THE AVERAGE LEAST CHARACTER NONRESIDUE

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Abstract. For each nonprincipal Dirichlet character χ, let n_χ be the least n with $\chi(n) \notin \{0,1\}$. We show that as $q \to \infty$, the average of n_χ over all nonprincipal characters χ modulo q is $\ell(q) + o(1)$, where $\ell(q)$ denotes the least prime not dividing q. Moreover, if one averages over all nonprincipal characters of moduli $\leq x$, the limiting value is $2.5305\ldots$ as $x \to \infty$.

1. Introduction

For χ a nonprincipal Dirichlet character modulo q, let n_χ denote the least positive integer n with $\chi(n) \notin \{0,1\}$. If $q = p$ is prime, then χ is a kth power residue character for some k dividing $p-1$, and the study of the maximal order of n_χ goes back to Vinogradov and Linnik in the early part of the twentieth century. Assuming the Riemann Hypothesis for Dirichlet L-functions, we know that

$$\max_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} n_\chi \leq 3(\log q)^2. \tag{1.1}$$

(The first result of this kind is due to Ankeny [Ank52]; as stated here, the result is due to Bach [Bac90, Theorem 3].) The best unconditional result in this direction, due to Norton (see [Nor98, eq. (1.22)]), asserts that the maximum in (1.1) is $\ll_\epsilon q^{\frac{1}{4\sqrt{e}}+\epsilon}$.

Short of a completely satisfactory "pointwise" result, one can study n_χ on average. The first to adopt this viewpoint was Erdős [Erd61], who treated quadratic characters modulo p: He showed that as $x \to \infty$,

$$\frac{1}{\pi(x)} \sum_{\substack{2<p\leq x \\ \chi(\cdot)=\left(\frac{\cdot}{p}\right)}} n_\chi \to \sum_{k=1}^{\infty} \frac{p_k}{2^k},$$

where p_k denotes the kth prime in increasing order. This result was extended to all real primitive characters by the second author [Pol11], who showed that

$$\frac{\sum_{|D|\leq x,\ \chi(\cdot)=\left(\frac{D}{\cdot}\right)} n_\chi}{\sum_{|D|\leq x} 1} \to \sum_{k=1}^{\infty} \frac{p_k^2}{2(p_k+1)},$$

where the sum on D is over fundamental discriminants of absolute value $\leq x$. It is the purpose of this note to evaluate the average of n_χ over all nonprincipal characters χ.

Let $\ell(q)$ denote the least prime not dividing q. If χ is any character modulo q, then $\chi(n) = 0$ whenever $1 < n < \ell(q)$. Hence, $n_\chi \geq \ell(q)$ for all nonprincipal χ. Our main theorem says that the average of n_χ is very close to $\ell(q)$:

Theorem 1.1. *For $q \geq 3$, we have*

$$\frac{1}{\phi(q)-1} \sum_{\chi\neq\chi_0} n_\chi = \ell(q) + O((\log\log q)^2/\log q),$$

where χ runs over all nonprincipal characters modulo q.

As a consequence, we have the following result for the average over all nonprincipal characters to moduli $\le x$:

Corollary 1.2. *As $x \to \infty$,*

$$\frac{\sum_{q\le x}\sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}} n_\chi}{\sum_{q\le x}\sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}} 1} \to \Delta, \quad \text{where} \quad \Delta := \sum_{\ell} \frac{\ell^2}{\prod_{p\le\ell}(p+1)}. \tag{1.2}$$

Here the right-hand sum is over all primes ℓ and the product in the denominator is over primes $p \le \ell$.

Remark 1.3. A quick calculation with Mathematica shows that

$$\Delta = 2.5350541803604388301655300071859083508611780138537 0\ldots$$

The proofs of Theorem 1.1 and Corollary 1.2, while similar in flavor to the arguments of [Erd61, Pol11], employ different tools. Our primary inspiration was a paper of Burthe [Bur97], which uses zero-density estimates and a theorem of Montgomery (Proposition 2.2 below) to prove that

$$\frac{1}{x}\sum_{q\le x}\max_{\substack{\chi \bmod q\\ \chi\ne\chi_0}} n_\chi \ll (\log x)^{97};$$

note that Burthe's result shows unconditionally that a bound of the same flavor as (1.1) holds on average.

