REMARKS ON DICKSON'S AMICABLE TUPLES

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Abstract. Let $k \geq 2$. In 1913, Dickson defined an *amicable k-tuple* as a collection of numbers $a_1, \dots, a_k$ (unordered, and not necessarily distinct) with the property that for each $1 \leq i \leq k$, we have

$$\sigma(a_i) = a_1 + \cdots + a_k.$$

When $k = 2$, this reduces to the familiar definition of an *amicable pair*. Such pairs have been the object of intense study from antiquity to the present day. Modern work on the distribution of amicable pairs traces back to Erdős, who in 1955 showed that almost all numbers do not belong to an amicable pair.

It seems that very little theoretical work has been done on amicable k-tuples when $k \geq 3$. We offer the following modest result, which has the same flavor as Erdős's but is not directly comparable: Among the numbers in the range of the σ-function, almost all of them (asymptotically 100%) do not appear as the common σ-value of an amicable tuple.

The key to the proof is a result of independent interest, proved using methods of Ford: For almost all values v belonging to the range of σ, the members of the preimage set $\sigma^{-1}(v)$ share the same largest prime factor.

1. Introduction

Two natural numbers m and n are said to form an *amicable pair* if each is the sum of the proper divisors of the other, so that $\sigma(m) - m = n$ and $\sigma(n) - n = m$. Equivalently, m and n form an amicable pair if $\sigma(m) = \sigma(n) = m + n$. According to Iamblichus, the example $m = 220$ and $n = 284$ was known already to Pythagoras in the sixth century BCE.

Nearly a century ago, Dickson [1] (cf. [2, p. 50]) proposed the following generalization: Say that the numbers $a_1, \dots, a_k$ (unordered, and with repetition allowed) form an *amicable k-tuple* if for each $1 \leq i \leq k$,

$$\sigma(a_i) = v, \quad \text{where} \quad v = a_1 + a_2 + \cdots + a_k. \tag{1}$$

Dickson gave a handful of examples of amicable triples ($k = 3$). Many other examples (with k as large as 6) were given by Mason [10], Poulet [15], and Mąkowski [9]. For example, if

$$m = 2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^5 \cdot 11^2 \cdot 13 \cdot 19^2 \cdot 31^2 \cdot 61 \cdot 83 \cdot 127 \cdot 223 \cdot 331 \cdot 7019 \cdot 112303 \cdot 898423 \cdot 616318177,$$

then

$$a_1 = 17 \cdot 1999 \cdot m, \quad a_2 = 23 \cdot 1499 \cdot m, \quad a_3 = 59 \cdot 599 \cdot m,$$
$$a_4 = 71 \cdot 499 \cdot m, \quad a_5 = 79 \cdot 449 \cdot m, \quad a_6 = 35999 \cdot m$$

is an amicable sextuple.

The first substantial result concerning the distribution of these k-tuples is due to Erdős [4], who showed that almost all numbers are not amicable; in other words, only $o(x)$ integers in the interval $[1, x]$ are part of an amicable pair, as $x \to \infty$. This result was sharpened by

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Erdős and Rieger [5] and then by Pomerance [11], [13]; in his 1981 paper, Pomerance shows that for large x, the number of $a \leq x$ which belong to an amicable pair is smaller than

$$x/\exp((\log x)^{1/3}). \tag{2}$$

For $k > 2$, it appears difficult to prove a direct analogue of Erdős's result. However, progress is possible if instead of counting numbers which appear in amicable tuples, we count the σ-values which appear as the common σ-value of an amicable tuple. Let $V'(x)$ be the number of natural numbers $v \leq x$ which occur in this way. Let $V(x)$ be the total number of integers in $[1, x]$ which belong to the range of σ. Our principal result is the following estimate, which has a similar flavor to the theorem of Erdős mentioned above:

Theorem 1. *Almost all σ-values do not correspond to any amicable tuple. In other words, as $x \to \infty$, we have $V'(x)/V(x) \to 0$.*

We give a short proof of Theorem 1 in §2, based on the following result of independent interest, whose proof is outlined in §3.

