Advanced Electromagnetics

EM field representation in rectangular waveguides

In this lecture we will introduce waveguides, a specific type of transmission line. Since a waveguide is bounded, a fixed number of modes is supported by the waveguide. These modes can be analysed, independently from any sources that are exciting the waveguide, by forcing the waveguide's boundary conditions on the Maxwell's equation. Solutions to the equations then represent the modes that can be supported by the transmission line. We will analyse more in depth one of the fundamental modes of a rectangular waveguide; the TE_{10}-mode.

Bounded Problems

In bounded problems, the boundary conditions of the problems lead to field representations written as a summation of modes:

$$\vec{E}(\vec{r}) = \sum_m V_m(z)\vec{e}(x, y, k_m)$$
$$\vec{H}(\vec{r}) = \sum_m I_m(z)\vec{h}(x, y, k_m)$$

The modes are the vector functions $\vec{e}(x, y, k_m), \vec{h}(x, y, k_m)$ that represents all the possible field distributions in the plane that is transverse to the direction of propagation. Depending on the external sources to the problem, the total field $\vec{E}(\vec{r}), \vec{H}(\vec{r})$ will be a summation of multiple modes weighted with the scalar amplitudes $V_m(z), I_m(z)$. Typically, only one mode is useful. This mode is referred to as the fundamental mode. Therefore, if we talk about voltages and currents (like with our AC-power grid), we are talking about the scalar amplitudes of the dominant mode. As mentioned before, the supported modes in a waveguide can be analysed in absence of any current sources by enforcing the boundary conditions and solve the Maxwell's equations.

Maxwell's Equations

Maxwell's equations can be written down in different forms. In Table 1, you can find Maxwell's equations in its time-domain, frequency-domain and phasor-domain form. Depending on the electromagnetic problem, one can choose the domain that is most suitable for the problem. It might be easier to solve the problem in the frequency domain. Subsequently, by using the according transformations shown below the table, the solution can then be converted back to time-domain solutions. Since only in a small amount of cases the solution is analytical, most electromagnetic problems are solved numerically using self-developed tools (e.g. method of moments) or already available commercial software tools. In any case, a deep understanding of electromagnetics is of utmost importance in order to judge the correctness of any results coming from numerical software.

Table 1: Maxwell's Equations

Time Domain	Frequency Domain	Phasor Domain
$\nabla \times \vec{e} = -\mu \frac{\partial \vec{h}}{\delta t} - \vec{j}_m$	$\nabla \times \vec{E}(\omega) = -j\omega\mu\vec{H}(\omega) - \vec{J}_m(\omega)$	$\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{J}_m$

$$\nabla \times \vec{h} = \epsilon \frac{\partial \vec{e}}{\delta t} - \vec{j} \qquad \nabla \times \vec{H}(\omega) = j\omega\epsilon\vec{E}(\omega) + \vec{J}(\omega) \qquad \nabla \times \vec{H} = j\omega\epsilon\vec{E} + \vec{J}_0$$

$$\nabla \cdot \vec{e} = \frac{\rho}{\epsilon} \qquad \nabla \cdot \vec{E}(\omega) = \frac{1}{\epsilon}P(\omega) \qquad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon}$$

$$\nabla \cdot \vec{h} = \frac{\rho_m}{\mu} \qquad \nabla \cdot \vec{H}(\omega) = \frac{1}{\mu}P_m(\omega) \qquad \nabla \cdot \vec{H} = \frac{\rho_m}{\mu}$$

$$\vec{e}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \vec{E}(\vec{r},\omega)e^{j\omega t}d\omega$$

$$\vec{e}(\vec{r},t) = \mathrm{Re}\left[\vec{E}(\vec{r})e^{j\omega t}\right]$$

Keep in mind the definition of a phasor, which is defined for an invariant angular frequency ω. Therefore the phasor is not a function of frequency; it is merely a complex vector representing amplitude and relative phase. Also keep in mind the different units of the different representations $\left(\vec{E}(\omega) \text{ in } \left[\frac{V}{m\cdot Hz}\right] \text{ versus } \vec{E} \text{ in } \left[\frac{V}{m}\right]\right)$.

The Maxwell's equations explain the relationships between the different electric- and magnetic- field components incorporating any boundary conditions (metallic surfaces / guided waves / free space etc.) and/or any enforced current sources. Recall that the magnetic current density $\vec{J}_m$ and magnetic charge density ρ_m are not physical quantities in simple media. We will continue with the Maxwell's equations in the phasor domain. Also, we would like to investigate the modes supported by the transmission line in absence of any sources ($\vec{J}_m = \vec{J}_0 = \rho = \rho_m = 0$).

Transmission line fields

As discussed before, the modes supported in a conventional transmission line are the vector functions $\vec{e}(x,y,k_m), \vec{h}(x,y,k_m)$ that represents field distributions in the plane that is transverse to the direction of propagation.

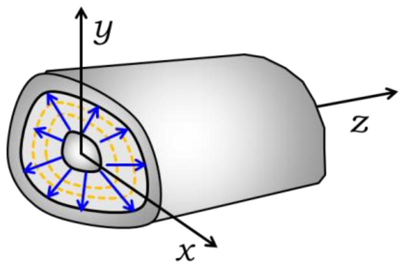

Figure 1: Transmission line with generic cross-section. The cross-section is invariant along the direction of propagation.

In other words, when analysing a transmission line with a generic cross section that is invariant along the direction of propagation (in our case $\hat{z}$ as shown in Figure 1), we can decompose the field phasors in their space dependencies:

$$\vec{E}(\mathrm{x,y,z}) = \vec{E}^0(\mathrm{x,y})e^{-jk_z z} \tag{1a}$$
$$\vec{H}(\mathrm{x,y,z}) = \vec{H}^0(\mathrm{x,y})e^{-jk_z z} \tag{1b}$$

In (1a) and (1b), k_z is the wavenumber in the direction of propagation. The wavenumber can be complex, $k = \mathrm{Re}[k] + j\mathrm{Im}[k] = \beta - j\alpha$, where β is the phase constant and α is the attenuation constant. As shown in Figure 2, the attenuation constant results in an exponential decay of the wave.

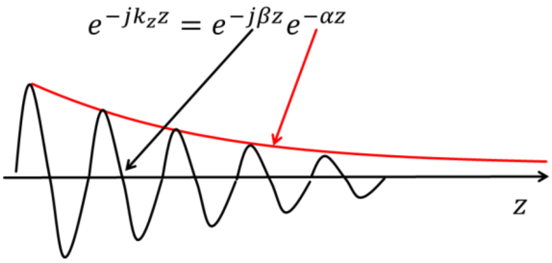

Figure 2: A complex wavenumber $k = \beta - j\alpha$, results in an attenuating wave.

