ON THE PARITY OF THE NUMBER OF MULTIPLICATIVE PARTITIONS AND RELATED PROBLEMS

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Abstract. Let $f(N)$ be the number of unordered factorizations of N, where a *factorization* is a way of writing N as a product of integers all larger than 1. For example, the factorizations of 30 are

$$2 \cdot 3 \cdot 5, \quad 5 \cdot 6, \quad 3 \cdot 10, \quad 2 \cdot 15, \quad 30,$$

so that $f(30) = 5$. The function $f(N)$, as a multiplicative analogue of the (additive) partition function $p(N)$, was first proposed by MacMahon, and its study was pursued by Oppenheim, Szekeres and Turán, and others.

Recently, Zaharescu and Zaki showed that $f(N)$ is even a positive proportion of the time and odd a positive proportion of the time. Here we show that for any arithmetic progression a mod m, the set of N for which

$$f(N) \equiv a \pmod{m}$$

possesses an asymptotic density. Moreover, the density is positive as long as there is at least one such N. For the case investigated by Zaharescu and Zaki, we show that f is odd more than 50% of the time (in fact, about 57%).

1. Introduction

Let $f(N)$ be the number of unordered factorizations of N, where a *factorization* of N is a way of writing N as a product of integers larger than 1. For example, $f(12) = 4$, corresponding to

$$2 \cdot 6, \quad 2 \cdot 2 \cdot 3, \quad 3 \cdot 4, \quad 12.$$

(We adopt the convention that $f(1) = 1$.) The function $f(N)$ is a multiplicative analogue of the (additive) partition function $p(N)$. Since its introduction by MacMahon [19], several authors have investigated properties of $f(N)$, such as its maximal order (Oppenheim [23], corrected by Canfield et al. [5]), its average order (Oppenheim [op. cit.], Szekeres and Turán [28], Luca et al. [16, Theorem 2]), and the size of its image (Luca et al. [ibid., Theorem 1], Balasubramanian and Luca [3]).

Motivated by unsolved problems on the parity distribution of $p(N)$ (see, e.g., [24], [2], [4], [22]), Zaharescu and Zaki [30] showed that $f(N)$ is even a positive proportion of the time (in the sense of asymptotic lower density) and odd a positive proportion of the time. Up to 10^7, about 57% of values of $f(N)$ are odd, but the arguments of [30] do not suffice to show that there is a limiting proportion of N for which $f(N)$ is odd.

Our purpose in this paper is to fill this gap. Our method applies not only to the parity of $f(N)$ but to the distribution of $f(N)$ modulo m for any modulus m.

Theorem 1.1. *Let a and m be any integers with $m \geq 1$. Then*

$$\lim_{x\to\infty} \frac{1}{x}\#\{N \leq x : f(N) \equiv a \pmod{m}\}$$

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exists. In other words, the set of N for which $f(N) \equiv a \pmod{m}$ possesses a natural density. Moreover, if there is a single N with $f(N) \equiv a \pmod{m}$, then this density is positive.

The proof of Theorem 1.1 is effective, in that it yields an algorithm for calculating these densities. As an example, we show at the end of §2.3 that the N with $f(N)$ odd comprise a set of density $> 50\%$, so that f is *not* uniformly distributed modulo 2.

It seems safe to conjecture that the densities appearing in Theorem 1.1 are always positive; we have verified this for every modulus $m \leq 1000$ (see the remark near the end of §2.3). This conjecture would follow from a well-known conjecture in the theory of partitions, that $p(n)$ hits every residue class to every modulus infinitely often (see [21] for the origin of this conjecture and [1] for recent progress). Indeed, for each prime power p^n, we have $f(p^n) = p(n)$.

Let us summarize briefly the approach of Zaharescu and Zaki. If N is squarefree, then $f(N)$ depends only on the number $\omega(N)$ of primes dividing N, not on N itself. In fact, writing $n = \omega(N)$, we see easily that $f(N)$ is the nth Bell number. (Recall that the nth *Bell number* is the number of ways to partition an n-element set into nonempty subsets.) It is known (see, e.g., [29]) that the Bell numbers are purely periodic modulo 2 with period 3, and so $f(N)$ is a function of the residue class of $n \bmod 3$. It is also known (see Lemma 2.2 below, and cf. [7]) that on squarefree numbers, $\omega(N)$ is uniformly distributed modulo 3. Since the sequence $\langle B_n \rangle$ of Bell numbers begins $B_0 = 1, B_1 = 1, B_2 = 2$, it follows that $f(N)$ is odd for $2/3$ of the squarefree numbers (a set of density $4/\pi^2$), and $f(N)$ is even for $1/3$ of them (a set of density $2/\pi^2$). The constants $4/\pi^2$ and $2/\pi^2$ improve the lower density bounds claimed in [30], which are obtained by more intricate elementary arguments.

