Analytic Number Theory (Fall 2018) – Homework #2
posted September 11, 2018; due September 20, 2018

Problems: References are to *Not always buried deep*; "Exercise A.B" means Exercise B at the end of Chapter A. Point values are listed in brackets. You *may* use outside resources, including published papers, but your write-up should mention which references you consulted.

1. [10] Recall that "$f(x) = O(g(x))$ as $x \to \infty$" means that there is some x_0 with $f(x) = O(g(x))$ on $[x_0, \infty)$. Now suppose $g(x)$ is a positive-valued function with $\lim_{x\to\infty} g(x) = 0$. Show that, as $x \to \infty$, one has
$$\frac{1}{1+O(g(x))} = 1 + O(g(x)), \quad e^{O(g(x))} = 1 + O(g(x)), \quad \log(1+O(g(x))) = O(g(x)).$$
[Here interpret the first equation to mean that if $f(x) = O(g(x))$ as $x \to \infty$, then $1/(1+f(x)) = 1 + O(g(x))$, as $x \to \infty$. Similarly for the others.]

2. [10] Show that as $x \to \infty$, we have
$$\left(1+\frac{1}{x}\right)^x = e - \frac{e}{2x} + O\left(\frac{1}{x^2}\right).$$

3. [5] Recall that with $S(x) := \sum_{p\le x} \frac{\log p}{p}$, we have $S(x) = \log x + O(1)$, for all $x \ge 2$. What goes wrong if you attempt to deduce an asymptotic formula for $\pi(x)$ from this, starting from
$$\pi(x) = \int_1^x \frac{t}{\log t}\, dS(t) \quad ?$$

4. [10] Exercise 3.8. Here "O_m" means that for different m you are allowed to choose different values of the implied constant.

5. [10] Exercise 3.9.

6. [15] Exercise 3.14. ("Full form of Bertrand's postulate" means that $(x, 2x]$ contains a prime for all real $x \ge 1$.)

7. [10] Exercise 3.18.

8. [20] Exercise 3.33.