Theorem 2. *For almost all numbers v in the range of σ, all the elements in the preimage set $\sigma^{-1}(v) := \{a : \sigma(a) = v\}$ share the same largest prime factor.*

The demonstration of Theorem 2 uses the structure theory of φ and σ-values developed by Ford [6] and follows closely the recent proof by Ford and Pollack [7] that almost all values in the range of Euler's φ-function are not σ-values, and vice versa.

Notation. Most of our number-theoretic notation is standard. A possible exceptions is the notation $P^+(n)$ to denote the largest prime factor of n (with the convention that $P^+(1) = 1$). We say that a number n is *y-smooth* if $P^+(n) \leq y$, and we write $\Psi(x, y)$ for the number of y-smooth integers $n \leq x$.

Little-oh symbols, big-Oh notation, and the related symbols "$\ll$," "$\gg$," and "$\asymp$" appear with their usual meanings. We also put $\log_1 x = \max\{1, \log x\}$ and we write $\log_k$ for the kth iterate of $\log_1$.

2. Proof of Theorem 1

We require a crude upper bound on $\Psi(x, y)$.

Lemma 1. *Uniformly for $x \geq y \geq 2$, we have $\Psi(x, y) \ll xe^{-u/2}$, where $u := \log x/\log y$.*

Lemma 1 appears, e.g., as [16, Theorem 1, p. 359].

Proof of Theorem 1. Suppose that v is counted by $V'(x)$; thus, there numbers $a_1, \dots, a_k$ with

$$a_1 + \cdots + a_k = v \quad \text{and} \quad \sigma(a_i) = v \quad \text{for all } 1 \leq i \leq k.$$

Clearly, $a_i \leq v \leq x$ for all $1 \leq i \leq k$. By Theorem 2, ignoring $o(V(x))$ values of v, we can assume that all the a_i have the same largest prime factor, say P. By Lemma 1, we can also assume that $P > z$, where $z := x^{1/(4 \log\log x)}$. Indeed, in the opposite case, each a_i is restricted to a set of size

$$\Psi(x, z) \leq x \exp(-2 \log\log x) = x/(\log x)^2.$$

Thus, the number of possible values of $v = \sigma(a_1)$ is also bounded by $x/(\log x)^2$. But $x/(\log x)^2 = o(V(x))$, since (e.g.) every number of the form $p + 1$ with p prime is a σ-value and so counted by $V(x)$.

For the remaining v, we have that $P \mid a_i$ for each i, and so $P \mid a_1 + \cdots + a_k = v = \sigma(a_1)$. Write $a_1 = Pm$. If $P \mid m$, then $P^2 \mid a_1$, and so the number of possibilities for a_1, and so also $v = \sigma(a_1)$, is bounded by $x\sum_{p>z} p^{-2} \ll x/z = o(V(x))$. So we can assume that $P \nmid m$. Since

$$P \mid \sigma(a_1) = (P+1)\sigma(m),$$

either some prime Q divides m with $Q \equiv -1 \pmod{P}$, or some proper prime power R divides m for which $P \mid \sigma(R)$. In the former case, $Q > P$, contradicting that P is the largest prime divisor of a_1. In the latter case, since P divides $\sigma(R)$ and $\sigma(R) \le 2R$, we have that $R \ge P/2 \ge z/2$, and so the number of possibilities for a_1 is at most

$$\sum_{\substack{R \text{ proper prime power} \\ R>z/2}} \frac{x}{R} \le x \sum_{\substack{R \text{ squarefull} \\ R>z/2}} \frac{1}{r} \ll x/\sqrt{z} = o(V(x)).$$

