THE INTEGRAL TEST AND LOWER BOUNDS ON THE NUMBER OF PRIMES UP TO N

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Abstract. Using nothing deeper than the integral test from calculus, we prove that the number of primes up to N is eventually larger than any fixed power of $\log N$.

1. Introduction

Let $\pi(N)$ denote the number of primes $p \leq N$. That $\pi(N) \to \infty$ as $N \to \infty$ is one of the oldest and best-known results in number theory. Unfortunately, the simplest proofs of this fact give only very weak lower bounds. For example, Euclid's famous argument can be adapted to show that the nth prime p_n is $< 2^{2^n}$; but this only proves that $\pi(x)$ grows at least as fast as a (constant multiple of the) doubly iterated logarithm of x.

A better bound follows from an argument of Perrott. His starting point was the fact that for each natural natural $N \geq 1$, the proportion of squarefree $n \leq N$ is bounded away from zero. (Recall that n is said to be *squarefree* if it is not divisible by the square of any prime.) Since each squarefree $n \leq N$ is the product of some subset of the primes $\leq N$, this shows that $\pi(N)$ grows at least as fast as a constant multiple of $\log N$. Happily, the positive-proportion result needed in this argument is quite easy to show. Indeed, the proportion of *non*-squarefree $n \leq N$ is clearly at most

$$\sum_p \frac{1}{p^2} < \sum_{n \geq 2} \frac{1}{n^2} < \int_2^\infty \frac{dt}{t^2} = 1,$$

using the familiar integral test from calculus to establish the final inequality.

Here we show that using nothing deeper than the integral test, one can get a much improved lower bound on $\pi(N)$. Suppose $0 < s < 1$. For any natural number $N \geq 1$, it is clear that

$$\sum_{\substack{n \leq N \\ n \text{ squarefree}}} \frac{1}{n^s} \leq \prod_{p \leq N} \left(1 + \frac{1}{p^s}\right).$$

The smallest term in the sum occurs when $n = N$, and so trivially

$$\sum_{n \leq N} \frac{1}{n^s} \geq N^{-s} \#\{n \leq N : n \text{ squarefree}\} \geq AN^{1-s},$$

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for some absolute constant $A > 0$, by the argument in the preceding paragraph. On the other hand, since $1 + x \leq \exp(x)$, we have

$$\prod_{p \leq N} \left(1 + \frac{1}{p^s}\right) \leq \exp\left(\sum_{p \leq N} \frac{1}{p^s}\right).$$

Comparing our bounds, we see that with $A' = \log A$,

$$A' + (1-s)\log N \leq \sum_{p \leq N} \frac{1}{p^s} \leq \sum_{1 \leq n \leq \pi(N)} \frac{1}{n^s}.$$

Comparing the last sum to an integral,

$$\sum_{1 \leq n \leq \pi(N)} \frac{1}{n^s} \leq 1 + \int_1^{\pi(N)} \frac{dx}{x^s} < \frac{\pi(N)^{1-s}}{1-s}.$$

Thus,

$$\pi(N)^{1-s} > (1-s)^2 \log N + A'(1-s).$$

Now take

$$s = 1 - \frac{2}{\sqrt{\log N}}.$$

This gives $\pi(N)^{1-s} > 3 > \mathrm{e}$ for large N, and so

$$\pi(N) > \exp\left(\frac{1}{1-s}\right) = \exp\left(\frac{1}{2}\sqrt{\log N}\right).$$

Since $\exp(u)$ grows faster than any fixed power of u (as $u \to \infty$), this lower bound on $\pi(N)$ grows faster than any fixed power of $\log N$.

References

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