Math 4000 – Learning objectives to meet for Exam #2

The exam will cover §2.2–§3.3 of the textbook (excluding §2.5). The proof of Gauss's lemma (to be covered on 10/24) is not examinable.

What to be able to state

Basic definitions

You should be able to give precise descriptions of all of the following:

- definition of $\mathbf{R}$ (a complete ordered field satisfying the conclusion of the monotone convergence theorem)
- the construction of $\mathbf{C}$ from $\mathbf{R}$ by ordered pairs
- complex conjugate of a complex number
- absolute value of a complex number
- polar form of a complex number
- definition of the ring $R[x]$ (starting with a commutative ring R) and allied concepts (such as the degree of a polynomial)
- definition of an irreducible polynomial in $F[x]$ (with F a field)
- the multiplicity of a root of a polynomial
- gcd of two elements of $F[x]$
- subring of a ring
- subfield and field extension
- definition of $F[\alpha]$, where F is a field and α is an element of a field extension of F
- definition of what it means for α to be algebraic over F
- what it means to say a polynomial $f(x) \in F[x]$ splits in an extension K of F
- what it means to say a field K is a splitting field of $f(x)$ over F

Big theorems

Give full statements of each of the following results, making sure to indicate all necessary hypotheses. For results proved in class, describe the components and main ideas of the proof.

- $\mathbf{Q}$ embeds into $\mathbf{R}$ (i.e., the map $a/b \mapsto ab^{-1}$ is well-defined, one-to-one, and preserves operations)
- theorems associated with multiplication of complex numbers in polar form, including de Moivre's theorem

- there are n distinct nth roots of 1 in $\mathbf{C}$, namely the numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$
- a nonzero complex number has exactly n distinct nth roots
- the quadratic formula
- description of the roots of a cubic polynomial
- $\deg(a(x)b(x)) = \deg a(x) + \deg b(x)$ when R is a domain
- the division algorithm in $F[x]$
- root-factor theorem
- Euclid's lemma for $F[x]$ and the unique factorization theorem for $F[x]$
- the Fundamental Theorem of Algebra (statement only)
- If K is a field extension of F and $\alpha \in K$, then $F[\alpha]$ is a field if α is algebraic over F
- rational root theorem
- Gauss's lemma about polynomial factorizations (statement only)
- theorem about irreducibility mod p vs. irreducibility over $\mathbf{Q}$ (Proposition 3.4 in the book)
- proofs of irreducibility via Eisenstein's criterion (as in the example of $x^{2016} - 7$ from class)

What to be able to compute

You are expected to know how to use the methods described in class to solve the following problems.

- Basic computations with complex numbers, in either rectangular (i.e., $a + bi$) or polar form
- Perform "long division" of polynomials with quotient and remainder; use this to perform the Euclidean algorithm, compute gcds, and express the gcd as a linear combination
- Explicitly compute (exactly) the complex roots of given quadratic or cubic polynomials
- Determine all rational roots of a given polynomial $f(x) \in \mathbf{Z}[x]$
- Argue that given polynomials are irreducible over $\mathbf{Q}$

Extra problems

Carefully review the HW solutions. I also recommend looking at the following problems:

§2.2: 8, 12

§2.3: 2, 7, 21, 22

§2.4: 9(b)

§3.1: 1, 2(b,c,d,e), 5, 9(c), 10(b,d), 11, 12, 13, 14

§3.2: 1, 6(a,b), 7

§3.3: 1(a,c,e,f,h), 3(a), 4, 5, 7

Here are two more problems to try.

1. Let K be a field extension of F. Prove that if $\alpha \in K$ and α is not algebraic over F, then $F[\alpha]$ is not a field. In fact, show that α has no inverse in $F[\alpha]$.
2. Recall that the notation "$F[\alpha, \beta]$" means $F[\alpha][\beta]$. Let

$$K = \mathbf{Q}[\sqrt[5]{2}, \cos(2\pi/5) + i\sin(2\pi/5)].$$

First, show that K is a field. Then show that K is a splitting field of $x^5 - 2$ over $\mathbf{Q}$.