MATH 8440 – Assignment #3

last updated February 10, 2023 (open)

Turn in <u>three</u> problems.

1. Recall that if $\mathcal{A}$ is a set of positive integers, the asymptotic density of $\mathcal{A}$ is the limit (if it exists) of $\frac{1}{x}\#\{n \le x : n \in \mathcal{A}\}$, as $x \to \infty$.

   Show that if $\mathcal{A}$ is a set of positive integers for which $\sum_{a\in\mathcal{A}} 1/a$ converges, then $\mathcal{A}$ has asymptotic density 0.

   Hint. One can write $\#\{n \le x : n \in \mathcal{A}\} = \sum_n 1_{n\in\mathcal{A}} 1_{n\le x}$. The majorization $1_{n \le x} \le \frac{x}{n}$ may be useful.

2. If $\mathcal{A}$ is a set of positive integers, the logarithmic density of $\mathcal{A}$ is the limit, if it exists, of $\frac{1}{\log x}\sum_{n\le x,\ n\in\mathcal{A}} 1/n$, as $x \to \infty$.

   Show that if $\mathcal{A}$ is a set of positive integers which has an asymptotic density, then it has a logarithmic density, and the logarithmic density is equal to the asymptotic density.

3. Let $\mathcal{A}$ be the set of positive integers whose leading (leftmost) digit is 1, in base 10. Show that $\mathcal{A}$ has logarithmic density $\frac{\log 2}{\log 10}$.

4. From arguments in class, we have that for all large real numbers x, and all integers $k = 1, 2, 3, \dots$,
$$\sum_{p\le x}\frac{1}{p} \le \left(k!(1+\log(x^k))\right)^{1/k}.$$
   (In fact, if you trace our estimation of the partial sums of the harmonic series, $x \ge 1$ is large enough.) By making a suitable choice of k, prove that as $x \to \infty$,
$$\sum_{p\le x}\frac{1}{p} \le \log\log x + O(\log\log\log x).$$
   You may assume as known that $k! = \sqrt{2\pi k}(k/e)^k(1 + O(1/k))$, which is the form of Stirling's formula shown in class (modulo the determination of the constant $\sqrt{2\pi}$, which was done on homework).

5. Let $\mathcal{N}$ be a set of positive integers. Suppose that there is a positive constant κ such that, as $x \to \infty$,
$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}}\frac{\log n}{n} = \kappa \log x + O(1). \qquad (*)$$
   (We saw in class that if $\mathcal{N}$ is the set of primes, then $\kappa = 1$ works.) Show that
$$\liminf_{x\to\infty}\frac{\#\{n \le x : n \in \mathcal{N}\}}{x/\log x} > 0 \quad\text{and}\quad \limsup_{x\to\infty}\frac{\#\{n \le x : n \in \mathcal{N}\}}{x/\log x} < \infty.$$
   Hint for the lim inf. Define $S(x) = \sum_{n\le x,\ n\in\mathcal{N}} \log n/n$. Fix a constant $K > 1$. Show that (for large x) $S(x) - S(x/K) \le \frac{\log(x/K)}{x/K}\#\{n \le x : n \in \mathcal{N}\}$. Conclude by picking K suitably large and using the assumption (*).

6. (a) Assume the prime number theorem in the form $\pi(x) \sim x/\log x$ as $x \to \infty$. Fix a positive real number A. Prove that $\pi(Ax)/\pi(x) \to A$ as $x \to \infty$. Deduce that for each fixed $\epsilon > 0$, there is a prime in the interval $(x, (1+\epsilon)x]$ for all large x.

(b) Assume the prime number theorem in the strong form stated in class: For each fixed integer $k \geq 2$, and all $x \geq 3$, we have $\pi(x) = \mathrm{Li}(x) + O_k(x/(\log x)^k)$. Find (and prove) an asymptotic formula for $2\pi(x) - \pi(2x)$. (That is, find a simple-looking smooth function $H(x)$ such that $2\pi(x) - \pi(2x) \sim H(x)$, as $x \to \infty$.)

7. Supply plausible values of positive constants C and K for which

$$\#\{n \leq x : n^2 + 1 \text{ prime}\} \sim Cx/(\log x)^K, \quad \text{as } x \to \infty.$$

Explain your reasoning. To show that the constant C you suggest 'makes sense', you may assume that

$$\sum_{\substack{\text{odd } p \leq x \\ -1 = \square \bmod p}} \frac{1}{p} - \frac{1}{2} \log\log x$$

tends to a limit, where the sum on the left runs over odd primes p for which -1 is congruent to a square mod p. (This last estimate can be viewed as a consequence of the Chebotarev density theorem, applied to the extension $\mathbb{Q}(i)/\mathbb{Q}$.)