MATH 8440 – Assignment #8

last updated May 3, 2023

These problems are for amusement only; this assignment is <u>not</u> to be handed in.

1. Let $d_k(n)$ be the number of ways to write n as an ordered product of positive integers (so $d(n) = d_2(n)$). Use Perron's formula to prove that for each fixed integer $k \geq 2$, and each fixed $\epsilon > 0$,
$$\sum_{n\leq x} d_k(n) = xP_k(\log x) + O(x^{1-\frac{1}{k+1}+\epsilon}),$$
where $P_k(T) \in \mathbf{R}[T]$ has degree $k-1$ and leading coefficient $\frac{1}{(k-1)!}$.

2. Recall that in a previous homework, you showed that $\sum_{n\leq x} |\mu(n)| = \frac{1}{\zeta(2)}x + O(x^{1/2})$. As a consequence, the set of squarefree numbers has asymptotic density $\frac{6}{\pi^2}$. Using Perron's formula and the bounds on $\zeta(s)$ shown in class (and HW), prove that
$$\sum_{n\leq x} \mu(n) = O(x \exp(-(\log x)^{1/15}))$$
for all large x. Deduce that the set of squarefree n with an even number of prime factors has density $\frac{1}{2} \cdot \frac{1}{\zeta(2)}$ and the same for the set of squarefree n with an odd number of prime factors.

3. Recall that $\Psi(x, y) = \sum_{n\leq x,\ P(n)\leq y} 1$, where $P(n)$ denotes the largest prime factor of n subject to the convention that $P(1) = 1$. In this exercise we outline a proof that if $x \geq 100$ and $x \geq y \geq (\log x)^3$, then
$$\Psi(x, y) \ll x(\log y) \exp(-u \log u + O(u \log\log(3u))),$$
where as usual $u = \frac{\log x}{\log y}$. This kind of uniform upper bound is very useful in applications. Here and below, all implied constants are intended as absolute.

   (a) Show that for any choice of ρ with $0 < \rho < 1$, we have
   $$\Psi(x, y) \leq x^{1-\rho} \sum_{n:\ P(n)\leq y} \frac{1}{n^{1-\rho}}.$$

   (b) Now choose $\rho = \frac{\log u}{\log y}$, so that $x^{1-\rho} = x \exp(-u \log u)$. Show that in our range of x and y, we have $\rho \leq \frac{1}{3}$ and
   $$\sum_{n:\ P(n)\leq y} \frac{1}{n^{1-\rho}} \ll \exp\left(\sum_{p\leq y} \frac{1}{p^{1-\rho}}\right).$$

   (c) Show that $\sum_{p\leq y} \frac{1}{p^{1-\rho}} \leq \log\log y + O(1) + \sum_{p\leq y} \frac{\exp(\rho \log p)-1}{p}$.

   (d) By splitting the sum at $2^{1/\rho}$, show that $\sum_{p\leq y} \frac{\exp(\rho \log p)-1}{p} \ll u \log\log(3u)$.

   (e) Conclude!

With more effort, one can remove the $\log y$ factor in the upper bound. With still more effort, one can show that $\Psi(x, y) = x \exp(-u \log u + O(u \log\log(3u)))$ in the range of x and y indicated above.