MATH 3100 – Learning objectives to meet for Exam #2

The exam will cover §1.6–§2.2 of the course notes, up to the very start of §2.3 (absolute convergence implies convergence).

What to be able to state

Basic definitions

You should be able to give complete and precise definitions of each of the following items:

- Completeness property of the real numbers
- Cauchy sequence
- Least upper bound of a set of real numbers
- Continuity of a function
- Sequence of partial sums of an infinite series
- Convergence of an infinite series
- Geometric series
- Absolute convergence

Big theorems

Be familiar with all of the following theorems, making sure that you can give full statements for those not already spelled out below. For results proved in class, describe the components and main ideas of the proof.

- Every sequence has a monotone subsequence.
- Every bounded sequence has a convergent subsequence.
- Cauchy sequences are bounded.
- A sequence converges if and only if it is a Cauchy sequence.
- Every nonempty subset of the real numbers that is bounded above has a least upper bound.
- Intermediate value theorem (both the simpler version appearing as Theorem 1.7.3 and the extension proved as Exercise 1.7.1)
- Maximum value theorem
- kth term divergence test (i.e., Proposition 2.1.10)
- Sum rule and constant multiple rule for series (Proposition 2.1.12)
- Convergence properties of geometric series (Proposition 2.1.14)

- Fundamental principle of nonnegative series: A series with nonnegative terms converges if and only if its sequence of partial sums is bounded above.
- Comparison test
- Eventual comparision test
- Limit comparison test
- Integral comparison test
- Absolute convergence implies convergence

What to expect on the exam

- At least one problem testing you on a basic definition and requiring you to give a proof using this definition (such as Practice Problem #1 below)
- At least one multipart problem asking you to test concrete examples of series for convergence/divergence
- At least one problem requiring you to apply the Intermediate Value Theorem.

Practice problems

1. (a) What does it mean to say that $\{a_n\}$ is a **Cauchy sequence**?
   (b) Without using the theorem from class that Cauchy sequences converge, prove that if $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then so is $\{a_n + b_n\}$.
2. Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely. Show that if $\{b_k\}$ is a sequence with $|b_k| \leq 2021$ for all k, then $\sum_{k=1}^{\infty} a_k b_k$ converges.
3. Show that if $\sum_{k=1}^{\infty} a_k$ is a convergent series of nonnegative terms, then so is $\sum_{k=1}^{\infty} a_{k^2}$. Note that k^2 is in the subscript here, so that $\sum_{k=1}^{\infty} a_{k^2}$ means $a_1 + a_4 + a_9 + a_{16} + \dots$.
4. For each natural number n, define
$$a_n = \frac{1}{n} - \int_n^{n+1} \frac{1}{t}\,dt.$$
   (a) Explain why each $a_n \geq 0$. (It might help to draw a picture.)
   (b) Show that the partial sums of $\sum_{n=1}^{\infty} a_n$ are bounded.
   (c) Does $\sum_{n=1}^{\infty} a_n$ converge? Justify your answer.
5. Suppose that f is a continuous function on the closed interval $[0, 2]$ and that
$$0 \leq f(x) \leq 4 \quad \text{for all } x \in [0, 2].$$
   Prove that there is an $x \in [0, 2]$ for which $f(x) = x^2$.