MATH 4400/6400 – Homework #2

posted January 30, 2023; due Feb. 8, by midnight

God made the integers, all else is the work of man.
– L. Kronecker

Directions. Give complete solutions, providing full justifications when appropriate. Your assignment must be stapled if it goes on beyond one page.

MATH 4400 problems

1. Give a careful proof that if π is prime in $\mathbb{Z}[i]$, so is $\epsilon\pi$ for each unit ϵ of $\mathbf{Z}[i]$.

2. Prove that if π is prime in $\mathbf{Z}[i]$, then $\pi \mid p$ for some ordinary prime p (of $\mathbf{Z}$).

3. Let $\alpha, \beta \in \mathbf{Z}[i]$.

   (a) Prove or disprove: If α, β have 1 as a greatest common divisor in $\mathbf{Z}[i]$, then $N(\alpha)$ and $N(\beta)$ have 1 as a greatest common divisor in $\mathbf{Z}$.

   (b) Prove or disprove: If $N(\alpha), N(\beta)$ have 1 as a greatest common divisor in $\mathbf{Z}$, then α and β have 1 as a greatest common divisor in $\mathbf{Z}[i]$.

4. Use the Euclidean algorithm to compute a greatest common divisor δ of $108 + i$ and $3 - 14i$. Use your work to find Gaussian integers μ, ν with $(108 + i)\mu + (3 - 14i)\nu = \delta$.

For each positive integer d, we can consider the ring $\mathbf{Z}[\sqrt{-d}] = \{a + b\sqrt{-n} : a, b \in \mathbf{Z}\}$. (The cases $n = 1$ and $n = 2$ were discussed in class.) Writing, as usual, $N(\alpha) = \alpha\bar{\alpha}$, we have

- $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in \mathbf{C}$,
- $N(\alpha) \in \mathbf{Z}_{\geq 0}$ for all $\alpha \in \mathbf{Z}[\sqrt{-d}]$, with $N\alpha = 0$ only if $\alpha = 0$,
- when $\alpha \in \mathbf{Z}[\sqrt{-d}]$, $\quad N(\alpha) = 1 \iff \alpha$ is a unit in $\mathbf{Z}[\sqrt{-d}]$.

(We omit the proofs, which are analogous to those in $\mathbf{Z}[i]$ and $\mathbf{Z}[\sqrt{-2}]$.)

5. Let d be a positive integer, $d > 1$. Show that ± 1 are the only units in $\mathbf{Z}[\sqrt{-d}]$.

6. Show that every nonzero, nonunit element of $\mathbf{Z}[\sqrt{-d}]$ can be written as a product of primes.

   Recall our definition of prime: π is prime if it is a nonzero, nonunit such that, whenever $\pi = \alpha\beta$, one of α or β is a unit.

7. If $\mathbf{Z}[\sqrt{-d}]$ obeys Euclid's lemma, then arguing as in class, one can prove that $\mathbf{Z}[\sqrt{-d}]$ obeys the unique factorization theorem. Give a careful proof of the converse. That is, show that if $\mathbf{Z}[\sqrt{-d}]$ obeys the unique factorization theorem, then it also obeys Euclid's lemma.[1]

8. Let d be a positive integer, $d \geq 3$.

[1] Here is a careful statement of what we mean when we say a domain D 'obeys the Unique Factorization Theorem': Every nonzero nonunit $\alpha \in D$ can be written as a product of primes in D. If $\alpha = \pi_1 \cdots \pi_k = \rho_1 \cdots \rho_\ell$ are two factorizations of the same nonzero nonunit α, then (a) $k = \ell$ and (b) after rearranging the ρ_j, there are units $\epsilon_1, \ldots, \epsilon_k$ of D with $\pi_i = \epsilon_i \rho_i$ for all $i = 1, 2, \ldots, k$.

(a) Show that 2 is prime in $\mathbf{Z}[\sqrt{-d}]$.

(b) Show that $2 \mid (d+\sqrt{-d})(d-\sqrt{-d})$ in $\mathbf{Z}[\sqrt{-d}]$ while $2 \nmid d+\sqrt{-d}$ and $2 \nmid d-\sqrt{-d}$.

(c) Conclude that $\mathbf{Z}[\sqrt{-d}]$ does not have unique factorization.

MATH 6400 problem

G1. Let $R = \{\frac{a+b\sqrt{-3}}{2} : a, b \in \mathbf{Z}, a \equiv b \pmod 2\}$. Prove that R contains 1 and is closed under multiplication and subtraction. (As you know from MATH 4000, it follows that R is a subring of $\mathbb{C}$.)

G2. (continuation)

(a) Show that the norm map, restricted to R, maps R into $\mathbf{Z}_{\geq 0}$.

(b) Prove the division algorithm holds in R: Given $\alpha, \beta \in R$ with $\beta \neq 0$, there are $\gamma, \rho \in R$ with $\alpha = \beta\gamma + \rho$ and $N\rho < N\beta$.