ON COMMON VALUES OF $\phi(n)$ AND $\sigma(m)$, I

KEVIN FORD AND PAUL POLLACK

ABSTRACT. We show, conditional on a uniform version of the prime k-tuples conjecture, that there are $x/(\log x)^{1+o(1)}$ numbers not exceeding x common to the ranges of ϕ and σ. Here ϕ is Euler's totient function and σ is the sum-of-divisors function.

1. INTRODUCTION

For each positive-integer valued arithmetic function f, let $\mathscr{V}_f \subset \mathbf{N}$ denote the image of f, and put $\mathscr{V}_f(x) := \mathscr{V}_f \cap [1, x]$ and $V_f(x) := \#\mathscr{V}_f(x)$. In this paper we are primarily concerned with the cases when $f = \phi$, the Euler totient function, and when $f = \sigma$, the usual sum-of-divisors function. When $f = \phi$, the study of the counting function V_f goes back to Pillai [14], and was subsequently taken up by Erdős [1, 2], Erdős and Hall [5, 6], Pomerance [15], Maier and Pomerance [12], and Ford [7] (with an announcement in [8]). From the sequence of results obtained by these authors, we mention Erdős's asymptotic formula (from [1]) for $\log \frac{V_f(x)}{x}$, namely

$$V_f(x) = \frac{x}{(\log x)^{1+o(1)}} \quad (x \to \infty) \tag{1}$$

and the much more intricate determination of the precise order of magnitude by Ford,

$$V_f(x) \asymp \frac{x}{\log x} \exp(C(\log_3 x - \log_4 x)^2 + D\log_3 x - (D + 1/2 - 2C)\log_4 x). \tag{2}$$

Here the constants C and D are defined as follows: Let

$$F(z) := \sum_{n=1}^{\infty} a_n z^n, \quad \text{where} \quad a_n = (n+1)\log(n+1) - n\log n - 1. \tag{3}$$

Since each $a_n > 0$ and $a_n \sim \log n$ as $n \to \infty$, it follows that $F(z)$ converges to a continuous, strictly increasing function on $(0, 1)$, and $F(z) \to \infty$ as $z \uparrow 1$. Thus there is a unique real number ρ for which

$$F(\rho) = 1 \quad (\rho = 0.542598586098471021959\ldots). \tag{4}$$

In addition, F' is strictly increasing, and $F'(\rho) = 5.697758\ldots$. Then $C = \frac{1}{2|\log \rho|} = 0.817814\ldots$ and $D = 2C(1 + \log F'(\rho) - \log(2C)) - 3/2 = 2.176968\ldots$. In [7], it is also shown that (2) holds for a wide class of ϕ-like functions, including $f = \sigma$. Consequently, $V_\phi(x) \asymp V_\sigma(x)$.

Erdős (see [3, 8, p. 172] or [4]) asked if it could be proved that infinitely many natural numbers appear in both $\mathscr{V}_\phi$ and $\mathscr{V}_\sigma$. This question was recently answered by Ford, Luca,

The first author was supported by NSF Grant DMS-0901339. The second author was supported by an NSF Postdoctoral Fellowship (award DMS-0802970). The research was conducted in part while the authors were visiting the Institute for Advanced Study, the first author supported by grants from the Ellentuck Fund and The Friends of the Institute For Advanced Study. Both authors thank the IAS for its hospitality and excellent working conditions.

and Pomerance [9]. Writing $V_{\phi,\sigma}(x)$ for the number of common values of $\mathscr{V}_\phi$ and $\mathscr{V}_\sigma$ up to x, they proved that

$$V_{\phi,\sigma}(x) \geqslant \exp((\log\log x)^c)$$

for some positive constant $c > 0$ and all large x (in [10] this is shown for *all* constants $c > 0$). This lower bound is probably very far from the truth. A naive guess, based on (1) and the hypothesis of independence, might be that $V_{\phi,\sigma}(x) = x/(\log x)^{2+o(1)}$. However, the analysis of [7] indicates that elements of $\mathscr{V}_\phi$ and $\mathscr{V}_\sigma$ share many structural features, which suggests that perhaps $V_{\phi,\sigma}$ is larger than this naive prediction.

In this paper we show that this is indeed the case, subject to the following plausible hypothesis:

Hypothesis UL. *Suppose $a_1, \dots, a_h$ are positive integers and $b_1, \dots, b_h$ are integers such that $\prod_{1\leqslant i<j\leqslant h}(a_ib_j - a_jb_i) \neq 0$. Assume that for some constant $A > 0$, we have*

$$\max_{1\leqslant i\leqslant h}\{|a_i|, |b_i|\} \leqslant x^A.$$

Then for large x, depending on A and h, the number of natural numbers $n \leqslant x$ for which $a_in + b_i$ is prime for every $1 \leqslant i \leqslant h$ is

$$\gg_{A,h} C\frac{x}{(\log x)^h}.$$

Here C is the singular series associated to $\{a_in + b_i\}_{i=1}^h$, defined by

$$C := \prod_p \frac{1-\nu(p)/p}{(1-1/p)^h}, \quad \textit{where} \quad \nu(p) := \#\{n \bmod p : \prod_{i=1}^h (a_in + b_i) \equiv 0 \pmod p\}.$$

This hypothesis is a quantitative form of Dickson's prime k-tuples conjecture. The name "Hypothesis UL" (with "L" for linear) is suggested by an analogous hypothesis proposed by Martin [13] to study smooth values of polynomials. His "Hypothesis UH" makes a somewhat stronger prediction in the more general context of Hypothesis H, in a similar range of uniformity. A very special case of Hypothesis UL, that the number of twin primes $p, p+2 \leqslant x$ is $\gg x/\log^2 x$, implies immediately that $V_{\phi,\sigma}(x) \gg x/\log^2 x$.

Theorem 1. *Assume Hypothesis UL. Then as $x \to \infty$,*

$$V_{\phi,\sigma}(x) \geqslant \frac{x}{(\log x)^{1+o(1)}}.$$

The proof, which proceeds along entirely different lines than [9], has its origin in the following simple observation: Write $R_f(v) := \#f^{-1}(v)$ for the number of preimages of v under the arithmetic function f. By Hölder's inequality, we have

$$\left(\sum_{v\leqslant x} R_\phi(v)R_\sigma(v)\right)^3 \leqslant V_{\phi,\sigma}(x)\left(\sum_{v\leqslant x} R_\phi(v)^2R_\sigma(v)\right)\left(\sum_{v\leqslant x} R_\phi(v)R_\sigma(v)^2\right). \tag{5}$$

In particular, to prove Theorem 1, it would suffice to show that the left-hand sum is bounded below by $x/(\log x)^{1+o(1)}$ while the two sums appearing on the right-hand side are bounded above by $x/(\log x)^{1+o(1)}$. Unfortunately these estimates are not so easy to obtain. It turns out that rather than count all preimages, as in our definition of R_f above, it is easier to obtain analogous estimates if we count only preimages belonging to certain specially constructed sets. The choice of these sets is motivated by the detailed structure theory of preimages developed in [12] and [7].

Notation. Most of our number-theoretic notation is standard. Possible exceptions include $P^+(n)$ for the largest prime factor of n, and $\Omega(n, U, T)$ for the total number of prime factors p of n with $U < p \leqslant T$, counted according to multiplicity.

Big-Oh notation and the related symbols "$\ll$," "$\gg$," and "$\asymp$" appear with their usual meanings, including subscripts to indicate the dependence of implied constants. We use $o_k(1)$ for a quantity that tends to zero for each fixed value of k. We also put $\log_1 x = \max\{1, \log x\}$ and we write $\log_k$ for the kth iterate of $\log_1$.

