MATH 3100 – Learning objectives to meet for Exam #1

The exam will cover §1.1–§1.5 of the course notes, through what is discussed by the end of class on Monday, February 5.

What to be able to state

Basic definitions

You should be able to give complete and precise definitions of each of the following items:

- sequence
- increasing, decreasing, monotone, and the "strictly" and "eventually" variants
- bounded above, bounded below, upper bound, lower bound, bounded
- subsequence
- $\{a_n\}$ converges to L, where L is a real number
- $\{a_n\}$ diverges
- geometric sequence
- $\{a_n\}$ diverges to ∞ or $-\infty$

Big theorems

Give full statements of each of the following results, making sure to indicate all necessary hypotheses. For results proved in class, describe the components and main ideas of the proof.

- Principal of mathematical induction, complete mathematical induction
- A convergent sequence has a **unique** limit
- Convergent sequences are bounded
- Subsequences of convergent sequences converge to the original limit
- Behavior of geometric sequences (Proposition 1.4.15)
- If $\{a_n\}$ is bounded above by U and $a_n \to L$, then $L \leq U$
- (Bounded) $\cdot$ (going to 0) goes to 0
- Triangle inequality for absolute values
- Sum rule for limits, product rule for limits, quotient rule for limits
- Ratio test for sequences

What to expect on the exam

You can expect 5 questions on the exam, including

- one problem testing mathematical induction
- one problem testing your ability to compute the limit of a specific sequence directly from the ϵ-N definition

The rest of the exam is designed to test your comfort level working with the basic definitions. I am not interested in having you regurgitate proofs of results from the notes; I want to know if you have internalized the ideas enough to solve similar problems.

Sample problems

1. (a) Let r be a real number with $0 < r < 1$. Use induction to prove that $(1-r)^n \geq 1 - nr$ for each natural number n.

   (b) Deduce, without difficult calculations, that $0.999^{500} \geq 0.5$.

2. (a) Prove that $\lim_{n\to\infty} \frac{n-4}{7n-9} = \frac{1}{7}$ directly from the definition of a limit.

   (b) Prove that $\lim_{n\to\infty} \frac{2n^2+3}{n^2+n} = 2$ directly from the definition of a limit.

3. (a) What does it mean to say a sequence is **bounded above**? **Bounded below**? **Bounded**?

   (b) Suppose $\{a_n\}$ is a sequence with the property that, for some natural number N,

   $$a_n = 2024 \cdot (-1)^n \quad \text{whenever } n \geq N.$$

   Show that $\{a_n\}$ is bounded.

4. Determine, with brief explanations, each of the following limits. Do **not** use the limit definition; rather, use known limits and limit rules established in class. For each part, explain which rules you use.

   (a) $\lim_{n\to\infty} \frac{n}{2n+1}$.

   (b) $\lim_{n\to\infty} (-2/3)^n \sin(n)$.

   (c) $\lim_{n\to\infty} (2^n/3^n - 4^n/5^n)$.

5. Suppose that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 1$. Show that there is an $N \in \mathbb{N}$ with

   $$a_n < b_n \quad \text{for all } n \geq N.$$

   Your proof should use the limit definition explicitly.

6. Let $\{a_n\}$ be a sequence.

   (a) Give the definition of "$\{a_n\}$ is **increasing**."

   (b) Give the definition of "$\lim a_n = \infty$."

   (c) Give a careful proof that if $\{a_n\}$ is increasing and not bounded above, then $\lim a_n = \infty$.