

# QUASI-AMICABLE NUMBERS ARE RARE

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**ABSTRACT.** Define a *quasi-amicable pair* as a pair of distinct natural numbers each of which is the sum of the nontrivial divisors of the other, e.g.,  $\{48, 75\}$ . Here *nontrivial* excludes both 1 and the number itself. Quasi-amicable pairs have been studied (primarily empirically) by Garcia, Beck and Najjar, Lal and Forbes, and Hagis and Lord. We prove that the set of  $n$  belonging to a quasi-amicable pair has asymptotic density zero.

## 1. INTRODUCTION

Let  $s(n) := \sum_{d|n, d < n} d$  be the sum of the proper divisors of  $n$ . Given a natural number  $n$ , what can one say about the *aliquot sequence at  $n$*  defined as  $n, s(n), s(s(n)), \dots$ ? From ancient times, there has been considerable interest in the case when this sequence is purely periodic. (In this case,  $n$  is called a *sociable number*; see Kobayashi et al. [11] for some recent results on such numbers.) An  $n$  for which the period is 1 is called *perfect* (see sequence A000396), and an  $n$  for which the period is 2 is called *amicable* (see sequence A063990). In the latter case, we call  $\{n, s(n)\}$  an *amicable pair*.

Let  $s^-(n) := \sum_{d|n, 1 < d < n} d$  be the sum of the nontrivial divisors of the natural number  $n$ , where *nontrivial* excludes both 1 and  $n$ . According to Lal and Forbes [12], it was Chowla who suggested studying *quasi-aliquot sequences* of the form  $n, s^-(n), s^-(s^-(n)), \dots$ . Call  $n$  *quasi-amicable* if the quasi-aliquot sequence starting from  $n$  is purely periodic of period 2 (see sequence A005276). Thus, a *quasi-amicable pair* is a pair of distinct natural numbers  $n$  and  $m$  with  $s^-(n) = m$  and  $s^-(m) = n$  (e.g.,  $n = 48$  and  $m = 75$ ). The numerical data, reproduced in Table 1 from sequence A126160, suggests that the number of such pairs with a member  $\leq N$  tends to infinity with  $N$ , albeit very slowly.

While quasi-amicable pairs have been studied empirically (see [8, 12, 1, 10, 2], and cf. [14, 13], [9, section B5]), it appears that very little theoretical work has been done. In this paper, we prove the following modest theorem, which is a quasi-amicable analogue of Erdős's 1955 result [4] concerning amicable pairs:

**Theorem 1.1.** *The set of quasi-amicable numbers has asymptotic density zero. In fact, as  $\epsilon \downarrow 0$ , the upper density of the set of  $n$  satisfying*

$$1 - \epsilon < \frac{s^-(s^-(n))}{n} < 1 + \epsilon \quad (1.1)$$

*tends to zero.*

**Remark.** With  $s$  replacing  $s^-$ , Theorem 1.1 follows from work of Erdős [4] and Erdős et al. [7, Theorem 5.1].

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$N$	# of quasi-amicable pairs with least member $\leq N$
$10^5$	9
$10^6$	17
$10^7$	46
$10^8$	79
$10^9$	180
$10^{10}$	404
$10^{11}$	882
$10^{12}$	1946

TABLE 1

**Notation.** Throughout,  $p$  and  $q$  always denote prime numbers. We use  $\sigma(n) := \sum_{d|n} d$  for the sum of all positive divisors of  $n$ , and we let  $\omega(n) := \sum_{p|n} 1$  stand for the number of distinct prime factors of  $n$ . We write  $P(n)$  for the largest prime divisor of  $n$ , with the understanding that  $P(1) = 1$ . We say that  $n$  is  $y$ -smooth if  $P(n) \leq y$ . For each  $n$ , its  $y$ -smooth part is defined as the largest  $y$ -smooth divisor of  $n$ .

The Landau–Bachmann  $o$  and  $O$ -symbols, as well as Vinogradov’s  $\ll$  notation, are employed with their usual meanings. *Implied constants are absolute unless otherwise specified.*

## 2. PROOF OF THEOREM 1.1

In this section we give the proof of Theorem 1.1, assuming two preliminary results whose proofs are deferred to §3 and §4.

**Proposition 2.1.** *For each  $\epsilon > 0$ , the set of natural numbers  $n$  with*

$$\frac{\sigma(n+1)}{n+1} - \epsilon < \frac{\sigma(s^-(n))}{s^-(n)} < \frac{\sigma(n+1)}{n+1} + \epsilon. \quad (2.1)$$

*has asymptotic density 1.*

**Remark.** If  $n$  is prime, then  $s^-(n) = 0$ , and the expression  $\sigma(s^-(n))/s^-(n)$  is undefined. This does not contradict Proposition 2.1, since the set of primes has asymptotic density zero.