2. Proof of Theorem 1

We now construct our surrogate representation functions. For a set $\mathscr{B}$ of natural numbers and f an arithmetic function, let

$$R_f(v; \mathscr{B}) := \#\{n \in \mathscr{B} : f(n) = v\}.$$

Then (5) continues to hold if we replace $R_\phi(v)$ by $R_\phi(v; \mathscr{B}_\phi)$ and $R_\sigma(v)$ by $R_\sigma(v; \mathscr{B}_\sigma)$. We now describe our choices of $\mathscr{B}_\phi$ and $\mathscr{B}_\sigma$.

It is convenient to work not with a single set $\mathscr{B}_\phi$, but with a family of such sets, and similarly for $\mathscr{B}_\sigma$. Our definition of these sets depends on a real parameter α, which we always suppose satisfies $1/2 < \alpha < \rho$ (with ρ as in (4)), on a natural number parameter k, and on x. We define $\mathscr{B}_\phi^{\alpha,k}(x)$ as the set of natural numbers n possessing all of the following properties:

(i) n is the product of k distinct primes and $\phi(n) \leqslant x$.

(ii) If $p_0 > \cdots > p_{k-1}$ is the decreasing list of the primes dividing n, then

$$v_i^{1/12} < P^+(p_i - 1) \leqslant p_i - 1 \leqslant v_i, \quad \text{and} \quad v_i = \exp((\log x)^{\alpha^i});$$

also, $P^+(p_i - 1)$ is the unique prime divisor of $p_i - 1$ exceeding $v_i^{1/12}$ for $1 \leqslant i \leqslant k-1$.

(iii) If $1 \leqslant j \leqslant i \leqslant k$, we have

$$\left|\Omega(p_{j-1} - 1, v_i, v_{i-1}) - (\alpha^{i-1} - \alpha^i) \log_2 x\right| \leqslant 2k\sqrt{(\alpha^{i-1} - \alpha^i) \log_2 x}.$$

(iv) 6 is the largest factor of $p_i - 1$ supported on the primes $\leqslant v_k$.

(v) If $p \mid \phi(n)$ and $p > v_k$, then $p \parallel \phi(n)$.

We define $\mathscr{B}_\sigma^{\alpha,k}(x)$ analogously, with ϕ replaced by σ in (i) and (v) and $p-1$ replaced by $p+1$ throughout in (ii)–(iv). If α, k, and x are all understood, we write simply $\mathscr{B}_\phi$ and $\mathscr{B}_\sigma$.

In order to establish Theorem 1, it is enough to prove the following two estimates:

Lemma 1. *Assume Hypothesis UL. Let $\epsilon > 0$. There is a real number $1/2 < \alpha_0 < \rho$ and a natural number k_0 with the following property: If $\alpha_0 < \alpha < \rho$ and $k \geqslant k_0$, then for all large enough x (depending on α and k),*

$$\sum_{v \leqslant x} R_\phi(v; \mathscr{B}_\phi) R_\sigma(v; \mathscr{B}_\sigma) \geqslant \frac{x}{(\log x)^{1+\epsilon}}.$$

In other words, there are at least $x/(\log x)^{1+\epsilon}$ solutions (n, m) to

$$\phi(n) = \sigma(m), \quad \textit{where} \quad (n, m) \in \mathscr{B}_\phi \times \mathscr{B}_\sigma.$$

Lemma 2. *Let $\epsilon > 0$. There is a natural number k_0 with the following property: If $1/2 < \alpha < \rho$ and $k \geqslant k_0$, then for all large enough x (depending on α and k),*

$$\sum_{v \leqslant x} R_\phi(v; \mathscr{B}_\phi)^2 R_\sigma(v; \mathscr{B}_\sigma) \leqslant \frac{x}{(\log x)^{1-\varepsilon}}.$$

In other words, there are at most $x/(\log x)^{1-\varepsilon}$ solutions (n, n', m) to

$$\phi(n) = \phi(n') = \sigma(m), \quad \text{where} \quad (n, n', m) \in \mathscr{B}_\phi \times \mathscr{B}_\phi \times \mathscr{B}_\sigma.$$

The same bound holds for $\sum_{v \leqslant x} R_\phi(v; \mathscr{B}_\phi) R_\sigma(v; \mathscr{B}_\sigma)^2$.

Note that Hypothesis UL is required for the proof of Lemma 1, while Lemma 2 is unconditional.

2.1. **Technical preliminaries.** We collect some technical results that will be used in the proofs of Lemmas 1 and 2. The first concerns the distribution of prime factors in a 'typical' factorization of a squarefree number N.

Lemma 3. *Let N be a squarefree natural number with I prime factors. Consider all i^I ways of writing N as a product of i natural numbers, say $N = d_1 \cdots d_i$, where the order of the factors is taken into account. For any $\Delta > 0$, the number of such decompositions with*

$$|\omega(d_1) - I/i| \geqslant \Delta \sqrt{I/i}$$

is at most $\Delta^{-2} i^I$, uniformly for $\Delta > 0$.

Proof. Let $\mathbf{X} = (X_1, \dots, X_i)$, where each $X_i = \omega(d_i)$. Viewing $\mathbf{X}$ as a random vector defined on the space of all decompositions of N into i factors, observe that $\mathbf{X}$ follows a multinomial distribution. The lemma now follows from Chebyshev's inequality, taking into account that $\mathbf{E}[X_1] = I/i$ and $\mathrm{var}(X_1) = (I/i)(1 - 1/i) \leqslant I/i$. $\square$

The following estimate is well-known from the study of sieve methods (see, e.g., [11, Theorem 4.2]).

Lemma 4. *Suppose $a_1, \dots, a_h$ are positive integers and $b_1, \dots, b_h$ are integers such that*

$$E := \prod_{i=1}^{h} a_i \prod_{1 \leqslant i < j \leqslant h} (a_i b_j - a_j b_i) \neq 0.$$

Then

$$\#\{n \leqslant x : a_i n + b_i \text{ prime } (1 \leqslant i \leqslant h)\} \ll_h \frac{x}{(\log x)^h} \prod_p \frac{1 - \nu(p)/p}{(1 - 1/p)^h} \ll_h \frac{x (\log_2(E+2))^h}{(\log x)^h},$$

where $\nu(p)$ is the number of solutions of the congruence $\prod(a_i n + b_i) \equiv 0 \pmod p$, and the implied constant may depend on h.

The next two lemmas concern the Poisson distribution.

Lemma 5. *If $z > 0$ and $\Delta > 0$, then*

$$\sum_{|k-z| > \Delta z} \frac{z^k}{k!} \leqslant \Delta^{-2} e^z.$$

Proof. This follows immediately from Chebyshev's inequality, once we recall that the Poisson distribution with parameter z has mean and variance both equal to z. $\square$

Lemma 6 (see e.g. [7, Lemma 2.1])**.** *If $z > 0$ and $0 < \alpha < 1 < \beta$, then*

$$\sum_{k \leqslant \alpha z} \frac{z^k}{k!} < \left(\frac{e}{\alpha}\right)^{\alpha z} \quad \text{and} \quad \sum_{k \geqslant \beta z} \frac{z^k}{k!} < \left(\frac{e}{\beta}\right)^{\beta z}.$$

2.2. **Proof of Lemma 1.** Suppose that $n = p_0 \cdots p_{k-1}$ and $m = q_0 \cdots q_{k-1}$ are squarefree numbers satisfying $\phi(n) = \sigma(m) \in [1, x]$, where the primes are ordered so that

$$p_0 > p_1 > \cdots > p_{k-1} \quad \text{and} \quad q_0 > q_1 > \cdots > q_{k-1}.$$

Then

$$(p_0 - 1)(p_1 - 1) \cdots (p_{k-1} - 1) = (q_0 + 1)(q_1 + 1) \cdots (q_{k-1} + 1). \tag{6}$$

