MATH 4000/6000 – Learning objectives to meet for Exam #2

The exam will cover §2.1–§3.2 (though what is covered by 3/18), excluding §2.5. The material on rational functions and partial fractions at the end of §3.1 is not examinable.

What to be able to state

Basic definitions

You should be able to give precise descriptions of all of the following:

- construction of $\mathbf{Q}$ from $\mathbf{Z}$ (starting with $\mathbf{Q}^{\text{pre}}$, introducing an equivalence relation, and taking equivalence classes)
- existence of $\mathbf{R}$ (that is: there is a complete ordered field satisfying the conclusion of the monotone convergence theorem)
- the construction of $\mathbf{C}$ from $\mathbf{R}$ by ordered pairs
- complex conjugate of a complex number
- absolute value of a complex number
- polar form of a complex number
- definition of the ring $R[x]$ (starting with a commutative ring R) and allied concepts (such as the degree of a polynomial)
- definition of an irreducible polynomial (prime polynomial) in $F[x]$, with F a field
- gcd of two elements of $F[x]$
- subring of a ring
- subfield of a field, and field extensions
- definition of $F[\alpha]$, where F is a field and α is an element of a field extension of F
- what it means to say a polynomial $f(x) \in F[x]$ splits in an extension K of F
- what it means to say a field K is a splitting field of $f(x)$ over F

Big theorems

Give full statements of each of the following results, making sure to indicate all necessary hypotheses. For results proved in class, describe the components and main ideas of the proof.

- $\mathbf{Q}$ is field containing $\mathbf{Z}$, possessing a positive set $\mathbf{Q}^+$
- theorems associated with multiplication of complex numbers in polar form, including de Moivre's theorem
- there are n distinct nth roots of 1 in $\mathbf{C}$, namely the numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$

- a nonzero complex number has exactly n distinct nth roots
- the quadratic formula
- description of the roots of a cubic polynomial
- $\deg(a(x)b(x)) = \deg a(x) + \deg b(x)$ when R is a domain
- the division algorithm in $F[x]$
- remainder theorem
- root-factor theorem
- the gcd of two polynomials in $F[x]$ is determined up to a nonzero constant
- if $a(x), b(x) \in F[x]$ and $d(x)$ is a gcd of $a(x)$ and $b(x)$, then $d(x) = a(x)X(x)+b(x)Y(x)$ for some $X(x), Y(x) \in F[x]$
- Euclid's lemma for $F[x]$ and the unique factorization theorem for $F[x]$
- if $f(x) \in F[x]$ has degree 2 or 3, then $f(x)$ is irreducible over F if and only if $f(x)$ has no root in F
- the Fundamental Theorem of Algebra (statement only)
- If F, K are fields with $F \subseteq K$, and $\alpha \in K$ is the root of a nonconstant polynomial in $F[x]$, then $F[\alpha]$ is a field
- Let F, K be fields with $F \subseteq K$. If $\alpha \in K$ is the root of an irreducible polynomial in $F[x]$ of degree n, then

  $$F[\alpha] = \{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} : a_0, a_1, \ldots, a_{n-1} \in F\}.$$

  In fact, every element of $F[\alpha]$ has a unique expression in the form $a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}$, with the $a_i \in F$.

What to be able to compute

You are expected to know how to use the methods described in class to solve the following problems.

- Basic computations with complex numbers, in either rectangular (i.e., $a + bi$) or polar form. Here "basic computations" includes computing nth roots in polar form.
- Perform "long division" of polynomials with quotient and remainder; use this to perform the Euclidean algorithm, compute gcds, and express your gcd as a linear combination
- Apply the remainder theorem and/or the root factor theorem to determine the remainder when $f(x)$ is divided by $x - c$
- You will **not** be asked to use the cubic formula to solve any cubic equations on this exam.

Extra problems

Carefully review the HW solutions. I also recommend looking at the following problems:

§2.1: 7(a,b). Also: Show that if $a, b, c, d \in \mathbf{Z}^+$ and $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

§2.2: 8, 12

§2.3: 2, 9(a,c,e), 20, 21(a), 22

§3.1: 1, 2(b,c,d,e), 5, 9(c), 10(b,d), 11, 12, 14

§3.2: 1, 13. Also: (a) Prove that $\mathbf{C}$ is a splitting field for x^2+1 over $\mathbf{R}$. (b) Prove that $\mathbf{Q}[i]$ is a splitting field for $x^2-8x+17$ over $\mathbf{Q}$. (c) Over $\mathbf{R}$, the polynomial x^4-10x^2+1 factors as $(x-(\sqrt{2}+\sqrt{3}))(x-(\sqrt{2}-\sqrt{3}))(x-(-\sqrt{2}+\sqrt{3}))(x-(-\sqrt{2}-\sqrt{3}))$. Use this to prove that $\mathbf{Q}[\sqrt{2},\sqrt{3}]$ is a splitting field for x^4-10x^2+1 over $\mathbf{Q}$.