MATH 3100 – Learning objectives to meet for Exam #3

The exam will cover §§2.3–3.2 of the course notes, up through the end of lecture on Wednesday, November 16, with a strong emphasis on §§2.4–3.1. Strong emphasis means you can expect at most one problem testing material from §2.3 or §3.2.

What to be able to state

Basic definitions

You should be able to give complete and precise definitions of each of the following items:

- Absolute convergence and conditional convergence
- Power series centered at 0, power series centered at a
- Domain of convergence, radius of convergence
- Pointwise convergence of a sequence of functions f_n to f on a set A
- Distance $d_A(f, g)$ between two functions f and g on a set A
- Uniform convergence of a sequence of functions f_n to f on a set A
- Convergence of a series of functions $\sum_{n=1}^{\infty} f_n$ on a set A
- Uniform convergence of a series of functions $\sum_{n=1}^{\infty} f_n$ on a set A
- Maclaurin series of a function f
- nth order Maclaurin polynomial $P_n^f(x)$

Big theorems

Be familiar with all of the following theorems, making sure that you can give full statements for those not already spelled out below. For results proved in class, describe the components and main ideas of the proof.

- Alternating series test
- Ratio test
- Key Lemma: If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_0$, it converges absolutely for all x with $|x| < |x_0|$
- Characterization of the domain of convergence of a power series centered at 0 (Proposition 2.4.4)
- A uniformly convergent sequence of continuous functions has a continuous limit (Theorem 3.1.11; in fact, we proved something a bit more general than the statement in the notes, since we didn't require that A be a closed interval $[a, b]$)
- Uniform convergence plays nice with integration on closed intervals (Proposition 3.1.12)

- If $\sum_{k=0}^{\infty} c_k x^k$ has radius of convergence $R > 0$, and $[a,b]$ is any closed interval with $[a,b] \subset (-R,R)$, then $\sum_{k=0}^{\infty} c_k x^k$ converges uniformly on $[a,b]$.
- A uniformly convergent series of continuous functions has a continuous sum.
- If $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[a,b]$, then $\int_a^b s_n(x)\,dx \to \int_a^b s(x)\,dx$, as $n \to \infty$. Here $s_n(x) = \sum_{k=1}^{n} f_k(x)$ and $s(x) = \sum_{k=1}^{\infty} f_k(x)$.
- Differentation and integration theorems for power series
- Weierstrass M-test
- If f is represented by a power series on an open interval $(-r,r)$ (with $r > 0$), that power series must be the Maclaurin series of f.
- sum and product rules for Maclaurin polynomials (no proofs)

What to expect on the exam

There will be five questions on the exam, possibly having multiple parts. These will include...

- A problem testing you on a basic definition and requiring you to give a proof using this definition
- A problem or problem part asking you to determine the domain and/or radius of convergence of (one or more) power series, possibly centered at a nonzero point
- At least one problem asking you to determine a pointwise limit of a sequence of functions and to further decide whether or not the convergence to the limit is uniform

Practice problems

1. Find the domain of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^{2022}}$. Do the same for $\displaystyle\sum_{n=1}^{\infty} \frac{n!^2}{2^n} x^n$.
2. Suppose f is a function on $A = [0,1]$ with $d_A(f,0) \le 10$. Assuming $\int_0^1 f(x)\,dx$ exists, prove that $-10 \le \int_0^1 f(x)\,dx \le 10$.
3. Consider the functions f_n defined on $A = [0,\infty)$ by $f_n(x) = \frac{x}{1+x^n}$. Determine the pointwise limit function f of the f_n. Does $f_n \to f$ uniformly on A? Does the answer to this last question change if $[0,\infty)$ is replaced by $[0,1/2]$?
4. State the Weierstrass M-test. Use that result to show that the series $\sum_{k=1}^{\infty} 2^{kx}\sin(kx)/3^k$ converges uniformly on $A = [0,1]$. If we set $s(x) = \sum_{k=1}^{\infty} 2^{kx}\sin(kx)/3^k$, explain briefly why $s(x)$ is continuous on A.
5. Recall from class that $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| < 1$. Assuming this, find the exact value of the series
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$
Simplify your answer as much as possible.

6. Find a power series representing $\displaystyle\int_0^x \frac{dt}{1-t^3}$ for $|x| < 1$. You **do not** have to evaluate $\displaystyle\int_0^x \frac{dt}{1-t^3}$ in closed form (though it is possible to do so!).