Maxwell's equations expanded for the Transmission line fields

Let us investigate the relationship between the different field components described by Maxwell's equations when the electric- and magnetic-field are defined as the transmission line fields in (1a) and (1b). Let us first expand the curl on the electric field:

$$\nabla \times \vec{E}(x,y,z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{y}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \tag{2}$$

The partial derivatives in x and y are yet undefined. However the partial derivative in z of the modal field in (1a) can be calculated as:

$$\frac{\partial \vec{E}(\mathrm{x,y,z})}{\partial z} = -jk_z \vec{E}^0(\mathrm{x,y}) e^{-jk_z z} = -jk_z \vec{E}(\mathrm{x,y,z}) \tag{3}$$

Substituting (3) in (2) leads to:

$$\nabla \times \vec{E}(x,y,z) = \hat{x}\left(\frac{\partial E_z}{\partial y} + jk_z E_y\right) - \hat{y}\left(\frac{\partial E_z}{\partial x} + jk_z E_x\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \tag{4}$$

Substituting the curl of the electric field (4) in the first Maxwell's equation from Table 1 (Faraday's law), we obtain the following field relationships:

$$\left(\frac{\partial E_z^0}{\partial y} + jk_z E_y^0\right) = -j\omega\mu H_x^0 \tag{5a}$$
$$\left(\frac{\partial E_z^0}{\partial x} + jk_z E_x^0\right) = j\omega\mu H_y^0 \tag{5b}$$
$$\left(\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y}\right) = -j\omega\mu H_z^0 \tag{5c}$$

Or equivalently in matrix form:

$$
\begin{bmatrix} H_x^0 \\ H_y^0 \\ H_z^0 \end{bmatrix} = -\frac{1}{j\omega\mu}\begin{bmatrix} 0 & jk_z & \frac{\partial}{\partial y} \\ -jk_z & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} E_x^0 \\ E_y^0 \\ E_z^0 \end{bmatrix} \tag{6}
$$

Similar, by following the same steps, we expand the curl on the magnetic field of the modal field (1b) and substitute it in the second Maxwell's equation (Ampere's law), leading to the relationships:

$$
\left(\frac{\partial H_z^0}{\partial y} + jk_z H_y^0\right) = j\omega\epsilon E_x^0 \tag{7a}
$$

$$
\left(\frac{\partial H_z^0}{\partial x} + jk_z H_x^0\right) = -j\omega\epsilon E_y^0 \tag{7b}
$$

$$
\left(\frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y}\right) = j\omega\epsilon E_z^0 \tag{7c}
$$

and in matrix form:

$$
\begin{bmatrix} E_x^0 \\ E_y^0 \\ E_z^0 \end{bmatrix} = \frac{1}{j\omega\epsilon}\begin{bmatrix} 0 & jk_z & \frac{\partial}{\partial y} \\ -jk_z & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} H_x^0 \\ H_y^0 \\ H_z^0 \end{bmatrix} \tag{8}
$$

Maxwell's equations expanded for the Transmission line fields, written in terms of longitudinal components.

We can characterize the supported modes in a waveguide in TE-modes and TM-modes.

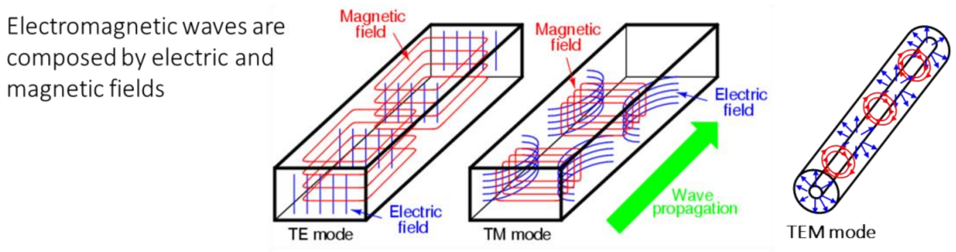

Figure 3: TE-, TM-, and TEM-modes for transmission lines

As is also illustrated in Figure 3, a mode is transverse electric (TE) w.r.t. $\hat{z}$, when the electric field is oriented in the plane transverse to the direction of propagation whereas a mode is referred to as transverse magnetic (TM) w.r.t. $\hat{z}$, when the magnetic field is oriented in the plane transverse to the direction of propagation.

Therefore, for a TE-mode, $E_z^0 = 0, H_z^0 \neq 0$, whereas for a TM-mode $E_z^0 \neq 0, H_z^0 = 0$. Because of these characteristics, we will see that it is convenient to write the fields (6) and (8) in terms of the longitudinal components only.

Consider the field relationships (5a) and (7b)

$$\left(\frac{\partial E_z^0}{\partial y} + jk_z E_y^0\right) = -j\omega\mu H_x^0 \qquad\qquad \left(\frac{\partial H_z^0}{\partial x} + jk_z H_x^0\right) = -j\omega\epsilon E_y^0$$

Let us substitute the expression on the right, E_y^0 (7b), into the expression on the left (5a):

$$\begin{aligned}
-j\omega\mu H_x^0 &= \frac{\partial E_z^0}{\partial y} - \frac{k_z}{\omega\epsilon}\left(\frac{\partial H_z^0}{\partial x} + jk_z H_x^0\right) \\
-j\omega\mu H_x^0 &= \frac{\partial E_z^0}{\partial y} - \frac{k_z}{\omega\epsilon}\frac{\partial H_z^0}{\partial x} - j\frac{k_z}{\omega\epsilon}k_z H_x^0 \\
H_x^0\left(-\frac{k_z^2}{j\omega\epsilon} - j\omega\mu\right) &= \frac{\partial E_z^0}{\partial y} - \frac{k_z}{\omega\epsilon}\frac{\partial H_z^0}{\partial x} \\
H_x^0(\omega^2\mu\epsilon - k_z^2)\frac{1}{j\omega\epsilon} &= \frac{\partial E_z^0}{\partial y} - \frac{k_z}{\omega\epsilon}\frac{\partial H_z^0}{\partial x}
\end{aligned}$$

Where we can recognise the wavenumber k in: $\omega^2\mu\epsilon = \left(\frac{\omega}{1/\sqrt{\mu\epsilon}}\right)^2 = \left(\frac{\omega}{c_0}\right)^2 = k^2$. Isolating H_x^0 leads to:

$$H_x^0 = -\frac{1}{(k^2 - k_z^2)}\left(jk_z\frac{\partial H_z^0}{\partial x} - j\omega\epsilon\frac{\partial E_z^0}{\partial y}\right)$$

