Math 4000/6000 – Homework #5

posted March 2, 2018; due at the **start of class** on March 9, 2018

> Algebra is nothing more than geometry, in words; geometry is nothing more than algebra, in pictures.
> – Sophie Germain (1776–1831)

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

1. (de Moivre's theorem)

   (a) Our rule from class for multiplying complex numbers implies that if we write z in polar form, say $z = r(\cos\theta + i\sin\theta)$, then
   $$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$
   for every positive integer n. Show that the same formula holds when $n = 0$ and when n is a negative integer.

   (b) By expanding $(\cos(\theta) + i\sin(\theta))^4$, find formulas for $\cos(4\theta)$ and $\sin(4\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

2. Let $n \in \mathbb{Z}^+$. We say that the complex number z is a *primitive nth root of* 1 if

   (i) $z^n = 1$, and

   (ii) there is no positive integer $m < n$ with $z^m = 1$.

   For example, -1 is a primitive 2nd root of 1, since $(-1)^2 = 1$ but $(-1)^1 \neq 1$.

   Show that a primitive nth root of 1 exists for every n. How many primitive nth roots of 1 are there for $n = 1, 2, 3, 4$?

3. Let $n \in \mathbb{Z}^+$. In this problem, we assume that z is a primitive nth root of 1.

   (a) Show that the elements of the list
   $$1, z, z^2, \ldots, z^{n-1}$$
   are distinct.

   (b) Prove that every element on the $1, z, z^2, \ldots, z^{n-1}$ is an nth root of 1, and that, conversely, every nth root of 1 is on this list.

   (c) Show that if m is an integer, then $z^m = 1$ if and only if n divides m.

   (d) Show that if m is an integer, then z^m is a primitive nth root of 1 if and only if $\gcd(m, n) = 1$.

   (e) How many primitive 10th roots of 1 are there?

4. Given a polynomial $f(z) = z^3 + pz + q$ (with p, q complex numbers), we set

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27}.$$

As shown in class, as long as $p \neq 0$, the complex roots of f are the numbers

$$v - \frac{p}{3v}, \quad \text{where } v^3 = A, \quad \text{for } A := -\frac{q}{2} + \sqrt{\Delta}, \qquad \text{(1st set of roots)}$$

along with the numbers

$$v' - \frac{p}{3v'}, \quad \text{where } v'^3 = B, \quad \text{for } B := -\frac{q}{2} - \sqrt{\Delta}. \qquad \text{(2nd set of roots)}$$

The goal of this exercise is for you to show that the second set of roots is redundant; every root in the second set is already in the first. (We claimed this in class.)

(a) Show that $B \neq 0$. Remember we are assuming $p \neq 0$.

(b) It follows from part (a) that B has three distinct (and nonzero) complex cube roots v'. Show that for each of these roots v', the number $-\frac{p}{3v'}$ is a cube root of A. Then show that if we let $v = -\frac{p}{3v'}$, then $v - \frac{p}{3v} = v' - \frac{p}{3v'}$. [Hence, every root in the second set is already in the first.]

5. Let $\omega = \cos(2\pi/5) + i\sin(2\pi/5)$. Here we describe how to express ω in terms of square roots.

(a) Show that ω is a root of the polynomial $z^4 + z^3 + z^2 + z + 1$. *Hint:* $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$.

(b) Show that $\omega + 1/\omega$ is a root of the polynomial $u^2 + u - 1$.

(c) Show that $\omega + 1/\omega = \frac{-1+\sqrt{5}}{2}$, where $\sqrt{5}$ means the positive square root of 5. *Hint:* Figure out the sign of $\omega + 1/\omega$ by adding the polar forms of ω and $1/\omega$.

(d) Put $\beta = \frac{-1+\sqrt{5}}{2}$. So in part (c), you showed $\omega + 1/\omega = \beta$. Now show that

$$\omega = \frac{\beta + i\sqrt{4-\beta^2}}{2},$$

where $\sqrt{4-\beta^2}$ means the positive square root of $4 - \beta^2$.

(e) Deduce from (d) that $\cos(2\pi/5) = \frac{\beta}{2}$ and $\sin(2\pi/5) = \frac{1}{2}\sqrt{4-\beta^2}$.

6. Exercise 2.4.6(a,b).

7. 3.1.2(a), and then
$f(x) = x^2 + 2x + 2$, $g(x) = x^2 + 1$, $F = \mathbb{Z}_3$

8. 3.1.6.

9. (*) Exercise 2.2.16. *Hint:* First, figure out what f does to rational numbers.