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Results in Mathematics

# On numbers $n$ for which the prime factors of $\sigma(n)$ are among the prime factors of $n$

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## Introduction

For any positive integer $n$ let

$$\operatorname{rad}(n) := \prod_{p|n} p. \tag{1}$$

At the West Coast Number Theory Conference in San Diego, December 2000, Jean-Marie De Koninck asked for all the positive integers $n$ for which

$$\sigma(n) = \operatorname{rad}(n)^2 \tag{2}$$

holds, where $\sigma(n)$ is the sum of divisors of $n$. Note that $n = 1$ and $n = 1782$ are solutions of the above equation. In this note, we look at the numbers $n$ such that all the prime factors of $\sigma(n)$ divide $n$. If we do not impose any condition on the exponents of the prime factors of $\sigma(n)$, i.e., if we only require $\operatorname{rad}(\sigma(n)) \mid \operatorname{rad}(n)$, then we cannot say much except for the fact that there are infinitely many such $n$ (take, for example, $n := \prod_{p<x} p$ where $x > 2$ is any large real number). We thus put an upper bound on the exponents of the prime factors of $\sigma(n)$, say $K$, and we investigate the set of positive integers $n$ such that $\sigma(n) \mid \operatorname{rad}(n)^K$.

Our main result shows that if $T \geq 1$ is any given positive integer, then there are only finitely many effectively computable values of $n$ satisfying the above divisibility relation and for which $\omega(n) \leq T$ holds, where $\omega(n)$ is the number of distinct prime factors of $n$.

**Theorem.**
*Let $K$, $L$, and $T$ be fixed positive integers. If $n$ is a positive integer with $\omega(n) = T$ and such that*

$$\sigma(n) = am, \tag{3}$$

*holds with some positive integer $a \leq L$ and some positive integer $m$ with $m \mid \operatorname{rad}(n)^K$, then*

$$n < \exp\left((M \cdot T!)^{2^T}\right), \tag{4}$$

*where $M$ is any positive constant such that $M \geq K + \log L$.*

In particular, for any $T \geq 1$, there exist only finitely many positive integers $n$ satisfying relation (2) and for which $\omega(n) \leq T$, and all such $n$ satisfy

$$n < \exp\left((2T!)^{2^T}\right).$$

Recall that a number $n$ is called *multiply perfect* if $n \mid \sigma(n)$. When $\sigma(n) = 2n$, the number $n$ is called *perfect*. It is known that if $n$ is a multiply perfect number having $\omega(n) \leq T$, then $n$ can be bounded in terms of $T$. In fact, from the comments on problem **B1** in [1], we learn that Pomerance showed that if $n$ is odd and perfect and has $\omega(n) \le T$, then

$$n < \exp\big((\log 4T)\cdot(4T)^{2^{T^2}}\big),$$

and that the above upper bound on $n$ has been improved by D.R. Heath-Brown (see [3]), who showed that if $n$ is an odd number with $\omega(n) \le T$ and $\sigma(n) = an$, where $a$ is a *rational number*, then $n < (4d)^{4^T}$, where $d$ is the denominator of $a$. Note that our upper bound (4) is quadruple exponential in $T$, while Heath-Brown's bound on the size of an odd perfect number with at most $T$ prime factors is only triple exponential in $T$. We also point out that the analogous problem of finding positive integers $n$ such that $\phi(n) = (\mathrm{rad}(n))^K$, where $\phi$ is the Euler function, was treated in [2]. There, it is shown that for each $K$ the above equation has finitely many effectively computable solutions $n$, and if $N_K$ denotes the number of such, then $N_K \ge \exp(cK\log K)$ holds for all $K \ge 1$, where $c$ is some positive constant.