Now H_x^0 is expressed only in terms of the longitudinal field components E_z^0 and H_z^0. Similarly we can express the other transverse field components only in terms of the longitudinal components, leading to the following field relationships:

$$E_x^0 = -\frac{1}{(k^2 - k_z^2)}\left(jk_z\frac{\partial E_z^0}{\partial x} + j\omega\mu\frac{\partial H_z^0}{\partial y}\right) \tag{9a}$$

$$E_y^0 = -\frac{1}{(k^2 - k_z^2)}\left(jk_z\frac{\partial E_z^0}{\partial y} - j\omega\mu\frac{\partial H_z^0}{\partial x}\right) \tag{9b}$$

$$H_x^0 = -\frac{1}{(k^2 - k_z^2)}\left(jk_z\frac{\partial H_z^0}{\partial x} - j\omega\epsilon\frac{\partial E_z^0}{\partial y}\right) \tag{9c}$$

$$H_y^0 = -\frac{1}{(k^2 - k_z^2)}\left(jk_z\frac{\partial H_z^0}{\partial y} + j\omega\epsilon\frac{\partial E_z^0}{\partial x}\right) \tag{9d}$$

Or equivalently in matrix form:

$$\begin{bmatrix} E_x^0 \\ E_y^0 \\ E_z^0 \end{bmatrix} = -\frac{1}{(k^2 - k_z^2)}\begin{bmatrix} jk_z\dfrac{\partial}{\partial x} & j\omega\mu\dfrac{\partial}{\partial y} \\ jk_z\dfrac{\partial}{\partial y} & -j\omega\mu\dfrac{\partial}{\partial x} \\ -(k^2 - k_z^2) & 0 \end{bmatrix}\begin{bmatrix} E_z^0 \\ H_z^0 \end{bmatrix} \tag{10a}$$

$$\begin{bmatrix} H_x^0 \\ H_y^0 \\ H_z^0 \end{bmatrix} = -\frac{1}{(k^2 - k_z^2)} \begin{bmatrix} -j\omega\epsilon\frac{\partial}{\partial y} & jk_z\frac{\partial}{\partial x} \\ j\omega\epsilon\frac{\partial}{\partial x} & jk_z\frac{\partial}{\partial y} \\ 0 & -(k^2 - k_z^2) \end{bmatrix} \begin{bmatrix} E_z^0 \\ H_z^0 \end{bmatrix} \tag{10b}$$

Why do we so much care about this representation in terms of the longitudinal components? Let us go back to our TE-mode/TM-mode discussion. In this document we will analyse the first TE-mode of a rectangular waveguide. If the fundamental mode of the rectangular waveguide is transverse electric (TE, as shown in the left of Figure 3), we know that $E_z^0 = 0$. In that scenario, the field dependencies from (10) reduces to (11):

$$\begin{bmatrix} E_x^0 \\ E_y^0 \\ E_y^0 \end{bmatrix} = -\frac{1}{(k^2 - k_z^2)} \begin{bmatrix} j\omega\mu\frac{\partial}{\partial y} \\ -j\omega\mu\frac{\partial}{\partial x} \\ 0 \end{bmatrix} H_z^0 \tag{11a}$$

$$\begin{bmatrix} H_x^0 \\ H_y^0 \\ H_z^0 \end{bmatrix} = -\frac{1}{(k^2 - k_z^2)} \begin{bmatrix} jk_z\frac{\partial}{\partial x} \\ jk_z\frac{\partial}{\partial y} \\ -(k^2 - k_z^2) \end{bmatrix} H_z^0 \tag{11b}$$

That means that we only need to solve the problem for H_z^0! If we know H_z^0, all the other components are known using (11a) and (11b)! The next step is to actually find a valid solution for H_z^0.

The electric and magnetic field are related to each other via:

$$\vec{E} = -Z_{TE}\left[\hat{z} \times \vec{H}\right]$$

with

$$Z_{TE} = \frac{\omega\mu}{k_z} = \frac{k\zeta}{k_z}$$

If one desires to evaluate the TM-modes of the transmission line, equivalent steps can be taken, enforcing $H_z^0 = 0$ and $E_z^0 \neq 0$.

Helmholtz equations

We want a solution of the Maxwell's equations for H_z^0 for our transmission line problem. Solving Maxwell's equations is often proven to be a difficult task. The relationships consist of partial differential equations not only in space, but also in time. It can be worthwhile to separate the equations in its different variables, making it easier to enforce the boundary conditions or solve the problem. We will show that Maxwell's equations can be written in the form of Helmholtz equations (12).

$$\nabla^2 A + k^2 A = 0 \tag{12}$$

Let us start with Faraday's law and Ampere's law in phasor domain, in absence of any sources:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{H} = j\omega\epsilon\vec{E}$$

We can combine both equations directly by taking an additional curl ($\nabla \times$) operation on both sides of Faraday's law:

$$\nabla \times \nabla \times \vec{E} = -j\omega\mu\nabla \times \vec{H} \tag{13}$$

On the right-hand side of (13) we can now substitute Ampere's law, leading to (14):

$$\nabla \times \nabla \times \vec{E} = \omega^2\mu\epsilon\vec{E} \tag{14}$$

Where we can recognise again the wavenumber k in: $\omega^2\mu\epsilon = \left(\frac{\omega}{1/\sqrt{\mu\epsilon}}\right)^2 = \left(\frac{\omega}{c_0}\right)^2 = k^2$. Furthermore, we will use the vectorial identity $a \times b \times c = b(a \cdot c) - c(a \cdot b)$, such that $\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E}$. Since we analyse the problem in absence of any sources, Gauss' Law from the Maxwell's equations tells us: $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} = 0$. Equation (14) now simplifies to (15):

$$-\nabla^2\vec{E} = k^2\vec{E} \tag{15}$$

Arranging the term to one side of the equation leads to the form of a Helmholtz equation. One can now repeat the steps by doing the additional curl operation on both sides of Ampere's law and substituting the result in Faraday's law. This will lead to the magnetic field version of the Helmholtz equation. The Helmholtz equations are shown in (16a) and (16b):

$$\nabla^2\vec{E} + k^2\vec{E} = 0 \tag{16a}$$
$$\nabla^2\vec{H} + k^2\vec{H} = 0 \tag{16b}$$

Both $\vec{E}$ and $\vec{H}$ are vectorial fields in $\hat{x}, \hat{y}, \hat{z}$ such that: $\nabla^2\vec{E} = \nabla^2 E_x\hat{x} + \nabla^2 E_y\hat{y} + \nabla^2 E_z\hat{z}$. Therefore, both equations (16a) and (16b) constitute actually three different equations in each Cartesian axis:

$$\nabla^2\vec{E} + k^2\vec{E} = 0 \qquad \begin{cases} \nabla^2 E_{\mathrm{x}} + k^2 E_x = 0 \\ \nabla^2 E_{\mathrm{y}} + k^2 E_y = 0 \\ \nabla^2 E_{\mathrm{z}} + k^2 E_z = 0 \end{cases} \tag{17a}$$

$$\nabla^2\vec{H} + k^2\vec{H} = 0 \qquad \begin{cases} \nabla^2 H_{\mathrm{x}} + k^2 H_x = 0 \\ \nabla^2 H_{\mathrm{y}} + k^2 H_y = 0 \\ \nabla^2 H_{\mathrm{z}} + k^2 H_z = 0 \end{cases} \tag{17b}$$

