NONNEGATIVE MULTIPLICATIVE FUNCTIONS WITH MODERATE DECAY ALONG THE PRIMES

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Abstract. In 1961, Wirsing proved an asymptotic formula for the partial sums of nonnegative multiplicative functions that possess a positive mean value on the primes. We prove an analogue of Wirsing's result when the mean value on the primes $p \le x$ decays like a positive power of $\log x$. As a consequence, we obtain estimates for the counting function of multiplicative semigroups generated by certain thin sets of primes.

The *Golomb primes* are the elements of the sequence $3, 5, 17, 23, \dots$, with each prime chosen as small as possible to not be 1 modulo a previous prime. While this set of primes is too thick for our main result to apply, we nevertheless show that the counting function of the corresponding 'Golomb semigroup' grows like $Cx/\log x$ for an explicit constant $C \approx 0.7$.

1. Introduction

Every working analytic number theorist is occasionally confronted with the task of estimating the partial sums of a multiplicative function f. When all of the values that f assumes are nonnegative, one can often apply the following general and elegant result of Wirsing [7] from 1961 (cf. [8, Satz 1.1]).

Theorem A. *Let f be a nonnegative multiplicative function. Suppose that along the sequence of primes, f has a (finite) positive mean value τ, in other words, $\sum_{p \le x} f(p) \sim \tau x/\log x$, as $x \to \infty$. Suppose also that at prime powers p^k with $k \ge 2$, we have*

$$f(p^k) \le \gamma_1 \cdot \gamma_2^k, \quad \text{where } \gamma_1 > 0 \text{ and } 0 < \gamma_2 < 2.$$

Then as $x \to \infty$,

$$\sum_{n \le x} f(n) \sim \tau \frac{x}{\log x} \sum_{m \le x} \frac{f(m)}{m}, \quad \text{while}$$

$$\sum_{m \le x} \frac{f(m)}{m} \sim \frac{e^{-\tau\gamma}}{\tau \cdot \Gamma(\tau)} \prod_{p \le x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right).$$

Here γ is the Euler–Mascheroni constant and $\Gamma(\cdot)$ is the gamma-function.

What if f does not have a positive mean value on the primes? If the average value of $f(p)$ along the primes $p \le x$ decays to zero, but does so very slowly, then the following theorem of Shikorov [6] may be used.

Theorem B. *Let f be a nonnegative multiplicative function. Suppose that for a constant $0 < \lambda < \frac{1}{2}$, we have $f(p^k) \ll p^{k\lambda}$ for all primes p and all $k \ge 1$. Suppose also that $\sum_{p \le x} f(p) \sim \tau(x) \frac{x}{\log x}$ as $x \to \infty$, where $\tau(x)$ is nonnegative, continuous for $x \ge 1$, and satisfies the following conditions:*

(i) *$\tau(x) \to 0$ as $x \to \infty$,*
(ii) *if $0 < \delta < 1$, and $x^\delta \le y \le x$, then $\tau(x) \sim \tau(y)$ uniformly in y, as $x \to \infty$,*

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(iii) *if we define* $\phi(x) = \int_2^x \frac{\tau(u)}{u}\,du$, *then* $\phi(x) \le \theta\tau(x)\log x$, *where* $\theta > 0$ *is a constant.*

Under these hypotheses,

$$\sum_{n\le x} f(n) \sim \tau(x)\frac{x}{\log x}\prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right). \tag{1}$$

Let $A, B > 0$. A typical example of a τ for which hypotheses (i)–(iii) hold is the function $\tau_{A,B}$ defined piecewise by $\tau_{A,B}(x) := A/(\log\log x)^B$ for $x \ge x_0 := \exp(\exp(A^{1/B}))$ and $\tau_{A,B}(x) = 1$ for $1 \le x \le x_0$. See [6, Theorem 2] for details of the verification.

Our main objective here is to establish an estimate of the same kind as Theorems A and B when f exhibits more pronounced (average) decay along the primes.

Definition. If $L(x)$ is a positive-valued function which is nondecreasing for $x \ge 1$, we say that L is *slowly increasing* if both of the following conditions hold:

(i) for fixed $u > 0$, we have $L(ux) \sim L(x)$ as $x \to \infty$,
(ii) for fixed $u > 0$, we have $L(x^u) \gg_u L(x)$ for all $x \ge 1$.

Theorem 1. *Let f be a nonnegative multiplicative function for which*

$$\sum_{p^k} \frac{f(p^k)}{p^k} < \infty, \tag{2}$$

where the sum is over all prime powers p^k with $k \ge 1$. Suppose that for a certain slowly increasing function $L(x)$,

(i) $\sum_{p^k\le x} f(p^k) \sim x/L(x)$,
(ii) *for each* fixed *prime p_0, we have* $\sum_{p_0^k\le x} f(p_0^k) = o(x/L(x))$.