To show that the set of N for which $f(N)$ is even possesses a density, it is no longer acceptable to limit one's attention to squarefree values of N. To proceed, we split up the natural numbers N according to their squarefull part M (i.e., their largest squarefull divisor). For each fixed M, we show that the parity of $f(N)$ is a purely periodic function of $\omega(N)$. It follows, as before, that a well-defined proportion of these N have $f(N)$ even. Then (as is easy to justify) the density of N with $f(N)$ even is obtained by summing the densities obtained for each squarefull number M. The most difficult part of the argument is establishing the periodicity, which leads us to study congruence properties of certain generalizations of the Bell numbers.

We conclude the paper with remarks concerning the analogous problems for $g(N)$, the number of *ordered* factorizations of N.

Notation. Most of our notation is standard. A possible exception is $\tau_k(n)$ (the k-fold Piltz divisor function), which denotes the number of ways of writing n as an ordered product of k natural numbers. We always reserve the letter p for a prime variable. We write $\mathbf{1}_S$ for the indicator function of a set or statement S; e.g., $\mathbf{1}_{3|n}$ is the characteristic function of the multiples of 3, and $\tau_0(n) = \mathbf{1}_{n=1}$. The Landau–Bachmann Big Oh and little oh notation, as well as the associated symbols "$\ll$" and "$\gg$", appear with their standard meanings. We use the term *period of a sequence* to refer to any multiple of the minimal period length.

2. Unordered factorizations

2.1. **Preliminaries for the proof of Theorem 1.1.** The following result of a type established by Halász (cf. [9]) appears in a stronger, more quantitative form in [10]. It is a useful criterion for a multiplicative function taking values in the unit disc to have mean value zero.

Lemma 2.1. *Let $\mathscr{D}$ be a closed, convex proper subset of the closed unit disc in* $\mathbf{C}$, *and assume that $0 \in \mathscr{D}$. Suppose that h is a complex-valued multiplicative function satisfying $|h(N)| \leq 1$ for all $N \in \mathbf{N}$ and $h(p) \in \mathscr{D}$ for all primes p. If the series*

$$\sum_{p} \frac{1 - \Re(h(p))}{p} \tag{1}$$

diverges, then h has mean value zero, i.e.,

$$\lim_{x\to\infty} \frac{1}{x} \sum_{N \leq x} h(N) = 0.$$

Lemma 2.2. *Let m and M be fixed natural numbers. Then $\omega(N)$ is uniformly distributed modulo m, as N runs over all squarefree natural numbers coprime to M.*

Proof. By a simple inclusion-exclusion (see, e.g., [15, p. 634]), the squarefree numbers coprime to M have asymptotic density $\frac{6}{\pi^2}\prod_{p|M}\frac{p}{p+1} > 0$. Now let ζ be any mth root of unity with $\zeta \neq 1$. By the standard orthogonality relations, it is enough to show that for each such ζ, the multiplicative function $h(N) := \mathbf{1}_{N \text{ squarefree}}\mathbf{1}_{\gcd(N,M)=1}\zeta^{\omega(N)}$ has mean value zero. Since $h(p) = 0$ or $h(p) = \zeta$, clearly $1 - \Re(h(p)) \gg_m 1$, and so the sum (1) diverges. □

Lemma 2.3. *Let M be a natural number. Suppose that $p_1, \dots, p_n$ are distinct primes not dividing M, where $n \geq 0$. Then*

$$f(Mp_1 \cdots p_n) = \sum_{k=0}^{n} S(n,k) \sum_{d|M} f(d)\tau_k(M/d).$$

Here the numbers $S(n,k)$ are *Stirling numbers of the second kind* (for background, see [6, Chapter 5]).

