MATH 3100/3100H – Learning objectives to meet for Exam #1

The exam will cover §1.1–§1.5 of the course notes, through what is discussed on Monday, September 15.

What to be able to state

Basic definitions

You should be able to give complete and precise definitions of each of the following items:

- sequence
- increasing, decreasing, monotone, and the "strictly" and "eventually" variants
- bounded above, bounded below, upper bound, lower bound, bounded
- subsequence
- $\{a_n\}$ converges to L, where L is a real number
- $\{a_n\}$ diverges
- geometric sequence
- $\{a_n\}$ diverges to ∞ or $-\infty$

Big theorems

Give full statements of each of the following results, making sure to indicate all necessary hypotheses. For results proved in class, describe the components and main ideas of the proof.

- Principle of mathematical induction, complete mathematical induction
- A convergent sequence has a **unique** limit
- Convergent sequences are bounded
- Subsequences of convergent sequences converge to the original limit
- Convergence/divergence behavior of geometric sequences (Proposition 1.4.15)
- If $\{a_n\}$ is bounded above by U and $a_n \to L$, then $L \leq U$
- If $\{a_n\}$ is increasing with limit U, then U is an upper bound on $\{a_n\}$
- (Bounded) $\cdot$ (going to 0) goes to 0
- Triangle inequality for absolute values
- Sum rule for limits, product rule for limits, quotient rule for limits
- Ratio tests for sequences. (Proof is not examinable.)

What to expect on the exam

You can expect 5 questions on the exam, including

- one problem testing mathematical induction
- one problem testing your ability to compute the limit of a specific sequence directly from the ϵ-N definition

The rest of the exam is designed to test your comfort level working with the basic definitions. For the most part, I am not interested in having you regurgitate proofs of results from the notes; I want to know if you have internalized the ideas enough to solve similar problems.

Sample problems

1. (a) Prove that $\lim_{n\to\infty} \frac{n+3}{2n-5} = \frac{1}{2}$ directly from the definition of a limit.

   (b) Prove that $\lim_{n\to\infty} \frac{2n^2+1}{2n^2-n+1} = 1$ directly from the definition of a limit.

2. (a) What does it mean to say a sequence is **bounded above**? **Bounded below**? **Bounded**?

   (b) A sequence $\{a_n\}$ is called **eventually constant** if there is a real number C and a natural number N such that

   $$a_n = C \quad \text{for all natural numbers } n \geq N.$$

   Prove that every eventually constant sequence is bounded. Do not use theorems proved in class: Proceed <u>directly</u> from the definitions of "eventually constant" and "bounded".

3. Determine, with brief explanations, whether or not each of the sequences below converges or diverges. For seqeuences that converge, determine the limit. Do **not** use the limit definition; rather, use relevant theorems proved in class. For each part, explain which results/rules you use.

   (a) $\lim_{n\to\infty} \frac{n}{n+2025}$.

   (b) $\lim_{n\to\infty} (7/8)^n \cos(n)$.

   (c) $\lim_{n\to\infty} (-3/2)^n/n^2$.

4. Assuming that $a_n \to 2$ and $b_n \to 1$, give a direct proof <u>from the limit definition</u> that

   $$\lim_{n\to\infty} (a_n - b_n) = 1.$$

   For this problem, you may **not** assume the sum or difference rule.

5. Let $\{a_n\}$ be a sequence.

   (a) What does it mean to say that $a_n \to -\infty$? $a_n \to \infty$?

   (b) Suppose $a_n \to \infty$. Prove that $a_n - 2025 \to \infty$.

   (c) Suppose $a_n \to \infty$. Prove that $-7a_n \to -\infty$.