# A perfect storm: variations on an ancient theme

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# Three types of natural numbers

*Among simple even numbers, some are superabundant, others are deficient: these two classes are as two extremes opposed one to the other; as for those that occupy the middle point between the two, they are said to be perfect.*

– Nicomachus (ca. 100 AD)

Let $s(n) = \sum_{d|n, d<n} d$ be the sum of the proper divisors of $n$.

**Abundant:** $s(n) > n$, e.g., $n = 12$.
**Deficient:** $s(n) < n$, e.g., $n = 5$.
**Perfect:** $s(n) = n$, e.g., $n = 6$.

*The superabundant number is . . . as if an adult animal was formed from too many parts or members, having "ten tongues", as the poet says, and ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands. . . . The deficient number is . . . as if an animal lacked members or natural parts . . . if he does not have a tongue or something like that.*

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*. . . In the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect.*

# From numerology to number theory

**Abundants:** 12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, 102, . . . .

**Deficients:** 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, . . . .

**Perfects:** 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 2658455991569831744654692615953842176, . . . .

Problem: Describe the distribution of each sequence.

# Densities

If $A$ is a subset of $\mathbb{N} = \{1, 2, 3, \dots\}$, define the *density* of $A$ as

$$\lim_{x \to \infty} \frac{\# A \cap [1, x]}{x}.$$

For example, the even numbers have density $1/2$, and the prime numbers have density 0. But the set of natural numbers with first digit 1 does not have a density.

**Question:** Does the set of abundant numbers have a density? What about the deficient numbers? The perfect numbers?

# A theorem of Davenport

### Theorem (Davenport, 1933)

*For each real $u \geq 0$, consider the set*

$$\mathcal{D}_s(u) = \{n : s(n)/n \leq u\}.$$

*This set always possesses an asymptotic density $D_s(u)$. Considered as a function of $u$, the function $D_s$ is continuous and strictly increasing, with $D_s(0) = 0$ and $D_s(\infty) = 1$.*

### Corollary

*The perfect numbers have density $0$, the deficient numbers have density $D_s(1)$, and the abundant numbers have density $1 - D_s(1)$.*

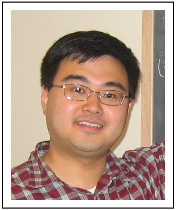
# Numerics

The following theorem improves on earlier work of Behrend, Salié, Wall, and Deléglise:

**Theorem (Kobayashi, 2010)**

*For the density of abundant numbers, we have*

$$0.24761 < 1 - D_s(1) < 0.24765.$$



So just under 1 in every 4 natural numbers is abundant, and just over 3 in 4 are deficient.

## Local distribution of abundant and deficient numbers

On “average”, an interval of length $y$ has about $\delta y$ deficient numbers and about $(1 - \delta)y$ abundants, where $\delta = D_s(1)$. But not every interval is average!

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**Theorem (I. M. Trivial)**

*For $n > 6$, the interval $(n, n+6]$ contains an abundant number.*

**Proof.**

If $n = 6k$ and $k > 1$, then $s(n) \geq 1 + k + 2k + 3k = 6k + 1 > n$.

So there is no gap of length $> 6$ between abundant numbers.

(It can be shown that each gap size $\leq 6$ occurs infinitely often.)

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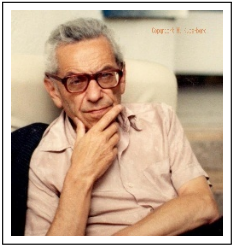
But we can be more precise:



### Theorem (Erdős, 1934)

*Let $G(x)$ be the largest gap $n' - n$ between two consecutive deficient numbers $n < n' \leq x$. There are positive constants $c_1$ and $c_2$ with*

$$c_1 \log\log\log x < G(x) < c_2 \log\log\log x.$$

Theorem (P., 2009)

*Let $G(x)$ be the largest gap $n' - n$ between two consecutive deficient numbers $n < n' \leq x$. As $x \to \infty$, we have*

$$\frac{G(x)}{\log\log\log x} \to C,$$

*where $C \approx 3.5$. In fact,*

$$C = \left( \int_0^1 \frac{D_s(u)}{u+1}\, du \right)^{-1}.$$

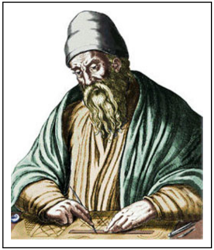
# Looking for perfect numbers

> *Just as ... ugly and vile things abound, so superabundant and deficient numbers are plentiful and can be found without a rule...*

What about perfect numbers?

