Proposition. *Suppose $\{a_n\}$ is Cauchy. Then $\{a_n\}$ converges.*

Proof. Since $\{a_n\}$ is bounded, there is a convergent subsequence $\{b_n\}$. Say $b_n \to L$.
Claim: $a_n \to L$.
Write $b_n = a_{g(n)}$, where $g\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing.
Since $b_n \to L$, we can pick $N_1 \in \mathbb{N}$ so that whenever $n > N_1$, we have $|b_n - L| < \epsilon$.
Since a_n is Cauchy, we can pick $N_2 \in \mathbb{N}$ so that whenever $n, m > N_2$, we have $|a_n - a_m| < \epsilon$.
Let $N = \max\{N_1, N_2\}$.
If $n > N$, we have $g(n) \geq n > N$ [remember that g is strictly increasing], and so

$$|a_n - a_{g(n)}| < \epsilon/2.$$

Also,

$$|a_{g(n)} - L| = |b_n - L| < \epsilon/2.$$

Hence,

$$|a_n - L| = |(a_n - a_{g(n)}) + (a_{g(n)} - L)| \leq |a_n - a_{g(n)}| + |a_{g(n)} - L| < \epsilon/2 + \epsilon/2 = \epsilon.$$

[What's the intuition? A subsequence converging to L means we get close to L if we allow ourselves to skip terms. But if all the terms are getting closer to each other anyway — as in a Cauchy sequence — then the non-skipped terms also have to be getting closer to L.] □

SUMMARY: Converging is equivalent to Cauchyness

Lecture #11, §1.6: What is reality, ctd.

Theorem. *Let S be a nonempty subset of the real numbers that is bounded above. Then S has a least upper bound.*

Example 37.
- If $S = (0, 1)$ or $S = [0, 1]$, then $\operatorname{lub}(S) = 1$. [So the least upper bound may or may not be in the set.]
- If $S = \{x : x^2 < 2\}$, then $\operatorname{lub}(S) = \sqrt{2}$.
- If $S = \emptyset$, then every $U \in \mathbb{R}$ is an upper bound for S. So S has no lub.

[Notice that if the only numbers we knew about were rational numbers, the set $\{x : x^2 < 2\}$ would be bounded above and nonempty, but have no lub.]

Lemma. *If $\{a_n\}$ is a convergent sequence and $a_n \geq L$ for all n, then* $\lim_{n\to\infty} a_n \geq L$.

Lemma. *If $\{a_n\}$ is a convergent decreasing sequence with limit L, then $a_n \geq L$ for all n.*

Compare these with Propositions 1.4.16, 1.4.17.

Proof of the theorem. Let $U = U_0$ be any upper bound for S. Define

$$U_1 = U_0 - n \cdot 1, \quad \text{where } n \in \{0, 1, 2, \dots\} \text{ is maximal with } U_1 \text{ still an upper bound.}$$

Define

$$U_2 = U_1 - n \cdot \frac{1}{2}, \quad \text{where } n \in \{0, 1, 2, \dots\} \text{ is maximal with } U_2 \text{ still an upper bound,}$$

$$U_3 = U_2 - n \cdot \frac{1}{3}, \quad \text{where } n \in \{0, 1, 2, \dots\} \text{ is maximal with } U_3 \text{ still an upper bound,}$$

etc. Then

$$U_1 \geq U_2 \geq U_3 \dots,$$

and $\{U_i\}$ is bounded below (by any element of S). So we can define

$$U := \lim_{n\to\infty} U_n.$$

(Draw a number line with U labeled, then $U-1$ and $U-2$, but with $U-3$ not an upper bound, then draw $U-2-\frac{1}{2}$, etc., until folks get the idea.)
Claim: $U = \text{lub}(S)$.
First we prove U is an upper bound on S. Let $x \in S$.
Then $U_n \geq x$ for every $n \in \mathbb{N}$.
Hence, $U = \lim_{n\to\infty} U_n \geq x$.
This holds for all x, so U is an upper bound on S.

