THE DISTRIBUTION OF NUMBERS WITH MANY FACTORIZATIONS

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Abstract. Let $f(m)$ denote the number of factorizations of the positive integer m, i.e., the number of ways of writing m as product of integers larger than 1, where the order of the factors is not taken into account. Let $\epsilon > 0$ and $\alpha \in (0,1)$. We prove that for all $x > x_0(\epsilon, \alpha)$ and every $\mathscr{S} \subset [1, x]$ with $\#\mathscr{S} \le x^{1-\alpha}$,

$$\sum_{m \in \mathscr{S}} f(m) \le x/L(x)^{\alpha-\epsilon},$$

where $L(x) = \exp(\log x \cdot \log\log\log x/\log\log x)$. This generalizes a recent result of the author concerning popular values of Euler's φ-function. We also estimate the β-th moment of $f(m)$, for all $\beta > 0$.

1. Introduction

By a *factorization* of m, we mean a representation of m as a product of integers larger than 1, where two factorizations are considered the same if they differ only in the order of the factors. (Another name for the same concept is a *multiplicative partition*.) We let $f(m)$ denote the number of factorizations of the positive integer m.

MacMahon [Mac24] introduced the function $f(m)$ in 1924, noting that it satisfies an identity resembling Euler's famous product formula for the Riemann zeta function,

$$\sum_{m=1}^{\infty} \frac{f(m)}{m^s} = \prod_{d=2}^{\infty} \frac{1}{1-\frac{1}{d^s}}.$$

Shortly thereafter, Oppenheim began investigating statistical properties of $f(m)$ [Opp26, Opp27]. In [Opp27], Oppenheim proved that

$$\sum_{m \le x} f(m) \sim \frac{x}{2\sqrt{\pi}} \frac{\exp(2\sqrt{\log x})}{(\log x)^{3/4}}, \quad \text{as } x \to \infty, \tag{1.1}$$

a result later rediscovered by Szekeres and Turán [ST33]. In [Opp26], Oppenheim claimed to prove that

$$f(m) \le m/L(m)^{2+o(1)}, \quad \text{as } m \to \infty,$$

where, here and below,

$$L(x) = \exp\left(\log x \cdot \frac{\log\log\log x}{\log\log x}\right).$$

More than 50 years later, Canfield, Erdős, and Pomerance [CEP83] (see also [Pom89]) disproved this "theorem" of Oppenheim, showing that in fact there is an infinite sequence

2010 *Mathematics Subject Classification.* 11N64 (primary), 11N25, 11N37 (secondary).

of m with $f(m) \geq m/L(m)^{1+o(1)}$. Moreover, they proved that this is best possible in that

$$\max_{m \leq x} f(m) = \frac{x}{L(x)^{1+o(1)}}, \quad \text{as } x \to \infty. \tag{1.2}$$

Here is another way of stating the upper-bound half of the Canfield–Erdős–Pomerance result: If $\mathscr{S}$ is a *singleton* subset of $[1, x]$, then $\sum_{m \in \mathscr{S}} f(m) \leq x/L(x)^{1+o(1)}$. Our main theorem is an analogous result under the much weaker restriction that $\#\mathscr{S} \leq x^{1-\alpha}$ for some fixed $\alpha \in (0, 1)$.

Theorem 1.1. *Fix $\epsilon > 0$ and $\alpha \in (0, 1)$. There is an $x_0 = x_0(\epsilon, \alpha)$ such that, for all $x > x_0$, and every subset $\mathscr{S} \subset [1, x]$ with $\#\mathscr{S} \leq x^{1-\alpha}$,*

$$\sum_{m \in \mathscr{S}} f(m) \leq x/L(x)^{\alpha-\epsilon}.$$

The same method used by Canfield–Erdős–Pomerance to prove the sharpness of their result implies that Theorem 1.1 is sharp for every α. Here we mean that the conclusion fails if the number α appearing in the exponent of $L(x)$ is replaced with anything larger. In fact, in a remark following the proof of Theorem 1.1, we show the following: If $\mathscr{S}$ is the set of numbers in $[1, x]$ all of whose prime factors do not exceed $(\log x)^{1/\alpha}$, then $\#\mathscr{S} = x^{1-\alpha+o(1)}$ as $x \to \infty$, and $\sum_{m \in \mathscr{S}} f(m) \geq x/L(x)^{\alpha+o(1)}$.

An easy consequence of Theorem 1.1, proved in §4, is an estimate for the β-th moment of $f(m)$, for each real $\beta > 1$.

Corollary 1.2. *Fix $\beta > 1$. Then*

$$\sum_{m \leq x} f(m)^\beta = x^\beta / L(x)^{\beta+o(1)}, \quad \textit{as } x \to \infty.$$

In view of Corollary 1.2, it is natural to wonder what one can prove about the moments when $0 < \beta < 1$. By very different methods, we prove the following result in §5.

