MATH 4000/6000 – Homework #1

posted January 11, 2019; due at the **start of class** on January 23, 2019

A biologist, a physicist and a mathematician were sitting in a street cafe watching the crowd. Across the street they saw a man and a woman entering a building. Ten minutes they reappeared together with a third person.
– They have multiplied, said the biologist.
– Oh no, an error in measurement, the physicist sighed.
– If exactly one person enters the building now, it will be empty again, the mathematician concluded.

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000

Fully explain your answers. In problems #1(a) and #2 **only**, you must justify which algebraic properties (properties A1–A4, M1–M3, D1 on the handout) you are using at every step of the proof. In your write-up, please refer to these properties by name rather than number. **You may assume that $a \cdot 0 = 0$ and $(-1)a = -a$ for all a, as already shown in class**. For all other problems, you do not have to justify those kinds of algebraic manipulations. Note: $\mathbb{Z}^+$ means the same as $\mathbb{N}$ (the book's notation).

1. Prove that for any $a, b \in \mathbb{Z}$, we have
   (a) $(-a)b = -(ab)$.
   (b) $(-a)(-b) = ab$.

2. Let $a, b \in \mathbb{Z}$ and suppose that $a < b$.
   (a) Prove that $a + c < b + c$ for every $c \in \mathbb{Z}$.
   (b) Prove that $ac < bc$ for every $c \in \mathbb{Z}^+$.

3. Let $a, b \in \mathbb{Z}$.
   (a) Prove that if $a < 0$ and $b < 0$, then $ab > 0$.
   (b) Show that if $a < 0$ and $b > 0$, then $ab < 0$.
   (c) Show that if $ab = 0$, then either $a = 0$ or $b = 0$.

4. (Laws of exponents) Let $a \in \mathbb{Z}$. Suppose that m, n belong to the set $\mathbb{Z}^+ \cup \{0\}$ of nonnegative integers.
   (a) Prove that $a^m \cdot a^n = a^{m+n}$.
   (b) Prove that $a^{mn} = (a^m)^n$.

   Hint: If $m = 0$ or $n = 0$, this is easy (why?). So you can suppose $m, n \in \mathbb{Z}^+$. Now think of m as fixed and proceed by induction on n.

5. In this exercise we outline a proof of the following statement, which we will be taking for granted in our proof of the division theorem: If $a, b \in \mathbb{Z}$ with $b > 0$, the set

   $$S = \{a - bq : q \in \mathbb{Z} \text{ and } a - bq \geq 0\}$$

   has a least element.
   (a) Prove the claim in the case $0 \in S$.
   (b) Prove the claim in the case $0 \notin S$ and $a > 0$.
   (c) Prove the claim in the case $0 \notin S$ and $a \leq 0$.

Hint: (a) is easy. To handle (b) and (c), first show that in these cases S is a nonempty set of natural numbers, so that the well-ordering principle guarantees S has a least element as long as S is nonempty. To prove S is nonempty, show that in case (b), the integer a is an element of S. You will have to work a little harder to prove S is nonempty in case (c).

6. Use the binomial theorem to find formulas for the following sums, as functions of n, where n is assumed to be a natural number.

   (a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}$.

   (b) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

7. (To be done after Monday's lecture) Use the Euclidean algorithm to find $\gcd(314, 159)$ and $\gcd(272, 1479)$. Show the steps, not just the final answer.

8. Show that if $a, b \in \mathbb{N}$ and $a \mid b$, then $a \le b$.

9. Let a, b be nonnegative integers, not both zero. Define the set

$$I(a, b) = \{ax + by : x, y \in \mathbb{Z}\}.$$

   (Thus, $I(a, b)$ is the set of all linear combinations of a, b, with coefficients from $\mathbb{Z}$. The letter I stands for *ideal*, which is a concept we will meet later in the course.)

   (a) Show that if a, b, q, r are integers with $a = bq + r$, then $I(a, b) = I(b, r)$.

   (b) Explain why (a) implies that $I(a, b) = I(0, \gcd(a, b))$.

   (c) Deduce from (b) that there are integers x and y with $\gcd(a, b) = ax + by$.

10. (*) We stated the binomial theorem under the assumption that $x, y \in \mathbb{Z}$. However, our proof only used that we could manipulate expressions in x and y by the usual algebraic rules. That assumption holds if x, y are formal symbols manipulated according to the usual rules for polynomials. Hence, the identity

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

   is valid as a *polynomial identity* in the variables x and y. (So far I have not asked you to prove anything, just to accept this as true!)

   Your mission: By computing $(x+y)^{2n}$ in two different ways and comparing coefficients, show that for every positive integer n,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

11. (*) Exercise 1.1.16 (this means Exercise 16 in §1 of Chapter 1).

   Hint: Start by writing each number in $\{1, 2, \ldots, 2n\}$ in the form $2^j \cdot q$, where j is a nonnegative integer and q is odd.