Math 4000/6000 – Homework #7
posted November 4, 2016; due November 15, 2016

> Examiner: What is a root of multiplicity m ?
> Examinee: Well, this is when we plug a number to a function, and obtain zero; then we plug it again, and obtain zero again…and this happens m times. But on the $(m+1)$-st time we do not obtain zero.
> – math joke of the day

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

In this assignment, "ring" always means "commutative ring."

1. Exercise 4.1.1. When answering part (b), assume neither of R and S is the zero ring.

2. Let R be a ring. Recall that if $x_1, \ldots, x_n$ are elements of R, then (by definition)
$$\langle x_1, \ldots, x_n \rangle = \{r_1 x_1 + \cdots + r_n x_n : \text{all } r_i \in R\}.$$
In other words, $\langle x_1, \ldots, x_n \rangle$ is the set of all linear combinations of $x_1, \ldots, x_n$ with coefficients from R. Prove that $\langle x_1, \ldots, x_n \rangle$ is an ideal of R by directly verifying the three definining properties.

3. Exercise 4.1.3. (In part (c), assume R is not the zero ring.)

4. Let $R = \mathbb{Z}$, and let $a_1, \ldots, a_n$ be positive integers. By Exercise 2, $\langle a_1, \ldots, a_n \rangle$ is an ideal of $\mathbb{Z}$. Since $\mathbb{Z}$ is a principal ideal ring, we know there is an integer d with
$$\langle a_1, \ldots, a_n \rangle = \langle d \rangle.$$
Show that d divides every a_i and that if d' is any integer dividing every a_i, then d' divides d. [Thus, d' is the "greatest common divisor" of $a_1, \ldots, a_n$.]

5. Let F be a field. Use the theorem on the division in algorithm in $F[x]$ to prove that $F[x]$ is a principal ideal ring.

6. Use the theorem on the division algorithm in $\mathbb{Z}[i]$ (from earlier homework) to prove that $\mathbb{Z}[i]$ is a principal ideal ring.

7. (a) Let R be an integral domain. Show that if $a, b \in R$, then $\langle a \rangle = \langle b \rangle$ if and only if $a = u \cdot b$ for some unit $u \in R$. *Hint:* First show that $\langle a \rangle = \langle b \rangle$ if and only if $a \mid b$ and $b \mid a$.
   (b) Now let $R = F[x]$. Show that $\langle a(x) \rangle = \langle b(x) \rangle$, where $a(x), b(x) \in F[x]$, if and only if $a(x) = c \cdot b(x)$ for some nonzero $c \in F$.

8. Prove that if F is a field and $f(x) \in F[x]$ has degree $n \geq 1$, then the elements of $F[x]/\langle f(x) \rangle$ all have the form $\overline{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}}$, where $a_0, \ldots, a_{n-1} \in F$. Moreover, show that this representation is unique; i.e., distinct choices of a_i lead to distinct elements of $F[x]/\langle f(x) \rangle$.

   Hint for the first half: For any $a(x) \in F[x]$, we can write $a(x) = f(x)q(x) + r(x)$, where $r(x) = 0$ or $\deg r(x) < n$. Argue that $\overline{a(x)} = \overline{r(x)}$, and that $\overline{r(x)}$ has the form appearing in the problem statement.

9. Exercise 4.1.14(c). Make sure to answer the two questions at the end (is it a field? is it an integral domain?).
10. Exercise 4.1.10. *Hint:* If you get stuck, try Exercise 4.1.9 first.
11. (*) Exercise 3.3.7.
12. (*) Exercise 3.3.10.