MATH 4000/6000 – Homework #4

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You did a number on me. But, honestly, baby, who's counting?
— Taylor Swift

Assignments are expected to be neat and stapled. **Illegible work may not be marked**. Starred problems (*) are required for those in MATH 6000 and extra credit for those in MATH 4000.

1. Let R be a ring, and let R' be a subset of R. We call R' a **subring** of R if
   (A) R' is a ring for the same operations $+$ and $\cdot$ as in R, *and*
   (B) R' contains the multiplicative identity 1_R of R.

   (For example, making the identification discussed in class, $\mathbb{Z}$ is a subring of $\mathbb{Q}$.)

   (a) Let R be a ring. Suppose that R' is a subset of R closed under the $+$ and $\cdot$ operations of R, that R' contains the additive inverse (in R) of each of its elements, and that R' contains 1_R. Show that R' is a subring of R.

   Hint. (B) holds by assumption. Check that all the ring axioms hold for R' in order to verify (A). To get started, show that 0_R must belong to R'.

   (b) Find a two-element subset R' of $R = \mathbb{Z}_6$ that satisfies condition (A) in the definition of a subring but not (B). You do **not** have to give a detailed proof that (A) holds.

2. (Introduction to the Gaussian integers) Let $\mathbb{Z}[i]$ be the subset of complex numbers defined by $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$.

   (a) Check that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$. (Exercise 1 above may be helpful.)

   (b) Define a function $N\colon \mathbb{Z}[i] \to \mathbb{R}$ by $N(z) = z \cdot \bar{z}$. This is called the **norm** of z. Explain why $N(z)$ is a nonnegative integer for every $z \in \mathbb{Z}[i]$. For which $z \in \mathbb{Z}[i]$ is $N(z) = 0$?

   (c) Prove that $N(zw) = N(z)N(w)$ for all $z, w \in \mathbb{Z}[i]$.

   (d) Using (c), show that $z \in \mathbb{Z}[i]$ is a unit $\iff N(z) = 1$. Then find (with proof) all units in $\mathbb{Z}[i]$.

3. Let F be a field in which $1 + 1 \neq 0$, and let a be a nonzero element of F. Show that the equation $z^2 = a$ has either no solutions in F or exactly two distinct solutions.

   Hint. If $z_1^2 = a$ and $z_2^2 = a$, how are z_1 and z_2 related?

4. (Quadratic Formula!) Let F be a field with $1 + 1 \neq 0$. Suppose $f(x) \in F[x]$ has degree 2, and write $f(x) = ax^2 + bx + c$, where $a, b, c \in F$. Define Δ by setting $\Delta = b^2 - 4ac$.

   (a) Show that if R is an element of F with $R^2 = \Delta$, then

   $$\frac{-b + R}{2a}$$

   is a root of f that belongs to F. (Interpret the fraction $\frac{-b+R}{2a}$ as $-(b + R)(2a)^{-1}$, which makes sense as an element of of F because $2a$ is a nonzero element of F.)

   (b) Prove the converse of (a). That is, show that every root of f that belongs to F has the form $\frac{-b+R}{2a}$ for some $R \in F$ satisfying $R^2 = \Delta$.

   Hint. Completing the square yields $4af(x) = (2ax + b)^2 - \Delta$.

5. Let F be a field, and let $f(x) \in F[x]$ be a polynomial of degree n. Show that f has at most n distinct roots in F. *Hint:* Use the Root-Factor theorem.

6. Decide whether each of the following polynomials is irreducible in $F[x]$ for the given field F.

   (a) $f(x) = x^2 + \bar{1}$, $F = \mathbb{Z}_5$,

   (c) $f(x) = x^2 + \bar{1}$, $F = \mathbb{Z}_{19}$,

   (e) $f(x) = x^3 + x + \bar{1}$, $F = \mathbb{Z}_2$.

7. (***; MATH 6000 problem**)

   (a) Is $\mathbb{Q}(x)$ Archimedean? That is: If $a(x), b(x) \in \mathbb{Q}(x)^+$, is there always a positive integer n such that

   $$\underbrace{a(x) + a(x) + \cdots + a(x)}_{n \text{ times}} > b(x)\,?$$

   Justify your answer.

   (b) Does $\mathbb{Q}(x)$ have the Least Upper Bound Property? That is, does every nonempty subset of $\mathbb{Q}(x)$ that is bounded above have a least upper bound? Justify your answer.