ON SUPER MULTIPLY PERFECT NUMBERS AND SOME GENERALIZATIONS OF SOCIABLE NUMBERS

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ABSTRACT. Let $\sigma(n)$ denote the sum of the positive divisors of n, and write $s(n) := \sigma(n) - n$ for the sum of the proper divisors.

1. Introduction

Let $\sigma(n)$ denote the sum of all the positive divisors of n. *Perfect numbers*, numbers n for which $\sigma(n) = 2n$, feature prominently in the historical development of number theory, and variations on the concept are ubiquitious in the number theoretic literature.

An example of such a variation are the *multiply perfect numbers*; these are numbers n which divide $\sigma(n)$. There are a number of open questions here, e.g., whether infinitely many multiply perfect numbers exist. Distant cousins to these are the *superperfect* numbers; these are numbers n for which $\sigma(\sigma(n)) = 2n$. The even superperfect numbers have been completely classified in terms of the Mersenne primes, but the odd superperfect numbers remain largely a mystery. Kanold has shown that an odd superperfect number is a perfect square. We communicate a proof, due to Pomerance, of the following result:

Theorem 1. *For some positive constant $c > 0$, the number of $m \leq x$ for which m^2 is superperfect is is at most $x/L(x)^{c+o(1)}$. Here*

$$L(x) := \exp(\sqrt{\log x \log\log x}),$$

and one may take $c = \frac{\sqrt{6}}{12}$.

In , Pomerance introduces the hybrid notion of *multiply superperfect numbers*; these are numbers n for which $\sigma(\sigma(n))$ is a multiple of n. One of the results of that paper is a complete determination all multiply superperfect n for which either n or $\sigma(n)$ is a prime power.

Define $\sigma_0(n) := n$, and for $k > 0$ put $\sigma_k(n) := \sigma(\sigma_{k-1}(n))$. Our first theorem is about the frequency with which $\sigma_k(n)/n$ obtains prescribed values. It is perhaps fitting that the statement of our result also relies

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on iteration; we need the functions $\log_k x$, defined by setting $\log_1 x := \max\{1, \log x\}$ and $\log_k x := \max\{1, \log(\log_{k-1}(x))\}$.

Theorem 2. *Let $k \geq 1$, and let $\alpha = a/b$ be a positive rational number, with $b > 0$ and $\gcd(a, b) = 1$. Then the number of $n \leq x$ for which*

$$\frac{\sigma_k(n)}{n} = \alpha \tag{1}$$

is at most $x/L(x)^{c+o_k(1)}$ as $x \to \infty$, where $c = \frac{\sqrt{2}}{4}$. Here the decay of the $o_k(1)$ term is uniform in α.

Remark. When $k = 1$, Wirsing shows that in fact the number of solutions to (1) is much smaller, namely $O(x^{C/\log\log x})$, where C is an absolute positive constant.

From this we quickly deduce the following corollary:

Corollary 1. *Fix $k \geq 1$. The number of $n \leq x$ for which n divides $\sigma_k(n)$ is at most $x/L(x)^{c+o_k(1)}$, for $c = \frac{\sqrt{2}}{4}$. In particular, for each fixed k, the sum of the reciprocals of the n of this type is convergent.*

This corollary is immediate once we recall that

$$\sigma(n) \leq (e^\gamma + o(1)) n \log_2 n \quad (\text{as } n \to \infty),$$

so that $\sigma_k(n) \ll_k x(\log_2 x)^k$ for all $n \leq x$. Note the case $k = 2$ is relevant to Pomerance's multiply superperfect numbers.

One is led to a different collection of problems if one defines a perfect number by the (mathematically, but not psychologically) equivalent relation $s(n) = n$, where $s(n) := \sigma(n) - n$ denotes the sum of only the proper divisors of n. Here perhaps *sociable numbers* provide the most natural generalization: Set $s_0(n) = n$ and for $k > 0$, define $s_k(n) = s(s_{k-1}(n))$ if $s_{k-1}(n) > 0$. (It is worth noting that because of this last condition, for $k > 1$ the function $s_k(\cdot)$ is only defined on a proper subset of the positive integers.) A number n is called *sociable* if $s_k(n) = n$ for some k, and the least such k is referred to as the order of n. For example, the sociable numbers of order $k = 1$ are exactly the perfect numbers, and those of order 2 are the *amicable numbers*. The distribution of these numbers has been investigated recently in .