Notation. The letters p and ℓ are reserved for prime variables. We write $P(n)$ for the largest prime factor of n. We say that n is *y-friable* (or *y-smooth*) if $P(n) \le y$, and we let $\Psi(x,y)$ denote the number of y-friable $n \le x$. We write $\omega(n) := \sum_{p\mid n} 1$ for the number of distinct prime factors of n and $\Omega(n) := \sum_{p^k\mid n} 1$ for the number of prime factors of n counted with multiplicity. We use $c_1, c_2, \ldots$ for absolute positive constants. We write $\log_1 x = \max\{1, \log x\}$, and we use $\log_k$ for the kth iterate of $\log_1$.

2. Proof of Theorem 1.1

We begin by quoting two theorems. The first, due to Baker and Harman [BH96, BH98], asserts that many shifted primes possess a large prime factor.

Proposition 2.1. *For each positive real number $\theta \le 0.677$, there is a constant $c_\theta > 0$ with the following property: For all large x, say $x > x_0(\theta)$, the number of primes $p \le x$ with $P(p-1) > x^\theta$ is $> c_\theta x/\log x$.*

The next result, due to Montgomery (see [Mon94, Theorem 1, p. 164], and cf. [LMO79]), relates the size of n_χ to a zero-free region for $L(s,\chi)$ near $s = 1$.

Proposition 2.2. *Let χ be a nonprincipal Dirichlet character modulo q. If $\frac{1}{\log q} < \delta \le 1/2$ and $N(1-\delta, \delta^2\log q, \chi) = 0$, then $n_\chi < (c_1\delta\log q)^{1/\delta}$. Here c_1 is an absolute positive constant.*

Proposition 2.2 allows us to establish the next lemma, which will eventually be used to show that characters χ with n_χ larger than about $(\log q)^5$ do not significantly affect the average of n_χ.

Lemma 2.3. *The number of nonprincipal characters χ modulo q with $n_\chi \ge (\frac{c_1}{5}\log q)^5$ is $\ll q^{9/20}$. Here c_1 has the same meaning as in Proposition 2.2.*

Proof. The proof uses Proposition 2.2 and the following zero-density estimate due to Jutila (see [Jut77, Theorem 1]): Let $\epsilon > 0$. For $4/5 \le \alpha \le 1$ and $T \ge 1$, we have

$$\sum_{\chi \bmod q} N(\alpha, T, \chi) \ll_\epsilon (qT)^{(2+\epsilon)(1-\alpha)}. \tag{2.1}$$

For the proof of the lemma, we may assume that q is large. By Proposition 2.2 (with $\delta = 1/5$), the number of nonprincipal χ with $n_\chi \ge (\frac{c_1}{5}\log q)^5$ is bounded above by

$$\sum_{\chi \bmod q} N\left(\frac{4}{5}, \frac{1}{25}\log q, \chi\right).$$

From (2.1) with $\epsilon = \frac{1}{10}$, this sum is $\ll (q\log q)^{21/50}$, which is crudely $\ll q^{9/20}$. $\square$

Lemma 2.4. *Assume $q > 1$, and write $\ell = \ell(q)$. Then*

$$\frac{1}{\phi(q)-1}\sum_{\substack{\chi \ne \chi_0 \\ n_\chi = \ell}} n_\chi = \ell + O(\ell/f), \tag{2.2}$$

while

$$\frac{1}{\phi(q)-1}\sum_{\substack{\chi \ne \chi_0 \\ n_\chi > \ell}} n_\chi \ll q^{-1/50} + \frac{1}{\phi(q)}\sum_{\substack{\chi\ne\chi_0 \\ \ell < n_\chi \le (\frac{c_1}{5}\log q)^5}} n_\chi. \tag{2.3}$$

Lemma 2.4 reduces the proof of Theorem 1.1 to the task of showing that both the O-term in (2.2) and the right-hand side of (2.3) are $\ll (\log_2 q)^2/\log q$.

