RESEARCH STATEMENT

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My research interests and activity have been primarily in two areas. In what follows, I describe my contributions to each and mention some remaining questions.

The distribution of primes, from $\mathbf{Z}$ to $\mathbf{F}_q[T]$

In 1792 or 1793, a young Gauss proposed, on the basis of careful examination of the numerical evidence, that the count of prime numbers below a large number x should be well-approximated by $x/\log x$. The *prime number theorem*, asserting that Gauss's guess is correct (interpreted as an asymptotic estimate) was established a century later by Hadamard and de la Vallée Poussin. Along with Dirichlet's 1837 theorem on primes that belong to a prescribed arithmetic progression, the prime number theorem secured the status of analytic number theory as a legitimate field of study.

Zooming ahead five years in Gauss's life, we find him preparing to publish his masterwork, the *Disquisitiones Arithmeticae*. It is not so well-known that the published version of this work, containing seven sections, is a truncation of Gauss's original vision. A planned eighth section, containing what Gauss described in a letter to J. Bolyai as "the most difficult matters", was omitted for reasons of length (see [Fre07]). These "most difficult matters" are, in modern language, the rudiments of a theory of finite fields. Among Gauss's many results here is the formula for the number of irreducible polynomials of degree n over the finite field $\mathbf{F}_q$. Denoting this quantity by $\pi(q;n)$, he showed that $\pi(q;n) \sim \frac{q^n}{n}$, whenever $q^n \to \infty$. The similarity to the prime number theorem is striking. Indeed, if we put $X = q^n$, then Gauss is telling us that $\pi(q;n) \sim X/\log_q X$ as $X \to \infty$, where the subscript indicates that the logarithm occurs with base q.

Since Gauss, there have been many results drawing out the analogies between the theory of rational primes and the distribution of irreducible polynomials over finite fields. For example, Landau's student Kornblum gave an $\mathbf{F}_q[T]$-analogue of Dirichlet's theorem on primes in progressions (see [Kor19]). A decade or so later, Artin gave a version of the prime number theorem for such progressions as part of his doctoral thesis [Art24]. More recently, effort has been put into generalizing the theory of sieves to the polynomial setting (e.g., Cherly [Che78], Webb [Web83], Car [Car84a]-[Car84b], Hsu [Hsu96]), developing a function field version of the circle method and exploring its consequences (e.g., Hayes [Hay66], Effinger and Hayes [EH91b], [EH91a] and the present author [Pola]), and generalizing Maier's matrix method (Thorne [Tho08]).

The work alluded to in the previous paragraph suggests a meta-theorem of the form "Every interesting result about the distribution of rational primes has an $\mathbf{F}_q[T]$-analogue." In my thesis, I asked the question: What about the interesting conjectures in prime number theory that are not (yet) theorems?

Let me focus the remaining discussion around polynomial analogues of two well-known conjectures in prime number theory. The first is the *twin prime conjecture*: There are infinitely many pairs of primes p and $p+2$ (e.g., $101, 103$). The second statement we consider was popularized by Landau, although one can trace it back to Euler's 18th century correspondence with Goldbach: There are infinitely many primes of the form $n^2 + 1$ (e.g., $110^2 + 1 = 12101$).

The first of these problems has been attacked with spectacular success by Chris Hall ([Hal03], [Hal06]). He shows that if $q > 3$, then there are infinitely many pairs of monic irreducible polynomials $A, A+1$ in $\mathbf{F}_q[T]$. The proof fits easily within a page; the key ingredient is Capelli's irreducibility criterion, which (in the needed form) goes back at least to the beginnings of the twentieth century. In [Polb], I extend Hall's methods to prove somewhat stronger results (e.g., relaxing the condition of his theorem to "$q > 2$," and proving an analogous result for "prime triples").

In the same paper, I show that Hall's methods can be tweaked to attack Landau's problem. For this, new input is needed, in the guise of a character sum estimate coming from the (proved) Riemann Hypothesis for curves. This allows one to show, e.g., that if -1 is not a square in $\mathbf{F}_q$, then there are infinitely many irreducible polynomials of the form $A^2 + 1$, as A ranges over $\mathbf{F}_q[T]$. And the method applies in much greater generality as well: Take any collection $F_1(T), \dots, F_r(T)$ of polynomials over $\mathbf{F}_q$ satisfying obvious necessary conditions; then for each fixed q which is sufficiently large compared to $\sum_{i=1}^{r} \deg F_i$, there are infinitely many $A \in \mathbf{F}_q[T]$ for which $F_1(A), \dots, F_r(A)$ are simultaneously irreducible. (This can be considered progress towards the polynomial analogue of the conjecture known in the rational setting as Schinzel's *Hypothesis H*.)

