Math 4000/6000 – Homework #7

posted October 19, 2015; due at the **start of class** on October 26, 2015

Examiner: What is a root of multiplicity m ?
Examinee: Well, this is when we plug a number to a function, and obtain zero; then we plug it again, and obtain zero again... and this happens m times. But on the $(m+1)$-st time we do not obtain zero.
– math joke of the day

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

1. Exercise 3.2.1.

2. Exercise 3.2.2(a,b), + extra part:
   Let n be an integer with $n \geq 2$, and let $\sqrt[n]{2}$ denote the positive nth root of 2. Let $\omega = \exp(2\pi i/n)$. Show that $\mathbb{Q}[\sqrt[n]{2}, \omega]$ is a splitting field for $x^n - 2$ over $\mathbb{Q}$.

   Hint for 3.2.2(a,b): First read Examples 2 on p. 97.

3. Exercise 3.2.6(a,c,e)

4. Exercise 3.2.7.

   Hint for (b): Given $f(x) \in \mathbb{R}[x]$, first write $f(x) = c(x - \alpha_1) \cdots (x - \alpha_n)$. Now find a way to use (a) to cleverly pair the factors $x - \alpha_i$.

5. Let F be a subfield of K. Let $\alpha \in K$, and suppose that α is the root of a nonconstant polynomial in $F[x]$. Under these assumptions, we showed that $F[\alpha]$ is a field. During the course of the proof, we argued that we could find an **irreducible** polynomial $p(x) \in F[x]$ with $p(\alpha) = 0$.

   Clearly, if $h(x) \in F[x]$ is divisible by $p(x)$, then $h(\alpha) = 0$. (You are not asked to prove this but you should make sure you see why this is true.) Prove the converse: If $h(x) \in F[x]$ satisfies $h(\alpha) = 0$, then $p(x) \mid h(x)$.

6. (continuation) We continue with the assumptions of the previous problem: α is an element of K that is a root of the irreducible polynomial $p(x) \in F[x]$. Assume now that $p(x)$ has degree n.

   (a) Show that every element of $F[\alpha]$ can be written in the form $a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}$, where $a_0, \ldots, a_{n-1} \in F$.

   (b) Show that the expression in (a) is unique. That is, if $\beta \in F[\alpha]$ and $\beta = a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} = a'_0 + a'_1\alpha + \cdots + a'_{n-1}\alpha^{n-1}$, with all a_i and a'_i in F, then $a_i = a'_i$ for all $i = 0, 1, 2, \ldots, n-1$.

7. Exercise 3.2.4. Justify your answers.

   Hint for (a): The smallest ring containing $\mathbb{Q}$ and $\sqrt[3]{2}$ is $\mathbb{Q}[\sqrt[3]{2}]$. Argue — perhaps using Exercise (6)(b) — that the set in (a) is not all of $\mathbb{Q}[\sqrt[3]{2}]$.

8. Let K/F be a field extension. Suppose $\alpha \in K$ and α is **not** the root of a nonconstant polynomial in $F[x]$. Prove that $F[\alpha]$ is **not** a field.

   Hint: Show that α is a nonzero element of $F[\alpha]$ that has no inverse in $F[\alpha]$.

   Example: It can be proved that π is not the root of any nonconstant polynomial in $\mathbb{Q}[x]$. (This is a strengthened form of the result that π is irrational.) Hence, $\mathbb{Q}[\pi]$ is not a field.

9. In Chapter 4, we will construct a field K with 4 elements containing $\mathbb{Z}_2$ as subfield. In this exercise, *assume* K is such a field. Then in addition to $0, 1$ (which belong to $\mathbb{Z}_2$), K has two extra elements; call these α and β.

   (a) Show that $\alpha + 1 = \beta$.

   (b) Show that $\alpha^2 = \beta$.

   (c) Prove that K is a splitting field over $\mathbb{Z}_2$ of the polynomial $x^2 + x + 1 \in \mathbb{Z}_2[x]$.

10. Exercise 3.3.4.

11. Exercise 3.3.7.

    Hint: Argue that the Eisenstein criterion can be applied to $f(x+1)$. Look at Examples 7(c) on p. 110.

12. (*) Exercise 3.2.18.

13. (*) Exercise 3.3.10.