Theorem 1.3. *Fix β with $0 < \beta < 1$. As $x \to \infty$,*

$$\sum_{m \leq x} f(m)^\beta = x \exp\left((1 + o(1))(1 - \beta)^{1/(1-\beta)} \log\log x \left(\frac{\log\log x}{\log\log\log x}\right)^{\beta/(1-\beta)}\right).$$

For example, $\sum_{m \leq x} f(m)^{1/2} = x \exp\left((\frac{1}{4} + o(1))(\log\log x)^2/\log\log\log x\right)$, as $x \to \infty$. In both Corollary 1.2 and Theorem 1.3, one observes a stark difference in behavior from the case $\beta = 1$ (Oppenheim's result (1.1)).

Theorem 1.1, with $f(m)$ replaced by the function $N(m) = \#\varphi^{-1}(m)$ (φ being Euler's totient), was the main result of the author's earlier paper [Polb]. These two results are connected as follows. If $\varphi(n) = m$, where $n = p_1^{e_1} \cdots p_k^{e_k}$, then

$$m = \prod_{i=1}^{k} \overbrace{p_i \cdots p_i \cdots p_i}^{e_i - 1 \text{ times}} (p_i - 1),$$

and so n induces a "factorization" of m into $\sum_{i=1}^{k}((e_i - 1) + 1) = \Omega(n)$ parts. The scare quotes are present because this expression for m may include 1 as a term, which is not allowed in an honest-to-goodness factorization. Developing this relationship, one may show that $N(m) \leq 4f(m)$ for all m. (See pp. 257–258 of [EPS87] for a sketch, and see Lemma 8 in [Pola] for a detailed proof.) Hence, the main theorem of [Polb] is entirely superseded by Theorem 1.1. The proofs of the two theorems are similar in spirit, but working with factorizations has certain advantages. The present arguments, in addition to yielding more general conclusions, are also somewhat streamlined over those of [Polb].

The reader interested in the study of integer factorizations might also wish to consult, in addition to the works already mentioned, the papers [Rie61, Kan83, Hen83, Hen84, Hen87, War93, Kim98, LMS10, Pol12, CJNW13, Bro14, BS17].

Notation and conventions. The letter p is reserved throughout for primes. We write $\omega(n)$ for the number of distinct prime factors of n, and we use $\Omega(n)$ for the number of prime factors of n counted with multiplicity. In addition, we write $\Omega_{>Y}(n)$ for the number of prime factors of n exceeding Y, with multiplicity; that is,

$$\Omega_{>Y}(n) = \sum_{\substack{p^e \| n \\ p>Y}} e.$$

We let $\tau_z(n)$ denote the z-fold divisor function, defined as the coefficient of n^{-s} in the Dirichlet series for $\zeta(s)^z$. Thus, when z is a nonnegative integer, $\tau_z(n)$ counts the number of ways of writing n as an ordered product of z positive integers. For every z, the function $\tau_z(n)$ is multiplicative and satisfies

$$\tau_z(p^k) = \frac{(z+k-1)(z+k-2)\cdots z}{k!}$$

on prime powers p^k.

We identify the space of factorizations with the space of multisets composed of integers at least 2, the elements of the multiset representing the terms of the factorization. Factorizations will be written in boldface letters such as $\boldsymbol{m}$. We denote the factorization with parts $m_1, \ldots, m_\ell$ as $\langle m_1, \ldots, m_\ell \rangle$, and we let

$$|\langle m_1, \ldots, m_\ell \rangle| = m_1 \cdots m_\ell,$$

so that $\boldsymbol{m}$ is a factorization of $|\boldsymbol{m}|$. By the product of two factorizations $\boldsymbol{m} \times \boldsymbol{n}$, we mean the factorization of $|\boldsymbol{m}| \cdot |\boldsymbol{n}|$ whose underlying multiset of terms is the union of the terms of $\boldsymbol{m}$ and $\boldsymbol{n}$. We say that $\boldsymbol{m}$ divides $\boldsymbol{n}$ if $\boldsymbol{n} = \boldsymbol{m} \times \boldsymbol{r}$ for some factorization $\boldsymbol{r}$.

2. Preliminaries for the proof of Theorem 1.1

In this section we collect some lemmas needed for the proof of Theorem 1.1, beginning with two exponential moment estimates. Roughly speaking, the first implies that few factorizations of numbers in $[1, x]$ are factorizations of numbers with many large prime factors. The second implies that, for somewhat different for definitions of "few" and "many", few factorizations of numbers in $[1, x]$ are factorizations of numbers with many prime factors in total.

