MATH 8440 – Assignment #5

last updated March 27, 2023

Turn in <u>three</u> problems.

1. Let F be a nonconstant polynomial with integer coefficients. In class, we showed (assuming the full strength of Brun's sieve) that $\liminf_{n\to\infty} \Omega(|F(n)|) < \infty$ when

$$\nu(p) < p \quad \text{for all primes } p, \tag{*}$$

where $\nu(p) := \#\{r \bmod p : F(r) \equiv 0 \pmod{p}\}$. Prove the same claim without the assumption (*).

Hint. Apply the known result to $F(AT+B)/C$ for suitably chosen integers A, B, C.

2. We recall the statement of the Brun–Hooley lower bound sieve: Assume our usual sieve setup. Partition $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_r$ (disjoint sets), and let $m_1, \ldots, m_r \in 2\mathbf{Z}_{\geq 0}$. Then

$$S(\mathcal{A}, \mathcal{P}) \geq \left(X \prod_{p \in \mathcal{P}} (1 - \alpha(p)) \right) \left(1 - \sum_{1 \leq i \leq r} \frac{\Sigma_i}{\Pi_i} \right) + O\left(\sum_{\theta_{d_1,\ldots,d_r}} |r(d_1 \cdots d_r)| \right).$$

Here

$$\Sigma_i = \sum_{\substack{d_i \mid P_i \\ \omega(d_i) = m_i + 1}} \alpha(d_i), \quad \Pi_i = \prod_{p \in \mathcal{P}_i} (1 - \alpha(p))$$

and the condition $\theta_{d_1,\ldots,d_r}$ indicates we sum over all r-tuples of positive integers $d_1, \ldots, d_r$ where (a) $d_i \mid P_i$ for all i and (b) $\omega(d_i) \leq m_i$ for all i, except for at most one exceptional index i, where $\omega(d_i) = m_i + 1$. (We also assume here that $\alpha(p) < 1$ for all $p \in \mathcal{P}$, so that division by Π_i makes sense.)

Use the Brun–Hooley lower bound sieve to prove that

$$\liminf_{n\to\infty} \Omega(n(n+2)) < \infty.$$

You can use the same basic setup as in the proof of our upper bound for $\pi_2(x)$. But you will want to choose z and $m_1, \ldots, m_{R+1}$ more carefully so that (e.g.) $1 - \sum_{1 \leq i \leq R+1} \frac{\Sigma_i}{\Pi_i} > 0$.

3. Let $a_1, a_2, \ldots, a_k$ be fixed positive integers for which the $\mathbf{Z}$-ideal $(a_1, \ldots, a_k) = \mathbf{Z}$. Let $C(n)$ be the number of tuples $(m_1, \ldots, m_k)$ of nonnegative integers with

$$m_1 a_1 + \cdots + m_k a_k = n.$$

Show that $C(n) \sim \frac{1}{a_1 \cdots a_k} \binom{n+(k-1)}{k-1}$, as $n \to \infty$.

This is a cleaner version of the HW problem threatened in class.

4. For each $A \subseteq \mathbf{Z}_{\geq 0}$, define a function $r_A \colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ by letting

$$r_A(n) = \#\{(a, a') : a, a' \in \mathcal{A}, a \leq a', a + a' = n\}.$$

That is, $r_A(n)$ is the number of unordered representations of n as a sum of two elements of A.

   (a) Let $A(x) = \sum_{n \in A} x^n$. Show that $\sum_{n \geq 0} r_A(n) x^n = \frac{1}{2}(A(x)^2 + A(x^2))$, as formal power series.

   (b) Assume A is infinite. Show that $r_A(n)$ cannot be eventually constant.[1]

[1] that is, there is no $N_0 \in \mathbf{Z}_{\geq 0}$ with $r_A(N_0) = r_A(N_0 + 1) = r_A(N_0 + 2) = \ldots$

5. As in class, we use $\mathcal{D}(\mathbb{C}, s)$ to denote the ring of formal Dirichlet series with complex coefficients.
   (a) Show that $\mathcal{D}(\mathbb{C}, s)$ is non-Noetherian.
   (b) Show that $\mathcal{D}(\mathbb{C}, s)$ is a local ring.
   (c) Show that $\mathcal{D}(\mathbb{C}, s)$ is atomic: that is, every nonzero nonunit factors as a finite product of irreducible elements.
   (d) Show that $\mathcal{D}(\mathbb{C}, s)$ is complete with respect to the metric induced by the absolute value $\|\cdot\|$ discussed (or to be discussed) in class.[2]

6. Let $\mathcal{M}(\mathbb{C}, s) = \{D(f, s) : f \text{ multiplicative}\}$ and, for each prime p, let

$$\mathcal{M}_p(\mathbb{C}, s) = \{1 + \textstyle\sum_{k\geq 1} a_{p^k}/p^{ks} : \text{ each } a_{p^k} \in \mathbb{C}\}.$$

   (a) Show that $\mathcal{M}_p(\mathbb{C}, s)$ is a group under the multiplication inherited from $\mathcal{D}(\mathbb{C}, s)$.
   To show inverses exist the discussion from class about inverting $1 + x$ will be helpful.
   (b) Suppose that for each prime p we are given an element $D(f_p, s) \in \mathcal{M}_p(\mathbb{C}, s)$. Show that $\prod_p D(f_p, s)$ converges in $\mathcal{D}(\mathbb{C}, s)$. Moreover, if we write the value of the product as $D(f, s)$, then f is multiplicative and $f(p^k) = f_p(p^k)$ for each prime power p^k (with p prime and k a nonnegative integer).
   (c) Now suppose f is multiplicative. For each prime p, define an arithmetic function f_p by setting $f_p(n) = 0$ unless $n = p^k$ (with k a nonnegative integer) and $f_p(p^k) = f(p^k)$. Show that $D(f, s) = \prod_p D(f_p, s)$.

7. Prove the following **formal** identities.
   (a) $\zeta(s)^2 = \sum_{n\geq 1} d(n)/n^s$, where $d(n)$ is the number of positive divisors of n.
   (b) $\zeta(s)^4/\zeta(2s) = \sum_{n\geq 1} d(n)^2/n^s$. Interpret $\zeta(2s)$ as $\sum_{n\geq 1} 1/n^{2s}$.
   (c) $\sum_{n\geq 1} \mu(n)z^n/(1 - z^n) = z$.

8. (a) Prove the formal identity $\sum_{n\geq 1} f(n)/n^s = 1/(2 - \zeta(s))$, where $f(n)$ counts the number of ordered factorizations of n. Here an ordered factorization is a representation of n as an ordered product of integers ≥ 2; by convention, $f(n) = 1$.
   (b) Show that there is a unique $\rho \in (1, \infty)$ with $\zeta(\rho) = 2$.
   (c) Let f be as in (a) and ρ be as in (b). Show that for each $\rho' > \rho$, one has

$$\lim_{n\to\infty} f(n)/n^{\rho'} = 0.$$

It was shown by Hille that if $\rho' < \rho$, then $f(n) > n^{\rho'}$ for infinitely many n.

[2] this was/will be defined on nonzero elements by $\|\sum_{n\geq 1} a_n/n^s\| = 1/n_0$, where n_0 is minimal with $a_{n_0} \neq 0$.