Proof. A character χ mod q has $\chi(\ell) = 1$ precisely when χ descends to a character on the quotient $(\mathbf{Z}/q\mathbf{Z})^\times/\langle \ell\rangle$. Hence, the proportion of characters χ mod q with $\chi(\ell) = 1$ is $\frac{1}{f}$, where f is the order of ℓ modulo q. So the contribution to the average of n_χ from those χ with $n_\chi = \ell$ is

$$\frac{\phi(q) - \phi(q)/f}{\phi(q)-1}\ell = \ell + O(\ell/f).$$

Turning to the contribution from the remaining χ, we have

$$\begin{aligned}\frac{1}{\phi(q)-1}\sum_{\substack{\chi\ne\chi_0\\ n_\chi>\ell}} n_\chi &= \frac{1}{\phi(q)-1}\sum_{\substack{\chi\ne\chi_0 \\ \ell<n_\chi\le(\frac{c_1}{5}\log q)^5}} n_\chi + \frac{1}{\phi(q)-1}\sum_{\substack{\chi\ne\chi_0\\ n_\chi > (\frac{c_1}{5}\log q)^5}} n_\chi \\ &= \frac{1}{\phi(q)-1}\sum_{\substack{\chi\ne\chi_0 \\ \ell<n_\chi\le(\frac{c_1}{5}\log q)^5}} n_\chi + O\left(\frac{\max_{\chi\ne\chi_0} n_\chi}{\phi(q)}\sum_{\substack{\chi\ne\chi_0\\ n_\chi > (\frac{c_1}{5}\log q)^5}} 1\right).\end{aligned}$$

By the Polya–Vinogradov inequality, the maximum over χ appearing here is $\ll q^{21/40}$ (say). (Norton's sharper result, quoted in the introduction, would not be any better for our purposes.) Using this in conjunction with Lemma 2.3, we find that the O-term term here is $\ll q^{39/40}/\phi(q) \ll q^{-1/50}$. $\square$

Proof of Theorem 1.1. We can assume that q is large. We first prove the theorem when $\ell > X$, where

$$X := (\log_2 q)^2/\log_3 q.$$

Fix $\theta := 2/3$. By Proposition 2.1, there are $\gg X/\log X$ primes $p \le X$ with $P(p-1) > X^\theta$. We claim that for almost all of these primes p, the order of ℓ modulo p is divisible by $P(p-1)$. To see this, note that if p does not have this property, then $\ell(p) \mid (p-1)/P(p-1)$, and so $\ell(p) < X^{1-\theta}$; hence,

$$p \mid \prod_{1\le j<X^{1-\theta}} (\ell^j - 1).$$

Now $\Omega(\ell^j-1) \ll j\log\ell$, and so summing over j, we see there are only $\ll X^{2(1-\theta)}\log\ell \ll X^{3/4}$ such exceptional p, and this number is $o(X/\log X)$.

Let S be the set of non-exceptional p constructed above, so that $\#S \gg X/\log X$. If $q > X^{\theta}$, the number of $p\in S$ for which $q = P(p-1)$ is clearly $\le \pi(x;q,1) < X/q < X^{1-\theta}$. Hence, the number of distinct values $P(p-1)$, as p ranges over S, is $\gg X^{\theta}/\log X$. Since f is divisible by all these values $P(p-1)$, it follows that

$$f \ge (X^{\theta})^{c_2X^{\theta}/\log X} \ge \exp(c_3X^{\theta})$$

for some $c_2, c_3 > 0$.

Consequently, for the O-term in (2.2), we have the estimate

$$\ell/f \le \ell/\exp(c_3(\log_2 q)^{2\theta}/(\log_3 q)^{\theta}) < 2\log q/\exp((\log_2 q)^{1.3}) < 1/\log q.$$

Moreover, the right-hand side of (2.3) is

$$\ll q^{-1/50} + \frac{(\log q)^5}{\phi(q)} \sum_{\substack{\chi\ne\chi_0\\ n_\chi>\ell}} 1 < q^{-1/50} + \frac{(\log q)^5}{f} < \frac{1}{\log q}.$$

So in the case when $\ell > X$, the average of n_χ is $\ell + O(1/\log q)$, which is sharper than what is claimed in the theorem.

In the above reasoning, it was not necessary to assume that q is divisible by *all* primes up to X; the same arguments apply if, in the notation of Proposition 2.1, q is divisible by all but at most $\frac12 c_\theta X/\log X$ primes $p\le X$. So in what follows, we assume not only that $\ell \le X$, but that there are more than $\frac12 c_\theta X/\log X$ primes $p \le X$ not dividing q. Under these hypotheses, the error term ℓ/f in (2.2) is trivially bounded. Indeed, from $q \mid \ell^f - 1$, it follows that

$$f \ge \log q/\log \ell, \tag{2.4}$$

and so

$$\ell/f \le \ell\log\ell/\log q \le X\log X/\log q \ll (\log_2 q)^2/\log q,$$

which is acceptable. Also, the right-hand side of (2.3) is

$$\ll q^{-1/50} + \frac{1}{\phi(q)} \sum_{\substack{\chi\ne\chi_0\\ \ell<n_\chi\le X}} n_\chi + \frac{1}{\phi(q)} \sum_{\substack{\chi\ne\chi_0\\ X<n_\chi\le(\frac{c_1}{5}\log q)^5}} n_\chi \ll q^{-1/50} + \frac{X}{f} + \frac{(\log q)^5}{\phi(q)} \sum_{\substack{\chi\ne\chi_0\\ n_\chi>X}} 1.$$

