MATH 3100 – Homework #8

posted November 28, 2022; due by 5 PM on Monday, December 5

Section and exercise numbers correspond to our course notes.

Required problems

1. Recall that functions f and g are said to **agree to order** n **(near** $x = 0$**)** if

$$\lim_{x \to 0} \frac{f(x) - g(x)}{x^n} = 0.$$

Let n be a nonnegative integer. Show that if f, g, and h are functions defined near 0, then ...

(a) f agrees with f to order n,

(b) if f agrees with g to order n, then g agrees with f to order n,

(c) if f agrees with g to order n and g agrees with h to order n, then f agrees with h to order n.

Hence, "agreement to order n" is an equivalencce relation. This was stated in class but not proved in full. Here you are being asked to supply the details.

2. Show that if f is a polynomial, then $P_n^f(x) = [f(x)]_n$ (for each $n = 0, 1, 2, 3, \dots$).

Recall that if $P(x)$ is a polynomial, $[P(x)]_n$ is the truncated polynomial keeping only the terms of degree not exceeding n. Explicitly, if we write $P(x) = c_0 + c_1 x + \dots + c_m x^m$ for some constants $c_0, \dots, c_m$, then $[P(x)]_n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$, with c_j interpreted as 0 when $j > m$.

For all remaining problems, assume the result stated in class that $\sin x$, $\cos x$, and e^x are everywhere equal to the sum of their Maclaurin series.

3. §3.2: 4

4. §3.2: 10

5. §3.2: 11

6. (a) Show that for every nonnegative real number x and every nonnegative integer k, we have $e^x \geq x^k/k!$.

   (b) Use (a) to show that for every nonnegative integer k, we have $k! \geq (k/e)^k$.

   Remark: A beautiful theorem of Stirling, discussed in courses on probability, asserts that $k!$ is very closely approximated by $\sqrt{2\pi k}(k/e)^k$. Here "very closely" means that the relative error in this approximation tends to 0 as k tends to infinity.

Recommended problems

§3.2: 1, 2, 5, 8, 15 (some of these you should only attempt after the discussion in class of Taylor's theorem)