MATH 3200 – "Homework" #7 (not to turn in!)

1. The relations on $\mathbf{R}$ described by $\{(x, x^2) : x \in \mathbf{R}\}$ and $\{(x, x^3) : x \in \mathbf{R}\}$ both represent functions from $\mathbf{R}$ to $\mathbf{R}$. The first is the function you wrote in calculus as x^2, while the second is the function you are used to writing as x^3.

   What about the relations $\{(x^2, x) : x \in \mathbf{R}\}$ and $\{(x^3, x) : x \in \mathbf{R}\}$? Are these functions from $\mathbf{R}$ to $\mathbf{R}$? Justify your answers.

2. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e, f\}$. How many functions are there from A to B? How many are one-to-one?

3. Suppose $f\colon X \to Y$. Let $B, C \subseteq Y$. Show that $f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C)$.

4. Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x - 5$. Prove that f is one-to-one and onto. Do the same for $f(x) = x^3$.

   Suggestions. To prove a function $f\colon X \to Y$ is one-to-one, suppose $x, x' \in X$ satisfy $f(x) = f(x')$, and deduce that $x = x'$. To prove $f\colon X \to Y$ is onto, let $y \in Y$ and prove there is an $x \in X$ with $f(x) = y$.

5. Let A, B, and C be nonempty sets and assume $f\colon A \to B$ and $g\colon B \to C$. Then $g \circ f$ maps A to C. Prove:

   (a) If $g \circ f$ is surjective, then g is surjective.

   (b) If $g \circ f$ is injective, then f is injective.

6. Suppose $f\colon X \to Y$, and let $A \subseteq X$ and $B \subseteq X$.

   (a) Give an example where $f(A \setminus B) \neq f(A) \setminus f(B)$. (Give an example means: Present choices of X, Y, A, B and f for which this fails.)

   (b) Suppose f is one-to-one. Prove that $f(A \setminus B) = f(A) \setminus f(B)$.

7. Suppose $f\colon X \to Y$ has inverse function $g\colon Y \to X$. Show that for every set $B \subseteq Y$, one has $f^{-1}(B) = g(B)$.

8. In class, we will see that if $f\colon X \to Y$ is a bijection, then f has an inverse function $g\colon Y \to X$. Conversely, if f has an inverse function, then f is a bijection.

   Suppose that $f\colon X \to Y$ is a bijection. Suppose further that f has inverse function $g\colon Y \to X$. Prove that g is a bijection.