MATH 3220 practice problems
Inequalities

Acknowledgements

This worksheet borrows from Larson's book and the text of Gelca and Andreescu. You may turn in any problem for credit. Especially interesting problems (to me!) are marked with an $*$.

Helpful results to keep in mind:

- **Squares of real numbers are nonnegative:** If x is any real number, then x^2 is nonnegative. As a consequence, any sum of squares of real numbers is also nonnegative. So if there is a complicated expression that you want to show is nonnegative, one way to prove this is to rewrite it as a sum of squares.

- **The AM-GM inequality:** Let $x_1, x_2, \ldots, x_n$ be any nonnegative real numbers. The **arithmetic mean** of these numbers is just the usual average, $\frac{x_1+\cdots+x_n}{n}$. The **geometric mean** is the nth root of their product, $\sqrt[n]{x_1 \cdots x_n}$. The AM-GM inequality says that the arithmetic mean is **always** at least as large as the geometric mean, i.e.,
$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}.$$
Equality holds if and only if all of the x_i are equal.

- **Cauchy–Schwarz inequality:** If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, then
$$|x_1y_1 + \cdots + x_ny_n| \leq \sqrt{x_1^2 + \cdots + x_n^2}\sqrt{y_1^2 + \cdots + y_n^2}.$$
In multivariable calculus, this says that the absolute value of the dot product of two vectors is never larger than the product of their lengths. The inequality itself has many applications outside of calculus.

- **One-variable calculus:** Oftentimes you can prove that an inequality is true by rephrasing it as saying that a certain function is always greater than 0. And you already learned in Calculus I some methods for proving claims of that type.

  For example, if $f(a) \geq 0$ and $f'(t) \geq 0$ for all t in the closed interval $[a, b]$, then $f(t) \geq 0$ for all $t \in [a, b]$.

Problems

Basic algebra OR sums of squares

1. Let $x_1, x_2, \ldots, x_n$ be real numbers in $[0, 1]$. Prove that
$$(1 - x_1)(1 - x_2) \cdots (1 - x_n) \geq 1 - (x_1 + x_2 + \cdots + x_n).$$
Hint: Try induction on n.

2. Find the minimum value of the function $f(x,y,z) = x^z + y^z - (xy)^{z/4}$ for $x, y, z > 0$. Give an example of a point (x,y,z) in $\mathbb{R}^3$ where this minimum value is attained.

   Hint: One can rewrite

$$f(x,y,z) = (x^{z/2} - y^{z/2})^2 + 2((xy)^{z/4} - \frac{1}{4})^2 - \frac{1}{8}.$$

   Now use the fact that sums of squares are nonnegative.

3. (*) If a and b are real numbers, what is the smallest that

$$\max\{a^2 + b, b^2 + a\}$$

   can be? Justify your answer.

   Example: If $a = 1$ and $b = -1$, this maximum is 2. But that's not the smallest possibility. For instance, if $a = b = 0$, the maximum is 0. Your task is to find the smallest possible maximum for any choice of a and b.

   Hint: Add $a^2 + b$ and $b^2 + a$ and try to "complete the square", in order to use that sums of squares are nonnegative.

4. Let $f(x) = 2 - \cos(x)^2 - x\sin(x)$. Show that $f(x) > 0$ for $0 < x < \frac{\pi}{2}$.

   Hint: Begin by noticing that $\cos(x)^2 = 1 - \sin(x)^2$, and use this to rewrite $f(x)$ entirely in terms of $\sin(x)$ and x.

5. (*) Prove that for all real numbers x, we have

$$2^x + 3^x - 4^x + 6^x - 9^x \leq 1.$$

   Hint: Begin by rewriting the left-hand side in terms of the new variables A and B, where $A = 2^x$ and $B = 3^x$.

Arithmetic-geometric mean inequality, and related

6. Let $x_1, x_2, \ldots, x_n$ be any real numbers, where $n \geq 2$. Prove that

$$x_1^2 + \cdots + x_n^2 \geq \frac{2}{n-1} \sum_{1 \leq i < j \leq n} x_i x_j.$$

   Hint: First show that $x_i^2 + x_j^2 \geq 2x_i x_j$ for each pair of i and j. What happens if you add all these inequalities? Maybe do an example with $n = 3$ first to see what's going on.

7. Use AM-GM to prove that $n! < \left(\frac{n+1}{2}\right)^n$.

   Hint: $n!^{1/n}$ is the geometric mean of $1, 2, 3, \ldots, n$. What is the arithmetic mean?