Lemma 2.1. *Let*

$$z = \exp((\log\log x)^{1/2}).$$

Fix any $\eta \in (0, 1)$, and let

$$A = (\log\log x)^{1-\eta}.$$

As $x \to \infty$,

$$\sum_{|\boldsymbol{m}|\le x} A^{\Omega_{>z}(|\boldsymbol{m}|)} \le x \cdot L(x)^{o(1)}.$$

Proof. Let $c \in (1,2)$ be a parameter to be specified more precisely momentarily. Then (Rankin's trick)

$$\sum_{|\boldsymbol{m}|\le x} A^{\Omega_{>z}(|\boldsymbol{m}|)} \le x^c \sum_{\boldsymbol{m}} \frac{A^{\Omega_{>z}(|\boldsymbol{m}|)}}{|\boldsymbol{m}|^c}$$

$$\le x^c \prod_{d\ge 2}\left(1 + \frac{A^{\Omega_{>z}(d)}}{d^c} + \frac{A^{\Omega_{>z}(d^2)}}{d^{2c}} + \dots\right). \tag{2.1}$$

Once x is large, $A < z^{1/3}$. Hence, for any $d \ge 2$, we have that $A^{\Omega_{>z}(d)} \le z^{\Omega_{>z}(d)/3} \le d^{1/3}$, and $A^{\Omega_{>z}(d)}/d^c \le d^{1/3-c} \le d^{-2/3} \le 2^{-2/3}$. Therefore,

$$\frac{A^{\Omega_{>z}(d)}}{d^c} + \frac{A^{\Omega_{>z}(d^2)}}{d^{2c}} + \dots = \frac{A^{\Omega_{>z}(d)}/d^c}{1 - A^{\Omega_{>z}(d)}/d^c} \ll \frac{A^{\Omega_{>z}(d)}}{d^c},$$

and

$$x^c \prod_{d\ge 2}\left(1 + \frac{A^{\Omega_{>z}(d)}}{d^c} + \frac{A^{\Omega_{>z}(d^2)}}{d^{2c}} + \dots\right) \le x \exp\left((c-1)\log x + O\left(\sum_{d\ge 2}\frac{A^{\Omega_{>z}(d)}}{d^c}\right)\right). \tag{2.2}$$

To estimate the final sum on d, we first consider the partial sums of $A^{\Omega_{>z}(d)}$. Let g be the arithmetic function determined by the identity $A^{\Omega_{>z}(d)} = \sum_{r\mid d} g(r)$. Then g is multiplicative and satisfies $g(p^e) = 0$ on primes $p \le z$, and $g(p^e) = A^e - A^{e-1}$ when $p > z$. So for any $T \ge 1$,

$$\begin{aligned}\sum_{d\le T} A^{\Omega_{>z}(d)} &= \sum_{r\le T} g(r)\left\lfloor \frac{T}{r}\right\rfloor \le T\sum_{r\le T}\frac{g(r)}{r}\\ &\le T\prod_{z<p\le T}\left(1 + \frac{A-1}{p} + \frac{A^2-A}{p^2} + \dots\right)\\ &\le T\exp\left((A-1)\sum_{z<p\le T}\frac{1}{p}\right)\exp\left(\sum_{z<p\le T}\left(\frac{A^2-A}{p^2} + \frac{A^3-A^2}{p^3} + \dots\right)\right)\end{aligned}$$

The final summand appearing here is a geometric series with ratio $A/p \le A/z < 1/2$ (if x is large), and so

$$\sum_{p>z}\left(\frac{A^2-A}{p^2} + \frac{A^3-A^2}{p^3} + \dots\right) < 2\sum_{p>z}\frac{A^2-A}{p^2} < 2A^2\sum_{p>z}\frac{1}{p^2} \ll A^2/z \ll 1.$$

By the prime number theorem with a standard error term, along with partial summation, we have for all $T \ge z$ that

$$\sum_{z<p\le T}\frac{1}{p} = \log\frac{\log T}{\log z} + O\left(\frac{1}{(\log z)^2}\right),$$

so that

$$(A-1)\sum_{z<p\le T}\frac{1}{p} = (A-1)\log\frac{\log T}{\log z} + O(1).$$

Collecting the above estimates yields

$$S(T) := \sum_{d \le T} A^{\Omega_{>z}(d)} \ll T \cdot \begin{cases} 1 & \text{if } T \le z, \\ (\log T/\log z)^{A-1} & \text{if } T > z. \end{cases}$$

By Abel summation (keeping mind that $1 < c < 2$),

$$\begin{aligned} \sum_{d \ge 2} \frac{A^{\Omega_{>z}(d)}}{d^c} &= \int_1^\infty t^{-c}\, dS(t) \le c \int_1^\infty \frac{S(t)}{t^{c+1}}\, dt \ll \int_1^z \frac{S(t)}{t^{c+1}}\, dt + \int_z^\infty \frac{S(t)}{t^{c+1}}\, dt \\ &\ll \int_1^z \frac{dt}{t} + \frac{1}{(\log z)^{A-1}} \int_z^\infty \frac{(\log t)^{A-1}}{t^c}\, dt. \end{aligned}$$

Now $\int_1^z dt/t = \log z$, and

$$\int_z^\infty \frac{(\log t)^{A-1}}{t^c}\, dt \le \int_1^\infty \frac{(\log t)^{A-1}}{t^c}\, dt = \frac{\Gamma(A)}{(c-1)^A}.$$

(To obtain the final expression, one should make the substitution $t = e^{u/(c-1)}$.)