**Proposition 2.2.** *As  $\epsilon \downarrow 0$ , the upper density of the set of natural numbers  $n$  for which*

$$1 - \epsilon < \left( \frac{\sigma(n)}{n} - 1 \right) \left( \frac{\sigma(n+1)}{n+1} - 1 \right) < 1 + \epsilon \quad (2.2)$$

*tends to zero.*

*Proof of Theorem 1.1.* It suffices to prove the upper density assertion of the theorem. Let  $\delta > 0$ . We will show that if  $\epsilon > 0$  is sufficiently small, then the upper density of the set of  $n$  for which (1.1) holds is at most  $2\delta$ . We start by assuming that both  $\sigma(n)/n \leq B$  and  $\sigma(n+1)/(n+1) \leq B$ , where  $B > 0$  is chosen so that these conditions exclude a set of  $n$  of upper density at most  $\delta$ . To see that such a choice is possible, we can use a first moment argument; indeed, since

$$\sum_{n \leq x} \frac{\sigma(n)}{n} = \sum_{n \leq x} \sum_{d|n} \frac{1}{d} \leq x \sum_{d \leq x} \frac{1}{d^2} < 2x,$$

we can take  $B = 4/\delta$ . Moreover, Proposition 2.1 shows that by excluding an additional set of density 0, we can assume that

$$\left| \frac{\sigma(s^-(n))}{s^-(n)} - \frac{\sigma(n+1)}{n+1} \right| < \frac{\epsilon}{2B}.$$

Now write

$$\begin{aligned} \frac{s^-(s^-(n))}{n} &= \frac{s^-(n)}{n} \frac{s^-(s^-(n))}{s^-(n)} \\ &= \left( \frac{\sigma(n)}{n} - 1 - \frac{1}{n} \right) \left( \frac{\sigma(s^-(n))}{s^-(n)} - 1 - \frac{1}{s^-(n)} \right). \end{aligned}$$

If  $n$  is a large natural number satisfying (1.1) and our above conditions, then a short computation shows  $\frac{s^-(s^-(n))}{n}$  is within  $\epsilon$  of the product  $(\frac{\sigma(n)}{n} - 1)(\frac{\sigma(n+1)}{n+1} - 1)$ . (Keep in mind that since  $n$  is composite, we have  $s^-(n) \geq \sqrt{n}$ .) Thus,

$$1 - 2\epsilon < \left( \frac{\sigma(n)}{n} - 1 \right) \left( \frac{\sigma(n+1)}{n+1} - 1 \right) < 1 + 2\epsilon.$$

Finally, Proposition (2.2) shows that if  $\epsilon$  is chosen sufficiently small, then these remaining  $n$  make up a set of upper density  $< \delta$ .  $\square$

### 3. THE PROOF OF PROPOSITION 2.1

**3.1. Preparation.** The proof of the proposition is very similar to the proof, due to Erdős, Granville, Pomerance, and Spiro, that  $s(s(n))/s(n) = s(n)/n + o(1)$ , as  $n \rightarrow \infty$  along a sequence of density 1 (see Erdős et al. [7, p. 195]). We follow their argument, as well as the author's adaptation [15], very closely.

We begin by recalling some auxiliary estimates. The first of these is due to Pomerance [16, Theorem 2].

**Lemma 3.1.** *Let  $D$  be a natural number, and let  $x \geq 2$ . The number of  $n \leq x$  for which  $D \nmid \sigma(n)$  is  $\ll x/(\log x)^{1/\varphi(D)}$ .*

For a given  $\alpha$ , we call the natural number  $n$  an  $\alpha$ -primitive number if  $\sigma(n)/n \geq 1 + \alpha$  while  $\sigma(d)/d < 1 + \alpha$  for every proper divisor  $d$  of  $n$ . The following estimate is due to Erdős [5, p. 6]:

**Lemma 3.2.** *Fix a positive rational number  $\alpha$ . There is a constant  $c = c(\alpha) > 0$  and an  $x_0 = x_0(\alpha)$  so that for  $x > x_0$ , the number of  $\alpha$ -primitive  $n \leq x$  is at most*

$$\frac{x}{\exp(c\sqrt{\log x \log \log x})}.$$

As a consequence of Lemma 3.2, we obtain the following convergence result, which we will need to conclude the proof of Proposition 2.1.

**Lemma 3.3.** *Fix a positive rational number  $\alpha$ . Then*

$$\sum_{a \text{ } \alpha\text{-primitive}} \frac{2^{\omega(a)}}{a} < \infty.$$

*Proof.* We split the values of  $a$  appearing in the sum into two classes, putting those  $a$  for which  $\omega(a) \leq 20 \log \log a$  in the first class and all other  $a$  in the second. If  $a$  belongs to the first class, then  $2^{\omega(a)} \leq (\log a)^{20 \log 2}$ , and Lemma 3.2 shows that the sum over these  $a$  converges (by partial summation). To handle the  $a$  in the second class, we ignore the  $\alpha$ -primitivity condition altogether and invoke a lemma of Pollack [15, Lemma 2.4], according to which  $\sum_{a: \omega(a) > 20 \log \log a} \frac{2^{\omega(a)}}{a} < \infty$ .  $\square$

**3.2. Proof proper.** We proceed to prove Proposition 2.1 in two stages; first we prove that the lower-bound holds almost always, and then we do the same for the upper bound. The following lemma is needed for both parts.

