MATH 4000/6000 – Learning objectives to meet for Exam #2

The exam will cover §2.2–§3.3, excluding §2.5. You will not be tested on finding roots of cubic polynomials. Moreover, the emphasis will be on material from Chapter 3 (cf. the practice problems).

What to be able to state

Basic definitions

You should be able to give precise descriptions of all of the following:

- ordered field
- the construction of $\mathbf{C}$ from $\mathbf{R}$ by ordered pairs, including operations on $\mathbf{C}$
- complex conjugate of a complex number
- absolute value of a complex number
- polar form of a complex number
- definition of the ring $R[x]$ (starting with a commutative ring R) and allied concepts (such as the degree of a polynomial)
- definition of an irreducible polynomial in $F[x]$ (with F a field)
- gcd of two elements of $F[x]$
- subring of a ring, subfield of a field
- definition of $F[\alpha]$, where F is a field and α is an element of a field extension of F
- definition of $F[\alpha_1, \alpha_2, \ldots \alpha_n]$
- what it means to say a polynomial $f(x) \in F[x]$ splits in an extension K of F
- what it means to say a field K is a splitting field of $f(x)$ over F

Big theorems

Give full statements of each of the following results, making sure to indicate all necessary hypotheses. For results proved in class, describe the components and main ideas of the proof.

- $\mathbf{Q}$ is an ordered field
- theorems associated with multiplication of complex numbers in polar form, including de Moivre's theorem
- there are n distinct nth roots of 1 in $\mathbf{C}$, namely the numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$
- a nonzero complex number has exactly n distinct nth roots

- the quadratic formula
- $\deg(a(x)b(x)) = \deg a(x) + \deg b(x)$ when R is a domain
- the division algorithm in $F[x]$
- root-factor theorem
- if $a(x), b(x) \in F[x]$ and $d(x)$ is a gcd of $a(x)$ and $b(x)$, then $d(x) = a(x)X(x)+b(x)Y(x)$ for some $X(x), Y(x) \in F[x]$
- Euclid's lemma for $F[x]$ and the unique factorization theorem for $F[x]$
- the Fundamental Theorem of Algebra (statement only)
- If F is a subfield of K, and $\alpha \in K$ is the root of a nonconstant polynomial in $F[x]$, then $F[\alpha]$ is a field
- If F is a subfield of K, and $\alpha \in K$ is a root of an irreducible polynomial $p(x) \in F[x]$ of degree n, then $F[\alpha] = \{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} : a_0, \ldots, a_{n-1} \in F\}$. Moreover, each element of $F[\alpha]$ has a unique representation in the form $a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}$.
- Gauss's lemma about polynomial factorizations: Let $f(x) \in \mathbf{Z}[x]$ be nonconstant. If $f(x)$ factors into two nonconstant polynomials in $\mathbf{Q}[x]$, it also factors into two nonconstant polynomials in $\mathbf{Z}[x]$ of those same degrees.
- irreducibility mod p implies irreducibility over $\mathbf{Q}$
- Eisenstein's irreducibility criterion

What to be able to do

You are expected to know how to use the methods described in class to solve the following problems.

- Basic computations with complex numbers, in either rectangular (i.e., $a + bi$) or polar form. Includes being able to apply de Movire's theorem.
- Perform "long division" of polynomials with quotient and remainder; use this to perform the Euclidean algorithm, compute gcds, and express the gcd as a linear combination
- Perform computations in $F[\alpha]$, such as addition, multiplication, and taking inverses
- Recognize fields as splitting fields of given polynomials
- Determine all rational roots of a given polynomial $f(x) \in \mathbf{Q}[x]$
- Argue that given polynomials are irreducible over $\mathbf{Q}$

Practice problems

1. (a) Let $z = 2(\cos(5\pi/8) + i\sin(5\pi/8))$. Find z^4. Express your answer in the form $a + bi$, with a, b simplified as much as possible.

   (b) Find all rational roots of the polynomial $2x^4 + x - 3$. Justify your answer, citing theorem(s) from class.

   (c) Show that $x^5 + 6x + 12$ is irreducible in $\mathbf{Q}[x]$. Justify your answer, citing theorem(s) from class.

2. (a) Let F be a field. Define what it means to say $f(x) \in F[x]$ is **irreducible**.

   (b) Now let $F = \mathbf{C}$. Show — citing theorems from class — that if $f(x)$ is irreducible over F, then $\deg f = 1$.

3. Let $K = \mathbf{Q}[\alpha]$, where α is a complex root of $x^5 + 6x + 12$.

   (a) **State a general theorem** guaranteeing that every element of $\mathbf{Q}[\alpha]$ can be written, uniquely, in the form $a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4$, where the $a_i \in \mathbf{Q}$.

   (b) Write α^5 in the form described in part (a). Then write α^{10}.

   (c) Write $(\alpha - 1)^{-1}$ in the form described in part (a).

4. (a) Let F be a subfield of K, and let $f(x)$ be a nonconstant polynomial in $F[x]$. What does it mean to say that $f(x) \in F[x]$ **split**s over K?

   (b) Let F be a subfield of K. What does it mean to say that K **is a splitting field of $f(x)$ over F**?

   (c) Show that $\mathbf{Q}[\sqrt{2}, \sqrt{5}]$ is a splitting field for $(x^2 - 2)(x^2 - 5)$ over $\mathbf{Q}$.

   (d) Prove or disprove: $\mathbf{Q}[\sqrt{2}, \sqrt{5}] = \mathbf{Q}[\sqrt{5}]$.

5. Let F be a field, and let $p(x), q(x)$ be irreducible polynomials in $F[x]$. Assume there is no $c \in F$ with $p(x) = cq(x)$. Let K be a field containing F as a subfield. Show that $p(x)$ and $q(x)$ do not have a common root in K.

6. Let F be a field. Let $f(x), g(x)$ be nonconstant polynomials in $F[x]$.

   (a) What does it mean to say $d(x) \in F[x]$ is **a greatest common divisor of $f(x), g(x)$ in $F[x]$?**

   (b) Suppose $d(x) \in F[x]$ is a greatest common divisor of $f(x), g(x) \in F[x]$.

   Let $D(x) \in F[x]$ be a common divisor of $f(x), g(x)$ with largest possible degree (among all common divisors of $f(x), g(x)$). Prove that $D(x) = c \cdot d(x)$ for some $c \in F$.