### Theorem (Euclid)

*If as many numbers as we please beginning from a unit be set out continuously in double proportion, until the sum of all becomes a prime, and if the sum multiplied into the last make some number, the product will be perfect.*



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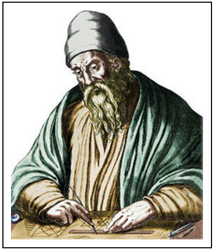

**Theorem (Euclid)**

*If $2^n - 1$ is a prime number, then*

$$N := 2^{n-1}(2^n - 1)$$

*is a perfect number.*

### Theorem (Euler)

*If $N$ is an even perfect number, then $N = 2^{n-1}(2^n - 1)$, where $2^n - 1$ is a prime number.*

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We know 47 primes of the form $2^n - 1$, and so 47 corresponding even perfect numbers, the largest being

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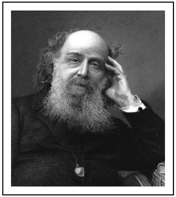
But we don't know if there are infinitely many primes of the form $2^n - 1$. We don't even know if $2^p - 1$ is composite for infinitely many primes $p$.

# The web of conditions



*. . . a prolonged meditation has satisfied me that the existence of [an odd perfect number] - its escape, so to say, from the complex web of conditions which hem it in on all sides - would be little short of a miracle. – J. J. Sylvester*

If $N$ is an odd perfect number, then:

1. $N$ has the form $p^e M^2$, where $p \equiv e \equiv 1 \pmod 4$,
2. $N$ has at least 9 distinct prime factors and at least 75 prime factors counted with multiplicity,
3. $N > 10^{300}$.

Conjecture

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There is probably no simple formula for odd perfect numbers.

### Theorem (Dickson, 1913)

*For each positive integer $k$, there are only finitely many odd perfect numbers $n$ with precisely $k$ distinct prime factors.*

## Theorem (Pomerance, 1977)

*If $n$ is an odd perfect number with $k$ distinct prime factors, then*

$$n < (4k)^{(4k)^{2^{k^2}}}.$$

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### Theorem (P., 2010)

*The number of odd perfect $n$ with $k$ distinct prime factors is at most*

$$4^{k^2}.$$

## Perfect numbers in prescribed sequences

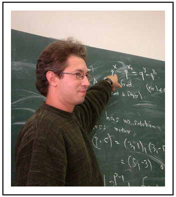
Many problems in number theory fit the following rubric:

*Let $\mathcal{A}$ and $\mathcal{B}$ be sets of natural numbers, each of which has a convenient arithmetic description. Say something "interesting" about $\mathcal{A} \cap \mathcal{B}$.*

Dickson's example is $\mathcal{A} = \{\text{odd perfect numbers}\}$ and $\mathcal{B} = \{n \text{ with } k \text{ prime factors}\}$.

**Theorem (Luca, 2000)**

*Take $\mathcal{A} = \{\text{perfect numbers}\}$ and $\mathcal{B} = \{\text{Fibonacci numbers}\}$. Then $\mathcal{A} \cap \mathcal{B} = \emptyset$.*



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Call a number $N$ *multiply perfect* if $N \mid s(N)$. For example, if $N = 120$, then $s(N) = 240$.

### Theorem (Luca–P., 2010)

*Take* $\mathcal{A} = \{$*multiply perfect numbers*$\}$ *and* $\mathcal{B} = \{$*repdigits in base* $g\}$*. Then* $\mathcal{A} \cap \mathcal{B}$ *is always finite, and if* $g = 10$*, equals* $\{1, 6\}$*.*

## Can we count perfect numbers?

The prototypical theorem in analytic number theory is probably ...

