THE SYLVESTER-SCHUR THEOREM

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1. Sylvester-Schur Theorem

1.1. **Introduction.**

Theorem 1.1 (Sylvester). Let k be a positive integer.

A product of k consecutive integers each larger than k always contains a prime factor $> k$. Equivalently, if $n \geq 2k$, then there is a number in the list

$$n-k+1, n-k+2, \dots, n$$

divisible by a prime $> k$.

Since for primes $p > k$,

$$p \mid n(n-1)\dots(n-k+1) \Longleftrightarrow p \mid \frac{n(n-1)\dots(n-k+1)}{k!} = \binom{n}{k}, \tag{1}$$

we may rephrase the theorem as the assertion that $\binom{n}{k}$ is always divisible by a prime $p > k$ whenever $n \geq 2k$.

First, let us dispose of the cases when $k \leq 10$. The case $k = 1$ is the assertion that every $n > 1$ always contains a prime factor > 1, which is clear. The case $k = 2$ asserts a product of two consecutive integers, each > 2, contains an odd prime factor. This is clear since one of factors is odd.

For the cases $k = 3$ suppose we have three consecutive integers whose product is divisible by no primes > 3. Exactly one of the integers is a multiple of 3, so the other two must be powers of 2. The only powers of 2 differing by less than 4 are $\{1, 2\}$ and $\{2, 4\}$. All these possibilities are excluded if we assume the integers are all > 3 to start with. This is the case $k = 3$. The case $k = 4$ follows from the case $k = 3$, since the prime factor guaranteed by that case is > 3, so must be $\geq 5 > 4$.

To handle $k = 5$ we observe that among five consecutive integers exactly one is divisible by 5 and at most two are divisible by 3. So if none have a prime factor > 5, there must be at least two that are powers of 2. But the only powers of 2 differing by less than 4 are $\{1, 2\}, \{2, 4\}$, and $\{4, 8\}$, and these cases are precluded if we assume all the integers are at least 5. The case $k = 6$ is a consequence of $k = 5$, since the smallest prime exceeding 5 also exceeds 6.

The cases $k = 8, 9$ and 10 all follow from the case $k = 7$. It is possible to argue this final case directly. But we choose to make a first application of the following lemma, instrumental in the proof of the main theorem:

Lemma 1.2. Suppose $0 \leq k \leq n$, and that $p^r \mid \binom{n}{k}$, with p prime. Then $p^r \leq n$.

Proof. The highest power of p dividing $\binom{n}{k}$ is

$$\sum_{j \geq 1} \left(\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{k}{p^j} \right\rfloor - \left\lfloor \frac{n-k}{p^j} \right\rfloor \right).$$

Each term of the sum is either 0 or 1, being an integer smaller than

$$\frac{n}{p^j} - \left(\frac{k}{p^j} - 1\right) - \left(\frac{n-k}{p^j} - 1\right) = 2$$

but larger than

$$\left(\frac{n}{p^j} - 1\right) - \frac{k}{p^j} - \frac{n-k}{p^j} = -1.$$

Moreover, the terms of this sum vanish except possibly when $p^j \leq n$. It follows that the highest power of p dividing $\binom{n}{k}$, does not exceed $\lfloor \log n / \log p \rfloor$, whence the theorem. □

Let us now dispose of the case $k = 7$. We take advantage of the lemma and of the equivalence (1). If none of $n, \dots, n-6$ were divisible by a prime > 7, then by the lemma we would have

$$\binom{n}{7} = \frac{n(n-1)\cdots(n-6)}{6!} = 2^a 3^b 5^c 7^d \leq n \cdot n \cdot n \cdot n = n^4.$$

As the left hand side is a polynomial of degree 7 in n, this inequality must fail for n sufficiently large. In fact, a computer-assisted check shows it fails already once $n \geq 24$ (see Exercise 1). This shows a product of 7 consecutive integers each at least 18 is divisible by a prime > 7. To handle the remaining cases we need only note that any list of 7 consecutive integers starting at a number between 8 and 17 contains at least one of $\{11, 13, 17\}$.