Hence, by (2.1) and (2.2),

$$\sum_{|\boldsymbol{m}| \le x} A^{\Omega_{>z}(|\boldsymbol{m}|)} \le x \cdot z^{O(1)} \cdot \exp\left((c-1)\log x + O\left(\frac{\Gamma(A)}{(c-1)^A (\log z)^{A-1}} \right) \right).$$

Clearly, the factor $z^{O(1)}$ is $L(x)^{o(1)}$, as $x \to \infty$. We complete the proof by showing that when

$$c = 1 + \frac{A}{(\log x)^{1/(A+1)} (\log z)^{(A-1)/(A+1)}},$$

then the final factor is also $L(x)^{o(1)}$. For this value of c,

$$\begin{aligned} (c-1)\log x + O\left(\frac{\Gamma(A)}{(c-1)^A (\log z)^{A-1}} \right) &\ll A(\log x/\log z)^{1-\frac{1}{A+1}} (\log z)^{\frac{1}{A+1}} \\ &\ll (\log\log x)^{O(1)} (\log x)^{1-\frac{1}{2A}} \ll \log x/\exp((\log\log x)^{\eta/2}). \end{aligned}$$

In particular, this is $o(\log L(x))$, as desired. □

Lemma 2.2. *Let B be a positive real number with $1 \le B < 2$, and let $x \ge 3$. Then*

$$\sum_{|\boldsymbol{m}| \le x} B^{\Omega(|\boldsymbol{m}|)} \le x \exp(O_B((\log x)^{3/4})).$$

Proof. Since $B < 2$, we have that

$$B^{\Omega(d)} = (2^{\Omega(d)})^{\log B/\log 2} \le d^{\log B/\log 2} = d^{1-\delta}$$

for the positive number $\delta = \log(2/B)/\log 2$. Now reasoning as in the proof of the last lemma, we find that for any choice of $c \in (1,2)$,

$$\begin{aligned} \sum_{|\boldsymbol{m}| \le x} B^{\Omega(|\boldsymbol{m}|)} &\le x^c \sum_{\boldsymbol{m}} \frac{B^{\Omega(|\boldsymbol{m}|)}}{|\boldsymbol{m}|^c} \\ &\le x \exp\left((c-1)\log x + O_B\left(\sum_{d \ge 2} \frac{B^{\Omega(d)}}{d^c} \right) \right). \end{aligned}$$

Recall that $S(T) := \sum_{d\le T} B^{\Omega(d)} \le \sum_{d\le T} 2^{\Omega(d)} \ll T(\log T)^2$ for all $T \ge 2$ (see [Gro56] or [Ten15, Exercise 57, p. 59] for the last estimate). It follows that

$$\sum_{d\ge 2} \frac{B^{\Omega(d)}}{d^c} \ll \frac{1}{(c-1)^3},$$

so that

$$\sum_{|\boldsymbol{m}|\le x} B^{\Omega(|\boldsymbol{m}|)} \le x \exp\left((c-1)\log x + O_B\left(\frac{1}{(c-1)^3}\right)\right).$$

Taking $c = 1 + \frac{1}{(\log x)^{1/4}}$ finishes the proof. □

Our next lemma is an upper bound for the average of $\tau_k(n)/n$, where k is a nonnegative integer. While the result is not new (see, e.g., Mardjanichvili [Mar39]), it is not so easy to locate a proof in the recent literature. So we include the short, simple argument for the convenience of the reader.

Lemma 2.3. *For each nonnegative integer k and real number $x \ge 1$,*

$$\sum_{e_1\cdots e_k \le x} \frac{1}{e_1\cdots e_k} \le \frac{(\log x + k)^k}{k!}.$$

Proof. When $k = 0$, there is precisely one tuple $(e_1, \dots, e_k)$ with $e_1 \dots e_k \le x$, viz. the empty tuple. Thus, the left-hand side is 1, and the inequality holds (with equality). Suppose the inequality is known for a given k and all $x \ge 1$. Then

$$\sum_{e_1\dots e_{k+1}\le x} \frac{1}{e_1\cdots e_{k+1}} = \sum_{e_{k+1}\le x} \frac{1}{e_{k+1}} \sum_{e_1\cdots e_k \le x/e_{k+1}} \frac{1}{e_1\cdots e_k} \le \sum_{e_{k+1}\le x} \frac{(\log(x/e_{k+1}) + k)^k}{k!\cdot e_{k+1}}.$$

Considered as a function of e_{k+1}, the summand is decreasing on $[1, x]$, and so

$$\begin{aligned}\sum_{e_{k+1}\le x} \frac{(\log(x/e_{k+1}) + k)^k}{k!\cdot e_{k+1}} &\le \frac{(\log x + k)^k}{k!} + \int_1^x \frac{(\log(x/t)+k)^k}{k!\cdot t}\,dt \\ &= \frac{(\log x + k)^k}{k!} + \frac{(\log x + k)^{k+1}}{(k+1)!} - \frac{k^{k+1}}{(k+1)!} \\ &\le \frac{(\log x + k + 1)^{k+1}}{(k+1)!}.\end{aligned}$$

To transition from the second to the third line, we used the mean value theorem, which guarantees that for some $k' \in (k, k+1)$,

$$\frac{(\log x + k + 1)^{k+1}}{(k+1)!} - \frac{(\log x + k)^{k+1}}{(k+1)!} = \frac{(\log x + k')^k}{k!} \ge \frac{(\log x + k)^k}{k!}.$$

Thus, the lemma follows by induction on k. □

Our final lemma provides a description of the divisors of a product.

