# ON DICKSON'S THEOREM CONCERNING ODD PERFECT NUMBERS

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Abstract. A 1913 theorem of Dickson asserts that for each fixed natural number $k$, there are only finitely many odd perfect numbers $N$ with $k$ distinct prime factors. We show that the number of such $N$ is bounded by $2^{(2k)^2}$.

## 1. Introduction

If $N$ is a natural number, we write $\sigma(N) := \sum_{d|N} d$ for the sum of the divisors of $N$. We call $N$ *perfect* if $\sigma(N) = 2N$, i.e., if $N$ is equal to the sum of its proper divisors. The even perfect numbers were completely classified by Euclid and Euler, but the odd perfect numbers remain utterly mysterious: Despite millennia of effort, we don't know of a single example, but we possess no argument ruling out their existence.

In 1913, Dickson [2] proved that for each fixed natural number $k$, there are only finitely many odd perfect numbers $N$ with $\omega(N) = k$. (Here and below, we write $\omega(N)$ for the number of distinct prime factors of the natural number $N$.) The first explicit bounds were given by Pomerance [5], who showed that an odd perfect $N$ with $\omega(N) \leq k$ satisfies

$$N \leq (4k)^{(4k)^{2^{k^2}}}.$$

After the work of Heath-Brown [3], and its subsequent refinements by Cook [1] and Nielsen [4], we know that any such $N$ satisfies

$$N < 2^{2^{2k}}. \tag{1}$$

In addition to an upper bound on the *size* of such $N$, it is sensible to ask for a bound on the *number* of such $N$. The purpose of this note is to prove the following estimate:

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**Theorem 1.** *For each positive integer $k$, the number of odd perfect numbers $N$ with $\omega(N) \le k$ is bounded by $2^{(2k)^2}$.*

Notice that the exponent $2^{2k}$ in (1) has been replaced with the typographically similar (but much smaller!) $(2k)^2$. Actually, Theorem 1 is a corollary of the following result that is perhaps of independent interest:

**Theorem 2.** *Suppose that $x > e^{12}$. The number of odd perfect $N \le x$ with $\omega(N) \le k$ is bounded by $(\log x)^{2k}$, uniformly in $k \ge 1$.*

The proofs of these results are essentially self-contained, except for the use in Theorem 1 of the upper bound (1). Most of our notation will be familiar to students of elementary number theory. A possible exception is the definition of "$\|$" (or *exactly divides*): If $p$ is a prime, we write $p^e \parallel n$ to mean that $p^e \mid n$ while $p^{e+1} \nmid n$.

## 2. Proofs

*Proof of Theorem 2.* We employ a modification of Wirsing's method from [6]. Suppose that $N \le x$ is odd perfect and $\omega(N) \le k$. Since no prime power is perfect, $k \ge 2$. Write $N = AB$, where $A := \prod_{\substack{p^e \parallel N \\ p > 2k}} p^e$ and $B := \prod_{\substack{p^e \parallel N \\ p \le 2k}} p^e$. We have

$$\sigma(A) = \prod_{p^e \parallel A} (1 + p + p^2 + \dots + p^e) \le A \prod_{p \mid A} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right),$$

and hence

$$\frac{A}{\sigma(A)} \ge \prod_{p \mid A} \left(1 - \frac{1}{p}\right) \ge 1 - \sum_{p \mid A} \frac{1}{p} \ge 1 - \frac{k}{2k+1} > \frac{1}{2}. \tag{2}$$

Since $N$ is perfect,

$$2AB = \sigma(A)\sigma(B), \tag{3}$$

and so

$$B < \frac{2A}{\sigma(A)} B = \sigma(B) \le 2B,$$

with equality on the right precisely when $A = 1$. Suppose $A \ne 1$. The preceding inequalities show that $\sigma(B) \nmid 2B$, and so there is a prime $p_1$ dividing $\sigma(B)$ to a higher power than that to which it divides $2B$; for definiteness, fix $p_1$ as the least such prime. It now follows from (3) that $p_1 \mid A$. Suppose $p_1^{e_1} \parallel A$, where $e_1 \ge 1$. Then if we put

$$A' := A/p_1^{e_1} \quad \text{and} \quad B' := Bp_1^{e_1},$$

it is clear that both (2) and (3) hold with $A'$ in place of $A$ and $B'$ in place of $B$. Arguing as above, we find that unless $A' = 1$, there is a prime $p_2$ dividing $\sigma(B')$ to a higher power than that to which it divides $2B'$. Again, for definiteness, let $p_2$ be the least such prime. Then $p_2^{e_2} \parallel A'$ for some $e_2 \geq 1$. We put

$$A'' := A'/p_2^{e_2} \quad \text{and} \quad B'' := B'p_2^{e_2}$$

and observe that (2) and (3) hold with $A''$ and $B''$ replacing $A$ and $B$. We continue in this manner; since the sequence $\omega(A), \omega(A'), \dots$ is strictly decreasing, it is clear that this process terminates, and we eventually arrive at a factorization

$$A = p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l},$$

where

$$l = \omega(A) = \omega(N) - \omega(B) \leq k - 1.$$

(Notice that (2) implies that $B \neq 1$, and hence $\omega(B) \geq 1$.)

The prime $p_1$ depends only on $B$, while for $i > 1$, the prime $p_i$ depends only $B$ and the exponents $e_1, \dots, e_{i-1}$. It follows that for a given $B$, the cofactor $A$ is entirely determined by the sequence of exponents $e_1, \dots, e_l$. Since $A \leq n \leq x$ and each $p_i \geq 2k+1 \geq 5$, each $e_i \in \{1, \dots, \lfloor \log x / \log 5 \rfloor\}$. Since $l \in \{0, 1, \dots, k-1\}$, the number of possibilities for the sequence of exponents is (crudely) bounded by

$$k\left(\log x / \log 5\right)^k \leq (\log x)^k. \tag{4}$$

To estimate the number of possibilities for $B$, we observe that the number of odd primes not exceeding $2k$ is smaller than $k$, while for each prime $p^e \parallel B$, we have $e \leq \log x / \log 3$. Hence the number of possibilities for $B$ is bounded by

$$(1 + \log x / \log 3)^k \leq (\log x)^k, \tag{5}$$

since we are assuming that $x > e^{12}$. Combining (4) and (5) gives the theorem. □

*Proof of Theorem 1.* We may suppose that $k \geq 3$, since an odd $N$ with $\omega(N) < 3$ satisfies

$$\begin{aligned}\sigma(N) &= N \prod_{p^e \parallel N} \left(1 + \frac{1}{p} + \dots + \frac{1}{p^e}\right) \\ &< N\left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + \dots\right) = \frac{15}{8}N < 2N,\end{aligned}$$

and so cannot be perfect. Then $x := 2^{2^{2k}} > e^{12}$, and by (1) and Theorem 2, the number of odd perfect $N$ with $\omega(N) \leq k$ is at most $(\log x)^{2k} < (2^{2k})^{2k} = 2^{(2k)^2}$. $\square$

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