MATH 3100 – Homework #4
posted September 29, 2023; due date TBA

Answer the questions, then question the answers. – Glenn Stevens

Section and exercise numbers correspond to the online notes. Assignments are expected to be **neat** and **stapled**, with problems submitted **in the order they appear below**. **Illegible work may not be marked.**

Required problems

1. OMIT. Start numbering with Problem #2.
2. §1.5: 6
3. §1.6: 5
4. Show that if A and B are nonempty sets of real numbers that are bounded above, and $A \subseteq B$, then $\operatorname{lub} A \leq \operatorname{lub} B$.

   Hint: There's a very short solution once you understand all the definitions.

5. Let $\{a_n\}$ be a bounded sequence. For each natural number k, define

$$T_k = \{a_n : n \geq k\}.$$

   We refer to T_k as the **k-tail set** of $\{a_n\}$: it is the collection of all real numbers that appear in the sequence at some index at least k. Since $\{a_n\}$ is bounded, each T_k is also bounded (above and below). Thus, the Least Upper Bound property implies that each T_k has a least upper bound. We let L_k denote the least upper bound of T_k; that is,

$$L_k = \operatorname{lub}\{a_n : n \geq k\}.$$

   (So far you are being told all of this; you are not asked to prove the above facts.)

   (a) Show that the sequence $L_1, L_2, L_3, \ldots$ is decreasing.
   (b) Show that the sequence $L_1, L_2, L_3, \ldots$ is bounded below.
   (c) Quickly explain why (a) and (b) imply that $\{L_k\}$ converges.

   Remark. The limit of the sequence $\{L_k\}$ in part (c) is denoted "$\limsup a_n$". That is,

$$\limsup a_n = \lim \operatorname{lub}\{a_n : n \geq k\}.$$

   (This looks less weird when you remember that sup is commonly used in place of lub.)

6. Let $\{a_n\}$ be a bounded sequence and let $L = \limsup a_n$.

   (a) Show that for every $\epsilon > 0$ and every positive integer k, there is natural number $n \geq k$ with $a_n > L - \epsilon$.

   Hint: Look back at the definition of L_k. Which is bigger, $L - \epsilon$ or L_k?

(b) Show that for every $\epsilon > 0$, there is a $K \in \mathbf{N}$ such that

$$a_n < L + \epsilon \quad \text{for all } n \geq K.$$

(c) By combining (a) and (b), show that for every $\epsilon > 0$, and every positive integer K, there is a natural number $n \geq K$ with $L - \epsilon < a_n < L + \epsilon$.

(d) Prove that there is a subsequence of $\{a_n\}$ converging to L.

Hint: Choose n_1 so that a_{n_1} is within 1 of L, then choose $n_2 > n_1$ with a_{n_2} within $\frac{1}{2}$ of L, then $n_3 > n_2$ with a_{n_3} within $\frac{1}{3}$ of L, etc.

Remark. With just a little more work, it can be proved that any convergent subsequence of $\{a_n\}$ converges to a number at most L. That is, $\limsup a_n$ is the largest limit of any convergent subsequence of $\{a_n\}$. Try showing this as practice!

7. §1.7: 1

8. §1.7: 3

Hint: If $r > 1$, show that the hypotheses of Theorem 1.7.3 hold with $f(x) = x^2 - r$ and the closed interval $[0, r]$. This choice of interval doesn't work if $0 < r \leq 1$. (Make sure you understand why!) Can you think of an interval which **does** work?

Recommended problems (NOT to turn in)

§1.6: 9, 10, 12
§1.7: 4, 5, 6