**Theorem**

*Let $\pi(x)$ be the number of prime numbers $p \leq x$. Then*

$$\pi(x) \sim x/\log x \quad \textit{as } x \to \infty.$$

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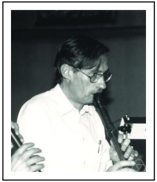
Theorem (Hornfeck–Wirsing, 1957)

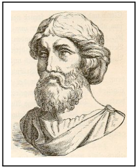
*Let $V_1(x)$ be the number of perfect numbers $n \leq x$. For each fixed $\epsilon > 0$, we have $V_1(x) < x^\epsilon$ for all $x > x_0(\epsilon)$.*



## Messing with perfection

Two natural numbers $n$ and $m$ are said to form an *amicable pair* if $s(n) = m$ and $s(m) = n$. For example,

$$s(220) = 284 \quad \text{and} \quad s(284) = 220.$$



Pythagoras, when asked what a friend was, replied:

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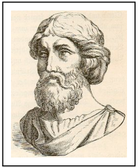

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According to Dickson's *History of the Theory of Numbers*, the 11th century Arab mathematician and astronomer al-Majriti

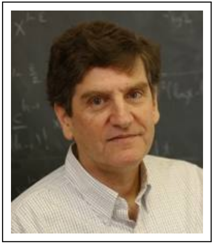
> *had himself put to the test the erotic effect of "giving any one the smaller number 220 to eat, and himself eating the larger number 284."*

## The distribution of amicable numbers

There are over ten million amicable pairs known, but we have no proof that there are infinitely many.

**Theorem (Erdős, 1955)**

*Almost all numbers are not amicable.*



**Theorem (Pomerance, 1981)**

*The number $V_2(x)$ of amicable numbers $n \le x$ satisfies*

$$V_2(x) \le x/\exp((\log x)^{1/3})$$

*for large $x$. In particular, the sum of the reciprocals of the amicable numbers converges.*

## Sociable numbers

Call $n$ a $k$-**sociable number** if $n$ starts a cycle of length $k$. (So perfect corresponds to $k = 1$, amicable to $k = 2$.) For example,

$$2115324 \mapsto 3317740 \mapsto 3649556 \mapsto 2797612 \mapsto 2115324 \mapsto \dots$$

is a sociable 4-cycle. We know 175 cycles of order $> 2$.

Let $V_k(x)$ denote the number of $k$-sociable numbers $n \leq x$.

Theorem (Erdős, 1976)

*Fix $k$. The set of $k$-sociable numbers has asymptotic density zero. In other words, $V_k(x)/x \to 0$ as $x \to \infty$.*

# Counting sociables

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$$V_k(x)/x \leq 1/\overbrace{\log\log\cdots\log}^{3k \text{ times}} x.$$

We (K.-P.-P.) obtain more reasonable bounds.
A further improvement is possible for odd $k$.

**Theorem (P., 2010)**

*Suppose $k$ is odd, and let $\epsilon > 0$. Then*

$$V_k(x) \leq x/(\log x)^{1-\epsilon}$$

*for all large $x$.*

## Counting sociables

What if we count all sociable numbers at once? Put

$$V(x) := V_1(x) + V_2(x) + V_3(x) + \dots$$

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**Theorem (K.–P.–P., 2009)**

$$\limsup V(x)/x \leq 0.0021.$$

**Theorem (K.–P.–P., 2009)**

*The number $V'(x)$ of natural numbers belonging to a cycle contained entirely in $[1, x]$ is $o(x)$. In other words, $V'(x)/x \to 0$.*

Our upper bound in the first theorem is the density of odd abundant numbers (e.g., $n = 945$).

# Another fine mess ...

Let $\sigma(n) := \sum_{d|n} d$ be the usual sum-of-divisors function. Then

$$n \text{ is perfect} \iff \sigma(n) = 2n.$$

Call a number **prime-perfect** if $n$ and $\sigma(n)$ have the same set of distinct prime factors. For example, if $n = 270$, then

$$n = 2 \cdot 3^3 \cdot 5, \quad \textit{and} \quad \sigma(n) = 2^4 \cdot 3^2 \cdot 5,$$

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Theorem (Pomerance–P., 2011)

*There are infinitely many prime-perfect numbers $n$; in fact, for each $k$, there are more than $(\log x)^k$ examples $n \leq x$ once $x$ is large. In the opposite direction, the number of examples up to $x$ is at most $x^{1/3+\epsilon}$ for all large $x$.*

# Bounding $V_3(x)$

Let me sketch the proof of the following result:

**Theorem**

*The number of sociable 3-cycles contained in the interval $[1, x]$ is at most*

$$\frac{x}{(\log x)^{1+o(1)}},$$

*as $x \to \infty$.*

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### Remark

1. Equivalent to $V_3(x) \leq x/(\log x)^{1+o(1)}$.
2. There are **no** known examples of 3-cycles!

# Bounding $V_3(x)$, ctd.

**Observation 1:** A sociable cycle of order $> 1$ must contain a deficient number. In fact, the largest member of the cycle is deficient.

