MATH 8440 – Assignment #7
last updated April 22, 2023

Turn in <u>three</u> problems.

1. Prove that for every $c > 0$,
$$\frac{1}{2\pi i}\int_{(c)} \frac{1}{s}\,\mathrm{d}s = \frac{1}{2}.$$

2. Let $a \in \mathbf{R}$. Expand $\zeta(s)^2\zeta(s+ia)\zeta(s-ia) = \sum_{n\geq 1} c_n/n^s$. Show that $c_{p^2} \geq 1$ for every prime number p.

3. (Mertens' theorem, redux) Earlier in the semester, we proved the existence of a positive constant c for which
$$\prod_{p\leq x}\left(1-\frac{1}{p}\right) = c/\log x + O(1/(\log x)^2) \quad \text{for all } x \geq 2. \tag{*}$$
In this problem, which builds on Exercise 7 of HW #6, you will show that $c = \mathrm{e}^{-\gamma}$, where γ is the familiar Euler–Mascheroni constant.

   Define $a_n = 1/k$ if $n = p^k$ is a prime power, and $a_n = 0$ otherwise.

   (a) Show that $\log\zeta(s) = \sum_{n\geq 1} a_n/n^s$, for $s > 1$. Deduce that $\sum_{n\geq 1} a_n/n^s = \log\frac{1}{s-1} + o(1)$, as $s \downarrow 1$.

   (b) For this part and the rest of the problem, assume (*) holds. Show that
   $$\sum_{n\leq x}\frac{a_n}{n} = \log\log x - \log c + o(1), \quad \text{as } x \to \infty.$$

   (c) Using (b) and partial summation, show that
   $$\sum_{n\geq 1}\frac{a_n}{n^s} = \log\frac{1}{s-1} + \int_0^\infty \mathrm{e}^{-v}\log v\,\mathrm{d}v - \log c + o(1), \quad \text{as } s \downarrow 1.$$
   (If you solved Exercise 7 of HW #6, you may refer back to your solution to justify certain steps.)

   (d) Conclude that $c = \mathrm{e}^{-\gamma}$.

4. (Dirichlet) As explained in class, $\sum_{n\leq x} d(n) = \sum_{ab\leq x} 1$. Inclusion-exclusion allows us to write
$$\sum_{ab\leq x} 1 = \sum_{\substack{ab\leq x\\ a\leq\sqrt{x}}} 1 + \sum_{\substack{ab\leq x\\ b\leq\sqrt{x}}} 1 - \sum_{\substack{ab\leq x\\ a,b\leq\sqrt{x}}} 1.$$
By estimating each sum on the right-hand side to within $O(x^{1/2})$, show that
$$\sum_{n\leq x} d(n) = x\log x + (2\gamma-1)x + O(x^{1/2}).$$

5. (Verifying some claims from class) Let $x, T \geq 10$ and suppose $0 < \epsilon < 1/10$. Put $c = 1 + 1/\log x$.

(a) Show that $\displaystyle\int_{c+iT}^{\epsilon+iT} \zeta(s)^2 \cdot \frac{x^s}{s}\,\mathrm{d}s \ll T(\log T)^2 \max\{1, x/T^2\}$.

(b) Show that $\displaystyle\int_{\epsilon+iT}^{\epsilon-iT} \zeta(s)^2 \cdot \frac{x^s}{s}\,\mathrm{d}s \ll x^{\epsilon} T^{2-2\epsilon}$.

As in class, all implied constants here are allowed to depend on ϵ.

You will want to use the estimates for $\zeta(s)$ established in class. Namely, if $\delta > 0$ is fixed and $s = \sigma + it$ with $\delta < \sigma \le 2$ and $|s-1| > \delta$, then $|\zeta(s)| \ll_\delta (1 + (1+|t|)^{1-\sigma}) \log(2+|t|)$. If also $\sigma \le 1 - \delta$, then $|\zeta(s)| \ll_\delta (1+|t|)^{1-\sigma}$.

6. Let $\psi(x) = \sum_{p^k \le x} \log p$. Assume that for all $x \ge 2$,

$$\psi(x) = x + O(x \exp(-(\log x)^{1/15})).$$

Show that for all $x \ge 2$,

$$\pi(x) = \int_2^x \frac{\mathrm{d}t}{\log t} + O(x \exp(-(\log x)^{1/15})).$$

It is of course enough to prove this estimate for $\pi(x)$ for large x. One way to proceed is to first prove that $\theta(x) = x + O(x\exp(-(\log x)^{1/15}))$, where $\theta(x) = \sum_{p \le x} \log p$, and then apply partial summation to remove the weight of $\log p$. Remember that we already proved $\psi(x) = \theta(x) + O(x^{1/2}(\log x)^2)$.

7. Assume $T \ge 10$. Suppose that $s = \sigma + it$, where $1 - \frac{1}{\log T} \le \sigma \le 2$ and $|t| \le T$. Prove that

$$|\zeta'(s)| \ll (\log T)^2 + \frac{\log T}{|s-1|} + \frac{1}{|s-1|^2}.$$

(Of course we assume here that $s \neq 1$.)

Hint. Use the expression $\zeta'(s) = -\sum_{n=1}^{N} \frac{\log n}{n^s} - N^{1-s}\frac{(1+(s-1)\log N)}{(s-1)^2} + s\int_N^\infty \frac{\{u\}\log u}{u^{s+1}}\,\mathrm{d}u - \int_N^\infty \frac{\{u\}}{u^{s+1}}\,\mathrm{d}u$, valid for all s with $\Re(s) > 0$ and all positive integers N. (This expression is corrected from the one stated in class.) You will want to estimate all of the terms appearing here with $N = \lfloor T \rfloor$.