Lemma 2.4. *Let d and $m_1, \dots, m_k$ be positive integers. If d divides $m_1\cdots m_k$, then one can write $d = d_1\cdots d_k$ where each d_i is a positive integer dividing m_i.*

Proof. Working prime-by-prime, the general result reduces to the case where d and the m_i are all powers of the same prime p, where it is clear. □

3. The average of $f(m)$ on small sets: Proof of Theorem 1.1

Throughout this section, we continue to use z for the quantity $\exp((\log\log x)^{1/2})$.

Our task is to estimate the number of $\boldsymbol{m}$ with $|\boldsymbol{m}| \in \mathscr{S}$. We will assume to start with that

$$|\boldsymbol{m}| > x/L(x); \tag{3.1}$$

by (1.1) (or by Lemma 2.2 with $B = 1$), the number of exceptional $\boldsymbol{m}$ is at most $x/L(x)^{1+o(1)}$, as $x \to \infty$, which is negligible compared to our target upper bound.

We also assume that

$$\Omega(|\boldsymbol{m}|) \le \frac{\log x}{(\log\log x)^{2/3}}. \tag{3.2}$$

The cardinality of the exceptional set can be estimated by Lemma 2.2 with $B = 3/2$. Since $\log(3/2) > 2/5$, we deduce that the number of exceptional $\boldsymbol{m}$ with $|\boldsymbol{m}| \le x$ is at most $x/\exp(\frac{2}{5}\log x/(\log\log x)^{2/3})$ (for large x), which is smaller than $x/L(x)$.

We fix $\eta \in (0,1)$ small enough that $(1-\eta)^2\alpha > \alpha - \frac{1}{2}\epsilon$. We will assume that

$$\Omega_{>z}(|\boldsymbol{m}|) \le (1-\eta)\alpha\frac{\log x}{\log\log x}. \tag{3.3}$$

By Lemma 2.1, the number of exceptions to (3.3) with $|\boldsymbol{m}| \le x$ is at most $x/L(x)^{(1-\eta)^2\alpha+o(1)}$, and so is smaller than $x/L(x)^{\alpha-\frac{1}{2}\epsilon}$ for large enough x.

Call a factorization $\boldsymbol{m}$ admissible if it satisfies (3.1), (3.2), and (3.3). We will show that the number of admissible $\boldsymbol{m}$ with $|\boldsymbol{m}| \in \mathscr{S}$ is $o(x/L(x))$. Together with the above estimates, this implies that the total number of $\boldsymbol{m}$ with $|\boldsymbol{m}| \in \mathscr{S}$ is smaller than $x/L(x)^{\alpha-\epsilon}$ for large x, completing the proof of Theorem 1.1.

Suppose that $\boldsymbol{m}$ is an admissible factorization with $|\boldsymbol{m}| \in \mathscr{S}$. Let m' and m'' be the z-rough and z-smooth parts of $m := |\boldsymbol{m}|$, respectively. (By the z-rough and z-smooth parts of m, we mean its largest divisors composed of primes $> z$, resp. $\le z$.) Since

$$m'' \le z^{\Omega(m)} \le \exp(\log x/(\log\log x)^{1/6}) = x^{o(1)},$$

we have

$$m' = m/m'' \ge \frac{x/L(x)}{m''} \ge x^{1-o(1)}.$$

We now switch perspectives. Viewing m' as given, we bound the number of admissible $\boldsymbol{m}$ for which m' is the z-rough part of $|\boldsymbol{m}|$. Since m' divides $|\boldsymbol{m}|$, Lemma 2.4 implies that we can find a factorization $\boldsymbol{d} = \langle d_1, \dots, d_k\rangle$ of m' such that, for some positive integers $e_1, \dots, e_k$ and some factorization $\boldsymbol{n}$,

$$\boldsymbol{m} = \langle d_1e_1, \dots, d_ke_k\rangle \times \boldsymbol{n}.$$

Clearly,

$$|\boldsymbol{n}| = \frac{|\boldsymbol{m}|}{d_1e_1d_2e_2\cdots d_ke_k} \le \frac{x}{m'e_1\cdots e_k}.$$

So by (1.1), the number of possibilities for $\boldsymbol{n}$ given $m', \boldsymbol{d}$ and the e_i is at most

$$\frac{x}{m'e_1\cdots e_k}\exp(O((\log x)^{1/2})) \le x^{o(1)}\frac{1}{e_1\cdots e_k},$$

as $x \to \infty$.

