MATH 4400/6400 – Homework #5

posted April 4, 2023; due April 10, by midnight

Any fool can know. The point is to understand.
– Albert Einstein

Directions. Give complete solutions, providing full justifications when appropriate. Your assignment must be stapled if it goes on beyond one page.

MATH 4400 problems

1. Let f, g be multiplicative functions. Define a new arithmetic function h by

$$h(n) = \sum_{d \mid n} f(d) g(n/d) \qquad \text{(for all positive integers } n\text{)}.$$

   Show that h is multiplicative.

2. Let a, b, c be positive integers. Show that $\gcd(ca, cb) = c \gcd(a, b)$.

   You don't need anything about multiplicative functions for this; this could have been a problem on HW #1.

3. Let n be a positive integer. Reduce each of the fractions $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}$ to lowest terms. Show that that for every positive integer d dividing n, there are exactly $\varphi(d)$ reduced fractions with denominator d.

   Hint. First, show that the number of reduced fractions with denominator d is the same as the number of integers $1 \le m \le n$ with $\gcd(m, n) = n/d$. Then find a way to apply Exercise 2.

4. Prove that for all positive integers n,

$$\sum_{e \mid n} \tau(e)^3 = \left(\sum_{e \mid n} \tau(e) \right)^2 .$$

   Hint. It suffices to show that the left and right-hand sides agree whenever $n = p^e$, with p^e a prime power. (Why?) You may assume the formula $\sum_{k=1}^{m} k^3 = (m(m+1)/2)^2$, which could be proved by induction.

5. Recall that Euler's φ-function is defined by

$$\varphi(n) = \#\{m : 1 \le m \le n \text{ and } \gcd(m, n) = 1\}.$$

   We showed in class that $\varphi(n)$ is multiplicative.

   Prove: $\varphi(n)\sigma(n) \le n^2$ for all n.

6. Define $\tau_k(n) = \#\{(d_1, \ldots, d_k) \in (\mathbf{Z}^+)^k : d_1 \cdots d_k = n\}$. Find a formula for $\tau_3(n)$ in terms of the prime factorization of n.

7. (a) Classify all n for which $\varphi(n)$ is an odd number. Justify your answer.

   (b) Classify all n for which $\tau(n)$ is an odd number. Justify your answer.

   (c) Classify all n for which $\sigma(n)$ is an odd number. Justify your answer.

8. Find and prove simple formulas for each of the functions

$$\sum_{e|n} \mu(e)\tau(n/e), \qquad \sum_{e|n} \mu(e)\tau(e), \qquad \sum_{e|n} \mu(e)^2\varphi(e).$$

For the second and third sums, express your answers in terms of the prime factorization of n.

9. A number n is called **perfect** if $\sigma(n) = 2n$. What (if anything) is wrong with the following "proof" that all perfect numbers are even?

   If n is a perfect number, then $\sigma(n) = 2n$. In other words, $2n = \sum_{d|n} d$. So if f and g are the arithmetic functions defined by $g(n) = 2n$ and $f(n) = n$, then $g(n) = \sum_{d|n} f(d)$. By Möbius inversion,

   $$n = f(n) = \sum_{d|n} \mu(n/d)g(d) = \sum_{d|n} \mu(n/d) \cdot (2d) = 2\left(\sum_{d|n} \mu(n/d)d\right).$$

   The final parenthesized expression is an integer, and so n is even.

MATH 6400 problems

10. (*) (Euler) Prove that if n is odd and $\sigma(n)$ is twice an odd number, then $n = p^\alpha m^2$ for some prime p and some positive integers α and m, where $p \nmid m$. Moreover, $p \equiv \alpha \equiv 1 \pmod 4$.

11. (*) Prove $\varphi(n)\sigma(n) \geq \frac{1}{2}n^2$ for all n.