Hence, the number of possibilities for $v = \sigma(a_1)$ is also $o(V(x))$. □

3. Proof of Theorem 2

Overview. We begin by surveying the method of [7] for showing that most φ-values are not σ-values, and vice versa. For $f \in \{\varphi, \sigma\}$, let $V_f(x)$ be the number of f-values belonging to $[1, x]$ (so that $V_\sigma(x) = V(x)$ in our earlier notation). From [6, Theorem 14], one knows that $V_\varphi(x) \asymp V_\sigma(x)$ for $x \ge 1$; thus, the main result of [7] follows if it is shown that the number of common φ–σ values in $[1, x]$ is $o(V_\varphi(x) + V_\sigma(x))$.

To this end, one begins by constructing [7, §3] sets $\mathscr{A}_\varphi$ and $\mathscr{A}_\sigma$ with the property that almost all f-values $\le x$ have all their f-preimages in $\mathscr{A}_f$. The precise definition of the sets $\mathscr{A}_f$ is quite technical and incorporates both "anatomical" and "structural" constraints. By "anatomical", we mean multiplicative constraints of the sort that often arise in elementary number theory. For example, we insist that for $a \in \mathscr{A}_f$, neither a nor $f(a)$ has an extraordinarily large squarefull divisor or "too many" prime divisors. Chief among the anatomical constraints is the requirement that every prime p dividing an element of $\mathscr{A}_f$ be a *normal* prime, meaning that the prime divisors of both $p-1$ and $p+1$ are roughly uniformly distributed on a double-logarithmic scale.

By "structural", we mean that extensive use is made of the results of [6] describing the fine structure of typical f-values and their preimages. As an example, precise inequalities are imposed on the prime divisors of elements of $\mathscr{A}_f$; the ordered list of such primes, after a double-logarithmic rescaling, must (up to a small error) correspond to a point in the *fundamental simplex* of [6, §3]. In addition, we require – and this is the main innovation of [7] – that a particular linear combination of renormalized prime factors be slightly less than 1 (see condition (8) of the definition of the sets $\mathscr{A}_f$ in [7]). This ensures that sieve bounds (such as those that feature in Lemma 7 below) eventually yield an estimate which is better than trivial.

Having constructed such sets $\mathscr{A}_f$, it is now enough to study how many φ–σ values appear as solutions to an equation of the form

$$\varphi(a) = \sigma(a'), \quad \text{where} \quad a \in \mathscr{A}_\varphi, a' \in \mathscr{A}_\sigma. \tag{3}$$

Write $a = p_0p_1p_2\cdots$ and $a' = q_0q_1q_2\cdots$, where the sequences of primes p_i and q_j are nonincreasing. The normality condition in the definition of the sets $\mathscr{A}_f$ implies that for small values of i, we have $p_i \approx q_i$, at least on a double logarithmic scale. We classify the primes p_i and q_i dividing a and a' into three categories: "large", "small", and "tiny" (as described in [7, §5.1]). Then equation (3) gives rise to an equation of the form

$$(p_0-1)(p_1-1)\cdots(p_{k-1}-1)fd = (q_0+1)(q_1+1)\cdots(q_{k-1}+1)e. \tag{4}$$

Here $p_0, \dots, p_{k-1}$ and $q_0, \dots, q_{k-1}$ are the large primes in a and a' (respectively), f is the contribution to $\varphi(a)$ of the small primes, d is the contribution to $\varphi(a)$ of the tiny primes, and e is the contribution to $\sigma(a')$ of both the small and tiny primes.