Sum over the possibilities for the e_i, using Lemma 2.3. Since $e_1 \cdots e_k \le x/m' \le x$, we find that the number of possibilities for $\boldsymbol{n}$ given $m', \boldsymbol{d}$ is at most

$$x^{o(1)} \cdot \frac{(\log x + k)^k}{k!}.$$

How large is the second factor? Since k is the number of parts in the factorization of a number not exceeding x, trivially $k \le \frac{\log x}{\log 2}$, and so $(\log x + k)^k/k! \le (3\log x)^k/k!$. In fact, we have a better upper bound on k. Since m' is z-rough and each $d_i > 1$,

$$k \le \sum_{i=1}^{k} \Omega(d_i) = \Omega(m') = \Omega_{>z}(|\boldsymbol{m}|) \le (1-\eta)\alpha \frac{\log x}{\log\log x}.$$

As $(3\log x)^k/k!$ is an increasing function of k for $k \le 3\log x$, we deduce from this stronger bound (and Stirling's formula) that

$$(\log x + k)^k/k! \le (3\log x)^k/k! \le L(x)^{(1-\eta)\alpha + o(1)} = x^{o(1)}.$$

Consequently, given m' and $\boldsymbol{d}$, there are only $x^{o(1)}$ corresponding $\boldsymbol{m}$.

Next we bound, for a given m', the number of possibilities for its factorization $\boldsymbol{d}$. As explained above, if m' is the z-rough part of $|\boldsymbol{m}|$ for some admissible $\boldsymbol{m}$, then $\Omega(m') \le (1-\eta)\alpha\frac{\log x}{\log\log x}$. Using the simple bound $f(m') \le \Omega(m')^{\Omega(m')}$, we conclude that the number of possibilities for $\boldsymbol{d}$ given m' is at most $x^{(1-\eta)\alpha}$.[1]

Piecing it all together, we see that given m' there are at most $x^{(1-\eta)\alpha+o(1)}$ possibilities for $\boldsymbol{m}$. But there are trivially at most $x^{1-\alpha}$ values of m', since each m' is the z-rough part of a number in $\mathscr{S}$. Hence, the number of $\boldsymbol{m}$ arising this way is at most $x^{1-\eta\alpha+o(1)}$, which is certainly $o(x/L(x))$.

Remark. Fix $\alpha \in (0,1)$. Put $y = (\log x)^{1/\alpha}$, and let $\mathscr{S}$ be the set of y-smooth numbers in $[1,x]$. Then (see, e.g., [Ten15, Theorem 5.2, p. 513]) $\#\mathscr{S} = x^{1-\alpha+o(1)}$, as $x \to \infty$. We now show that

$$\sum_{m \in \mathscr{S}} f(m) \ge x/L(x)^{\alpha+o(1)};$$

it follows that Theorem 1.1 is best possible, for each $\alpha \in (0,1)$.

Let $k = \lfloor \log x/(\log\log x)^2 \rfloor$, and let $X = x^{1/k}$. Let $\mathcal{M}$ be the set of y-smooth numbers contained within $[2, X]$. Setting $u = \log X/\log y$, we have (see, e.g., [Ten15, Corollary 5.19, p. 534])

$$\begin{aligned}\#\mathcal{M} &= X\exp(-(1+o(1))u\log u) \\ &= X\exp(-(1+o(1))\alpha\log\log x \cdot \log\log\log x).\end{aligned}$$

If $\{m_1, \dots, m_k\}$ is a k-element subset of $\mathcal{M}$, then $\langle m_1, \dots, m_k\rangle$ is a factorization of a number in $\mathscr{S}$. Moreover, distinct subsets correspond to distinct factorizations. Using the

[1]To see that $f(n) \le \Omega(n)^{\Omega(n)}$ for every positive integer n, one can proceed as follows: Let $p_1, \dots, p_w$ be the prime factors of n, with multiplicities, so that $w = \Omega(n)$. To each set partition $\mathscr{S}_1, \dots, \mathscr{S}_k$ of $\{1, 2, 3, \dots, w\}$, associate the factorization $\langle s_1, \dots, s_k\rangle$ of n, where $s_I := \prod_{i \in \mathscr{S}_I} p_i$. This association establishes a surjective map onto the factorizations of n. One concludes by noting that the number of set partitions of $\{1, 2, 3, \dots, w\}$ is at most w^w (see [BT10] for explicit bounds on counts of set partitions, and compare with the asymptotic results quoted in our Proposition 5.1).

above estimate for $\#\mathcal{M}$, we deduce that

$$\sum_{m\in\mathscr{S}} f(m) \geq \binom{\#\mathcal{M}}{k} \geq \left(\frac{\#\mathcal{M}}{k}\right)^k = x/L(x)^{\alpha+o(1)},$$

as desired. (This proof is very similar to that of Theorem 2.1 in [CEP83].)

4. The β-th moment of $f(m)$ when $\beta > 1$: Proof of Corollary 1.2

We argue similarly to the proof of [Polb, Corollary 2], where the second moment of $N(m) = \#\varphi^{-1}(m)$ was estimated.