Then as $x\to\infty$,

$$\sum_{n\le x} f(n) \sim C_f\frac{x}{L(x)}, \quad \textit{where} \quad C_f := \sum_{m=1}^{\infty}\frac{f(m)}{m}.$$

Remark. The convergence of the sum defining C_f follows from (2) and the Euler product expansion $\sum_{m=1}^{\infty}\frac{f(m)}{m} = \prod_p\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right)$.

In contrast to Theorem B, our Theorem 1 requires the convergence of the series (2). However, the hypotheses on $L(x)$ are fairly weak, and Theorem 1 applies in many cases when the partial sums of $f(p^k)$ are smaller than allowed by Theorem B. One such scenario is when $\sum_{p^k\le x} f(p^k) \sim Ax/(\log x)^B$ for constants $A > 0$ and $B > 1$. Here the convergence criterion (2) follows by partial summation, and we may take $L = L_{A,B}$, where $L_{A,B}(x) := \frac{1}{A}(\log x)^B$ for $x \ge \mathrm{e}$ and $L(x) = \frac{1}{A}$ for $x \le \mathrm{e}$.

In practice, Theorems A, B and Theorem 1 together cover most of the naturally occurring cases where $\sum_{p\le x} f(p)$ possesses a reasonable asymptotic formula of the shape $x^{1-o(1)}$. If we apply iteratively condition (ii) in the definition of a slowly increasing function, we find that $L(x) \le (\log x)^{O_L(1)}$ for large x. So none the theorems we have discussed apply when $\sum_{p^k\le x} f(p^k)$ is very small, say smaller than $x^{1-\delta}$; here much remains to be understood.

We now give two examples of Theorem 1.

Example. Consider the completely multiplicative function f specified by setting $f(p) = 1/\log p$ for all primes p. It is simple to prove, using the prime number theorem, that

$\sum_{p^k\le x} f(p^k) \sim x/(\log x)^2$. Moreover, for any fixed prime p_0, one has

$$\sum_{p_0^k\le x} f(p_0^k) \le \sum_{k\le \log x/\log 2} (\log 2)^{-k} \ll x^{\log(\frac{1}{\log 2})/\log 2} < x^{0.53}.$$

We conclude that the hypotheses of Theorem 1 are satisfied with $L(x) = L_{1,2}(x)$. Now replacing C_f by its Euler product expansion, we deduce from Theorem 1 that

$$\sum_{n\le x} f(n) \sim \frac{x}{(\log x)^2}\prod_p \left(1+\sum_{k=1}^{\infty}\frac{1}{(p\log p)^k}\right).$$

Example. An arbitrary set of primes $\mathscr{P}$ generates a multiplicative semigroup $S_{\mathscr{P}} := \{n \in \mathbf{N} : p \mid n \Rightarrow p \in \mathscr{P}\}$. Suppose that $\sum_{p\in\mathscr{P}} \frac{1}{p} < \infty$ and that the counting function $\pi_{\mathscr{P}}(x)$ of $\mathscr{P}$ satisfies $\pi_{\mathscr{P}}(x) \sim x/L(x)$ for a slowly increasing function $L(x)$. Then the characteristic function of $S_{\mathscr{P}}$ satisfies the hypotheses of Theorem 1, and we find that

$$\#\{n \le x : p \mid n \Rightarrow p \in \mathscr{P}\} \sim \frac{x}{L(x)}\prod_{p\in\mathscr{P}}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right), \quad \text{as } x\to\infty.$$

We now turn to our second topic. By a *Golomb prime*, we mean a member of the sequence $3, 5, 17, 23, 29, 53, 83, \dots$. Here the rule for producing the next term in the sequence is to take the least prime that is not 1 modulo any previous term. Golomb introduced this sequence in 1955 [2], noting that the semigroup generated by these primes has the property that each of its squarefree elements n satisfies $\gcd(n, \varphi(n)) = 1$. (Cf. [5, Chapter 7.3].) We call this the *Golomb semigroup.*

In 1961, Erdős [1] gave an asymptotic formula for the number of Golomb primes $p \le x$. Our second theorem describes the asymptotic behavior of the counting function of the Golomb semigroup.

Theorem 2. *Let $\mathscr{P}$ be the collection of Golomb primes. As $x \to \infty$, there are asymptotically $Gx/\log x$ members of the Golomb semigroup in $[1, x]$, where the constant $G := \prod_{p\in\mathscr{P}}(1 - 1/(p-1)^2)$.*

Theorem 2 may be quickly derived from Erdős's results in [1] and Shirokov's Theorem B. However, we have chosen instead to present a simple argument independent of Shirokov's work (but still using some of Wirsing's ideas). We believe that our short proof may be of some independent interest.

