MATH 3220 practice problems
Finite sums and infinite series

Acknowledgements

This worksheet borrows from Larson's book, the text of Gelca and Andreescu, and from material published online by Ravi Vakil and Cecil Rousseau.

Helpful concepts to keep in mind:

- Telescoping series
- Geometric series
- Power series and tricks for manipulating them (e.g., differentiation and integration)
- Reversing the order of summation
- Tests for convergence from `MATH 2260/3100`: **comparison test**, **integral test**, **ratio/root test**, **a series with nonnegative terms converges if and only if the partial sums are bounded**.

Problems

1. The following problems can be solved by the method of telescoping.

   (a) Find the sum of the series
   $$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}}.$$

   (b) Find the sum of the finite series
   $$\frac{1}{2!}+\frac{2}{3!}+\cdots+\frac{n-1}{n!}.$$

   (c) Find the sum of the infinite series
   $$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\frac{1}{10\cdot 13}+\cdots.$$

2. Let F_n be the nth Fibonacci number. Recall that this means $F_0=0$, $F_1=1$, and $F_{n+2}=F_{n+1}+F_n$ for each nonnegative integer n.

   (a) Prove that $F_1+F_3+\cdots+F_{2n-1}=F_{2n}$.

   (b) Prove that $\displaystyle\sum_{n=2}^{\infty}\frac{1}{F_{n-1}F_{n+1}}=1$.

3. (*) Consider the sequence defined by $x_0 = 0$, $x_1 = 1$, and

$$x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n}$$

for each positive integer n. Determine $\lim_{n\to\infty} x_n$.

Hint: There's a telescoping series hiding here!

4. Evaluate the sum of the series

$$\sum_{n=1}^{\infty} 3^{n-1} \sin^3\left(\frac{x}{3^n}\right).$$

Hint: First establish the identity

$$\sin^3(\theta) = \frac{3}{4}\sin(\theta) - \frac{1}{4}\sin(3\theta).$$

5. Show that for every real number $A \in (0,1)$, there is a sequence of positive real numbers $x_1, x_2, x_3, \ldots$ where

$$x_1 + x_2 + x_3 + \cdots = 1$$

and

$$x_1^2 + x_2^2 + x_3^2 + \cdots = A.$$

Show, in addition, that one cannot find such a sequence if $A > 1$.

6. Evaluate the sum of the series

$$\sum_{n=0}^{\infty} \arctan \frac{1}{n^2+n+1}.$$

Hint: It helps to remember the addition formula for tangent:

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.$$

Using this, compute the first few partial sums of the series, expressing them as values of the arctangent function. For example, show that the sum of the terms corresponding to $n = 0$, $n = 1$, and $n = 2$ is $\arctan(3)$.

7. (*)

   (a) Explain why

$$\int_0^1 x^x \, dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (x \log x)^n \, dx.$$

   (b) Find and prove a closed form expression for

$$\int_0^1 (x \log x)^n \, dx.$$

   Hint: Integration by parts is your friend.

(c) Combining the results of (a) and (b), prove the remarkable identity

$$\int_0^1 x^x \, dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \dots.$$

8. Find the sum of the series

$$\frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \dots.$$

Hint: Start with the series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Manipulate this to get a closed form formula for $\sum_{n=0}^{\infty} \frac{(n+1)^2 x^{n+1}}{n!}$. Then plug in $x = 1$.

9. (*) For each positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evauate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

Hint: Given a positive integer m, show that $\langle n \rangle = m$ precisely when $m^2 - m < n \leq m^2 + m$. Then rewrite the sum as a sum over the variable m.

10. Suppose that n is a positive integer whose prime factorization has the form $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where the p_i are distinct primes and the e_i are positive integers. Show that

$$\sum_{d|n} d = \prod_{i=1}^{k} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Here the left-hand side denotes the sum of all of the positive integer divisors of d, For example, since

$$60 = 2^2 \cdot 3 \cdot 5, \quad \text{we have} \quad \sigma(60) = \frac{2^3 - 1}{2 - 1} \frac{3^2 - 1}{3 - 1} \frac{5^2 - 1}{5 - 1} = 7 \cdot 4 \cdot 6 = 18.$$

11. For each real number $s > 1$, the Riemann zeta function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(From Calculus II, this converges as long as $s > 1$.) Evaluate

$$\sum_{k=2}^{\infty} (\zeta(k) - 1).$$

12. (*) Evaluate the sum of the double series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2 n + m n^2 + 2mn}.$$

Your answer will be a rational number.

Hint: Start by rewriting

$$\frac{1}{m^2n + mn^2 + 2mn} = \frac{1}{n(n+2)}\left(\frac{1}{m} - \frac{1}{m+n+2}\right);$$

this makes the inner sum over m telescope.

