MATH 8440 – Assignment #4

last updated February 24, 2023 (open)

Turn in <u>three</u> problems.

1. Let k be an integer with $k > 1$. We say that the positive integer n is kth-powerfree if there is no prime p for which $p^k \mid n$. For example, $90 = 2 \cdot 3^2 \cdot 5$ is not squarefree ($k = 2$) but is cubefree ($k = 3$).

   For the rest of this problem, $k > 1$ is fixed.

   (a) Show that $1_{n \text{ is } k\text{th-powerfree}} = \sum_{d^k \mid n} \mu(d)$. Here the right-hand sum is extended over all positive integers d for which $d^k \mid n$.

   (b) Let $x \geq 1$. Show that
   $$\#\{k\text{th-powerfree } n \leq x\} = x \sum_{d \leq x^{1/k}} \mu(d)/d^k + O(x^{1/k}).$$

   (c) Show that $\sum_{d \geq 1} \mu(d)/d^k$ converges to a constant M_k (say) and that, for all $x \geq 1$,
   $$\#\{k\text{th-powerfree } n \leq x\} = M_k x + O(x^{1/k}).$$
   In particular, the set of kth-powerfree numbers has asymptotic density M_k.

   (d) Prove that $M_k = 1/\zeta(k)$.

2. Fix integers a, q with $q > 0$. Let $\mathcal{P}$ be a set of primes none of which divide q. Show that $\mathcal{S}(\{n \in \mathbf{Z}^+ : n \equiv a \pmod{q}\}, \mathcal{P})$ has density $\frac{1}{q} \prod_{p \in \mathcal{P}} (1 - 1/p)$. [1]

3. Let q be a positive integer. Suppose that H is a proper subgroup of $(\mathbf{Z}/q\mathbf{Z})^\times$. Prove that $\sum_{p \bmod q \notin H} \frac{1}{p}$ diverges. In the sum, p ranges over all primes whose reductions mod q do not belong to H.

4. If p is a prime and e is a nonnegative integer, we write "$p^e \parallel n$" if $p^e \mid n$ and $p^{e+1} \nmid n$. That is, p^e is the power of p that appears in the canonical factorization of n as a product of prime powers.

   Give a careful proof of the following statement: Let $\mathcal{P}$ be a set of primes (possibly infinite). Then
   $$\{n \in \mathbf{Z}^+ : \text{there is no } p \in \mathcal{P} \text{ for which } p \parallel n\}$$
   has asymptotic density $\prod_{p \in \mathcal{P}} (1 - 1/p + 1/p^2)$.

   Remarks. The set in question cannot be realized as $\mathcal{S}(\mathcal{A}, \mathcal{P})$ for any $\mathcal{A}$ and $\mathcal{P}$, and so one cannot directly apply our sieve estimates. But a similar idea can be made to work. One plan of attack: Fix a large z (which will tend to infinity later). First estimate the number of $n \leq x$ for which there is no $p \in \mathcal{P} \cap [2, z]$ for which $p \parallel n$. In the case when $\sum_{p \in \mathcal{P}} 1/p < \infty$, bound the number of $n \leq x$ for which $p \parallel n$ for some $p \in \mathcal{P}$ with $p > z$.

5. Show that as $z \to \infty$,
   $$\frac{1}{2} \prod_{2 < p \leq z} (1 - 2/p) \sim 2C_2 e^{-2\gamma}/(\log z)^2.$$

[1] I flubbed this in class, but $\mathcal{S}(\mathcal{A}, \mathcal{P})$ should stand for the <u>collection</u> of elements of $\mathcal{A}$ not divisible by any prime $p \in \mathcal{P}$, while — in the case when $\mathcal{A}$ and $\mathcal{P}$ are finite — $S(\mathcal{A}, \mathcal{P})$ means the <u>size</u> of $\mathcal{S}(\mathcal{A}, \mathcal{P})$.

6. Let k be a positive integer. Show that for each nonnegative integer $0 \le m \le k$,

$$\sum_{0 \le j \le m} (-1)^j \binom{k}{j} = (-1)^m \binom{k-1}{m}.$$

7. Let $\sigma_k(X_1, \dots, X_n) \in \mathbf{Z}[X_1, \dots, X_n]$ be the kth elementary symmetric function in the independent indeterminates $X_1, \dots, X_n$. That is, $\sigma_k(X_1, \dots, X_n)$ is the sum of all $\binom{n}{k}$ products of k of the X_i. Or if you prefer: $\sigma_k(X_1, \dots, X_n)$ is the coefficient of Z^k in the formal expansion

$$\prod_{i=1}^{n}(1 - X_i Z) = \sum_{k=0}^{n}(-1)^k \sigma_k(X_1, \dots, X_n) Z^k.$$

Let $a_1, \dots, a_n$ be real numbers in $[0, 1]$, where n is a nonnegative integer. Show that for every nonnegative integer m, the difference

$$\sum_{k=0}^{m}(-1)^k \sigma_k(a_1, \dots, a_n) - \prod_{i=1}^{n}(1 - a_i)$$

is nonnegative or nonpositive according to whether m is even or odd, respectively. (If $n = 0$, interpret empty products as 1, so that $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > 0$.)

Hint. Try induction on n.

8. Assume the result of Exercise 7. Show that, with notation as in our usual sieve setup,

$$0 \le \sum_{\substack{d \mid P \\ \omega(d) \le m}} \mu(d)\alpha(d) - \prod_{p \in \mathcal{P}}(1 - \alpha(p)) \le \sum_{\substack{d \mid P \\ \omega(d) = m+1}} \alpha(d),$$

for every nonnegative even integer m.

This result will be needed in our discussion of the Brun–Hooley sieve.