We can prove an analogue of Theorem 2 for the iterates of s:

Theorem 3. *Let $k \geq 1$, and let α be a positive rational number. Then for $x \geq 3$, the number of solutions $n \leq x$ to*

$$\frac{s_k(n)}{n} = \alpha$$

is $O_k(x/\log_3 x)$. The implied constant here depends only on k, and in particular is independent of α.

Remark. As a special case ($\alpha = 1$) of Theorem 2, we see that the number of sociable numbers of order k is $O_k(x/\log_3 x)$ for each fixed k. Stronger (and more uniform) estimates for this quantity are established in KPP.

The estimate of Theorem 1 is much weaker than that of Theorem 2, and so we do not immediately obtain from Theorem 3 that the n which divide $s_k(n)$ have density zero. We prove this result by a somewhat more involved argument. First notice that by the immediately preceding remark, we can assume $s_k(n)/n \in \{0\} \cup \{2, 3, 4, \dots\}$.

It is implicit in the literature that the n for which $s_k(n) = 0$ have density zero. Indeed, Erdős has shown that for each fixed $k \geq 1$ and $\epsilon > 0$ the following statement holds for all n outside a set of density zero: Each of the numbers $n, s(n), \dots, s_k(n)$ exist and moreover,

$$\frac{s_{j+1}(n)}{s_j(n)} > \frac{s(n)}{n} - \epsilon \quad \text{for all } 1 \leq j < k. \tag{2}$$

Since $s_{k+1}(n)$ fails to exist when $s_k(n) = 0$, Erdős's result implies immediately that the latter can happen only on a density zero set of n. Unfortunately it seems difficult to extract a satisfactory explicit estimate for the number of such exceptional $n \leq x$; his argument gives an upper bound of the form $O_k(x/\log_r x)$, where r grows linearly with k (e.g., $r = 3k$ is permissible).

We complete the proof that the n dividing $s_k(n)$ have density zero by establishing the following result:

Theorem 4. *For each fixed $k \geq 1$, the number of $n \leq x$ for which n divides $s_k(n)$ and $s_k(n)/n \geq 2$ is*

$$\ll_k \frac{x \log_3 x}{\log_2 x} (2 \log_4 x)^{2k}. \tag{3}$$

Rather than asking for n to divide $s_k(n)$, we could ask for $s_k(n)$ to divide n. We conclude with the following easy consequence of Erdős's result (2) and Theorem 3.

Theorem 5. *For each fixed k, the set of n for which $s_k(n)$ divides n has density zero.*

2. Proof of Theorem 1

Proof (Pomerance). Let α be a fixed positive number, which will be specified shortly. We assume that for the largest prime divisor p of m, we have $p > L(x)^\alpha$ and p^2 not dividing m; together these conditions exclude

$$\ll x/L(x)^{(1+o(1))/(2\alpha)} + x/L(x)^\alpha$$

values of $m \leq x$. We also assume that $\sigma(m^2)$ does not have any divisors of the form r^a with r prime, $a \geq 3$ and $r^a > L(x)^{\alpha/2}$. To estimate the number of m this condition excludes, notice that if $r^a \parallel \sigma(m^2)$ then

$$\sigma(r^a) \mid \sigma(\sigma(m^2)) = 2m^2 \mid (2m)^2. \tag{4}$$

Define $\mathrm{rad}_2(n)$ as the smallest number R for which $n \mid R^2$; then rad_2 is a multiplicative function satisfying $\mathrm{rad}_2(n) = \prod_{p^{e_p} \parallel n} p^{\lceil e_p/2 \rceil}$. It follows from (4) that $\mathrm{rad}_2(\sigma(r^a)) \mid 2m$, so that the number of excluded values of $m \leq x$ is

$$\ll \sum_{\substack{r^a \geq L(x)^{\alpha/2} \\ r \text{ prime},\ a \geq 3}} \frac{x}{\mathrm{rad}_2(\sigma(r^a))} \leq \sum_{\substack{r^a \geq L(x)^{\alpha/2} \\ r \text{ prime},\ a \geq 3}} \frac{x}{r^{a/2}} \ll x/L(x)^{\alpha/12},$$

where we have made use of the trivial lower bound $\mathrm{rad}_2(n) \geq n^{1/2}$.

