A SIMPLE PROOF OF A THEOREM OF HAJDU–JARDEN–NARKIEWICZ

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Abstract. Let K be an algebraic number field, and let G be a finitely generated subgroup of $K^{\times}$. We give a short proof that for every positive integer n, there is an element of $\mathcal{O}_K$ not expressible as a sum of n elements of G.

1. Introduction

Let K be an algebraic number field. The following theorem was proved independently (and almost simultaneously) by Jarden and Narkiewicz [6] and Hajdu [5].

Theorem 1. *Let K be a number field. Let G be a finitely generated subgroup of $K^{\times}$. For each positive integer t, there is an $\alpha \in \mathcal{O}_K$ not expressible as a sum of t elements of G.*

The proofs in [5] and [6] depend crucially on the modern theory of S-unit equations. It is the purpose of this note to outline an entirely different, very short, and seemingly more elementary proof of Theorem 1.

We let $\lambda(n)$ denote Carmichael's function, defined as the exponent of the group $U(\mathbb{Z}/n\mathbb{Z})$. The following lemma — which seems possibly of some independent interest — is the key ingredient in our proof of Theorem 1.

Lemma 2. *Let $\mathcal{P}$ be a set of primes of positive upper (relative) density. For each $\kappa > 0$, there are infinitely many squarefree natural numbers n which are divisible only by primes in $\mathcal{P}$ and which satisfy $\lambda(n) < n^{\kappa}$.*

If we do not restrict the prime factors of n, then $\lambda(n)$ is occasionally as small as $(\log n)^{O(\log\log\log n)}$, as shown by Erdős–Pomerance–Schmutz [4]. That estimate has been applied in a context similar to the present one by several authors (beginning in work of Ádám, Hajdu, and Luca [1]), but only when $K = \mathbb{Q}$. The upper bound of Lemma 2 on the values of $\lambda(n)$ is weaker than that of [4], but the ability to restrict the support of n facilitates applications to arbitrary number fields.

Without further ado, we show how to deduce Theorem 1 from Lemma 2.

Proof of Theorem 1. Suppose that $\eta_1, \dots, \eta_m$ generate G. Let $\mathcal{P}$ be the set of rational primes that split completely in K and are not below any prime ideal appearing in the factorizations of the η_i. Then $\mathcal{P}$ has positive upper density; in fact, by Landau's prime ideal theorem [7] applied to the Galois closure L (say) of $K/\mathbb{Q}$, the density of $\mathcal{P}$ is $\frac{1}{[L:\mathbb{Q}]}$. So by Lemma 2, there are infinitely many squarefree n composed of primes from $\mathcal{P}$ that satisfy $\lambda(n) < n^{1/mt}$. Since n is a squarefree product of split completely primes, $\mathcal{O}_K/n\mathcal{O}_K \cong (\mathbb{Z}/n\mathbb{Z})^{[K:\mathbb{Q}]}$, and so the

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group $U(\mathcal{O}_K/n\mathcal{O}_K)$ has exponent $\lambda(n)$. By the choice of $\mathcal{P}$, it is sensible to reduce the η_i modulo n, and (with the obvious notation)

$$\#G \bmod n\mathcal{O}_K \leq \lambda(n)^m < n^{1/t}.$$

Hence, any sum of t elements of G falls into one of $< (n^{1/t})^t = n$ residue classes mod $n\mathcal{O}_K$. But $\#\mathcal{O}_K/n\mathcal{O}_K = n^{[K:\mathbb{Q}]} \geq n$. So the set of elements of $\mathcal{O}_K$ that cannot be written as a sum of t elements of G includes an entire residue class modulo $n\mathcal{O}_K$, and in particular is nonempty! □

2. Proof of Lemma 2

The proof of Lemma 2 rests on the following simple consequence of Brun's sieve first noticed by Erdős [3].

Lemma 3. *Let $\delta > 0$. There is an $\epsilon > 0$ such that, for all $X > X_0(\delta, \epsilon)$,*

$$\#\{\text{primes } p \leq X : p-1 \text{ has a prime factor} > X^{1-\epsilon}\} < \delta \frac{X}{\log X}.$$

Proof (sketch). In fact, if $\epsilon > 0$ is fixed, Erdős's arguments show that for all $X > X_0(\epsilon)$,

$$\#\{\text{primes } p \leq X : p-1 \text{ has a prime factor} > X^{1-\epsilon}\} \leq C\epsilon \frac{X}{\log X},$$

where C is an absolute constant. (See p. 213 of [3]. A reference with notation more similar to that used here is [2]; see the second display on p. 192.) So we may choose any $\epsilon < \delta/C$. □

Proof of Lemma 2. By assumption, there is a constant $\delta > 0$ and a sequence of X tending to infinity with $\#\{p \in \mathcal{P} : p \leq X\} > \delta \frac{X}{\log X}$. If ϵ is fixed sufficiently small in terms of δ, then for all large enough X in our sequence,

$$\#\{p \in \mathcal{P} : p \leq X, \text{ all prime factors } \ell \text{ of } p-1 \text{ are } \leq X^{1-\epsilon}\} > \frac{\delta}{2} \frac{X}{\log X}.$$

For these X, we set

$$n = \prod_{\substack{p \in \mathcal{P} \cap [\frac{\delta}{8}X, X] \\ \ell \mid p-1 \Rightarrow \ell \leq X^{1-\epsilon}}} p.$$

Assuming X is large, the total number of primes up to $\frac{\delta}{8}X$ is smaller than $\frac{\delta}{4}X/\log X$, by the prime number theorem. Hence, the number of prime factors of n is at least $\frac{\delta}{4}\frac{X}{\log X}$, and

$$n \geq \left(\frac{\delta}{8}X\right)^{\frac{\delta}{4}\frac{X}{\log X}} > \exp\left(\frac{\delta}{8}X\right),$$

once X is large enough. We now turn attention to $\lambda(n)$. Since $\lambda(n) = \mathrm{lcm}_{p \mid n}[p-1]$, each prime power divisor of $\lambda(n)$ is smaller than X. Moreover, if ℓ divides $\lambda(n)$, then $\ell \leq X^{1-\epsilon}$. Thus, there are (very crudely) no more than $X^{1-\epsilon}$ such primes ℓ. It follows that

$$\lambda(n) < X^{X^{1-\epsilon}} = \exp(X^{1-\epsilon} \log X).$$

Comparing this upper bound for $\lambda(n)$ with the displayed lower bound for n, it is clear that $\lambda(n) < n^{\kappa}$ once X is sufficiently large. (In fact, $\lambda(n) < \exp((\log n)^{1-\frac{1}{2}\epsilon})$.) □

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