MATH 3200 – Learning objectives to meet for the final exam

The final exam is cumulative: **All of the material we have covered from the start of the semester up through our class on April 24 is potentially examinable**. However, since we did not have a chance to discuss one-to-one and onto functions in depth, that material will **not** appear on your final exam.

Due to the special circumstances we find ourselves in, the exam will **not be timed or proctored**. The exam will also be **open book/open notes/open HW solutions/open worksheet**. However, you **may not** discuss the exam questions with your others, and you **may not** look up answers in outside sources.

You will have 24 hours to turn in the exam, starting from Friday, May 1, 12 PM Eastern Time. The exam will be distributed by email, via Piazza; **you will need to scan your solutions and upload them through Gradescope**.

Please refer to the review sheets for Exams #1 and #2 (and the exams themselves) for guidance on what material to study from Chapters 1–4. The learning outcomes below refer only to the Chapter 5 material, covered after Exam #2.

What to be able to state

Basic definitions

Be able to give concise, complete, and precise definitions of each of the following terms.

- **Relation** on sets S and T, and **relation** on a single set S
- what it means to say $a \in S$ is **related** to $b \in T$, given a relation R on sets S and T
- the **class** of x with respect to a relation R, denoted C_x (or $\mathcal{C}_x$)
- what it means for a relation to be **reflexive**, **symmetric**, or **transitive**
- equivalence relation
- the relation on $\mathbf{Z}$ we denoted "$\equiv_m$"
- **function** from a set X to a set Y, and the associated terms **domain**, **range**, **codomain**
- the image set $f[A]$ and preimage set $f^{-1}[B]$

What to be able to do

You may be called on to perform any or all of the following tasks. Examples of all of these tasks can be found on the class worksheets.

- Recognize examples and non-examples of the concepts appearing in the "basic definitions" section above.
- Determine if a relation is reflexive, symmetric, and/or transitive. This includes relations that are given to you explicitly as sets of ordered pairs, as well as relations described in words.

- Write proofs showing your ability to work with the concepts of reflexive, symmetric and transitive.
- Use properties of $\equiv_m$ to compute remainders when you divide by m.
- Determine if a given relation on two sets represents a function.
- Write proofs showing your ability to work with the concepts of image and preimage.

What to expect on the final exam

You can expect roughly 10 questions on the final exam. These will include:

- at least one problem assessing your negation skills
- a proof by induction or strong induction
- at least one problem asking you to use properties of $\equiv_m$ to determine a remainder or remainders
- at least one problem asking you to prove/disprove that a relation is reflexive, symmetric, and/or transitive

Extra practice problems

1. Find the remainder when $98765 + 9876 + 987 + 98 + 9$ is divided by 3, without first doing the sum. Explain your reasoning. (For you to receive credit on the exam, I need to be able to follow your explanation.)

2. What is the remainder when 11^{1000} is divided by 7? Explain your reasoning.

3. Let $X = \{1, 2, 3, 4, 5, 6\}$.

   (a) Let
   $$R = \{((1,2),(2,1),(1,1),(3,2),(3,1),(1,3),(2,3),(3,3),(4,1),(1,4),(4,4),(4,2),(2,4),(2,2)\}.$$
   Is R reflexive? symmetric? transitive? List all of the distinct classes of elements of X under the relation R.

   (b) Let
   $$R = \{(1,1),(1,6),(2,2),(2,3),(2,4),(3,3),(3,2),(3,4),(4,4),(4,2),(4,3),(5,5),(6,6),(6,1)\}.$$
   Is R reflexive? symmetric? transitive? List all of the distinct classes of elements of X under the relation R.

4. Let R be the relation on the real numbers $\mathbf{R}$ defined by saying x is related to y if $\sin x = \sin y$. Prove that R is an equivalence relation on $\mathbf{R}$. List 5 different elements in the set $\mathcal{C}_0$ (the class of 0).

5. Let $\mathbf{N}$ be the set of natural numbers, and let $X = \mathbf{N} \times \mathbf{N}$. We define a relation R on X by saying that (a, b) is related to (c, d) if $ad = bc$.

   Example: $(2, 4)$ is related to $(3, 6)$ since $2 \cdot 6 = 4 \cdot 3$. However, $(2, 4)$ is not related to $(3, 5)$ since $2 \cdot 5$ is not the same as $4 \cdot 3$.[1]

   (a) Prove that $\mathcal{C}_{(1,2)} = \{(a, 2a) : a \in \mathbf{N}\}$.

   (b) Prove that R is an equivalence relation on X.

6. (a) Is the relation $\{(a,1),(a,-1),(b,1),(b,-1),(c,1),(c,-1),(d,1),(d,2)\}$ a function from $A = \{a,b,c,d\}$ to $B = \mathbf{Z}$? Why or why not?

   (b) Is the relation $\{(a,1),(b,2),(c,3)\}$ a function from $A = \{a,b,c,d\}$ to $B = \mathbf{Z}$? Why or why not?

   (c) Is the relation $\{(a,1),(b,1),(c,1),(d,1)\}$ a function from $A = \{a,b,c,d\}$ to $B = \mathbf{Z}$? Why or why not?

7. Let f be the relation on $\mathbf{R}$ defined by $R = \{(x, \sin x) : x \in \mathbf{R}\}$. Is f a function from $\mathbf{R}$ to $\mathbf{R}$? What about $f = \{(\sin x, x) : x \in \mathbf{R}\}$? What about $f = \{(x^2, x^4) : x \in \mathbf{R}\}$? Explain.

8. Let $f\colon X \to Y$ be a function. Let B be a subset of Y, and suppose that B is contained in the range of f. Prove that $f[f^{-1}[B]] = B$.

[1] This may require a little time to wrap your mind around. Notice that the elements of X are ordered pairs. So the elements of R are ordered pairs of ordered pairs! For example, $((2,4),(3,6)) \in R$.

9. Let $f\colon X \to Y$ be a function.

   (a) Let $A \subseteq X$. Is it necessarily true that $A \subseteq f^{-1}[f[A]]$? If so, prove it. If not, give a counterexample.

   (b) Let $B \subseteq Y$. Is it necessarily true that $B \subseteq f[f^{-1}[B]]$? If so, prove it. If not, give a counterexample.

10. Let $f_1\colon \mathbf{Z} \to \mathbf{Z}$ be the function defined by $f_1(x) = 3x - 1$, and let $f_2\colon \mathbf{Z} \to \mathbf{Z}$ be the function defined by $f_2(x) = 3x + 8$. Prove that the range of f_1 is equal to the range of f_2. Use element chasing.