By our hypotheses, we may pick six primes $p_1,\dots,p_6 \le X$ not dividing q. If $n_\chi > X$, then χ vanishes on the subgroup of $(\mathbf{Z}/q\mathbf{Z})^\times$ generated by (the images of) the p_i. The order of this subgroup is not less than the number of $n\le q$ which factor as a product of the p_i, which is $\gg (\log q/\log X)^6 \gg (\log q/\log_3 q)^6$. It follows that

$$\frac{1}{\phi(q)} \sum_{\substack{\chi\ne\chi_0\\ n_\chi>X}} 1 \ll \frac{(\log_3 q)^6}{(\log q)^6}.$$

With (2.4), this gives that the right-hand side of (2.3) is $\ll (\log_2 q)^2/\log q$ and completes the proof of the theorem. $\square$

3. Proof of Corollary 1.2

Lemma 3.1. *Let m be a natural number. For $x \geq 1$, we have that*

$$\sum_{\substack{n \leq x \\ \gcd(n,m)=1}} \phi(n) = \frac{3x^2}{\pi^2} \prod_{p|m} (1+1/p)^{-1} + O(2^{\omega(m)} x \log(\mathrm{e}x)),$$

uniformly in m.

Proof. Let χ_0 be the principal character modulo m, so that we seek to estimate the partial sums of $\phi\chi_0$. For each natural number d, let $h(d)$ denote the largest divisor of d coprime to m. One checks easily that $\frac{\phi(n)}{n}\chi_0(n) = \sum_{d|n} \mu(d)/h(d)$, so that

$$\sum_{\substack{n \leq x \\ \gcd(n,m)=1}} \phi(n) = \sum_{n \leq x} n \sum_{d|n} \mu(d)/h(d) = \sum_{d \leq x} \frac{\mu(d)}{h(d)} \sum_{e \leq x/d} (de)$$

$$= \sum_{d \leq x} \frac{\mu(d)}{h(d)} \left(\frac{x^2}{2d} + O(x) \right) = \frac{x^2}{2} \sum_{d \leq x} \frac{\mu(d)}{h(d)d} + O\left(x \sum_{\substack{d \leq x \\ d \text{ squarefree}}} \frac{1}{h(d)} \right).$$

Now the infinite sum

$$\sum_{d} \frac{\mu(d)}{h(d)d} = \left(\prod_{p \nmid m} \left(1 - \frac{1}{p^2}\right) \right) \left(\prod_{p|m} \left(1 - \frac{1}{p}\right) \right)$$
$$= \left(\prod_{p} \left(1 - \frac{1}{p^2}\right) \right) \prod_{p|m} \left(1 + \frac{1}{p}\right)^{-1} = \frac{6}{\pi^2} \prod_{p|m} \left(1 + \frac{1}{p}\right)^{-1},$$

and so
(3.1)

$$\sum_{\substack{n \leq x \\ \gcd(n,m)=1}} \phi(n) = \frac{3x^2}{\pi^2} \prod_{p|m} \left(1 + \frac{1}{p}\right)^{-1} + O\left(x^2 \sum_{\substack{d > x \\ d \text{ squarefree}}} \frac{1}{dh(d)} \right) + O\left(x \sum_{\substack{d \leq x \\ d \text{ squarefree}}} \frac{1}{h(d)} \right).$$

For each $y \geq 1$, we have

$$S(y) := \sum_{\substack{d \leq y \\ d \text{ squarefree}}} \frac{1}{h(d)} \leq \prod_{p \leq y} \left(1 + \frac{1}{h(p)}\right) \leq 2^{\omega(m)} \prod_{p \leq y} \left(1 + \frac{1}{p}\right) \ll 2^{\omega(m)} \log(\mathrm{e}y).$$