Now we prove U is the least upper bound. If not, then $U' = U - \frac{1}{m}$ is also an upper bound for some $m \in \mathbb{N}$. Since $\{U_n\}$ is decreasing, each $U_n \geq U$. In particular, $U_m \geq U$, and

$$U_m - \frac{1}{m} \geq U - \frac{1}{m}.$$

Hence,

$$U_m - \frac{1}{m}$$

is an upper bound on S. But this contradicts the choice of U_m. □

§1.7: More results from calculus

Definition. We say $f\colon \mathbb{R} \to \mathbb{R}$ is **right-continuous** at $x = a$ if for every $\epsilon > 0$, there is a $\delta > 0$ so that if $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ and $x \geq a$. Similarly for **left-continuous**. [Note that continuous is equivalent to being both left and right continuous at the given point.]

(Draw graph of a function which is right-continuous at $x = 0$ but not continuous there.)

Definition. We say f is **continuous on the closed interval** $[a, b]$ if f is right-continuous at a, left-continuous at b, and continuous at each point of (a, b).

(Draw graph of $\sin(x)$ on $[0, \pi]$.)

Theorem (Intermediate value theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$. Suppose $f(a) < 0$ and $f(b) > 0$, or vice versa. Then there is some $c \in (a, b)$ with $f(c) = 0$.*

Example 38. Show that there is some c for which $c^3 = c + 1$.

Solution. Consider the function $f(x) = x^3 - (x + 1)$. Then $c^3 = c + 1$ iff $f(c) = 0$.
Choose the closed interval $[0, 2]$. We have $f(0) = -1 < 0$ and $f(2) = 5 > 0$. So there is some $c \in (0, 2)$ with $f(c) = 0$. [There was some freedom in choosing the interval; in general, you have to experiment to find an interval that works for you.] □

[The other big theorem we want to prove in this section is the following.]

Theorem (Maximum value theorem)**.** *Suppose f is continuous on $[a, b]$. Then f assumes a maximum value on $[a, b]$. In other words, there is an $c \in [a, b]$ with $f(c) \geq f(x)$ for all $x \in [a, b]$.*

[Similarly, f assumes a minimum value. The proof is similar.]
(Draw complicated, random continuous function on $[0, 1]$, and label a value of c.) [Note that the proof only shows that such a maximum exists — it doesn't tell you how to find it. For that, you use calculus!]

Lemma. *Suppose f is continuous on $[a, b]$. Suppose that $a_n \in [a, b]$ converges to a limit L. (Then $a \leq L \leq b$.) Then*

$$f(a_n) \to f(L).$$

(Compare with Proposition 1.5.10.)

Theorem (Intermediate value theorem)**.** *Suppose that f is continuous on the closed interval $[a,b]$. Suppose $f(a) < 0$ and $f(b) > 0$, or vice versa. Then there is some $c \in (a,b)$ with $f(c) = 0$.*

Proof of the intermediate value theorem. Let $S = \{x \in [a,b] : f(x) \leq 0\}$.
(Draw a picture.)
Then S is nonempty, since $a \in S$.
Also S is bounded above, for example, by b.
So S has a least upper bound, say c, and $a \leq c \leq b$.
Claim: $f(c) = 0$.

Step 1: Show $f(c) \leq 0$.
This is clear if $c = a$.
Choose $x_1 \in [a,b]$ with $c - 1 < x_1 \leq c$ and $f(x_1) \leq 0$.
This has to be possible, otherwise $\max\{a, c-1\}$ would be an upper bound on S, contradicting that c is the lub.
Choose $x_2 \in [a,b]$ with $c - \frac{1}{2} < x_1 \leq c$ and $f(x_1) \leq 0$.
Choose $x_3 \in [a,b]$ with $c - \frac{1}{3} < x_1 \leq c$ and $f(x_1) \leq 0$,
etc. Then $x_i \to c$, so $f(x_i) \to f(c)$.
Each $f(x_i) \leq 0$, so $f(c) = \lim f(x_i) \leq 0$.