**Lemma 3.4.** *Fix a natural number  $T$ . For each composite value of  $n$  with  $1 \leq n \leq x$ , write*

$$n + 1 = m_1 m_2 \quad \text{and} \quad s^-(n) = M_1 M_2,$$

*where  $P(m_1 M_1) \leq T$  and every prime dividing  $m_2 M_2$  exceeds  $T$ . Then, except for  $o(x)$  (as  $x \rightarrow \infty$ ) choices of  $n$ , we have  $m_1 = M_1$ .*

*Proof.* At the cost of excluding  $o(x)$  values of  $n \leq x$ , we may assume that

$$m_1 \leq (\log \log x)^{1/2} \left( \prod_{p \leq T} p \right)^{-1} =: R.$$

Indeed, in the opposite case,  $n + 1$  has a  $T$ -smooth divisor exceeding  $R$ , and the number of such  $n \leq x$  is

$$\ll x \sum_{\substack{e \text{ } T\text{-smooth} \\ e > R}} \frac{1}{e} = o(x),$$

as  $x \rightarrow \infty$ . Here we use that the sum of the reciprocals of the  $T$ -smooth numbers is  $\prod_{p \leq T} (1 - 1/p)^{-1} < \infty$ . Hence,  $m_1 \prod_{p \leq T} p \leq (\log \log x)^{1/2}$ , and so Lemma 3.1 shows that excluding  $o(x)$  values of  $n \leq x$ , we can assume that  $m_1 \prod_{p \leq T} p$  divides  $\sigma(n)$ . Since

$$s^-(n) = \sigma(n) - (n + 1),$$

it follows that  $m_1$  is the  $T$ -smooth part of  $s^-(n)$ . That is,  $m_1 = M_1$ .  $\square$

*Proof of the lower bound half of Proposition 2.1.* Fix  $\delta > 0$ . We will show that the number of  $n \leq x$  for which the left-hand inequality in (2.1) fails is smaller than  $3\delta x$ , once  $x$  is large.

Fix  $B$  large enough that  $\sigma(n + 1)/(n + 1) \leq B$  except for at most  $\delta x$  exceptional  $n \leq x$ . That this is possible follows from the first moment argument used in the proof of Theorem 1.1 (e.g., we may take  $B = 4/\delta$  again). Next, fix  $T$  large enough so that with  $m_2$  defined as in Lemma 3.4, we have

$$\frac{\sigma(m_2)}{m_2} \leq \exp(\epsilon/B)$$

except for at most  $\delta x$  exceptional  $n \leq x$ . To see that a suitable choice of  $T$  exists, observe that

$$\begin{aligned} \sum_{n \leq x} \log \frac{\sigma(m_2)}{m_2} &\leq \sum_{n \leq x} \sum_{\substack{p|n+1 \\ p > T}} \log \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \dots \right) \\ &\leq \sum_{n \leq x} \sum_{\substack{p|n+1 \\ p > T}} \frac{1}{p-1} \leq 2x \sum_{p > T} \frac{1}{p(p-1)} < \frac{2x}{T}. \end{aligned}$$

Hence, we may take  $T = \lceil 2B/(\delta\epsilon) \rceil$ .

For large  $x$ , we have that  $n$  is composite (so that  $M_1$  is defined) and that  $m_1 = M_1$ , except for at most  $\delta x$  values of  $n \leq x$ . This follows from Lemma 3.4 and the fact that the primes have density 0.

If  $n$  is not in any of the exceptional classes defined above, then

$$\begin{aligned} \frac{\sigma(s^-(n))}{s^-(n)} &= \frac{\sigma(M_1 M_2)}{M_1 M_2} \geq \frac{\sigma(M_1)}{M_1} = \frac{\sigma(m_1)}{m_1} = \frac{\sigma(n+1)/(n+1)}{\sigma(m_2)/m_2} \\ &\geq \frac{\sigma(n+1)}{n+1} \exp\left(-\frac{\epsilon}{B}\right) > \frac{\sigma(n+1)}{n+1} \left(1 - \frac{\epsilon}{B}\right) \geq \frac{\sigma(n+1)}{n+1} - \epsilon, \end{aligned}$$

which is the desired lower bound. Note that at most  $3\delta x$  values of  $n \leq x$  are exceptional, as claimed.  $\square$

