MATH 4000/6000 – Learning objectives to meet for Exam #3

The exam will be over §§3.3, 4.1, and 4.2 of the textbook, as well as material covered on your HW assignments.

What to be able to state

Basic definitions

You should be able to give precise descriptions of all of the following:

- homomorphism
- kernel of a homomorphism
- ideal of a commutative ring
- principal ideal
- the notation $\langle a \rangle$ and $\langle a_1, \ldots, a_k \rangle$, as well as aR
- definition of the quotient ring R/I (including what the elements are and how the operations are defined)
- isomorphism of rings (you are also expected to remember the definition of the terms one-to-one and onto)
- direct product of rings
- the definition of the ring of Gaussian integers $\mathbf{Z}[i]$, and the definition of the norm map $N(\cdot)$ on $\mathbf{Z}[i]$ (see HW)
- earlier definitions regarding splitting of polynomials and splitting fields (see the review sheet for Exam #2)

Big theorems

Give full statements of each of the following results, making sure to indicate all necessary hypotheses. For results proved in class, describe the components and main ideas of the proof.

- rational root theorem
- Gauss's lemma about polynomial factorizations: Let $f(x) \in \mathbf{Z}[x]$ be nonconstant. If $f(x)$ factors into two polynomials in $\mathbf{Q}[x]$, it also factors into two polynomials in $\mathbf{Z}[x]$ of those same degrees.
- theorem about irreducibility mod p vs. irreducibility over $\mathbf{Q}$ (Proposition 3.4 in the book)
- Eisenstein's criterion for ireducibility
- the kernel of a homomorphism is an ideal
- statement of the division algorithm in $\mathbf{Z}[i]$ (see HW)

- In each of the rings $\mathbf{Z}$, $F[x]$, and $\mathbf{Z}[i]$, every ideal is principal
- If F is a subfield of K and $\alpha \in K$ is the root of a nonconstant polynomial in $F[x]$, then α is the root of an irreducible polynomial $p(x) \in F[x]$. Moreover, if I is the collection of polynomials in $F[x]$ that vanish at α, then $I = \langle p(x) \rangle$.
- If $\phi\colon R \to S$ is a homomorphism, then ϕ is one-to-one if and only if $\ker(\phi) = \{0\}$.
- If $\phi\colon R \to S$ is an isomorphism, then a is a unit in R if and only if $\phi(a)$ is a unit in S. Same statement for zero divisors instead of units.
- Fundamental Homomorphism Theorem
- If $f(x) \in F[x]$ is irreducible, then $K = F[x]/\langle f(x) \rangle$ is a field containing F, and $f(x)$ has a root in K.
- If $f(x) \in F[x]$ is any nonconstant polynomial, there is an extension K/F in which $f(x)$ has a root.
- If $f(x) \in F[x]$ is any nonconstant polynomial, there is an extension K/F in which $f(x)$ splits.
- If $f(x) \in F[x]$ is any nonconstant polynomial, there is a splitting field for $f(x)$ over F.

What to be able to do

You are expected to know how to use the methods described in class/developed on HW to solve the following problems (not comprehensive!).

- Show that you understand the fundamental concepts by being able to combine them in simple proofs.
- Determine all rational roots of a given polynomial $f(x) \in \mathbf{Z}[x]$
- Argue that given polynomials are irreducible over $\mathbf{Q}$
- Establish properties of quotient rings R/I by relating these back to the properties of the original ring R
- Perform computations in quotient rings R/I. (E.g., multiplying two given elements of $\mathbf{Q}[x]/\langle x^3 + x \rangle$.)
- Establish isomorphisms between rings using the Fundamental Homomorphism Theorem