MATH 4000/6000 – Homework #7
posted April 4, 2022; due April 11, 2022

You can observe a lot by just looking. – Yogi Berra

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

In this assignment, "ring" always means "commutative ring."

1. Let R be a ring. Recall that if $x_1, \dots, x_n$ are elements of R, then (by definition)

$$\langle x_1, \dots, x_n \rangle = \{r_1 x_1 + \cdots + r_n x_n : \text{all } r_i \in R\}.$$

   In other words, $\langle x_1, \dots, x_n \rangle$ is the set of all linear combinations of $x_1, \dots, x_n$ with coefficients from R. Prove that $\langle x_1, \dots, x_n \rangle$ is an ideal of R by directly verifying the three defining properties of an ideal.

2. Exercise 4.1.3. (In part (c), assume R is not the zero ring.)

3. Suppose R be a ring in which every ideal is principal. That is, every ideal of R has the form $\langle r \rangle$ for some $r \in R$.

   Let $x_1, \dots, x_n \in R$. Since $\langle x_1, \dots, x_n \rangle$ is an ideal of R, there is some $d \in R$ with $\langle x_1, \dots, x_n \rangle = \langle d \rangle$. Prove that d divides all of $x_1, \dots, x_n$ and that if d' is any element of R dividing all of $x_1, \dots, x_n$, then $d' \mid d$.

4. Let F be a field. Prove that if I is any ideal of $F[x]$, then $I = \langle f(x) \rangle$ for some $f(x) \in F[x]$.

5. (a) Let R be an integral domain. Show that if $a, b \in R$, then $\langle a \rangle = \langle b \rangle$ if and only if $a = u \cdot b$ for some unit $u \in R$.

   Hint. First show that $\langle a \rangle = \langle b \rangle$ if and only if $a \mid b$ and $b \mid a$.

   (b) Now let $R = F[x]$. Show that $\langle a(x) \rangle = \langle b(x) \rangle$, where $a(x), b(x) \in F[x]$, if and only if $a(x) = c \cdot b(x)$ for some nonzero $c \in F$.

6. Let F be a field and suppose that $f(x) \in F[x]$ has degree $n \geq 1$. In class, we showed that the elements of $F[x]/\langle f(x) \rangle$ all have the form $\overline{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}}$, where $a_0, \dots, a_{n-1} \in F$. Show that this representation is unique; that is, distinct choices of a_i lead to distinct elements of $F[x]/\langle f(x) \rangle$.

7. Exercise 4.1.14(c). Make sure to answer the two questions at the end (is it a field? is it an integral domain?).

8. Exercise 4.1.10.

   Hint. If you get stuck, try Exercise 4.1.9 first.

9. Let F be a field, and let $f(x) \in F[x]$ be irreducible. Show that $F[x]/\langle f(x) \rangle$ is a field.

   Hint. If $f(x) \nmid a(x)$, then there are $X(x), Y(x) \in F[x]$ with $a(x)X(x) + f(x)Y(x) = 1$. What does this equation tell you in $F[x]/\langle f(x) \rangle$?

10. (*) Let $R = \mathbb{Z}[x]$, and let I be the set of elements of R with even constant term. Show that I is an ideal of R but that I is not principal: there is no $f(x) \in \mathbb{Z}[x]$ with $I = \langle f(x) \rangle$.