MATH 3220 practice problems

**Induction/pigeonhole**

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## Acknowledgements

This worksheet borrows heavily from Larson's book, the text of Gelca and Andreescu, and from material published online by Stanford professors Kannan Soundararajan and Ravi Vakil.

More challenging problems are marked with an $*$.

## Strategies to keep in mind:

Try small examples to get a feel for what's going on. Look for extremal cases. Look for patterns. Discuss your ideas with others in groups. Proof by contradication can be your friend. Look for hidden symmetry. It's sometimes easier to prove a stronger statement!

## Problems

1. Given 9 lattice points in three dimensional space, show that for some pair of these points, the line segment connecting them also contains a lattice point (other than the two endpoints).

   *Hint:* Show that the midpoint of some pair of points is also a lattice point.

2. Show that in any group of $n \geq 2$ people, there are two who have an identical number of friends within the group.

3. Given $n + 1$ integers belonging to the interval $[1, 2n]$, show that there are distinct $a$ and $b$ with $\gcd(a, b) = 1$.

   *Hint:* This is an example where it's easier to prove a stronger statement.

4. Given 9 positive integers, each of which has only prime factors $\leq 5$, show that two of them must multiply to a perfect square.

   *Hint:* Each of your numbers can be written in the form $2^a 3^b 5^c$; when is the product of two such numbers a square? Use the Pigeonhole principle to finish this problem off.

5. (*) Given $n+1$ integers belonging to the interval $[1, 2n]$, show that there are distinct $a$ and $b$ for which $a$ divides $b$.

   *Hint:* Look at the odd part of the numbers in your collection; here the **odd part** of $n$ means the largest odd divisor of $n$. For example, 100 has odd part 25.

6. Show that for every positive integer $n$,

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

7. (a) Show that if $a$ and $b$ are positive real numbers, then

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

(b) Show that if $a$, $b$, $c$, and $d$ are positive real numbers, then

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}.$$

*Hint:* Use the result of (a).

(c) Show that if $a$, $b$, $c$ are positive real numbers, then

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}.$$

*Hint:* Use the result of (b).

8. (*) Generalizing the result of the preceding problem, show that if $a_1, \ldots, a_n$ are positive real numbers, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

This is known as the **Arithmetic-Geometric Mean Inequality** (or **AM-GM**).

*Hint:* First prove this whenever $n$ is a power of 2, imitating the way you deduced (b) from (a) in the preceding problem. Then show that if the statement holds for $n$ numbers, it also holds for $n-1$ numbers; this is analogous to deducing (c) from (b) above. Finally, convince yourself that you're done.

9. (*) Suppose that a function $D(n)$, with domain the set of positive integers, possesses the following two properties:

(I) $D(1) = 0$ and $D(p) = 1$ if $p$ is prime,

(II) $D(uv) = uD(v) + vD(u)$ for all positive integers $u$, $v$.

We will call the function $D$ the **integer derivative**.

(a) Show that if the canonical prime factorization of $n$ is given by $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then

$$D(n) = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i}.$$

Moreover, show that if we take this as the definition of $D(n)$, then $D$ has properties (I) and (II). Thus, the **integer derivative** is uniquely defined.

*Hint:* Conditions (I) and (II) become simpler to play with if you rewrite them as rules governing the behavior of $D(n)/n$.

(b) Determine all positive integers $n$ which are their own integer derivatives.

(c) Compute the limit $\lim_{n\to\infty} D^{(n)}(63)$, where $D^{(n)}$ is the $n$th iterate of the integer derivative.

10. Suppose $a_1, a_2, \ldots, a_{26}$ are 26 positive integers, and that none of the $a_i$ have a prime factor larger than 100. Show that there is a subset of $\{a_1, \ldots, a_{26}\}$ whose product is a perfect square.

    *Hint:* There are 25 primes up to 100. I will warn you that this this problem is easiest if you use some linear algebra.

11. (*) Suppose that $x_1, \ldots, x_{n^2+1}$ is a sequence of $n^2+1$ distinct real numbers. Prove that there is a monotone (i.e., strictly increasing or strictly decreasing) subsequence of length $n+1$.

    *Hint:* Assume otherwise. Define a function $f$ on $\{1, 2, 3, \ldots, n^2+1\}$ by setting $f(i)$ to be the length of the longest increasing subsequence that ends with $x_i$. Start by showing that there are indices $i_1 < i_2 < \cdots < i_{n+1}$ with $f(x_{i_1}) = \cdots = f(x_{i_{n+1}})$. What can you say about the sequence $x_{i_1}, \ldots, x_{i_n}$?

12. (*) Show that in any group of six people, you can find either (a) a set of three people all of whom know each other (mutual friends), or (b) a set of three people none of whom know each other (mutual strangers).

    *Hint:* Picture the six people as dots, draw a red line between any two dots if the two people know each other, and a blue line between the two dots otherwise. You need to show that there is a monochromatic triangle in your picture. This will require some Pigeonholing, perhaps more than once.

13. Recall that the **Fibonacci sequence** is the sequence defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for all $n = 1, 2, 3, \ldots$.

    (a) Show that every positive integer can be written as a sum of distinct terms of the Fibonacci sequence.