1.2. **The Cases $k < n^{2/3}$.** We will prove Theorem 1.1 by first proving the result holds for all n and k (with $n \geq 2k$) provided $n \geq 257$, and then examining directly the remaining cases. Here we carry out this plan for n and k with k in the range $k < n^{2/3}$.

Lemma 1.3. For $k \geq 8$, we have $\pi(k) \leq k/2$. For $k \geq 33$, we have $\pi(k) \leq k/3$.

Proof. We can directly verify the assertion for $k = 8$ and $k = 9$; the claim then follows inductively from the relation

$$\pi(k+2) \leq \pi(k) + 1 \quad (k \geq 8),$$

which is immediate upon observing that either k or $k+1$ is even. Similarly, after verifying directly that $\pi(k) \leq k/3$ for $33 \leq k \leq 38$, the second of the stated relations follows inductively from

$$\pi(k+6) \leq 2 + \pi(k) \quad (k \geq 33).$$

This inequality derives from the fact that in any list of six consecutive integers only two are coprime to 6. □

Lemma 1.4. If the assertion of Theorem 1.1 fails for n and k (with $n \geq 2k$), then

$$\left(\frac{n}{k}\right)^k \leq n^{\pi(k)}.$$

Proof of Lemma 1.4. Suppose $n \geq 2k$ but $\binom{n}{k}$ is divisible by no prime $p > k$. Then by Lemma 1.2,

$$\binom{n}{k} = \prod_{p \leq k, p^{r_p} \| \binom{n}{k}} p^{r_p} \leq \prod_{p \leq k} n = n^{\pi(k)},$$

while

$$\binom{n}{k} = \frac{n}{k}\frac{n-1}{k-1}\cdots\frac{n-k+1}{1} \geq \left(\frac{n}{k}\right)^k.$$

So

$$\left(\frac{n}{k}\right)^k \leq n^{\pi(k)}. \qquad \square$$

Lemma 1.5. Theorem 1.1 is true whenever $k < n^{2/3}$ and $n \geq 257$.

Proof. The cases $k < 11$ were handled in the introduction. Suppose now that $k \geq 33$. Then $\pi(k) \leq k/3$, so if Theorem 1.1 fails for k and n we must have

$$(n/k)^k \leq n^{\pi(k)} \leq n^{k/3};$$

taking kth roots shows $k \geq n^{2/3}$, which we are assuming is not the case.

It remains to handle the range $11 < k \leq 32$. Rearranging the inequality $(n/k)^k \leq n^{\pi(k)}$ shows that in any counterexample,

$$n \leq k^{1+\frac{\pi(k)}{k-\pi(k)}}.$$

The proof will be finished if we can show the right hand side never exceeds $16^2 = 256$ for the values of k under consideration. First of all, since $\pi(k) \leq k/2$ in our range, the right hand side above is bounded by k^2, which proves the assertion for $11 \leq k \leq 16$.

In the remaining cases,

$$k^{1+\frac{\pi(k)}{k-\pi(k)}} \leq \begin{cases} 18^{1+\frac{7}{17-7}} \approx 136.1 & \text{if } 17 \leq k < 19, \\ 22^{1+\frac{8}{19-8}} \approx 208.3 & \text{if } 19 \leq k < 23, \\ 28^{1+\frac{9}{23-9}} \approx 238.5 & \text{if } 23 \leq k < 29, \\ 30^{1+\frac{10}{29-10}} \approx 179.7 & \text{if } 29 \leq k < 31, \\ 32^{1+\frac{11}{31-11}} \approx 215.3 & \text{if } 31 \leq k < 33. \end{cases} \qquad \square$$

1.3. **The Cases $k \geq n^{2/3}$.** We now assume $k \geq n^{2/3}$. We consider separately the two cases when $2k \leq n < 3k$ and $3k \leq n$. The estimation of $\binom{n}{k}$ in these two cases (respectively) will be accomplished by the following explicit lower bounds on binomial coefficients:

Lemma 1.6. For $k \geq 1$,

$$\binom{2k}{k} \geq \frac{4^k}{2k} \quad \text{and} \quad \binom{3k}{k} \geq \left(\frac{3^3}{2^2}\right)^k \frac{1}{3k}.$$

Proof. The proofs are straightforward induction arguments.

