THE PICARD THEOREMS

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Theorem 1 (Big Picard)**.** *Let f have an essential singularity at z_0; then f assumes every value with at most one exception in every deleted neighborhood of z_0 (contained in its domain).*

Remark. We cannot omit "with at most one exception" as the example $f(z) = \exp(1/z)$ shows.

Though it will not be used in the sequel, we note that a weaker result of the same flavor has a much easier proof:

Theorem 2 (Casorati-Weierstrass)**.** *Suppose we are in the situation of big Picard. For any complex number w, there is a sequence $z_1, z_2, \ldots$ tending to z_0 on which $f(z_i) \to w$.*

Proof. Suppose not; then there is an $\epsilon > 0$ and an $r > 0$ with the property that $|f(z) - w| \geq \epsilon$ for $|z - z_0| < r$. Define $F(z) = 1/(g(z) - w)$. Then z_0 is an isolated singularity of F, but $|F(z)| \leq 1/\epsilon$ for $|z - z_0| < r$. As a consequence, z_0 is a removable singularity, and F is analytic on $B(z_0, r)$. But then $g(z) = w + 1/F(z)$ is either analytic at z_0 (if $F(z_0) \neq 0$) or has a pole there. Either case contradicts the assumption that g is an essential singularity. □

We shall prove Picard's great theorem following a method of Landau.

Proposition 1 (Landau)**.** *There is a positive constant C with the following property: If f is a function analytic on the open unit ball with $|f'(0)| \geq 1$, then $f(B(0, r))$ contains a ball of radius at least C. In fact, we may take $C = 1/13$.*

Lemma 1. *Let $F \in \mathrm{Hol}(B(0, 1))$ with $F(0) = 0, F'(0) = 1$ and $|F'| \leq M$. Then*

$$F(B(0, 1)) \supset F(B(0, 1/(M + 1)) \supset B(0, 1/(2M + 1)).$$

Proof. Consider the function

$$z \to \frac{F'(z) - 1}{M + 1}.$$

This function is holomorphic in $B(0, 1)$ and maps $B(0, 1)$ into $B[0, 1]$; since this function also vanishes at 0, Schwarz's lemma implies

$$|F'(z) - 1| \leq (M + 1)|z|$$

for $|z| < 1$. Integrate along $[0, z]$ to get (using $F(0) = 1$)

$$\begin{aligned} |F(z) - z| &= \left|\int_0^z F'(t) - 1\, dt\right| = \left|\int_0^1 (F'(uz) - 1)z\, du\right| \\ &\leq |z| \int_0^1 |F'(uz) - 1|\, du = (M + 1)|z|^2 \int_0^1 u\, du = \frac{M + 1}{2}|z|^2. \end{aligned}$$

Suppose $w \in B(0, 1/(2M+1))$. We will consider the functions

$$G_1(z) = F(z) - w, \quad G_2(z) = z - w$$

defined for, say, $z \in B(0, 1/(M+1))$. By what we proved above,

$$|G_1(z) - G_2(z)| = |F(z) - z| \leq \frac{M+1}{2}|z|^2 = \frac{1}{2(M+1)} < |z-w| = |G_2(z)|$$

on the boundary of $B(0, 1/(M+1))$. Consequently, Rouche's theorem implies $G_1(z)$ and $G_2(z)$ have the same number of zeros inside $B(0, 1/(M+1))$. But G_2 has exactly one zero here, so so does G_1; consequently, $w \in F(B(0, 1/(M+1))$. But w was just any element of $B(0, 1/2(M+1))$, so $F(B(0, 1/(M+1)) \supset B(0, 1/2(M+1))$. $\square$

Proof of Proposition 1. It suffices to consider the case $|f'(0)| = 1$.