Now we embark on the proof proper. Since $p^2 \parallel 2m^2 = \sigma(\sigma(m^2))$, either there is a prime $q \parallel \sigma(m^2)$ for which $q \equiv -1 \pmod{p}$, or

$$p^2 \parallel \sigma(R), \quad \text{where} \quad R := \prod_{\substack{r^a \parallel \sigma(m^2) \\ r \text{ prime},\ a \geq 2,\ p \mid \sigma(r^a)}} r^a.$$

Write $m = up$, so that $u = m/p \leq x/L(x)^\alpha$. We show that for a given u, each of these two cases described above can hold for only $O(\log x)$ values of p. Combining this with our earlier estimates for the exceptional sets proves that the number of possibilities for $m \leq x$ is

$$\ll x/L(x)^{(1+o(1))/(2\alpha)} + x/L(x)^{\alpha/12} + (\log x)x/L(x)^\alpha,$$

which gives the statement of the theorem upon choosing $\alpha = \sqrt{6}$.

Suppose we are in the first case, so that $q \parallel \sigma(m^2) = \sigma(u^2)\sigma(p^2)$ for some $q \equiv -1 \pmod{p}$. It is easy to check that $\sigma(p^2)$ has no prime factors $\equiv -1 \pmod{p}$, so it must be that $q \parallel \sigma(u^2)$. Thus, setting $v := \sigma(u^2)$, we have that

$$p \mid \sigma(q) \mid \sigma(v).$$

But (estimating rather crudely),

$$\sigma(v) \leq v^2 \leq u^8 \leq x^8,$$

and so there are only $O(\log x)$ possibilities for p.

Thus we can suppose we are in the second case. Suppose that $r^a \parallel R$. Since p divides $\sigma(r^a)$, it follows that

$$(r^a)^2 \geq \sigma(r^a) \geq p > L(x)^\alpha,$$

so that $r^a > L(x)^{\alpha/2}$. Since $r^a \parallel \sigma_2(m^2)$, our previous assumptions force $a = 2$ and thus

$$R = \prod_{\substack{r^2 \| \sigma(m^2) \\ r \text{ prime},\, p \mid \sigma(r^2)}} r^2.$$

In this case we have

$$p^2 \parallel \sigma(R),$$

and

$$R \parallel \sigma(m^2) = \sigma(p^2)\sigma(u^2).$$

It cannot be the case that R divides $\sigma(p^2)$ here: If it does, then

$$p^2 \mid \sigma(R) \mid \sigma(\sigma(p^2)),$$

and since

$$1 < \frac{\sigma(\sigma(p^2))}{p^2} \leq \frac{\sigma(\sigma(p^2))}{p^2}\frac{\sigma(\sigma(u^2))}{u^2} \leq \frac{\sigma(\sigma(m^2))}{m^2} \leq 2,$$

it follows that p^2 is superperfect. But it is known that no prime power is superperfect. So we must have $\gcd(R, v) > 1$, where as above v denotes $\sigma(u^2)$. Let r be a prime dividing both R and v; then either $r \parallel v$ or $r^2 \parallel v$. If $r^2 \parallel v$, then

$$p \mid \sigma(r^2) \mid \sigma(v^2),$$

and if $r^2 \parallel v$ then

$$p \mid \sigma(r^2) \mid \sigma(v).$$

Arguing as above, we see that either case leads to only $O(\log x)$ possible values of p. □

3. Proof of Theorem 2

Proof. Suppose $n \leq x$ and $\sigma_k(n) = \alpha n$, so that $b\sigma_k(n) = an$. Since $\gcd(a, b) = 1$, we have that b divides n, and so we can assume $b \leq L(x)^{\sqrt{2}}$. We put $\eta = \sqrt{2\log_2 x/\log x}$. The number of $n \leq x$ for which $P(n) \leq x^\eta$ is at most $x/L(x)^{\sqrt{2}/4+o(1)}$, and so we may assume that

$$P(n) > x^\eta = L(x)^{\sqrt{2}}.$$

We may also assume that for all $0 \leq j < k$, the squarefull part of $\sigma_j(n)$ does not exceed $x^{\eta/2}$. To see this, fix $0 \leq j < k$. Note that for large x,

$$\sigma_j(n) \leq 2^j x(\log_2 x)^j =: X_j,$$

say. Let $\mathcal{A}$ be the set of $m \le X_j$ possessing a squarefull divisor exceeding $x^{\eta/2}$. Then

$$\#\mathcal{A} \le X_j \sum_{\substack{l > x^{\eta/2} \\ l \text{ squarefull}}} \frac{1}{l} \ll X_j x^{-\eta/4} \ll_k x^{1-\eta/4} (\log_2 x)^k$$