Step 2: Show $f(c) \geq 0$.
This is clear if $c = b$.
Now let x_i be a sequence of terms in $[a,b]$ with each $x_i > c$ and $x_i \to c$. For example,

$$x_1 = \min\{c+1, b\}, \quad x_2 = \min\{c + \frac{1}{2}, b\}, \quad x_3 = \min\{c + \frac{1}{3}, b\}, \ldots.$$

Each $x_i > c$, so $x_i \notin S$. Hence, $f(x_i) > 0$.
So each $f(x_i) \geq 0$.
Hence, $f(c) = \lim f(x_i) \geq 0$.
Finally, notice that since $f(a) < 0$ and $f(b) > 0$, we must have $c \in (a,b)$. □

Corollary (Intermediate value theorem, v2.0)**.** *Suppose f is continuous on $[a,b]$. For every α between $f(a)$ and $f(b)$, there is some $c \in (a,b)$ with $f(c) = y$.*

Proof: HW.

Theorem (Maximum value theorem)**.** *Suppose f is continuous on $[a,b]$. Then f assumes a maximum value on $[a,b]$. In other words, there is an $c \in [a,b]$ with $f(c) \geq f(x)$ for all $x \in [a,b]$.*

[As a prelude, we prove that its values there are bounded.]

Lemma. *Suppose that f is continuous on $[a,b]$. There is a real number M with $f(x) \leq M$ for all $x \in [a,b]$.*

Proof. Suppose not.
There is an $a_1 \in [a,b]$ with $f(a_1) > 1$, as otherwise $M = 1$ works.
There is an $a_2 \in [a,b]$ with $f(a_2) > 2$.
There is an $a_3 \in [a,b]$ with $f(a_3) > 3$, etc.
Now $\{a_n\}$ is bounded, so $\{a_n\}$ has a convergent subsequence $\{b_n\}$.
Write $b_n = a_{g(n)}$, where $g\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing.

$$f(b_1) > g(1) \geq 1, \quad f(b_2) > g(2) \geq 2, \quad f(b_3) > g(3) \geq 3, \quad \text{etc..}$$

In general, $f(b_n) > n$.
So $\lim_{n\to\infty} f(b_n) = \infty$.
But if $L = \lim_{n\to\infty} b_n$, then

$$\lim_{n\to\infty} f(b_n) = f(L),$$

and $f(L)$ is a finite real number. $\square$

Proof of the maximum value theorem. Let $S = \{f(x) : x \in [a,b]\}$. Then S is bounded above (by the lemma). Let $U = \operatorname{lub}(S)$.
Claim: There is some $c \in [a,b]$ with $f(c) = U$.
Then for any $x \in [a,b]$, we have $f(x) \le U = f(c)$.
Since $U = \operatorname{lub}(S)$, we know $U - 1$ is not an upper bound on S.

So we can pick $a_1 \in [a,b]$ with $f(a_1) > U - 1$.
Similarly, we can pick $a_2 \in [a,b]$ with $f(a_2) > U - \frac{1}{2}$.
So we can pick $a_3 \in [a,b]$ with $f(a_3) > U - \frac{1}{3}$, etc.
Let $\{c_n\}$ be a convergent subsequence of $\{c_n\}$, and let $c = \lim_{n\to\infty} c_n$.
Say $c_n = a_{g(n)}$, where $g\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing.
Then $f(c_1) = f(a_{g(1)}) > U - \frac{1}{g(1)} \ge U - 1$.
Then $f(c_2) = f(a_{g(2)}) > U - \frac{1}{g(2)} \ge U - \frac{1}{2}$, etc.
In general, $f(c_n) > U - \frac{1}{n}$.
So for all natural numbers n,

$$f(c_n) + \frac{1}{n} \ge U.$$

Hence,

$$\lim_{n\to\infty} \left(f(c_n) + \frac{1}{n} \right) \ge U.$$

But LHS is $f(\lim_{n\to\infty} c_n) = f(c)$.
So $f(c) \ge U$.
But by definition of U, we also have $f(c) \le U$.
So $f(c) = U$. $\square$

Is there a minimum value theorem?

