REFINEMENTS OF LAGRANGE'S FOUR-SQUARE THEOREM

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ABSTRACT. A well-known theorem of Lagrange asserts that every nonnegative integer n can be written in the form $a^2 + b^2 + c^2 + d^2$, where $a, b, c, d \in \mathbb{Z}$. We characterize the values assumed by a + b + c + d as we range over all such representations of n.

Our point of departure is the following signature result from a first course in number theory.

Lagrange's four-square theorem. Every nonnegative integer can be written as the sum of four integer squares. That is, for every $n \in \mathbb{N}$, there are $a, b, c, d \in \mathbb{Z}$ with

$$n = a^2 + b^2 + c^2 + d^2. (1)$$

For instance (n = 2017), we have

$$2017 = 18^2 + 21^2 + 24^2 + 26^2.$$

Twenty years before Lagrange's proof, Euler had already conjectured a refinement of the four-square theorem for odd numbers n. The following statement can be found in a letter to Goldbach dated June 9, 1750.

Conjecture 1. Every odd positive integer n has a representation in the form (1) satisfying the extra constraint a + b + c + d = 1.

Picking back up our earlier example, when n = 2017, Euler's conjecture is satisfied with a = -18, b = 21, c = 24, and d = -26. A proof of Euler's conjecture was posted to MathOverflow by Franz Lemmermeyer in September 2010.¹

Much more recently, Sun & Sun (apparently unaware of Euler's conjecture) presented a number of related refinements of Lagrange's theorem [3] (cf. [4]). One of their many results is that for every $n \in \mathbb{N}$, there is a representation (1) with a + b + c + d a square, as well as one with a + b + c + d a cube [3, Theorem 1.1(a)].

We can unify all the above assertions by introducing the sum spectrum

$$\mathcal{S}(n) = \{a+b+c+d : a^2+b^2+c^2+d^2=n\}.$$

Lagrange's theorem is equivalent to $\mathscr{S}(n) \neq \emptyset$; Euler's conjecture asserts that $1 \in \mathscr{S}(n)$ for all odd $n \in \mathbb{N}$; and Sun & Sun's theorem asserts that $\mathscr{S}(n)$ contains a perfect square and a perfect cube for every n. Our goal in this note is to completely describe the set $\mathscr{S}(n)$.

We have not found our results stated anywhere in the literature, but we do not claim they are novel. In the introduction to his resolution [1] of Fermat's polygonal number conjecture, Cauchy poses the following problem: Décomposer un nombre entier donné en

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¹See https://mathoverflow.net/questions/37278/euler-and-the-four-squares-theorem.

n	exceptional T
1	Ø
2	Ø
3	Ø
4	Ø
5	Ø Ø Ø
6	Ø
7	
8	{±2} ∅
9	Ø
10	Ø
11	Ø
12	Ø
13	Ø
14	
15	Ø
16	$\{\pm 2, \pm 6\}$

n	exceptional T
17	Ø
18	Ø
19	Ø
20	Ø
21	Ø
22	Ø
23	Ø
24	$\{\pm 2, \pm 6\}$
25	Ø
26	Ø
27	Ø
28	{0}
29	Ø
30	Ø
31	Ø
32	$\{\pm 2, \pm 4, \pm 6, \pm 10\}$

n	exceptional T
33	Ø
34	Ø
35	Ø
36	Ø
37	Ø
38	Ø
39	Ø
40	$\{\pm 2, \pm 6, \pm 10\}$
41	Ø
42	Ø
43	Ø
44	{±8} ∅
45	
46	Ø
47	Ø
48	$\{\pm 2, \pm 6, \pm 10\}$

TABLE 1. Values of T satisfying (2) and (3) but not belonging to $\mathcal{S}(n)$.

quatre quarrés dont les racines fassent une somme donnée.² For Cauchy, "racines" are nonnegative; hence, he is asking for a description of

$$\mathscr{S}^+(n) := \{a+b+c+d : a,b,c,d \in \mathbb{N} \text{ and } a^2+b^2+c^2+d^2=n\}.$$

Cauchy goes on to prove a partial characterization of $\mathscr{S}^+(n)$ (see Remark 1 below for a summary of his results), by essentially the same methods we describe below. Despite being anticipated, we believe an explicit description of $\mathscr{S}(n)$ is sufficiently interesting (and Cauchy's work on $\mathscr{S}^+(n)$ sufficiently underappreciated) to warrant popularization here. Moreover, we will show how our characterization of $\mathscr{S}(n)$ immediately implies both Sun & Sun's theorems and a generalization of Euler's conjecture.