If $\sigma_j(n)$ has squarefull part exceeding $x^{\eta/2}$, then $m := \sigma_j(n) \in \mathcal{A}$ and so $\sigma_k(n) \in \mathcal{B}$, where $\mathcal{B} := \{\sigma_{k-j}(m) : m \in \mathcal{A}\}$. But this means the number of possibilities for $n = (b/a)\sigma_k(n)$ is at most

$$\#\mathcal{B} \le \#\mathcal{A} \ll_k x^{1-\eta/4} (\log_2 x)^k \ll x/L(x)^{\sqrt{2}/4+o(1)},$$

which fits within the bound of the theorem.

We shall show that there are no n satisfying (1) satisfying all these assumptions. Put $p_0 := P(n)$, so that

$$p_0 \mid an = b\sigma_k(n).$$

Since $p_0 > x^\eta \ge b$, we must have that

$$p_0 \mid \sigma_k(n) = \sigma(\sigma_{k-1}(n)) = \sigma(\sigma_{k-1}(n)')\sigma(\sigma_{k-1}(n)''),$$

where here and below we use $'$ to denote the squarefree part of an integer and $''$ to denote its squarefull part. Using the assumption of the previous paragraph, we have $\sigma_{k-1}(n)'' \le x^{\eta/2}$, so that

$$\sigma(\sigma_{k-1}(n)'') < x^{2\eta/3} < x^\eta < p_0,$$

and so it must be that

$$p_0 \mid \sigma(\sigma_{k-1}(n)') = \prod_{p \| \sigma_{k-1}(n)} (p+1).$$

Hence $p_1 \parallel \sigma_{k-1}(n)$ for some $p_1 \equiv -1 \pmod{p_0}$. Note that $p_1 > p_0 > x^\eta$. If $k > 1$, we can continue: From

$$p_1 \mid \sigma_{k-1}(n) = \sigma(\sigma_{k-2}(n)')\sigma(\sigma_{k-2}(n)''),$$

we deduce in the same way as above that there exists a prime $p_2 \parallel \sigma_{k-2}(n)$ for which $p_2 \equiv -1 \pmod{p_1}$. Continuing in this way we obtain a sequence of primes $p_0 < p_1 < p_2 < \cdots < p_k$ with

$$p_j \mid \sigma_{k-j}(n) \quad \text{and} \quad p_j \equiv -1 \pmod{p_{j-1}}$$

for each $1 \le j \le k$. In particular, p_k divides $\sigma_0(n) = n$; but this contradicts the maximality of p_0. $\square$

4. Proof of Theorem 3

Lemma 1 (KPP). *For all sufficiently large x, there are sets $\mathcal{A}_1(x)$ and $\mathcal{A}_2(x)$ with*

$$\max\{\#\mathcal{A}_1(x), \#\mathcal{A}_2(x)\} \ll \frac{x}{(\log_2 x)^{1/4}}$$

and for which the following holds: If $n \le x$, then

$$\left|\frac{s(s(n))}{s(n)} - \frac{s(n)}{n}\right| \le \frac{(\log_3 x)^2}{(\log_2 x)^{1/4}}$$

or $n \in \mathcal{A}_1(x)$ or $s(n) \in \mathcal{A}_2(x)$.

Lemma 2 (Erdős). *Let ρ be any real number and $t > 1$. If $x > t$, then the number of $n \le x$ for which $\sigma(n)/n \in [\rho, \rho + 1/t)$ is $O(x/\log t)$. Here the implied constant is absolute.*

Lemma 3 (Erdős). *For every $x > 0$, the number of positive integers $n \le x$ with $\sigma(n)/n > y$ is*

$$\le x/\exp(\exp((e^{-\gamma} + o(1))y)), \quad \text{as } y \to \infty,$$

uniformly in x, with γ the Euler-Mascheroni constant.