*Proof of the upper bound half of Proposition 2.1.* We may suppose that  $0 < \epsilon < 1$ . Let  $\delta > 0$  be given. Fix  $\eta \in (0, 1)$  so small that the number of  $n \leq x$  which are either prime or which fail to satisfy

$$P(n) > x^\eta \quad \text{and} \quad P(n)^2 \nmid n \tag{3.1}$$

is smaller than  $\delta x$ , once  $x$  is large. The existence of such an  $\eta$  follows either from Brun's sieve or well-known work of Dickman on smooth numbers. Next, using the first moment argument from the proof of Theorem 1.1, choose a fixed number  $B \geq 1$  so that all but at most  $\delta x$  of the numbers  $n \leq x$  satisfy

$$\frac{\sigma(n+1)}{n+1} \leq B. \tag{3.2}$$

We fix rational numbers  $\alpha_1$  and  $\alpha_2$  satisfying

$$0 < \alpha_1 \leq \frac{\epsilon}{4B}, \quad 0 < \alpha_2 \leq \frac{\alpha_1 \eta}{12}.$$

Finally, we fix a natural number  $T$  which is sufficiently large, depending only on the  $\alpha_i$ ,  $\delta$ ,  $\eta$ , and  $B$ . The precise meaning of “sufficiently large” will be specified in the course of the proof.

Suppose that the right-hand inequality (2.1) fails for  $n$ , where we assume that  $n$  is composite and satisfies both (3.1) and (3.2). Write

$$n+1 = m_1 m_2 \quad \text{and} \quad s^-(n) = M_1 M_2,$$

where  $P(m_1 M_1) \leq T$  and every prime dividing  $m_2 M_2$  exceeds  $T$ . By Lemma 3.4, we can assume  $m_1 = M_1$ , excluding at most  $\delta x$  values of  $n \leq x$ . Thus,

$$\frac{\sigma(M_2)/M_2}{\sigma(m_2)/m_2} = \frac{\sigma(s^-(n))/s^-(n)}{\sigma(n+1)/(n+1)} \geq 1 + \frac{\epsilon}{\sigma(n+1)/(n+1)} \geq 1 + \frac{\epsilon}{B} \geq 1 + 4\alpha_1.$$

In particular,

$$\frac{\sigma(M_2)}{M_2} \geq 1 + 4\alpha_1. \quad (3.3)$$

We can assume our choice of  $T$  was such that, apart from at most  $\delta x$  exceptional  $n \leq x$ , we have

$$\frac{\sigma(m_2)}{m_2} \leq 1 + \alpha_1. \quad (3.4)$$

Indeed, the argument for the analogous claim in the proof of the lower-bound shows it is sufficient that  $T > 2(\delta \log(1 + \alpha_1))^{-1}$ . Henceforth, we assume (3.4). Now write  $M_2 = M_3 M_4$ , where every prime dividing  $M_3$  divides  $n + 1$ , while  $M_4$  is coprime to  $n + 1$ . Note that every prime dividing  $M_3$  divides  $m_2$ . Hence,

$$\begin{aligned} \frac{\sigma(M_3)}{M_3} &\leq \prod_{p|M_3} \left(1 + \frac{1}{p-1}\right) = \left(\prod_{p|M_3} \frac{p^2}{p^2-1}\right) \prod_{q|M_3} \frac{q+1}{q} \\ &\leq \left(\prod_{p>T} \frac{p^2}{p^2-1}\right) \frac{\sigma(m_2)}{m_2} \leq 1 + 2\alpha_1, \end{aligned}$$

using (3.4) and assuming an initial appropriate choice of  $T$ . So from (3.3),

$$\frac{\sigma(M_4)}{M_4} = \frac{\sigma(M_2)/M_2}{\sigma(M_3)/M_3} \geq \frac{1 + 4\alpha_1}{1 + 2\alpha_1} \geq 1 + \alpha_1.$$

It follows that there is an  $\alpha_1$ -primitive number  $a_1$  dividing  $M_4$ , where each prime dividing  $a_1$  exceeds  $T$ .

We claim next that there is a squarefree,  $\alpha_2$ -primitive number  $a_2$  dividing  $a_1$  with

$$a_2 \leq a_1^{\eta/2}.$$

List the distinct prime factors of  $a_1$  in increasing order, say  $T < q_1 < q_2 < \dots < q_t$ , and put  $a_0 := q_1 q_2 \dots q_{\lfloor \eta t/2 \rfloor}$ , so that

$$a_0 \leq (q_1 \dots q_t)^{\lfloor \eta t/2 \rfloor / t} \leq a_1^{\eta/2}.$$

We will show that  $\sigma(a_0)/a_0 \geq 1 + \alpha_2$ ; then we can take  $a_2$  as any  $\alpha_2$ -primitive divisor of  $a_0$ . First, observe that  $\lfloor \eta t/2 \rfloor \geq \eta t/3$ . Otherwise,  $t < 6/\eta$  and