We begin by recording two easy observations. First, since an integer and its square have the same parity, every $T \in \mathcal{S}(n)$ satisfies

$$T \equiv n \pmod{2}. \tag{2}$$

Second, for any real numbers a, b, c, d, the Cauchy-Schwarz inequality yields

$$(a+b+c+d)^2 \le (a^2+b^2+c^2+d^2)(1^2+1^2+1^2+1^2)$$

= $4(a^2+b^2+c^2+d^2)$;

it follows that every $T \in \mathcal{S}(n)$ satisfies

$$T^2 \le 4n. \tag{3}$$

As shown in Table 1, the necessary conditions (2) and (3) are quite often (but not always) sufficient for membership in $\mathcal{S}(n)$. The following theorem, which is our main result, tells the full story.

²Decompose a given whole number into four squares whose roots make a given sum.

Theorem 2. Suppose n and T are integers satisfying (2). Then $T \in \mathcal{S}(n)$ if and only if $4n - T^2$ is a sum of three integer squares.

Note that (3) is implied by the condition on $4n - T^2$ and so does not need to be included explicitly as a hypothesis in Theorem 2.

To convince the reader that Theorem 2 qualifies as a complete description of $\mathcal{S}(n)$, we recall the following classical result (see the Appendix to Chapter IV of [2] for a proof).

Legendre–Gauss three-squares theorem. Let $n \in \mathbb{N}$. Then n can be written as a sum of three squares if and only if $n \neq 4^k(8\ell+7)$ for any $k, \ell \in \mathbb{N}$.

Proof of Theorem 2. We begin by recording the easily-verified identity

$$(2(a+b)-T)^{2} + (2(a+c)-T)^{2} + (2(b+c)-T)^{2} + T^{2}$$

$$= 4a^{2} + 4b^{2} + 4c^{2} + 4(T-a-b-c)^{2}. (4)$$

Thus if $T \in \mathcal{S}(n)$, say with $a^2 + b^2 + c^2 + d^2 = n$ and a + b + c + d = T, then

$$(2(a+b)-T)^{2} + (2(a+c)-T)^{2} + (2(b+c)-T)^{2} = 4n - T^{2},$$

so that $4n - T^2$ is a sum of three squares.

Conversely, suppose that $4n - T^2$ is a sum of three squares, say

$$4n - T^2 = A^2 + B^2 + C^2. (5)$$

In view of (4), it is enough to show that—after possibly swapping the signs of A, B, C—there are $a, b, c \in \mathbb{Z}$ with

$$2a + 2b - T = A$$
, $2a + 2c - T = B$, $2b + 2c - T = C$. (6)

Indeed, in that case setting d = T - (a + b + c), we have

$$a+b+c+d=T$$
.

and

$$4(a^{2} + b^{2} + c^{2} + d^{2}) - T^{2}$$

$$= (2a + 2b - T)^{2} + (2a + 2c - T)^{2} + (2b + 2c - T)^{2}$$

$$= A^{2} + B^{2} + C^{2} = 4n - T^{2}.$$

so that

$$a^2 + b^2 + c^2 + d^2 = n.$$

Thus, we focus our attention on (6).

Solving for a, b, c in terms of A, B, C gives

$$a = \frac{1}{4}(A+B-C+T), \ b = \frac{1}{4}(A-B+C+T), \ c = \frac{1}{4}(-A+B+C+T).$$

We claim that A, B, C, and T must all have the same parity. To see this, note that (5) gives $A^2 + B^2 + C^2 \equiv -T^2 \pmod{4}$. If T is odd then $A^2 + B^2 + C^2 \equiv 3 \pmod{4}$, and a moment's thought shows that all of A, B, and C must be odd. Similarly, if T is even, then $A^2 + B^2 + C^2 \equiv 0 \pmod{4}$, and this forces A, B, and C to all be even. In either case, the difference between any pair of A, B, A and C is even, so the difference between

any pair of a, b, and c is an integer. It follows that if any of $a, b, c \in \mathbb{Z}$, then all three are in \mathbb{Z} . Moreover,

$$A + B - C + T \equiv A + B + C + T \equiv A^2 + B^2 + C^2 + T^2 \equiv 4n \equiv 0 \pmod{2},$$

and so the only way we can fail to have $a \in \mathbb{Z}$ (and hence all of $a, b, c \in \mathbb{Z}$) is if

$$A + B - C + T \equiv 2 \pmod{4}. \tag{7}$$

If T is odd, then A is odd, and so if necessary we can replace A with -A to avoid (7). If T is even, we will show that (7) cannot occur. Indeed, (7) implies that $8 \nmid (A + B - C + T)^2$. But

$$(A + B - C + T)^{2}$$

$$= A^{2} + B^{2} + C^{2} + T^{2} + 2(AB - AC + AT - BC + BT - CT)$$

$$= 4n + 2(AB - AC + AT - BC + BT - CT)$$

$$\equiv 0 \text{ (mod 8)};$$

here we used that $n \equiv T \equiv 0 \pmod{2}$ and that all of A, B, C, and T are even.

