MATH 4400/6400 – Homework #5

posted April 1, 2022; due April 8, by midnight

Any fool can know. The point is to understand.
– Albert Einstein

Directions. Give complete solutions, providing full justifications when appropriate. Your assignment must be stapled if it goes on beyond one page.

MATH 4400 problems

1. For every positive integer a, let $N(a)$ denote the number of integers in the half-open interval $(a, a^2]$ which are divisible by a prime $p > a$.

   (a) Prove that $N(a) = \sum_{a<p\le a^2} \lfloor a^2/p \rfloor$.

   (b) Deduce from (a) and the trivial inequality $N(a) \le a^2$ that $\sum_{a<p\le a^2} \frac{1}{p} \le 2$.

   (c) By using part (b) several times, find an upper bound on $\sum_{2<p\le 2^{32}} \frac{1}{p}$.

2. Let $\mathcal{D}(n) = \{d \in \mathbf{Z}^+ : d \mid n\}$. Let n_1 and n_2 be relatively prime positive integers. Show that the map

$$\begin{aligned} M \colon \mathcal{D}(n_1) \times \mathcal{D}(n_2) &\to \mathcal{D}(n_1 n_2) \\ (d_1, d_2) &\mapsto d_1 d_2 \end{aligned}$$

   is a bijection.

3. Suppose f is a multiplicative function.

   (a) Show that $f(1) = 1$ or $f(1) = 0$.

   (b) Show that if $f(1) = 0$, then $f(n) = 0$ for all positive integers n.

4. Show that if f, g are multiplicative functions with $f(1) = g(1) = 1$, and $f(p^e) = g(p^e)$ for all primes p and all positive integers e, then $f(n) = g(n)$ for all positive integers n.

5. Prove that for all positive integers n,

$$\sum_{e\mid n} d(e)^3 = \left(\sum_{e\mid n} d(e)\right)^2.$$

   Hint. You may assume the formula $\sum_{k=1}^{m} k^3 = (m(m+1)/2)^2$, which could be proved by induction.

6. Recall that Euler's ϕ-function is defined by

$$\phi(n) = \#\{m : 1 \le m \le n \text{ and } \gcd(m, n) = 1\}.$$

   We will show in class that $\phi(n)$ is multiplicative.

   Prove: $\phi(n)\sigma(n) \le n^2$ for all n.

7. Recall from class that $d_k(n) = \#\{(d_1, \ldots, d_k) \in (\mathbf{Z}^+)^k : d_1 \cdots d_k = n\}$. Find a formula for $d_3(n)$ in terms of the prime factorization of n.

8. (a) Classify all n for which $\phi(n)$ is an odd number. Justify your answer.
   (b) Classify all n for which $d(n)$ is an odd number. Justify your answer.
   (c) Classify all n for which $\sigma(n)$ is an odd number. Justify your answer.

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9. (*) (Euler) Prove that if n is odd and $\sigma(n)$ is twice an odd number, then $n = p^\alpha m^2$ for some prime p and some positive integers α and m, where $p \nmid m$. Moreover, $p \equiv \alpha \equiv 1 \pmod 4$.

   Remark. The assumptions of the problem hold, in particular, if n is an *odd perfect number*. This was the context in which Euler proved the result.

10. (*) Prove $\phi(n)\sigma(n) \geq \frac{1}{2}n^2$ for all n.