Notation. Throughout, we use O and o-notation, as well as the associated Vinogradov $\ll$ and $\gg$ notations, with their standard meanings. We also use $F \lesssim G$ to mean that $\limsup F/G \le 1$. We write $P(n)$ for the largest prime factor of n, with the convention that $P(1) = 1$. If p is a prime, we write $p^e \parallel n$ to mean that $p^e \mid n$ but that $p^{e+1} \nmid n$. We use $\omega(n)$ for the number of distinct primes dividing n, so that $\omega(n) = \sum_{p\mid n} 1$.

2. Proof of Theorem 1

We begin by demonstrating an upper bound for $\sum_{n\le x} f(n)$ of the correct order of magnitude. This lemma and its proof were inspired by recent work of Gottschlich [3, Lemma 2.3].

Lemma 3. *Let f be a nonnegative multiplicative function satisfying* (2). *Suppose that for a certain slowly increasing function $L(x)$, we have*

$$\sum_{p^k\le x} f(p^k) \ll x/L(x) \quad \textit{for all } x \ge 1. \tag{3}$$

Then for $x \geq 1$,

$$\sum_{n \leq x} f(n) \ll x/L(x).$$

The implied constant in this last estimate depends on the implied constant in (3)*, an upper bound for the sum of the series in* (2)*, and the function L.*

Proof. By hypothesis, we can choose C_1 so that $\sum_{p^k \leq x} f(p^k) \leq C_1 x/L(x)$ holds for all $x \geq 1$. We choose C_2 so that $L(x) \leq C_2 L(\sqrt{x})$ for all $x \geq 1$, and we put $C_3 = \sum_{p^e} \frac{f(p^e)}{p^e}$. Finally, we set

$$C := \max\{C_1, C_2 C_3\}.$$

We prove by induction that for every natural number k,

$$\sum_{\substack{n \leq x \\ \omega(n) = k}} f(n) \leq \frac{C^k}{(k-1)!} \frac{x}{L(x)}. \tag{4}$$

When $k = 1$, this holds since $C \geq C_1$. Now suppose the claim to be proved for k. Let $n \leq x$ denote a generic natural number with $k + 1$ distinct prime factors. At most one of the prime power components of n can exceed $\sqrt{x}$. We pivot on the other components to discover that

$$\begin{aligned} k \sum_{\substack{n \leq x \\ \omega(n) = k+1}} f(n) \leq \sum_{p^e \leq \sqrt{x}} \sum_{\substack{n \leq x \\ p^e \| n}} f(n) &= \sum_{p^e \leq \sqrt{x}} f(p^e) \sum_{\substack{m \leq x/p^e \\ p \nmid m}} f(m) \\ &\leq \frac{C^k}{(k-1)!} x \sum_{p^e \leq \sqrt{x}} \frac{f(p^e)}{p^e} \frac{1}{L(x/p^e)}. \end{aligned}$$

Using that $L(x/p^e) \geq L(\sqrt{x}) \geq C_2^{-1} L(x)$, we find upon rearranging that

$$\sum_{\substack{n \leq x \\ \omega(n) = k+1}} f(n) \leq \frac{C^k C_2 C_3}{k!} \frac{x}{L(x)} \leq \frac{C^{k+1}}{k!} \frac{x}{L(x)}.$$

This completes the proof of (4). Summing the relation (4) over $k \geq 1$ shows that $\sum_{n \leq x} f(n) \leq 1 + C\mathrm{e}^C x/L(x) \ll x/L(x)$, as desired. $\square$

Lemma 4. *Suppose that f satisfies the hypotheses of Theorem 1 for a certainly slowly increasing function $L(x)$. For each natural number $m \geq 1$, put*

$$\mathscr{A}_m := \{mp^e : p > P(m), e \geq 1\}.$$

For every fixed natural number m,

$$\sum_{\substack{n \leq x \\ n \in \mathscr{A}_m}} f(n) = \left(\frac{f(m)}{m} + o(1)\right) \frac{x}{L(x)}, \quad \text{as } x \to \infty.$$

Proof. Observe that

$$\sum_{\substack{n \leq x \\ n \in \mathscr{A}_m}} f(n) = f(m) \sum_{p^e \leq x/m} f(p^e) - f(m) \sum_{\substack{p^e \leq x/m \\ p \leq P(m)}} f(p^e).$$

By condition (ii) in the statement of Theorem 1, the subtracted term is $o(x/L(x))$. Also,

$$\sum_{p^e \leq x/m} f(p^e) \sim \frac{x/m}{L(x/m)} \sim \frac{x}{mL(x)},$$

using that f is slowly increasing. Collecting these estimates completes the proof. $\square$