Helmholtz Equations for the Transmission line fields

Let us see what happens when we apply de Laplace operator (i.e. ∇^2) on the transmission line field expression $E_x(\mathrm{x,y,z}) = E_x^0(\mathrm{x,y})e^{-jk_z z}$ from (1a):

$$
\begin{aligned}
&\nabla^2 E_x(\mathrm{x,y,z}) \\
&= \nabla^2 E_x(\mathrm{x,y}) e^{-jk_z z} \\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_x^0(\mathrm{x,y}) e^{-jk_z z} \\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_x^0(\mathrm{x,y}) e^{-jk_z z} + \left(\frac{\partial^2 e^{-jk_z z}}{\partial z^2}\right) E_x^0(\mathrm{x,y}) \\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_x^0(\mathrm{x,y}) e^{-jk_z z} - k_z^2 E_x^0(\mathrm{x,y}) e^{-jk_z z} \\
&= \nabla_t^2 E_x(\mathrm{x,y,z}) - k_z^2 E_x(\mathrm{x,y,z}) \\
&= [\nabla_t^2 - k_z^2] E_x(\mathrm{x,y,z})
\end{aligned}
$$

The separation in space dependency results in an orthogonalized version of Helmholtz equations (17a) and (17b):

$$
[\nabla_t^2 + k^2 - k_z^2]\vec{E}^0(\mathrm{x,y}) = 0 \qquad \begin{cases} [\nabla_t^2 + k^2 - k_z^2]E_\mathrm{x}^0(\mathrm{x,y}) = 0 \\ [\nabla_t^2 + k^2 - k_z^2]E_\mathrm{y}^0(\mathrm{x,y}) = 0 \\ [\nabla_t^2 + k^2 - k_z^2]E_\mathrm{z}^0(\mathrm{x,y}) = 0 \end{cases} \tag{18a}
$$

$$
[\nabla_t^2 + k^2 - k_z^2]\vec{H}^0(\mathrm{x,y}) = 00 \qquad \begin{cases} [\nabla_t^2 + k^2 - k_z^2]H_\mathrm{x}^0(\mathrm{x,y}) = 0 \\ [\nabla_t^2 + k^2 - k_z^2]H_\mathrm{y}^0(\mathrm{x,y}) = 0 \\ [\nabla_t^2 + k^2 - k_z^2]H_\mathrm{z}^0(\mathrm{x,y}) = 0 \end{cases} \tag{18b}
$$

Remember from the previous discussion that we want to calculate the TE-modes of a rectangular waveguide. Therefore we said we want to solve the Maxwell's equation for $H_z^0(x,y)$. If we know $H_z^0(x,y)$, we know all other field components of the modal field. Therefore, we can now focus on solving the single Helmholtz equation (19) as function of $H_z^0(x,y)$:

$$
[\nabla_t^2 + k^2 - k_z^2]H_\mathrm{z}^0(\mathrm{x,y}) = 0 \tag{19}
$$

To find a solution, we have to incorporate the boundary conditions of our rectangular waveguide in the solution to the Maxwell's equations.

Rectangular Waveguide – TE-modes

Finally we come to the essence of the problem. Consider the waveguide configuration as is shown in Figure 4.

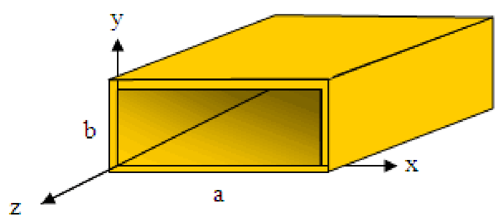

Figure 4: Rectangular waveguide

What are the boundary conditions on this bounded problem? The boundary conditions are that the electric field, tangential to the metallic surfaces, are zero:

$$E_x^0(x; y = 0, b) = 0 \quad (20a)$$
$$E_y^0(x = 0, a; y) = 0 \quad (20b)$$

Remember from the orthogonalization of the Maxwell's equation for the transmission line fields from (11a), that these boundary conditions directly translates to boundary conditions on H_z^0:

$$E_x^0 = -\frac{1}{(k^2 - k_z^2)}\left(j\omega\mu\frac{\partial H_z^0}{\partial y}\right)$$
$$E_y^0 = -\frac{1}{(k^2 - k_z^2)}\left(-j\omega\mu\frac{\partial H_z^0}{\partial x}\right)$$

So if (20a) and (20b) are the boundary conditions, it directly follows that:

$$\left.\frac{\partial H_z^0(x,y)}{\partial y}\right|_{(x;y=0,b)} = 0 \quad (21a)$$
$$\left.\frac{\partial H_z^0(x,y)}{\partial x}\right|_{(x=0,a;y)} = 0 \quad (21b)$$

So we are trying to find solutions to the Helmholtz equation (19), $[\nabla_t^2 + k^2 - k_z^2]H_z^0(\mathrm{x},\mathrm{y}) = 0$, where the boundary conditions (21) are satisfied. Standard basis functions for waveguides are cosine and sine based. Let us verify if (22) verifies both Helmholtz equation (19) as the boundary conditions:

$$H_z^0(x,y) = H_{mn}^z \cos\left(\frac{m\pi x}{a}\right)\cos(\frac{n\pi y}{b}) \quad (22)$$

In (22), H_{mn}^0 is the amplitude of the transverse electric mode TE_{mn}. The partial derivatives of (22) w.r.t. x and y are:

$$\frac{\partial H_z^0(x,y)}{\partial y} = -H_{mn}^z \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad \rightarrow \quad \left.\frac{\partial H_z^0(x,y)}{\partial y}\right|_{(x;y=0,b)} = 0$$
$$\frac{\partial H_z^0(x,y)}{\partial x} = -H_{mn}^z \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \quad \rightarrow \quad \left.\frac{\partial H_z^0(x,y)}{\partial x}\right|_{(x=0,a;y)} = 0$$

Indeed, it satisfies the boundary conditions (21) and therefore (20). Does the solution (22) also satisfy our Helmholtz equation (19)?