Let us assume for the moment that f is holomorphic on an open set containing $B[0,1]$. Define $a : [0,1] \to [0, \infty)$ by

$$a(r) := (1-r) \sup_{|z|=r} |f'(z)| \quad (0 \leq r \leq 1).$$

One checks that a is a continuous function. Noting that $a(0) = 1$ and $a(1) = 0$, it follows that there is a largest $s \in [0,1)$ with $a(s) = 1$, and for this s we have $a(t) < 1$ when $s < t \leq 1$. Choose ζ with $|\zeta| = s$ and $|f'(\zeta)| = (1-s)^{-1}$, and define for $R = (1-s)/2$

$$F(z) := 2(f(Rz + \zeta) - f(\zeta)) \qquad (|z| \leq 1).$$

Let us check the hypotheses of the previous lemma are satisfied with $M = 2$. F is analytic in $B(0,1)$, and it's easily checked that $F(0) = 0$ and $F'(0) = 1$. Let us bound $F'(z)$ for $z \in B(0,1)$:

$$\begin{aligned} |F'(z)| = 2R|f'(Rz+\zeta)| \leq 2R \sup_{|w| \leq R+s} |f'(w)| &= 2R \sup_{|w|=R+s} |f'(w)| \\ &= \frac{2R}{1-(R+s)} a(R+s) = 2a(R+s) \leq 2, \end{aligned}$$

since s is the largest value at which the function a is at least 1. Consequently, $F(B(0,1))$ contains a ball of radius of $1/6$, and referring to the definition of F shows that $f(B(0,1))$ contains a ball of radius at least $1/12$.

In the general case, consider $f(\rho z)/\rho$, where $\rho = 12/13$. This function satisfies the above hypotheses; it follows that $f(B(0,1))$ contains a ball of radius at least $12/13 \cdot 1/13 = 1/13$. $\square$

Proposition 2 (Schottky)**.** *Let $M > 0$ and $r \in (0,1)$. There exists a constant $C = C(M, r)$ with the following property: If $f \in \mathrm{Hol}(B(0,1))$ and omits the values $0, 1$, and if $|F(0)| \leq M$, then $|F(z)| \leq C$ for $|z| \leq r$.*

Proof. Let $\log F$ denote an analytic logarithm of F (which exists since f is nonvanishing and $B(0,1)$ is simply connected), normalized so that $|\Im(\log F(0))| \leq \pi$. Let $A := (2\pi i)^{-1} \log F$. Then A never assumes integer values because F omits the value 1.

Thus A and $A-1$ are nonvanishing functions. By simply-connectedness, we can choose holomorphic square roots $\sqrt{A}$ and $\sqrt{A-1}$. Define

$$B := \sqrt{A} - \sqrt{A-1}.$$

Then B is holomorphic in $B(0,1)$ and nonvanishing. Moreover, B doesn't assume any of the values $\sqrt{n} \pm \sqrt{n-1}$ for positive integral n– a simple computation shows that if it did, then we would have $A(z) = n$.

Now let H be an analytic logarithm of B, normalized so that $|\Im(H(0))| \leq \pi$. Then H never assumes any of the values

$$a_{n,m} = \mathrm{Log}(\sqrt{n} + \sqrt{n-1}) + 2\pi i m \qquad (n = 1, 2, \ldots, \text{ and } m \in \mathbf{Z}).$$

But these points in the plane are such that every open disc of sufficiently large radius contains at least one of them. In fact, one checks radius 10 works here. It follows that the range of H doesn't cover any disc of radius 10.

Let $z \in B(0,1)$ and suppose $H'(z) \neq 0$. Then the function

$$\zeta \to \frac{H(\zeta) - H(z)}{H'(z)} \qquad (\zeta \in B(z, 1 - |z|))$$

must fill of a disc of radius $(1-|z|)/13$, by our proof of Landau's theorem. So the values of H fill a disc of radius $|H'(z)|(1-|z|)/13$; thus

$$|H'(z)| \leq 130/(1-|z|).$$

This remains valid when $H'(z) = 0$. If we integrate the estimate over $[0, z]$, we see

$$(1) \quad |H(z)| \leq |H(0)| + 130\mathrm{Log}(1-|z|)^{-1} \leq |H(0)| - 130\mathrm{Log}(1-r) \quad (|z| \leq r).$$

Since

$$\exp H = \sqrt{\frac{\log F}{2\pi i|}} - \sqrt{\frac{\log F}{2\pi i} - 1},$$

we have

$$\sqrt{\frac{\log F(z)}{2\pi i}} = \frac{e^{H(z)} + e^{-H(z)}}{2},$$

so that

$$(2) \qquad F = -\exp\left(\frac{\pi i}{2}(e^{2H} + e^{-2H})\right) \quad \text{and} \quad |F| \leq \exp(\pi e^{2|H|}).$$