We consider separately the prime factors of each shifted prime lying in each interval $(v_i, v_{i-1}]$, where $v_i = \exp((\log x)^{\alpha^i})$. For $0 \leqslant j \leqslant k-1$ and $0 \leqslant i \leqslant k$, let

$$s_{i,j} := \prod_{\substack{p^a \| (p_j - 1) \\ p \leqslant v_i}} p^a, \qquad s'_{i,j} := \prod_{\substack{p^a \| (q_j + 1) \\ p \leqslant v_i}} p^a, \qquad s_i := \prod_{j=0}^{k-1} s_{i,j} = \prod_{j=0}^{k-1} s'_{i,j}.$$

Also, for $0 \leqslant j \leqslant k-1$ and $1 \leqslant i \leqslant k$, let

$$t_{i,j} := \frac{s_{i-1,j}}{s_{i,j}}, \qquad t'_{i,j} := \frac{s'_{i-1,j}}{s'_{i,j}}, \qquad t_i := \prod_{j=0}^{k-1} t_{i,j} = \prod_{j=0}^{k-1} t'_{i,j}.$$

Let

$$\sigma_i = \{s_i; s_{i,0}, \ldots, s_{i,k-1}; s'_{i,0}, \ldots, s'_{i,k-1}\}, \tag{7}$$

$$\tau_i = \{t_i; t_{i,0}, \ldots, t_{i,k-1}; t'_{i,0}, \ldots, t'_{i,k-1}\}. \tag{8}$$

Observe that if we define multiplication of $(2k+1)$-tuples component-wise, then we have

$$\sigma_{i-1} = \sigma_i \tau_i. \tag{9}$$

Suppose we are given a collection of squarefree solutions (n, m) to (6) for which $m, n \leqslant x$. Let $\mathfrak{S}_i$ denote the set of σ_i that arise from these solutions, and let $\mathfrak{T}_i$ denote the corresponding set of τ_i. For $1 \leqslant i \leqslant k$, let

$$\mathfrak{U}_i := \{(\sigma, \tau) : \sigma \in \mathfrak{S}_i, \tau \in \mathfrak{T}_i, \sigma\tau \in \mathfrak{S}_{i-1}\}.$$

The given set of solutions (n, m) is in one-to-one correspondence with the set $\mathfrak{S}_0$, since

$$\sigma_0 = (\phi(n); p_0 - 1, \ldots, p_{k-1} - 1, q_0 + 1, q_1 + 1, \ldots, q_{k-1} + 1)$$

both determines the pair (n, m) and is determined by it. Also, from (9) we see that the set $\mathfrak{S}_0$ is completely determined once we know $\mathfrak{S}_k$ and each of the sets $\mathfrak{U}_k, \mathfrak{U}_{k-1}, \ldots, \mathfrak{U}_1$. To construct a set of solutions, we can reverse the process, first picking a set $\mathfrak{S}_k$ and then successively constructing $\mathfrak{U}_k, \ldots, \mathfrak{U}_1$. We carry out this plan, verifying that $(n, m) \in \mathscr{B}_\phi \times \mathscr{B}_\sigma$ for all the solutions constructed in this way.

We begin by putting $\mathfrak{S}_k := \{\sigma\}$, where $\sigma := (6^k; 6, \ldots, 6; 6, \ldots, 6)$.

Suppose that $\mathfrak{S}_i$ has been determined, where $2 \leqslant i \leqslant k$. For each $\sigma_i \in \mathfrak{S}_i$, write σ_i in the form (7). As part of the induction hypothesis, suppose that each σ_i satisfies

$$s_i = \prod_{j=0}^{k-1} s_{i,j} = \prod_{j=0}^{k-1} s'_{i,j}, \tag{10}$$

$$\max_{0 \leqslant j \leqslant k-1} \{s_{i,j}, s'_{i,j}\} \leqslant v_i. \tag{11}$$

Moreover, suppose also that for $j = i, i+1, \dots, k-1$, we have

$$p_j := s_{j,j} + 1 \quad \text{and} \quad q_j := s_{j,j} - 1 \tag{12}$$

all prime. Clearly all of these hypotheses hold when $i = k$ (the last condition being vacuous).

Now we construct $\mathfrak{U}_i$ and so determine $\mathfrak{S}_{i-1}$. Let t_i^* range over all numbers satisfying

(a) t_i^* is squarefree,

(b) every prime divisor of t_i^* belongs to $(v_i, v_{i-1}^{\frac{1}{12\log_2 x}}]$,

(c) t_i^* has exactly $N_i := \lfloor i(\alpha^{i-1} - \alpha^i)\log_2 x \rfloor$ prime divisors,

and suppose that the variables $t_{i,0}, \dots, t_{i,i-2}, t'_{i,0}, \dots, t'_{i,i-2}, u_i, u'_i$ range over all (ordered) dual factorizations of t_i^* of the shape

$$t_i^* = t_{i,0} \dots t_{i,i-2} u_i = t'_{i,0} \cdots t'_{i,i-2} u'_i \tag{13}$$

for which each of the variables $t_{i,j}, t'_{i,j}, u_i, u'_i$ satisfies

$$|\Omega(\cdot, v_i, v_{i-1}) - (\alpha^{i-1} - \alpha^i)\log_2 x| < k\sqrt{(\alpha^{i-1} - \alpha^i)\log_2 x}. \tag{14}$$

Let Q_i range over all primes in the interval

$$v_{i-1}^{1/12} < Q_i \leqslant v_{i-1}^{1/6} \tag{15}$$

for which also

$$p_{i-1} := s_{i,i-1} u_i Q_i + 1 \quad \text{and} \quad q_{i-1} := s'_{i,i-1} u'_i Q_i - 1 \tag{16}$$

are prime. We put $t_{i,i-1} := u_i Q_i$, $t'_{i,i-1} := u'_i Q_i$, and $t_i := t_i^* Q_i$ (so that $t_i = \prod_j t_{i,j} = \prod_j t'_{i,j}$) and we add to $\mathfrak{U}_i$ all pairs of the form (σ_i, τ_i), where

$$\tau_i := (t_i; t_{i,0}, \dots, t_{i,i-1}, 1, \dots, 1; t'_{i,0}, \dots, t'_{i,i-1}, 1, \dots, 1).$$

For the set $\mathfrak{S}_{i-1} = \{\sigma_i \tau_i : (\sigma_i, \tau_i) \in \mathfrak{U}_i\}$ determined this way, our induction hypotheses (10)-(12) continue to hold. Indeed, (10) and (12) hold by construction. To verify (11) for $i-1$ in place of i, observe that if $j \neq i-1$, then

$$s_{i-1,j} = s_{i,j} t_{i,j} \leqslant v_i \left((v_{i-1})^{\frac{1}{12\log_2 x}} \right)^{\Omega(t_{i,j})} < v_i v_{i-1}^{1/6} < v_{i-1}$$

for large x, by our induction hypothesis and the inequality $\Omega(t_{i,j}) \leqslant N_i \leqslant 2\log_2 x$. (Throughout this proof, the meaning of "large" x is allowed to depend on α and k.) If $j = i-1$, then

$$s_{i-1,j} = s_{i,j} t_{i,j} \leqslant v_i u_i Q_i \leqslant v_i v_{i-1}^{1/6} v_{i-1}^{1/6} < v_{i-1}.$$

If $j > i-1$, then $s_{i-1,j} = s_{i,j}$, and so $s_{i-1,j} \leqslant v_i \leqslant v_{i-1}$. Analogous estimates hold for $s'_{i-1,j}$ in place of $s_{i-1,j}$, giving (11).