This shows immediately that the second O-term in (3.1) is acceptable for us. By partial summation, $\sum_{d > x,\ \text{squarefree}} \frac{1}{dh(d)} = \int_x^\infty t^{-1} dS(t) \leq \int_x^\infty S(t)/t^2\, dt \ll 2^{\omega(m)} x^{-1} \log(\mathrm{e}x)$. Hence, the first O-term in (3.1) is $\ll 2^{\omega(m)} x \log(\mathrm{e}x)$, which is again acceptable. This completes the proof of the lemma. □

Proof of Corollary 1.2. By Lemma 3.1, the denominator in (1.2) is

$$\sum_{q \leq x} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} 1 = \sum_{q \leq x} (\phi(q) - 1) = \frac{3x^2}{\pi^2} + O(x \log x),$$

and so it suffices to show that the numerator in (1.2) is $\sim \frac{3}{\pi^2}\Delta x^2$ as $x \to \infty$. By Theorem 1.1, we have that as $x \to \infty$,

$$\begin{aligned}\sum_{q\le x}\sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}} n_\chi &= \sum_{q\le x}(\phi(q)-1)(\ell(q)+O((\log_2 q)^2/\log q))\\ &= \sum_{1<q\le x}\phi(q)\ell(q) + O\left(\sum_{1<q\le x}\left(\ell(q)+\frac{(\log_2 q)^2}{\log q}\right)\right)\\ &= \sum_{1<q\le x}\phi(q)\ell(q)+o(x^2).\end{aligned}$$

To work through the above, it is helpful to keep in mind that with $y := 2\log x$, we have $\ell(q) \le y$ uniformly for $q \le x$ (assuming x is large). To estimate the remaining sum $\sum_{1<q\le x}\phi(q)\ell(q)$, we let M be the y-friable part of q and partition the sum according to the value of M. Observe that since $q > 1$, we have that $M > 1$ and $\ell(q) = \ell(M)$.

We can assume that $M \le x^{1/2}$. Indeed, the number of $q \le x$ divisible by a y-friable number $M > x^{1/2}$ is at most

$$x\sum_{\substack{M>x^{1/2}\\ p\mid M\Rightarrow p\le y}}\frac{1}{M} = x\int_{x^{1/2}}^{\infty}\frac{d\Psi(t,y)}{t} \le x\int_{x^{1/2}}^{\infty}\frac{\Psi(t,y)}{t^2}\,dt \le x^{2/3},$$

say, once x is large. (We use here that $\Psi(t,y) \le \Psi(t, 4\log t) \le t^{o(1)}$ as $t \to \infty$; see, e.g., [Ten95, Theorem 2, p. 359].) Since $\phi(q)\ell(q) \le xy$ for all $q \le x$, those q corresponding to values $M > x^{1/2}$ contribute $\le x^{5/3}y = o(x^2)$, which is negligible.

Again invoking Lemma 3.1, we find that each remaining value of $M > 1$ contributes

$$\begin{aligned}\ell(M)\sum_{\substack{1<q\le x\\ \ell(q)=M}}\phi(q) &= \ell(M)\phi(M)\sum_{\substack{q'\le x/M\\ \gcd(q',\prod_{p\le y}p)=1}}\phi(q')\\ &= \frac{3x^2}{\pi^2}\frac{\ell(M)\phi(M)}{M^2}\prod_{p\le y}\left(1+\frac{1}{p}\right)^{-1} + O\left(2^{\pi(y)}\frac{\ell(M)\phi(M)}{M}x\log(\mathrm{e}x)\right).\end{aligned}$$

Now sum this estimate over y-friable M from the interval $(1, x^{1/2}]$. The O-error is

$$\ll 2^{\pi(y)}\sum_{M\le x^{1/2}}(yx\log(\mathrm{e}x)) \ll x^{3/2}(\log x)^2\exp(O(\log x/\log\log x)),$$

which is $o(x^2)$. The main term is given by

$$\frac{3x^2}{\pi^2}\left(\prod_{p\le y}\left(1+\frac{1}{p}\right)^{-1}\right)\sum_{\ell\le y}\ell\sum_{\substack{M\le x^{1/2},\ y\text{-friable}\\ \ell(M)=\ell}}\frac{\phi(M)}{M^2}. \tag{3.2}$$

