Reading Reflection Week 2

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Works considered this week:

1. A Survey of the Stable Marriage Problem and Its Variants
2. How do I marry thee? Let me count the ways

1 A Survey of the Stable Marriage Problem and Its Variants

1.1 Summary

The paper first introduces the most basic stable matching problem with some mathematics notions, then discuss different variants and corresponding recent results.

1.2 Important Notes

1.2.1 Mathematics Symbols

1. **A bipartite graph** G contains three sets: U and V are sets of vertices and E is a set of edges. In the Stable Marriage Problem, U represents the set of Men and V represents the set of Women.
2. $w1 \succ_m w2$ means a man m prefers $w1$ to $w2$.
3. In a matching M, a man m matches with a woman w is written in $M(m) = w$ and $M(w) = m$.
4. A blocking pair or a unstable pair needs meet the following three conditions: (i) $M(m) \neq w$; (ii) $w \succ_m M(m)$; and (iii) $m \succ_w M(w)$.

1.2.2 Incomplete Preference Lists

1. **Setting changed:** A person can exclude some members whom he/she does not want to be matched with.

2. There will be a set of men (women) who have partners in all stable matchings and the other is the set of men (women) who are single in all stable matchings. All stable matchings for a single instance are of the same size.

3. **A blocking pair:** (i) $M(m) \neq w$ but m and w are acceptable to each other; (ii) $w \succ_m M(m)$ or m is single in M; and (iii) $m \succ_w M(w)$ or w is single in M.

4. **Result:** The Gale-Shapley algorithm with a slight modification can be applied to find a stable matching.

1.2.3 Preference Lists with Ties

1. **Setting changed:** A person can include two or more persons with the same preference in a tie in his ordered list. Use $w1 =_m w2$ or $w1 \succeq_m w2$.

2. **Super-stability:** A blocking pair: (i) $M(m) \neq w$ (ii) $w \succeq_m M(m)$ (iii) $m \succeq_w M(w)$ or w.

3. **Strong-stability:** A blocking pair: (i) $M(m) \neq w$ (ii) $w \succ_m M(m)$ (iii) $m \succeq_w M(w)$ or w.

4. **Weak-stability:** A blocking pair: (i) $M(m) \neq w$ (ii) $w \succ_m M(m)$ (iii) $m \succ_w M(w)$ or w.

5. A super-stable matching is strongly stable, and a strongly stable matching is weakly stable.

6. **Result:** A weakly stable matching always exist. Some instances don't have super-stable or strongly stable matching. An algorithm can decide if a super-stable matching exists and finds one if any $O(n^2)$.

1.2.4 Incomplete Preference Lists with Ties

1. **Setting changed:** Both incompleteness and ties in preference lists.

2. **Results:**
   Super-stable matching: An algorithm can decide if a super-stable matching exists and finds one if any $O(a)$.
   Strongly stable matching: An algorithm can decide if a strongly stable matching exists and finds one if any $O(na)$.
   Under both super and strong stabilities, all stable matchings for a single instance have the same size. **Weak stable matching:** A stable matching exists for any instance, and can be found in time $O(a)$; one instance can have stable matchings of different sizes, and the problem of finding a largest one is NP-hard.

1.2.5 The Number of Stable Matchings

1. **Setting:** Find the maximum number of stable of matchings an instance of size n can have.

2. **Result:** For any n power of 2, there are at least $2.28^n/(1+\sqrt{3})$ stable matchings; For n = 4, the maximum if 10.

1.2.6 Optimal Stable Matchings

1. $p_m(w)$ denotes the position of woman w in man m's preference list and $p_w(m)$ denotes the position of man m in woman w's preference list

2. **Regret cost:** $r(M) = \max_{(m,w)\in M} \max\{p_m(w), p_w(m)\}$

3. **egalitarian cost:** $c(M) = \sum_{(m,w)\in M} p_m(w) + \sum_{(m,w)\in M} p_w(m)$

4. **sex-equalness cost:** $d(M) = \sum_{(m,w)\in M} p_m(w) - \sum_{(m,w)\in M} p_w(m)$

5. **Results:** A polyno-mial time algorithms is proposed to minimize first two costs by exploiting a lattice structure which is of polynomial size but contains information of all stable matchings; In contrast, the sex-equal stable matching problem is NP-hard.

