NOTES ON NEAR-PERFECT NUMBERS

Theorem 1. *The number of near-perfect* $n \le x$ *is at most* $x^{5/6+o(1)}$, *as* $x \to \infty$.

The proof of Theorem 1 requires some preparation. We begin by recalling Gronwall's determination of the maximal order of the sum-of-divisors function [1, Theorem 323].

Lemma 1 (Gronwall). *As* $n \to \infty$, *we have* $\limsup \frac{\sigma(n)}{n \log\log n} = e^{\gamma}$, *where* $\gamma = 0.57721\dots$ *is the Euler–Mascheroni constant.*

The next proposition, which asserts that $\gcd(n, \sigma(n))$ is small on average, is extracted from [2, Theorem 1.3].

Proposition 1. *For each* $x \ge 3$, *we have*

$$\sum_{n \le x} \gcd(n, \sigma(n)) \le x^{1+C/\sqrt{\log\log x}},$$

where C *is an absolute positive constant.*

The next lemma concerns solutions to the congruence $\sigma(n) \equiv a \pmod{n}$. For a given a, we divide the solutions n to this congruence into two classes: by a *trivial solution*, we mean a natural number

$$n = pm, \quad \text{where } p \text{ is a prime not dividing } m, \quad m \mid \sigma(m), \quad \text{and} \quad \sigma(m) = a. \tag{1}$$

(It is straightforward to check that all such n satisfy $\sigma(n) \equiv a \pmod{n}$.) All other solutions are called *sporadic*. Pomerance [5, Theorem 3] showed that for each fixed a, the number of sporadic solutions to $\sigma(n) \equiv a \pmod{n}$ with $n \le x$ is at most

$$x/\exp((1/\sqrt{2} + o(1))\sqrt{\log x \log\log x}), \tag{2}$$

as $x \to \infty$. Theorem 1 requires a stronger bound, with attention paid to uniformity in a.

Proposition 2. *Let* $x \ge 3$, *and let* a *be an integer with* $|a| < x^{2/3}$. *Then the number of sporadic solutions* $n \le x$ *to the congruence* $\sigma(n) \equiv a \pmod{n}$ *at most* $x^{2/3+o(1)}$. *Here the* $o(1)$ *term decays to* 0 *as* $x \to \infty$, *uniformly in* a.

Remark. *In addition to the congruence* $\sigma(n) \equiv a \pmod{n}$, *Pomerance* [5] *also treats the congruence* $n \equiv a \pmod{\phi(n)}$, *proving the same upper bound* (2) *for the number of non-trivial solutions* $n \le x$. *He returned to this latter congruence in the papers* [3], [4], *which sharpen the upper bound to* $x^{2/3+o(1)}$ *and* $x^{1/2+o(1)}$ *(again, for each fixed* a*). Our proof of Proposition 2 relies on the method of* [3]. *It would be interesting to improve the exponent to* $2/3$ *to* $1/2$, *as in* [4], *but this seems somewhat more difficult than might be expected.*

Proof. We may assume that the squarefull part of n is bounded by $x^{2/3}$, since the number of $n \le x$ for which this condition fails is

$$\ll x \sum_{\substack{m > x^{2/3} \\ \text{squarefull}}} \frac{1}{m} \ll x^{2/3}.$$

(We use here that the counting funciton of the squarefull numbers is $\ll x^{1/2}$.) We also assume, as is clearly permissible, that $n > x^{2/3}$.

Consider first the case when the largest prime factor p of n satisfies $p > x^{1/3}$. Say that $n = mp$, so that $m \leq x^{2/3}$. By our condition on the squarefull part of n, we see that $p \nmid m$. Write $\sigma(n) = nq + a$, where q is a nonnegative integer; from Lemma 1, $q \ll \log\log x$. Observe that

$$\sigma(m)(p+1) = \sigma(mp) = qmp + a,$$

so that

$$p(\sigma(m) - qm) = a - \sigma(m). \tag{3}$$

If $\sigma(m) - qm = 0$, then (3) implies that $a = \sigma(m)$; referring back to the definitions we see that n is a trivial solution to the congruence $\sigma(n) \equiv a \pmod n$, contrary to hypothesis. Thus, $\sigma(m) - qm \neq 0$, and now (3) shows that p is uniquely determined given m and q. Since the number of possibilities for m is at most $x^{2/3}$, while $q \ll \log\log x$, the number of n that arise in this manner is $\ll x^{2/3} \log\log x$, which is acceptable for us.