13. (*) Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \dots$$

where the denominators are the positive integers not divisible by any prime outside of the set $\{2, 3, 5\}$.

14. Evaluate the sum of the infinite series

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \dots.$$

Hint: Geometric series gives

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \dots.$$

This can be considered a warm-up for the next problem.

15. (*) Find the sum of the infinite series

$$\sum_{n,m} \frac{(1/2)^{n+m}}{n+m},$$

where the sum is over all pairs of positive integers n and m. *Hint:* Define the power series

$$F(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{x^{n+m}}{n+m}.$$

You want $F(1/2)$. To start with, find a closed form for $F'(x)$.

16. (a) Find a closed-form formula for the sum of the finite series

$$\cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta).$$

Hint: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

(b) Show that for every real θ,

$$\sum_{n=0}^{\infty} \frac{\sin(n\theta)}{n!} = \sin(\sin(\theta))e^{\cos(\theta)}.$$

17. Does the double series $\sum_{m,n\geq 1} \frac{1}{m^2+n^2}$ converge or diverge? Here the sum is over all pairs of positive integers m and n. Justify your answer.

18. (*) Let $d(n)$ be the number of positive divisors of the natural number n. For example, $d(12) = 6$, since the divisors of 12 are $1, 2, 3, 4, 6, 12$.

   (a) Show that for every n, we have $d(n) \leq 2\sqrt{n}$.
   Hint: Consider the pairing of divisors of n where each d is paired with n/d.

   (b) Let $a_1 < a_2 < a_3 < \ldots$ be any sequence of positive integers, and let u_n be the least common multiple of $a_1, a_2, \ldots, a_n$. (For example, if $\{a_i\}$ is the sequence $1, 2, 3, 4, 5, \ldots$ of positive integers, then $u_1 = 1, u_2 = 2, u_3 = 6, u_4 = 12, u_5 = 60, \ldots$.) Show that

$$\sum_{n=1}^{\infty} \frac{1}{u_n}$$

   converges.
   Hint: First find the connection to part (a).

19. The *fractional part* of a real number x is defined as $x - \lfloor x \rfloor$; in other words, it's the amount by which x exceeds its integer part. For example, 17 has fractional part 0, 2.673 has fractional part 0.673, and -1.4 has fractional part 0.6.

   Consider the familiar constant

$$\begin{aligned} e &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \\ &= 2.71828182845904523536\ldots \end{aligned}$$

   (a) For each n, find an infinite series expression for the fractional part of $n!e$. Use this to show that the fractional part of $n!e$ is always a nonzero number.
   Example: Consider the case $n = 1$. Since the integer part of e is 2, the fractional part of e is $e - 2 = \frac{1}{2!} + \frac{1}{3!} + \ldots$. So this series gives an answer to the problem when $n = 1$.

   (b) Show that if e were rational, then the fractional part of $n!e$ would be 0 for all large enough values of n. Deduce that e is an irrational number.

20. Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(1/3)^{2^n}}{1 - (1/3)^{2^{n+1}}}.$$

   Hint: The 1/3 is a red herring; first do this for a generic number x with $0 < x < 1$.

21. (*) Does the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n}$ converge or diverge? Justify your answer.

22. Suppose that $a_1, a_2, a_3, \ldots$ is an infinite sequence of positive real numbers. Prove that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} a_n^{1+1/n} \text{ converges.}$$

   Hint: The direction $\Longrightarrow$ is easy. For the other direction, you may want to partition n according to whether or not $a_n^{1/n} > 2$.

23. (*) Let $\{a_n\}$ be a sequence of positive real numbers. Show that if the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, then so does the series

$$\sum_{n=1}^{\infty} \frac{n^2}{(a_1 + a_2 + \cdots + a_n)^2} a_n.$$

24. (*) Find the sum of the series

$$\sum_{m,n} \frac{1}{m^2 n^2},$$

where the sum is over all pairs (m, n) of positive integers where the largest power of 2 dividing m is different from the largest power of 2 dividing n. Express your answer as a rational number multiplied by a power of π. You may assume the famous result of Euler that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

25. Prove that every positive rational number can be written as the sum of distinct terms from the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots.$$

For example, $13/17 = \frac{1}{2} + \frac{1}{4} + \frac{1}{68}$.

26. (*) For each natural number n, let $B(n)$ be the number of 1s in the binary expansion of n. For example, $B(6) = 2$, since $6 = 2^2 + 2^1$, and $B(100) = 3$, since $100 = 2^6 + 2^5 + 2^2$. Show that

$$\sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)} = \ln 4.$$

Hint: First show that $B(2n) = B(n)$ and $B(2n+1) = B(n)+1$. Then find a way to use these identities! At some point you will need to use that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$.