MATH 4400/6400 – Homework #4

posted March 17, 2023; due March 27, by midnight

Number theorists are like lotus-eaters — having once tasted of this food they can never give it up.
– Leopold Kronecker

Directions. Give complete solutions, providing full justifications when appropriate. Your assignment must be stapled if it goes on beyond one page.

MATH 4400 problems

1. (a) Show that if P, Q are odd integers, then $\frac{P^2-1}{8} + \frac{Q^2-1}{8} \equiv \frac{(PQ)^2-1}{8} \pmod 2$.
   (b) Prove (as claimed in class) that $(-1)^{(P^2-1)/8} = 1$ if $P \equiv \pm 1 \pmod 8$ and that $(-1)^{(P^2-1)/8} = -1$ if $P \equiv \pm 3 \pmod 8$.

2. (a) Find $\left(\frac{82}{365}\right)$. Given that 365 is not prime, what — if anything — can you conclude (without further calculation) from this about whether 82 is a square mod 365?
   (b) Find $\left(\frac{82}{367}\right)$. Noting that 367 is prime, what — if anything — can you conclude from this (without further calculation) about whether 82 is a square mod 367?

3. Let a be a nonzero integer. Let $M = 4|a|$. Show that if P, P' are odd positive integers with $P \equiv P' \pmod M$, then $\left(\frac{a}{P}\right) = \left(\frac{a}{P'}\right)$. (This says that the value of the symbol $\left(\frac{a}{\cdot}\right)$ depends only the "denominator" modulo M.)

   Hint. Write $a = (\pm 1) \cdot 2^k \cdot b$ where b is an odd positive integer. Show that $\left(\frac{\pm 1}{P}\right) = \left(\frac{\pm 1}{P'}\right)$, $\left(\frac{2^k}{P}\right) = \left(\frac{2^k}{P'}\right)$, and $\left(\frac{b}{P}\right) = \left(\frac{b}{P'}\right)$.

4. Let N be a positive integer. Prove that $k(N+1-k) \geq N$ for each integer $k = 1, 2, 3, \ldots, N$. Deduce that $(N!)^2 \geq N^N$.

5. Use calculus to show that $\frac{x}{\log x} > \sqrt{x}$ for every real number $x > 1$.

   Hint. What does the graph of $\frac{x/\log x}{\sqrt{x}}$ look like? Remember that for us, $\log x$ means $\ln x$, the log base e.

6. Prove that for all real numbers α and β,
$$\lfloor \alpha + \beta \rfloor - \lfloor \alpha \rfloor - \lfloor \beta \rfloor = 0 \text{ or } 1.$$

7. Recall that $\mathrm{ord}_p(m)$ denotes the exponent on the largest power of the prime p dividing the positive integer m. In class, we will show that if p is a prime, and n is a positive integer, then
$$\mathrm{ord}_p(n!) = \sum_{k \geq 1} \lfloor n/p^k \rfloor.$$
   Using this formula, determine the number of zeros at the end of 2023! (written in decimal, as usual).

8. Use Exercise 6 to prove that if p is prime and n is a positive integer, then
$$\mathrm{ord}_p(n!) \geq \mathrm{ord}_p(k!(n-k)!) \quad \text{for all integers } 0 \leq k \leq n.$$

   [It follows that $k!(n-k)!$ divides $n!$. This shows that the binomial coefficients $\binom{n}{k}$ are integers!]

9. Define $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$; this function is called the logarithmic integral. Compute

$$\lim_{x\to\infty} \frac{\mathrm{Li}(x)}{x/\log x}.$$

MATH 6400 problems

G1. Let p be a prime number. Show that if p divides a number of the form $x^4 - x^2 + 1$, where $x \in \mathbf{Z}$, then $p \equiv 1 \pmod{12}$.

Hint. First show that both -1 and -3 are squares mod p.

G2. Recall from class that $\pi(x)/x \to 0$ as $x \to \infty$. Using this result, show that for every integer $k > 1$, there is a positive integer n with $n/\pi(n) = k$.