Assume the first inequality holds for $k-1 \geq 1$; then

$$\begin{aligned}\binom{2k}{k} = \frac{2k}{k}\frac{2k-1}{k}\binom{2(k-1)}{k-1} &\geq 2 \cdot 2 \cdot \left(1 - \frac{1}{2k}\right)\frac{4^{k-1}}{2(k-1)} \\ &\geq (1-1/k)\frac{4^k}{2(k-1)} = \frac{4^k}{2k},\end{aligned}$$

and the first is proved. Similarly, if the second holds for a certain $k-1 \geq 1$, then

$$\binom{3k}{k} = \frac{3k}{2k}\frac{3k-1}{2k-1}\frac{3k-2}{k}\binom{3(k-1)}{k-1} \geq \frac{3}{2}\frac{3}{2}(3-2/k)\left(\frac{3^3}{2^2}\right)^{k-1}\frac{1}{3(k-1)}$$
$$\geq \frac{3}{2}\frac{3}{2}3(1-1/k)\left(\frac{3^3}{2^2}\right)^{k-1}\frac{1}{3(k-1)} = \left(\frac{3^3}{2^2}\right)^{k}\frac{1}{3k}. \qquad \square$$

We also need the following good Chebyshev-type estimate of Hanson. The proof, which is elementary, is nonetheless sufficiently complicated to be deferred to the end of this section.

Theorem 1.7 (Hanson). For each $x > 0$,

$$3^x > \prod_{p \leq x} p \prod_{p \leq \sqrt{x}} p \prod_{p \leq \sqrt[3]{x}} p \cdots .$$

Equivalently, $\psi(x) < x \log 3$.

Corollary 1.8. Suppose $n \geq 2k$ but $\binom{n}{k}$ has no prime divisor exceeding k (i.e., that n and k are a counterexample to Theorem 1.1). Suppose also that $k \geq n^{2/3}$. Then

$$\binom{n}{k} < 3^{k+n^{1/2}}.$$

Proof. We first show that

$$\binom{n}{k} \leq \prod_{p \leq k} p \prod_{p \leq \sqrt{n}} p \prod_{p \leq \sqrt[3]{n}} p \cdots \tag{2}$$

by showing the left hand side divides the right. Indeed, if p^r exactly divides the left hand side, then $p \leq k$ by hypothesis and $p^r \leq n$ by Lemma 1.2. This means p shows up in each of the r first factors on the right and implies the claim.

Next we observe that for $l \geq 2$,

$$k \geq n^{2/3} \Longrightarrow k^{1/l} \geq n^{\frac{1}{2l+1}},$$

so that

$$\prod_{p \leq k} \prod_{p \leq \sqrt[3]{n}} p \prod_{p \leq \sqrt[5]{n}} p \cdots \leq \prod_{p \leq k} \prod_{p \leq k^{1/2}} p \prod_{p \leq k^{1/3}} p \cdots < 3^k$$

by Theorem 1.7. Another application of the same theorem shows

$$\prod_{p \leq \sqrt{n}} p \prod_{p \leq \sqrt[4]{n}} p \prod_{p \leq \sqrt[6]{n}} p \cdots < 3^{\sqrt{n}}.$$

Multiplying the two preceding estimates and referring to (2) yields the corollary. $\square$

Take now the (sub)case when $n \geq 3k$. If Theorem 1.1 failed for this n and k, we would have by Lemma 1.6 and Corollary 1.8 that

$$\left(\frac{3^3}{2^2}\right)^k \frac{1}{3k} \leq \binom{3k}{k} \leq \binom{n}{k} \leq 3^{k+n^{1/2}}.$$

Taking logarithms and rearranging we obtain

$$k \log \frac{9}{4} \leq n^{1/2} \log 3 + \log 3k.$$

Using that $k \geq n^{2/3}$ and $3k \leq n$ we obtain

$$n^{2/3} \log \frac{9}{4} \leq n^{1/2} \log 3 + \log n.$$

As a function of n, the left hand side grows more quickly than the right, so this inequality fails eventually. In Exercise 2 we indicate a proof that the inequality fails for $n \geq 257$, which is all we use in the sequel, though a computer check shows it fails already for $n \geq 64$.