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**Observation 2:** Suppose $n_1, n_2, n_3$ form a sociable 3-cycle, numbered so that

$$s(n_1) = n_2, \quad s(n_2) = n_3, \quad s(n_3) = n_1.$$

Let $d$ be the greatest common divisor of the numbers $\sigma(n_i)$. Let $D = d$ if $d$ is odd, and let $D = d/2$ otherwise. Then $D$ is a common divisor of the $n_i$:

$$D \mid n_1, \quad D \mid n_2, \quad D \mid n_3.$$

Proof of Observation 2.

Since $d \mid \sigma(n_i)$, we have

$$s(n_i) = \sigma(n_i) - n_i \equiv -n_i \pmod{d}.$$

Looking at this for $i = 1, 2, 3$, and using that $s(n_3) = n_1$, we get

$$n_1 \equiv (-1)^3 n_1 \equiv -n_1 \pmod{d}.$$

So $d \mid 2n_1$, and so $D \mid n_1$.
By symmetry, $D \mid n_2$ and $D \mid n_3$.

**Observation 3:** The number $D$ is deficient: If $D$ is nondeficient, all multiples of $D$ are nondeficient, and so all the members of the 3-cycle are nondeficient.

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## Proof idea.

Suppose $n_1$ is squarefree, say $n_1 = p_0 p_1 \cdots p_r$. Then

$$\sigma(n_1) = (p_0 + 1) \cdots (p_r + 1)$$

is divisible by $2^r$. So $2^k \mid \sigma(n_1)$ if $r = \omega(n_1) - 1 > \log\log\log\log x$. Repeat for $n_2$ and $n_3$.

**Observation 5:** $D$ is divisible by $2^{k-1}$, and $2^{k-1}$ is *almost* nondeficient. We have

$$s(2^{k-1}) = 1 + 2 + \cdots + 2^{k-2} = 2^{k-1} - 1.$$

If $q$ is an odd prime with $q < 2^k$, then $2^{k-1}q$ is nondeficient.
Hence, $D$ can't be divisible by any odd prime $q < 2^k$.
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Let

$$\mathcal{Q}_1 := \{q < 2^k : q \nmid \sigma(n_1)\},$$

$$\mathcal{Q}_2 := \{q < 2^k : q \nmid \sigma(n_2)\},$$

$$\mathcal{Q}_3 := \{q < 2^k : q \nmid \sigma(n_3)\}.$$

Then one of $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ is thick: $\sum_{q \in \mathcal{Q}_i} \frac{1}{q} \geq \frac{1}{3} \sum_{q < 2^k} \frac{1}{q}$.

So $\mathcal{Q}_1 := \{q \nmid \sigma(n_1)\}, \mathcal{Q}_2 := \{q \nmid \sigma(n_2)\}, \mathcal{Q}_3 := \{q \nmid \sigma(n_3)\}$. For one of $i = 1, 2, 3$, we have $\sum_{q \in \mathcal{Q}_i} \frac{1}{q} \to \infty$.

### Lemma (Scourfield, Luca–Pomerance)

*Let $\mathcal{Q}$ be a set of odd primes contained in the interval $[1, \log\log x]$. If $x \to \infty$ and $\sum_{q \in \mathcal{Q}} \frac{1}{q} \to \infty$, and we put*

$$\mathcal{E}_Q := \{n \le x : \sigma(n) \textit{ is coprime to all } q \in \mathcal{Q}\},$$

*then $\#\mathcal{E}_Q \le x/(\log x)^{1+o(1)}$.*

There is a thick $\mathcal{Q}$ and an $n$ in our cycle for which $n \in \mathcal{E}_{\mathcal{Q}}$.
So the number of possible cycles is at most

$$\frac{x}{(\log x)^{1+o(1)}}(\#\text{of possible } \mathcal{Q}).$$

But $\mathcal{Q}$ is a subset of $[1, 2^k]$, so $\#$ of possible $\mathcal{Q}$ is $\le 2^{2^k} \le (\log x)^{o(1)}$.

Thank you!