To finish the argument, we require an estimate for the number of solutions to equations of the form (4). We prove a lemma ([7, Lemma 4.1], cf. Lemma 2 below) counting the number of solutions $p_0, \dots, p_{k-1}, q_0, \dots, q_{k-1}, e, f$ to possible equations of the form (4), given d and given intervals encoding the rough location of the primes p_i and q_i. (The phrase "possible equations" means that there are many further technical hypotheses in the lemma, but that these hypotheses are automatically satisfied because of our choice of the sets $\mathscr{A}_f$.) Finally, we sum the estimate of the lemma over all possible values of d and all possible selections of intervals; this allows us to show that

$$\#\{\varphi(a) : (a, a') \in \mathscr{A}_\varphi \times \mathscr{A}_\sigma \text{ and } \varphi(a) = \sigma(a')\} \ll \frac{x}{\log x} \exp\left(-\frac{1}{4}(\log_2 x)^{1/2}\right), \tag{5}$$

which is $o(V_\varphi(x) + V_\sigma(x))$ with much room to spare.

3.1. **Overview of the proof of Theorem 2.** Say that the σ-value v is *exceptional* if it possesses two preimages a and a' for which $P^+(a) \neq P^+(a')$. We describe how to modify the argument sketched above to show that almost all σ-values are not exceptional. Let $\mathscr{A}_\sigma$ be as above (i.e., as chosen in [7, §3]).

Suppose that $v \le x$ is an exceptional σ-value. We may assume that v is such that all of its preimages belong to $\mathscr{A}_\sigma$. Picking preimages a and a' with $P^+(a) \neq P^+(a')$, the equation $\sigma(a) = \sigma(a')$ gives rise to an equation of the shape

$$(p_0+1)(p_1+1)\cdots(p_{k-1}+1)fd = (q_0+1)(q_1+1)\cdots(q_{k-1}+1)e, \tag{6}$$

where $p_0 \neq q_0$. The strategy, as before, is to count solutions to this equation corresponding to given values of d and given intervals containing the p_i and q_i, and then to sum over all possibilities.

3.2. **Filling out the sketch.** The details of the proof follow very closely those given in [7, §§4, 5], with which we assume some familiarity. In place of [7, Lemma 4.1], we need the following variant. As the required changes to the proof of [7, Lemma 4.1] are little more than typographical, we omit the demonstration.

Lemma 2. *Let y be large, $k \ge 1$, $e^e \le S \le v_k \le v_{k-1} \le \ldots \le v_0 = y$, and $u_j \le v_j$ for $0 \le j \le k-1$. Suppose that $1 \le r \le y^{1/10}$, and put $\delta = \sqrt{\log_2 S / \log_2 y}$. Set $\nu_j = \log_2 v_j / \log_2 y$ and $\mu_j = \log_2 u_j / \log_2 y$. Suppose that d is a natural number for which $P^+(d) \le v_k$. Moreover, suppose that both of the following hold:*

(a) *For $2 \le j \le k-1$, either $(\mu_j, \nu_j) = (\mu_{j-1}, \nu_{j-1})$ or $\nu_j \le \mu_{j-1} - 2\delta$. Also, $\nu_k \le \mu_{k-1} - 2\delta$.*
(b) *For $1 \le j \le k-2$, we have $\nu_j > \nu_{j+2}$.*

The number of solutions of

$$(p_0+1)\cdots(p_{k-1}+1)fd = (q_0+1)\cdots(q_{k-1}+1)e \le y/r, \tag{7}$$

in $p_0, \dots, p_{k-1}, q_0, \dots, q_{k-1}, e, f$ satisfying

(i) *$\gcd(\prod_{i=0}^{k-1} p_i, \prod_{j=0}^{k-1} q_j) = 1$;*
(ii) *p_i and q_i are S-normal primes;*
(iii) *$u_i \le P^+(p_i+1), P^+(q_i+1) \le v_i$ for $0 \le i \le k-1$;*
(iv) *no p_i+1 or q_i+1 is divisible by r^2 for a prime $r \ge v_k$;*
(v) *$P^+(ef) \le v_k$; $\Omega(f) \le 4\ell \log_2 v_k$;*