Proof of Theorem 1. We have $\sum_{1<n\le x} f(n) = \sum_{m\ge 1}\sum_{n\in\mathscr{A}_m\cap[1,x]} f(n)$. So for each fixed y, Lemma 4 yields $\sum_{n\le x} f(n) \ge (C_{f,y}+o(1))x/L(x)$ as $x\to\infty$, where we put $C_{f,y} := \sum_{m\le y} f(m)/m$. Letting $y\to\infty$ proves the lower bound implicit in the theorem. On the other hand, we also have from Lemma 4 that for each fixed y,

$$\limsup_{x\to\infty}\frac{\sum_{n\le x} f(n)}{x/L(x)} \le C_f + \limsup_{x\to\infty}\frac{\sum_{m>y}\sum_{\substack{n\le x\\ n\in\mathscr{A}_m}} f(n)}{x/L(x)}.$$

Let us show that the lim sup on the right-hand side tends to zero as $y\to\infty$. We first estimate the contribution from those n corresponding to $m\le\sqrt{x}$. Since $L(x/m)\ge L(\sqrt{x})\gg L(x)$,

$$\sum_{y<m\le\sqrt{x}}\sum_{\substack{n\le x\\ n\in\mathscr{A}_m}} f(n) \le \sum_{y<m\le\sqrt{x}} f(m)\sum_{p^e\le x/m} f(p^e) \ll \sum_{m>y} f(m)\frac{x/m}{L(x/m)} \ll \frac{x}{L(x)}\sum_{m>y}\frac{f(m)}{m}.$$

Since the final sum tends to zero as $y\to\infty$, this contribution is acceptable. Next, consider the contribution from values of n with $m>\sqrt{x}$ and $P(n)>y$. We have

$$\sum_{\sqrt{x}<m\le x}\sum_{\substack{n\le x\\ n\in\mathscr{A}_m,\ P(n)>y}} f(n) \le \sum_{\substack{p^e\le x\\ p>y}} f(p^e)\sum_{\sqrt{x}<m\le x/p^e} f(m).$$

We use Lemma 3 to estimate the inner sum and find, since $L(x/p^e)\ge L(\sqrt{x})\gg L(x)$, that

$$\sum_{\substack{p^e\le x\\ p>y}} f(p^e)\sum_{\sqrt{x}<m\le x/p^e} f(m) \ll x\sum_{\substack{p^e\le x\\ p>y}}\frac{f(p^e)}{p^e}\frac{1}{L(x/p^e)} \ll \frac{x}{L(x)}\sum_{p^e>y}\frac{f(p^e)}{p^e}.$$

Again, the remaining sum tends to zero as $y\to\infty$. To complete the proof, it suffices to show that

$$\sum_{\substack{x^{1/2}<n\le x\\ P(n)\le y}} f(n) = o(x/L(x)), \quad \text{as } x\to\infty. \tag{5}$$

Write $n=p_0^e h$, where p_0^e is the largest prime power component of n. Then $p_0^e\ge n^{1/\omega(n)}\ge x^{1/2y}$, and we find that

$$\sum_{\substack{x^{1/2}<n\le x\\ P(n)\le y}} f(n) \le \sum_{h\le x^{1-\frac{1}{2y}}} f(h)\sum_{\substack{x^{1/2y}\le p_0^e\le x/h\\ p_0\le y}} f(p_0^e). \tag{6}$$

Using assumption (ii) in the statement of Theorem 1, the inner sum is $o(\frac{x}{hL(x/h)})$, which is $o(\frac{x}{hL(x)})$ by condition (ii) in the definition of slowly increasing. (Recall that $x/h\ge p_0^e\ge x^{1/2y}$ and y is *fixed*.) Putting this estimate back into (6) and noting that $\sum_{h\le x^{1-1/2y}} f(h)/h \le C_f \ll_f 1$ completes the proof of (5) and also of Theorem 1. $\square$

Remark. In some applications, we may only have an upper bound $\sum_{p^k\le x} f(p^k)\lesssim x/L(x)$ and not a precise asymptotic formula. In this case, our proof of Theorem 1 shows that (under all of the remaining hypotheses of that theorem) $\sum_{n\le x} f(n)\lesssim C_f x/L(x)$. This result is similar in spirit to an upper-bound variant of Wirsing's Theorem A put forward by Halberstam and Richert [4, Theorem 2].

3. Proof of Theorem 2

Throughout this section, $\mathscr{P}$ denotes the collection of Golomb primes. The following estimates are due to Erdős [1].

Theorem C. *As $x \to \infty$,*

$$\pi_{\mathscr{P}}(x) \sim \frac{x}{\log x \log\log x}$$

and

$$\prod_{\substack{p\in\mathscr{P}\\ p\le x}} \left(1-\frac{1}{p-1}\right)^{-1} \sim \log\log x.$$

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