1.2.7 Stable Roommates Problem

1. **Setting:** There is an even number 2n of persons, and each has a preference list over the other 2n1 people.

2. **Results:** A polynomial time algorithm is proposed to decide if an instance admits a stable matching, and finds one if exists. The problem of finding a matching with minimum number of blocking pairs is NP-hard and hard to approximate.

3. **Allow ties:** Determining whether an instance admits a weakly stable matching is NP-complete even in complete preference lists, while super-stability and the strong stability are solved in polynomial time.

1.2.8 Hospitals/Resident Problem

1. **Setting:** Each hospital declares the number of residents it will accept and each student can also choose one hospital.

2. **Results:** It can be solved by reducing HR into SM by replacing each hospital with a quota q by its q copies.

3. **Property:** Any stable matching assigns the same number of residents to all hospitals. Furthermore, if a hospital obtains residents fewer than its quota in one stable matching, then the hospital gets the same set of residents in any stable matching.

1.2.9 3-Dimensional Stable Matching

1. **Setting:** There are three sets of agents
2. **Results:** One model proposed by Ng and Hirschberg and Subramanian is proved NP-completeness result.

1.3 Interesting sections

1.3.1 Hospitals/Residents Problem

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1.3.2 3-Dimensional Stable Matching

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3. C. C. Haung, "Two's company, three's a crowd: stable family and threesome roommates problems," Proc.ESA 2007, LNCS 4698, pp. 558-569, 2007.
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2 How do I marry thee? Let me count the ways

2.1 Summary

A stable marriage problem of size 2n is constructed which contains at least $2^n\sqrt{n}$stable matchings by using the special properties of the latin marriage problem.

2.2 Important notes

1. **Latin marriages:** An $n * n$ matrix where every row and column is a permutation of the numbers 0,1,...,n-1. a_{ij} denotes the man i's rank of woman j, and n-1-a_ij to be woman j's rank of man i, where 0 is the best rank and n-1 is the worst rank. A sequence $X_0, X_1, ..., X_{n-1}$, denotes the numbers selected in column j or which mam a woman chooses. By contrast, a sequence $X^0, ..., X^{n-1}$ denotes the numbers selected in row i, or which woman a man chooses.

2. A matching on A is stable if and only if there do not exist $i, j \in \{0, \ldots, n-1\}$ such that $X^i > a_{ij} > X_j$ or $X^i < a_{ij} < X_j$

3. Size n latin marriage problems must possess at least n stable problem, namely the constant matchings where every man receives his mth choice for some m.

4. The matrix S_n defined to have (i,j) entry $s_{ij} = (i+j) \bmod n$ achieves this minimum. S_n has only n stable matchings, the constant ones.

5. DS_n is a $2n \times 2n$ latin square possessing many stable matchings. For $0 \le i, j \le n-1$, $a_{ij} = a_{i+n,j+n} = 2s_{ij}$. The other quadrants are defined by the vertical reflection relation $a_{ij} + a_{i,2n-1-j} = 2n-1$.

6. If (i,j) and (k,l) are in opposite quadrants of DS_n and $a_{ij} = a_{kl}$, then $a_{il} = a_{kj}$.

7. The number of stable matchings in DS_n is equal to the number of valid sequence $X_0, ..., X_n$, where $X_n = X_0$ and $0 \le X_j \le 2n-1$for all j.

8. $I(2n)$ is the number of stable matchings of DS_n, and for $0 \le k \le n-1$, $h(n,k)$ to be the number of valid sequences $X_0, ..., X_)n$ where $X_n = X_0 = k$, then $I(2n) = 2\sum_{k=0}^{n-1} h(n,k)$.

9. $h(n,k) = \begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix} - \begin{pmatrix} 2n-1 \\ n-2-k \end{pmatrix}$

10. $I(2n) = (n+1)\begin{pmatrix} 2n \\ n \end{pmatrix} - 2^{2n-1}$