Math 4000/6000 – Homework #7

posted November 6, 2018; due November 13, 2018

Many who have never had occasion to learn what mathematics is confuse it with arithmetic, and consider it a dry and arid science. In reality, however, it is the science which demands the utmost imagination.
– Sofia Kovalevskaya (1850–1891)

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

In this assignment, "ring" always means "commutative ring."

1. Let $\phi\colon R \to S$ be a homomorphism. Prove that $\phi(R)$ is a subring of S. Recall that, by definition, $\phi(R) = \{\phi(r) : r \in R\}$.

2. Let R be a ring, and let I be an ideal of R. Prove that R/I is the zero ring if and only if $I = R$.

3. In this exercise you will show that isomorphism is an equivalence relation on the class of rings. Since every ring is isomorphic to itself (use the identity map), it is enough to show that isomorphism is symmetric and transitive.

   (a) (Isomorphism is symmetric) Suppose $\phi\colon R \to S$ is an isomorphism. Since ϕ is a bijection, it has an inverse; in other words, there is a map $\psi\colon S \to R$ satisfying

   $$(\psi \circ \phi)(r) = r \text{ for all } r \in R, \quad \text{and} \quad (\phi \circ \psi)(s) = s \text{ for all } s \in S.$$

   Prove that ψ is an isomorphism from S to R.

   Note: You may take as known that ψ is a bijection.

   (b) (Isomorphism is transitive) Suppose $\phi\colon R \to S$ and $\psi\colon S \to T$ are isomorphisms. Prove that $\psi \circ \phi$ is an isomorphism from R to T.

   Note: You may take as known that the composition of bijections is a bijection.

4. (a) Let R be a ring, not the zero ring. We call an ideal $I \subseteq R$ a **prime ideal** if

   (i) $I \neq R$,

   (ii) whenever a and b are elements of R for which $ab \in I$, either $a \in I$ or $b \in I$ (or both).

   Show that for every ideal I of R,

   $$R/I \text{ is an integral domain} \iff I \text{ is a prime ideal of } R.$$

   (b) What are all of the prime ideals of $\mathbb{Z}$? Justify your answer.

5. Exercise 4.2.1.

6. Exercise 4.2.6(b).

7. Let R be a ring.

   (a) If I and J are ideals of R, we let $I + J = \{i + j : i \in I, j \in J\}$. Show that $I + J$ is an ideal of R and that $I + J$ contains both I and J.

   (b) Let $a \in R$, and let I be an ideal of R. Suppose that $\langle a \rangle + I = R$, where $+$ is addition of ideals as defined in part (a). Show that $\overline{a}$ is a unit in R/I.

8. Use the Fundamental Homomorphism Theorem to establish the following ring isomorphisms.

   (a) $R/\langle 0\rangle \cong R$ for every ring R.

   (b) $R[x]/\langle x\rangle \cong R$ for every ring R.

   (c) $\mathbb{R}[x]/\langle x^2+6\rangle \cong \mathbb{C}$.

   Hint: Consider the "evaluation at $i\sqrt{6}$" homomorphism taking $f(x) \in \mathbb{R}[x]$ to $f(i\sqrt{6}) \in \mathbb{C}$.

   (d) $\mathbb{Q}[x]/\langle x^2-1\rangle \cong \mathbb{Q}\times\mathbb{Q}$. *Hint:* Consider the homomorphism from $\mathbb{Q}[x]$ to $\mathbb{Q}\times\mathbb{Q}$ given by $f(x) \mapsto (f(1), f(-1))$.

9. Let p be a prime number. Recall from earlier homework that $1\cdot 2\cdot 3\cdots(p-1) \equiv -1 \pmod{p}$.

   (a) Show that when p is odd, we have $(\frac{p-1}{2}!)^2 \equiv (-1)^{(p+1)/2} \pmod{p}$.

   (b) Deduce from part (a) that if $p \equiv 1 \pmod{4}$, then there is an integer r for which $p \mid r^2+1$.

10. Let m and n be positive integers. Show that if $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$, then $\gcd(m,n) = 1$. (This is the converse of an assertion proved in class.)

    Hint: Consider the smallest number of times you have to add 1 to itself to get 0.

11. (*) How many elements are there in the ring $\mathbb{Z}[i]/\langle 4+3i\rangle$?

    Hint: One way to answer this is to first show that $\mathbb{Z}[i]/\langle 4+3i\rangle$ is isomorphic to a more familiar ring.

12. (*) Let R be a ring, and let I and J be ideals of R with $I+J = R$. Show that the map $\phi\colon R \to R/I \times R/J$ is a surjective (i.e., onto) homomorphism. Use this to deduce that $R/(I\cap J) \cong R/I \times R/J$.