At this stage we have determined all of $\mathfrak{S}_k, \dots, \mathfrak{S}_1$. It remains to construct $\mathfrak{U}_1$ and so determine $\mathfrak{S}_0$. Let $\sigma_1 \in \mathfrak{S}_1$, and write σ_1 in the form

$$\sigma_1 = (s_1; s_{1,0}, p_1 - 1, \dots, p_{k-1} - 1; s'_{1,0}, q_1 + 1, \dots, q_{k-1} + 1).$$

Let t' range over all natural numbers satisfying

(a) t' is squarefree,

(b) $t' \leqslant x^{1/3}/s_1$,

(c) every prime dividing t' belongs to $(v_1, v_0]$,

(d) $|\Omega(t', v_1, v_0) - (1-\alpha)\log_2 x| \leqslant k\sqrt{(1-\alpha)\log_2 x}$.

For each t', let Q_1 range over primes with

$$x^{1/2} \leqslant Q_1 \leqslant \frac{x}{s_1 t'} \quad \text{for which} \quad p_0 := s_{1,0} t' Q_1 + 1, \quad q_0 := s'_{1,0} t' Q_1 - 1 \text{ are prime.} \tag{17}$$

We set $t_1 := t'Q_1$, and let $\mathfrak{U}_1$ consist of all tuples of the form (σ_1, τ_1), where

$$\tau_1 := (t_1; t_1, 1, 1, \dots, 1; t_1, 1, 1, \dots, 1).$$

Finally, we put $\mathfrak{S}_0 = \{\sigma_1\tau_1 : (\sigma_1, \tau_1) \in \mathfrak{U}_1\}$.

The remainder of the proof consists of verifying that the set $\mathfrak{S}_0$ determined by this construction is as large as claimed and that the solutions corresponding to the elements of $\mathfrak{S}_0$ belong to $\mathscr{B}_\phi \times \mathscr{B}_\sigma$. The lower bound for $\#\mathfrak{S}_0$ will be made to depend on a lower bound for $\sum_{\sigma_1 \in \mathfrak{S}_1} 1/s_1$. First, observe that

$$\sum_{\sigma_1 \in \mathfrak{S}_k} \frac{1}{s_k} = \frac{1}{6^k} \gg_k 1. \tag{18}$$

For $2 \leqslant i \leqslant k$, we have

$$\sum_{\sigma_{i-1} \in \mathfrak{S}_{i-1}} \frac{1}{s_{i-1}} = \sum_{\sigma_i \in \mathfrak{S}_i} \frac{1}{s_i} \sum_{\tau_i :\ (\sigma_i, \tau_i) \in \mathfrak{U}_i} \frac{1}{t_i} = \sum_{\sigma_i \in \mathfrak{S}_i} \frac{1}{s_i} \sum_{t_i^*} \frac{h(t_i^*)}{t_i^*} \sum_{Q_i} \frac{1}{Q_i}. \tag{19}$$

Here t_i^*, t_i, and Q_i are as the quantities appearing in the description of $\mathfrak{U}_i$ for $2 \leqslant i \leqslant k$ (see (13)–(16)), and $h(t_i^*)$ is the number of dual factorizations of t_i^* of the form (13), where the factors $u_i, u'_i, t_{i,j}, t'_{i,j}$ all satisfy (14).

By our choice of $\mathfrak{S}_k$, we have that 6 divides both $s_{i,i-1}$ and $s'_{i,i-1}$. It follows that the singular series corresponding to the affine linear forms $T, s_{i,i-1}u_i T + 1$, and $s'_{i,i-1}u'_i T - 1$ is bounded away from zero (using $\nu(p) \leqslant \min(p-1, 3)$). Moreover, all the coefficients of these forms are bounded by v_{i-1}. Hence Hypothesis UL and partial summation shows that for the inner sum in (19),

$$\sum_{Q_i} \frac{1}{Q_i} \gg \frac{1}{\log^2 v_{i-1}^{1/12}} \gg \frac{1}{(\log x)^{2\alpha^{i-1}}}, \tag{20}$$

where the implied constant is absolute. Inserting this estimate in (19), we find that

$$\sum_{\sigma_{i-1} \in \mathfrak{S}_{i-1}} \frac{1}{s_{i-1}} \gg \frac{1}{(\log x)^{2\alpha^{i-1}}} \sum_{\sigma_i \in \mathfrak{S}_i} \frac{1}{s_i} \sum_{t_i^*} \frac{h(t_i^*)}{t_i^*}. \tag{21}$$

Recalling the definition of $h(\cdot)$, we see that for each t_i^*,

$$\begin{aligned} h(t_i^*) &\geqslant \#\{\text{dual } i\text{-fold factorizations of } t_i^*\} - \#\{\text{dual } i\text{-fold factorizations of } t_i^* \text{ failing (14)}\} \\ &\geqslant i^{2N_i} - 2i^{N_i} h'(t_i^*), \end{aligned}$$

where $h'(t_i^*)$ is the number of (single) i-fold decompositions of t_i^* where (at least) one of the factors fails to satisfy (14). By Lemma 3, we have $h'(t_i^*) \ll i(i^{N_i}/k^2) \leqslant i^{N_i}/k$, and thus $h'(t_i^*) < i^{N_i}/4$, assuming (as we may) that k is sufficiently large. Hence

$$h(t_i^*) \geqslant \frac{1}{2} i^{2N_i}, \tag{22}$$

uniformly in t_i^*. Moreover, by the multinomial theorem, if we put

$$S := \sum_{v_i < p \leqslant v_{i-1}^{\frac{1}{12\log_2 x}}} \frac{1}{p} \quad \text{and} \quad S' := \sum_{v_i < p \leqslant v_{i-1}} \frac{1}{p^2},$$

then

$$\sum_{t_i^*} \frac{1}{t_i^*} \geqslant \frac{S^{N_i}}{N_i!} - \frac{S^{N_i-2}S'}{(N_i-2)!} \geqslant \frac{S^{N_i}}{N_i!}\left(1 - O(N_i^2 S'/S^2)\right) \geqslant \frac{1}{2}\frac{S^{N_i}}{N_i!} \tag{23}$$

for large x. Combining the results of (22) and (23) with (21), we find that

$$\begin{aligned}\sum_{\sigma_{i-1}\in\mathfrak{S}_{i-1}} \frac{1}{s_{i-1}} &\gg \frac{1}{(\log x)^{2\alpha^i}} \frac{(i^2 S)^{N_i}}{N_i!} \sum_{\sigma_i\in\mathfrak{S}_i} \frac{1}{s_i} \\ &\gg \frac{1}{(\log x)^{2\alpha^i}} \frac{1}{\sqrt{N_i}} \left(\frac{ei^2 S}{N_i}\right)^{N_i} \sum_{\sigma_i\in\mathfrak{S}_i} \frac{1}{s_i}. \end{aligned} \tag{24}$$

(Here we have used Stirling's formula to estimate $N_i!$.) A routine computation, making use of the estimates

$$S = (\alpha^{i-1} - \alpha^i)\log_2 x + O(\log_3 x), \qquad N_i = i(\alpha^{i-1} - \alpha^i)\log_2 x + O(1),$$

shows that as $x \to \infty$,

$$\sum_{\sigma_{i-1}\in\mathfrak{S}_{i-1}} \frac{1}{s_{i-1}} \geqslant (\log x)^{(\alpha^{i-1}-\alpha^i)(i+i\log i)-2\alpha^{i-1}+o_k(1)} \sum_{\sigma_i\in\mathfrak{S}_i} \frac{1}{s_i}.$$

Starting with (18) and then descending from $i = k$ down to $i = 2$, we obtain that

$$\sum_{\sigma_1\in\mathfrak{S}_1} \frac{1}{s_1} \geqslant (\log x)^{\sum_{i=2}^k\left((\alpha^{i-1}-\alpha^i)(i+i\log i)-2\alpha^{i-1}\right)+o_k(1)}.$$