8. (*) Show that if a, b, and c are the three sides of a triangle, then

$$3(ab + bc + ac) \leq (a + b + c)^2 \leq 4(ab + bc + ca).$$

Hint: It will help to multiply out $(a + b + c)^2$ first. It will also help to remember that in any triangle, each side length is smaller than the sum of the other two sides.

9. A rectangular box (parallellepiped) has edge lengths a, b, and c, so that its surface area and volume are given respectively by

$$A = 2(ab + bc + ca), \quad \text{and} \quad V = abc.$$

Show that the minimum surface area of a box with volume $V = 1$ is $A = 6$. (This can be considered a warm-up for the next problem.)

Hint: What is the arithmetic mean of ab, bc, and ca? What is their geometric mean?

10. (*) If a, b, and c are positive real numbers with $(1+a)(1+b)(1+c) = 8$, show that

$$abc \leq 1.$$

11. Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that

$$s_n > n((n + 1)^{1/n} - 1).$$

Hint: Apply AM-GM to the $n + 1$ real numbers $1 + \frac{1}{1}$, $1 + \frac{1}{2}$, $\ldots$, $1 + \frac{1}{n}$.

12. (*) Find the minimum value of the expression

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

Cauchy–Schwarz

13. Show that if you have n positive real numbers $a_1, a_2, \ldots, a_n$, then

$$a_1 + \cdots + a_n \leq \sqrt{n}(a_1^2 + \cdots + a_n^2)^{1/2}.$$

Deduce that

$$a_1^2 + \cdots + a_n^2 \geq \frac{1}{n}(a_1 + \cdots + a_n)^2.$$

Hint: $a_1 + \cdots + a_n = a_1 \cdot 1 + \cdots + a_n \cdot 1$. So you can apply Cauchy–Schwarz with $\mathbf{x} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{y} = (1, 1, \ldots, 1)$.

14. If x, y, and z are three positive real numbers with $xyz = 1$, prove that

$$x^2 + y^2 + z^2 \geq x + y + z.$$

Hint: First use Cauchy-Schwarz to show $x^2 + y^2 + z^2 \geq \frac{1}{3}(x + y + z)^2$, then use AM-GM to bound $x + y + z$ from below in terms of xyz.

15. (*) If a, b, c, and d are positive real numbers with $a+b+c+d+e=1$ and $a^2+b^2+c^2+d^2+e^2=2$, what is the largest that e can be? Give an example of a, b, c, d, e where your maximum value is attained.

16. (*) If $a_1+\cdots+a_n=n$ in positive real numbers a_i, prove that $a_1^4+\cdots+a_n^4 \geq n$.

Calculus

17. Show that if a and b are positive real numbers and $0 \leq p \leq 1$, then

$$|a+b|^p \leq |a|^p + |b|^p.$$

Show that the inequality is reversed if $p > 1$.

Hint: Assuming that $0 \leq p \leq 1$, use calculus to prove that $(1+x)^p \leq 1+x^p$ for all $x > 0$. If $p > 1$, show the reverse inequality.

18. (*) Let $0 < a < 1$. Prove that

$$(1+x)^a \leq 1+ax \quad \text{for all} \quad x \geq -1.$$

This is known as **Bernoulli's inequality**.

19. Show that if $x > 0$, then

$$\log\left(1+\frac{1}{x}\right) > \frac{1}{1+x}.$$

Hint: Let $f(x)$ be the function defined by the difference of the left and right-hand sides. Use calculus to show that $f'(x) < 0$ and that $\lim_{x\to\infty} f(x) = 0$. Why does this imply that $f(x) > 0$ for all $x > 0$?

20. Show that $n! > (n/e)^n$ for every positive integer n.

Hint: Write down the Taylor series for $f(x) = e^x$ evaluated at $x = n$.

21. (A classic) Without computing approximations to either number, decide which is larger, e^π or π^e.

Hint: This is a question about the maximum value of $x^{1/x}$ for $x > 0$, in disguise.

22. (*) Show that $(n-1)! < e(n/e)^n$ for every positive integer n.

Hint: When $n \geq 2$, we have $\log(n-1)! = \log(1)+\log(2)+\log(3)+\cdots+\log(n-1)$. Do you remember how the integral test was proved in Calculus II? If so, use the same idea to compare the sum $\sum_{1\leq k<n} \log k$ to $\int_1^n \log t\, dt$.