Proof. Each unordered factorization of $Mp_1 \cdots p_n$ arises precisely once from the following construction: Given $0 \leq k \leq n$, choose an unordered factorization of $p_1 \cdots p_n$ into k parts; this can be done in $S(n,k)$ ways. Order the parts and call them $D_1, \dots, D_k$. Choose a divisor d of M, and choose any of the $\tau_k(M/d)$ ways of writing M/d as a product of k natural numbers, say $M/d = d_1' d_2' \cdots d_k'$. Then the corresponding factorization of $Mp_1 \cdots p_n$ is obtained by appending to any of the $f(d)$ unordered factorizations of d the k-term factorization

$$(d_1' D_1)(d_2' D_2) \cdots (d_k' D_k)$$

of $Mp_1 \cdots p_n/d$. □

The following lemma is the key technical result used in the proof of Theorem 1.1. As explained in §2.3, it is a special case of results of Mazouz [20].

Lemma 2.4. *Let a be any integer, and let $q \geq 1$. Let m be a natural number. Consider the sequence whose nth term, for $n \geq 0$, is given by*

$$\sum_{\substack{0 \leq k \leq n \\ k \equiv a \pmod{q}}} S(n,k).$$

When reduced modulo m, this sequence is purely periodic.

Remark 2.5. If $q = 1$, then the sum considered in the lemma is the nth Bell number B_n, and the statement of the lemma is contained in the results of [25] (see also [17]).

Finally, it is convenient to have a simple criterion for a union of disjoint sets to have the expected asymptotic density.

Lemma 2.6. *Let $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \dots$ be a sequence of disjoint subsets of the natural numbers with respective asymptotic densities $d_1, d_2, d_3, \dots$. Suppose that as $n \to \infty$, the upper density of the set $\mathscr{A}^{(n)} := \cup_{i=n+1}^{\infty} \mathscr{A}_i$ tends to zero. Then $\mathscr{A} := \cup_{i=1}^{\infty} A_i$ has asymptotic density $d := \sum_{i=1}^{\infty} d_i$.*

Proof. Since $\mathscr{A}$ contains each finite union $\mathscr{A}_{(n)} := \cup_{i=1}^{n} \mathscr{A}_i$, the lower density of $\mathscr{A}$ is bounded below by $d_1 + \dots + d_n$, and so (letting $n \to \infty$) is at least as large as d. Similarly, since $\mathscr{A} \subset \mathscr{A}_{(n)} \cup \mathscr{A}^{(n)}$, the upper density of $\mathscr{A}$ is bounded above by $d_1 + \dots + d_n + o(1)$, and so is at most d (again, letting $n \to \infty$). $\square$

2.2. **Proof of Theorem 1.1.** Fix an arithmetic progression $a \bmod m$. Let $1 = M_1 < M_2 < M_3 < \dots$ be the sequence of squarefull integers, and define $\mathscr{A}_i$ as the set of N with squarefull part M_i for which $f(N) \equiv a \pmod m$. Clearly, the upper density of $\mathscr{A}^{(n)} = \cup_{i=n+1}^{\infty} \mathscr{A}_i$ is bounded above by $\sum_{i=n+1}^{\infty} \frac{1}{M_i}$. We have

$$\sum_{i=1}^{\infty} \frac{1}{M_i} = \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \dots\right) < \infty,$$

so that by Lemma 2.6, it suffices to show that each $\mathscr{A}_i$ has a natural density. For the remainder of the argument we fix i and write $M = M_i$.

Each number N with squarefull part M has the form $N = Mp_1 \cdots p_n$, where the p_j are distinct primes not dividing M and $n = \omega(N) - \omega(M)$. The number $f(N)$ depends only on n and not the individual primes p_j, so it makes sense to define $\hat{f}(n)$ for $n \ge 0$ by

$$\hat{f}(n) = f(Mp_1p_2 \cdots p_n). \tag{2}$$

It is sufficient to show that the reduction modulo m of the sequence $\langle \hat{f}(n) \rangle_{n=0}^{\infty}$ is purely periodic; indeed, if the period length is R, then by Lemma 2.2, the set $\mathscr{A}_i$ will have asymptotic density

$$\frac{6J}{MR\pi^2} \prod_{p \mid M} \frac{p}{p+1}, \quad \text{where} \quad J := \#\{0 \le j < R : \hat{f}(j) \equiv a \pmod m\}. \tag{3}$$