The second order partial derivatives are

$$\frac{\partial^2}{\partial x^2}H_z^0(x,y) = -H_{mn}^z\left(\frac{m\pi}{a}\right)^2\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$
$$\frac{\partial^2}{\partial y^2}H_z^0(x,y) = -H_{mn}^z\left(\frac{n\pi}{b}\right)^2\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Let us substitute these partial derivatives in our Helmholtz equation:

$$[\nabla_t^2 + k^2 - k_z^2]H_z^0(\mathrm{x},\mathrm{y}) = 0$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2-k_z^2\right]H_z^0(\mathrm{x,y})=0$$

$$[k^2-k_z^2]H_z^0(\mathrm{x,y})=-\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]H_z^0(\mathrm{x,y})$$

$$[k^2-k_z^2]H_{mn}^z\cos\left(\frac{m\pi x}{a}\right)\cos(\frac{n\pi y}{b})=\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]H_{mn}^z\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

From this equality, we can deduct that this can only be true when (23) holds:

$$[k^2-k_z^2]=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \tag{23}$$

From here we define (24):

$$k_{xm}=\frac{m\pi}{a} \tag{24a}$$

$$k_{yn}=\frac{n\pi}{b} \tag{24b}$$

So, the solution (22) holds for TE-modes of a rectangular waveguides with a transverse propagation constant (23).

Propagation of the TE_{10}-mode

Let us investigate the TE_{10}-mode of the rectangular waveguide from Figure 4. The subscript 10 in TE_{10} indicates that $m=1$ and $n=0$. The solution to the longitudinal magnetic field then becomes (25):

$$H_z^0(x,y)|_{1,0}=H_{10}^z\cos\left(\frac{\pi x}{a}\right) \tag{25}$$

with

$$[k^2-k_z^2]|_{1,0}=k_{x1}^2=\left(\frac{\pi}{a}\right)^2 \tag{26}$$

Since we now know the solution for the longitudinal magnetic field, we can deduct all other field components using (11). This leads to the field expression in (27). Remember that the relationship $\vec{E}=-Z_{TE}\left[\hat{z}\times\vec{H}\right]$ holds.

$$E_x^0(x,y)=0 \tag{27a}$$

$$E_y^0(x,y)=-j\frac{k\zeta}{k_{x1}}H_{10}^z\sin(k_{x1}x) \tag{27b}$$

$$E_z^0(x,y)=0 \tag{27c}$$

$$H_x^0(x,y)=j\frac{k_z}{k_{x1}}H_{10}^z\sin(k_{x1}x) \tag{27d}$$

$$H_y^0(x,y)=0 \tag{27e}$$

$$H_z^0(x,y)=H_{10}^z\cos(k_{x1}x) \tag{27f}$$

Let us define the amplitudes of field components in (27) as:

$$E_y^0(x,y) = E_{10}^y \sin(k_{x1}x) \tag{28a}$$
$$H_x^0(x,y) = H_{10}^x \sin(k_{x1}x) \tag{28b}$$
$$H_z^0(x,y) = H_{10}^z \cos(k_{x1}x) \tag{28c}$$

With:

$$H_{10}^x = j\frac{k_z}{k_{x1}} H_{10}^z \tag{29a}$$
$$E_{10}^y = -Z_{TE} H_{10}^x = -j\zeta \frac{k}{k_{x1}} H_0^z \tag{29b}$$

Let us go more in depth into the longitudinal propagation constant k_z. Fort this TE_{10}-mode, we observed that $k^2 - k_z^2 = k_{x1}^2$. The longitudinal propagation constant then follows as:

$$k_z = \sqrt{k^2 - k_{x1}^2} \tag{30}$$

Now, this square-root is always tricky in EM-analysis. A square-root is a two-valued function, the solution can be positive or negative. We need to make sure that the solution we take is actually physically valid.

What happens when $(k^2 > k_{x1}^2)$? The solution is real. We do need to make sure that we take the desired solution to the square-root. When we take the positive solution,

$$k_z = +\beta$$

In this case $Re\left[e^{-j\beta z} e^{j\omega t}\right] = \cos(\omega t - \beta z)$ we have a **progressive wave**. It means that for increasing time t, the wave moves in the $+z$ direction. In the case we have had chosen the solution to be negative, $k_z = -\beta$, the wave is regressive.

In the case that $(k^2 < k_{x1}^2)$, the solution to the square root will be complex. It is of utmost importance that we chose the **negative solution** in association with a progressive wave. In that case

$$k_z = -j\alpha$$

With this propagation constant $e^{-jk_z z} = e^{-\alpha z}$. The wave is attenuating as is shown again in Figure 5. If the positive solution would have been chosen, the wave was exponentially increasing in amplitude when the wave progresses. This is NOT a physical solution.

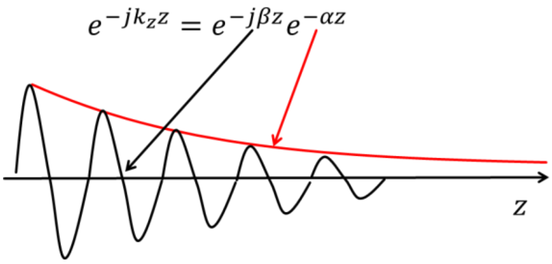

Figure 5: A complex wavenumber $k = \beta - j\alpha$, results in an attenuating and outgoing wave. Different solutions to the square root in (28) results in field solutions that are not physically valid.

There is a way to force the square-root to the solution you desire, by following this mathematical trick:

$$k_z = \sqrt{k^2 - k_{x1}^2} = \sqrt{(-1)(k_{x1}^2 - k^2)} = \sqrt{-1}\sqrt{(k_{x1}^2 - k^2)} = \pm j\sqrt{(k_{x1}^2 - k^2)}$$

Now we can simply select the solution with negative imaginary part:

$$k_z = -j\sqrt{k_{x1}^2 - k^2} \tag{31}$$

Also note that for $(k^2 > k_{x1}^2)$: $k_z = (-j)(j\beta) = \beta$, still holds.

When the longitudinal propagation constant k_z is purely imaginary ($k_z = -j\alpha$ for $k^2 < k_{x1}^2$), the wave is not propagating in the waveguide ($\beta = 0$), there is only attenuation. We say that this mode is **in cut-off**. The frequency at which this happens, we refer to as the **cut-off frequency**. The cut-off wavenumber, wavelength and frequency can be derived by equating $k = k_{x1}$:

$$k_c = \frac{\pi}{a}$$
$$f_c = \frac{c}{2a}$$
$$\lambda_c = 2a$$

One conclusion we can deduct from this, is that the waveguide must be at least half a wavelength in width in order to allow propagation of this fundamental TE_{10}-mode. Using k_c, f_c and λ_c, different ways to express k_z are:

$$k_z = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2} \qquad k_z = \sqrt{k^2 - k_{x1}^2}$$

$$k_z = k\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2} \qquad k_z = k\sqrt{1 - \left(\frac{c}{2af}\right)^2}$$

$$k_z = k\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2} \qquad k_z = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

The longitudinal wavenumber can be illustrated geometrically as shown in Figure 6 using (32):

$$k_z = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} = k\cos(\theta) \tag{32}$$

