Math 4000/6000 – Homework #5

posted October 5, 2016; due at the **start of class** on October 10, 2016

The essence of mathematics lies in its freedom. – Georg Cantor

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

1. (de Moivre's theorem)

   (a) In class, we noted that our rule for multiplying complex numbers implies that if we write z in polar form, say $z = r(\cos\theta + i\sin\theta)$, then

   $$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

   for every positive integer n. Show that the same formula holds when $n = 0$ and when n is a negative integer.

   (b) Find formulas for $\cos(4\theta)$ and $\sin(4\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$. The binomial theorem may be helpful.

2. (more on nth roots of complex numbers) Let $n \in \mathbb{Z}^+$, and let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. Let A be a nonzero complex number. Show that if $\sqrt[n]{A}$ is any fixed nth root of A, then the set of all nth roots of A consists precisely of the n numbers

   $$\sqrt[n]{A}, \quad \omega\sqrt[n]{A}, \quad \dots, \quad \omega^{n-1}\sqrt[n]{A}.$$

   (This generalizes a result from class for the case $n = 3$.)

3. Exercise 2.3.13.

4. Given a polynomial $f(z) = z^3 + pz + q$ (with p, q complex numbers), we set $\Delta = \frac{q^2}{4} + \frac{p^3}{27}$. As shown in class, as long as $p \neq 0$, the complex roots of f are the numbers

   $$v - \frac{p}{3v}, \quad \text{where } v \text{ runs over the cube roots of } \quad A := -\frac{q}{2} + \sqrt{\Delta}.$$

   Here $\sqrt{\Delta}$ denotes any fixed square root of Δ.

   (a) Show that $A \neq 0$. (Remember we are assuming $p \neq 0$.)

   (b) It follows from (a) that A has three distinct (and nonzero) cube roots v. Show that for each of these roots v, the number $-\frac{p}{3v}$ is a cube root of $-\frac{q}{2} - \sqrt{\Delta}$. (This explains why our derivation for the roots of a cubic equation yields three roots and not six!)

5. (A lemma for problem 6) Let p be a nonzero complex number. Show that if v and v' are nonzero complex numbers, then

   $$v - \frac{p}{3v} = v' - \frac{p}{3v'} \iff \text{either } v = v' \text{ or } v = -\frac{p}{3v'}.$$

6. Let $f(z) = z^3 + pz + q$. In this exercise, you will show that

$$f \text{ has fewer than 3 distinct complex roots} \Longrightarrow \Delta = 0.$$

(The reverse implication is also true but we will not show that here.)

We adopt the notation $A := -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ and $B := -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$.

   (a) Prove that if $p = 0$ and f has fewer than 3 distinct roots, then $q = 0$ and $\Delta = 0$.

   (b) Now assume $p \neq 0$. Using the formula for the roots of f and the result of problem 5, show that if f has fewer than 3 distinct roots, then there are two cube roots v and v' of A for which $v = -\frac{p}{3v'}$.

   (c) (continuation) With v, v' as in part (b), use problem 4(b) to show that v is a cube root of both A and B. Conclude that $\Delta = 0$.

7. Exercise 2.4.6(a,b).

8. Let R be a ring. A subset $R' \subseteq R$ is called a *subring* of R if (A) R' is a ring for the same operations $+$ and $\cdot$ as in R, **and** (B) R' contains the multiplicative identity 1_R of R.

   (For example, making the identifications discussed in class, $\mathbb{Z}$ is a subring of $\mathbb{Q}$ and $\mathbb{Q}$ is a subring of $\mathbb{R}$.)

   (a) Let R be a ring. Suppose that R' is a subset of R closed under $+$ and $\cdot$, that R' contains the additive inverse of each of its elements, and that R' contains 1_R. Show that R' is a subring of R.

   Hint: (B) holds by assumption. Check that all the ring axioms hold for R' in order to verify (A). To get started, show that the additive identity of R — call this 0_R — must belong to R'.

   (b) Find a two-element subset R' of $R = \mathbb{Z}_6$ that satisfies condition (A) in the definition of a subring but not (B). (You do **not** have to give a detailed proof that (A) holds.)

9. (*) Exercise 2.2.16.

10. (*) Suppose distinct complex numbers z_1, z_2, z_3 satisfy $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_1 z_3$. Show that z_1, z_2, z_3 are the vertices of an equilateral triangle.

    Hint: The constraint equation can also be written in the form $(z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2 = 0$.