To expose the periodicity, observe that by Lemma 2.3,

$$\hat{f}(n) = \sum_{k=0}^{n} S(n,k) \sum_{d \mid M} f(d) \tau_k(M/d). \tag{4}$$

Let

$$I_k := \sum_{d \mid M} f(d) \tau_k(M/d)$$

denote the inner sum in (4). We claim that modulo m, the function I_k is purely periodic as a function of k. Since M is fixed, the claim follows if we show that at a fixed prime power p^e, the function $\tau_k(p^e)$ is purely periodic modulo m. But

$$\tau_k(p^e) = \binom{e+k-1}{e} = \frac{k(k+1)\cdots(k+e-1)}{e!},$$

and this is clearly purely periodic modulo m with period $e!m$. Let J be a period of $\langle I_k \rangle$ mod m, and observe that from (4),

$$\hat{f}(n) = \sum_{k=0}^{n} S(n,k) I_k \equiv \sum_{0 \le j < J} I_j \sum_{\substack{0 \le k \le n \\ k \equiv j \pmod{J}}} S(n,k) \pmod{m}.$$

But by Lemma 2.4, for each fixed j, the remaining inner sum taken modulo m is purely periodic in n. Hence, $\hat{f}(n)$ is also purely periodic modulo m.

It remains to prove the last assertion of Theorem 1.1. Suppose that $f(N) \equiv a \pmod{m}$, and let be M the squarefull part of N. Write $M = M_i$. In the notation of (3), we have $J > 0$, and so the density of $\mathscr{A}_i$ is positive.

2.3. **Proof of Lemma 2.4 (sketch).** Recall (see, e.g., [6, §3.3]) that the one-variable Bell polynomials $B_n(x)$ are defined by the formal identity

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \mathrm{e}^{(\mathrm{e}^t - 1)x}, \tag{5}$$

or, equivalently but more explicitly, by $B_n(x) = \sum_{k=0}^{n} S(n,k) x^k$. (Thus, the nth Bell number B_n is given by $B_n(1)$.) Now given a fixed arithmetic progression a mod q, as in Lemma 2.4, the orthogonality relations for additive characters show that

$$\sum_{\substack{0 \le k \le n \\ k \equiv a \pmod{q}}} S(n,k) = \frac{1}{q} \sum_{\omega\colon \omega^q = 1} \omega^{-a} B_n(\omega).$$

So to prove the assertion of Lemma 2.4 that the left-hand side here is purely periodic modulo m, it is enough to show that for each fixed qth root of unity ω, the sequence $\langle B_n(\omega) \rangle_{n=0}^{\infty}$ is purely periodic taken modulo qm. (Here the congruences are understood as holding in the ring $\mathbf{Z}[\omega]$ of integers of $\mathbf{Q}[\omega]$.)

This is a special case of the results of Mazouz [20, §2], who studies p-adic properties of the numbers $B(n, \lambda, \omega)$ defined by the exponential generating function

$$\sum_{n \ge 0} B(n, \lambda, \omega) t^n / n! = \mathrm{e}^{\lambda t + \omega(\mathrm{e}^t - 1)}.$$

Here λ and ω are elements of (the Tate field) $\mathbf{C}_p$, algebraic over $\mathbf{Q}_p$, and assumed to satisfy $|\lambda|, |\omega| \le 1$. When $\lambda = 0$ and $|\omega| = 1$, his results [20, §3, (1)–(3)] imply that the sequences $\langle B(n, 0, \omega) \rangle_{n=0}^{\infty}$ are purely periodic modulo every power of p.[1] To obtain the claimed pure periodicity of $\langle B_n(\omega) \rangle_{n=0}^{\infty}$ mod qm (and an explicit expression for the period length), for each prime p dividing qm we view $\mathbf{Q}(\omega)$ as sitting inside $\mathbf{C}_p$ by completing $\mathbf{Q}(\omega)$ at a prime lying above p.

Remark 2.7. It is the author's opinion that Lemma 2.4 is of independent interest. However, one may prove Theorem 1.1 without it: Starting from the relation $kS(n,k) = S(n+1,k) - S(n,k-1)$ (see [6, §5.3, Theorem A]), induction on r shows that $k^r S(n,k)$ is always expressible as an integer linear combination of terms of the form $S(n+c_1, k-c_2)$, where c_1 and c_2 are nonnegative integers. By linearity, if $h(x)$ is any polynomial, then $\sum_k h(k) S(n,k)$ can be rewritten as a linear combination of Bell numbers of the form B_{n+c}. For fixed M, the terms I_k appearing in the proof of Theorem 1.1 are polynomials

[1] The reference [20] contains some misprints; in case (2) of §3, the condition should be that the trace is *non*zero, while the trace should be assumed to vanish in case (3).

