MATH 3100 – Learning objectives to meet for Exam #3

The exam will cover §2.4 of the course notes, the portions of sections 17, 20, 28, and 29 from the Ross text that were treated in lecture, and our subsequent discussion of manipulating power series. No material from after Wednesday, November 15 is examinable.

What to be able to state

Basic definitions

You should be able to give complete and precise definitions of each of the following items:

- power series
- domain of convergence and radius of convergence
- continuity of a function f at x_0
- f is continuous (with no point specified)
- reach of a subset $S \subseteq \mathbf{R}$
- $\lim_{x \to a^S} f(x)$
- $\lim_{x \to a} f(x)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, where a is a real number
- $\lim_{x \to -\infty} f(x)$, $\lim_{x \to \infty} f(x)$
- f is differentiable at a

Big theorems

Be familiar with all of the following theorems, making sure that you can give full statements for those not already spelled out below. For results proved in class, describe the components and main ideas of the proof.

- Ratio test
- Key Lemma: If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_0$, it converges absolutely for all x with $|x| < |x_0|$
- Characterization of the domain of convergence of a power series (Proposition 2.4.4)
- Sequential criterion for continuity
- Constant multiples, sums, products, and ratios of continuous functions are continuous
- The composition of continuous functions is continuous
- Limit laws for sums, products, and quotients
- Differentiability at a implies continuity at a
- Derivatives of constant multiples, sums, products, and quotients

- Chain rule
- Rolle's theorem
- Mean Value Theorem
- If f has a maximum at $x_0 \in (a, b)$, and f is differentiable at x_0, then $f'(x_0) = 0$. And similarly for minimum.
- Differentiation and integration theorems for power series

What to expect on the exam

There will be five questions on the exam, possibly having multiple parts. These will include...

- A problem testing you on a basic definition and requiring you to give a proof using this definition
- A problem or problem part asking you to determine the domain and/or radius of convergence of (one or more) power series
- A problem requiring you to apply our differentiation and/or integration rules for manipulating power series

Practice problems

1. Find the domain of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n!^2}{2^n} x^n$. Do the same for $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^n n(n-1)} x^n$.

2. (a) What does it mean to say that f is continuous at x_0? Make sure to state all of the assumptions on f.

   (b) Prove, directly from the ϵ-δ definition, that $f(x) = 1/x^2$ is continuous at $x = 1$. You may **not** use the sequential criterion for continuity or the continuity rules proved in class.

3. (a) Give a careful proof that if $\{a_n\}$ is any sequence where $a_n \to \infty$, then $a_n - 10 \to \infty$.

   (b) Suppose that f is a real-valued function with the property that $\lim_{x\to\infty} f(x) = \infty$. Prove that $\lim_{x\to\infty}(f(x) - 10) = \infty$.

4. (a) What does it mean to say that $f(x)$ is **differentiable** at $x = c$?

   (b) Suppose $f(x)$ is a real-valued function defined for all real numbers x and satisfying

   $$|f(x) - f(y)| \leq |x - y|^2 \quad \text{for all real } x, y.$$

   Prove that f is a constant function: That is, there is a real number C such that $f(x) = C$ for all real numbers x.

5. (a) State the **Mean Value Theorem**.

(b) Suppose f is differentiable on all of $\mathbf{R}$, that $f(0) = 0$, and that $f'(x) \leq 2$ for every real number x. Prove that $f(10) \leq 20$.

6. Let $s(x) = \sum_{n=2}^{\infty} \frac{1}{2^n n(n-1)} x^n$.

   (a) Show that $s(x)$ converges for $|x| < 1$.

   (b) Find a closed form for $s''(x)$ (the second derivative), valid when $|x| < 1$.

   (c) Find $s'(1)$.