It is natural to wonder if the qualitative assertions of the last paragraph can be quantitatively refined. To see what this might mean, recall that in the classical setting, Hardy & Littlewood [HL23] conjectured that the number of twin prime pairs $p, p+2 \leq x$ is $\sim 2Cx/(\log x)^2$, where C is a certain infinite product over primes. In the appendix to [Polb], we make the following prediction, which is an analogue of the conjectures of [HL23] and [BH62]. (Another analogue of the Hardy-Littlewood conjectures, without uniformity in q but more general in other respects, appears in [CCG08].)

Conjecture 1. *Let $F_1, \dots, F_r$ be nonassociate irreducible one-variable polynomials over $\mathbf{F}_q$ with the degree of $F_1 \cdots F_r$ bounded by B. Suppose that there is no prime P of $\mathbf{F}_q[T]$ for which the map*

$$A(T) \mapsto F_1(A(T)) \cdots F_r(A(T)) \bmod P$$

is identically zero. Let $N(n)$ denote the number of monic polynomials $A(T) \in \mathbf{F}_q[T]$ of degree n for which all of $F_1(A(T)), \dots, F_r(A(T))$ are irreducible. Then

$$N(n) = (1 + o_B(1)) \frac{\mathfrak{S}(F_1, \dots, F_r)}{\prod_{i=1}^{r} \deg F_i} \frac{q^n}{n^r} \quad \text{as } q^n \to \infty. \tag{1}$$

Here the local factor $\mathfrak{S}(F_1, \dots, F_r)$ is defined by

$$\mathfrak{S}(F_1, \dots, F_r) := \prod_{m=1}^{\infty} \prod_{\substack{\deg P = m \\ P \text{ monic, prime}}} \frac{1 - \omega(P)/q^m}{(1 - 1/q^m)^r},$$

where $\omega(P) := \#\{R \bmod P : F_1(R) \cdots F_r(R) \equiv 0 \pmod{P}\}$.

It is important to observe that this conjecture makes a prediction not simply when q is fixed and n is large (corresponding most directly to the classical setting), but whenever q^n is large. So far there has been little success confirming Conjecture 1 for fixed q. But this conjecture is not entirely unattackable; the following is the main result of [Pol08b]:

Theorem 2. *Let n be a positive integer. Let $F_1(T), \dots, F_r(T)$ be nonassociate irreducible polynomials over $\mathbf{F}_q$ with the degree of the product $F_1 \cdots F_r$ bounded by B. Then with $N(n)$ defined as above,*

$$N(n) = \frac{\mathfrak{S}(F_1, \dots, F_r)}{\prod_{i=1}^{r} \deg F_i} \frac{q^n}{n^r} + O(Bn \cdot n!^B q^{n-1/2}), \tag{2}$$

provided that $\gcd(q, 2n) = 1$. *Here the O-constant is absolute.*

Theorem 2 confirms Conjecture 1 when q is much larger than n and coprime to $2n$. Its method of proof admits several other applications, e.g., to counting "smooth" specializations of polynomials (cf. [Mar02]) and to investigating the distribution of irreducibles in "short intervals" (cf. [Gal76], and see [Pol09] for details). In my thesis [Pol08a, Chapter 7], the method is also applied to prove the expected asymptotic estimates in various ranges for both a polynomial version of the Goldbach conjecture and for a more general version of the twin prime conjecture than that discussed above.

Much remains to be done. Here are a few possibilities that I intend to explore:

- Fix a prime power q. Is it true that for almost all natural numbers d (asymptotically 100%), every polynomial F of degree d in $\mathbf{F}_q[T]$ admits infinitely many irreducible specializations? (The latter phrase means that $F(A)$ is irreducible for infinitely many $A \in \mathbf{F}_q[T]$.) It is shown in [Polb] that this holds under a technical hypothesis on generalized Wieferich primes.
- In our Theorem 2, we require that $\gcd(q, 2n) = 1$. Bary-Soroker [BS] has shown that this restriction on the characteristic of $\mathbf{F}_q$ can be relaxed. Andreas Bender and I [BP10] are currently exploring the possibility of relaxing this condition also in the analogous theorems concerning the twin prime and Goldbach problems.
- No version of the twin prime conjecture has yet been proved in $\mathbf{F}_2[T]$. Are there infinitely many twin prime pairs of the form $P, P + (T^2 + T)$ there? Effinger [Eff] observes that $P = T^n + T^3 + T^2 + T + 1$ starts such a pair whenever P is irreducible. I plan to investigate generalizations of this curious construction.

