Math 4000/6000 – Learning objectives to meet for Exam #2

The exam will cover §2.2–§3.2 of the textbook (everything covered through the class of 10/16, excepting section 2.5).

What to be able to state

Basic definitions

You should be able to give precise descriptions of all of the following:

- definition of $\mathbf{R}$ (a complete ordered field satisfying the conclusion of the monotone convergence theorem)
- the construction of $\mathbf{C}$ from $\mathbf{R}$ by ordered pairs
- complex conjugate of a complex number
- absolute value of a complex number
- polar form of a complex number
- definition of the ring $R[x]$ (starting with a commutative ring R) and allied concepts (such as the degree of a polynomial)
- definition of an irreducible polynomial in $F[x]$ (with F a field)
- gcd of two elements of $F[x]$
- subring of a ring
- subfield and field extension
- definition of $F[\alpha]$, where F is a field and α is an element of a field extension of F
- what it means to say a polynomial $f(x) \in F[x]$ splits in an extension K of F
- what it means to say a field K is a splitting field of $f(x)$ over F

Big theorems

Give full statements of each of the following results, making sure to indicate all necessary hypotheses. For results proved in class, describe the components and main ideas of the proof.

- theorems associated with multiplication of complex numbers in polar form, including de Moivre's theorem
- there are n distinct nth roots of 1 in $\mathbf{C}$, namely the numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$
- a nonzero complex number has exactly n distinct nth roots
- the quadratic formula
- description of the roots of a cubic polynomial
- $\deg(a(x)b(x)) = \deg a(x) + \deg b(x)$ if $a(x), b(x) \in R[x]$ and R is a domain
- the division algorithm in $F[x]$

- remainder theorem and the root-factor theorem
- if $f(x) \in F[x]$ has degree 2 or 3, then $f(x)$ is irreducible over F if and only if $f(x)$ has no root in F
- Euclid's lemma for $F[x]$ and the unique factorization theorem for $F[x]$
- the Fundamental Theorem of Algebra (*statement only!*)
- If K is a field extension of F and $\alpha \in K$, then $F[\alpha]$ is a field if α is the root of a nonconstant polynomial in $F[x]$
- If K is a field extension of F and $\alpha \in K$ is the root of an irreducible polynomial in $F[x]$ of degree n, then every element of $F[\alpha]$ can be expressed in the form $a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}$, where $a_0, a_1, \ldots, a_{n-1} \in F$. Moreover, this representation is unique.

What to be able to compute

You are expected to know how to use the methods described in class to solve the following problems.

- Basic computations with complex numbers, in either rectangular (i.e., $a + bi$) or polar form, including finding nth roots of complex numbers
- Perform "long division" of polynomials with quotient and remainder; use this to perform the Euclidean algorithm, compute gcds, and express the gcd as a linear combination
- Explicitly compute (exactly) the complex roots of given quadratic or cubic polynomials
- Find inverses of elements in $F[\alpha]$, if α is specified as the root of a given irreducible polynomial over F

Extra problems

Carefully review the HW solutions. I also recommend looking at the following problems:

§2.2: 8, 12

§2.3: 2, 7, 20, 21(a), 22

§3.1: 1, 2(b,c,d,e), 5, 9(c), 10(b,d), 11, 12, 14

§3.2: 1, 7, 13

Here is another problem to try:

> Let K be a field extension of F. Prove that if $\alpha \in K$ and α is **not** the root of a nonconstant polynomial in $F[x]$, then $F[\alpha]$ is not a field. In fact, show that α has no inverse in $F[\alpha]$.