MATH 3220 practice problems
Algebra: Polynomials and complex numbers

Acknowledgements

This worksheet borrows from Larson's book, the text of Gelca and Andreescu, and from material published online by Kannan Soundararajan, Ravi Vakil, and Miguel Lerma.

Helpful results to keep in mind:

- **Root-factor theorem:** Suppose $f(x)$ is a polynomial with complex coefficients. If α_1 is a complex root of $f(x)$, then $f(x) = (x - \alpha_1)g(x)$ for some polynomial $g(x)$ with complex coefficients, and vice versa. The same equivalence holds with the word "complex" replaced everywhere by "real" or "integer".

- **Fundamental theorem of algebra:** If $f(x)$ is a polynomial of degree $n \geq 1$ over the complex numbers, then $f(x)$ can be factored as

$$A(x-\alpha_1)^{e_1}(x-\alpha_2)^{e_2}\cdots(x-\alpha_m)^{e_m},$$

  where the distinct complex roots of f are precisely $\alpha_1, \ldots, \alpha_m$, and $e_1+e_2+\cdots+e_m = n$. We refer to e_i as the **multiplicity** of the root α_i.

- **Vieta's formulas:** Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is a monic polynomial with complex coefficients. (Here **monic** means that the leading coefficient is 1.) If the roots of f are $\alpha_1, \ldots, \alpha_n$, listed with multiplicity, then

$$\begin{aligned}
-a_{n-1} &= \sum_{1\leq i\leq n} \alpha_i, \\
a_{n-2} &= \sum_{1\leq i<j\leq n} \alpha_i\alpha_j, \\
-a_{n-3} &= \sum_{1\leq i<j<k\leq n} \alpha_i\alpha_j\alpha_k, \\
&\vdots \\
(-1)^n a_0 &= \alpha_1\cdots\alpha_n.
\end{aligned}$$

- **Identity theorem:** Two polynomials of degree $\leq n$ that agree for $n+1$ different values of the variable must be the same polynomial.

- **Rational root test:** If $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a degree n polynomial with integer coefficients, and p/q is a root of $f(x)$ expressed in lowest terms, then $q \mid a_n$ and $p \mid a_0$.

Often it is helpful to know not just these results themselves, but also their proofs.

Problems

1. Let a, b, and c be real numbers. Show that the following two statements are equivalent:

   (a) $a, b, c \geq 0$,

   (b) $a + b + c \geq 0$, $ab + bc + ac \geq 0$, $abc \geq 0$.

   Hint: It's easy to see that (a) implies (b). To go the other way, show that the polynomial $(x + a)(x + b)(x + c)$ has only nonpositive roots.

2. Let $\mathcal{L}$ be a line that meets the graph of $y = 2x^4 + 7x^3 + 3x - 5$ at four distinct points, say (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4). Show that the value of

   $$x_1 + x_2 + x_3 + x_4$$

   does not depend on the particular choice of line $\mathcal{L}$, and find this value.

   Hint: Start by writing down the equation of the line $\mathcal{L}$.

3. If q, r, and s are the solutions to $x^3 - 3x^2 + 1 = 0$, find $q + r + s$, $q^2 + r^2 + s^2$, and $q^3 + r^3 + s^3$.

4. ($*$) (Goldbach) Let $f(x)$ be a nonconstant polynomial with integer coefficients. Show that at least one of the numbers $f(1), f(2), f(3), \ldots$ is not prime.

   Hint: Proceed by contradiction.

5. What is the remainder when one uses the polynomial long division algorithm to divide $x^{2013} - x + 1$ by $x^3 - x$?

   Hint: Division with remainder gives $x^{2013} - x + 1 = (x^3 - x)Q(x) + R(x)$, where $R(x)$ is the remainder you're after, and $\deg R(x) < 3$ (or $R(x) = 0$). Determine $R(x)$ by computing its values for three convenient choices of x.

6. Let $f(x)$ and $g(x)$ be nonzero polynomials with real coefficients satisfying

   $$f(x^2 + x + 1) = f(x)g(x).$$

   Show that $f(x)$ has even degree.

   Hint: It's enough to show that $f(x)$ has no real roots. (Make sure you see why this is enough!)

7. ($*$) Show that there is no nonzero polynomial $f(x)$ with $x \cdot f(x-1) = (x+1) \cdot f(x)$ for all real x.