(vi) p_0+1 *has a divisor* $\geq y^{1/2}$ *which is composed of primes* $\geq v_1$;

is

$$\ll \frac{y}{dr}(c\log_2 y)^{6k}(k+1)^{\Omega(d)}(\log v_k)^{8(k+l)\log(k+l)+1}(\log y)^{-2+\sum_{i=1}^{k-1}a_i\nu_i+E},$$

where $E=\delta\sum_{i=2}^{k}(i\log i+i)+2\sum_{i=1}^{k-1}(\nu_i-\mu_i)$. *Here* c *is an absolute positive constant.*

The application of Lemma 2 requires some preparation. Up to now, we have not discussed the term S-*normal*, which appears in Lemma 2(ii). S-normality is a way of quantifying the condition mentioned earlier that the prime factors of $p+1$ be roughly uniformly distributed on a double logarithmic scale. The precise definition does not concern us here, except to note that if, following [7], we put

$$S := \exp((\log_2 x)^{36}),$$

then all the prime divisors of elements of $\mathscr{A}_\sigma$ are S-normal. We now describe, more precisely than in our sketch above, how to associate to each solution of $\sigma(a)=\sigma(a')$, with $(a,a')\in \mathscr{A}_\sigma\times\mathscr{A}_\sigma$, a bona fide equation of the form (6).

We adopt the notation of [7, §5]. Fixing $\epsilon=1/10$, we choose our "large/small" cutoff k as described in [7, §5.1], and we choose the "small/tiny" cutoff L as in [7, eq. (3.1)]. For $0\leq i<k$, the primes p_i and q_i are all large enough to guarantee (from the anatomical constraints on $\mathscr{A}_\sigma$) that $p_i^2\nmid a$ and $q_i^2\nmid a'$. We put

$$e := \sigma(p_k p_{k+1} p_{k+2}\cdots).$$

The choice of d and f is slightly more delicate. If $p_{L-1}\neq p_L$, then we put

$$f := \sigma(p_k p_{k+1}\cdots p_{L-1}), \quad d := \sigma(p_L p_{L+1}\cdots).$$

In the general case, we let A be the largest divisor of a supported on the primes $p_k,\dots,p_{L-1}$, and we put $f:=\sigma(A)$ and $d:=\sigma(a/(p_1\cdots p_{k-1}A))$. Then (6) holds.

Now choose the the intervals $[\mu_i,\nu_i]$, the intervals $[u_i,v_i]$, and ν_k and v_k by the procedure described in [7, §5.2]. Then the arguments of [7, §5.3] show that all the hypotheses of Lemma 7 are satisfied with $y=x$, $r=1$, and $l=L-k$, except possibly for (i); it may well be that a and a' share some large prime factors.

In order to work around this, put

$$m := \gcd(p_0\cdots p_{k-1}, q_0\cdots q_{k-1}).$$

By hypothesis, $p_0\neq q_0$. It follows that neither p_0 nor q_0 can divide m. Indeed, the proof of [7, Lemma 3.2] shows that

$$p_0, q_0 > x^{1/3},$$

while condition (7) in the definition of $\mathscr{A}_\sigma$ states that

$$p_i, q_i \leq x^{\frac{1}{100\log_2 x}} \quad \text{for} \quad i\geq 1.$$

For each prime p dividing m, we cancel the factors of $p-1$ from both sides of (6). Relabeling, we obtain an equation of the form

$$(\tilde{p}_0+1)\cdots(\tilde{p}_{K-1}+1)df = (\tilde{q}_0+1)\cdots(\tilde{q}_{K-1}+1)e \tag{8}$$

where $K=k-\omega(m)$ and the common value of both sides of (8) is at most $x/\sigma(m)$. Assuming (as we may) that the $\tilde{p}_i$ and $\tilde{q}_i$ are in nondecreasing order, write each

$$\tilde{p}_i = p_{j_i}, \quad \text{and} \quad \tilde{q}_i = q_{j_i'},$$

where the indices i and i' satisfy $i\leq j_i, j_i' < k$ for $0\leq i<K$. Moreover, since $\gcd(m,p_0q_0)=1$, we have $\tilde{p}_0=p_0$ and $\tilde{q}_0=q_0$. In particular, $K\geq 1$.

