Math 347 – Exam #2 practice problems
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Definitions etc.

1. What is meant by a *k-arrangement* of a set? What about a *k-selection*? If S is a set with n elements and $0 \leq k \leq n$, how many k-arrangements of S are there? How many k-selections are there? What if $k > n$?

2. What is the formula for $\binom{n}{k}$ in terms of factorials? How does one prove it?

3. What is Pascal's identity? How does one prove it?

4. State the binomial theorem.

5. What is meant by a *combinatorial proof*?

6. What is the definition of a *prime number*? Is 1 a prime number?

7. What do we mean by the *greatest common divisor* of two integers a and b? Does every pair of integers a and b have a greatest common divisor?

8. State the well-ordering principle.

9. Use the Euclidean algorithm to find the greatest common divisor of 63 and 39. Show your work.

10. What does it mean to say that two integers are *relatively prime*? Does it make sense to ask whether 91 is relatively prime? (Why or why not?) Which integers are relatively prime to 1? If p is a prime number, which integers are relatively prime to p?

11. What is the *fundamental theorem of arithmetic*?

Other problems

The first 3 problems below are taken from Chapter 1 of the text *Enumerative Combinatorics through Guided Discovery* by Ken Bogart.

1. Five teams are going to send their baseball teams to a tournament, in which each team must play each other team exactly once. How many games are required? What if there are n teams?

2. One of the schools sending teams to the tournament has to send its players from far away, and so is making sandwiches for them. There are three choices for the kind of bread and five choices for the kind of filling. How many different kinds of sandwiches are available?

3. The coach of one of these teams knows of an ice cream parlor where she plans to stop to buy each team member a triple decker cone. There are 12 different flavors of ice cream, and triple decker cones are made in homemade waffle cones. Having chocolate as the bottom scoop is different from having chocolate from the top scoop. How many possible ice cream cones are going to be available to the team members? How many cones with three different kinds of ice cream will be available?

4. Suppose m and n are positive integers. How many functions are there from the set $[m]$ to the set $[n]$? How many bijections are there from $[n]$ to itself?

5. How many solutions $x_1, \ldots, x_5$ are there in *positive* integers to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 100$?

6. There are 624 five-card poker hands consisting of 'four-of-a-kind'; explain why.

7. (Exercise 5.24 from the text) A *bridge hand* consists of 13 cards from a standard 52-card deck. How many such hands consist of five cards in one suit and four cards in each of two other suits? (So you can check yourself, the answer is 7895358900.) How many consist of three cards in three suits and four cards in one suit? (The answer is 66905856160.) How many consist of eight cards in one suit, three cards in one suit and two cards in one suit? (The answer is 689049504.)

8. In class we gave a combinatorial proof that the elements in the nth row of Pascal's triangle sum to 2^n. Give a combinatorial proof that the sum of the squares of the elements in the nth row of Pascal's triangle is $\binom{2n}{n}$. That is, prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Hint: Recall from class that $\binom{n}{k} = \binom{n}{n-k}$; hence we can rewrite $\binom{n}{k}^2$ on the left hand side as $\binom{n}{k}\binom{n}{n-k}$.

9. Give a combinatorial proof that for integers n, k, and j satisfying $n \geq k \geq j \geq 0$,

$$\binom{n}{k}\binom{k}{j} = \binom{n}{k-j}\binom{n-(k-j)}{j}.$$

10. Give a combinatorial proof of the identity

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1},$$

which you proved by induction in your homework.

11. Prove that if a and b are positive integers and a divides b, then $a \leq b$. (We used this fact repeatedly in class.)

12. Prove, without using the results in class that we derived from the Euclidean algorithm, that if d is the greatest common divisor of the positive integers a and b, then the numbers a/d and b/d are relatively prime.

13. Prove or give a counterexample: If a and b are positive integers and d is the greatest common divisor of a and b, then a/d and b are relatively prime.

14. Prove that the positive integers a and b are relatively prime if and only if there is no prime number that divides both a and b.

15. Let a, b, and c be natural numbers. Prove that if a is relatively prime to both b and c, then a is relatively prime to bc. *Hint:* You may find it useful to use the result of the previous problem.

16. Determine whether each of the following statements is true or false. In each case give a proof or a counterexample.

    (a) Suppose a, b, x, y are integers. If d is an integer which divides a and b, then d divides $ax + by$.

    (b) Suppose a, b, and n are positive integers. If n divides a or n divides b, then n divides ab.

    (c) Suppose a, b, and n are positive integers. If n divides ab and $\gcd(a, b) = 1$, then n divides a or n divides b.

17. Suppose the integer $n > 1$ is not prime. Prove that there exist integers a and b for which n divides $a \cdot b$ while n does not divide either a or b.

18. How many positive divisors does the number $7!$ have? Justify your answer.

19. Prove that every integer $n > 1$ has a prime divisor. (Don't assume the theorem we proved in class that every integer > 1 has a prime factorization!)

20. Let n be a natural number. Show that the binomial coefficient $\binom{2n}{n}$ is divisible by every prime number p in the interval $n < p < 2n$.

21. Prove that for any positive integers a and b, the pair of numbers a and b has the same gcd as the pair $5a + 3b$ and $3a + 2b$.