It suffices to prove the result for functions satisfying $|F(0)| \geq 1/2$, for otherwise apply the result to $1 - F$. In this case there is a constant C_2 depending only on M such that

$$C_2 \geq \left| \frac{e^{H(0)} + e^{-H(0)}}{2} \right| \geq \sinh(\Re(H(0))).$$

This bounds $\Re(H(0))$ above and below, by a constant depending only on M, while $\Im(H(0))$ is bounded (in abs. value) by π, by our initial choice. Consequently, equation (1) implies $|H|$ is bounded on $B[0,r]$ by constants depending only on M and r, so that F is as well, by (2). □

Proof of Big Picard. By translation and dilation, we can assume $z_0 = 0$ and that f is analytic on $B(0, 2\pi)$. Suppose f omits a, b. Then

$$z \to \frac{f(z) - a}{b - a}$$

omits $0, 1$. So it will suffice to show there is no function f analytic on $B(0, 2\pi) \setminus \{0\}$ which omits $0, 1$ and has an essential singularity at 0.

If f is such a function, then $\lim_{z\to 0} |f(z)|$ isn't infinite (otherwise 0 would be a pole), so there is an infinite sequence of $z_i \to 0$ with

$$1 > |z_1| > |z_2| > \cdots > |z_n| > \dots$$

with $|f(z_i)| \le M$, say. For fixed i, consider the function

$$G(\zeta) := f(z_i e^{2\pi i\zeta}) \qquad (\zeta \in B(0,1)).$$

This is holomorphic on $B(0,1)$, omits the values 0 and 1 and satifies $|G(0)| = |f(z_i)| \le M$. Schottky's theorem says that there is a constant C (depending only on M – so in particular not on i) with the property that

$$|G(\zeta)| = |f(z_i e^{2\pi i\zeta})| \le C \qquad (\zeta \in B[0,1/2]).$$

In particular, it implies

$$|f(z_i e^{2\pi i t})| \le C \qquad (-1/2 \le t \le 1/2).$$

Thus f is bounded by C on the circle $|z| = |z_i|$.

This holds for every i, so f is bounded by C on the boundary of each concentric annulus (with centers at 0 and radii $|z_i|, |z_{i-1}|$). By the maximum modulus principle, f is bounded by C on the interiors of these annuli as well. But every point of $B(0,|z_1|) \setminus \{0\}$ lies in one of these annuli. It follows that f remains bounded on a neighborhood of 0, so 0 is a removable singularity. □

Corollary 1 (Little Picard)**.** *Let f be a nonconstant entire function. Then f assumes every value with at most one exception.*

Remark. Again, the example $f(z) = \exp(z)$ shows one cannot hope to prove f assumes every value.

Proof. Let $g(z) = f(1/z)$. If g has an essential singularity at $z = 0$, then we're done by Big Picard. If g has a pole or a removable singularity at 0, then $H(z) := z^h g(z)$ will be entire for some nonnegative integer h, and $f(z) = g(1/z) = z^h(g(1/z)z^{-h}) = z^h H(1/z)$. Consequently,

$$|f(z)| \le |z|^h(|H(0)| + 1)$$

if $|z|$ is sufficiently large. By Liouville, f is a polynomial; as f is nonconstant, its range is all of $\mathbf{C}$ by the fundamental theorem of algebra. □

1. Unproved Lemmata

Lemma 2 (Rouche's Theorem)**.** *Let f, g be defined and analytic on Ω, and let K be a compact subset of Ω. Suppose that for all z along ∂K, one has $|f - g| < |f| + |g|$; then f and g have the same number of zeros inside K.*

Lemma 3 (Maximum Modulus Principle)**.** *Let K be a compact set and suppose f is analytic on the interior of K and continuous on all of K. Then $|f|$ assumes its maximum value on the boundary of K.*

Lemma 4 (Liouville's Theorem)**.** *Let f be an entire function and suppose there exists a nonnegative integer h with $|f| \ll |z|^h$ for all large $|z|$. Then f is a polynomial of degree at most h.*