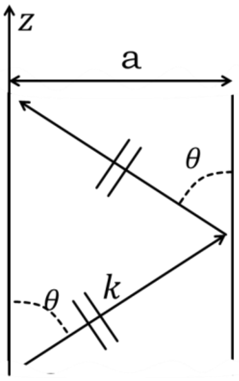

Figure 6: Propagating mode above its cut-off frequency

A mode above cut-off frequency, still travels with propagation constant k. However it bounces within the waveguide with an angle θ. The longitudinal propagation constant is then $k_z = k\cos(\theta)$. As the frequency approaches the cut off frequency of the mode, the angle approaches $\theta = 90°$. An illustrative picture of the wavenumber k_z as function of frequency is shown in Figure 7. Above the cut-off frequency, $f > f_c$, the wavenumber is real and approaches the wavenumber of the medium. Remember from Figure 6 that although $k_z < k$, the actual phase velocity is still k; the mode is bouncing in the waveguide. For frequencies below the cut-off frequency, $f < f_c$, the wavenumber is purely imaginary and approaches $\frac{\pi}{a}$.

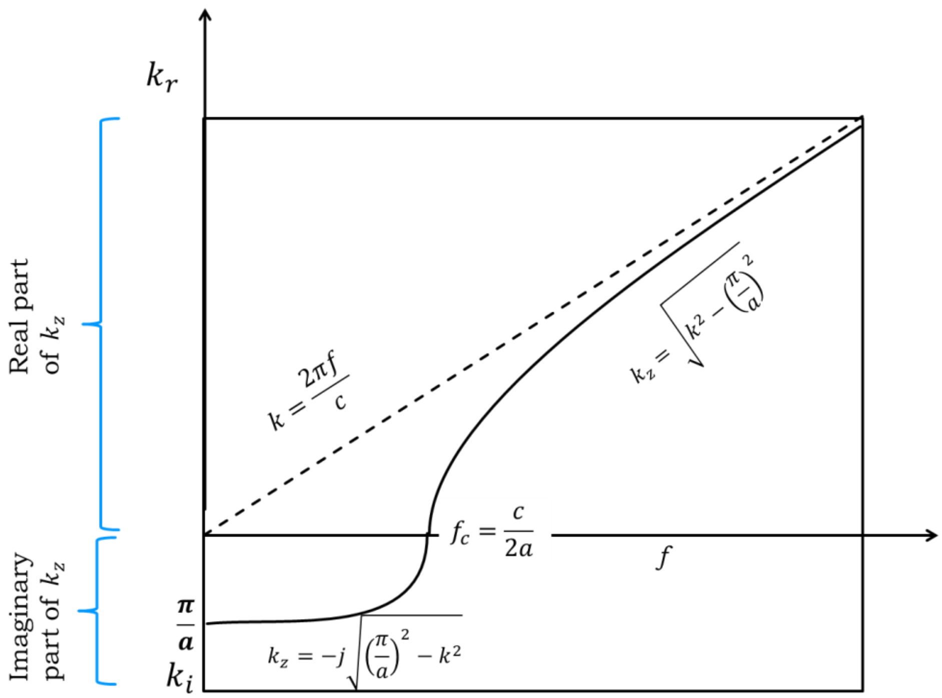

Figure 7: Illustrative picture of the longitudinal wavenumber as function of frequency. Above the cut-off frequency, the wavenumber is real and approaches the wavenumber in free-space whereas for frequencies below the cut-off frequency, the wavenumber is purely imaginary and approaches $\frac{\pi}{a}$.

Higher order modes in the rectangular waveguide

We have derived the cut-off frequency from which the fundamental mode of a rectangular waveguide, the TE_{10}-mode when $a > b$, starts propagating. Now one has to be aware that, as the frequency increases, more modes can be excited in the rectangular waveguide. Depending on the transverse dimensions a and b, this can be a TE_{20}-mode, or a TE_{01}-mode. In general we can write the propagation constant as:

$$k_z = \sqrt{k^2 - k_{xm}^2 - k_{yn}^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

The propagation constant of the TE_{20}-mode is:

$$k_z^{TE_{20}} = \sqrt{k^2 - \left(\frac{2\pi}{a}\right)^2}$$

and it will start propagating when $k^2 > \left(\frac{2\pi}{a}\right)^2$ such that the cut-off frequency is

$$f_c^{TE_{20}} = \frac{c}{a}$$

The propagation constant and cut-off frequency of the TE_{01}-mode is:

$$k_z^{TE_{01}} = \sqrt{k^2 - \left(\frac{\pi}{b}\right)^2}$$

$$f_c^{TE_{01}} = \frac{c}{2b}$$

So if $b < a/2$, the second mode will be the TE_{20}-mode. If $b > a/2$, the next mode will be the TE_{01}-mode.

Power flow of the TE10-mode in the transmission line

Let us calculate the average power that is transmitted in the waveguide. Recall the Poynting theorem describing the average power transported by the field:

$$\vec{S} = \vec{E} \times \vec{H}^* \tag{33a}$$

$$P = \frac{1}{2}\mathrm{Re}\left[\iint_S \vec{S} \cdot \hat{n}\, ds\right] \tag{33b}$$

Let us first calculate $\vec{S}^0$ using the field expressions for the TE10-mode:

$$\vec{S}^0 = \vec{E}^0 \times \overrightarrow{H^0}^* = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & E_y^0 & 0 \\ H_x^{0*} & 0 & H_z^{0*} \end{vmatrix} = \left[E_y^0 H_z^{0*}\hat{x} - E_y^0 H_x^{0*}\hat{z}\right]$$

Only real valued fields will contribute to the propagating power. In the multiplication $E_y^0 H_z^{0*}\hat{x}$ we have E_y^0 (27b), which is a real field. However, H_z^0 (27f) is complex, making the multiplication complex.

This term will not contribute to any power propagating. The normal of the surface for which we want to calculate the pointing vector is $\hat{n} = \hat{z}$. The transmitted power, P_0, can thus be calculated to be:

$$
\begin{aligned}
P_0 &= \frac{1}{2}\mathrm{Re}\left[\iint_S \vec{S}\cdot\hat{n}\,ds\right] = -\frac{1}{2}\mathrm{Re}\left[\iint_S E_y^0 H_x^{0*}\,ds\right] \\
&= -\frac{1}{2}\mathrm{Re}\left[\iint_S E_y^0\left(-\frac{E_y^{0^*}}{Z_{TE}}\right)ds\right] \\
&= \frac{1}{2Z_{TE}}\mathrm{Re}\left[\iint_S \left|E_y^0\right|^2 ds\right] \\
&= \frac{\left|E_{10}^y\right|^2}{2Z_{TE}}\mathrm{Re}\left[\iint_S \sin^2(k_{x1}x)\,ds\right] \\
&= \frac{\left|E_{10}^y\right|^2}{2Z_{TE}}\int_0^b dy\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx
\end{aligned}
$$

Solving the integrals leads to the expression for average transmitted power (34):

$$
P_0 = \frac{1}{4Z_{TE}}\left|E_{10}^y\right|^2 \mathrm{ab} = \frac{1}{4\zeta}\left|E_{10}^y\right|^2 \mathrm{ab}\sqrt{1-\left(\frac{f_c}{f}\right)^2} \qquad (34)
$$

This number presents the average power that is transmitted in the waveguide.