$$1 + \alpha_1 \leq \frac{\sigma(a_1)}{a_1} \leq \prod_{1 \leq i \leq t} \left(1 + \frac{1}{q_i - 1}\right) \leq \left(1 + \frac{1}{T}\right)^{6/\eta} \leq \exp\left(\frac{6}{\eta T}\right),$$

which is false, assuming a suitable initial choice of  $T$ . It follows that

$$\frac{\sigma(a_0)}{a_0} = \prod_{1 \leq i \leq \lfloor \eta t/2 \rfloor} \frac{q_i + 1}{q_i} \geq \left(\prod_{p>T} \frac{p^2 - 1}{p^2}\right) \prod_{1 \leq i \leq \lfloor \eta t/2 \rfloor} \frac{q_i}{q_i - 1},$$

while

$$\begin{aligned} \prod_{1 \leq i \leq \lfloor \eta t/2 \rfloor} \frac{q_i}{q_i - 1} &\geq \left(\prod_{1 \leq i \leq t} \frac{q_i}{q_i - 1}\right)^{\lfloor \eta t/2 \rfloor / t} \\ &\geq \left(\frac{\sigma(a_1)}{a_1}\right)^{\eta/3} \geq (1 + \alpha_1)^{\eta/3} \geq 1 + \frac{\alpha_1 \eta}{6}. \end{aligned}$$

Thus,

$$\frac{\sigma(a_0)}{a_0} \geq \left( \prod_{p>T} \frac{p^2-1}{p^2} \right) \left( 1 + \frac{\alpha_1 \eta}{6} \right) \geq 1 + \frac{\alpha_1 \eta}{12} \geq 1 + \alpha_2,$$

again assuming a suitable choice of  $T$  to justify the middle inequality.

Observe that  $a_2$  satisfies

$$a_2 \leq a_1^{\eta/2} \leq (s^-(n))^{\eta/2} < x^{2\eta/3},$$

for large  $x$ . Write  $n = Pr$ , where  $P = P(n)$ . Then  $r > 1$  (since  $n$  is composite) and also, by (3.1),

$$r \leq x/P \leq x^{1-\eta}.$$

Moreover,  $a_2$  divides

$$s^-(Pr) = P(\sigma(r) - r) + \sigma(r) - 1,$$

and so

$$P(\sigma(r) - r) \equiv 1 - \sigma(r) \pmod{a_2}.$$

We view this as a linear congruence condition on  $P$  modulo  $a_2$ . If there are any solutions, then  $D := \gcd(\sigma(r) - r, a_2) \mid 1 - \sigma(r)$ , and in this case there are exactly  $D$  solutions modulo  $a_2$ . Note that if there are any solutions, then  $D \mid r - 1$ . Also note that  $D$  is squarefree, since  $a_2$  is squarefree.

We now sum over pairs  $a_2$  and  $r$ , for each pair counting the number of possible values of  $P \leq x/r$ . By the Brun–Titchmarsh inequality and the preceding remarks about  $D$ , we have that the number of possible values of  $n = Pr$  is

$$\begin{aligned} &\ll \sum_{\substack{a_2 \text{ } \alpha_2\text{-primitive} \\ T < a_2 \leq x^{2\eta/3}}} \sum_{1 < r \leq x^{1-\eta}} \sum_{\substack{D \mid (a_2, r-1) \\ D \text{ squarefree}}} D \frac{x/r}{\varphi(a_2) \log(x/(a_2 r))} \\ &\ll \frac{x}{\eta \log x} \sum_{\substack{a_2 \text{ } \alpha_2\text{-primitive} \\ T < a_2 \leq x^{\eta/3}}} \frac{1}{\varphi(a_2)} \sum_{\substack{D \mid a_2 \\ D \text{ squarefree}}} D \sum_{\substack{1 < r \leq x^{1-\eta} \\ D \mid r-1}} \frac{1}{r}. \end{aligned}$$

The sum on  $r$  is  $\ll \frac{1}{D} \log x$ . Moreover, since  $a_2$  is  $\alpha_2$ -primitive, we have

$$\frac{a_2}{\varphi(a_2)} \ll \frac{\sigma(a_2)}{a_2} \leq \frac{3}{2}(1 + \alpha_2) \ll 1,$$

and so  $\varphi(a_2) \gg a_2$ . Thus, the remaining sum is

$$\ll \frac{x}{\eta} \sum_{\substack{a_2 \text{ } \alpha_2\text{-primitive} \\ T < a_2 \leq x^{2\eta/3}}} \frac{1}{a_2} \sum_{\substack{D \mid a_2 \\ D \text{ squarefree}}} 1 \ll \frac{x}{\eta} \sum_{\substack{a_2 \text{ } \alpha_2\text{-primitive} \\ a_2 \geq T}} \frac{2^{\omega(a_2)}}{a_2}.$$

But if  $T$  was chosen sufficiently large, then this last sum is bounded by  $\eta \delta x$  (by Lemma 3.3), leading to an upper bound of  $\ll \delta x$ . Since the number of exceptional  $n$  appearing earlier in the argument is also  $\ll \delta x$ , and  $\delta > 0$  was arbitrary, the proof is complete.  $\square$

## 4. PROOF OF PROPOSITION 2.2

We start by quoting two lemmas. The first was developed by Erdős [3] to estimate the decay of the distribution function of  $\sigma(n)/n$  near infinity. We state the lemma in a slightly stronger form which is supported by his proof.