Let us see how Theorem 2 makes quick work of both the conjecture of Euler and the theorems of Sun & Sun. We begin with the latter. If 4n itself is a sum of three squares, then Theorem 2 shows that $T=0\in \mathcal{S}(n)$, and 0 is both a square and a cube. Otherwise, by the Legendre–Gauss theorem, $4n=4^{k+1}(8\ell+7)$, where k and ℓ are nonnegative integers. Then

$$4n - (2^k)^2 = 4^k (32\ell + 27), \quad 4n - (2^{k+1})^2 = 4^{k+1} (8\ell + 6),$$

and $4n - (2^{k+2})^2 = 4^{k+1} (8\ell + 3);$

invoking the Legendre–Gauss theorem once more, we see that all three of these numbers are sums of three squares. By Theorem 2, all of $2^k, 2^{k+1}, 2^{k+2}$ must belong to $\mathcal{S}(n)$. Clearly, the set $\{2^k, 2^{k+1}, 2^{k+2}\}$ contains both a square and a cube.

As for Euler's conjecture, we prove the following generalization (which, for most n, gives a very simple description of $\mathcal{S}(n)$):

Proposition 3. Suppose $n \in \mathbb{N}$ is not a multiple of 4. Then

$$\mathscr{S}(n) = \{ T \equiv n \pmod{2} : |T| \le 2\sqrt{n} \}.$$

Remark 1. Cauchy proves that if $T \in \mathcal{S}^+(n)$, then $4n - T^2$ is a sum of three squares, and that when $4 \nmid n$,

$$\mathscr{S}^+(n) \supseteq \{ T \equiv n \pmod{2} : \sqrt{3n-2} - 1 \le T \le 2\sqrt{n} \}.$$

See [1, Corollary I of Theorem I, Theorem IV, and Corollary II of Theorem III].

Proof of Proposition 3. In view of Theorem 2, our task is to show that $4n-T^2$ is a sum of three squares whenever $4 \nmid n$. Suppose first that n is odd, so that T is also odd. Then $4n \equiv 4 \pmod 8$ and $T^2 \equiv 1 \pmod 8$, whence $4n-T^2 \equiv 3 \pmod 8$. By the Legendre-Gauss theorem, $4n-T^2$ is a sum of three squares, and we're done. Now suppose instead that n is twice an odd integer. Then T is even, say T=2t, so that $4n-T^2=4(n-t^2)$. It will suffice to show that $n-t^2$ is a sum of three squares, for then $4(n-t^2)$ is as well. Since $n \equiv 2 \pmod 4$ and $t^2 \equiv 0$ or $1 \pmod 4$, we have $n-t^2 \equiv 1$ or $2 \pmod 4$. In

particular, $n-t^2$ is not of the form $4^k(8\ell+7)$, and so the desired conclusion follows from the Legendre–Gauss theorem. This completes the proof.

We conclude this note with a few remarks about the structure of $\mathcal{S}(n)$ for general n. When $8 \mid n$, it is easy to see that any integer solution to

$$a^2 + b^2 + c^2 + d^2 = n$$

has all of a, b, c, d even. Thus, there is a bijection $(a, b, c, d) \leftrightarrow (a/2, b/2, c/2, d/2)$ between representations of n as a sum of four squares and representations of n/4. Consequently,

$$\mathscr{S}(n) = 2\mathscr{S}(n/4),$$

where the notation on the right-hand side means dilation by a factor of 2. Iterating, if k is the largest nonnegative integer for which $2^{2k+3} \mid n$, we find that

$$\mathscr{S}(n) = 2^{k+1} \mathscr{S}(n/4^{k+1}).$$

We have from our choice of k that $2 \mid n/4^{k+1}$ while $8 \nmid n/4^{k+1}$.

The observations of the last paragraph show that to describe $\mathcal{S}(n)$, it is enough to consider those cases where $8 \nmid n$. When $4 \nmid n$, Proposition 3 tells us the answer. However, when $4 \mid n$, it does not seem that there is much to be said beyond what follows immediately from Theorem 2 and the Legendre–Gauss theorem.

The situation becomes both clearer and a bit cleaner if one is willing to shift perspective. Rather than first picking n and asking for a description of the elements of $\mathcal{S}(n)$, we may pick T and ask for which n we have $T \in \mathcal{S}(n)$.

Proposition 4. Let $T \in \mathbb{Z}$. Assume that $n \geq T^2/4$ and $n \equiv T \pmod{2}$.

- (1) If T is odd, then $T \in \mathcal{S}(n)$.
- (2) If T is twice an odd integer, then $T \in \mathscr{S}(n)$ if and only if $n \not\equiv 0 \pmod{8}$.
- (3) Suppose that $4 \mid T$. Then $T \in \mathcal{S}(n)$ if and only if

$$n \notin \bigcup_{k \ge 1} \{ T^2 - 4^k \pmod{2^{2k+3}} \}.$$

Here the right-hand side is an infinite union of disjoint residue classes modulo 2^{2k+3} , over positive integers k.

We leave the (routine) proof of this proposition to the interested reader.

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