The lower bound in Corollary 1.2 is trivial: By (1.2), we can choose $m \leq x$ with $f(m) = x/L(x)^{1+o(1)}$, and this single m makes a contribution to $\sum_{m\leq x} f(m)^\beta$ of size $x^\beta/L(x)^{\beta+o(1)}$. So we focus on the upper bound.

For any set $\mathscr{S}$ of positive integers, it is clear that

$$\sum_{m\in\mathscr{S}} f(m)^\beta \leq \left(\max_{m\in\mathscr{S}} f(m)\right)^{\beta-1} \sum_{m\in\mathscr{S}} f(m). \tag{4.1}$$

We partition the set of integers $n \in [1,x]$ into two sets $\mathscr{S}_1$ and $\mathscr{S}_2$, according to whether or not $f(n) \leq x/L(x)^{\beta/(\beta-1)}$. Applying (4.1) with $\mathscr{S} = \mathscr{S}_1$, we find that

$$\sum_{m\in\mathscr{S}_1} f(m)^\beta \leq \frac{x^{\beta-1}}{L(x)^\beta} \sum_{m\leq x} f(m) \leq \frac{x^\beta}{L(x)^{\beta+o(1)}};$$

here we used that

$$\sum_{m\leq x} f(m) \leq x\cdot L(x)^{o(1)}, \tag{4.2}$$

which follows from (1.1). Turning to $\mathscr{S}_2$, notice that (4.2) implies that

$$\#\mathscr{S}_2 \leq L(x)^{\beta/(\beta-1)+o(1)} \leq x^{o(1)},$$

as $x \to \infty$. By Theorem 1.1,

$$\sum_{m\in\mathscr{S}_2} f(m) \leq x/L(x)^{1+o(1)}.$$

Since $f(m) \leq x/L(x)^{1+o(1)}$ for all $m \leq x$, taking $\mathscr{S} = \mathscr{S}_2$ in (4.1) reveals that

$$\sum_{m\in\mathscr{S}_2} f(m)^\beta \leq \frac{x^{\beta-1}}{L(x)^{\beta-1+o(1)}}\cdot\frac{x}{L(x)^{1+o(1)}} = \frac{x^\beta}{L(x)^{\beta+o(1)}}.$$

Adding our estimates for $\mathscr{S}_1$ and $\mathscr{S}_2$ finishes the proof.

5. The β-th moment of $f(m)$ when $0 < \beta < 1$: Proof of Theorem 1.3

We begin with the lower bound. To start off, notice that if m is squarefree with $\omega(m) = k$, then $f(m)$ is the kth Bell number B_k, which counts the number of set partitions of a k-element set. Since $f(m) \geq f(n)$ when $n \mid m$, we see that $f(m) \geq B_k$ whenever $\omega(m) = k$, whether or not m is squarefree. Hence, for any choice of k,

$$\sum_{m\leq x} f(m)^\beta \geq {B_k}^\beta \cdot \#\{m \leq x : \omega(m) = k\}. \tag{5.1}$$

We will select k to make the right-hand side essentially as large as possible. To carry out this plan, we require estimates for both right-hand factors in (5.1).

The following estimate for the Bell numbers is a weakened form of a result of de Bruijn (see eq. (6.27) on p. 108 of [dB81]).

Proposition 5.1. *As $k \to \infty$,*

$$\log B_k = k \log k - k \log\log k - k + O\left(k \frac{\log\log k}{\log k}\right).$$

Pomerance [Pom85, Theorem 3.1] showed the following lower bound for the number of $n \le x$ for which $\omega(n)$ assumes a prescribed value.

Proposition 5.2. *There is an absolute constant x_0 such that for all $x \ge x_0$ and all integers k with*

$$\log\log x \cdot (\log\log\log x)^2 \le k \le \frac{\log x}{3 \log\log x}, \tag{5.2}$$

we have uniformly

$$\#\{m \le x : \omega(m) = k\} \ge \frac{x}{k! \log x} \exp\left(k\left(\log L + \frac{\log L}{L} + O\left(\frac{1}{L}\right)\right)\right),$$

where $L = \log\log x - \log k - \log\log k$.

We will choose k to be of size $(\log x)^{o(1)}$ and to fall in the range of validity (5.2) of Proposition 5.2. Then, in the notation of that proposition, $L = (1 + o(1)) \log\log x$ and $\log L = \log\log\log x + o(1)$. Inserting the estimate for B_k from Proposition 5.1, and recalling that $\log k! = k \log k - k + O(\log k)$, we deduce from (5.1) that

$$\sum_{m \le x} f(m)^\beta \ge \exp(k \log\log\log x + (\beta - 1)k \log k - \beta k \log\log k + (1 - \beta)k + o(k)). \tag{5.3}$$

