Math 4000/6000 – Homework #6
posted October 30, 2018; due November 6, 2018

Examiner: What is a root of multiplicity m ?

Examinee: Well, this is when we plug a number to a function, and obtain zero; then we plug it again, and obtain zero again... and this happens m times. But on the $(m+1)$-st time we do not obtain zero.

– math joke of the day

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

Throughout this assignment, "ring" always means "commutative ring."

1. Exercise 3.3.2(b,c,e,h).

2. Exercise 3.3.4.

3. Exercise 4.1.1. When answering part (b), assume neither of R and S is the zero ring.

4. (a) Let R be a ring. Recall that if $x_1, \ldots, x_n$ are elements of R, then (by definition)
$$\langle x_1, \ldots, x_n \rangle = \{r_1 x_1 + \cdots + r_n x_n : \text{all } r_i \in R\}.$$
That is, $\langle x_1, \ldots, x_n \rangle$ is the set of all R-linear combinations of $x_1, \ldots, x_n$. Prove that $\langle x_1, \ldots, x_n \rangle$ is an ideal of R by directly verifying the three definining properties.

   (b) Let $R = \mathbb{Z}$, and let $a_1, \ldots, a_n$ be positive integers. From part (a), $\langle a_1, \ldots, a_n \rangle$ is an ideal of $\mathbb{Z}$. From class, we know there is an integer d with
$$\langle a_1, \ldots, a_n \rangle = \langle d \rangle.$$
Show that $d \mid a_i$ and that if d' is any integer dividing every a_i, then $d' \mid d$.

5. Exercise 4.1.3. (In part (c), assume R is not the zero ring.)

6. (a) Let R be an integral domain. Show that if $a, b \in R$, then $\langle a \rangle = \langle b \rangle$ if and only if $a = u \cdot b$ for some unit $u \in R$. *Hint:* First show that $\langle a \rangle = \langle b \rangle$ if and only if $a \mid b$ and $b \mid a$.

   (b) Now let $R = F[x]$. Show that $\langle a(x) \rangle = \langle b(x) \rangle$, where $a(x), b(x) \in F[x]$, if and only if $a(x) = c \cdot b(x)$ for some nonzero $c \in F$.

7. (a) Let F be a field. Use the division algorithm in $F[x]$ to prove that every ideal in $F[x]$ has the form $\langle m(x) \rangle$ for some $m(x) \in F[x]$.

   (b) Use the division algorithm in $\mathbb{Z}[i]$ (from earlier homework) to prove that every ideal in $\mathbb{Z}[i]$ has the form $\langle \mu \rangle$ for some $\mu \in \mathbb{Z}[i]$.

8. Prove that if F is a field and $f(x) \in F[x]$ has degree $n \geq 1$, then the elements of $F[x]/\langle f(x) \rangle$ admit a unique expression in the form $\overline{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}}$, where $a_0, \ldots, a_{n-1} \in F$.

   Hint: We (will have) proved existence in class. Your task is to show uniqueness.

9. Exercise 4.1.14(c). Make sure to answer the two questions at the end.

10. Let F be a field, and let $f(x)$ be a nonconstant polynomial in $F[x]$. Prove that if $f(x)$ is reducible, then $F[x]/\langle f(x) \rangle$ is not an integral domain. When $f(x)$ is irreducible, show that $F[x]/\langle f(x) \rangle$ is a field.

11. Let $R = \mathbb{Z}[x]$. Let I be the collection of elements of R with even constant term. Show that $I = \langle 2, x \rangle$, and that I cannot be expressed as $\langle f(x) \rangle$ for any $f(x) \in \mathbb{Z}[x]$.

12. (*) Exercise 3.3.7.

13. (*) Let R be the subring of $\mathbb{Q}[x]$ consisting of those polynomials in $\mathbb{Q}[x]$ with integer constant term. Let I be the subset of R containing those elements with zero constant term. Show that I is an ideal of R, but that there is no finite list of elements $r_1, \ldots, r_k \in R$ with $I = \langle r_1, \ldots, r_k \rangle$.