M	$\hat{f}(n) = f(Mp_1 \cdots p_n)$
1	B_n
p^2	$\frac{1}{2}(B_{n+2} + B_{n+1} + B_n)$
p^3	$\frac{1}{6}(B_{n+3} + 3B_{n+2} + 5B_{n+1} + 2B_n)$
p^4	$\frac{1}{24}(B_{n+4} + 6B_{n+3} + 17B_{n+2} + 20B_{n+1} + 21B_n)$
p^5	$\frac{1}{120}(B_{n+5} + 10B_{n+4} + 45B_{n+3} + 100B_{n+2} + 169B_{n+1} + 44B_n)$
p^6	$\frac{1}{720}(B_{n+6} + 15B_{n+5} + 100B_{n+4} + 355B_{n+3} + 874B_{n+2} + 869B_{n+1} + 1045B_n)$
p^2q^2	$\frac{1}{4}(B_{n+4} + 2B_{n+3} + 3B_{n+2} + 2B_{n+1} + 3B_n)$

Table 1. Some values of M and the associated functions $\hat{f}(n)$. Here p and q are primes, and the p_i are distinct primes not dividing M.

in k with rational coefficients. Thus, the function $\hat{f}(n) = \sum_k I_k S(n,k)$ is a rational combination of terms of the form B_{n+c}; see Table 1 for some examples.

Given such an expression for $\hat{f}$, the (pure) periodicity of $\hat{f}$ modulo m, as well as a period length, can be read off directly from the results of [25] on the classical Bell numbers. These expressions are also useful for computation. For example, `MAPLE` can compute that every residue class a mod m, with $m \leq 1000$, is represented by at least one of the sequences $\langle \hat{f}(n) \rangle_{n=0}^{3000}$, where M has one of the forms in Table 1.

Example 2.8 (the parity of $f(N)$ revisited)**.** To illustrate the effectivity of our methods, we conclude by sketching a proof that $f(N)$ is odd more than half of the time. As already mentioned in the introduction, $f(N)$ is odd for 2/3 of all squarefree numbers N. We now calculate the corresponding proportion for numbers N of the form p^2N' or p^3N', where p is a prime and N' is a squarefree number coprime to p. Numbers of the first form correspond to the choice $M = p^2$, in the notation of Table 1. Using the corresponding formula in this table and the results of [25], we may calculate that for these M, the function $\hat{f}(n)$ is purely periodic modulo 2, with period $0, 0, 1, 0, 1, 0$. So asymptotically 1/3 of the numbers N of the form p^2N' have $f(N)$ odd. Similarly, taking $M = p^3$, we find that $\hat{f}(n)$ is purely periodic mod 2 with period $1, 1, 1, 0, 0, 1$. Thus, asymptotically 2/3 of the numbers N of the form p^3N' have $f(N)$ odd. It follows that the density of N for which $f(N)$ is odd is at least

$$\frac{2}{3}\left(\frac{6}{\pi^2}\right) + \frac{1}{3}\left(\frac{6}{\pi^2}\sum_p \frac{1}{p(p+1)}\right) + \frac{2}{3}\left(\frac{6}{\pi^2}\sum_p \frac{1}{p^2(p+1)}\right) = 0.52165\ldots.$$

More extensive calculations show that to the nearest tenth of a percent, $f(n)$ is odd 57.1% of the time.

3. Ordered factorizations

Let $g(N)$ denote the number of ordered factorizations of N, so that now two factorizations are considered different whenever the order of the factors is different. Thus, $g(N)$ is to additive compositions what $f(N)$ is to additive partitions. For example, $g(12) = 8$, corresponding to the eight ordered factorizations

$$2 \cdot 2 \cdot 3, \quad 2 \cdot 3 \cdot 2, \quad 3 \cdot 2 \cdot 2, \quad 3 \cdot 4, \quad 4 \cdot 3, \quad 2 \cdot 6, \quad 6 \cdot 2, \quad 12.$$

While a formula for g in terms of the prime factorization of N appears in 19th century work of MacMahon [18, §2], the study of $g(N)$ as a function of N (instead of the factorization pattern of N) is due to Kalmár, who investigated its average order ([12],

[13]). The maximal order of $g(N)$ has been the subject of recent work by Luca and Klazar [14] and by Deléglise et al. [8]. Just as with $f(N)$, one can ask about the parity distribution of $g(N)$ or, more generally, its distribution in arithmetic progressions.