**Lemma 4.1.** *For  $x > 0$ , the number of positive integers  $n \leq x$  with  $\sigma(n)/n > y$  is*

$$\leq x / \exp(\exp((e^{-\gamma} + o(1))y)), \quad \text{as } y \rightarrow \infty,$$

*uniformly in  $x$ , where  $\gamma$  is the Euler–Mascheroni constant.*

The next lemma, also due to Erdős [6], supplies an estimate for how often  $\sigma(n)/n$  lands in a short interval; note the uniformity in the parameter  $a$ .

**Lemma 4.2.** *Let  $x > t \geq 2$  and let  $a \in \mathbb{R}$ . The number of  $n \leq x$  with  $a < \sigma(n)/n < a + 1/t$  is  $\ll x / \log t$ .*

The next two lemmas develop the philosophy that the rough size of  $\sigma(n)/n$  is usually determined by the small prime factors of  $n$ . Put  $h(n) := \sum_{d|n} \frac{1}{d}$ , so that  $h(n) = \sigma(n)/n$ . For each natural number  $T$ , set  $h_T(n) := \sum_{d|n, P(d) \leq T} \frac{1}{d}$ . The next lemma says that  $h$  and  $h_T$  are usually close once  $T$  is large.

**Lemma 4.3.** *Let  $\epsilon > 0$  and  $x \geq 1$ . The number of  $n \leq x$  with  $h(n) - h_T(n) > \epsilon$  is  $\ll x/(T\epsilon)$ .*

*Proof.* Again, we use a first moment argument. We have

$$\sum_{n \leq x} (h(n) - h_T(n)) \leq \sum_{n \leq x} \sum_{\substack{d|n \\ d > T}} \frac{1}{d} \leq x \sum_{d > T} \frac{1}{d^2} \ll x/T,$$

from which the result is immediate.  $\square$

**Lemma 4.4.** *Let  $T$  be a natural number. Let  $S$  be any set of real numbers, and define  $\mathcal{E}(S)$  as the set of  $T$ -smooth numbers  $e$  for which  $h_T(e) - 1 \in S$ . Then for  $n \in \mathbb{N}$ , we have  $h_T(n) - 1 \in S$  precisely when  $n$  has  $T$ -smooth part  $e$  for some  $e \in \mathcal{E}(S)$ . Moreover, the density of such  $n$  exists and is given by*

$$\sum_{e \in \mathcal{E}(S)} \frac{1}{e} \prod_{p \leq T} (1 - 1/p). \quad (4.1)$$

*Proof.* It is clear that  $h_T(n)$  depends only on the  $T$ -smooth part of  $n$ . So it suffices to prove that the density of  $n$  with  $T$ -smooth part in  $\mathcal{E}(S)$  is given by (4.1).

For each set of  $T$ -smooth numbers  $\mathcal{E}$ , let  $\bar{d}_{\mathcal{E}}$  and  $\underline{d}_{\mathcal{E}}$  denote the upper and lower densities of the set of  $n$  whose  $T$ -smooth part belongs to  $\mathcal{E}$ . If  $\bar{d}_{\mathcal{E}} = \underline{d}_{\mathcal{E}}$ , then the density of this set exists; denote it by  $d_{\mathcal{E}}$ .

For each  $T$ -smooth number  $e$ , a natural number  $n$  has  $T$ -smooth part  $e$  precisely when  $e$  divides  $n$  and  $n/e$  is coprime to  $\prod_{p \leq T} p$ , so that the set of such  $n$  has density  $\frac{1}{e} \prod_{p \leq T} (1 - 1/p)$ . Since density is finitely additive, it follows that for any finite subset  $\mathcal{E} \subset \mathcal{E}(S)$ ,

$$d_{\mathcal{E}} = \sum_{e \in \mathcal{E}} \frac{1}{e} \prod_{p \leq T} (1 - 1/p).$$

Now let  $x > 0$ , and put  $\mathcal{E}(S) = \mathcal{E}_1 \cup \mathcal{E}_2$ , where  $\mathcal{E}_1 = \mathcal{E}(S) \cap [1, x]$  and  $\mathcal{E}_2 = \mathcal{E}(S) \setminus \mathcal{E}_1$ . Then  $\underline{d}_{\mathcal{E}(S)} \geq \underline{d}_{\mathcal{E}_1}$  for all  $x$ , and so letting  $x \rightarrow \infty$ , we find that  $\underline{d}_{\mathcal{E}(S)}$  is bounded below by