8. Suppose that d and n are positive integers. Show that if $x^d - 1$ divides $x^n - 1$ over the complex numbers, then $d \mid n$, and vice versa. For example, $x^{27} - 1$ divides $x^{81} - 1$ but not $x^{40} - 1$.

9. Determine all prime numbers that can be written in the form $n^4 + 4$. For example, $5 = 1^4 + 4$ is such a prime.

10. ($*$) Given that $P(x)$ is a polynomial of degree 2013, and that

$$P(n) = \frac{n}{n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots, 2013,$$

find a closed form expression for $P(2014)$.

11. ($*$) Let $f(x)$ be a nonconstant polynomial over the complex numbers.

(a) Show that if α is a root of $f(x)$ of multiplicity $m \geq 1$, then α is a root of the derivative $f'(x)$ of multiplicity $m-1$. (If $m-1 = 0$, this should be understood to mean that α is not a root of $f'(x)$.)

(b) Show that if $f'(x)$ divides $f(x)^{2013}$, then $f(x)$ has exactly one complex root.

12. ($*$) Suppose that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$ is a polynomial with first coefficient 1, last coefficient 1, and all inbetween coefficients a_i nonnegative ($i = 1, 2, \ldots, n-1$). Suppose that f has n distinct roots and that all of these are real. Show that $f(2) \geq 3^n$.

13. A complex number is called **algebraic** if it is the root of a nonzero polynomial $f(x)$ with integer coefficients. For example, $\sqrt{2}$ is algebraic, since it is a root of $x^2 - 2$.

(a) Show that $\sqrt{2} + \sqrt{3}$ is algebraic.

(b) Show that $\sqrt{2} + \sqrt{3}$ is not a rational number.

(c) Show that $\cos(1^\circ)$ is algebraic. (Here $\circ$ indicates degrees.)

Hint for the last part: Show that for each fixed n, $\cos(nx)$ can be written as a polynomial in $\cos(x)$. For example, $\cos(2x) = 2\cos(x)^2 - 1$.

14. ($*$) Let $f(x)$ be a polynomial with real coefficients. Suppose that

$$f(x) + f'(x) > 0$$

for all real x. Show that then $f(x) > 0$ for all real x.

Hint: Suppose for a contradiction that f vanishes at $x = a$ for some real a.

15. ($*$) Classify (with proof) all solutions in real numbers x, y, z, w to the system of equations

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}$$

and

$$x + y + z = w.$$

16. Find all nonconstant polynomials $P(x)$ with the property that $P(P(x)) = P(x)^{2013}$ for all real numbers x.

17. ($*$) Find all polynomials $P(x)$ with real coefficients having the property that $P(x)^{2013} = P(x^{2013})$ for all real x.

18. According to the **binomial theorem**, we have that for every positive integer n,

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Using this result, prove that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Hint: $(1+x)^{2n} = (1+x)^n(1+x)^n$.

19. Show that none of the terms of the sequence

$$10001, \quad 100010001, \quad 1000100010001, \ldots$$

are prime numbers.

Hint: First find a formula for $1 + x^4 + x^8 + \cdots + x^{4n}$.

20. Consider distinct complex numbers z_1, z_2, z_3 thought of as points in the plane. Show that these points form an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

21. ($*$) Suppose that $f(x)$ is a polynomial with real coefficients having $f(x) \geq 0$ for all real x. Show that there are two polynomials $g(x)$ and $h(x)$, also with real coefficients, satisfying

$$f(x) = g(x)^2 + h(x)^2.$$

Hint: The roots of a polynomial with real coefficients come in complex conjugate pairs.

22. ($*$) Let $a, b,$ and c be three distinct integers. Let P be a polynomial with integer coefficients. Show that we **cannot** have $P(a) = b$, $P(b) = c$, and $P(c) = a$.

23. Show that over the complex numbers, the polynomial $x^{2013}+1$ divides a polynomial where the coefficients on the powers of x are all multiples of 10000 (in other words, a polynomial in the variable x^{10000}).

Hint: It might help to first factor $x^{2013} + 1$ as a product of linear factors.

24. Let $F(x)$ be a polynomial with integer coefficients. Suppose that there are distinct integers $a, b, c,$ and d with $F(a) = F(b) = F(c) = F(d) = 5$. Prove that there is no integer k with $F(k) = 8$.

25. (*) A **repunit** is a positive integer all of whose digits in base 10 are 1. For example, 111 and 111111111 are repunits. Find all polynomials f with the property that whenever n is a repunit, then $f(n)$ is also a repunit.