The parity is easy to address: If we let $G(s)$ be the formal Dirichlet series defined by $G(s) := \sum_{N=1}^{\infty} g(N)N^{-s}$, then

$$G(s) = \sum_{k\geq 0}\left(\sum_{d\geq 2}\frac{1}{d^s}\right)^k = \frac{1}{2-\zeta(s)}, \quad \text{where, as usual,} \quad \zeta(s) := \sum_{N=1}^{\infty}\frac{1}{N^s}. \tag{6}$$

Reducing modulo 2 in the ring of formal Dirichlet series with integer coefficients, we find that

$$G(s) \equiv \frac{1}{\zeta(s)} = \prod_p (1-p^{-s}) \equiv \prod_p (1+p^{-s}) = \sum_{N \text{ squarefree}} \frac{1}{N^s}.$$

Hence, $g(N)$ is odd precisely when N is squarefree. The author owes this observation to F. Luca (private communication).

We now prove the g-analogue of the first half of Theorem 1.1.

Theorem 3.1. *Let a and m be any integers with $m \geq 1$. Then*

$$\lim_{x\to\infty}\frac{1}{x}\#\{N \leq x : g(N) \equiv a \pmod{m}\}$$

exists. In other words, the set of N for which $g(N) \equiv a \pmod{m}$ possesses a natural density.

We imitate the proof of Theorem 1.1. As in that theorem, it is enough to show that the density exists for the set of N with $g(N) \equiv a \pmod{m}$ and possessing a *fixed* squarefull part M. Define, in analogy with (2),

$$\hat{g}(n) := g(Mp_1 \cdots p_n),$$

where the p_i are distinct primes not dividing M. It suffices to show that $\hat{g}(n)$, taken modulo m, is eventually periodic. Indeed, that implies that for the N under consideration, $g(N)$ modulo m is an ultimately periodic function of $\omega(N)$, and we can apply Lemma 2.2 as before. (We can ignore the preperiod because the set of N for which $\omega(N)$ is bounded is a set of density zero; see, e.g., [11, §22.11].)

So let us prove this periodicity property. Write $g(N;k)$ for the number of ordered factorizations of N into exactly k parts, and call two factorizations counted in $g(N;k)$ *associates* if one is a permutation of the other. The number of associates of a given factorization is $\frac{k!}{e_1!e_2!\cdots e_r!}$, where $e_1, \dots, e_r$ are the multiplicities of the repeated factors. For N with squarefull part M, we have

$$e_1 + e_2 + \cdots + e_r \leq \Omega(M),$$

and so $e_1! \cdots e_r! \mid \Omega(M)!$. Choosing k_0 large enough that $k_0!$ is a multiple of $\Omega(M)!m$, it follows that for N with squarefull part M,

$$\begin{aligned} g(N) &= \sum_k g(N;k) \\ &\equiv \sum_{0\leq k<k_0} g(N;k) \pmod{m}. \end{aligned}$$

Hence,

$$\hat{g}(n) = g(Mp_1\cdots p_n) \equiv \sum_{0\leq k<k_0} g(Mp_1\dots p_n;k) \pmod{m}.$$

Fix k with $0 \le k < k_0$. As formal Dirichlet series, we have $\sum_{N\ge 1} g(N;k)N^{-s} = (\zeta(s)-1)^k$, and so

$$\begin{aligned} g(Mp_1\cdots p_n;k) &= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\tau_j(Mp_1\cdots p_n) \\ &= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\tau_j(M)j^n. \end{aligned} \tag{7}$$

Taken modulo m, (7) represents an ultimately periodic function of n with period length $\varphi(m)$ (cf. [27, Theorem 8a, p. 261]). Since $\hat{g}(n) \bmod m$ is a finite sum of such terms, $\hat{g}(n)$ is also ultimately periodic modulo m with period $\varphi(m)$.