Proof of Theorem 3. By Theorem 2 (or Wirsing's theorem), we may assume $k \ge 2$. We may also assume that for all $0 \le j < k$, we have

$$\frac{s_{j+1}(n)}{s_j(n)} \le 2\log_4 x.$$

To see this, suppose this inequality fails for n, and let j be the first index for which it fails. Then

$$m := s_j(n) \le x(2\log_4 x)^j \le x(2\log_4 x)^k,$$

and $s(m)/m > 2\log_4 x$. By Lemma 3, the number of possibilities for m is

$$\le x\frac{(2\log_4 x)^k}{\exp(\exp((e^{-\gamma} + o(1))(2\log_4 x)))} \le x\frac{(2\log_4 x)^k}{\log_2 x}$$

if x is large (which we may assume). But m determines $\alpha n = s_k(n) = s_{k-j}(m)$, which determines n (since $\alpha \ne 0$); thus the same bound holds on the number of possibilities for n. Summing over j we see that a negligible number of n can arise this way, and so our assumption is validated.

In particular, we may assume that $\{n, s(n), \dots, s_k(n)\} \subset [1, X]$, for $X := x(2\log_4 x)^k$. We now consider two cases, according to whether $s(n)/n$ is particularly close to $\alpha^{1/k}$ or not. Suppose first that

$$\left|s(n)/n - \alpha^{1/k}\right| < k\frac{(\log_3 x)^2}{(\log_2 x)^{1/4}}; \tag{5}$$

then by Erdős's theorem, n belongs to a set of size $\ll x/\log_3 x$, and we are done.

So we may suppose that (5) does not hold. By repeated application of Lemma 1, either one of $n, s(n), \dots, s_{k-2}(n)$ belongs to $\mathcal{A}_1(X)$, one of $s(n), s_2(n) \dots, s_{k-1}(n)$ belongs to $\mathcal{A}_2(X)$, or

$$\left|\frac{s_{j+1}(n)}{s_j(n)} - \frac{s_j(n)}{s_{j-1}(n)}\right| \leq \frac{(\log_3 X)^2}{(\log_2 X)^{1/4}} < \frac{(\log_3 x)^2}{(\log_2 x)^{1/4}} \tag{6}$$

for all $1 \leq j < k$. As above, each of $n, \dots, s_{k-1}(n)$ determines n, so that the former possibilities give rise to at most

$$\ll_k \#\mathcal{A}_1(X) + \#\mathcal{A}_2(X) \ll \frac{X}{(\log_2 X)^{1/4}} \ll_k x\frac{(\log_2 x)^k}{(\log_2 x)^{1/4}}$$

values of n, which is negligible. If (6) holds for all $1 \leq j < k$, then by the triangle inequality, we have

$$\left|\frac{s_{j+1}(n)}{s_j(n)} - \frac{s(n)}{n}\right| \leq j\frac{(\log_3 x)^2}{(\log_2 x)^{1/4}}$$

for all $0 \leq j < k$. Since (5) fails, it follows that all the ratios $s_{j+1}(n)/s_j(n)$ lie strictly on the same side of $\alpha^{1/k}$. But then $s_k(n)/n = \prod_{0\leq j<k}(s_{j+1}(n)/s_j(n))$ cannot equal α. □

5. Proof of Theorem 4

Lemma 4 (Pomerance)**.** *Let $x \geq 3$ and m any positive integer. The number of $n \leq x$ for which $m \nmid \sigma(n)$ is $\ll x/(\log x)^{1/\phi(m)}$, where the implied constant is absolute.*

Proof of Theorem 4. Let Z denote expression appearing on the right of (3). Clearly we may assume $n > Z$. We may also assume that for all $0 \leq j < k$, we have

$$\frac{s_{j+1}(n)}{s_j(n)} \leq 2\log_4 x.$$

To see this, suppose this inequality fails for n, and let j be the first index for which it fails. Then $m := s_j(n) \leq x(2\log_4 x)^j$ and $s(m)/m > 2\log_4 x$. Let $\mathcal{A}$ be the set of $m \leq x(2\log_4 x)^j$ for which $s(m)/m > 2\log_4 x$. By Lemma 3,

$$\#\mathcal{A} \leq x\frac{(2\log_4 x)^k}{\exp(\exp((e^{-\gamma}+o(1))(2\log_4 x)))} \ll_k \frac{x}{(\log_2 x)^{k+2}}. \tag{7}$$