Letting t' and Q_1 denote the quantities appearing in the definition of $\mathfrak{U}_1$, we have from Hypothesis UL and the above lower bound on $\sum 1/s_1$ that

$$\begin{aligned}\#\mathfrak{S}_0 = \sum_{\sigma_1\in\mathfrak{S}_1} \sum_{\tau_1:(\sigma_1,\tau_1)\in\mathfrak{U}_1} 1 &= \sum_{\sigma_1\in\mathfrak{S}_1} \sum_{t'} \sum_{Q_1} 1 \gg \sum_{\sigma_1\in\mathfrak{S}_1} \sum_{t'} \frac{x}{s_1 t' \log^3 x} \\ &= \frac{x}{\log^3 x} (\log x)^{\sum_{i=2}^k\left((\alpha^{i-1}-\alpha^i)(i+i\log i)-2\alpha^{i-1}\right)+o_k(1)} \sum \frac{1}{t'}. \end{aligned} \tag{25}$$

(Note that $\max\{s_{1,0}t', s'_{1,0}t'\} \leqslant v_1 t' \leqslant x^{1/3}$ and that 6 divides both $s_{1,0}$ and $s'_{1,0}$.) We now estimate $\sum 1/t'$ from below. Let us temporarily ignore the restriction (d) on $\Omega(t')$, and for brevity write $T = \{t' \leqslant x^{1/3}/s_1 : t' \text{ squarefree}, p|t' \implies v_1 < p \leqslant v_0\}$. Then for large x,

$$\sum_{t'\in T} \frac{1}{t'} \prod_{p\leqslant v_1}\left(1 + \frac{1}{p}\right) \geqslant \sum_{t'\leqslant x^{1/3}/s_1,\ \text{squarefree}} \frac{1}{t'} \gg \log(x^{1/3}/s_1) \gg \log x.$$

(Note that $s_1 \leqslant v_1^k = x^{o_k(1)}$.) So by Mertens' theorem,

$$\sum_{t'\in T} \frac{1}{t'} \gg \frac{\log x}{\log v_1} = (\log x)^{1-\alpha}. \tag{26}$$

To obtain a corresponding lower bound incorporating (d), we show those t' for which (d) is violated make a negligible contribution to (26). Redefine

$$S := \sum_{v_1 < p \leqslant v_0} \frac{1}{p} = (1-\alpha)\log_2 x + O(1),$$

and observe that by Lemma 5, for large x,

$$\begin{aligned} \sum_{\substack{t' \in T \\ |\Omega(t') - (1-\alpha)\log_2 x| \geqslant k\sqrt{(1-\alpha)\log_2 x}}} \frac{1}{t'} &\leqslant \sum_{\substack{t' \in T \\ |\Omega(t') - S| \geqslant \frac{1}{2}k\sqrt{S}}} \frac{1}{t'} \\ &\leqslant \sum_{|j-S| \geqslant \frac{k}{2}\sqrt{S}} \frac{S^j}{j!} \leqslant \frac{4}{k^2}\exp(S) \ll \frac{1}{k^2}(\log x)^{1-\alpha}. \end{aligned} \tag{27}$$

So assuming that k is large, we have from (26) and (27) that for the final sum in (25),

$$\sum \frac{1}{t'} \gg (\log x)^{1-\alpha}.$$

So by (25), we have that as $x \to \infty$,

$$\#\mathfrak{S}_0 \geqslant \frac{x}{(\log x)^{2+\alpha-\sum_{i=2}^{k}((i\log i + i)(\alpha^{i-1}-\alpha^i)-2\alpha^{i-1})+o_k(1)}}.$$

Ignoring the $o_k(1)$ term, the exponent on $\log x$ in the denominator simplifies under Abel summation to

$$2 - \sum_{i=1}^{k-1} a_i\alpha^i + (k\log k + k)\alpha^k = 2 - F(\alpha) + O((k\log k)\alpha^k).$$

Using (4), we can now fix $\alpha_0 \in (1/2, \rho)$ with $F(\alpha_0) > 1 - \epsilon/2$. Then if we begin the argument with $\alpha > \alpha_0$ and k large enough (say $k > k_0$), we find that

$$\#\mathfrak{S}_0 > \frac{x}{(\log x)^{1+\epsilon}} \tag{28}$$

once x is large.

It remains to show that the elements of $\mathfrak{S}_0$ correspond to solutions $(n,m) \in \mathscr{B}_\phi \times \mathscr{B}_\sigma$ to $\phi(n) = \sigma(m)$. Let $\sigma \in \mathfrak{S}_0$, and write σ in the form

$$\sigma = (s_0; p_0 - 1, \dots, p_{k-1} - 1; q_0 + 1, \dots, q_{k-1} + 1).$$

We associate to σ the pair (n,m), where $n := p_0 \cdots p_{k-1}$ and $m := q_0 \cdots q_{k-1}$. At this point, we know that

$$\prod (p_i - 1) = s_0 = \prod (q_j + 1),$$

but we cannot conclude (yet) that $\phi(n) = \sigma(m)$, because we have not proved that n and m are squarefree. This is, of course, contained in showing that $(n,m) \in \mathscr{B}_\phi \times \mathscr{B}_\sigma$, and so we now turn to that proof. It will be enough to show that $n \in \mathscr{B}_\phi$, since the proof that $m \in \mathscr{B}_\sigma$ is entirely analogous.

We first establish properties (i) and (ii) in the definition of $\mathscr{B}_\phi$. From (17), we have

$$p_0 - 1 \geqslant P^+(p_0 - 1) = Q_1 \geqslant x^{1/2} > v_0^{1/12}$$

and (in the notation of (17))

$$p_0 - 1 = s_{1,0} t' Q_1 \leqslant s_1 t' Q_1 \leqslant x = v_0.$$

Also, for $2 \leqslant i \leqslant k$, we have

$$p_{i-1} - 1 = s_{i-1,i-1} \leqslant v_{i-1}$$

and, in the notation used to define $\mathfrak{U}_2, \ldots, \mathfrak{U}_k$,

$$p_{i-1} - 1 \geqslant P^+(p_{i-1} - 1) = Q_i > v_{i-1}^{1/12},$$

using (15). Thus, for each $1 \leqslant i \leqslant k$,

$$p_{i-1} - 1 > v_{i-1}^{1/12} > v_i \geqslant p_i - 1,$$

which shows that $p_0 > p_1 > \cdots > p_{k-1}$ and so proves (i). The only statement of (ii) not shown above is that $P^+(p_{i-1} - 1)$ is the unique prime divisor of $p_{i-1} - 1$ exceeding $v_{i-1}^{1/12}$, for $2 \leqslant i \leqslant k$. In fact, any prime p dividing $p_i - 1$ other than $P^+(p_i - 1)$ satisfies (in the notation of (16))

$$p \leqslant P^+(s_{i,i-1} u_i) \leqslant \max\{s_{i,i-1}, P^+(u_i)\} \leqslant \max\{v_i, v_{i-1}^{\frac{1}{12 \log_2 x}}\} = v_{i-1}^{\frac{1}{12 \log_2 x}} < v_{i-1}^{1/12}.$$

So we have (ii). Property (iv) follows from the definition of $\mathfrak{S}_k$ and the observation that each τ_i has all of its components supported on primes $> v_i \geqslant v_k$. To see (v), notice that the part of $\phi(n)$ supported on primes $> v_k$ can be written as the first component of $\tau_k \cdots \tau_1$, so as the k-fold product

$$t_k t_{k-1} \cdots t_1.$$

But in our construction, the k factors appearing here are squarefree and supported on pairwise disjoint sets of primes. Lastly we turn to (iii): If $i = 1$, then also $j = 1$, and in the notation used to define $\mathfrak{U}_1$, we have

$$\Omega(p_0 - 1, v_1, v_0) = \Omega(s_{1,0} t' Q_1, v_1, v_0) = \Omega(t' Q_1) = 1 + \Omega(t');$$

the result (ii) in this case follows from (d) in the definition of t'. If $2 \leqslant i \leqslant k$, and $j = i$, then (in the notation used to define $\mathfrak{U}_2, \ldots, \mathfrak{U}_k$)

$$\begin{aligned}\Omega(p_{j-1} - 1, v_i, v_{i-1}) &= \Omega(s_{i,i-1} t_{i,i-1}, v_i, v_{i-1}) \\ &= \Omega(t_{i,i-1}) = \Omega(u_i Q_i) = 1 + \Omega(u_i),\end{aligned}$$

and the result follows from (14). If $2 \leqslant i \leqslant k$ and $j < i$, then (in the same notation)

$$\Omega(p_{j-1} - 1, v_i, v_{i-1}) = \Omega(t_{i,j-1}),$$

and the result again follows from (14). This completes the proof that $n \in \mathscr{B}_\phi$.