For $i \geq 1$, the intervals $[u_i, v_i]$ and $[u_{i+1}, v_{i+1}]$ either coincide or are disjoint. It follows that for $1 \leq i \leq K-1$, we have

$$u_{j_i} = u_{j'_i} \leq P^+(p_{j_i}+1), P^+(q_{j'_i}+1) \leq v_{j_i} = v_{j'_i}.$$

For $0 \leq i \leq K-1$, put $\tilde{u}_i = u_{j_i}, \tilde{v}_i = v_{j_i}$, and put $\tilde{\nu}_K = \nu_k$.

We now apply Lemma 7 to count solutions to (8), taking K in place of k, the $\tilde{u}_i$ and $\tilde{v}_i$ in place of u_i and v_i, and $y = x$, $r = \sigma(m)$, and $l = L - k$. Note that from condition (7) in the definition of $\mathscr{A}_f$ and [7, eq. (3.3)],

$$r = \sigma(m) \leq m^2 \leq p_1^2 p_2^2 \cdots p_{k-1}^2 \leq x^{\frac{1}{50 \log_2 x}} (p_2 p_3 p_4 \cdots)^2 \leq x^{\frac{1}{50 \log_2 x}} x^{1/50} < x^{1/10}$$

for large x. Since all the hypotheses of Lemma 7 except (i) held above, it follows easily that every hypothesis of the lemma is now satisfied.

The calculations of [7, §5.4] show that, in exact analogy with (5), the number of exceptional σ-values corresponding to some solution $(a, a') \in \mathscr{A}_\sigma \times \mathscr{A}_\sigma$ with $m = \gcd(p_0 \cdots p_{k-1}, q_0 \cdots q_{k-1})$ is bounded by

$$\frac{x}{\sigma(m) \log x} \exp\left(-\frac{1}{4}(\log_2 x)^{1/2}\right).$$

It remains to sum over m. Since m divides $p_1 \cdots p_{k-1}$, we have that m is squarefree and (see [7, Lemma 5.3])

$$\omega(m) < k < 0.9 \log_3 x$$

for large x. Hence,

$$\sum_m \frac{1}{\sigma(m)} \leq \sum_m \frac{1}{m} \leq \sum_{j \leq 0.9 \log_3 x} \frac{1}{j!} \left(\sum_{p \leq x} \frac{1}{p}\right)^j < \exp((\log_3 x)^2),$$

by a short calculation using Mertens's estimate $\sum_{p \leq x} p^{-1} = \log_2 x + O(1)$. We conclude that the number of exceptional σ-values with all their preimages in $\mathscr{A}_\sigma$ is bounded by (say)

$$\frac{x}{\log x} \exp\left(-\frac{1}{5}(\log_2 x)^{1/2}\right),$$

which is certainly $o(V_\sigma(x))$.

Remarks.

(i) *In* [7, Lemma 3.2], *it is shown that for any fixed* $\epsilon > 0$, *one can choose the set* $\mathscr{A}_\sigma$ *so that the number of* σ*-values* $v \leq x$ *with a preimage not in* $\mathscr{A}_\sigma$ *is* $\ll V_\sigma(x)/(\log_2 x)^{1/2-\epsilon}$. *Using this in our argument above, it follows that the number of exceptional* σ*-values up to* x *is bounded by* $V_\sigma(x)/(\log_2 x)^{1/2+o(1)}$, *as* $x \to \infty$.

