Math 4000/6000 – Homework #1

posted August 23, 2015; due at the **start of class** on August 28, 2015

> A biologist, a physicist and a mathematician were sitting in a street cafe watching the crowd. Across the street they saw a man and a woman entering a building. Ten minutes they reappeared together with a third person.
> – They have multiplied, said the biologist.
> – Oh no, an error in measurement, the physicist sighed.
> – If exactly one person enters the building now, it will be empty again, the mathematician concluded.

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

1. ($\mathbb{C}$ cannot be ordered) Let S be a subset of the complex numbers. Show that it is impossible for S to have all of the following three properties:

   (i) the sum of two elements of S is always in S,

   (ii) the product of two elements of S is always in S,

   (iii) for each complex number x, exactly one of the following holds: $x = 0$, $x \in S$, or $-x \in S$.

2. (Laws of exponents) Let $a \in \mathbb{Z}$. Suppose that m, n belong to the set $\mathbb{N} \cup \{0\}$ of nonnegative integers.

   (a) Prove that $a^m \cdot a^n = a^{m+n}$.

   (b) Prove that $a^{mn} = (a^m)^n$.

   Hint: If $m = 0$ or $n = 0$, this is easy (why?). So you can suppose $m, n \in \mathbb{N}$. Now think of m as fixed and proceed by induction on n.

3. Prove that for any $a, b \in \mathbb{Z}$, we have

   (a) $(-a)b = -(ab)$.

   (b) $(-a)(-b) = ab$,

   In this problem **only**, you must point out exactly which algebraic properties you use at EVERY step of the proof. You may assume $-(-a) = a$ and $(-1)a = -a$, as these results were already discussed in class.

4. Recall that for integers u and v, we defined "$u < v$" to mean that $v - u \in \mathbb{N}$. Let $a, b \in \mathbb{Z}$.

   (a) Prove, using the order properties of $\mathbb{Z}$ discussed in class, that if $a < 0$ and $b < 0$, then $ab > 0$.

   (b) Show that if $a < 0$ and $b > 0$, then $ab < 0$.

   (c) Show that if $ab = 0$, then either $a = 0$ or $b = 0$.

Note that you do **not** have to point out when you use algebraic properties like associativity or the distributive law.

5. In this exercise we outline a proof of the following statement, which was left as a "missing step" in our proof of the division theorem: If $a, b \in \mathbb{Z}$ with $b > 0$, the set

$$S = \{a - bq : q \in \mathbb{Z} \text{ and } a - bq \geq 0\}$$

has a least element.

(a) Prove the claim in the case $0 \in S$.

(b) Prove the claim in the case $0 \notin S$ and $a > 0$.

(c) Prove the claim in the case $0 \notin S$ and $a \leq 0$.

Hint: (a) is easy. To handle (b) and (c), first show that in these cases S is a nonempty set of natural numbers, so that the well-ordering principle guarantees S has a least element as long as S is nonempty. To prove S is nonempty, show that in case (b), the integer a is an element of S. You will have to work a little harder to prove S is nonempty in case (c).

6. Use the binomial theorem to find formulas for the following sums, as functions of n, where n is assumed to be a natural number.

(a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}$.

(b) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

7. (*) We stated the binomial theorem under the assumption that $x, y \in \mathbb{Z}$. However, our proof only used that we could manipulate expressions in x and y by the usual algebraic rules. That assumption holds if x, y are variables. Hence, the identity

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

is valid as a *polynomial identity* in the variables x and y. (So far I have not asked you to prove anything, just to accept this as true!)

Your mission: By computing $(x+y)^{2n}$ in two different ways and comparing coefficients, show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

8. (To be done after Monday's lecture) Use the Euclidean algorithm to find $\gcd(314, 159)$ and $\gcd(272, 1479)$. Show the steps, not just the final answer.

9. Show that if $a, b \in \mathbb{N}$ and $a \mid b$, then $a \leq b$.

10. Let a, b be nonnegative integers, not both zero. Define the set

$$I(a, b) = \{ax + by : x, y \in \mathbb{Z}\}.$$

(Thus, $I(a, b)$ is the set of all linear combinations of a, b, with coefficients from $\mathbb{Z}$. The letter I stands for *ideal*, which is a concept we will meet later in the course.)

(a) Show that if a, b, q, r are integers with $a = bq + r$, then $I(a, b) = I(b, r)$.

(b) Explain why (a) implies that $I(a, b) = I(0, \gcd(a, b))$.

(c) Deduce from (b) that there are integers x and y with $\gcd(a, b) = ax + by$.

11. (*) Exercise 1.1.16 (this means Exercise 16 in §1 of Chapter 1)