Arithmetic functions

Summing divisors. Let $s(n) := \sum_{d \mid n, d<n} d$ denote the sum of the proper divisors of the natural number n. It is possible that the oldest unsolved problems in number theory center around behavior of the iterates of $s(n)$. As an example, call a number n *perfect* if $s(n) = n$ (e.g., $n = 6$). Interest in such numbers traces back to the Pythagorean school five centuries B.C.E. (for some of the history, see [CS97]). Yet we still do not have answers to some of the simplest questions, perhaps the most natural example being

- Are there infinitely many perfect numbers?

Much of my recent work can be seen as attempting to mitigate the embarrassment to modern mathematics presented by such simple-seeming open questions.

The even perfect numbers were completely classified by Euclid and Euler. For various reasons, it is widely conjectured that no odd examples exist. Short of establishing this, one could ask for a proof that there are not too many odd perfect numbers. After a flurry of activity in the 1950s, we know from work of Wirsing [Wir59] that the number of odd perfect $n \leq x$ is bounded by $x^{W/\log\log x}$ for some absolute constant W and all $x > 3$. I find myself constantly returning to the problem of improving this 50-year old bound. Even showing that W can be taken arbitrarily small, for large enough x, appears very challenging.

As shown by Dickson [Dic13], we get a finiteness result for odd perfect numbers if we also restrict the number of distinct prime factors. Explicit versions of Dickson's theorem were given by Pomerance [Pom77], Heath-Brown [HB94], Cook [Coo99], and Nielsen [Nie03]; we now know that if N is an odd perfect number with at most k prime factors, then $N < 2^{4^k}$. My contribution to this subject consists in bounding not the size of such N but the number of such N. In a short note to appear in the *Monthly* [Pol11], I bound this count by 4^{k^2}, via an adaptation of Wirsing's method alluded to above.

The *amicable numbers* are close cousins of the perfect numbers. Pythagoras is alleged to have known that $s(220) = 284$ and $s(284) = 220$. We call $\{220, 284\}$ an *amicable pair*, and both 220 and 284 are called *amicable numbers*. While we know over 12 million amicable pairs, a proof that there are infinitely many seems out of reach. In the opposite direction, Erdős [Erd55] showed in 1955 that the set of amicable numbers has asymptotic density zero. The sharpest known form of this last result is due to Pomerance [Pom81]. Perhaps after 30 years, Pomerance's upper bound is now ripe for improvement; I intend to investigate this.

Both perfect numbers and amicable numbers can be seen as special cases of *sociable numbers*. A number n is called *sociable* if the sequence $n, s(n), s(s(n)), s(s(s(n))), \dots$ is purely periodic; if the period length is k, then n is called *k-sociable*. Thus a perfect number is a 1-sociable number, while an amicable number is 2-sociable. Let $N_k(x)$ be the number of k-sociable numbers $n \le x$. What can be said about $N_k(x)$ when $k \ge 3$? It is implicit in the work of Erdős [Erd76] that $N_k(x)/x \to 0$ for every fixed k. However, if one asks for an explicit upper bound on $N_k(x)$, Erdős's method gives only a very poor result. In [KPP09], we substantially improve his bound for all $k \ge 3$. The case of odd k is somewhat easier, and in [Pol10], I give a further improvement in this case, showing that $N_k(x) \le x/(\log x)^{1+o(1)}$, as $x \to \infty$. I hope to improve this last estimate a tad bit more, and so obtain that the sum of the reciprocals of the k-sociable numbers diverges.

Let $N(x)$ denote the counting function of all sociable numbers, clumping together sociable numbers of every order. Thus,

$$N(x) := N_1(x) + N_2(x) + N_3(x) + \dots.$$

Kobayashi, Pomerance, and I conjecture (op. cit.) that $N(x)/x \to 0$, i.e., that the set of sociable numbers has asymptotic density zero. The following is our main result in this direction; recall that a number n is *abundant* if $s(n) > n$.

Theorem 3. *The set of sociable numbers which are not both odd and abundant has asymptotic density zero.*

It is heartening to note that, as we show in [KPP09], the odd abundant numbers form a set of asymptotic density about 1/500; so our theorem gets us about 99.8% of the way towards our conjecture! We do not know how to show that a positive proportion of odd abundant numbers are nonsociable. A related open problem, perhaps attackable, is to show that the set of nonsociable numbers meets every arithmetic progression.