Finally, notice that distinct $\sigma \in \mathfrak{S}_0$ induce distinct solutions (n, m), by unique factorization. Thus, the number of solutions (n, m) to $\phi(n) = \sigma(m)$ which we find in this way is precisely $\#\mathfrak{S}_0$, and the lemma follows from (28).

2.3. **Proof of Lemma 2.** Suppose $\alpha, k,$ and x are given. Take a solution $(n, n', m) \in \mathscr{B}_\phi \times \mathscr{B}_\phi \times \mathscr{B}_\sigma$ to $\phi(n) = \phi(n') = \sigma(m)$. Write $n = \prod_{i=0}^{k-1} p_i$, $n' = \prod_{i=0}^{k-1} p'_i$, and $m = \prod_{i=0}^{k-1} q_i$, where the p_i, p'_i, and q_i are decreasing. Put

$$\mathcal{I} = \{0 \leqslant i \leqslant k-1 : p_i = p'_i\}, \quad \text{so that} \quad \gcd(n, n') = \prod_{i \in \mathcal{I}} p_i.$$

Given k, there are only $O_k(1)$ possibilities for $\mathcal{I}$, and so we may (and do) carry out all the estimates below assuming that $\mathcal{I}$ is fixed. We have

$$(p_0 - 1) \cdots (p_{k-1} - 1) = (p'_0 - 1) \cdots (p'_{k-1} - 1) = (q_0 + 1) \cdots (q_{k-1} + 1). \tag{29}$$

For each nonnegative integer i, let $v_i := \exp((\log x)^{\alpha^i})$. We consider separately the prime factors of each shifted prime lying in each interval $(v_i, v_{i-1}]$. For $0 \leqslant j \leqslant k-1$ and $0 \leqslant i \leqslant k$, let

$$s_{i,j}(n) := \prod_{\substack{p^a \| p_i - 1 \\ p \leqslant v_i}} p^a, \quad s'_{i,j}(n) := \prod_{\substack{p^a \| p'_i - 1 \\ p \leqslant v_i}} p^a, \quad s''_{i,j}(n) := \prod_{\substack{p^a \| q_i + 1 \\ p \leqslant v_i}} p^a,$$

and put

$$s_i := \prod_{j=0}^{k-1} s_{i,j} = \prod_{j=0}^{k-1} s'_{i,j} = \prod_{j=0}^{k-1} s''_{i,j}.$$

Also, for $0 \leqslant j \leqslant k-1$, let

$$t_{i,j} := \frac{s_{i-1,j}}{s_{i,j}}, \quad t'_{i,j} := \frac{s'_{i-1,j}}{s'_{i,j}}, \quad t''_{i,j} := \frac{s''_{i-1,j}}{s''_{i,j}},$$

and put

$$t_i := \prod_{j=0}^{k-1} t_{i,j} = \prod_{j=0}^{k-1} t'_{i,j} = \prod_{j=0}^{k-1} t''_{i,j}.$$

For each solution $(n, n', m) \in \mathscr{B}_\phi \times \mathscr{B}_\phi \times \mathscr{B}_\sigma$ to $\phi(n) = \phi(n') = \sigma(m)$, put

$$\sigma_i := (s_i; s_{i,0}, \ldots, s_{i,k-1}; s'_{i,0}, \ldots, s'_{i,k-1}; s''_{i,0}, \ldots, s''_{i,k-1}),$$
$$\tau_i := (t_i; t_{i,0}, \ldots, t_{i,k-1}; t'_{i,0}, \ldots, t'_{i,k-1}; t''_{i,0}, \ldots, t''_{i,k-1}).$$

Note that with multiplication of $(3k+1)$-tuples defined componentwise, we have $\sigma_{i-1} = \sigma_i \tau_i$. Let $\mathfrak{S}_i$ denote the set of σ_i arising from solutions $(n, m, m') \in \mathscr{B}_\phi \times \mathscr{B}_\phi \times \mathscr{B}_\sigma$, and let $\mathfrak{T}_i$ denote the corresponding set of τ_i. The number of solutions of (29) is

$$\#\mathfrak{S}_0 = \sum_{\sigma \in \mathfrak{S}_1} \sum_{\substack{\tau \in \mathfrak{T}_1 \\ \sigma\tau \in \mathfrak{S}_0}} 1.$$

To estimate this quantity, we apply an iterative procedure based on the identity

$$\sum_{\sigma_{i-1} \in \mathfrak{S}_{i-1}} \frac{1}{s_{i-1}} = \sum_{\sigma_i \in \mathfrak{S}_i} \frac{1}{s_i} \sum_{\substack{\tau_i \in \mathfrak{T}_i \\ \sigma_i \tau_i \in \mathfrak{S}_{i-1}}} \frac{1}{t_i}. \tag{30}$$

First, fix $\sigma_1 \in \mathfrak{S}_1$. If $\tau_1 \in \mathfrak{T}_1$ is such that $\sigma_1 \tau_1 \in \mathfrak{S}_0$, then $t_1 = t_{1,0} = t'_{1,0} = t''_{1,0} \leqslant x/s_1$, t_1 is composed of primes $> v_1$, and all of

$$s_{1,0} t_1 + 1, \quad s'_{1,0} t_1 + 1, \quad s''_{1,0} t_1 - 1$$

are prime. Write $t_1 = t_1' Q$, where $Q = P^+(t_1)$. Then $Q = P^+(p_0 - 1) \geqslant x^{1/12}$ by property (ii) in the definition of $\mathscr{B}_\phi$. Hence

$$\sum_{\substack{\tau_1 \in \mathfrak{T}_1 \\ \sigma_1\tau_1 \in \mathfrak{S}_0}} 1 \leqslant \sum_{\substack{t_1' \leqslant x/s_1 \\ p \mid t_1' \Rightarrow p > v_1}} \sum_{x^{1/12} \leqslant Q \leqslant \frac{x}{s_1 t_1'}} 1,$$

where the final sum is over primes Q for which $s_{1,0} t_1' Q + 1, s_{1,0}' t_1' Q + 1$, and $s_{1,0}'' t_1' Q - 1$ are also prime. By Lemma 4, the inner sum over Q is

$$\ll \begin{cases} \frac{x}{s_1 t_1' (\log x)^4} (\log_2 x)^4 & \text{if } 0 \notin \mathcal{I}, \\ \frac{x}{s_1 t_1' (\log x)^3} (\log_2 x)^3 & \text{otherwise.} \end{cases}$$

Moreover,

$$\sum_{t_1'} \frac{1}{t_1'} \leqslant \prod_{v_1 < p \leqslant x/s_1} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) \ll \frac{\log x}{\log v_1} = (\log x)^{1-\alpha}.$$

It follows that

$$\sum_{\substack{\tau_1 \in \mathfrak{T}_1 \\ \sigma_1\tau_1 \in \mathfrak{S}_0}} 1 \ll \frac{x}{s_1 (\log x)^{2+\alpha+\chi(0)}} (\log_2 x)^4, \tag{31}$$

where here and below, χ denotes the characteristic function of $\mathcal{I}^c = [0, k-1] \setminus \mathcal{I}$.

