MATH 8440 – Assignment #2

last updated January 30, 2023 (CLOSED)

Turn in <u>three</u> problems.

1. Let $\{a_k\}_{k\geq 2}$ be a sequence of positive real numbers satisfying $\frac{1}{2k} \leq a_k \leq \frac{1}{2(k-1)}$ for all $k \geq 2$. Show that there is a constant C such that

$$\sum_{2\leq n\leq N} a_n = \frac{1}{2}\log N + C + O(1/N), \quad \text{as } N \to \infty.$$

2. Let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers. Let $x \geq 1$, and let f be a C^1 function on $[1, x]$. Show that

$$\sum_{n\leq x} f(n) = S(x)f(x) - \int_1^x S(t)f'(t)\,\mathrm{d}t,$$

where $S(t) := \sum_{n\leq t} a_n$.

We proved this in class when $x \in \mathbf{Z}^+$.

3. (a) Prove that for every nonnegative integer n,

$$\int_0^{\pi/2} \sin^{2n} x\,\mathrm{d}x = \frac{\pi}{2} \cdot \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)},$$
$$\int_0^{\pi/2} \sin^{2n+1} x\,\mathrm{d}x = \frac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}.$$

(b) Show that $\int_0^{\pi/2} \sin^{2n} x\,\mathrm{d}x / \int_0^{\pi/2} \sin^{2n+1} x\,\mathrm{d}x \to 1$ as $n \to \infty$. Deduce that $\frac{\pi}{2} = \prod_{k=1}^{\infty}(\frac{2k}{2k-1} \cdot \frac{2k}{2k+1})$ (Wallis's product formula).

4. In class, we showed that $N! \sim C\sqrt{N}(N/e)^N$ (as $N \to \infty$), for a certain positive constant C. Use Wallis's product formula to show that $C = \sqrt{2\pi}$.

The estimate $N! \sim \sqrt{2\pi N}(N/e)^N$ is known as Stirling's formula.

5. Suppose $f(x), g(x)$ are real-valued functions defined for $x \geq 2$ and that $g(x) > 0$ for all $x \geq 2$. Suppose also that the Riemann integrals $\int_a^b f(t)\,\mathrm{d}t, \int_a^b g(t)\,\mathrm{d}t$ exist whenever $2 \leq a < b$.

Show that if $f(x) = o(g(x))$ as $x \to \infty$, and $\int_2^x g(t)\,\mathrm{d}t \to \infty$ as $x \to \infty$, then $\int_2^x f(t)\,\mathrm{d}t = o(\int_2^x g(t)\,\mathrm{d}t)$ as $x \to \infty$.

6. The following is <u>incorrect</u>: Let $c(t) = \int_0^t \{t\}\,\mathrm{d}t$ for $0 \leq t < 1$ and extend $c(t)$ to all of $\mathbf{R}$ by 1-periodicity. Then $c'(t) = \{t\}$ for $t \notin \mathbf{Z}$. Thus, for each positive integer N,

$$\int_N^{\infty} \frac{\{t\}}{t^2}\,\mathrm{d}t = \int_N^{\infty} \frac{c'(t)}{t^2}\,\mathrm{d}t = \frac{c(\infty)}{\infty} - \frac{c(N)}{N} + 2\int_N^{\infty} \frac{c(t)}{t^3}\,\mathrm{d}t$$
$$= 2\int_N^{\infty} \frac{c(t)}{t^3}\,\mathrm{d}t.$$

This looks suspiciously similar to what we worked out in class. There we defined $b_2(t) = \int_0^t (\{t\} - 1/2)\,\mathrm{d}t$ for $0 \le t < 1$, extended $b_2(t)$ by 1-periodicity, and claimed (justified by a calculation completely analogous to the above) that

$$\int_N^\infty \frac{\{t\} - 1/2}{t^2} = 2\int_N^\infty \frac{b_2(t)}{t^3}.$$

Why is the first argument incorrect but the second OK (after filling in a detail)?

7. Let $\mathcal{A}$ be the set of positive integers with leading (leftmost) digit 1. Show that

$$\limsup_{N\to\infty} \frac{1}{N}\#\{n \le N : n \in \mathcal{A}\} \ge 5/9.$$

but that

$$\liminf_{N\to\infty} \frac{1}{N}\#\{n \le N : n \in \mathcal{A}\} \le 1/9.$$

8. (a) Let $P_n(x)$ be the nth Taylor polynomial for e^x about $x = 0$. For example, $P_1(x) = 1 + x$. Use Taylor's theorem to show that if n is an odd positive integer, then $P_n(x) \le e^x$ for all real numbers x.

   (b) Let $\{a_n\}_{n\ge1}$ be a sequence of real numbers with $0 \le a_n < 1$ for all n. Show that if $\sum_{n=1}^\infty a_n$ diverges, then $\prod_{n=1}^\infty (1 - a_n) = 0$.

   Suggestion. Use (a) with $n = 1$.

   (c) Let $\{a_n\}_{n\ge1}$ be a sequence of real numbers with $0 \le a_n < 1$ for all n, but now suppose $\sum_{n=1}^\infty a_n$ converges. Show that if the positive integer M is chosen with $\sum_{n=M+1}^\infty a_n < 1/2$, then

$$\prod_{n=1}^\infty (1 - a_n) \ge \frac{1}{2}\prod_{n=1}^M (1 - a_n) > 0.$$