(4.1). On the other hand,  $\bar{d}_{\mathcal{E}(S)} \leq \bar{d}_{\mathcal{E}_1} + \bar{d}_{\mathcal{E}_2}$ . But  $\bar{d}_{\mathcal{E}_1}$  is bounded above by (4.1) for all  $x$ , while  $\bar{d}_{\mathcal{E}_2} \leq \sum_{\substack{e \text{ } T\text{-smooth} \\ e > x}} e^{-1} = o(1)$ , as  $x \rightarrow \infty$ . Thus, letting  $x \rightarrow \infty$ , we obtain that  $\bar{d}_{\mathcal{E}(S)}$  is bounded above by (4.1).  $\square$

*Proof of Proposition 2.2.* Let  $\delta > 0$  be sufficiently small. We will show that for

$$\epsilon < \exp(-4/\delta), \quad (4.2)$$

the number of  $n \leq x$  satisfying (2.2) is  $\ll \delta(\log \log \frac{1}{\delta})x$ , for large  $x$ . Note that since  $\delta \log \log \frac{1}{\delta} \rightarrow 0$  as  $\delta \downarrow 0$ , this proves the proposition. In what follows, we fix  $\delta$  and  $\epsilon$ , always assuming that  $\delta$  is small and that  $\epsilon > 0$  satisfies (4.2).

Put  $T := \epsilon^{-1}\delta^{-1}$ . We can assume that both  $n$  and  $n+1$  have  $T$ -smooth part  $\leq \log x$ . Indeed, for large  $x$ , this excludes a set of  $n$  size  $< \delta x$ , since

$$\sum_{\substack{e \text{ } T\text{-smooth} \\ e > \log x}} \frac{1}{e} = o(1),$$

as  $x \rightarrow \infty$ .

Let  $I$  be the closed interval defined by  $I := [\exp(-1/\delta), 2 \log \log \frac{1}{\delta}]$ . For large  $x$ , Lemmas 4.1 and 4.2 imply that all but  $\ll \delta x$  values of  $n \leq x$  are such that  $h(n) - 1 \in I$  and  $h(n+1) - 1 \in I$ . By Lemma 4.3, excluding  $\ll \delta x$  additional values of  $n \leq x$ , we can assume that  $h_T(n) \geq h(n) - \epsilon$  and  $h_T(n+1) \geq h(n+1) - \epsilon$ . Recalling the upper bound (4.2) on  $\epsilon$ , we see that both  $h_T(n) - 1$  and  $h_T(n+1) - 1$  belong to the interval  $J$ , where

$$J := \left[ \frac{1}{2} \exp(-1/\delta), 2 \log \log \frac{1}{\delta} \right].$$

Moreover (always assuming  $\delta$  sufficiently small),

$$(h_T(n) - 1)(h_T(n+1) - 1) \geq ((h(n) - 1) - \epsilon)((h(n+1) - 1) - \epsilon) \geq 1 - 5\epsilon \log \log \frac{1}{\delta}, \quad (4.3)$$

and

$$(h_T(n) - 1)(h_T(n+1) - 1) \leq (h(n) - 1)(h(n+1) - 1) \leq 1 + \epsilon. \quad (4.4)$$

Write  $J$  as the disjoint union of  $N := \lceil 1/\epsilon \rceil$  consecutive intervals  $J_0, J_1, \dots, J_{N-1}$ , each of length  $1/N$ . We estimate, for each  $0 \leq i < N$ , the number of  $n$  for which  $h_T(n) - 1$  belongs to  $J_i$ . Fix  $0 \leq i < N$ . Since  $h_T(n) - 1$  belongs to  $J_i$ , (4.3) and (4.4) show that

$$h_T(n+1) - 1 \in \left[ \frac{1 - 5\epsilon \log \log \frac{1}{\delta}}{x_{i+1}}, \frac{1 + \epsilon}{x_i} \right] =: J'_i, \quad (4.5)$$

where  $x_i$  and  $x_{i+1}$  are the left and right endpoints of  $J_i$ , respectively. So in the notation of Lemma 4.4,  $n$  has  $T$ -smooth part  $e \in \mathcal{E}(J_i)$  and  $n+1$  has  $T$ -smooth part  $e' \in \mathcal{E}(J'_i)$ . Clearly,  $\gcd(e, e') = 1$ . That  $n$  and  $n+1$  have  $T$ -smooth parts  $e$  and  $e'$ , respectively, amounts to a congruence condition on  $n$  modulo  $M := ee' \prod_{p \leq T} p$ , where the number of allowable residue classes is  $\prod_{p|ee'} (p-1) \prod_{p \nmid ee', p \leq T} (p-2)$ . For large  $x$ ,

