ANALYTIC NUMBER THEORY OF POLYNOMIALS OVER FINITE FIELDS: THESIS OUTLINE

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Chapter I: History and Overview of Results

- Early history: Gauss, Kornblum, Artin, Weil.
- Dirichlet characters in the function field setting and the prime number theorem for arithmetic progressions: description and overview of the proofs
- Application of these results to a stronger 'lexicographic' version of the prime number theorem: i.e., the estimate

$$\pi_p(x) := \#\{A : |A| \leq x, A \text{ irreducible over } \mathbf{F}_p\} \sim x/\log_p x.$$

- Brief overview of work in multiplicative number theory over $\mathbf{F}_q[T]$ (applications of sieve methods, Car's work, Thorne's work on the Maier matrix method, Rhoades's adaptation of Granville-Sound's paper on Erdős-Kac)
- Summary of the work of Conrad, Conrad and Gross on irreducible specializations in genus zero (no proofs)
- Announcement of our new results
- Include application to perfect polynomials

Chapter II: Substitution Method

Essentially the contents of the Montreal paper:

- Overview of the substitution method
- A uniform version of the Bateman-Horn conjecture
- Extension of Hall's results on twin primes
- Proof of "Hypothesis H" for q large
- Proof of Bunyakovsky's conjecture for "most" single-polynomial families

Chapter III: Lightweight Twin Prime Conjecture

- Elementary attack via Selberg's upper-bound sieve and exponential sums
- A complete proof via the circle method
- Fill in the gaps in Hayes's proof of an asymptotic formula (as $q \to \infty$) for the number of (not necessarily) prime pairs P, Q of degree $n+1$ whose difference is a prescribed polynomial H of degree n. (This improves Lemma 4 in my old draft of the Lightweight Twin Prime paper.) A similar argument allows one to count (as $q \to \infty$) the number of monic prime pairs P, Q of degree n over $\mathbf{F}_q$ whose difference belongs to $\mathbf{F}_q$, a quantity previously considered by Effinger, Hicks and Mullen.

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Chapter IV: Chebotarev Density Method

- Description of the method, including a summary of the results of S. D. Cohen and those of Bender & Wittenberg
- Proof of the main theorem of the counting paper

Chapter V: Applications of the Chebotarev Density Method

- Application to twin prime pairs of odd/even degree, etc.
- Proof that Brun's constant over $\mathbf{F}_p$ tends to $\pi^2/6$ as $p \to \infty$
- A Hilbert Irreducibility analogue and the proof that infinitely many primes are sums of three prime cubes in characteristic > 2
- Proof that for $q \gg_n 0$ and $\gcd(q, 2n) = 1$, every polynomial of degree n over $\mathbf{F}_q$ with leading coefficient 2 is the sum of two monic primes
- A conjecture about the Poisson distribution of prime gaps and a proof (a la Gallagher) assuming a suitably uniform version of the k-tuples conjecture
- Proof that the gaps are Poisson over prime fields $\mathbf{F}_p$ when p is much larger than the degree over which we sample
- Other statistics on prime values of polynomials and the connection with random permutations (e.g., smoothness, number of prime factors, etc.)
- Other examples of the method (e.g., infinitely many twin prime polynomial pairs?)

Chapter VI: Open Questions

- Obvious items of interest: remove restrictions on the characteristic, improve the implied constants so as to widen the range of the results
- Mention Effinger's result that if $T^n + T^3 + T^2 + T + 1$ is irreducible over $\mathbf{F}_2$ infinitely often, then there are infinitely many "twin prime pairs" $f, f + T^2 + T$ over $\mathbf{F}_2$.
- What about some of these questions for primitive polynomials? Mention the progress on the counting problem, and quote Lenstra's result about primitive polynomials in arithmetic progressions. (Maybe prove this last result, since no proof appears in print?)

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