Let me quickly mention several other problems of this nature near to my heart and brain. First, in [Polc], I take up some questions raised by Erdős [Erd56] concerning the behavior of the function $N(x, A) := \#\{n \le x : \gcd(n, \sigma(n)) > A\}$, which measures how often n and $\sigma(n)$ have a large common factor. Erdős's claims about $N(x, A)$ appear to have gone unnoticed for fifty years; I prove corrected versions of these claims and in some cases get substantially stronger results. In [Pold], I show that the k-sociable numbers still have density zero if the defining condition is relaxed from equality to divisibility (in either direction). In [Polg], I prove that the quasi-amicable numbers (defined in analogy with the amicable numbers but with respect to the function $s^{-}(n) := \sum_{d|n, 1<d<n} d$) have asymptotic density zero. I have also investigated the appearance of sociable numbers within prescribed sequences of arithmetic interest; see [Pole], [LP11], and [Polf].

Totients. Another arithmetic function that has attracted a great deal of interest is Euler's totient function $\phi(n)$, defined as the number of units in the ring $\mathbf{Z}/n\mathbf{Z}$. Here again, there are many unsolved problems. For example: For each m, must the equation $\phi(n) = \phi(m)$ have a solution $n \neq m$? This innocuous-seeming question was posed by Carmichael ([Car07], [Car22]) in the early part of the 20th century.

Many of the strongest results on the distribution of ϕ-values appear in a 1998 study of Ford [For98]. It is proved there that any counterexample m to the above conjecture of Carmichael satisfies $m > 10^{10^{10}}$, and that if $Z(x)$ is any function tending to infinity with x, then almost all (asymptotically 100%) ϕ-values $v \leq x$ have fewer than $Z(x)$ preimages. This last result contrasts with a recent theorem of Luca and myself [LP], that for almost all m, the number of solutions n of $\phi(n) = \phi(m)$ is sandwiched between two fixed positive powers of $(\log m)^{(\log\log m)(\log\log\log m)}$.

In [For98], Ford also determines the precise order of magnitude of $V_\phi(x)$, the number of ϕ-values belonging to $[1, x]$. This nearly settles an old problem of Pillai and Erdős (see [Erd35], [EH73], [EH76], [Pom86], [MP88] for earlier results). The same method shows that if $V_\sigma(x)$ is defined as the number of σ-values $\leq x$, then $V_\phi(x)/V_\sigma(x)$ is bounded between two positive constants for all $x \geq 1$. Define $V_{\phi,\sigma}(x)$ as the number of $n \leq x$ belonging to the image of both ϕ and σ. A fifty-year old problem of Erdős (see, e.g., [Erd59, p. 172]) asks one to show that $V_{\phi,\sigma}(x)$ tends to infinity with x. Since $\phi(p) = p-1$ while $\sigma(p) = p+1$, resolving Erdős's question requires grappling with the multiplicative structure of shifted primes.

Quite recently, Erdős's question was answered in the affirmative by Ford, Luca, and Pomerance [FLP10]; they prove that $V_{\phi,\sigma}(x) \geq \exp((\log\log x)^c)$ for some $c > 0$ and all large x. It is expected that this lower bound is quite far from the truth. For example, if p and $p+2$ are both prime, then $\phi(p+2) = p+1 = \sigma(p)$, and so the quantitative version of the twin prime conjecture previously alluded to would imply immediately that $V_{\phi,\sigma}(x) \gg x/(\log x)^2$ for large x. Ford and I have shown how to deduce [FPa], conditional on a generalization of the twin prime conjecture, the stronger bound

$$V_{\phi,\sigma}(x) \geq x/(\log x)^{1+o(1)} \quad (x \to \infty).$$

Since (as known since Erdős [Erd35]) both $V_\phi(x)$ and $V_\sigma(x)$ themselves have the shape $x/(\log x)^{1+o(1)}$, this suggests that values common to the range of ϕ and σ are more numerous than one might naively expect. Once one is thinking in this direction, it is natural to wonder if perhaps common values make up a positive proportion of all ϕ-values? In recent work [FPb], Ford and I show that this is not the case; as $x \to \infty$,

$$V_{\phi,\sigma}(x) \leq \frac{V_\phi(x)}{(\log\log x)^{1/2+o(1)}}.$$

The proof makes use of the detailed structure theory of ϕ-values developed in [For98]. Ford and I are currently exploring whether this result reflects the true rate of decay of $V_{\phi,\sigma}(x)/V_\phi(x)$, as $x \to \infty$.

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