MATH 4400/6400 – Homework #6

posted April 12, 2023; due April 19, 2023

The Möbius Inversion was a small but disruptive wormhole used in the Antarian Trans-stellar Rally, a race held in the Delta Quadrant to commemorate the signing of a treaty which brought four warring races to peace.
– Memory Alpha, describing the *Star Trek: Voyager* episode "Drive"

Directions. Give complete solutions, providing full justifications when appropriate. Your assignment must be stapled if it goes on beyond one page. Starred problems are required for MATH 6400 students and extra credit for 4400 students.

MATH 4400 problems

1. Let $\alpha, \beta \in \mathbb{H}$. Prove that $\overline{\alpha\beta} = \overline{\beta} \cdot \overline{\alpha}$.

2. Show that if $\alpha = a + bi + cj + dk \in \mathbb{H}$, then $N(\alpha) = a^2 + b^2 + c^2 + d^2$.

3. The center of a ring R, denoted $Z(R)$, is defined by

$$Z(R) = \{r \in R : rs = sr \text{ for all } s \in R\}.$$

   That is, $Z(R)$ is the collection of elements of R that commute under multiplication with everything. Determine $Z(\mathbb{H})$. Justify your answer.

MATH 6400 problem

4. (*) Consider the set $\tilde{H}$ of all matrices of the form $\left[\begin{smallmatrix} z & w \\ -\overline{w} & \overline{z} \end{smallmatrix}\right]$, where z and w run through all complex numbers.

   (a) Recall from MATH 4000/6000 that a nonempty subset of a ring is a subring whenever it contains the multiplicative identity of the ring and is closed under multiplication and under subtraction. Using this criterion, verify that $\tilde{H}$ is a subring of the ring of all 2×2 complex matrices.

   (b) Consider the elements of $\tilde{H}$ defined by

$$\mathbb{1} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \quad I = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right], \quad J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], \quad K = \left[\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right].$$

   Show that $\mathbb{1}, I, J, K$ form a basis for $\tilde{H}$, viewed as a vector space over $\mathbb{R}$. In other words, every element of $\tilde{H}$ is uniquely expressible as an $\mathbb{R}$-linear combination of $\mathbb{1}, I, J, K$.

Remark: Since $\mathbb{1}$ is the identity element for 2×2 complex matrices, it is also the multiplicative identity for $\mathbb{H}$. One can check directly that $I^2 = J^2 = K^2 = -\mathbb{1}$, that $IJ = K$, $JK = I$, and $KI = J$, and that I, J, K anticommute. These properties (along with the distributive law) imply $\tilde{H}$ is isomorphic to the ring $\mathbb{H}$ of real quaternions.