Now suppose $2k \leq n < 3k$. Failure of Theorem 1.1 in this case implies

$$\frac{4^k}{2k} \leq \binom{2k}{k} \leq \binom{n}{k} \leq 3^{k+n^{1/2}},$$

whence

$$\frac{n}{3} \log \frac{4}{3} < k \log \frac{4}{3} \leq n^{1/2} \log 3 + \log 2k \leq n^{1/2} \log 3 + \log n.$$

This inequality also fails for $n \geq 257$; again, see Exercise 2. (It actually fails as soon as $n \geq 231$.)

1.4. **The Exceptional Cases.** We have completely settled the cases where $n \geq 257$. So we can suppose now that $n \leq 256$ and hence $k \leq n/2 \leq 128$.

The majority of these cases may be eliminated using the following simple observation:

Lemma 1.9. Let $k \geq 8$ and $n \geq 2k$. If

$$p_{i+1} - p_i \leq k$$

for all primes $p_i < n$, then the assertion of Theorem 1.1 holds for this pair k, n.

Proof. We need to show that among the numbers

$$n - k + 1, n - k + 2, \ldots, n$$

there is one with a prime divisor $> k$. In fact, this list always contains a prime p, where necessarily $p \geq n - k + 1 \geq 2k - k + 1 > k$, so the assertion to follows.

To see this, suppose otherwise and let p' be the largest prime smaller than $n - k + 1$. (Observe $n - k + 1 > k > 8$, so such a prime certainly exists.) If p is the smallest prime exceeding p', then

$$n - k + 1 \leq p \leq p' + k < (n - k + 1) + k = n + 1,$$

so p is on the list, a contradiction. □

The following list of primes, each of which differs from the former by at most 14, shows the hypotheses of the preceding lemma are satisfied for $k \geq 14$ and $n \leq 256$:

$$(3) \quad 2, 13, 23, 37, 47, 61, 73, 83, 97, 107, 113, 127, 139, 151,$$
$$163, 173, 181, 193, 199, 211, 223, 233, 241, 251, 263.$$

It therefore suffices to prove the theorem in the cases $11 \leq k \leq 13$. Since $\pi(k) \leq k/2$ in this range, any counterexample would have to satisfy

$$(n/k)^k \leq n^{\pi(k)} \leq n^{k/2},$$

so that $k \geq n^{1/2}$ and $n \leq k^2 \leq 169$.

Consider the increasing sequence

$$17, 23, 34 = 2 \cdot 17, 43, 53, 62 = 2 \cdot 31, 73, 83,$$
$$94 = 2 \cdot 47, 103, 114 = 2 \cdot 3 \cdot 19, 124 = 2^2 \cdot 31, 134 = 2 \cdot 67,$$
$$145 = 5 \cdot 29, 155 = 5 \cdot 31, 166 = 2 \cdot 83, 177 = 3 \cdot 59.$$

Each number on this list is divisible by a prime > 13 and differs from the preceding by at most 11. Any list of ≥ 11 consecutive positive integers, all exceeding 11 but not exceeding 169, contains a number on this list (cf. the argument for the above lemma). This implies the remaining cases and finishes the proof.

1.5. **Proof of Hanson's Theorem 1.7.** We now give Hanson's proof that $\psi(n) < n \log 3$ for all positive integral n (the extension to nonintegral n following immediately).

