MATH 4400/6400 – Homework #1

posted January 14, 2019; due Jan. 23, 2019

God made the integers, all else is the work of man.
– L. Kronecker

Directions. Give complete solutions, providing full justifications when appropriate. Your assignment must be stapled if it goes on beyond one page. Starred problems are required for MATH 6400 students and extra credit for 4400 students.

1. (a) Show that if m is an odd positive integer, then

$$x^m + 1 = (x+1)(x^{m-1} - x^{m-2} + \cdots + 1).$$

   (Is it important that m is odd?)

   (b) Prove that if n is a positive integer for which $2^n + 1$ is prime, then $n = 2^k$ for some nonnegative integer k.

   (c) Show that if n is a positive integer for which $2^n - 1$ is prime, then n is prime.

2. Fix a positive integer k. Show that for all positive integers n,

$$n \text{ is a } k\text{th power} \iff k \mid v_p(n) \text{ for every prime } p.$$

   Here a "kth power" means a number of the form m^k, where $m \in \mathbf{Z}$.

3. Fix $k \in \mathbf{Z}^+$. Suppose that $u \cdot v$ is a kth power, where u, v are relatively prime positive integers. Show that both u and v are kth powers.

4. Show that the product of any two consecutive positive integers is never a square. Then show the same for the product of any three consecutive positive integers and the product of any four consecutive positive integers.

   Much more is true: It is a deep theorem, due to Paul Erdős and John Selfridge, that the product of at least two consecutive integers is never a kth power, for any $k \geq 2$.

5. Define a sequence of prime numbers recursively as follows: Let $p_1 = 2$, and for each positive integer n, let p_{n+1} be the largest prime dividing $p_1 \cdots p_n + 1$.

   (a) Show that the p_i are all distinct. (Hence, there are infinitely many primes!)

   (b) Compute (perhaps with the aid of a calculator or computer) the first five terms $p_1, \ldots, p_5$ of this sequence.

   (c) Prove or disprove: The prime number 5 appears somewhere in the list $p_1, p_2, \ldots$.

6. Let a, b be positive integers, and let p be a prime number. Prove that

$$v_p(a+b) \geq \min\{v_p(a), v_p(b)\},$$

   and that equality holds here if $v_p(a) \neq v_p(b)$.

7. (a) Show that if $a, b \in \mathbf{Z}^+$, then $\mathrm{lcm}[a,b] = \frac{ab}{\gcd(a,b)}$.

   (b) Find, with proof, all pairs of positive integers a and b that satisfy $a + b = 85$ and $\mathrm{lcm}[a,b] = 546$.

8. (Goldbach) The nth **Fermat number** is defined by $F_n := 2^{2^n} + 1$ (compare with Problem 1(b) above). For example, $F_3 = 257$. These numbers were discussed by Fermat, who mistakenly claimed that all of the F_n are prime. In fact, as pointed out by Euler, $F_5 = 641 \cdot 6700417$.

(a) Show that for any pair of distinct nonnegative integers i, j, we have $\gcd(F_i, F_j) = 1$.

(b) Using the result of (a), give another proof that there are infinitely many primes.

9. (Euclid) Suppose that n is a positive integer for which $2^n - 1$ is prime. (Note that by Problem 1(c) above, n itself must be prime.) Let $N = (2^n - 1)2^{n-1}$. Show that the sum of the positive divisors of N is precisely $2N$. (That is, N is a *perfect number*.)

10. (*) Our definitions of the lcm and gcd make sense for any finite collection of positive integers, not merely pairs. Prove that for all positive integers a, b, and c, we have

$$\operatorname{lcm}[a, b, c] = \frac{abc \cdot \gcd(a, b, c)}{\gcd(a, b)\gcd(b, c)\gcd(c, a)}.$$

11. (*) The nth harmonic number is defined by

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Show that the only n for which H_n is an integer is $n = 1$.

Hint: Look at the highest power of 2 dividing the denominators on the right-hand side.