

Math 4000/6000 – Homework #1

posted August 17, 2016; due at the **start of class** on August 26, 2016

An engineer, a physicist, and a mathematician are shown a pasture with a herd of sheep, and told to put them inside the smallest possible amount of fence.

The ENGINEER is first. She herds the sheep into a circle and then puts the fence around them, declaring, “A circle will use the least fence for a given area, so this is the best solution.”

The PHYSICIST is next. He creates a circular fence of infinite radius around the sheep, and then draws the fence tight around the herd, declaring, “This will give the smallest circular fence around the herd.”

The MATHEMATICIAN is last. After giving the problem a little thought, she puts a small fence around herself and then declares, “I define myself to be on the outside!”

Assignments are expected to be **neat** and **stapled**. Illegible work may not be marked. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

Carefully explain your answers. In problems #1(a) and #2 **only**, you must justify which algebraic properties (properties A1–A4, M1–M3, D1 on the handout) you are using at every step of the proof. In your write-up, please refer to these properties by name rather than number. For all other problems, you do not have to justify those kinds of algebraic manipulations. Note: \mathbb{Z}^+ means the same as \mathbb{N} (the book’s notation).

1. (Trichotomy for inequalities) Let $a, b \in \mathbb{Z}$.
 - (a) Prove that $-(a + -b) = b + -a$.
 - (b) Using the result of (a), show that exactly one of the following holds: $a < b$, $a = b$, or $a > b$. (Recall that “ $a < b$ ” means that $b + -a \in \mathbb{Z}^+$, while “ $a > b$ ” means the same as “ $b < a$ ”.)
2. Prove that for any $a, b \in \mathbb{Z}$, we have
 - (a) $(-a)b = -(ab)$.
 - (b) $(-a)(-b) = ab$.
3. Let $a, b \in \mathbb{Z}$ and suppose that $a < b$.
 - (a) Prove that $a + c < b + c$ for every $c \in \mathbb{Z}$.
 - (b) Prove that $ac < bc$ for every $c \in \mathbb{Z}^+$.
4. Let $a, b \in \mathbb{Z}$.
 - (a) Prove that if $a < 0$ and $b < 0$, then $ab > 0$.
 - (b) Show that if $a < 0$ and $b > 0$, then $ab < 0$.
 - (c) Show that if $ab = 0$, then either $a = 0$ or $b = 0$.
5. (Laws of exponents) Let $a \in \mathbb{Z}$. Suppose that m, n belong to the set $\mathbb{Z}^+ \cup \{0\}$ of nonnegative integers.
 - (a) Prove that $a^m \cdot a^n = a^{m+n}$.
 - (b) Prove that $a^{mn} = (a^m)^n$.

Hint: If $m = 0$ or $n = 0$, this is easy (why?). So you can suppose $m, n \in \mathbb{N}$. Now think of m as fixed and proceed by induction on n .

6. In this exercise we outline a proof of the following statement, which was left as a “missing step” in our proof of the division theorem: If $a, b \in \mathbb{Z}$ with $b > 0$, the set

$$S = \{a - bq : q \in \mathbb{Z} \text{ and } a - bq \geq 0\}$$

has a least element.

- (a) Prove the claim in the case $0 \in S$.
- (b) Prove the claim in the case $0 \notin S$ and $a > 0$.
- (c) Prove the claim in the case $0 \notin S$ and $a \leq 0$.

Hint: (a) is easy. To handle (b) and (c), first show that in these cases S is a nonempty set of natural numbers, so that the well-ordering principle guarantees S has a least element as long as S is nonempty. To prove S is nonempty, show that in case (b), the integer a is an element of S . You will have to work a little harder to prove S is nonempty in case (c).

7. Use the binomial theorem to find formulas for the following sums, as functions of n , where n is assumed to be a natural number.

(a) $\sum_{k=0}^n \binom{n}{k}.$

(b) $\sum_{k=0}^n (-1)^k \binom{n}{k}.$

8. (To be done after Monday’s lecture) Use the Euclidean algorithm to find $\gcd(314, 159)$ and $\gcd(272, 1479)$. Show the steps, not just the final answer.

9. Show that if $a, b \in \mathbb{N}$ and $a \mid b$, then $a \leq b$.
10. Let a, b be nonnegative integers, not both zero. Define the set

$$I(a, b) = \{ax + by : x, y \in \mathbb{Z}\}.$$

(Thus, $I(a, b)$ is the set of all linear combinations of a, b , with coefficients from \mathbb{Z} . The letter I stands for *ideal*, which is a concept we will meet later in the course.)

- (a) Show that if a, b, q, r are integers with $a = bq + r$, then $I(a, b) = I(b, r)$.
 - (b) Explain why (a) implies that $I(a, b) = I(0, \gcd(a, b))$.
 - (c) Deduce from (b) that there are integers x and y with $\gcd(a, b) = ax + by$.
11. (*) (\mathbb{C} cannot be ordered) Let S be a subset of the complex numbers. Show that it is impossible for S to have all of the following three properties:
- (i) the sum of two elements of S is always in S ,
 - (ii) the product of two elements of S is always in S ,
 - (iii) for each complex number x , exactly one of the following holds: $x = 0$, $x \in S$, or $-x \in S$.
12. (*) Exercise 1.1.16 (this means Exercise 16 in §1 of Chapter 1). *Hint:* Write each number in $\{1, 2, \dots, 2n\}$ in the form $2^j \cdot q$, where j is a nonnegative integer and q is odd. How many different possibilities for q are there?