MATH 4400/6400 – Homework #5

posted April 22, 2019; due April 26, 2019

The Möbius Inversion was a small but disruptive wormhole used in the Antarian Trans-stellar Rally, a race held in the Delta Quadrant to commemorate the signing of a treaty which brought four warring races to peace.
– Memory Alpha, describing the *Star Trek: Voyager* episode "Drive"

Directions. Give complete solutions, providing full justifications when appropriate. Your assignment must be stapled if it goes on beyond one page. Starred problems are required for MATH 6400 students and extra credit for 4400 students.

We use the standard notation for arithmetic functions from class. In particular, μ is the Möbius function, while

$$\tau(n) = \sum_{d\mid n} 1, \quad \text{and} \quad \sigma(n) = \sum_{d\mid n} d.$$

1. Prove or disprove: For all positive integers n, we have that $\sigma(n)\phi(n) \le n^2$.

2. Suppose that f and g are multiplicative arithmetic functions. Define a new arithmetic function h by setting
$$h(n) = \sum_{d\mid n} f(d)g(n/d) \qquad (\text{for all } n \in \mathbf{Z}^+).$$
Prove that h is also multiplicative.

3. Find and prove simple formulas for each of the functions
$$\sum_{d\mid n} \mu(d)\tau(n/d), \qquad \sum_{d\mid n} \mu(d)\tau(d), \qquad \sum_{d\mid n} \mu(d)^2\phi(d).$$
For the second and third sums, express your answers in terms of the prime factorization of n.

4. Prove that for all positive integers n,
$$\sum_{d\mid n} \tau(d)^3 = \left(\sum_{d\mid n} \tau(d)\right)^2.$$
Hint: You may assume the identity $\sum_{k=1}^{m} k^3 = (\sum_{k=1}^{m} k)^2$, valid for all positive integers m.

5. Let $\lambda(n)$ be the arithmetic function defined as follows. If $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, then
$$\lambda(n) = (-1)^{e_1+e_2+\cdots+e_k}.$$
For example, since $12 = 2^2 \cdot 3$, we have $\lambda(12) = (-1)^{2+1} = -1$. Show that if n is not a perfect square, then
$$\sum_{d\mid n} \lambda(d) = 0.$$
What happens if n is a square?

6. (a) Classify all n for which $\phi(n)$ is an odd number. Justify your answer.
   (b) Classify all n for which $\tau(n)$ is an odd number. Justify your answer.
   (c) Classify all n for which $\sigma(n)$ is an odd number. Justify your answer.

7. (Euler) Prove that if n is odd and $\sigma(n)$ is twice an odd number, then $n = p^\alpha m^2$ for some prime p and some positive integers α and m, where $p \nmid m$. Moreover, $p \equiv \alpha \equiv 1 \pmod 4$.

   In particular, every odd perfect number n has the stated form!

8. What (if anything) is wrong with the following "proof" that all perfect numbers are even?

   > If n is a perfect number, then $\sigma(n) = 2n$. In other words, $2n = \sum_{d\mid n} d$. So if f and g are the arithmetic functions defined by $g(n) = 2n$ and $f(n) = n$, then $g(n) = \sum_{d\mid n} f(d)$. So by Möbius inversion,
   >
   > $$n = f(n) = \sum_{d\mid n} \mu(n/d) g(d) = \sum_{d\mid n} \mu(n/d)\cdot(2d) = 2\left(\sum_{d\mid n}\mu(n/d)d\right).$$
   >
   > The final parenthesized expression is an integer, and so n is even.

9. (*) Let $d_1(n)$ denote the number of positive divisors of n that are congruent to 1 modulo 4, and let $d_3(n)$ denote the number of positive divisors of n that are congruent to 3 modulo 4. For example, $d_1(45) = 4$, and $d_3(45) = 2$.

   Show that $d_1(n) \ge d_3(n)$ for every positive integer n.

   Hint: Let χ be the arithmetic function which takes the value 0 on even integers, and which on odd numbers n is given by $\chi(n) = 1$ for $n \equiv 1 \pmod 4$, and $\chi(n) = -1$ for $n \equiv -1 \pmod 4$. Prove that χ is multiplicative, and then relate χ to the difference function $d_1 - d_3$.