MATH 4000/6000 – Homework #6

posted April 8, 2019; due by 5 PM on April 15, 2019

Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. – Bertrand Russell

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

In this assignment, assume that all rings mentioned are commutative.

1. Exercise 4.1.1(a)

2. Exercise 4.1.8.

3. Let F be a field, and let $f(x) \in F[x]$ be nonconstant. Let $n = \deg f(x)$. In class, we proved that all elements of $F[x]/\langle f(x)\rangle$ have the form $\overline{a_0 + a_1x + \cdots + a_{n-1}x^{n-1}}$, where $a_0, \ldots, a_{n-1} \in F$.

   Show that this representation is unique; that is, distinct choices of the a_i correspond to distinct elements of $F[x]/\langle f(x)\rangle$.

4. Suppose that $\phi\colon R \to S$ is a homomorphism of rings. Prove that $\phi(R)$ is a subring of S. (Recall that, by definition, $\phi(R) = \{\phi(r) : r \in R\}$.)

   Hint: Use the criterion you proved in Problem 1(a) on HW #4.

5. Let F be a field and suppose that $p(x) \in F[x]$ is irreducible.

   (a) Prove that $F[x]/\langle p(x)\rangle$ is a field.

   Hint: Imitate our proof that $\mathbb{Z}_p$ is a field when p is a prime. Namely, suppose $\overline{a(x)}$ is not $\overline{0}$ in $F[x]/\langle p(x)\rangle$. Then $p(x)$ does not divide $a(x)$. What does this mean about $\gcd(p(x), a(x))$? Go from there.

   (b) Prove that if I is an ideal of $F[x]$ containing $p(x)$, then either $I = \langle p(x)\rangle$ or $I = F[x]$.

6. Let R be a ring, and let I be an ideal of R. Prove that R/I is the zero ring $\Longleftrightarrow I = R$.

7. Let R be a ring, not the zero ring. We say that an ideal $I \subseteq R$ is a **prime ideal** if

   (i) $I \neq R$,

   (ii) whenever a and b are elements of R for which $ab \in I$, either $a \in I$ or $b \in I$ (or both).

   Show that for every ideal I of R,

   $$R/I \text{ is a domain} \Longleftrightarrow I \text{ is a prime ideal of } R.$$

8. Let R, S be rings.

   (a) (Isomorphism is symmetric) Suppose $\phi\colon R \to S$ is an isomorphism. Since ϕ is a bijection, you know from MATH 3200 that it has an inverse; in other words, there is a map $\psi\colon S \to R$ satisfying

   $$(\psi \circ \phi)(r) = r \text{ for all } r \in R, \quad \text{and} \quad (\phi \circ \psi)(s) = s \text{ for all } s \in S.$$

   Prove that ψ is an isomorphism from S to R.

   Hint: You may assume as known that ψ is a bijection.

(b) (Isomorphism is transitive) Suppose $\phi\colon R \to S$ and $\psi\colon S \to T$ are isomorphisms. Prove that $\psi \circ \phi$ is an isomorphism from R to T.

Hint: You may take as known that the composition of bijections is a bijection.

9. Exercise 4.2.1.

10. Let $m, n \in \mathbb{Z}^+$ with $\gcd(m, n) = 1$. Use the Fundamental Homomorphism Theorem to prove that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

    Hint: Generalize the argument from class when $m = 3, n = 5$.

11. Use the Fundamental Homomorphism Theorem to establish the following ring isomorphisms.

    (a) $\mathbb{R}[x]/\langle x^2 + 6\rangle \cong \mathbb{C}$.

    Hint: Consider the "evaluation at $i\sqrt{6}$" homomorphism taking $f(x) \in \mathbb{R}[x]$ to $f(i\sqrt{6}) \in \mathbb{C}$.

    (b) $R[x]/\langle x\rangle \cong R$ for every ring R.

    (c) $\mathbb{Q}[x]/\langle x^2 - 1\rangle \cong \mathbb{Q} \times \mathbb{Q}$.

    Hint: Consider the homomorphism from $\mathbb{Q}[x]$ to $\mathbb{Q} \times \mathbb{Q}$ given by $f(x) \mapsto (f(1), f(-1))$.

12. (*) Let m and n be positive integers. Show that if $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$, then $\gcd(m, n) = 1$. This is the converse of Exercise 10.