How I messed with perfection. . .
and lived to write papers about it

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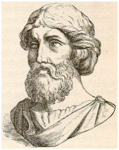
Backstory

Let $s(n) := \sum_{d|n, d<n} d$ be the sum of the proper divisors of n, and let $\sigma(n) = \sum_{d|n} d$ be the sum of all positive divisors of n. So, e.g.,

$$s(4) = 1 + 2 = 3, \qquad \sigma(4) = 1 + 2 + 4 = 7.$$

A natural number n is called **perfect** if $s(n) = n$ (equivalently, $\sigma(n) = 2n$), and **amicable** if $s(n) \neq n$ and $s(s(n)) = n$. For example, $s(6) = 6$, so 6 is perfect. Also,

$$s(220) = 284, \quad \text{and} \quad s(284) = 220.$$



Pythagoras, when asked what a friend was, replied:

One who is the other I, such are 220 and 284.

Is there a perfect number theorem?

Euclid–Euler: The number of even perfect numbers $N \le x$ is $O(\log x)$.

Conjecture

There are no odd perfect numbers.

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If you can't show a set is empty, maybe you can show it's not too big.

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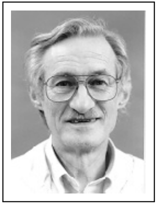
Theorem (Hornfeck–Wirsing, 1957)

Let $V_1(x)$ be the number of perfect numbers $n \le x$. For each fixed $\epsilon > 0$, we have $V_1(x) < x^{\epsilon}$ for all $x > x_0(\epsilon)$.

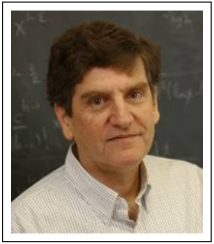


The distribution of amicable numbers

There are over ten million amicable pairs known, but we have no proof that there are infinitely many.

Theorem (Erdős, 1955)

Almost all numbers are not amicable.



Theorem (Pomerance, 1981)

The number $V_2(x)$ of amicable numbers $n \leq x$ satisfies

$$V_2(x) \leq x/\exp((\log x)^{1/3})$$

for large x. In particular, the sum of the reciprocals of the amicable numbers converges.

Sociable numbers

More generally, we call n a k-**sociable number** if n starts a cycle of length k. (So perfect corresponds to $k = 1$, amicable to $k = 2$.) For example,

$$2115324 \mapsto 3317740 \mapsto 3649556 \mapsto 2797612 \mapsto 2115324 \mapsto \dots$$

is a sociable 4-cycle. We know 175 cycles of order > 2.

Let $V_k(x)$ denote the number of k-sociable numbers $n \leq x$.

Theorem (Erdős, 1976)

Fix k. The set of k-sociable numbers has asymptotic density zero. In other words, $V_k(x)/x \to 0$ as $x \to \infty$.

Counting sociables

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$$V_k(x)/x \le 1/\overbrace{\log\log\cdots\log}^{3k \text{ times}} x.$$

We (K.-P.-P.) obtain more reasonable bounds. A further improvement is possible for odd k.

Theorem (P., 2010)

Suppose k is odd, and let $\epsilon > 0$. Then

$$V_k(x) \le x/(\log x)^{1-\epsilon}$$

for all large x.

Counting sociables

What if we count all sociable numbers at once? Put

$$V(x) := V_1(x) + V_2(x) + V_3(x) + \ldots$$

Is it still true that most numbers are not sociable numbers?

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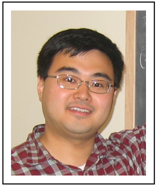
Theorem (K.–P.–P., 2009)

$$\limsup V(x)/x \leq 0.0021.$$



Our upper bound is the density of **odd abundant numbers**, odd numbers n for which $s(n) > n$ (e.g., $n = 945$).

Another fine mess . . .

Pomerance and I have recently considered another variant of the perfect numbers. Call a number **prime-perfect** if n and $\sigma(n)$ have the same set of distinct prime factors. For example, if $n = 270$, then

$$n = 2 \cdot 3^3 \cdot 5, \quad \textit{and} \quad \sigma(n) = 2^4 \cdot 3^2 \cdot 5,$$

so n is prime-perfect.

Theorem (Pomerance–P., 2011)

There are infinitely many prime-perfect numbers n; in fact, for each k, there are more than $(\log x)^k$ examples $n \leq x$ once x is large. In the opposite direction, the number of examples up to x is at most $x^{1/3+\epsilon}$ for all large x.

Thank you!