Losses in a rectangular waveguide

In a previous section we have seen that when a mode is above its cut-off frequency, the longitudinal wavenumber is purely real. This implies that there is no attenuation of the mode in the waveguide. However, until now we have neglected the fact that perfect electric conductors do not exist. All metals have a finite conductivity. Let us investigate the ohmic losses. In order to simplify the analysis, to calculate the losses we will make use of the Leontovich method (as has been discussed in Lecture 1) and the perturbation theory.

As explained in Lecture 1, when a plane wave is incident onto a good conductor a part of the field is transmitted inside. This field induces conductive currents inside the material which in turn generates ohmic losses. Since the attenuation constant in a good conductor is very large, the field is only significant in a small volume close to the interface. The Leontovich method was introduced in order to simplify the analysis of such type of problems. The idea is to model the conductor as a PEC (without calculating the fields inside it) to derive the distribution of the fields inside the waveguide as we have done in the previous section. Then one can use this magnetic field on the air-PEC boundary to calculate the surface currents that will introduce the losses that are generated inside the good conductor.

The Leontovich method is used together with the perturbation technique to evaluate the attenuation constant of the mode. We assume that the fields of the lossy line are not greatly different form the fields of the lossless line. For infinite transmission lines (i.e. without any reflections), the power flow in the longitudinal direction can be expressed as:

$$
P(z) = P_0 e^{-2\alpha z}
$$

where P_0 is the power at reference plane $z = 0$. We can now calculate the power loss per unit length as:

$$P_{loss}(z) = -\frac{\partial P(z)}{\partial z} = 2\alpha P_0 e^{-2\alpha z} = 2\alpha P(z)$$

The minus sign ensures that the power lost is a positive quantity (The derivative will be negative). So if we want to calculate the attenuation constant α for the transmission line (35):

$$\alpha = \frac{P_{loss}(z)}{2P(z)} = \frac{P_{loss}(z=0)}{2P(z=0)} = \frac{P_{loss}(z=0)}{2P_0} \tag{35}$$

we need to calculate the power loss per unit length P_{loss}, evaluated at our reference plane $z = 0$, and the power transmitted along the transmission line (i.e. at $P(z = 0)$). Since we make use of the Leontovich method, we can calculate the power lost using the fields of the lossline as we have calculated before!

Power loss per unit length for the $\mathbf{TE_{10}}$-mode

From lecture 1 you know that the power loss due to the ohmic losses in the walls can be calculated using the Leontovich method in which:

$$P_{loss} = \frac{1}{2}\iint_S R_s(\vec{r}) \left|\vec{J_s}(\vec{r})\right|^2 ds \tag{36}$$

where $R_s \approx \sqrt{\frac{\omega\mu}{2\sigma}}$ is the surface resistance and we approximate the surface current density $\vec{J_s}$ as:

$$\vec{J_s}(\vec{r}) \approx \hat{n} \times \vec{H}^{pec}(\vec{r}) \tag{37}$$

Let us calculate the surface currents on the different walls from the magnetic fields calculated from (28bc) using (37). This leads to the surface current densities in (38) as are shown in Figure 8.

$$\vec{J_s}(\vec{r})\Big|_{x=0} = -\hat{y}H_{10}^z \tag{38a}$$
$$\vec{J_s}(\vec{r})\Big|_{x=a} = -\hat{y}H_{10}^z \tag{38b}$$
$$\vec{J_s}(\vec{r})\Big|_{y=0} = \hat{x}\, H_{10}^z \cos(k_{x1}x) - \hat{z}H_{10}^x \sin(k_{x1}x) \tag{38c}$$
$$\vec{J_s}(\vec{r})\Big|_{y=b} = -\hat{x}\, H_{10}^z \cos(k_{x1}x) + \hat{z}H_{10}^x \sin(k_{x1}x) \tag{38d}$$

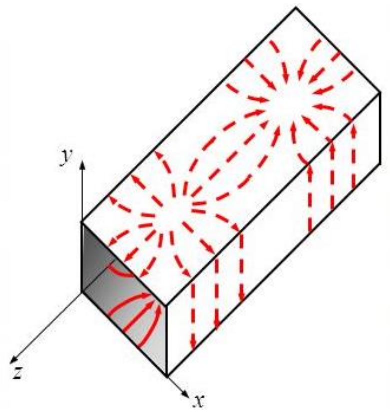

Figure 8: Surface current distribution on the metallic walls for the TE10-mode.

Conclusions we can draw by observing the currents:

1. Currents on lateral sides are vertical
2. Currents on the top and bottom are vectors in x and z
3. Currents on the center of the top and bottom are longitudinal in z

Because of the symmetry we only need to calculate the losses for the bottom wall and one lateral wall and multiply the result by two.

The magnitude of the surface current on the lateral walls are:

$$\left|\vec{J_s}(\vec{r})\big|_{x=0,a}\right|^2 = |H_{10}^z|^2$$

Whereas the magnitude of the surface current on the top or bottom wall are:

$$\left|\vec{J_s}(\vec{r})\big|_{y=0,b}\right|^2 = |H_{10}^z|^2 \cos^2(k_{x1}x) + |H_{10}^x|^2 \sin^2(k_{x1}x)$$

Let us calculate the dissipation losses using (36). To calculate the total losses over a surface, we need to do the integration in (36) over x and z. However, in order to apply the perturbation method we are interested in the power loss per unit length. Therefore we can skip the integration in z. The power loss per unit length can therefore be evaluated as:

$$\begin{aligned}
P_{loss} &\approx 2\frac{1}{2}R_s \int_0^a \left|\vec{J_s}(\vec{r})\big|_{y=0,b}\right|^2 dx + 2\frac{1}{2}R_s \int_0^b \left|\vec{J_s}(\vec{r})\big|_{x=0,a}\right|^2 dy \\
&= R_s \left[\int_0^a |H_{10}^z|^2 \cos^2(k_{x1}x)\, dx + \int_0^a |H_{10}^x|^2 \sin^2(k_{x1}x)\, dx + \int_0^b |H_{10}^z|^2 dy\right] \\
&= R_s \left[|H_{10}^z|^2 \int_0^a \cos^2(k_{x1}x)\, dx + |H_{10}^x|^2 \int_0^a \sin^2(k_{x1}x)\, dx + |H_{10}^z|^2 \int_0^b dy\right] \\
&= R_s \left[|H_{10}^z|^2 \frac{a}{2} + |H_{10}^x|^2 \frac{a}{2} + |H_{10}^z|^2 b\right] \\
&= \frac{R_s a}{2}\left[|H_{10}^z|^2 + |H_{10}^x|^2 + |H_{10}^z|^2 \frac{2b}{a}\right]
\end{aligned}$$

