

## MATH 4000/6000 – Learning objectives to meet for Exam #3

The exam will be over §§4.1–4.2; this includes all material covered since your previous exam through the lecture on Friday, 4/15.

### What to be able to state

#### Basic definitions

You should be able to give precise descriptions of all of the following:

- homomorphism
- kernel of a homomorphism
- ideal of a commutative ring
- principal ideal
- the notation  $\langle x \rangle$  and (more generally)  $\langle x_1, \dots, x_n \rangle$
- definition of the quotient ring  $R/I$  (including what the elements are and how the operations are defined)
- prime ideal
- isomorphism of rings
- direct product of rings
- earlier definitions regarding splitting of polynomials and splitting fields (see the review sheet for Exam #2)

#### Big theorems

Give full statements of each of the following results, making sure to indicate all necessary hypotheses. For results proved in class or on homework, describe the components and main ideas of the proof.

- the kernel of a homomorphism is an ideal
- every ideal of  $\mathbf{Z}$  is principal
- every ideal of  $F[x]$  (with  $F$  a field) is principal
- If  $\phi: R \rightarrow S$  is a homomorphism, then  $\phi$  is one-to-one if and only if  $\ker(\phi) = \langle 0 \rangle$ .
- Fundamental Homomorphism Theorem, as stated in class
- $R/I$  is a domain if and only if  $I$  is a prime ideal
- If  $f(x) \in F[x]$  has degree  $n \geq 1$ , then  $F[x]/\langle f(x) \rangle = \overline{\{a_0 + a_1x + \dots + a_{n-1}x^{n-1} : \text{all } a_i \in F\}}$ . Moreover, each element of  $F[x]/\langle f(x) \rangle$  has a unique expression as  $\overline{a_0 + a_1x + \dots + a_{n-1}x^{n-1}}$ .

- If  $p(x) \in F[x]$  is irreducible, then  $K = F[x]/\langle p(x) \rangle$  is a field.
- If  $f(x) \in F[x]$  is reducible, then  $K = F[x]/\langle f(x) \rangle$  has zero divisors.
- If  $p(x) \in F[x]$  is irreducible, then  $p(x)$  has the root  $\alpha = \bar{t}$  in  $K = F[t]/\langle p(t) \rangle$ .
- If  $f(x) \in F[x]$  is any nonconstant polynomial, there is a field  $K$  containing  $F$  in which  $f(x)$  has a root.
- If  $f(x) \in F[x]$  is any nonconstant polynomial, there is a field  $K$  containing  $F$  over which  $f(x)$  splits.
- If  $f(x) \in F[x]$  is any nonconstant polynomial, there is a splitting field for  $f(x)$  over  $F$ .

## What to be able to do

You are expected to know how to use the methods described in class/developed on HW to solve the following problems (not comprehensive!).

- Establish properties of quotient rings  $R/I$  by relating these back to the properties of the original ring  $R$ .
- Perform computations in quotient rings  $R/I$  (for example, multiplying two elements  $\mathbb{Q}[x]/\langle x^3 + x \rangle$ ).
- Find inverses of elements in  $F[x]/\langle f(x) \rangle$ , via the Euclidean algorithm and back substitution.
- Establish isomorphisms between rings using the Fundamental Homomorphism Theorem.

# Practice problems

1. Let  $R$  and  $S$  be commutative rings.
  - (a) What is meant by a **homomorphism** from  $R$  to  $S$ ?
  - (b) Let  $J$  be an ideal of  $S$ . Define  $I = \{r \in R : \phi(r) \in J\}$ . Show that  $I$  is an ideal of  $R$ .
  - (c) Prove or disprove: If  $J$  is a prime ideal of  $S$ , then  $I$  (as defined in part b) is a prime ideal of  $R$ .
2. How many homomorphisms  $\phi : \mathbf{Z}_5 \rightarrow \mathbf{Z}_7$  are there? Justify your answer.
3.
  - (a) If  $R$  is a (commutative ring) and  $I$  is an ideal of  $R$ , what do we mean by the **quotient ring**  $R/I$ ? Describe both the elements of  $R/I$  and the definition of addition and multiplication in  $R/I$ .
  - (b) Since  $\mathbf{Z}_{17}[t]$  is a domain, there are no solutions to  $x^2 = 0$  in  $\mathbf{Z}_{17}[t]$  other than  $x = 0$ . What are all solutions to  $x^2 = 0$  in the ring  $\mathbf{Z}_{17}[t]/\langle t^3 \rangle$ ? How many are there? Prove your answer is correct.
4.
  - (a) State the **Fundamental Homomorphism Theorem**, as given in class.
  - (b) Is  $\mathbf{Z}_7[x]/\langle x^2 + 1 \rangle$  isomorphic to  $\mathbf{Z}_7 \times \mathbf{Z}_7$ ? If the answer is no, explain why not. If yes, prove there is an isomorphism by using the Fundamental Homomorphism Theorem.
  - (c) Is  $\mathbf{Z}_7[x]/\langle x^2 - 2 \rangle$  isomorphic to  $\mathbf{Z}_7 \times \mathbf{Z}_7$ ? Justify your answer. If the answer is no, explain why not. If yes, prove there is an isomorphism by using the Fundamental Homomorphism Theorem.
5. Let  $K = \mathbf{Z}_3[t]/\langle t^3 - t + 1 \rangle$ . Let  $\alpha = \bar{t} \in K$ .
  - (a) Is  $K$  a domain? Is  $K$  a field?
  - (b) Find  $\alpha^{-1}$  in  $K$ . Express your answer in the form  $c_0 + c_1\alpha + c_2\alpha^2$ , where  $c_0, c_1, c_2 \in \mathbf{Z}_3$ .
  - (c) With  $f(x) = x^3 - x + 1$ , find all of  $f(\alpha)$ ,  $f(\alpha^2)$  and  $f(\alpha^3)$ . Express your answers in the form  $c_0 + c_1\alpha + c_2\alpha^2$ , where  $c_0, c_1, c_2 \in \mathbf{Z}_3$ .