## The Proof

**The Proof of the Theorem.** We proceed by induction on $\omega(n) = T$ to show that if $n$ satisfies (3), then the inequality

$$L\mathrm{rad}(n)^K < \exp\big((M\cdot T!)^{2^T}\big) \tag{5}$$

holds. When $T = 1$, then $n = q^e$ holds with a prime number $q$ and a positive integer $e$, and since $\sigma(n)$ and $q$ are coprime, relation (3) implies that $q \le q^e \le \sigma(n) \le L$. Thus,

$$L\mathrm{rad}(n)^K = Lq^K \le \exp(\log L + K\log L) < \exp(M^2). \tag{6}$$

Assume now that $T \ge 2$ is a fixed positive integer, and that the inequality asserted at (5) holds for all positive integers $n'$ with $\omega(n') = T' < T$, and for which $\sigma(n')$ can be written under the form $a'm'$ with some $a' \le L'$, and some $m'$ with $m' \mid \mathrm{rad}(n')^{K'}$, and with $M$ replaced by $M' := K' + \log L'$. We write $n$ as

$$n := q_1^{e_1}\cdot\ldots\cdot q_T^{e_T}, \tag{7}$$

with the convention that $q_1 > q_2 > \ldots > q_T$. From now on, we split the argument into several steps.

**Step I.** *Assume that* $q_T \le \max\{2, L\}$.

Note that not both $K$ and $L$ can be 1, because if this were the case, then (3) would imply that $\sigma(n) \mid \mathrm{rad}(n)$, which is impossible for $n > 1$ because $\sigma(n) > n \ge \mathrm{rad}(n)$ holds. Since $q_T \le \max\{2, L\}$, it follows that $q_T \le L + 1$. Write

$$L' := L\cdot(L+1)^K = \exp(\log L + K\log(L+1)).$$

One can easily check that since $K$ and $L$ are positive integers, then the inequality

$$\log L + K\log(L+1) < (\log L + K)^2 \le M^2$$

holds. Thus,

$$Lq_T^K \le L\cdot(L+1)^K = L' < \exp(M^2).$$

Writing $n' := n/q_T^{e_T}$, we infer from (3) that

$$\sigma(n') = a'm'$$

holds, where $a' \leq aq_T^K \leq L' < \exp(M^2)$, $m' \mid \mathrm{rad}(n')^K$, and $\omega(n') = \omega(n) - 1 = T - 1$. With $M' := \log L' + K < M^2 + K \leq 2M^2$ and the induction hypothesis, it follows that

$$L\mathrm{rad}(n)^K = Lq_T^K \mathrm{rad}(n')^K = L'\mathrm{rad}(n')^K < \exp\big((2M^2)(T-1)!)^{2^{T-1}}\big) <$$

$$\exp\big(M^{2^T}(2(T-1)!)^{2^{T-1}}\big) < \exp\big((M \cdot T!)^{2^T}\big), \tag{8}$$

which is precisely the desired inequality. In the above inequality (8), we used the obvious inequality $2(T-1)! \leq T!$, which holds for all $T \geq 2$.

From now on, we assume that $q_T > \max\{2, L\}$. In what follows, we shall use induction on $i$ for $i = 1,\ 2,\ \ldots,\ T$ to find upper bounds on the exponents $e_i$ for $i = 1, \ldots, T$.

**Step II.** *An upper bound on $e_1$.*

This is trivial to find by the following argument. We write

$$q_1^{e_1} < \sigma(q_1^{e_1}) = a_1 \cdot \prod_{j=2}^{T} q_j^{f_{1j}}, \tag{9}$$

where $a_1 \leq L$, and $f_{1j} \leq K$ holds for all $j = 2,\ \ldots,\ T$. Thus,

$$q_1^{e_1} < Lq_1^{(T-1)K},$$

or

$$q_1^{e_1-(T-1)K} < L.$$

Since $q_1 > 3 > e$ (because $q_T > 2$ and $T \geq 2$), it follows that

$$e_1 - (T-1)K < \log L,$$

therefore

$$e_1 < (T-1)K + \log L \leq MT < M \cdot T!.$$

Let $E_1 := M \cdot T!$ be the upper bound on the exponent $e_1$ of $q_1$. Assume now that $1 \leq i < T$, and that we know the upper bounds $E_1,\ E_2,\ \ldots,\ E_i$ on $e_1,\ e_2,\ \ldots,\ e_i$, respectively. We shall derive an upper bound $E_{i+1}$ on the exponent $e_{i+1}$.

**Step III.** *An upper bound on $e_{i+1}$: the nondegenerate case.*

We shall first show, by induction on the paramater $i$, that as long as a certain determinant which we shall denote by $\Delta_i$ and we shall define below does not vanish, then one may take $E_i := (M \cdot T!)^{2^{i-1}}$, and we shall return to the case in which this determinant vanishes later. From what we have just said above, this is certainly so when $i = 1$.