For all large x, there is a unique real number $k_0 = k_0(x)$ satisfying

$$\log\log\log x = (1 - \beta) \log k_0 + \beta \log\log k_0.$$

Note that k_0 tends to infinity with x. Dividing by $1 - \beta$ and exponentiating,

$$(\log\log x)^{1/(1-\beta)} = k_0 (\log k_0)^{\beta/(1-\beta)}.$$

Taking the logarithm of this last displayed equation, we deduce that

$$\log k_0 \sim \frac{1}{1-\beta} \log\log\log x.$$

Substituting this back reveals that, as $x \to \infty$,

$$\begin{aligned} k_0 &\sim (1-\beta)^{\beta/(1-\beta)} \frac{(\log\log x)^{1/(1-\beta)}}{(\log\log\log x)^{\beta/(1-\beta)}} \\ &= (1-\beta)^{\beta/(1-\beta)} \log\log x \left(\frac{\log\log x}{\log\log\log x}\right)^{\beta/(1-\beta)}. \end{aligned}$$

We let k be the integer nearest to k_0, noting that this choice of k has size $(\log x)^{o(1)}$ and falls in the range (5.2). Then

$$k \log\log\log x + (\beta - 1)k \log k - \beta k \log\log k = o(k).$$

(To see this, note that the left-hand side vanishes with k_0 in place of k, and apply the mean value theorem.) Thus, from (5.3),

$$\sum_{m\le x} f(m)^\beta \ge \exp((1-\beta+o(1))k)$$
$$= \exp\left((1+o(1))(1-\beta)^{1/(1-\beta)}\log\log x\left(\frac{\log\log x}{\log\log\log x}\right)^{\beta/(1-\beta)}\right),$$

which is the lower bound of Theorem 1.3.

The upper bound argument employs the following observation of Oppenheim (see eq. (5.2) in [Opp26]).

Proposition 5.3. *There is an absolute positive constant C such that for every integer $m \le x$ (where $x \ge 16 > e^e$),*

$$f(m) \le \sum_{k\le \frac{\log x}{\log 2}} (C\log\log\log x)^k \cdot \frac{\tau_k(m)}{k!}.$$

We will also use the following estimate of Norton [Nor92, Theorem 1.11].

Proposition 5.4. *Let $x \ge 3$, let $z \ge 1$, and let $0 < \beta \le 1$. Then*

$$\sum_{m\le x} \tau_z(m)^\beta \le x\exp((z^\beta-1)\log\log x + z\log\log(3z) + O(z)).$$

The implied constant is uniform is all parameters.

(Note that Norton excludes $z = 1$ in his statement, but the proposition is trivially valid when $z = 1$.)

By Proposition 5.3, for all $x \ge 16$,

$$\sum_{m\le x} f(m)^\beta \le \sum_{m\le x}\left(\sum_{k\le\frac{\log x}{\log 2}} (C\log\log\log x)^k\cdot\frac{\tau_k(m)}{k!}\right)^\beta$$
$$\le \sum_{k\le\frac{\log x}{\log 2}} (C\log\log\log x)^{k\beta} k!^{-\beta}\sum_{m\le x}\tau_k(m)^\beta$$
$$\ll \log x\cdot\max_{k\le\frac{\log x}{\log 2}}\left((C\log\log\log x)^{k\beta}k!^{-\beta}\sum_{m\le x}\tau_k(m)^\beta\right). \tag{5.4}$$

By Proposition 5.4, for a certain absolute constant D and every positive integer k,

$$(C\log\log\log x)^{k\beta}k!^{-\beta}\sum_{m\le x}\tau_k(m)^\beta \tag{5.5}$$
$$\le x\exp((k^\beta-1)\log\log x + k\log\log(3k) - \beta k\log k + k\beta\log\log\log\log x + Dk).$$

View the expression inside the exponential as a function of a real variable $k \ge 1$. For large x, this function of k assumes a maximum on $[1,\infty)$, at a place where the derivative vanishes. Let $k = k(x)$ be a spot where the maximum is attained. Computing the derivative explicitly, one sees quickly from its vanishing at k that $k \to \infty$ as $x \to \infty$, and in fact that $k > \log\log x$ for large x. Considering again at what it means for the derivative to vanish, keeping in mind the lower bound $k > \log\log x$, we find that

$$\beta k^{\beta-1}\log\log x \sim \beta \log k,$$

as $x \to \infty$. Hence, $(1-\beta)\log\log x \sim k^{1-\beta}\log(k^{1-\beta})$, so that

$$\begin{aligned} k^{1-\beta} &\sim (1-\beta)\log\log x/\log((1-\beta)\log\log x) \\ &\sim (1-\beta)\log\log x/\log\log\log x, \end{aligned}$$

and

$$k \sim (1-\beta)^{1/(1-\beta)}\left(\frac{\log\log x}{\log\log\log x}\right)^{1/(1-\beta)}.$$

Putting this estimate back into (5.5), we find after some computation that the maximum in (5.4) is at most

$$x\exp\left((1+o(1))(1-\beta)^{1/(1-\beta)}\log\log x\left(\frac{\log\log x}{\log\log\log x}\right)^{\beta/(1-\beta)}\right).$$

The factor of $O(\log x)$ outside the maximum in (5.4) fits inside the error term already present, completing the proof of the upper bound half of Theorem 1.3.

Acknowledgments

The author is supported by NSF award DMS-1402268. He thanks Carl Pomerance for helpful comments.

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