Math 4000/6000 – Homework #4

posted October 1, 2018; due at the **start of class** on October 9, 2018

The essence of mathematics lies in its freedom. – Georg Cantor

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

1. (de Moivre's theorem)
   (a) In class, we noted that our rule for multiplying complex numbers implies that if we write z in polar form, say $z = r(\cos\theta + i\sin\theta)$, then
   $$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$
   for every positive integer n. Show that the same formula holds when $n = 0$ and when n is a negative integer.
   (b) Find formulas for $\cos(4\theta)$ and $\sin(4\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$. The binomial theorem may be helpful.
2. Exercise 2.3.13.
3. Given a polynomial $f(z) = z^3 + pz + q$ (with p, q complex numbers), we set $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$. As shown in class, as long as $p \neq 0$, the complex roots of f are the numbers
   $$v - \frac{p}{3v}, \quad \text{where } v \text{ runs over the cube roots of} \quad A := -\frac{q}{2} + \sqrt{\Delta}.$$
   Here $\sqrt{\Delta}$ denotes any fixed square root of Δ.
   (a) Show that $A \neq 0$. (Remember we are assuming $p \neq 0$.)
   (b) It follows from (a) that A has three distinct (and nonzero) cube roots v. Show that for each of these roots v, the number $-\frac{p}{3v}$ is a cube root of $-\frac{q}{2} - \sqrt{\Delta}$.
   (This explains why our derivation for the roots of a cubic equation yields three roots and not six!)
4. Exercise 2.4.6(a,b).
5. 3.1.2(a), and then
   $f(x) = x^2 + 2x + 2$, $g(x) = x^2 + 1$, $F = \mathbb{Z}_3$
6. 3.1.6.
7. 3.1.8.
8. 3.1.10(a,c,e).
9. Let R be a ring. A subset $R' \subseteq R$ is called a *subring* of R if (A) R' is a ring for the same operations $+$ and $\cdot$ as in R, **and** (B) R' contains the multiplicative identity 1_R of R.

   (For example, making the identifications discussed in class, $\mathbb{Z}$ is a subring of $\mathbb{Q}$ and $\mathbb{Q}$ is a subring of $\mathbb{R}$.)

   (a) Let R be a ring. Suppose that R' is a subset of R closed under $+$ and $\cdot$, that R' contains the additive inverse of each of its elements, and that R' contains 1_R. Show that R' is a subring of R.
   Hint: (B) holds by assumption. Check that all the ring axioms hold for R' in order to verify (A). To get started, show that the additive identity of R — call this 0_R — must belong to R'.

(b) Find a two-element subset R' of $R = \mathbb{Z}_6$ that satisfies condition (A) in the definition of a subring but not (B). (You do **not** have to give a detailed proof that (A) holds.)

10. (The Gaussian integers) Let $\mathbb{Z}[i]$ be the subset of complex numbers defined by $\mathbb{Z}[i] := \{a+bi : a, b \in \mathbb{Z}\}$.

    (a) Check that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$.

    (b) Define a function $N\colon \mathbb{Z}[i] \to \mathbb{R}$ by $N(z) = z \cdot \bar{z}$. Explain why $N(z)$ is a nonnegative integer for every $z \in \mathbb{Z}[i]$. For which $z \in \mathbb{Z}[i]$ is $N(z) = 0$?

    (c) Prove that $N(zw) = N(z)N(w)$ for all $z, w \in \mathbb{Z}[i]$.

    (d) Using your work in (b) and (c), find (with proof) all units in $\mathbb{Z}[i]$.
    Hint: First show that $z \in \mathbb{Z}[i]$ is a unit if and only if $N(z) = 1$.

11. (*) Exercise 2.2.16.

12. (*) Suppose distinct complex numbers z_1, z_2, z_3 satisfy $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_1 z_3$. Show that z_1, z_2, z_3 are the vertices of an equilateral triangle.

    Hint: The constraint equation can also be written as $(z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2 = 0$.