{\log x} \exp((C + o(1))(\log_3 x)^2),$$

where $C = 0.817...$ is the expected constant.

Of course Theorem B is too imprecise to say anything about the proportion of totients that are σ-values, but I think it's an interesting step in that direction. The proofs of Theorems A and B are based on the same idea, so I will only sketch what I have in mind for B. I haven't included all the details (such as the precise definition of 'normal prime' below) but hopefully it's enough to follow – please let me know if anything looks suspicious!

Proof of Theorem B (sketch). Put $y = \exp((\log x)^{1/4})$. Consider the integers $m \leq x$ of the form

$$m = (p_0 - 1)T,$$

where $p_0 > 2y$ and $T < y$ is a totient with a not too abnormally large number of prime factors, say

$$\Omega(T) < 0.6 \log_2 x.$$

Note that each m constructed in this way is itself a totient. Lemma 2 from the 1976 Erdos–Hall paper guarantees that the number of distinct m is

$$\gg \frac{x}{\log x} \sum \frac{1}{T},$$

which from Maier–Pomerance (or your work) is at least

$$\frac{x}{\log x}\exp((C+o(1)(\log_3 x)^2).$$

(Here we use that $\log_3 y$ is asymptotic to $\log_3 x$.) We claim that such an m is almost never a σ-value. In fact, the argument below shows that the number of such m that are σ-values is $\ll x/(\log x)^{1+\delta}$ for some $\delta > 0$.

Indeed, suppose $m = \sigma(n)$. Let q_0 be the largest prime dividing n; we can assume $q_0 \geq x^{\frac{1}{10\log_2 x}}$ and that $q_0^2 \nmid n$. Write

$$n = q_0 n',$$

and put $S = \sigma(n')$, so that

$$m = (p_0 - 1)T = (q_0 + 1)S. \tag{1}$$

I claim we can assume that S is small, say smaller than $\exp((\log x)^{0.26})$. We prove this claim in two steps: We bound the largest prime factor $P(S)$ of S and then bound the total number $\Omega(S)$ of prime factors of S.

First observe we can assume that each prime q exactly dividing n' is at most $\exp((\log x)^{0.251})$: Indeed, if there is a q which exceeds this bound, then it should be that the number of prime factors of $\sigma(n)$ in

$$[\exp((\log x)^{0.25}), \exp((\log x)^{0.251})]$$

coming from q_0+1 and $q+1$ exceeds the number of prime factors of $(p_0-1)T$ in the same interval by about $0.001\log_2 x$ – otherwise either q_0+1, $q+1$, or $p-1$ is abnormal for a shifted prime. But then either m is a rare σ-value or m is a rare totient, and so we can assume that doesn't happen. In particular, for any q exactly dividing n',

$$P(\sigma(q)) = P(q+1) \leq \max\{3, q\} \leq \exp((\log x)^{0.251}).$$

Moreover, we can assume that the squarefull part of n (and so *a fortiori* n') is smaller than $(\log x)^2$. Thus if q is a prime for which $q^e \parallel n'$ with $e > 1$, then $q^e \leq (\log x)^2$, and so

$$P(\sigma(q^e)) \leq \sigma(q^e) \leq q^{2e} \leq (\log x)^4.$$

Hence

$$P(S) = \max_{q^e \| n'} P(\sigma(q^e)) \le \max\{(\log x)^4, \exp((\log x)^{0.251})\} = \exp((\log x)^{0.251}).$$

We obviously have

$$\Omega(S) \le \Omega(m) = \Omega(p_0 - 1) + \Omega(T) < 2 \log_2 x.$$

(Assuming, as we may, that p_0 is normal in the appropriate sense, and using our assumption on $\Omega(T)$.) So

$$S \le P(S)^{\Omega(S)} \le \exp(2 \log_2 x (\log x)^{0.251}) < \exp((\log x)^{0.26}).$$

Now we are home free! We fix $T, S \le \exp((\log x)^{0.26})$ and count the number of corresponding p_0, q_0 giving equality in (1). The usual sieve argument shows that the number of such p_0, q_0 is

$$\ll \frac{x}{(\log x)^2} \frac{(T, S)}{TS}.$$

Summing over all integers T, S in our range and using a trivial bound, we get

$$\ll \frac{x}{(\log x)^2}((\log x)^{0.26})^3 = \frac{x}{(\log x)^{1.22}}.$$

□

What do you think?

Best regards,
Paul