Now suppose that $2 \leqslant i \leqslant k$, $\sigma_i \in \mathfrak{S}_i$, $\tau_i \in \mathfrak{T}_i$ and $\sigma_i\tau_i \in \mathfrak{S}_{i-1}$. Observe that

$$t_{i,0} \cdots t_{i,i-1} = t_{i,0}' \cdots t_{i,i-1}' = t_{i,0}'' \cdots t_{i,i-1}'' = t_i. \tag{32}$$

Let Q_1, Q_2, and Q_3 be the largest prime factors of $t_{i,i-1}, t_{i,i-1}'$, and $t_{i,i-1}''$, respectively, and let b, b', and b'' be the corresponding cofactors. Recall that $Q_1, Q_2, Q_3 > v_{i-1}^{1/12}$ by (ii) in the definitions of $\mathscr{B}_\phi$ and $\mathscr{B}_\sigma$. Now fix $\mathcal{J} \subset \{1, 2, 3\}$ indexing the distinct Q_j. Then $\prod_{j \in \mathcal{J}} Q_j \mid t_i$. Moreover, t_i is squarefree (by property (v) in the definition of $\mathscr{B}_\phi$), and so each Q_j divides exactly one term from each of the three i-fold factorizations exhibited in (32). Dividing all such terms by their corresponding factor Q_j, we obtain an induced identity of the shape

$$\hat{t}_{i,0} \cdots \hat{t}_{i,i-1} = \hat{t}_{i,0}' \cdots \hat{t}_{i,i-1}' = \hat{t}_{i,0}'' \cdots \hat{t}_{i,i-1}'' = \frac{t_i}{\prod_{j \in J} Q_j}, \tag{33}$$

where

$$\hat{t}_{i,j} = \hat{t}_{i,j}' \quad \text{for all} \quad i \in \mathcal{I} \cap [0, i-1], \tag{34}$$

and

$$\left|\Omega(\cdot, v_i, v_{i-1}) - (\alpha^{i-1} - \alpha^i) \log_2 x\right| \leqslant 3k\sqrt{(\alpha^{i-1} - \alpha^i) \log_2 x} \tag{35}$$

for each of the $3i$ factors in the triple i-fold factorization (33). Here we use (iii) from the definitions of $\mathscr{B}_\phi$ and $\mathscr{B}_\sigma$. Also, the uniqueness statement in (ii) allows us to deduce that $b = \hat{t}_{i,i-1}$, $b' = \hat{t}_{i,i-1}$, $b'' = \hat{t}_{i,i-1}$. Putting $t = t_i / \prod_{j \in \mathcal{J}} Q_j$, we can expand

$$\sum_{\substack{\tau_i \in \mathfrak{T}_i \\ \sigma_i\tau_i \in \mathfrak{S}_{i-1}}}^{(\mathcal{J})} \frac{1}{t_i} = \sum_t \frac{1}{t} \sum_{\substack{\text{3-fold} \\ \text{factorizations}}} \sum_{\text{posns. of } Q_j} \prod_{j \in \mathcal{J}} \left(\sum_{Q_j} \frac{1}{Q_j}\right). \tag{36}$$

The superscript in the left-hand sum indicates that the sum is only taken over τ_i which correspond to the index set $\mathcal{J}$. The second right-hand sum is over factorizations (33) corresponding to t (which necessarily satisfy (34) and (35)), and the third right-hand sum is over the original positions in (32) of the Q_j, before they were divided out to produce (33).

Now we estimate the innermost sum in (36). Observe that

$$p_{i-1} = s_{i,i-1} b Q_1 + 1, \quad p'_{i-1} = s'_{i,i-1} b' Q_2 + 1, \quad q_{i-1} = s''_{i,i-1} b'' Q_3 - 1$$

are all prime. For each $j \in \mathcal{J}$, let n_j be the number of distinct linear forms among these which involve the prime Q_j. Since Q_j itself is also prime, the sieve (Lemma 4) implies that the number of possibilities for $Q_j \leqslant z$ is $\ll z(\log_2 x)^{n_j+1}/(\log z)^{n_j+1}$, and so

$$\sum_{Q_j \geqslant v_{i-1}^{1/12}} \frac{1}{Q_j} \ll \frac{1}{(\log v_{i-1})^{n_j}} (\log_2 x)^{n_j+1} = \frac{1}{(\log x)^{\alpha^{i-1} n_j}} (\log_2 x)^{n_j+1}.$$

We have $\sum_{j\in\mathcal{J}} n_j = 2 + \chi(i-1)$, and so

$$\prod_{j\in\mathcal{J}} \left(\sum_{Q_j \geqslant v_{i-1}^{1/12}} \frac{1}{Q_j} \right) \ll \frac{1}{(\log x)^{2\alpha^{i-1} + \alpha^{i-1}\chi(i-1)}} (\log_2 x)^6.$$

We insert this into (36). Note that the number of possible original postions of the Q_j is bounded by $i^{2|\mathcal{J}|} \leqslant i^6 \ll_k 1$. Thus, letting $h(t)$ denote the number of triple i-fold factorizations of the form (33) satisfying (34) and (35), we find that

$$\sum_{\substack{\tau_i \in \mathfrak{T}_i \\ \sigma_i\tau_i \in \mathfrak{S}_{i-1}}}^{(\mathcal{J})} \frac{1}{t_i} \ll_k \frac{1}{(\log x)^{2\alpha^{i-1} + \alpha^{i-1}\chi(i-1)}} (\log_2 x)^6 \sum_t \frac{h(t)}{t}. \tag{37}$$

For each value of t that can appear here, we have that t is squarefree, supported on primes in $(v_i, v_{i-1}]$, and satisfies

$$|\Omega(t) - i(\alpha^{i-1} - \alpha^i)\log_2 x| \leqslant 3ki\sqrt{(\alpha^{i-1} - \alpha^i)\log_2 x}.$$

(The last inequality is a consequence of (35).) For all such t, we have for $N := \#\mathcal{I} \cap [0, i-1]$,

$$h(t) \leqslant \sum_{\substack{i_0,\ldots,i_N \\ i_l \text{ satisfies (35) for } 0 \leqslant l \leqslant N \\ i_{N+1} := \Omega(t) - \sum_{0\leqslant l\leqslant N} i_l}} \binom{\Omega(t)}{i_0, \ldots, i_N, i_{N+1}} ((i-N)^{\Omega(t) - i_{N+1}})^2 i^{\Omega(t)};$$

here the multinomial coefficient accounts for the common portion of the first two factorizations of (33), the factor $((i-N)^{\Omega(t)-i_{N+1}})^2$ bounds the number of possibilities for the uncommon portion, and the factor $i^{\Omega(t)}$ bounds the total number of possibilities for the third factorization (which is unrestricted by (34)). A calculation with Stirling's formula now shows that

$$h(t) \leqslant (i^2 (i-N)^{1-N/i})^{\Omega(t)} \exp(O_k(\log_3 x \sqrt{\log_2 x})),$$

uniformly in t. Put

$$I := i(\alpha^{i-1} - \alpha^i)\log_2 x + 3k^2\sqrt{(\alpha^{i-1} - \alpha^i)\log_2 x}.$$

Then with

$$S := \sum_{v_i < p \leqslant v_{i-1}} \frac{1}{p} = (\alpha^{i-1} - \alpha^i)\log_2 x + O(1),$$

the multinomial theorem gives us that

$$\sum_t \frac{h(t)}{t} \leqslant \exp(O_k(\log_3 x\sqrt{\log_2 x}))\sum_{j\leqslant I}(i^2(i-N)^{1-N/i})^j\frac{S^j}{j!}. \tag{38}$$

