the multinomial theorem gives us that

$$\sum_t \frac{h(t)}{t} \leqq \exp\big(O_k\big(\log_3 x\sqrt{\log_2 x}\,\big)\big)\sum_{j\leqq I}\big(i^2(i-N)^{1-N/i}\big)^j\frac{S^j}{j!}. \tag{38}$$

For large $x$, we have $I < i^2 S \leqq i^2 S(i-N)^{1-N/i}$, and so by Lemma 6,

$$\begin{aligned}\sum_{j\leqq I}\big(i^2(i-N)^{1-N/i}\big)^j\frac{S^j}{j!} &\leqq \left(\frac{ei^2(i-N)^{1-N/i}S}{I}\right)^I \\ &\leqq \exp\big(O_k\big(\sqrt{\log_2 x}\,\big)\big)\big(ei(i-N)^{1-N/i}\big)^I \\ &= (\log x)^{(\alpha^{i-1}-\alpha^i)(i+i\log i)+(i-N)\log(i-N)(\alpha^{i-1}-\alpha^i)+o_k(1)}.\end{aligned} \tag{39}$$

From (37), (38), and (39), we deduce that

$$\sum_{\substack{\tau_i\in\mathfrak{T}_i\\ \sigma_i\tau_i\in\mathfrak{S}_{i-1}}}\frac{1}{t_i} \tag{40}$$

$$\leqq (\log x)^{(\alpha^{i-1}-\alpha^i)(i+i\log i)-2\alpha^{i-1}+(i-N)\log(i-N)\left(\alpha^{i-1}-\alpha^i\right)-\chi(i-1)\alpha^{i-1}+o_k(1)};$$

we use here that there are only finitely many possibilities for $\mathcal{J}$, so that we can drop the superscript $(\mathcal{J})$ on the sum.

By (30), (31), and (40), we see that

$$\#\mathfrak{S}_0 \leqq (\log x)^{o_k(1)}\frac{x}{(\log x)^{2+\alpha-\sum_{i=2}^k\left((\alpha^{i-1}-\alpha^i)(i+i\log i)-2\alpha^{i-1}\right)}}$$

$$\times(\log x)^{\sum_{i=2}^k\left(i-\#\mathcal{I}\cap[0,i-1]\right)\log\left(i-\#\mathcal{I}\cap[0,i-1]\right)\left(\alpha^{i-1}-\alpha^i\right)-\sum_{i=0}^{k-1}\chi(i)\alpha^i}\times\sum_{\sigma_k\in\mathfrak{S}_k}\frac{1}{s_k},$$

with the convention that $0\log 0 = 0$. From (iv) in the definitions of $\mathscr{B}_\phi$ and $\mathscr{B}_\sigma$, the final sum is $6^{-k}\leqq 1$. Also,

$$2+\alpha-\sum_{i=2}^k\left(\left(\alpha^{i-1}-\alpha^i\right)(i+i\log i)-2\alpha^{i-1}\right)$$

$$=2-\sum_{i=1}^{k-1}a_i\alpha^i+(k\log k+k)\alpha^k = 2-F(\alpha)+O\big((k\log k)\alpha^k\big).$$