Example 3.2 ($g(N)$ modulo 4)**.** We have seen already that $g(N)$ is odd if and only if N is squarefree. We now determine $g(N)$ modulo 4. We first suppose that N is squarefree, which corresponds to taking $M = 1$. The proof of Theorem 3.1 will show that with $M = 1$ and $m = 4$, the sequence $\hat{g}(n) \bmod m$ has the form $1, 1, -1, 1, -1, 1, -1, \dots$, where the preperiod consists only of the first term $\hat{g}(0) = 1$. In other words, for squarefree $N > 1$, we have

$$g(N) \equiv -\mu(N) \pmod 4. \tag{8}$$

We now use (8) to show that $g(N) \equiv 2 \pmod 4$ precisely when N is the square of a squarefree number larger than 1. Put $G_0(s) := \sum_{\substack{N\ge 1\\ N \text{ squarefree}}} g(N)N^{-s}$, so that from (8),

$$G_0(s) \equiv 2 - \zeta(s)^{-1} \pmod 4. \tag{9}$$

Write $G(s) = G_0(s) + G_1(s)$. It is sufficient to show that modulo 4, we have $G_1(s) \equiv 2G_2(s)$, where $G_2(s)$ is a Dirichlet series with integral coefficients whose reduction modulo 2 has coefficients supported precisely on the squares of the squarefree numbers $m > 1$. From (6) and (9), we obtain that

$$G_1(s) = G(s) - G_0(s) \equiv 2\frac{\zeta(s) + 1/\zeta(s)}{2-\zeta(s)} \pmod 4,$$

and modulo 2,

$$\frac{\zeta(s)+1/\zeta(s)}{2-\zeta(s)} \equiv 1 + \frac{1}{\zeta(s)^2} = 1 + \prod_p (1-p^{-s})^2 \equiv 1 + \prod_p (1+p^{-2s}) \equiv \sum_{\substack{m \text{ squarefree}\\ m>1}} \frac{1}{m^{2s}}.$$

This shows that the second half of Theorem 1.1 does *not* hold for g. Indeed, there are infinitely many N with $g(N) \equiv 2 \pmod 4$, but the set of such N has density zero.

Just as in the unordered case, it is sensible to ask for a classification of those residue classes for which the density appearing in Theorem 3.1 is positive. The following result is a first step towards answering this question.

Proposition 3.3. *Suppose m is squarefree. If the progression $a \bmod m$ contains an even integer, then the density appearing in Theorem 3.1 is positive.*

The condition that there be some even number $N \equiv a \pmod m$ cannot be removed. For example, there are *no* integers N for which $g(N) \equiv 5 \pmod 6$.

Proof of Proposition 3.3. We start by observing that for all nonnegative integers h,

$$g(2^h \cdot 3) = (h+2)2^{h-1} = \frac{1}{2^3}(h+2)2^{h+2}.$$

This follows (e.g.) by induction on h, via the recurrence relation $g(N) = \sum_{d|N, d<N} g(d)$, valid for $N > 1$.

We now prove the proposition for odd m, where the restriction on a is vacuous. By a result of Rieger [26, Théorème 2], the sequence $\langle h \cdot 2^h \rangle_{h=0}^{\infty}$ taken modulo m is purely periodic and uniformly distributed among the residue classes mod m. Hence, we may fix an $h \geq 2$ with $g(2^h \cdot 3) \equiv a \pmod{m}$. Put $M := 2^h$. Then

$$\hat{g}(1) = g(Mp_1) \equiv a \pmod{m},$$

in the notation of the proof of Theorem 3.1. To show that the set of N with $f(N) \equiv a \pmod{m}$ has positive density, it is enough to argue that $\langle \hat{g}(n) \rangle$, which we know is eventually periodic modulo m, is periodic starting already from $n = 1$. From (7), we have that $\hat{g}(n)$ is congruent, modulo m, to an integer linear combination of terms of the form j^n. But m is squarefree; hence, for each fixed $j \geq 0$, the sequence $\langle j^n \rangle$ mod m is periodic starting already from $n = 1$. (This is obvious if m is prime, and the general case follows from the Chinese remainder theorem.)

Now suppose that $m = 2m'$, where m' is odd. Then $2 \mid a$. By the argument in the preceding paragraph, we can fix $h \geq 2$ so that a positive proportion of numbers N with squarefull part 2^h satisfy $f(N) \equiv a \pmod{m'}$. Since N is not squarefree, these N also satisfy $f(N) \equiv 0 \equiv a \pmod{2}$. Hence, $f(N) \equiv a \pmod{m}$. □

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