MATH 4400/6400 – Homework #7

posted April 22, 2022; due May 2, 2022

MATH 4400/6400 problems.

1. Show that $\mathcal{I}$ is a subring of $\mathbb{H}$.

   Hint. The tricky part is closure under multiplication. For this, let $\delta = \frac{1}{2}(1+i+j+k)$ and argue that $\mathcal{I} = \{A + Bi + Cj + D\delta : A, B, C, D \in \mathbb{Z}\}$. Then verify that the product of any pair of $1, i, j, \delta$ belongs to $\mathcal{I}$.

2. (a) Show that the units of $\mathcal{I}$ are precisely the elements of $\mathcal{I}$ of norm 1.

   (b) Show that the units in $\mathcal{I}$ are precisely $\pm 1, \pm i, \pm j, \pm k$ and $\frac{\pm 1 \pm i \pm j \pm k}{2}$.

3. Suppose that $\alpha \in \mathcal{I}$. Show that one can find a unit ϵ of $\mathcal{I}$ for which $\alpha \cdot \epsilon \in \mathcal{L}$.

   Hint. If $\alpha \in \mathcal{L}$, take $\epsilon = 1$. Otherwise, $\alpha = \frac{1}{2}(a + bi + cj + dk)$ for odd integers a, b, c, d. Try $\epsilon = \frac{1}{2}(A - Bi - Cj - Dk)$ where A, B, C, D are all ± 1, with the signs chosen to make $A \equiv a,\ B \equiv b,\ C \equiv c,\ D \equiv d \pmod 4$.