Here is the plan: Define the sequence of a_i inductively by $a_1 = 2$ and $a_{n+1} = a_1 a_2 \dots a_n + 1$; thus $a_1 = 2, a_2 = 3, a_3 = 7, a_4 = 43$, etc. Let

$$C(n) = \frac{n!}{\lfloor n/a_1 \rfloor! \lfloor n/a_2 \rfloor! \cdots}. \tag{4}$$

We will show that $C(n)$ is an integer smaller than 3^n and that

$$B(n) := \prod_{p^k \leq n} p \mid C(n). \tag{5}$$

This implies, in particular, that

$$\psi(n) = \log B(n) \leq \log C(n) \leq n \log 3,$$

as desired.

To get started we need a few elementary properties of the sequence given above:

Lemma 1.10. The sequence $\{a_i\}$ defined above has the following properties:

(1) For $n \geq 1$ we have
$$a_{n+1} = a_n^2 - a_n + 1.$$
(2) For $n \geq 1$,
$$\sum_{i=1}^{n} \frac{1}{a_i} = 1 - \frac{1}{a_{n+1} - 1}.$$
In particular, $\sum_{i=1}^{\infty} 1/a_i = 1$.
(3) For $n \geq 3$,
$$a_n > 2^{2^{n-2}} + 1.$$

Proof. We have

$$a_{n+1} = (a_1 \dots a_{n-1}) a_n + 1 = (a_n - 1) a_n + 1 = a_n^2 + a_n + 1.$$

This proves (i).

For the proof of (ii), observe the claim holds for $n = 1$ and that if it holds for $n = k - 1$ then

$$\sum_{i=1}^{k} \frac{1}{a_i} = \frac{1}{a_k} + 1 - \frac{1}{a_k - 1} = 1 - \frac{1}{a_k^2 - a_k} = 1 - \frac{1}{a_{k+1} - 1},$$

so that it holds also for $n = k$. So we are finished by induction.

For the third claim observe that by (i),

$$a_{n+1} > (a_n - 1)^2$$

for each n. The claim follows inductively from this inequality and the observation that

$$a_3 = 7 > 2^{2^1} + 1. \qquad \square$$

For the remainder of this proof let $r = r(n)$, defined for $n \geq 2$, denote the largest index r with $a_r \leq n$. If $n \geq 7$, then $r(n) \geq 3$, so by (iii) above we have

$$r \leq \log_2 \log_2 n - 1 + 2 < \log_2 \log_2 n + 2.$$

Property (ii) allows us to show simultaneously that $C(n)$, as defined by (4), is an integer and is divisible by $B(n)$ (as defined in (5)). Write

$$C(n) = \frac{n!}{\lfloor n/a_1 \rfloor! \lfloor n/a_2 \rfloor! \cdots \lfloor n/a_r \rfloor!};$$

the highest power of a prime p occurring in $C(n)$ is

$$\sum_{j \geq 1} \left(\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{n}{a_1 p^j} \right\rfloor - \cdots - \left\lfloor \frac{n}{a_r p^j} \right\rfloor \right). \tag{6}$$

We estimate the subtracted terms by observing that

$$\begin{aligned} \left\lfloor \frac{n}{a_1 p^j} \right\rfloor + \cdots + \left\lfloor \frac{n}{a_r p^j} \right\rfloor &= \left\lfloor \frac{\lfloor n/p^j \rfloor}{a_1} \right\rfloor + \cdots + \left\lfloor \frac{\lfloor n/p^j \rfloor}{a_r} \right\rfloor \\ &\leq \lfloor n/p^j \rfloor \left(\frac{1}{a_i} + \ldots \frac{1}{a_r} \right) \leq \lfloor n/p^j \rfloor \left(1 - \frac{1}{a_{r+1} - 1} \right). \end{aligned} \tag{7}$$

In particular, the total of the subtracted terms is bounded by $\lfloor n/p^j \rfloor$, which means by (6) that the power of p occurring in $C(n)$ is nonnegative. This implies $C(n)$ is an integer. Moreover, (7) implies the total of subtracted terms is strictly less than $\lfloor n/p^j \rfloor$ provided $\lfloor n/p^j \rfloor \geq 1$, i.e., provided $p^j \leq n$. It follows that $p^{\lfloor \log_p n \rfloor} \mid C(n)$. Since $p^{\lfloor \log_p n \rfloor} \| B(n)$, the claim follows.

