Two thousand years of summing divisors

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Messing with perfection

Let $s(n) := \sum_{d|n, d<n} d$ denote the sum of the proper divisors of n.
So if $\sigma(n) = \sum_{d|n} d$ is the usual sum-of-divisors function, then

$$s(n) = \sigma(n) - n.$$

For example,

$$s(4) = 1 + 2 = 3, \qquad \sigma(4) = 1 + 2 + 4 = 7.$$

The ancient Greeks said that n was ...
deficient if $s(n) < n$, for instance $n = 5$;
abundant if $s(n) > n$, for instance $n = 12$;
perfect if $s(n) = n$, for example $n = 6$.

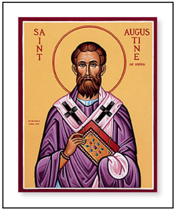
Nicomachus (60-120 AD) and the Goldilox theory

The superabundant number is . . . as if an adult animal was formed from too many parts or members, having "ten tongues", as the poet says, and ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands. . . . The deficient number is . . . as if an animal lacked members or natural parts . . . if he does not have a tongue or something like that.

. . . In the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect.

Iamblichus (245-325) and St. Augustine (354-430) on perfect numbers

The number Six .... which is said to be perfect ... was called Marriage by the Pythagoreans, because it is produced from the intermixing of the first meeting of male and female; and for the same reason this number is called Holy and represents Beauty, because of the richness of its proportions.



Six is a number perfect in itself, and not because God created all things in six days; rather, the converse is true. God created all things in six days because the number is perfect.

A deep thought

We tend to scoff at the beliefs of the ancients.

A deep thought

We tend to scoff at the beliefs of the ancients.
But we can't scoff at them personally, to their faces, and this is what annoys me.

– Jack Handey

From numerology to number theory

Perfect numbers are solutions to the equation $\sigma(N) = 2N$. What do these solutions look like?

Theorem (Euclid)

If $2^n - 1$ is a prime number, then $N := 2^{n-1}(2^n - 1)$ is a perfect number.

For example, $2^2 - 1$ is prime, so $N = 2 \cdot (2^2 - 1) = 6$ is perfect.
A slightly larger example ($\approx$ 35 million digits) corresponds to $n = 57885161$.

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Problem

Are there any **odd** *perfect numbers?*

Anatomy of an odd perfect integer

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If N is an odd perfect number, then:

1. N has the form $p^e M^2$, where $p \equiv e \equiv 1 \pmod 4$ (Euler),
2. N has at least 10 distinct prime factors (Nielsen, 2014) and at least 101 prime factors counted with multiplicity (Ochem and Rao, 2012),
3. $N > 10^{1500}$ (Ochem and Rao, 2012).

Conjecture

There are no odd perfect numbers.

Counting perfects

Let $V'(x)$ denote the number of odd perfect numbers $n \leq x$.

Theorem (Hornfeck)

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Proof.

Each odd perfect N has the form $p^e M^2$. If $N \le x$, then $M \le \sqrt{x}$.
We will show that each M corresponds to at most one N.
In fact, since $\sigma(p^e)\sigma(M^2) = \sigma(N) = 2N = 2p^e M^2$, we get

$$\frac{\sigma(p^e)}{p^e} = \frac{2M^2}{\sigma(M^2)}.$$

The right-hand fraction depends only on M.
The left-hand side is already a reduced fraction, since $p \nmid 1 + p + \cdots + p^e = \sigma(p^e)$. Thus, p^e depends only on M.

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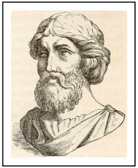
Best result: $V_1(x) \leq x^{c/\log\log x}$ (Wirsing, 1959).

Iterate, then iterate again

Consider the map $s\colon \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, extended to have $s(0) = 0$. A perfect number is nothing other than a positive integer fixed point.

We say n is **amicable** if n generates a two-cycle: in other words, $s(n) \neq n$ and $s(s(n)) = n$. For example,

$$s(220) = 284, \quad \text{and} \quad s(284) = 220.$$



Pythagoras, when asked what a friend was, replied:
One who is the other I, such are 220 and 284.

The distribution of amicable numbers

There are over a billion amicable pairs known, but we have no proof that there are infinitely many.
But we can still guess!

Let $V_2(x)$ denote the number of $n \leq x$ that belong to some amicable pair.

Conjecture (Bratley–Lunnon–McKay, 1970)

$V_2(x) = o(\sqrt{x})$.

They based this on a complete list of amicable pairs to 10^7.

In contrast with B-L-McK, Erdős suggests that for each $\epsilon > 0$ and each positive integer K, one has

$$x^{1-\epsilon} < V_2(x) < x/(\log x)^K.$$

Erdős's conjectured upper bound has been proved by Pomerance. In fact, Pomerance has shown that

$$V_2(x) < x/\exp(\sqrt{\log x})$$

for all large enough x.

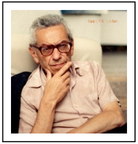
Sociable numbers

More generally, we call n a k-**sociable number** if n starts a cycle of length k. (So perfect corresponds to $k = 1$, amicable to $k = 2$.) For example,

$$2115324 \mapsto 3317740 \mapsto 3649556 \mapsto 2797612 \mapsto 2115324 \mapsto \dots$$

is a sociable 4-cycle.

Let $V_k(x)$ denote the number of k-sociable numbers $n \le x$.



Theorem (Erdős, 1976)

Fix k. The set of k-sociable numbers has asymptotic density zero. In other words, $V_k(x)/x \to 0$ as $x \to \infty$.

Counting sociables

What if we count all sociable numbers at once? Put

$$V(x) := V_1(x) + V_2(x) + V_3(x) + \dots$$

Is it still true that most numbers are not sociable numbers?

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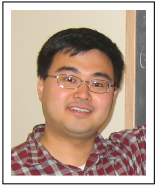
Theorem (K.–P.–P., 2009)

$$\limsup V(x)/x \leq 0.0021.$$



Theorem (K.–P.–P., 2009)

The number of $n \leq x$ which belong to a cycle entirely contained in $[1, x]$ is $o(x)$, as $x \to \infty$.

Here 0.0021 is standing in for the density of **odd abundant numbers**, odd numbers n for which $s(n) > n$ (e.g., $n = 945$).


THANKS
GRAZIAS
TANGKIU
MAULIATE
SHUKRAN
DANKE
MAKASIH
KASIH
TERIMA
Merci
TACK
KAMSAHAMNIDA
KOMAPSUMNIDA
OBRIGADU
NANDRI
ARIGATŌ
XIÈXIE
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thanks