Theorem. *Let $f(x)$ be continuous on $[a,b]$. There is a $c \in [a,b]$ with $f(c) \le f(x)$ for all $x \in [a,b]$.*

Proof. Apply the maximum value theorem to $-f(x)$, which is also continuous on $[a,b]$. $\square$

Lecture #13, §2.1: Introduction to series

Recall **Sigma-notation**: If $a_1, \ldots, a_n$ are real numbers

$$a_1 + \cdots + a_n = \sum_{j=1}^{n} a_j.$$

[The subscript $j = 1$ indicates that we start at 1, and the superscript $j = n$ indicates that we stop at n]

Example 39.

(1) $1 + 1/2 + \cdots + 1/71 = \sum_{j=1}^{71} \frac{1}{j}$.
(2) $1^2 + 2^2 + \cdots + 100^2 = \sum_{j=1}^{100} j^2$.
(3) $1^2 + \cdots + 100^2$ also $= \sum_{j=0}^{100} (j+1)^2$, and $= \sum_{j=10}^{109} (j-9)^2$.

Definition. Let $\{a_n\}$ be a sequence. Any formal expression of the form $\sum_{j=1}^{\infty} a_j$ is called an **infinite series**.

Example 40. Examples: $\sum_{j=1}^{\infty} 1 = 1 + 1 + 1 + \dots$, $\sum_{j=1}^{\infty}(-1)^{j+1} = 1 - 1 + 1 - 1 + \dots$, $\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots$, $\sum_{j=1}^{\infty} \frac{1}{2^{j-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \dots$.

Motivating question: Right now, this is all notation.
Can we assign a number to a real number to an expression of the form $\sum_{j=1}^{\infty} a_j$?

Definition. Let $\{a_n\}$ be a sequence. The sequence of **partial sums** associated to $\{a_n\}$ is the sequence $\{s_n\}$ defined by $s_n = \sum_{j=1}^{n} a_j$. Alternatively, s_n is defined recursively by $s_1 = a_1$ and $s_{n+1} = s_n + a_{n+1}$ for all $n \in \mathbb{N}$.

Definition. If $\{a_n\}$ is a sequence and $\{s_n\}$ is the associated sequence of partial sums, we say $\{a_n\}$ is **summable** if $\{s_n\}$ is convergent. In this case, we define

$$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} s_n.$$

In other words, $\displaystyle\sum_{j=1}^{\infty} a_j \overbrace{=}^{\text{by definition}} \lim_{n\to\infty} \sum_{j=1}^{n} a_j$. We say the infinite series $\sum_{j=1}^{\infty} a_j$ **converges** if $\{a_n\}$ is summable. Otherwise, we say $\{a_n\}$ **diverges**.

Example 41.

(1) Suppose $a_n = 1$ for all n. Then $s_n = \sum_{j=1}^{n} 1 = n$. So the sequence $\{s_n\}$ diverges. Hence, $\{a_n\}$ is **not sumamable**, and $\sum_{j=1}^{\infty} a_j$ **diverges**.

(2) Suppose $a_n = (-1)^n$ for all n. Then $s_1 = 1$, $s_2 = 0$, $s_3 = 1$, $s_4 = 0$; in general, $s_{2k} = 1$ and $s_{2k-1} = 0$. So $\{s_n\}$ diverges. So again, $\sum_{j=1}^{\infty} a_j$ **diverges**.

(3) Suppose $a_n = \frac{1}{n(n+1)}$ for all n. Then

$$\begin{aligned} s_1 &= \frac{1}{2}, \\ s_2 &= \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\ s_3 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4} \end{aligned}$$

In general, $s_n = 1 - \frac{1}{n+1}$. (Easy induction!) Thus,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

So $\{a_n\}$ is summable and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

(4) Suppose $a_n = \frac{1}{2^{n-1}}$. Then

$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\ s_3 &= 1 + \frac{1}{2} + \frac{1}{4} == \frac{7}{4}. \end{aligned}$$