To prove what we have just said, we write

$$\sigma(q_l^{e_l}) = a_l \prod_{j=1}^{T} q_j^{f_{lj}}, \qquad \text{with } l := 1,\ 2,\ \ldots,\ i, \tag{10}$$

where, of course, $f_{lj}$ are nonnegative integers for all $l = 1,\ 2,\ \ldots,\ i$, and all $j = 1,\ 2,\ \ldots,\ T$ and with

$$\sum_{l=1}^{i} f_{lj} \leq K$$

for all $l = 1, 2, \ldots, i$, and $\prod_{l=1}^{i} a_l \leq L$. In particular, both $a_l \leq L$, and $f_{lj} \leq K$ hold for all $l = 1, 2, \ldots, i$, and all $j = 1, 2, \ldots, T$. The observation that we employ now is that

$$q^e < \sigma(q^e) < 2q^e$$

holds for all prime numbers $q \geq 2$ and all positive integers $e$. In particular, by (9) and (10), we conclude that

$$\frac{a_l}{2} \cdot \prod_{j=1}^{T} q_j^{f_{lj}} < q_l^{e_l} < a_l \prod_{l=1}^{T} q_j^{f_{jl}} \tag{11}$$

holds for all $l = 1, 2, \ldots, i$. Taking logarithms in (11) and writing

$$X_l := \log q_l, \qquad A_l := \log\big(a_l \prod_{j=i+1}^{T} q_j^{f_{lj}}\big) - \log 2, \qquad B_l := A_l + \log 2,$$

we get that the following inequalities hold

$$A_l < e_l X_l - \sum_{j=1}^{i} f_{lj} X_j < B_l, \qquad \text{for all } l = 1, 2, \ldots, i. \tag{12}$$

For each value of $l$ between 1 and $i$, let $L_l(x_1, \ldots, x_i)$ be the linear form on $\mathbf{R}^i$ given by

$$L_l(x_1, \ldots, x_i) := e_l x_l - \sum_{j=1}^{i} f_{lj} x_j. \tag{13}$$

Notice that since $q$ does not divide $\sigma(q^e)$ for any prime number $q$ and any positive integer $e$, it follows that $f_{ll} = 0$ holds for all $l = 1, 2, \ldots, i$. Let $D_i$ be the $i \times i$ matrix whose entries are $D_i(l,l) := e_l$ for $l = 1, 2, \ldots, i$, and $D_i(l,j) := -f_{lj}$ for all $l \neq j$ with both $l$ and $j$ in the set $\{1, 2, \ldots, i\}$. That is, the $l$th row of the matriz $D_i$ is

$$r_l := (-f_{l1}, -f_{l2}, \ldots, -e_l, \ldots, -f_{li}),$$

i.e., is the normal vector to the linear form $L_l(x_1, x_2, \ldots, x_i)$ for all $l = 1, 2, \ldots, i$. Let $\Delta_i$ be the determinant of $D_i$.

Assume that $\Delta_i \neq 0$. Obviously, $|\Delta_i| \geq 1$ because all the entries of $D_i$ are integers. We may therefore rewrite system (12) under the form

$$e_l X_l - \sum_{j=1}^{i} f_{lj} X_j = \lambda_l, \qquad \text{for all } l = 1, 2, \ldots, i, \tag{14}$$

where $\lambda_l$ is some number satisfying

$$|\lambda_l| < \sum_{j=i+1}^{T} f_{lj} \log q_j \leq M(T-i) \log q_{i+1}, \tag{15}$$

for all $l = 1, 2, \ldots, i$. Since $\Delta_i \neq 0$, it follows that system (14) is nonsingular, therefore we can solve it, by using Cramer's rule, for example. Since $|\Delta_i| \geq 1$, we use Cramer's rule together with the assumed values of $E_i$ on $e_i$ (notice that $E_i > E_{i-1} > \ldots > E_1 > K$), and we bound each involved determinant by the value of the permanent of the corresponding matrix constructed with the absolute values of the entries of the original determinant, to get

$$X_l < i! \cdot E_2 \cdot \ldots \cdot E_i \cdot \max_{l=1}^{i}\{|\lambda_l|\} \leq i! \cdot E_2 \cdot \ldots \cdot E_i \cdot \left(\max_{l=1}^{i}\{\log a_l\} + K(T-i) \cdot \log q_{i+1}\right). \quad (16)$$

In particular, we get that

$$\sum_{l=1}^{i} \log q_l < i! \cdot i \cdot E_2 \cdot \ldots \cdot E_i \cdot \left(\max_{l=1}^{i}\{\log a_l\} + K(T-i)\log q_{i+1}\right). \quad (17)$$

Thus,

$$q_{i+1}^{e_{i+1}} = a_{i+1} \prod_{l=1}^{i} q_l^{f_{l,i+1}} \prod_{l>i+1} q_l^{f_{l,i+1}},$$

or

$$e_{i+1}\log q_{i+1} < \log a_{i+1} + K\sum_{l=1}^{i}\log q_l + K\sum_{l=i+2}^{T}\log q_l <$$

$$\log a_{i+1} + K \cdot i! \cdot i \cdot E_2 \cdot \ldots \cdot E_i \cdot \left(\max_{l=1}^{i}\{\log a_l\} + K(T-i)\log q_{i+1}\right) + K(T-i-1)\log q_{i+1}. \quad (18)$$

But is is easily seen that the right hand side of (18) is bounded above by

$$M^2 \cdot \log q_{i+1} \cdot E_2 \cdot \ldots \cdot E_i \cdot \left((T-i+1)i \cdot i! + (T-i)\right) < M^2 \cdot \log q_{i+1} \cdot E_2 \cdot \ldots \cdot E_i \cdot T \cdot T!,$$

where in order to deduce the above inequality we used the fact that $q_{i+1} > 2$, and the inequality

$$(T-i+1)i \cdot i! + (T-i) \leq T \cdot T!,$$

which holds for all $i \leq T$, and all $T \geq 2$. Thus, we get

$$e_{i+1} \leq M^2 T \cdot T! \cdot E_2 \cdot \ldots \cdot E_i.$$

Recalling now that $E_1 = M \cdot T!$ and that $T \leq T!$, we simply get

$$e_{i+1} \leq M \cdot T! \cdot E_1 \cdot \ldots \cdot E_i,$$

or, after taking logarithms, we arrive at

$$\log e_{i+1} \leq \log(M \cdot T!) + \log E_1 + \ldots + \log E_i.$$

Since the induction hypothesis is that $E_l := (M \cdot T!)^{2^{l-1}}$ for all $l = 1, 2, \ldots, i$, we simply get

$$\log e_{i+1} \leq (1 + 1 + 2 + \ldots + 2^{i-1})\log(M \cdot T!) = 2^i \log(M \cdot T!).$$

Thus, we may indeed choose $E_{i+1} := (M \cdot T!)^{2^i}$.

If also $\Delta_T \neq 0$, then in this case the numbers $\lambda_l$ simply satisfy $|\lambda_l| \leq \log L \leq M$ for all $l = 1, 2, \ldots, T$, and now inequality (16) simply tells us that

$$X_l = \log q_l < M \cdot T! \cdot E_2 \cdot \ldots \cdot E_T = E_1 \cdot E_2 \cdot \ldots \cdot E_T,$$

therefore

$$L\mathrm{rad}(n)^K \leq \exp\bigl(\log L + KE_1E_2\cdot\ldots\cdot E_T\bigr) < \exp\bigl(M\cdot T!\cdot E_1\cdot E_2\cdot\ldots\cdot E_T\bigr) = \exp((M\cdot T!)^{2^T}),$$

which is the desired inequality.

It remains to study what happens when $\Delta_i = 0$ holds for some $i = 1,\ 2,\ \ldots,\ l$.

**Step IV.** *An upper bound on $e_{i+1}$: the degenerate case.*

Let $i$ be the first such index for which $\Delta_i = 0$. It is obvious that $i \geq 2$. In particular, the bounds $E_l \leq (M\cdot T!)^{2^{l-1}}$ hold for all $l = 1,\ 2,\ \ldots,\ i$. Since $\Delta_i = 0$, it follows that there exists a linear combination $(\mu_1,\ \mu_2,\ \ldots,\ \mu_i)$, with not all the $\mu_l = 0$ for $l = 1,\ldots,i$, and such that if we denote by $\mathbf{r}_l$ the $l$th row of the matrix $D_i$, then

$$\sum_{l=1}^{i} \mu_l \mathbf{r}_l = 0.$$

In particular, this leads to

$$\sum_{l=1}^{i} \mu_l L_l(X_1,\ \ldots,\ X_l) = 0. \tag{19}$$

Since

$$L_l(X_1,\ \ldots,\ X_l) = e_lX_l - \sum_{j=1}^{i} f_{ji}X_j,$$

it follows that

$$L_l(X_1,\ \ldots,\ X_l) = -\log\Bigl(\frac{\sigma(q_l^{e_l})}{q_l^{e_l}}\Bigr) + B_l, \qquad \text{for all } l = 1,\ 2,\ \ldots,\ i. \tag{20}$$

So, relation (19) becomes

$$0 = -\sum_{l=1}^{i} \mu_l \log\Bigl(\frac{\sigma(q_l^{e_l})}{q_l^{e_l}}\Bigr) + \sum_{l=1}^{i} \mu_l B_l,$$

or

$$\prod_{l=1}^{i}\Bigl(\frac{\sigma(q^{e_l})}{q^{e_l}}\Bigr)^{\mu_l} = \exp\Bigl(\sum_{l=1}^{i} \mu_l B_l\Bigr), \tag{21}$$

and we know that not all the numbers $\mu_l$ are zero. We split the set $\{1,\ 2,\ \ldots,\ i\}$ into two subsets $I$ and $J$ such that $\mu_l \geq 0$ for $l \in I$ and $\mu_l < 0$ for $l \in J$. It is clear that $I$ and $J$ partition $\{1,\ 2,\ \ldots,\ i\}$, but one of them might be empty. Assume that $I \neq \emptyset$, for if not, we may change all the signs of the $\mu_l$s simultaneously. We may rewrite relation (21) as

$$\prod_{l\in I}\Bigl(\frac{\sigma(q_l^{e_l})}{q_l^{e_l}}\Bigr)^{\mu_l} = \prod_{l\in J}\Bigl(\frac{\sigma(q_l^{e_l})}{q_l^{e_l}}\Bigr)^{-\mu_l}\cdot u_i, \tag{22}$$

where $u_i$ is a rational number which in reduced form can be represented as $\alpha_i/\beta_i$, with $\alpha_i$ and $\beta_i$ positive integers such that all their prime divisors are either smaller than $L$, or belong to the set

$\{q_{i+1}, \ldots, q_T\}$. Since $q_T > L$, and $\sigma(q_l^{e_l})$ are positive integers for $l = 1, 2, \ldots, i$, relation (22) implies that both

$$\prod_{l\in I}\Big(\frac{\sigma(q_l^{e_l})}{q_l^{e_l}}\Big)^{\mu_l} \quad \text{and} \quad \prod_{l\in J}\Big(\frac{\sigma(q_l^{e_l})}{q_l^{e_l}}\Big)^{-\mu_l}$$

are positive integers.

We now recall that since $i$ is the first index where $\Delta_i = 0$, it follows that $\mu_i \neq 0$. So, we may assume, up to simultaneously changing all the signs of the $\mu_l$ for $l = 1, 2, \ldots, i$, that $i \in I$. So, by replacing all $\mu_l$ with $\min\{\mu_l, 0\}$ for $l = 1, 2, \ldots, i$, it follows that

$$\prod_{l=1}^{i}\Big(\frac{\sigma(q_l^{e_l})}{q_l^{e_l}}\Big)^{\mu_l} = v_i \in \mathbf{Z}, \tag{23}$$

where now all the $\mu_l$'s are nonnegative. We first find a better upper bound on $E_i$. Let $E \geq \max_{l=1}^{i}\{e_l\}$, and assume that $E \geq K$. We apply the absolute value inequality to bound $|\Delta_i|$ by

$$|\Delta_i| > E^i - (i! - 1)K^2E^{i-2},$$

and since $\Delta_i = 0$, we get

$$E < i!^{1/2}K. \tag{24}$$

We may now choose the vector $(\mu_1, \ldots, \mu_l)$ to be one of the nonzero rows of the matrix $D_i^*$ which is constructed with all the $i-1$-minors of the matrix $D_i$. In particular, we get that

$$\max_{l=1}^{i}\{\mu_l\} < (i-1)!E^{i-1} < (i-1)!(i!)^{(i-1)/2}K^{(i-1)}. \tag{25}$$