(Here we use that $2e^{-\gamma} > 1$.) Hence $s_k(n) = s_{k-j}(s_j(n)) \in \mathcal{B}$, where $\mathcal{B} := \{s_{k-j}(m) : m \in \mathcal{A} \text{ and } s_{k-j}(m) \text{ exists}\}$. Moreover, $s_k(n) \leq X$, where $X := 2^k x(\log_2 x)^k$. Since n divides $s_k(n) > 0$, the number of possibilities for n is at most $\sum_{r\in\mathcal{B}\cap[1,X]} d'(r)$, where

$$d'(r) := \sum_{\substack{d|r\\ r>Z}} 1.$$

Trivially $d'(r) \le r/Z$, so that by (3) and (7),

$$\sum_{r\in\mathcal{B}\cap[1,X]} d'(r) \le (X/Z)\#\mathcal{B} \le (X/Z)\#\mathcal{A} \ll_k \frac{x}{\log_2 x \log_3 x (2\log_4 x)^{2k}}.$$

This is negligible in comparison with the upper bound of the theorem. Summing over $0 \le j < k$, we see our assumption is justified. Consequently, if $s_k(n) = \alpha n$ for some integer $\alpha \ge 2$, we may assume that $\alpha \le (2\log_4 x)^k$.

We now fix an integer $2 \le \alpha \le (2\log_4 x)^k$ and estimate the number of $n \le x$ which satisfy $s_k(n) = \alpha n$. Let L be a power of α chosen to satisfy

$$\frac{\log_2 x}{\log_3 x}(2\log_4 x)^{-k} < L \le \frac{\log_2 x}{\log_3 x}.$$

We can assume that L divides $\sigma(s_j(n))$ for all $0 \le j < k$. Indeed, if this is false for a certain value of $0 \le j < k$, then by Lemma 4 there are $\ll x(2\log_4 x)^k/\log_2 x$ possibilities for $s_j(n)$. But $s_j(n)$ determines αn through the relation $\alpha n = s_k(n) = s_{k-j}(s_j(n))$. Summing over the k possibilities for j and the at most $(2\log_4 x)^k$ possibilities for α, we find that the number of n that can arise in this way is

$$\ll_k \frac{x(2\log_4 x)^{2k}}{\log_2 x}. \tag{8}$$

Assuming now that L divides each $\sigma(s_j(n))$, it follows that

$$s_{j+1}(n) = \sigma(s_j(n)) - s_j(n) \equiv -s_j(n) \pmod{L}$$

for all $0 \le j < k$. Hence

$$\alpha n = s_k(n) \equiv (-1)^k s_0(n) = (-1)^k n \pmod{L},$$

so that L divides $(\alpha + (-1)^{k+1})n$. Since L is coprime to $\alpha + (-1)^{k+1}$, it must be that L divides n, and so the number of possibilities for n is

$$\ll x/L \ll \frac{x\log_3 x}{\log_2 x}(2\log_4 x)^k. \tag{9}$$

Summing over $2 \le \alpha \le (2\log_4 x)^k$, we obtain a total of

$$\ll_k \frac{x\log_3 x}{\log_2 x}(2\log_4 x)^{2k}$$

possible values of n. □

6. Proof of Theorem 5

Proof of Theorem 5. Fix $\epsilon > 0$. Choose $u > 0$ so that the n for which $s(n)/n < u$ form a set of upper density at most ϵ. (To see that such a choice is possible, note that if $s(n)/n < u$, then n has no prime factors up to u^{-1}; the result now follows from an elementary sieve. Alternatively, one can apply Lemma 2.) We claim that if $s(n)/n > u$ and $s_k(n)$ divides n, then n belongs to a set of density zero. It follows that the n for which $s_k(n)$ divides n comprise a set of upper density at most ϵ.

To prove the claim, suppose that $s(n)/n \geq u$. By Erdős's result (5), throwing away a set of density zero, we may assume that $s_{j+1}(n)/s_j(n) > u/2$ for all $1 \leq j < k$, so that

$$\frac{n}{s_k(n)} = \prod_{j=0}^{k-1} \frac{s_j(n)}{s_{j+1}(n)} \leq (2/u)^k.$$

Thus if $s_k(n)$ divides n, then $s_k(n)/n \in \{1/1, 1/2, \dots, 1/B\}$, where $B := \lfloor (2/u)^k \rfloor$. But Theorem 3 implies that the set of n with this property has density zero. ☐

7. Concluding remarks

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