It remains to prove the estimate $C(n) < 3^n$. This requires some technical finagling. We first prove that

$$C(n) < \frac{n^n}{\lfloor n/a_1 \rfloor^{\lfloor n/a_1 \rfloor} \lfloor n/a_2 \rfloor^{\lfloor n/a_2 \rfloor} \ldots \lfloor n/a_r \rfloor^{\lfloor n/a_r \rfloor}}. \tag{8}$$

To see this, abbreviate $\alpha_i = \lfloor n/a_i \rfloor$, and let

$$m := \sum_{i=1}^{r} \alpha_i \leq n \sum_{i=1}^{r} \frac{1}{a_i} < n.$$

Then

$$\begin{aligned} \alpha_1^{\alpha_1} \ldots \alpha_r^{\alpha_r} C(n) &= (n(n-1) \ldots m) \alpha_1^{\alpha_1} \ldots \alpha_r^{\alpha_r} \frac{m!}{\alpha_1! \alpha_2! \ldots \alpha_r!} \\ &\leq n^{n-m} (\alpha_1 + \alpha_2 + \cdots + \alpha_r)^m = n^{n-m} m^m < n^{n-m} n^m = n^n, \end{aligned}$$

which is the assertion of (8). Note that the transition from the first to the second line comes from replacing one term in the multinomial expansion of $(\alpha_1 + \cdots + \alpha_1)^m$ with the entire quantity.

To make the analysis easier we would like to obtain an analogous inequality without the greatest integer signs. This is made possible by the inequality

$$\frac{(n/a_i)^{n/a_i}}{\lfloor n/a_i \rfloor^{\lfloor n/a_i \rfloor}} \leq \left(\frac{en}{a_i}\right)^{(a_i-1)/a_i},$$

valid whenever $n \geq a_i$. For the proof we may suppose $n > a_i$, the case $n = a_i$ being clear. Then the left hand side is bounded by

$$\begin{aligned}
&\frac{(n/a_i)^{n/a_i}}{((n-a_i+1)/a_i)^{(n-a_i+1)/a_i}} \\
&\qquad = \left(\frac{n}{a_i}\right)^{(a_i-1)/a_i} \left(1 + \frac{1}{(n-a_i+1)/(a_i-1)}\right)^{\frac{n-a_i+1}{a_i-1}\cdot\frac{a_i-1}{a_i}} \\
&\qquad \leq \left(\frac{n}{a_i}\right)^{(a_i-1)/a_i} e^{\frac{1}{(n-a_i+1)/(a_i-1)}\cdot\frac{n-a_i+1}{a_i-1}\cdot\frac{a_i-1}{a_i}} = \left(\frac{en}{a_i}\right)^{(a_i-1)/a_i},
\end{aligned}$$

and the inequality is proven. Using this in (8) we obtain

$$C(n) < n^n \prod_{i=1}^{r} \left(\frac{en}{a_i}\right)^{(a_i-1)/a_i} \prod_{i=1}^{r} \left(\frac{n}{a_i}\right)^{-n/a_i}. \tag{9}$$

Now assume, as we shall justify below, that the limit

$$c := \lim_{k\to\infty} a_1^{1/a_1} a_2^{1/a_2} \dots a_k^{1/a_k}$$

exists as a finite number. Note that the quantity inside the limit is an increasing function of k, so c is simply the supremum of this quantity (which could, a priori, be infinite). With this assumption, we can use the bound (9) to obtain

$$\begin{aligned}
C(n) &< n^n \prod_{i=1}^{r} \left(\frac{en}{a_i}\right)^{(a_i-1)/a_i} \prod_{i=1}^{\infty} \left(\frac{n}{a_i}\right)^{-n/a_i} \\
&= c^n \prod_{i=1}^{r} \left(\frac{en}{a_i}\right)^{(a_i-1)/a_i} \leq c^n \prod_{i=1}^{r} \left(\frac{en}{2}\right)^{(a_i-1)/a_i}
\end{aligned}$$

