MATH 3200 – Homework #3

posted February 12, 2020; due at the **start of class** on February 19, 2020

All numbering corresponds to the course textbook, A TeXas-Style Introduction to Proof. Assignments are expected to be **neat** and **stapled**. **Illegible work may not be marked.** Assume — unless explicitly told otherwise — that **you are expected to include clear and concise explanations of your reasoning, expressed in complete sentences.**

When doing the induction proofs below, make sure to identify the statements $P(n)$ being proven! See the "proof" in Exercise 9 below for a sample of how to work this in to your write-ups.

1. Statement 3.29.

2. Statement 3.31.

3. Statement 3.39.

   [As suggested in the paragraph preceding Statement 3.39, your numbering for this problem should start with 0 instead of 1: that is, your statements are now numbered $P(0), P(1), \ldots$. Your base case is now $P(0)$. And in the induction step, you should prove that $P(n) \Rightarrow P(n+1)$ for all integers n at least 0 (instead of all natural numbers n).]

4. Prove that $n! > 3^n$ for all natural numbers $n \geq 7$.

   [This time, start your statement numbering at $n = 7$.]

5. Define real numbers α and β by $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

   (a) Check that α and β are roots of the polynomial $x^2 - x - 1$.

   (b) Using (a), deduce that $\alpha^{n+1} = \alpha^n + \alpha^{n-1}$ and $\beta^{n+1} = \beta^n + \beta^{n-1}$, for every integer n. (You don't need induction for this step, just algebra.)

   (c) Recall that the Fibonacci sequence $\{F_n\}$ is defined by $F_1 = 1$, $F_2 = 1$, and the recurrence $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.
   Use complete induction to prove that $\frac{\alpha^n - \beta^n}{\sqrt{5}} = F_n$ for all natural numbers n.

   [*Hint:* The result of (b) will be useful.]

6. Consider the following statement.

   For every natural number n, the number $n^2 + n + 41$ is prime.

   Either give a counterexample, or prove the statement by induction.

7. Statement 3.45.

8. Statement 3.47.

   [*Hint*: In the induction step, when you assume $P(1), \ldots P(n)$ and try to prove $P(n+1)$, it may help to take two cases: $n+1$ is even, or $n+1$ is odd.]

9. The following argument is an *alleged* proof by induction that any finite group of people all have the same height:

   For every natural number n, let $P(n)$ be the statement "every group of n people share the same height".
   Base case: $P(1)$ is true, since if there is just one person, they all have the same height!

Induction step: We now suppose that $P(n)$ is true and we prove $P(n+1)$. Consider any group of $n+1$ people, say $A_1, \ldots, A_{n+1}$. Since $P(n)$ holds, it must be that $A_1, \ldots, A_n$ all share the same height, and similarly for $A_2, \ldots, A_{n+1}$. But these two groups overlap; for instance, the second person A_2 is in both. So all of our $n+1$ people have the same height (indeed, everyone is the same height as A_2). Thus, $P(n+1)$ holds.

By induction, $P(n)$ is true for all natural numbers n.

Exactly where is the mistake in this proof?