Let us express all terms in terms of $\left|E_{10}^{y}\right|^{2}$. That is $|H_{10}^{z}|^{2}=\left(\frac{k_{x1}}{\zeta k}\right)^{2}\left|E_{10}^{y}\right|^{2}$ and $|H_{10}^{x}|^{2}=\frac{1}{Z_{TE}^{2}}\left|E_{10}^{y}\right|^{2}$. With $Z_{TE}=\frac{k\zeta}{k_z}$. The total loss per unit length then becomes:

$$P_{loss}=\frac{R_s a\left|E_{10}^{y}\right|^{2}}{2}\left[\left(\frac{k_{x1}}{\zeta k}\right)^{2}+\left(\frac{k_{z}}{\zeta k}\right)^{2}+\left(\frac{k_{x1}}{\zeta k}\right)^{2}\frac{2b}{a}\right]$$
$$P_{loss}=\frac{R_s a\left|E_{10}^{y}\right|^{2}}{2\zeta^{2}k^{2}}\left[k_{x1}^{2}+k_{z}^{2}+\frac{2bk_{x1}^{2}}{a}\right]$$
$$P_{loss}=\frac{R_s a\left|E_{10}^{y}\right|^{2}}{2\zeta^{2}}\left[1+\frac{2b}{a}\frac{k_{x1}^{2}}{k^{2}}\right]$$

Knowing that $\frac{k_{x1}^{2}}{k^{2}}=\left(\frac{f_c}{f}\right)^{2}$, the expression of the dissipation losses in the walls per unit length becomes (38):

$$P_{loss}=\frac{R_s a\left|E_{10}^{y}\right|^{2}}{2\zeta^{2}}\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^{2}\right] \tag{38}$$

Now we have the power loss per unit length and the average power flow in the line. Let us now calculate attenuation constant using the perturbation theory.

Attenuation constant for TE10 mode in a rectangular waveguide

We now have the power loss per unit length (38) where we have made use of the Leontovich method. Also we now the power that is transmitted in the mode (34). Using the perturbation theorem (35), we can now calculate the attenuation constant of the TE_{10}-mode along the line.

$$\alpha=\frac{P_{loss}(z=0)}{2P_0}$$
$$\alpha=\frac{\frac{R_s a\left|E_{10}^{y}\right|^{2}}{2\zeta^{2}}\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^{2}\right]}{2\frac{1}{4\zeta}\left|E_{10}^{y}\right|^{2}\mathrm{ab}\sqrt{1-\left(\frac{f_c}{f}\right)^{2}}}$$
$$\alpha=\frac{\frac{R_s a\left|E_{10}^{y}\right|^{2}}{2\zeta^{2}}\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^{2}\right]}{\frac{2\left|E_{10}^{y}\right|^{2}\mathrm{ab}\sqrt{1-\left(\frac{f_c}{f}\right)^{2}}}{4\zeta}}\frac{\frac{4\zeta}{2\left|E_{10}^{y}\right|^{2}\mathrm{ab}\sqrt{1-\left(\frac{f_c}{f}\right)^{2}}}}{\frac{4\zeta}{2\left|E_{10}^{y}\right|^{2}\mathrm{ab}\sqrt{1-\left(\frac{f_c}{f}\right)^{2}}}}$$

$$\alpha = \frac{R_s a\left|E_{10}^{y}\right|^2}{2\zeta^2}\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right]\frac{4\zeta}{2\left|E_{10}^{y}\right|^2 \mathrm{ab}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

Rearranging all terms gives us the expression for the attenuation constant along the line (39):

$$\alpha = \frac{R_s}{\zeta \mathrm{b}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right] \tag{39}$$

The unit of α is in $\frac{Np}{m}$ (Nepers per meter). This is not a very informative figure. Instead, we would like to convert this in power loss in dB/m.

$$L_{dB} = 10\log\left(\frac{P(z)}{P_0}\right) = 10\log(e^{-2\alpha z}) = 10\frac{\ln(e^{-2\alpha z})}{\ln(10)} = \frac{10}{\ln(10)}(-2\alpha z) = -8.686\alpha z$$

So the loss in dB/m is

$$L_{dB/m} = -8.686\alpha = -8.686\alpha\frac{R_s}{b\zeta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[1+2\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right] \tag{40}$$

An important last conclusion we have to make is that a waveguide that is operating close to the cut-off frequency has much more losses than a waveguide operated at very high frequency! Of course, this conclusion is intuitive given the fact that the path length of the mode is much longer when operating close to cut-off frequency. This is illustrated in Figure 9.

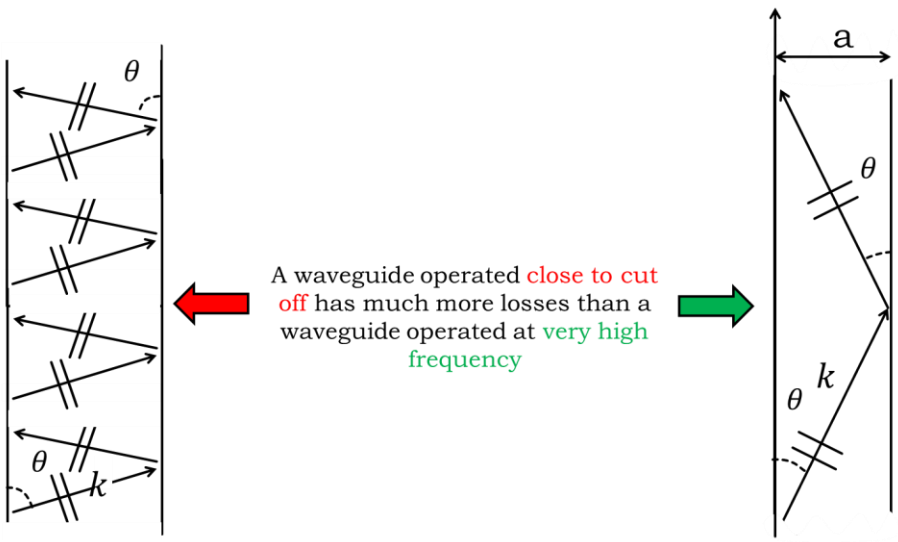

Figure 9: Propagation path of TE10-mode close to cut-off frequency and far away from cut-off frequency.