For large x, we have $I < i^2 S \leqslant i^2 S(i-N)^{1-N/i}$, and so by Lemma 6,

$$\begin{aligned}\sum_{j\leqslant I}(i^2(i-N)^{1-N/i})^j\frac{S^j}{j!} &\leqslant \left(\frac{ei^2(i-N)^{1-N/i}S}{I}\right)^I \\ &\leqslant \exp(O_k(\sqrt{\log_2 x}))(ei(i-N)^{1-N/i})^I \\ &= (\log x)^{(\alpha^{i-1}-\alpha^i)(i+i\log i)+(i-N)\log(i-N)(\alpha^{i-1}-\alpha^i)+o_k(1)}. \end{aligned}\tag{39}$$

From (37), (38), and (39), we deduce that

$$\sum_{\substack{\tau_i\in\mathfrak{T}_i\\ \sigma_i\tau_i\in\mathfrak{S}_{i-1}}}\frac{1}{t_i} \leqslant (\log x)^{(\alpha^{i-1}-\alpha^i)(i+i\log i)-2\alpha^{i-1}+(i-N)\log(i-N)(\alpha^{i-1}-\alpha^i)-\chi(i-1)\alpha^{i-1}+o_k(1)}; \tag{40}$$

we use here that there are only finitely many possibilities for $\mathcal{J}$, so that we can drop the superscript $(\mathcal{J})$ on the sum.

By (30), (31), and (40), we see that

$$\begin{aligned}\#\mathfrak{S}_0 \leqslant (\log x)^{o_k(1)} &\frac{x}{(\log x)^{2+\alpha-\sum_{i=2}^k((\alpha^{i-1}-\alpha^i)(i+i\log i)-2\alpha^{i-1})}} \\ &\times (\log x)^{\sum_{i=2}^k (i-\#\mathcal{I}\cap[0,i-1])\log(i-\#\mathcal{I}\cap[0,i-1])(\alpha^{i-1}-\alpha^i)-\sum_{i=0}^{k-1}\chi(i)\alpha^i} \times \sum_{\sigma_k\in\mathfrak{S}_k}\frac{1}{s_k},\end{aligned}$$

with the convention that $0\log 0 = 0$. From (iv) in the definitions of $\mathscr{B}_\phi$ and $\mathscr{B}_\sigma$, the final sum is $6^{-k} \leqslant 1$. Also,

$$\begin{aligned}2+\alpha-\sum_{i=2}^k((\alpha^{i-1}-\alpha^i)(i+i\log i)-2\alpha^{i-1}) &= 2-\sum_{i=1}^{k-1}a_i\alpha^i + (k\log k + k)\alpha^k \\ &= 2 - F(\alpha) + O((k\log k)\alpha^k).\end{aligned}$$

Since $F(\alpha) \leqslant F(\rho) = 1$, for sufficiently large k, this last expression is at least $1-\varepsilon/2$. So the desired upper bound on $\mathfrak{S}_0$ follows if it is shown that

$$\sum_{i=2}^k (i-\#\mathcal{I}\cap[0,i-1])\log(i-\#\mathcal{I}\cap[0,i-1])(\alpha^{i-1}-\alpha^i) \leqslant \sum_{i=0}^{k-1}\chi(i)\alpha^i. \tag{41}$$

For brevity, write $M(i) := i - \#\mathcal{I}\cap[0,i-1] = \#\mathcal{I}^c\cap[0,i-1]$. Abel summation implies that for the left-hand side of (41), we have

$$\sum_{i=2}^k M(i)\log M(i)(\alpha^{i-1}-\alpha^i) \leqslant \sum_{i=1}^{k-1}(M(i+1)\log M(i+1) - M(i)\log M(i))\alpha^i.$$

Suppose that

$$1 \leqslant i_1 < i_2 < \cdots < i_L \leqslant k-1$$

is a list of the elements of $\mathcal{I}^c$ in $[1, k-1]$. If $0 \notin \mathcal{I}^c$, then

$$\begin{aligned}
&\sum_{i=1}^{k-1} (M(i+1)\log M(i+1) - M(i)\log M(i))\alpha^i \\
&\qquad = \sum_{l=2}^{L} (a_{l-1}+1)\alpha^{i_l} = \sum_{l=2}^{L} a_{l-1}\alpha^{i_l} + \sum_{i=i_2}^{k-1} \chi(i)\alpha^i \\
&\qquad\qquad \leqslant \alpha^{i_1} \sum_{i=1}^{\infty} a_i\alpha^i + \sum_{i=i_2}^{k-1} \chi(i)\alpha^i \leqslant \alpha^{i_1} + \sum_{i=i_2}^{k-1} \chi(i)\alpha^i = \sum_{i=0}^{k-1} \chi(i)\alpha^i.
\end{aligned}$$

If $0 \in \mathcal{I}^c$, then

$$\begin{aligned}
&\sum_{i=1}^{k-1} (M(i+1)\log M(i+1) - M(i)\log M(i))\alpha^i \leqslant \sum_{l=1}^{L} (a_l+1)\alpha^{i_l} \\
&\qquad\qquad \leqslant \sum_{i=1}^{L} a_l\alpha^l + \sum_{i=1}^{k-1} \chi(i)\alpha^i < 1 + \sum_{i=1}^{k-1} \chi(i)\alpha^i = \sum_{i=0}^{k-1} \chi(i)\alpha^i.
\end{aligned}$$

So (41) holds in either case.

This concludes the proof of the first half of Lemma 2. The estimate for the number of solutions to $\phi(n) = \sigma(m) = \sigma(m')$, where $(n, m, m') \in \mathscr{B}_\phi \times \mathscr{B}_\sigma \times \mathscr{B}_\sigma$, is entirely analogous.

Remarks. The term $(\log x)^{1+o(1)}$ in Theorem 1 can be sharpened by allowing k, α to depend on x in the above argument, or by using the fine structure theory of totients from [7].

References

[1] P. Erdős, *On the normal number of prime factors of* $p-1$ *and some related problems concerning Euler's* ϕ*-function*, Quart J. Math **6** (1935), 205–213.

[2] ———, *Some remarks on Euler's* ϕ*-function and some related problems*, Bull. Amer. Math. Soc. **51** (1945), 540–544.

[3] ———, *Remarks on number theory. II. Some problems on the* σ *function*, Acta Arith. **5** (1959), 171–177.

[4] P. Erdős and R. L. Graham, *Old and new problems and results in combinatorial number theory*, Monographies de L'Enseignement Mathématique, vol. 28, Université de Genève, Geneva, 1980.

[5] P. Erdős and R. R. Hall, *On the values of Euler's* ϕ*-function*, Acta Arith. **22** (1973), 201–206.

[6] ———, *Distinct values of Euler's* ϕ*-function*, Mathematika **23** (1976), 1–3.

[7] K. Ford, *The distribution of totients*, Ramanujan J. **2** (1998), 67–151.

[8] ———, *The distribution of totients*, Electron. Res. Announc. Amer. Math. Soc. **4** (1998), 27–34 (electronic).

[9] K. Ford, F. Luca, and C. Pomerance, *Common values of the arithmetic functions* ϕ *and* σ, Bull. London Math. Soc. **42** (2010), 478-488.

[10] M. Garaev, *On the number of common values of arithmetic functions* ϕ *and* σ *below* x, (2010), preprint.

[11] H. Halberstam and H.-E. Richert, *Sieve methods*, Academic Press, London, 1974.

[12] H. Maier and C. Pomerance, *On the number of distinct values of Euler's* ϕ*-function*, Acta Arith. **49** (1988), 263–275.

[13] G. Martin, *An asymptotic formula for the number of smooth values of a polynomial*, J. Number Theory **93** (2002), 108–182.

[14] S. S. Pillai, *On some functions connected with $\phi(n)$*, Bull. Amer. Math. Soc. **35** (1929), 832–836.
[15] C. Pomerance, *On the distribution of the values of Euler's function*, Acta Arith. **47** (1986), 63–70.

Department of Mathematics, 1409 West Green Street, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA