MATH 3100 – Homework #7

posted November 3, 2023; due Friday, November 10

Limits, like fears, are often just an illusion. – Michael Jordan

Required problems

1. For this problem, you may assume that $\sin x$ and $\cos x$ are defined and continuous on all of $\mathbf{R}$. You may also assume that the function x^π, with domain $(0, \infty)$, is continuous on $(0, \infty)$. Let
$$f(x) = (\sin^2 x + \cos^6 x)^\pi.$$
   (a) What is the domain of $f(x)$? Justify your answer.
   (b) Using theorems stated in class or in the textbook, explain why $f(x)$ is continuous.

2. Let $f(x) = 2x \cos(1/x)$ for $x \neq 0$ and set $f(0) = 0$. Show that f is continuous at 0 by directly verifying the ϵ-δ definition of continuity. Do *not* use the sequential criterion.

3. Prove that $f(x) = x^3$ is continuous at $x = 2$ by directly verifying the ϵ-δ definition of continuity. Do *not* use the sequential criterion.

4. (Limits of constants, function version) Let f be a real-valued function defined on a set of real numbers S. Suppose that f is constant on S, say $f(x) = c$ for all $x \in S$. Prove that for every $a \in \bar{S}$,
$$\lim_{x \to a^S} f(x) = c.$$

5. (Squeeze Lemma for sequences) Suppose $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are three sequences satisfying
$$a_n \leq b_n \leq c_n \quad \text{for all } n \in \mathbf{N}.$$
Suppose there is a real number L such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Prove that $\lim_{n\to\infty} b_n = L$.

   Remark. This problem could have been assigned in Chapter 1; no new theory is required.

6. (Squeeze Lemma for functions) Let f_1, f_2, and f_3 be three functions defined on S, and let a be a real number belonging to the reach of S. Suppose that
$$f_1(x) \leq f_2(x) \leq f_3(x) \quad \text{for all } x \in S.$$
Suppose also that there is a real number L for which
$$\lim_{x \to a^S} f_1(x) = \lim_{x \to a^S} f_3(x) = L.$$
Prove that
$$\lim_{x \to a^S} f_2(x) = L.$$

7. Let a and L be a real numbers. Suppose there is an open interval I containing a such that f is defined on $I \setminus \{a\}$ and, with $S = I \setminus \{a\}$,

$$\lim_{x \to a^S} f(x) = L.$$

Prove that if I' is *any* open interval containing a for which f is defined on $I' \setminus \{a\}$, then with $S' = I' \setminus \{a\}$,

$$\lim_{x \to a^{S'}} f(x) = L.$$

Remark. This shows our definition of "$\lim_{x \to a} f(x) = L$" is independent of the choice of interval I, at least in the case when $L \in \mathbf{R}$. A similar argument could be given when $L = \pm\infty$.

Hint. We are told that $\lim_{x \to a^S} f(x) = L$. Thus, if the x_n are elements of $I \setminus \{a\}$ for which $x_n \to a$, then $f(x_n) \to L$. Now let y_n be elements of $I' \setminus \{a\}$ for which $y_n \to a$. We have to show that $f(y_n) \to L$. Start by showing that $y_n \in I$ eventually.

8. Let a be a real number. Suppose that f is a real-valued function defined on $I \setminus \{a\}$ for some open interval I containing a. Suppose that

$$\lim_{x \to a} f(x) = L,$$

where L is a real number. Give a careful proof that

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L.$$

[In the next problem, you will be asked to prove the converse!]

9. Let a be a real number. Suppose that f is a real-valued function defined on $I \setminus \{a\}$ for some open interval I containing a. Suppose that

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L,$$

where L is a real number. Prove that

$$\lim_{x \to a} f(x) = L.$$

Hint. With I as in the problem statement, let x_n be a sequence of elements of $I \setminus \{a\}$ for which $x_n \to L$. Suppose for a contradiction that $f(x_n) \not\to L$. Prove that for some $\epsilon > 0$, there are infinitely many n with $|f(x_n) - L| \geq \epsilon$. Now take two cases, according to whether infinitely many of *those* n have $x_n < a$ or infinitely many have $x_n > a$.

Recommended problems (from Ross)

§17: 3(a,c,f,g,h), 8, 11, 15
§20: 1, 9, 11, 13