Now suppose that the largest prime factor of n does not exceed $x^{1/3}$. We claim that n has a unitary divisor m from the interval $(x^{1/3}, x^{2/3}]$. The claim obviously holds if every prime power divisor of n is bounded by $x^{1/3}$. Otherwise, $p^e \parallel n$ for some prime power $p^e > x^{1/3}$ (with $e > 1$). In this case, $p^e \leq x^{2/3}$ by our restriction on the squarefull part of n, and so we can take $m = p^e$.

Since m is a unitary divisor of n, it follows that

$$\sigma(n) \equiv 0 \pmod{\sigma(m)} \quad \text{and} \quad \sigma(n) \equiv a \pmod m.$$

This places $\sigma(n)$ is a uniquely-defined residue class modulo $[m, \sigma(m)]$. Thus, summing over $m \in (x^{1/3}, x^{2/3}]$, we have that the number of values $\sigma(n)$ that can arise this way is at most

$$\begin{aligned} \sum_{x^{1/3} < m \leq x^{2/3}} \left(\frac{x}{\operatorname{lcm}[m, \sigma(m)]} + 1 \right) &\leq x^{2/3} + x \sum_{x^{1/3} < m \leq x^{2/3}} \frac{\gcd(m, \sigma(m))}{m\sigma(m)} \\ &\leq x^{2/3} + x \sum_{x^{1/3} < m \leq x^{2/3}} \frac{\gcd(m, \sigma(m))}{m^2}. \end{aligned} \tag{4}$$

Letting $A(t) = \sum_{m \leq t} \gcd(m, \sigma(m))$, the final sum in (4) is given by

$$\begin{aligned} \int_{x^{1/3}}^{x^{2/3}} \frac{1}{t^2}\, dA(t) &\leq A(x^{2/3}) x^{-4/3} + 2 \int_{x^{1/3}}^{x^{2/3}} A(t) t^{-3}\, dt \\ &\leq x^{-2/3+o(1)} + x^{-1/3+o(1)} = x^{-1/3+o(1)}, \end{aligned}$$

where we use the estimate of Proposition 1 for $A(t)$. Referring back to (4), we see that the number of values $\sigma(n)$ that can arise is at most $x^{2/3+o(1)}$. Since $\sigma(n) = qn + a$, the values $\sigma(n)$ and q uniquely determine n. Since the number of possible values of q is $\ll \log\log x = x^{o(1)}$ (as above), and there are only $x^{2/3+o(1)}$ possible values of $\sigma(n)$, there are also only $x^{2/3+o(1)}$ possible values of n. $\square$

Proof of Theorem 1. We can assume that $n > x^{5/6}$. Write $\sigma(n) = 2n + d$, where d is a proper divisor of n. If $d > x^{1/6}$, then $\gcd(n, \sigma(n)) = d > x^{1/6}$. By Proposition 1, the number of such $n \leq x$ is at most $x^{5/6+o(1)}$.

So suppose that $d \leq x^{1/6}$. In this case, we observe that $\sigma(n) \equiv d \pmod{n}$ and apply Proposition 2. Let us check that our near-perfect number n is not a trivial solution to this congruence. If it were, then we could write n in the form (1), with 'd' in place of 'a'. Then

$$(p+1)d = (p+1)\sigma(m) = \sigma(mp) = 2mp + d,$$

so that $d = 2m$. But then d and pm have the same number of prime factors (counted with multiplicity), contradicting that d is a proper divisor of n. So n is a sporadic solution, and thus the number of possibilities for n, given d, is at most $x^{2/3+o(1)}$. Summing over values of $d \leq x^{1/6}$, we see the number of n that arise in this way is at most $x^{5/6+o(1)}$. $\square$

References

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