We now notice that equation (23) can be written as

$$v_i = \prod_{l=1}^{i}\Big(1 + \frac{1}{q_1} + \ldots + \frac{1}{q_1{}^{e_1}}\Big)^{\mu_1} \cdot \ldots \cdot \Big(1 + \frac{1}{q_i} + \ldots + \frac{1}{q_i^{e_i}}\Big)^{\mu_i} = 1 + \sum_{j=1}^{N}\frac{1}{w_j}, \tag{26}$$

where

$$N := \prod_{l=1}^{i}(e_l + 1)^{\mu_l} - 1, \tag{27}$$

and the numbers $w_j$ are positive integers strictly larger than 1, with the smallest one of them being $q_i$. It now follows right away from (26), that

$$q_i < N, \tag{28}$$

or

$$\log q_i < \log N < i \cdot \max_{l=1}^{i}\{\mu_l\} \cdot \log(E + 1). \tag{29}$$

So,

$$\log L + K\sum_{j=i}^{T}\log q_i < \log L + K(T - i + 1) \cdot i \cdot \max_{l=1}^{i}\{\mu_l\} \cdot \log(E + 1). \tag{30}$$

**Step V.** *The reduction to the induction hypothesis.*
Combining inequalities (25), (26) and (30), we get that with

$$M' := M^i (i!)^{(i+1)/2}(T-i+1)\log(Ki!^{1/2}+1), \tag{31}$$

we have that if

$$a' := a\prod_{l=i}^{T} q_i^K,$$

then

$$\log a' \le M'. \tag{32}$$

So, with

$$n' := \frac{n}{\prod_{l=i}^{T} q_l^{e_l}},$$

we then have that

$$n' = \prod_{l=1}^{i-1} q_l^{e_l},$$

and

$$\sigma(n') = a'm',$$

where $a'$ satisfies inequality (32), and $m' \mid \mathrm{rad}(n')^K$. At this point, the number $n'$ has $\omega(n') = i-1$, but, moreover, none of the determinants $\Delta_l$ for $l = 1, 2, \ldots, i-1$ vanishes (because these determinants are the same as the ones coresponding to $n$), and also $E_1 \le E_2 \le \ldots \le E_{i-1} \le E$, where $E$ satisfies inequality (24). With inequality (16), we now get that

$$X_1 = \log q_1 < (i-1)! \cdot E_2 \cdot \ldots \cdot E_{i-1} \cdot M' \le (i-1)! \cdot E^{i-2} \cdot M',$$

or

$$X_1 < (i-1)! M^{i-2}(i!)^{(i-2)/2} \cdot M^i (i!)^{(i+1)/2}(T-i+1)\log(Ki!^{1/2}+1) <$$
$$(i-1)! M^{2i-2}(i!)^{(2i-1)/2}(T+i-1)\log(Ki!^{1/2}+1).$$

Since the inequality

$$i!^{1/2} + 1 < i^{(i+2)/2} \le i^i$$

obviously holds for all $i \ge 2$, we get

$$\log(Ki!^{1/2}+1) < i\log i + \log K < M \cdot i\log i,$$

and therefore we have

$$X_1 < M^{2i-1} \cdot i!^{i+1/2} \cdot (T-i+1) \cdot \log i.$$

Thus,

$$L \cdot \mathrm{rad}(n)^K < \exp(\log L + KTX_1) < \exp\left(M^{2i} \cdot i!^{i+1/2} \cdot T \cdot (T-i+1) \cdot \log i\right). \tag{33}$$

Since $\log i < i \le T$, and since $(T-i+1) \le T$, we get that

$$i!^{i+1/2} \cdot T \cdot (T-i+1) \cdot \log i \le i!^{i+1} \cdot T^3 < (T-1)!^{i+1} \cdot T^{i+2} < T!^{i+2},$$

because $T \geq i+1$ and $i \geq 2$. Finally, since $i \geq 2$, we get that both inequalities

$$i+2 \leq 2^i, \quad \text{and} \quad 2i \leq 2^i$$

hold, so from inequality (33) we conclude that

$$L \cdot \mathrm{rad}(n)^K \leq \exp\big((M \cdot T!)^{2^i}\big). \tag{34}$$

Since $i \leq T$, from inequality (34) we get that

$$L \cdot \mathrm{rad}(n)^K \leq \exp\big((M \cdot T!)^{2^T}\big)$$

holds, and the Theorem is therefore completely proved.

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