(ii) *Using (i) and further ideas from* [6] *and* [7], *one can establish the following strengthening of Theorem 2: For each fixed* K, *almost all* σ*-values in* $[1, x]$ *are such that all of their preimages share the same largest* $K+1$ *prime factors.*

(iii) *All of our results remain valid for* φ*-values (totients) in place of* σ*-values.*

4. Concluding observations

We conclude with some observations about counting amicable k-tuples. Let $W_k(x)$ be the number of amicable k-tuples contained in $[1, x]$. For $W_2(x)$, we have the result (2) of Pomerance, so we suppose that $k > 2$.

Trivially, $W_k(x) \lesssim x^{k-1}/(k-1)!$ (as $x \to \infty$), since any $k-1$ members determine the tuple. In fact, it is easy to show that

$$W_k(x) \le x^{k-1}/L(x)^{k-2+o(1)}, \quad \text{where} \quad L(x) := \exp\left(\log x \frac{\log\log\log x}{\log\log x}\right). \tag{9}$$

This is because, by a result of Pomerance (see [12] or [14]),

$$m(v) := \#\{n \le x : \sigma(n) = v\} \le x/L(x)^{1+o(1)},$$

as $x \to \infty$, uniformly in v. (Pomerance states his results for φ, but the arguments hold for σ also with obvious modifications.) Thus, the number of $(k-1)$-tuples $a_1, \dots, a_{k-1}$ in $[1, x]$ with

$$\sigma(a_1) = \cdots = \sigma(a_{k-1})$$

is bounded by

$$\sum_{v \le x} m(v)^{k-1} \le \max_{v \le x} (m(v)^{k-2}) \sum_{v \le x} m(v) \le x (\max_{v \le x} m(v))^{k-2} \le x^{k-1}/L(x)^{k-2+o(1)};$$

again, these $k-1$ numbers determine the tuple, and so we have (9).

While this argument is crude, the following heuristic argument suggests that it is best possible up to the precise value of the exponent of $L(x)$. As first shown by Erdős [3], there is an absolute constant $\eta > 0$ so that $\max_{v \le x} m(v) \ge x^\eta$ for large x. Under reasonable hypotheses about smooth shifted primes (such as what would follow from the Elliott–Halberstam conjecture), one can take any $\eta < 1$ here. A more refined heuristic (see again [12] or [14]) suggests that the upper bound on $m(v)$ quoted above is sharp, i.e., that

$$\max_{v \le x} m(v) = x/L(x)^{1+o(1)} \qquad (x \to \infty).$$

In fact, the argument suggests that one can choose a v which attains this maximum with $v \ge x/L(x)^{o(1)}$. Then the number of ways of writing v as a sum of k natural numbers $a_1, \dots, a_k$ is $\asymp_k x^{k-1}/L(x)^{o(1)}$. Now notice that by the choice of v, the probability that a natural number $a \le x$ (chosen uniformly at random) satisfies $\sigma(a) = v$ is at least $L(x)^{-1+o(1)}$; so we might guess that the number of solutions to (1) is at least

$$x^{k-1}/L(x)^{k+o(1)}.$$

Each solution corresponds to an amicable k-tuple, so that this serves as a lower bound on $W_k(x)$. This heuristic goes back essentially to Erdős and underlies his conjecture in [4] that $W_2(x) = x^{1+o(1)}$.

We leave the reader with a formal statement of the most natural unsolved problem in this area, that of generalizing the result of [4] to amicable k-tuples:

Problem. *Fix $k > 2$. Show that the set of natural numbers which appear in some amicable k-tuple has asymptotic density* 0.

This appears difficult; even the following much weaker variant seems resistant to obvious attacks:

Problem. *Fix $k > 2$.* ***Disprove*** *that the set of natural numbers which appear in some amicable k-tuple has asymptotic density* 1.

One can at least show, borrowing ideas from [8], that the number of $a \le x$ which do not belong to any amicable tuple is $\gg x/\log\log\log x$. In fact, most numbers a with no prime factors up to $(\log\log x)^3$ have this property. We suppress the simple proof.

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