MATH 4400 – Learning objectives to meet for Exam #2

The exam covers everything from your last exam (starting with the 3/14 lecture) through class on Friday, 4/15.

What to be able to state

Basic definitions

You should be able to give complete and precise definitions of each of the following items:

- Jacobi symbols
- the prime counting function $\pi(x)$
- arithmetic function
- multiplicative function
- $d(n)$, $d_k(n)$, $\varphi(n)$, $\sigma(n)$, $\mu(n)$
- $\mathbf{H}$ (the real quaternions) and $\mathcal{L}$ (the Lipschitz integers inside $\mathbb{H}$); you should know both the elements and the definitions of addition and multiplication
- conjugation in $\mathbf{H}$, norm on $\mathbf{H}$

Big theorems

Give full statements of each of the following results, making sure to indicate all necessary hypotheses. For results proved in class, describe the components and main ideas of the proof.

- Basic properties of the Jacobi symbol
- Quadratic reciprocity for the Jacobi symbol
- The rules for Jacobi symbols $\left(\frac{-1}{P}\right)$ and $\left(\frac{2}{P}\right)$
- There are infinitely many primes.
- Legendre's formula $\mathrm{ord}_p(n!) = \sum_{k\geq 1} \lfloor n/p^k \rfloor$.
- Chebyshev's theorem: There are constants $c_1, c_2 > 0$ and a real number x_0 such that $c_1 x/\log x \leq \pi(x) \leq c_2 x/\log x$ for all $x > x_0$.
- If $f(n)$ is multiplicative, so is $g(n) = \sum_{d|n} g(d)$.
- If $f(n)$ and $f'(n)$ are multiplicative, so is $g(n) = \sum_{d|n} f(d) f'(n/d)$.
- $\sum_{d|n} \mu(d) = 1$ if $n = 1$ and 0 otherwise
- Möbius inversion formula
- If $g(n) = \sum_{d|n} f(d)$ and g is multiplicative, then f is multiplicative.
- Euler's two square theorem
- $\overline{\alpha\beta} = \overline{\beta} \cdot \overline{\alpha}$ for $\alpha, \beta \in \mathbf{H}$
- $N(\alpha\beta) = N(\alpha)N(\beta)$ for $\alpha, \beta \in \mathbf{H}$

What to be able to compute

You should know how to use methods discussed in class to perform the following computations.

- compute Jacobi symbols and know how to interpret the results (e.g., whether or not they tell you that a is a square mod P)
- compute the highest power of a prime dividing a factorial and related quantities (e.g., the number of zeros at the end of a factorial)
- compute values of multiplicative functions by exploiting multiplicativity
- perform basic arithmetic operations in $\mathbf{H}$ (addition, multiplication, taking inverses)

Format

Pretty muchu the same as Exam #1: You can expect $\leq$ five (multi-part) questions on the exam. At least 40% of the exam will test you on definitions or computations. The rest of the exam is designed to test your comfort level manipulating the concepts and theorems from class. For the most part, I am not interested in having you regurgitate proofs of results from class/HW; I want to know if you have internalized the ideas enough to solve similar problems.

Practice problems

1. (Computations)
   (a) Evaluate $\left(\frac{21}{311}\right)$. Without knowing whether 311 is prime, what can you conclude about whether $x^2 \equiv 21 \pmod{311}$ has a solution?
   (b) Determine $\mathrm{ord}_3(500!)$.
   (c) Determine $\sum_{d|60} \mu(d)\sigma(d)$. Show your work.

2. Define an arithmetic function χ by setting

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{3}, \\ -1 & \text{if } n \equiv -1 \pmod{3}, \\ 0 & \text{if } 3 \mid n, \end{cases} .$$

   (a) Show that χ is a multiplicative function.
   (b) Define $r(n) = \sum_{d|n} \chi(d)$. Show that $r(n) \geq 0$ for all positive integers n.

3. (a) State Legendre's sum formula for $\mathrm{ord}_p(n!)$.
   (b) Use part (a) to derive a sum formula for $\mathrm{ord}_p(6n)! - \mathrm{ord}_p((3n)!(2n)!n!)$.
   (c) Prove that $\frac{(6n)!}{(3n)!(2n)!n!}$ is an integer for every positive integer n.

4. Suppose $p \equiv 1 \pmod{4}$ is prime. We know from class that $p = a^2 + b^2$ for some integers a, b, which we can assume to be positive (by swapping the signs of a and b if necessary).
   (a) Show that exactly one of a, b is odd.
   (b) Show that $2p = (a+b)^2 + (a-b)^2$.
   (c) Assuming a is odd, prove that $\left(\frac{a}{p}\right) = 1$ and $\left(\frac{a+b}{p}\right) = (-1)^{((a+b)^2-1)/8}$.

5. Assume f is an arithmetic function with $f(1) \neq 0$. Prove the existence of an arithmetic function g for which

$$\sum_{d|n} f(d)g(n/d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$