Extending the inner sum over all M, we find that

$$\begin{aligned}\sum_{\substack{M\ y\text{-friable}\\ \ell(M)=\ell}}\frac{\phi(M)}{M^2} &= \prod_{p<\ell}\left(\frac{\phi(p)}{p}+\frac{\phi(p^2)}{p^4}+\dots\right)\prod_{\ell<p\le y}\left(1+\frac{\phi(p)}{p}+\frac{\phi(p^2)}{p^4}+\dots\right)\\ &= \left(\prod_{p<\ell}\frac{1}{p}\right)\left(\prod_{\ell<p\le y}\left(1+\frac{1}{p}\right)\right);\end{aligned}$$

moreover, the error incurred by extending the sum is (for large x) at most

$$\sum_{\substack{M\ y\text{-friable}\\ M>x^{1/2}}} \frac{1}{M} = \int_{x^{1/2}}^{\infty} \frac{d\Psi(t,y)}{t} < \frac{1}{x^{1/3}}.$$

This shows that (3.2) is

$$\frac{3x^2}{\pi^2}\sum_{\ell\le y}\frac{\ell^2}{\prod_{p\le\ell}(p+1)} + O\left(x^2\sum_{\ell\le y}\ell x^{-1/3}\right).$$

The error here is $\ll x^{5/3}y^2$, and so is again $o(x^2)$. Since the sum over ℓ appearing in the main term tends to Δ as $x\to\infty$, collecting our estimates we find that the numerator in (1.2) is indeed $\sim \frac{3}{\pi^2}\Delta x^2$ as $x\to\infty$, as desired. □

Remark 3.2. Let $q>1$, let $\ell=\ell(q)$, and let $\mathscr{X}(q)$ be a nonempty collection of nonprincipal Dirichlet characters mod q. The set of $\chi\in\mathscr{X}(q)$ with $n_\chi>\ell$ is obviously a subset of the set of all χ mod q with $n_\chi>\ell$. This triviality, taken together with the estimates occurring in the proof of Theorem 1.1, shows that

$$\frac{\sum_{\chi\in\mathscr{X}(q)} n_\chi}{\sum_{\chi\in\mathscr{X}(q)} 1} = \ell + O\left(\frac{\phi(q)}{\#\mathscr{X}(q)}(\log_2 q)^2/\log q\right). \tag{3.3}$$

To take an example of special interest, let $\mathscr{X}(q)$ be the set of primitive characters modulo q, and let $\phi'(q) := \#\mathscr{X}(q)$. From Möbius inversion applied to the relation $\sum_{d\mid q}\phi'(d)=\phi(q)$, we find that

$$\phi'(q) = q\prod_{p\| q}\left(1-\frac{2}{q}\right)\prod_{p^2\mid q}\left(1-\frac{1}{q}\right)^2.$$

Hence, $\phi'(q)>0$ precisely when $q\not\equiv 2 \pmod 4$, and whenever $\phi'(q)$ is nonvanishing, we have

$$\phi'(q)\gg\phi(q)\prod_{p\mid q}(1-1/p)\gg\phi(q)/\log_2 q.$$

So when $q\not\equiv 2\pmod 4$, estimate (3.3) shows that the average of n_χ taken over primitive characters χ modulo q is $\ell(q)+O((\log_2 q)^3/\log q)$. From this, one can deduce a corollary similar to Corollary 1.2. One replaces Lemma 3.1 with the following estimate, which can be proved by a similar argument:

Lemma 3.3. *Let m be a natural number. For $x\ge 1$, we have that*

$$\sum_{\substack{n\le x\\ \gcd(n,m)=1}}\phi'(n) = \frac{18x^2}{\pi^4}\left(\prod_{p\mid m}\frac{p^3}{(p+1)(p^2-1)}\right) + O(2^{\omega(m)}x(\log(\mathrm{e}x))^2),$$

uniformly in m.

Imitating the proof of Corollary 1.2 but using Lemma 3.3 as input, one eventually finds that as $x\to\infty$,

$$\frac{\sum_{1<q\le x}\sum_{\chi}^{*} n_\chi}{\sum_{1<q\le x}\sum_{\chi}^{*} 1}\to\sum_{\ell}\frac{\ell^4}{(\ell+1)(\ell^2-1)}\prod_{p<\ell}\frac{p^2-p-1}{(p+1)(p^2-1)},$$

where $\sum^*$ indicates a sum over primitive characters modulo q. MATHEMATICA evaluates the right-hand sum as

$$2.1514351056861465486242810050965840532633045718584 5\ldots.$$

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