$$M \leq (\log x)^2 \prod_{p \leq T} p < (\log x)^3 \leq x.$$

(Recall that  $e, e' \leq \log x$ .) Thus, the Chinese remainder theorem shows that the number of such  $n \leq x$  is

$$\begin{aligned} &\ll \frac{x}{ee'} \prod_{p|ee'} (1 - 1/p) \prod_{\substack{p \nmid ee' \\ p \leq T}} (1 - 2/p) \\ &\leq x \left( \frac{1}{e} \prod_{p \leq T} (1 - 1/p) \right) \left( \frac{1}{e'} \prod_{p \leq T} (1 - 1/p) \right) \prod_{p|ee'} (1 - 1/p)^{-1}. \end{aligned}$$

But

$$\prod_{p|ee'} (1 - 1/p)^{-1} = \frac{e}{\varphi(e)} \frac{e'}{\varphi(e')} \ll \frac{\sigma(e)}{e} \frac{\sigma(e')}{e'} \ll \left( \log \log \frac{1}{\delta} \right)^2,$$

since  $h(e) - 1, h(e') - 1 \leq 2 \log \log \frac{1}{\delta}$ . Summing over  $e \in \mathcal{E}(J_i)$  and  $e' \in \mathcal{E}(J'_i)$ , we find that the number of  $n$  under consideration is

$$\ll x \left( \log \log \frac{1}{\delta} \right)^2 \left( \sum_{e \in \mathcal{E}(J_i)} \frac{1}{e} \prod_{p \leq T} (1 - 1/p) \right) \left( \sum_{e' \in \mathcal{E}(J'_i)} \frac{1}{e'} \prod_{p \leq T} (1 - 1/p) \right).$$

Now sum over  $0 \leq i < N$ . We obtain that the number of remaining  $n$  satisfying (2.2) is  $\ll Lx(\log \log \frac{1}{\delta})^2$ , where

$$\begin{aligned} L &:= \sup_{0 \leq i < N} \left\{ \sum_{e' \in \mathcal{E}(J'_i)} \frac{1}{e'} \prod_{p \leq T} (1 - 1/p) \right\} \left( \sum_{0 \leq i < N} \left\{ \sum_{e \in \mathcal{E}(J_i)} \frac{1}{e} \prod_{p \leq T} (1 - 1/p) \right\} \right) \\ &\leq \sup_{0 \leq i < N} \left\{ \sum_{e' \in \mathcal{E}(J'_i)} \frac{1}{e'} \prod_{p \leq T} (1 - 1/p) \right\}; \end{aligned}$$

we use here that the  $J_i$  are disjoint, so that

$$\sum_{0 \leq i < N} \sum_{e \in \mathcal{E}(J_i)} \frac{1}{e} \leq \sum_{e \text{ } T\text{-smooth}} \frac{1}{e} = \prod_{p \leq T} (1 - 1/p)^{-1}.$$

The proof will be completed by showing that  $L \ll \delta$ . It is enough to argue that each sum

$$\sum_{e' \in \mathcal{E}(J'_i)} \frac{1}{e'} \prod_{p \leq T} (1 - 1/p)$$

is  $\ll \delta$ , uniformly for  $0 \leq i < N$ . By Lemma 4.4, this sum describes the density of those natural numbers  $m$  for which  $h_T(m) - 1 \in J'_i$ . We split these  $m$  into two classes, according to whether  $h(m) - h_T(m) > \epsilon$  or not. The set of  $m$  in the former class has upper density  $\ll \delta$ , by Lemma 4.3. Suppose now that  $h(m)$  belongs to the second class. From the expression (4.5) defining  $J'_i$  and a short computation, we see that  $h_T(m)$  is trapped within a specific interval of length

$$\ll \exp(2/\delta) \left( \log \log \frac{1}{\delta} \right)^2 \epsilon \ll \exp(3/\delta) \epsilon.$$

Since  $m$  belongs to the second class,  $h(m)$  is also trapped within a specific interval of length  $\ll \exp(3/\delta)\epsilon$ . By (4.2),  $\exp(3/\delta)\epsilon \leq \exp(-1/\delta)$ , and so by Lemma 4.2, the upper density of the set of those  $m$  in the second class is

$$\ll \frac{1}{\delta^{-1} + O(1)} \ll \delta,$$

assuming again that  $\delta$  is sufficiently small.  $\square$

**Remark.** Our argument also shows that the set of *augmented amicable numbers* has density zero (see sequences A007992, A015630). Here an augmented amicable number is an integer which generates a 2-cycle under iteration of the function  $s^+(n) := 1 + \sum_{d|n, d < n} d$ , e.g.,  $n = 6160$ .

**Sequences discussed.** A000396, A005276, A007992, A015630, A063990, A126160

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