

## MATH 4000/6000 – Homework #1

posted January 14, 2022; due at the **start of class** on January 24, 2022

You know, for a mathematician, he did not have enough imagination. But he has become a poet and now he is fine. — David Hilbert (1862–1943), talking of an ex-student

Assignments are expected to be neat and stapled. **Illegible work may not be marked.** Starred problems (\*) are *required* for those in MATH 6000 and *extra credit* for those in MATH 4000.

Fully explain your answers. In problems #1 and #2 you must explain which algebraic properties (properties A1–A4, M1–M3, D1 on the handout) you are using at every step of the proof. I recommend a “two column” format with each line showing, on the left, a step in the proof and on the right a justification. For example:

$$a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \quad (\text{Distributive law}).$$

Refer to properties by name rather than number (write “Distributive law” and not “D1”).

For all other problems, you may do “familiar” algebraic manipulations without citing the algebraic properties on the handout. **Throughout this problem set, you may also assume that  $a \cdot 0 = 0 = 0 \cdot a$  and  $(-1)a = -a = a(-1)$  for all  $a$ , as already shown in class.** You may also assume that  $a > 0$  is equivalent to  $a \in \mathbb{Z}^+$ .

Note:  $\mathbb{Z}^+$  means the same as  $\mathbb{N}$  (the book’s notation).

1. Prove that for all  $a, b \in \mathbb{Z}$ , we have

(a)  $(-a)b = -(ab)$ .

(b)  $(-a)(-b) = ab$ .

2. (a) Let  $a, b \in \mathbb{Z}$  and suppose that  $a < b$ . Prove that  $a + c < b + c$  for every  $c \in \mathbb{Z}$ .

(b) Let  $a, b \in \mathbb{Z}$  and suppose that  $a < b$ . Prove that  $ac < bc$  for every  $c \in \mathbb{Z}^+$ .

3. Let  $a, b \in \mathbb{Z}$ . Show that  $a < b$ ,  $a = b$ , or  $a > b$ , and in fact exactly one of these three holds.

4. Let  $a, b \in \mathbb{Z}$ .

(a) Prove that if  $a < 0$  and  $b < 0$ , then  $ab > 0$ .

(b) Show that if  $a < 0$  and  $b > 0$ , then  $ab < 0$ .

(c) Show that if  $ab = 0$ , then either  $a = 0$  or  $b = 0$ .

5. Use the Well-Ordering Principle to prove the following version of the Principle of Mathematical Induction.

Let  $S$  be a subset of  $\mathbb{Z}^+$ . Assume:

(1)  $1 \in S$ ,

(2) for all  $n \in \mathbb{Z}^+$ , if  $n \in S$  then also  $n + 1 \in S$ .

Then  $S = \mathbb{Z}^+$ .

*Hint to get you started:* If  $S \subsetneq \mathbb{Z}^+$ , then there is a least positive integer *not* in  $S$ .

6. (Laws of exponents) Let  $a \in \mathbb{Z}$ . Suppose that  $m, n$  belong to the set  $\mathbb{Z}^+ \cup \{0\}$  of nonnegative integers.

- (a) Prove that  $a^m \cdot a^n = a^{m+n}$ .
- (b) Prove that  $a^{mn} = (a^m)^n$ .

*Hint:* If  $m = 0$  or  $n = 0$ , this is easy (why?). So you can suppose  $m, n \in \mathbb{Z}^+$ . Now think of  $m$  as fixed and proceed by induction on  $n$ .

7. Use the binomial theorem to find formulas for the following sums, as functions of  $n$ , where  $n$  is assumed to be a natural number.

(a)  $\sum_{k=0}^n \binom{n}{k}.$

(b)  $\sum_{k=0}^n (-1)^k \binom{n}{k}.$

8. Show that if  $a, b \in \mathbb{Z}^+$  and  $a \mid b$ , then  $a \leq b$ .
9. In this exercise we outline a proof of the following statement, which we will be taking for granted in our proof of the Division Theorem: If  $a, b \in \mathbb{Z}$  with  $b > 0$ , the set

$$S = \{a - bq : q \in \mathbb{Z} \text{ and } a - bq \geq 0\}$$

has a least element.

- (a) Prove the claim in the case  $0 \in S$ .
- (b) Prove the claim in the case  $0 \notin S$  and  $a > 0$ .
- (c) Prove the claim in the case  $0 \notin S$  and  $a \leq 0$ .

*Hint:* (a) is easy. To handle (b) and (c), first show that in these cases that  $S$  is a nonempty set of natural numbers, so that the well-ordering principle guarantees  $S$  has a least element as long as  $S$  is nonempty. To prove  $S$  is nonempty, show that in case (b), the integer  $a$  is an element of  $S$ . You will have to work a little harder to prove  $S$  is nonempty in case (c).

10. Use the Euclidean algorithm to find  $\gcd(314, 159)$  and  $\gcd(272, 1479)$ . Show the steps, not just the final answer.

## 6000 problems

11. (\*) Prove that there is no subset  $S$  of the complex numbers  $\mathbb{C}$  satisfying all three of the following properties.
- (1) If  $a, b \in S$ , then  $a + b \in S$ .
  - (2) If  $a, b \in S$ , then  $a \cdot b \in S$ .
  - (3) For every  $a \in \mathbb{C}$ , exactly one of the following holds: (a)  $a \in S$ , (b)  $a = 0$ , (c)  $-a \in S$ .

You may assume familiar properties of  $\mathbb{C}$  for this problem.

12. (\*) Prove that the properties of the set  $\mathbb{Z}^+$  on the handout uniquely determine  $\mathbb{Z}^+$  as a set. Precisely: Assume all of A1–N1, as usual. Furthermore, assume O1 and WOP hold with two subsets  $P, P'$  of the integers in place of  $\mathbb{Z}^+$ . Prove that  $P = P'$ .