MATH 8440 – Assignment #1
last updated January 19, 2023 (CLOSED)
due Monday, January 23

1. (Schur) For each $F(T) \in \mathbf{Z}[T]$, define
$$\mathcal{P}_F = \{\text{primes } p : p \mid F(n) \text{ for some integer } n\}.$$
Show that if F is nonconstant, then $\mathcal{P}_F$ is an infinite set.

2. Fix an odd integer m. Show that there are infinitely many primes p for which the congruence $2^n \equiv m \pmod p$ has a positive integer solution n.

   Hints. Here are some ideas to get you started, where for concreteness we assume $m = 2023$. Consider numbers of the form $2^{n!} - 2023$. Show that the only primes that can divide infinitely many of these numbers are 3 and 337; then show that for large n, neither 3^2 nor 337^2 divide $2^{n!} - 2023$. Conclude from there, and generalize to all odd m.

3. Euler showed not only that $\zeta(2) = \pi^2/6$ but that $\zeta(4) = \pi^4/90$.
   (a) Assuming these series identities of Euler, show that $\prod_{p \text{ prime}} \frac{p^2+1}{p^2-1} = \frac{5}{2}$.
   (b) Use (a) to give another proof that there are infinitely many primes.

   Hint. If the product on p in part (a) were finite, where would the denominator 3 (arising from the term $p = 2$) 'go'?

4. (Erdős) In this exercise you are asked to fill in the details of a proof of Erdős that $\sum_{p \text{ prime}} \frac{1}{p}$ diverges. In what follows, p always denotes a prime.

   Suppose for a contradiction that $\sum_p \frac{1}{p}$ converges and fix a natural number M with $\sum_{p>M} \frac{1}{p} < \frac{1}{2}$. For each natural number N, we define quantities N_1, N_2 by
$$N_1 = \#\{\text{positive integers } n \le N : p \mid n \text{ for some } p > M\},$$
$$N_2 = \#\{\text{positive integers } n \le N : p \mid n \Rightarrow p \le M\}.$$
   Clearly, $N = N_1 + N_2$.
   (a) Show that $N_1 < \frac{1}{2}N$.

   Hint. For each prime p, how many multiples p are there that are $\le N$?

   (b) Show that $N_2 \le 2^{\pi(M)} N^{1/2}$.

   Hint. Write each n in the set counted by N_2 in the form $n = rs^2$, where $r, s \in \mathbf{Z}^+$ and r is squarefree. How many possibilities are there for r? For s?

   (c) Derive a contradiction from (a) and (b) if N is large enough.

5. By elementary calculus, the function $w \mapsto we^w$ is a strictly increasing function of w for $w \ge -1$. Hence, for each $x \ge -e^{-1}$, there is a unique $w \ge -1$ with $we^w = x$. Below, w denotes this (implicitly defined) function of x. Note that
$$w + \log w = \log x.$$
   (a) Show that for all large x, we have $w = \log x + O(\log\log x)$.

(b) Using (a), show that for large x we have $\log w = \log\log x + O(\log\log x/\log x)$. Deduce that in fact

$$w = \log x - \log\log x + O(\log\log x/\log x).$$

(c) Using (b), show that for large x we have $\log w = \log\log x - \frac{\log\log x}{\log x} + O((\log\log x/\log x)^2)$. Deduce that

$$w = \log x - \log\log x + \frac{\log\log x}{\log x} + O((\log\log x/\log x)^2).$$

(We could take this expansion even further, but . . . you get the idea.)

6. (a) Show that for each fixed positive integer k, and all $x \geq 4$,

$$\int_2^x \frac{dt}{(\log t)^k} = O_k(x/(\log x)^k).$$

Here the subscript in O_k means the implied constant is allowed to depend on k.

Hint. Split the integral at $t = \sqrt{x}$.

(b) Show that for each fixed positive integer k, and all $x \geq 4$,

$$\int_2^x \frac{dt}{\log t} = \frac{x}{\log x}\sum_{j=0}^{k} \frac{j!}{(\log x)^j} + O_k\left(\frac{x}{(\log x)^{k+2}}\right).$$