Observing that

$$\sum_{i=1}^{r} \frac{a_i - 1}{a_i} = r - \sum_{i=1}^{r} \frac{1}{a_i} = r - 1 + \frac{1}{a_{r+1}-1} \leq r - 5/6.$$

for $r \geq 3$, and using the bound on r derived before, we obtain

$$C(n) < c^n \left(\frac{en}{2}\right)^{r-5/6} \leq c^n \left(\frac{en}{2}\right)^{\log_2 \log_2 n + 7/6}. \tag{10}$$

for $n \geq 7$, say.

We now investigate the existence and value of c. It suffices to investigate the series

$$\sum_{i=1}^{\infty} \log a_i^{1/a_i},$$

for if this series is finite then c exists and is its exponential. In fact, this series converges rather rapidly. Since

$$(a_i - 1)^2 < a_{i+1} = a_i^2 - a_i + 1 < a_i^2,$$

we have

$$\frac{\log a_{i+1}^{1/a_{i+1}}}{\log a_i^{1/a_i}} = \frac{a_i \log a_{i+1}}{a_{i+1}\log a_i} < \frac{2a_i}{a_{i+1}} < 2\frac{a_i}{(a_i-1)^2} < 2\frac{7}{6^2} < 1/2$$

if $i \geq 3$. The series therefore converges by the 'ratio test', and in fact we have

$$\log c = \sum_{i=5}^{\infty} \log a_i^{1/a_i} + \sum_{i=6}^{\infty} \log a_i^{1/a_i} < \sum_{i=1}^{5} \log a_i^{1/a_i} + 2\log a_6^{1/a_6} < 1.0824,$$

whence

$$c < e^{1.0824} < 2.952.$$

Using this in (10) we see that $C(n) < 3^n$ for $n \geq 2400$ (see Exercise 3). In the remaining range,

$$C(n) = \frac{n!}{\lfloor n/2\rfloor!\lfloor n/3\rfloor!\lfloor n/7\rfloor!\lfloor n/43\rfloor!\lfloor n/1807\rfloor!},$$

and the inequality $C(n) < 3^n$ can be verified quickly on a system such as `MAPLE`. Alternately, in this range one can verify $\psi(n) < n\log 3$ by tables, such as ...

1.6. **Exercises.**

Exercise 1. Check that $\binom{n}{7} > n^4$ for $n \geq 24$. Suggestion: Prove that for $n \geq 24$,

$$\binom{n}{7}/n^4 \geq \frac{n(n-1)(n-2)}{7!}(3/4)^4.$$

Show that for $n \geq 27$ the right hand side exceeds 1 (note that because the right hand side is increasing, it suffices to consider $n = 27$). Now check the cases $n = 24, 25$ and 26 separately.

Exercise 2. Prove that for $n \geq 257$ we have both

$$n^{2/3}\log\frac{9}{4} > n^{1/2}\log 3 + \log n \quad \text{and} \quad \frac{n}{3}\log\frac{4}{3} > n^{1/2}\log 3 + \log n.$$

Suggestion: Observe that $\log(n)/n^{1/2}$ is decreasing for $n \geq e^2$, so that it suffices to prove these inequalities with $\log n$ replaced by $(n/257)^{1/2}\log 257$.

Exercise 3. Prove that for $n \geq 1300$, we have

$$(2.952)^n \left(\frac{en}{2}\right)^{\log_2\log_2 n + 7/6} < 3^n.$$

Suggestion: Prove this in the form

$$\left(\frac{en}{2}\right)^{\frac{\log_2\log_2 n}{n} + \frac{7}{6n}} < \frac{3}{2.952}$$

by verifying the inequality for $n = 2400$ and showing the left hand side is decreasing for $n \geq 2400$.