    (b) Show that for every nonnegative integer $n$,

    $$F_{2n+1} = F_{n+1}^2 + F_n^2.$$

14. (*) Again we discuss the Fibonacci numbers.

    (a) Show that there is some term of the Fibonacci sequence divisible by 2013.

    (b) Show that in part (a), the number 2013 can be replaced with any positive integer.

    *Hint:* Here it's helpful to know something about modular arithmetic. For example, part (a) is equivalent to saying that there is some Fibonacci number that is 0 modulo 2013. If you experiment with small numbers $m$, it appears that the Fibonacci sequence $F_0, F_1, F_2, \ldots$ mod $m$ is periodic modulo and that the period starts at the beginning. (Such a sequence is called **purely periodic**.) For example, modulo 3, the sequence is $0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, \ldots$, and the sequence has begun to repeat at its 8th term. Since the sequence starts at 0, there will be infinitely many zero terms, and so infinitely many Fibonacci numbers divisible by 3. It would suffice for (a) to have the same fact with 3 replaced by 2013.

To get you started, notice that any two consecutive terms in the Fibonacci sequence determine everything else, because of the recursion. (You run the recursion forwards to determine the succeeding terms, and run it backwards to determine the preceding terms.) Use the fact that there are only $m$ distinct residue classes modulo $m$ to see that some block of two terms has to repeat (infinite Pigeonhole). Go from there.

15. Prove that there exist integers $a, b, c$ (not all zero) with $|a|, |b|, |c| < 10^6$ such that $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.

    *Hint:* If $0 \leq a, b, c < 10^6$, then $0 \leq a + b\sqrt{2} + c\sqrt{3} \leq (1 + \sqrt{2} + \sqrt{3}) \cdot 10^6$. Divide the interval $[0, (1 + \sqrt{2} + \sqrt{3}) \cdot 10^6]$ into $10^{18} - 1$ consecutive intervals of equal length; these are the holes which you can apply Pigeonhole to.

16. (*) The points of the plane are colored either red, blue, or green. Show that one can always find a rectangle all of whose vertices are the same color.

    *Hint:* The whole plane is a distraction. In fact, all that you need to think about are the colorings of the points $(x, 1), (x, 2), (x, 3), (x, 4)$, where $x$ is an integer. We'll call such a collection of points the $x$**th column**. What can you say if two of these columns are colored the same way?

17. Determine the number of ways that a $2 \times n$ rectangle can be tiled with $2 \times 1$ dominos.

    *Hint:* Experiment with small $n$ to guess the answer. Then try to prove it by establish a familiar recurrence relation.

18. Prove that $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer for every $n = 0, 1, 2, 3, \ldots$.

19. Let $R_1 = 1$, and for $n \geq 1$, let $R_{n+1} = 1 + n/R_n$. Show that for every integer $n \geq 1$, we have
$$\sqrt{n} \leq R_n \leq \sqrt{n} + 1.$$

20. (*) Show that for any five points placed on a sphere, there is a closed hemisphere containing (at least) four of the five points.

    *Hint:* Two points on a sphere determine a great circle. Find a way to finish this off using the Pigeonhole principle.

21. Let $f(x) = \sqrt{x^2 - 1}$ for $x > 1$. Show that $f^{(n)}(x) > 0$ if $n$ is odd, and $f^{(n)}(x) < 0$ if $n$ is even.

22. (*) Let $a_1 = 2$, and suppose $a_{n+1} = a_n^2 - a_n + 1$ for all positive integers $n$. Prove that:

    (a) If $n < m$, then $\gcd(a_n, a_m) = 1$.

    (b) The infinite series $\displaystyle\sum_{i=1}^{\infty} \frac{1}{a_i} = 1$.

    *Hint:* Experiment! Work out the first several of the $a_i$, and work out the first few partial sums of the series in part (b).

23. (*) Let $\alpha$ be a real number, and let $N$ be a positive integer. Show that one of the numbers $\alpha, 2\alpha, 3\alpha, \ldots, N\alpha$ is within $\frac{1}{N+1}$ units from the nearest integer.

    *Hint:* Think about the wrapping the real number line around the unit interval over and over, so that it becomes a circle with circumference 1. Divide the circle into $N+1$ neighboring arcs, each of length $\frac{1}{N+1}$. Where might $0 = 0\alpha,\ \alpha,\ 2\alpha,\ \ldots,\ N\alpha$ fall on the circle? Think Pigeonhole!

    **Remark:** This result, important for its applications in number theory, is known as **Dirichlet's approximation principle**.

24. (*) Show that any positive integer can be written in the form $\pm 1^2 \pm 2^2 \pm 3^2 \pm \cdots \pm n^2$ for some positive integer $n$ and some choice of signs.

25. (*) Suppose that $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ is a **multiplicative function**, in the sense that $f(ab) = f(a)f(b)$ whenever $a$ and $b$ are relatively prime. For example, $f(15) = f(3)f(5)$. Suppose that the sequence $\{f(n)\}$ is strictly increasing, i.e., $f(1) < f(2) < f(3) < \ldots$. Finally, suppose that $f(2) = 2$. Prove that $f(n) = n$ for all positive integers $n$.

    *Hint:* First prove $f(3) = 3$. (This is already hard!) Bootstrap this to show that $f(p) = p$ for all primes $p$.