MATH 4000/6000 – Homework #5

posted March 15, 2022; due by midnight on March 23, 2022

Algebra is but written geometry and geometry is but written algebra. – Sophie Germain

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

1. 3.1.6.

2. 3.1.10(a,c,e).

3. Let F be a field. Prove that the units in $F[x]$ are precisely the nonzero elements of F.

4. Let F be a field. Recall the definition of the gcd in $F[x]$: a gcd of $a(x), b(x)$ is a common divisor of $a(x)$ and $b(x)$ in $F[x]$ that is divisible by every common divisor in $F[x]$.

   Show that if $d(x) \in F[x]$ is a gcd of $a(x), b(x)$, then so is $c \cdot d(x)$ for every nonzero $c \in F$. Conversely, show that every gcd of $a(x), b(x)$ has the form $c \cdot d(x)$ for some nonzero $c \in F$.

5. Let F be a field. Give a detailed proof that every nonconstant polynomial in $F[x]$ can be written as a product of irreducible polynomials. (You are not asked to prove uniqueness in this problem.)

6. In Chapter 4, we will construct a field K with 4 elements containing $\mathbb{Z}_2$ as subfield. In this exercise, *assume* K is such a field. Then in addition to $0, 1$ from $\mathbb{Z}_2$, the field K has two extra elements; call these α and β.

   (a) Show that $\alpha + 1 = \beta$.

   Hint. Try process of elimination.

   (b) Show that $\alpha^2 = \beta$.

   (c) Show that both α and β are roots of x^2+x+1 and deduce that $x^2+x+1 = (x-\alpha)(x-\beta)$ in $K[x]$.

7. Let F be a subfield of K, and let $\alpha \in K$. Suppose that α is a root of the irreducible polynomial $p(x) \in F[x]$. Let n be the degree of $p(x)$. Show that every element of $F[\alpha]$ has a *unique* representation in the form

$$a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1},$$

   where $a_0, a_1, \ldots, a_{n-1} \in F$.

   Hint: We [will have] proved this in class without the uniqueness requirement. So your job is (only) to prove uniqueness.

8. (a) Let $\sqrt{2}, \sqrt{3}$ denote the positive real square roots of 2 and 3, respectively. Prove that $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$.

   (b) Prove that $\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\sqrt{2} + \sqrt{3}]$.

   Hint: Show containment both ways. One direction is fairly easy: Since $\sqrt{2}, \sqrt{3} \in \mathbb{Q}[\sqrt{2}, \sqrt{3}]$, and $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$ is closed under addition (being a ring), we have $\sqrt{2}+\sqrt{3} \in \mathbb{Q}[\sqrt{2}, \sqrt{3}]$. Since $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$ contains both $\mathbb{Q}$ and $\sqrt{2}+\sqrt{3}$, and is closed under addition and multiplication (being a ring), it follows that $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$ contains $\mathbb{Q}[\sqrt{2} + \sqrt{3}]$. Can you find a similar argument for the other containment?

9. Let F be a subfield of K, and suppose $\alpha \in K$ is not algebraic over F. Prove that α has no multiplicative inverse in $F[\alpha]$. Deduce that $F[\alpha]$ is not a field.

10. (*) Let F be a field and let $a \in F$ be nonzero.

(a) Prove that there cannot be **exactly two** distinct solutions z in F to the equation $z^3 = a$.

(b) Write down an example of an equation $z^3 = a$ (with a nonzero) that has no solutions. Then write down an example with 1 solution and an example with 3 solutions. (An example consists of both a specific field F and a nonzero element $a \in F$. You will probably want to use different choices of F for different examples!)