Name: ______________________

MATH 3100H Exam 1

February 3, 2017

Instructions: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam. **None of the following are allowed**: notes, formula sheets, or electronic devices of any kind (including calculators, cell phones, etc.).

- **Print** your name clearly in the space provided.
- Read all of the questions carefully before starting to work.
- Give complete arguments and explanations unless otherwise indicated.
- Continue on the back of the **previous** page if you run out of space.
- Enjoy!

Honor Statement:

I have neither given nor received help on this exam, and all of the answers are my own.

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Signature

Question	Points	Score
1	15	
2	25	
3	15	
4	20	
5	25	
Total:	100	

1. [15 points] Consider the sequence $\langle a_n \rangle$ defined recursively by $a_1 = 1$ and

$$a_{n+1} = a_n + \frac{1}{2^n} \quad \text{for all } n \in \mathbb{N}.$$

Use induction to prove that $a_n = 2 - \frac{1}{2^{n-1}}$ for all $n \in \mathbb{N}$.

2. (a) [10 points] Carefully state the definition of convergence of a sequence $\langle a_n \rangle$ to the real number L.

(b) [15 points] Suppose that $\langle a_n \rangle$ is the sequence given by $a_n = \frac{2n^2+n}{n^2-10}$. Use **the definition of convergence** to prove that $\lim_{n\to\infty} a_n = 2$. (No credit will be given for a solution using the limit rules of §1.5.)

3. [15 points] Suppose $\langle a_n \rangle$, $\langle b_n \rangle$ are sequences with

$$\lim_{n\to\infty} a_n = 10, \qquad \lim_{n\to\infty} b_n = 10.$$

Prove that there is an $N \in \mathbb{N}$ such that

$$|a_n - b_n| < 1 \qquad \text{for all } n \geq N.$$

4. Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be sequences of real numbers.

   (a) [5 points] Give the definition of what it means to say $\langle a_n \rangle$ is **bounded**.

   (b) [5 points] Prove that the product of two bounded sequences is also bounded. You may use any facts about bounded sequences proved in class.

   (c) [10 points] Suppose that $\langle a_n \rangle$ is a bounded sequence. Suppose that there exists a natural number N such that

$$a_n = b_n \qquad \text{whenever } n > N.$$

   Show that $\langle b_n \rangle$ is also bounded.

5. The goal of this problem is to prove a special case of the sequence ratio test. Suppose that $\langle a_n \rangle$ is a sequence of *positive* real numbers satisfying

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}.$$

(a) [5 points] Show that there is an $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} < 2/3 \qquad \text{for all } n \geq N.$$

(b) [12 points] Show that there is a real number $K > 0$ such that

$$a_n \leq K(2/3)^n \qquad \text{for all } n \in \mathbb{N}.$$

(c) [8 points] Use the result of (b) to prove that $\lim_{n\to\infty} a_n = 0$. You may use any theorem discussed in class (except the ratio test for sequences!), but make sure to indicate which results you use.

You may receive credit for this part regardless of whether you solved (b) correctly.