MATH 8440 – Assignment #6
last updated April 7, 2023

Turn in <u>three</u> problems.

1. Let f be an arithmetic function. Suppose $D(f,s)$ converges for $s = s_0$. Fix $\delta \in (0,\pi/2)$. Prove that $D(f,s)$ converges uniformly in the open region of s satisfying $\mathrm{Arg}(s-s_0) \leq \frac{\pi}{2} - \delta$.

   Again, you can show "uniform Cauchyness", starting from $\sum_{N_1<n\leq N_2} f(n)n^{-s} = S(N_2)N_2^{s_0-s} - S(N_1)N_1^{s_0-s} - \int_{N_1}^{N_2} S(t)(t^{s_0-s})'\,\mathrm{d}t$, where $S(x) = \sum_{n\leq x} f(n)n^{-s_0}$. It will be helpful that if $\mathrm{Arg}(s-s_0) \leq \delta$, then $\Re(s-s_0) \geq c|s-s_0|$ for a positive constant $c = c(\delta)$.

2. Let f and g be arithmetic functions. Suppose that $D(f,s)$ and $D(g,s)$ converge and represent the same function in the half plane $\Re(s) \geq \sigma$ (where $\sigma \in \mathbf{R}$). Prove that $f = g$.

   In class we proved $f(1) = g(1)$ and sketched an inductive approach to show $f(n) = g(n)$ for all positive integers n. You are being asked to fill in the details.

3. Let f be an arithmetic function and let $S(x) = \sum_{n\leq x} f(n)$ be the corresponding summatory function. Suppose that for a fixed real number α, we have $S(x) = O(x^\alpha)$ (for all $x \geq 1$). Show that $D(f,s)$ converges uniformly on compact subsets of $\Re(s) > \alpha$.

   As a special case ($\alpha = 0$), if the partial sums of $f(n)$ are bounded, then $D(f,s)$ converges uniformly on compact subsets of $\Re(s) > 0$.

4. Let χ be a nontrivial Dirichlet character modulo q. For $\Re(s) > 1$, we defined "$\mathrm{Log}\, L(s,\chi)$" as the sum of a certain double series,

$$\mathrm{Log}\, L(s,\chi) = \sum_p \sum_k \frac{\chi(p^k)}{kp^{ks}}.$$

   (a) Show that this double series converges absolutely for $\Re(s) > 1$.

   (b) Show that $\sum_{p^k} \frac{\chi(p^k)}{kp^{ks}}$ converges uniformly on compact subsets of $\Re(s) > 1$. Here the notation means that the sum is taken over all prime powers p^k (p prime, $k \geq 1$), with the prime powers p^k arranged in increasing order.

   It follows that $\mathrm{Log}\, L(s,\chi)$ is analytic for $\Re(s) > 1$.

   (c) Argue that $\exp(\mathrm{Log}\, L(s,\chi)) = L(s,\chi)$ for $\Re(s) > 1$.

   Thus, $\mathrm{Log}\, L(s,\chi)$ is a genuine logarithm of $L(s,\chi)$.

5. (Dirichlet density = logarithmic density, for sets of primes)

   (a) Show that $\sum_{p>x} p^{-1-1/\log x} = O(1)$, for $x \geq 2$.

   Hint. Apply partial summation along with Chebyshev's upper bound $\pi(x) \ll x/\log x$.

   (b) Show that $\sum_{p\leq x}(p^{-1} - p^{-1-1/\log x}) \ll 1$, for $x \geq 2$.

   Hint. Find a way to use the bound $\sum_{p\leq x} \log p/p \ll \log x$, also shown earlier in class.

(c) Let $\mathcal{P}$ be any set of primes. By applying the results of (a) and (b) with $x = \exp(1/(s-1))$, show that whenever $1 < s < 3/2$,

$$\sum_{p \in \mathcal{P}} \frac{1}{p^s} = \sum_{\substack{p \in \mathcal{P} \\ p \leq \exp(1/(s-1))}} \frac{1}{p} + O(1).$$

(d) Show that a set of primes $\mathcal{P}$ has Dirichlet density[1] δ if and only if

$$\lim_{x \to \infty} \frac{1}{\log \log x} \sum_{\substack{p \leq x \\ p \in \mathcal{P}}} \frac{1}{p} = \delta.$$

6. Let n be a positive integer, say $n \geq 100$. Put $y = \log n / (\log \log n)^3$. Decompose $n = AB$, where[2]

$$A = \prod_{\substack{p^e \| n \\ p \leq y}} p^e, \quad \text{and} \quad B = \prod_{\substack{p^e \| n \\ p > y}} p^e.$$

That is, A is the largest piece of n composed only of primes up to y and B is everything else. In this exercise you will prove a useful upper bound on $d(n) = d(A)d(B)$.[3]

(a) Show that if $p^e \mid n$, then $e \leq \log n / \log 2$. Use this to prove that $d(A) \leq \exp(O(\log n / (\log \log n)^2))$.

(b) Explain why $d(B) \leq 2^{\sum_{p^e \| B} e}$.

(c) Show that $B \geq y^{\sum_{p^e \| B} e}$. Deduce that $\sum_{p^e \| B} e \leq \log n / \log y$.

(d) Prove that for every fixed $\epsilon > 0$ and all large enough values of n,

$$d(n) \leq \exp((\log 2 + \epsilon) \log n / \log \log n).$$

It can be shown that the constant $\log 2$ in part (d) is optimal, in that there are infinitely many n with $d(n) \geq \exp((\log 2 - \epsilon) \log n / \log \log n)$. As $\log n / \log \log n$ is of smaller order than $\log n$, the bound in (d) implies that $d(n) \ll_\delta n^\delta$ for each fixed $\delta > 0$ and all positive integers n; this weaker bound on $d(n)$ is often sufficient for applications.

7. For real $s > 1$, define

$$F(s) = (s-1) \int_1^\infty \left(\sum_{n \leq \log t} 1/n \right) t^{-s} \, \mathrm{d}t.$$

(a) Show that $F(s) = \log \frac{1}{1 - e^{1-s}}$. Deduce that $F(s) = \log \frac{1}{s-1} + o(1)$, as $s \downarrow 1$.

(b) We know from earlier in the semester that $\sum_{n \leq \log t} 1/n = \log \log t + \gamma + O(1/\log t)$ whenever $t \geq \mathrm{e}$. Using this, show that as $s \downarrow 1$,

$$F(s) = \log \frac{1}{s-1} + \gamma + \int_0^\infty \mathrm{e}^{-v} \log v \, \mathrm{d}v + o(1).$$

Suggestion. To estimate $\int_{\mathrm{e}}^\infty (\log \log t) t^{-s} \, \mathrm{d}t$, first substitute $t = \mathrm{e}^u$, and then substitute $u = v/(s-1)$.

[1] Recall that the Dirichlet density of $\mathcal{P}$ is the limit, as $s \downarrow 1$, of $\frac{1}{\log \frac{1}{s-1}} \sum_{p \in \mathcal{P}} \frac{1}{p^s}$.

[2] The notation $p^e \| n$ means that $p^e \mid n$ while $p^{e+1} \nmid n$.

[3] Both $d(n)$ and $\tau(n)$ are used (interchangeably) for the number of positive divisors of n.

(c) Conclude that

$$\gamma = -\int_0^\infty \mathrm{e}^{-v} \log v \, \mathrm{d}v.$$

8. Following the method from class, derive an analytic continuation of $\zeta(s)$ to $\Re(s) > -2$ (apart from the pole at $s = 1$). Use your answer to show $\zeta(-1) = -1/12$.