MATH 4400/6400 – Homework #4

posted March 22, 2019; due March 27, 2019

Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the mind will never penetrate.
– L. Euler

Directions. Give complete solutions, providing full justifications when appropriate. Your assignment must be stapled if it goes on beyond one page. Starred problems are required for MATH 6400 students and extra credit for 4400 students.

Throughout this problem set, p always denotes a prime number, even if this is not explicitly specified.

1. This exercise outlines another proof that there are infinitely many primes. In what follows, n denotes a positive integer.
   (a) Prove that $n!^2 \geq n^n$.
   (b) Show that $v_p(n!) \leq \frac{n}{p-1}$ for every prime p.
   (c) Prove that $n!^{1/n} \leq \prod_{p \leq n} p^{1/(p-1)}$.
   (d) Conclude!
2. For every positive integer a, let $N(a)$ denote the number of integers in the interval $(a, a^2]$ which are divisible by a prime $p > a$.
   (a) Prove that $N(a) = \displaystyle\sum_{a<p\leq a^2} \lfloor a^2/p \rfloor$.
   (b) Deduce that $\displaystyle\sum_{a<p\leq a^2} \frac{1}{p} \leq 2$.
   (c) By using part (b) several times, find an upper bound on $\sum_{p \leq 2^{32}} \frac{1}{p}$.
3. A positive integer is called *k-semiprime* if it has the form $p_1 p_2 \cdots p_k$, for some primes $p_1, \ldots, p_k$. (The p_i do not have to be distinct.) For example, $9 = 3 \cdot 3$ is a 2-semiprime, and $210 = 2 \cdot 3 \cdot 5 \cdot 7$ is a 4-semiprime.

   Using Bertrand's postulate, prove the following theorem: For every postitive integer k, there is a positive integer n_0 such that, for all positive integers $n \geq n_0$, the interval $(n, 2n]$ contains a k-semiprime.
4. Prove that $\sqrt{x} \leq 2x/\log x$ for all $x \geq 2$. [We needed this to prove our upper bound on $\pi(x)$.]

   Hint: Use your calculus-fu to minimize the ratio $\frac{2x/\log x}{\sqrt{x}}$ on $[2, \infty)$.
5. Show that $\binom{2n}{n}$ is the largest of the numbers $\binom{2n}{k}$ for $k = 0, 1, 2, \ldots, 2n$.

   Hint: Consider the ratio of $\binom{2n}{k+1}$ to $\binom{2n}{k}$.
6. In this exercise, you will show that the inequality

$$4^n/(2n+1) \leq (2n)^{\sqrt{2n}} 4^{2n/3}$$

   is **false** for all $n \geq 2^{12}$.

   Proceed by filling in the details of the following proof sketch. (This way of completing the proof is taken from notes of modern-day mathematician Robin Chapman.) We suppose for a contradiction that the inequality holds.

(a) Prove that $(2n+1) < (2n)^2$ and that $2 + \sqrt{2n} < \frac{4}{3}\sqrt{2n}$. Deduce that

$$4^{n/3} \leq (2n)^{\frac{4}{3}\sqrt{2n}}.$$

(b) Taking logarithms, show that

$$\sqrt{2n}\log 2 \leq 4\log(2n).$$

(c) Write $2n = 4^t$, where t is a real number. Deduce from (b) that

$$\frac{2^t}{t} \leq 8.$$

(d) Since $2n > n \geq 2^{12} = 4^6$, we know that the number t in part (c) satisfies $t > 6$. Derive a contradiction by computing the minimum of $2^u/u$ for $u \in [6, \infty)$.

7. Let $\mathbb{H}$ denote the set of matrices of the form $\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$, where $\alpha, \beta \in \mathbf{Z}[i]$.

   (a) Show that $\mathbb{H}$ is closed under multiplication.

   (b) Show that if $a, b \in \mathbf{Z}$ are both sums of four integer squares, then ab is also a sum of four integer squares. *Hint*: Determinants!

8. Using Bertrand's postulate, prove that every integer $n > 6$ can be written as a sum of distinct prime numbers.

   Hint: Start by observing that if $6 < n \leq 19$, then n is a sum of distinct primes that are at most 11.

9. (*) Prove that for every $k > 1$, there is some positive integer n with $n/\pi(n) = k$.