MATH 3200 – Homework #5

posted October 23, 2024; due at the **start of class** on October 31, 2024

Assignments are expected to be **neat** and **stapled**. **Illegible work may not be marked.** Assume — unless explicitly told otherwise — that **you are expected to include clear and concise explanations of your reasoning, expressed in complete sentences.**

1. Recall that $\emptyset$ denotes the empty set. How many elements are there in the set $\mathbf{Z} \times \emptyset$? Explain your answer.

2. Which of the following are `TRUE` or `FALSE`? No justification necessary.

   (a) $\{\text{red}\} \subseteq \{\text{red}, \text{green}, \text{blue}\}$,

   (b) $\{\text{red}\} \subseteq \{\{\text{red}, \text{green}\}, \text{blue}\}$,

   (c) $\{\text{red}\} \in \mathcal{P}(\{\text{red}, \text{green}, \text{blue}\})$,

   (d) $\{\text{red}\} \subseteq \mathcal{P}(\{\text{red}, \text{green}, \text{blue}\})$,

   (e) $\emptyset \in \{\text{red}, \text{green}, \text{blue}\}$,

3. Write each of the following sets in the form $\{\cdots\}$, where $\cdots$ is a complete list of elements. No justification necessary.

   (a) the powerset of $\emptyset$,

   (b) the powerset of $\{\emptyset\}$,

   (c) $\mathcal{P}(\mathcal{P}(\{\emptyset\}))$.

   (d) $\mathcal{P}(\{a, b\} \times \{0\})$.

4. (a) Let A and B be any two sets. Show that if $U \subseteq A$ and $V \subseteq B$, then $U \times V$ is a subset of $A \times B$.

   (b) Now let $A = \{1, 2\}$ and $B = \{\text{red}, \text{blue}, \text{green}\}$. Write down a subset of $A \times B$ that **cannot** be expressed as $U \times V$ with U a subset of A and V a subset of B. Explain fully (that is, prove!) that your subset cannot be written in that way.

5. Suppose $A \neq \emptyset$. Prove that $A \times B \subseteq A \times C$ if and only if $B \subseteq C$. Remember that an "if and only if" claim requires a proof for both directions. That is, you must to prove that if $A \times B \subseteq A \times C$, then $B \subseteq C$, *and* vice versa.

6. Prove that for any two sets A and B,

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B).$$

7. We showed in class that $\mathcal{P}(A) = \mathcal{P}(B)$ if and only if $A = B$. Prove or disprove: For every pair of sets A and B, we have $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

8. (a) Prove that if $A \subseteq \mathcal{P}(A)$, then $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$.

   (b) Give an example of a nonempty set A for which $A \subseteq \mathcal{P}(A)$.

   (c) Is there an example of a set A with $|A| > 2024$ and $A \subseteq \mathcal{P}(A)$? Justify your answer.

9. Recall from class that $\mathsf{MULT}_n = \{m \in \mathbf{Z} : n \mid m\}$. That is, MULT_n is the collection of all integer multiples of n. Prove that

$$\bigcup_{n \in \mathbf{N}} \mathsf{MULT}_n = \mathbf{Z}$$

and that

$$\bigcap_{n \in \mathbf{N}} \mathsf{MULT}_n = \{0\}.$$

Suggestion. You may assume the following fact. If a and b are integers with $b \neq 0$, and $a \mid b$, then $|a| \leq |b|$. Here $|\cdot|$ denotes absolute value.

10. (Extra challenge problem; not to turn in!) Let a, b, c, d be elements of some universal set U. Prove that if $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$, then $a = c$ and $b = d$. In courses on set theory, the ordered pair (a, b) is sometimes **defined** as the set $\{\{a\}, \{a, b\}\}$.

Hint: Here is a way to get started. First consider the case when $a = b$. Argue that $\{\{a\}, \{a, b\}\}$ then has a single element, namely $\{\{a\}\}$. Deduce from $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ that $c = d$, and then argue that $a = c$. Conclude that $a = b = c = d$ in this case, which confirms the claim in the problem (in this situation). Next, try the case when $a \neq b$ .... If you get stuck, feel free to discuss this at office hours!