

Computer Applications in Water Supply

VOLUME 1 – SYSTEMS ANALYSIS
AND SIMULATION

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SECTION 1
NETWORK MODELLING AND SOLUTIONS

1. A Gradient Algorithm for the Analysis of Pipe Networks

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ABSTRACT. The following paper deals with the derivation of a gradient algorithm to be used for flow analysis in pipe networks. The Newton-Raphson technique is applied both in terms of nodal heads and pipe flows to the simultaneous solution of the system of partly linear and partly non-linear equations expressing mass and energy balance. The problem is analytically reconducted to the iterative solution of a system of linear equations, which size equals the number of unknown heads. The solution is then efficiently obtained using the Incomplete Choleski Factorization/Modified Conjugate Gradient (ICF/MCG) algorithm.

1. INTRODUCTION

The design of a water distribution system, defined in terms of links (pipes, pumps, valves) and nodes once its topological structure is given, generally consists of two different stages:

- network definition and optimization
- network flow analysis

The first stage, given the network topology, the physical properties and constraints, the nodal demands and the costs involved in each link consists in the determination of the sizes of the linking elements that minimize a certain objective function, subject to a number of constraints.

In the case of looped networks the objective function should incorporate actual costs as well as a measure of hydraulic reliability (Pilati and Todini, 1984).

The constraints are generally introduced in terms of:

- pipe diameters
- minimum head at nodes
- maximum velocity in pipes.
- etc.

The use of more complex optimization schemes where network topology is also unknown, although interesting from a scientific point of view, can hardly be justified in practice, where the layout of the network is generally an *a priori* constraint to the problem.

The second stage, the network flow analysis, is used to assess the consistency of the designed network with service requirements and conditions or to analyze the effects of system partializations. The problem consists of determining both the flows in every linking element of the network, and the piezometric head in every node of the system, under the assumptions that the steady state flow has been reached.

The solution of network flow analysis problems has given rise to a number of alternative methods that can be classified as:

- Local gradient
- Newton-Raphson
- Linearization
- Numerical minimization

The first three classes include methods used for the solution of systems of non-linear equations, while the fourth deals with the search of minimum of a non-linear convex function under linear equality and inequality constraints.

The first class includes methods on the line of those developed by Hardy Cross, 1936, who first systematized the network analysis problem. These methods solve the original system of equations in terms of a local gradient, either in the sub-domain of independent loops present in the network or in the sub-domain of nodes. Cao, 1963, proved that according to the choice of the independent loops sub-domain (which is not unique) the method may diverge, while the nodal H.Cross method has slow rate of convergence and sometimes lack of convergence.

The Newton-Raphson derivation, also carried out either in the independent loops sub-domain (Warga, 1954, Martin and Peters, 1963) or in the node sub-domain (Shamir and Howard, 1968), may be viewed as an extension of Cross local corrections technique by means of a simultaneous

multidimensional correction algorithm. These methods, although more convergent than the original local gradient techniques, still reveal some deficiencies depending upon both the complexity of the network and the choice of the externally provided initial solution.

The linear theory method (Wood and Charles, 1972) can also be regarded as an application of the Newton-Raphson technique in the sub-domain of loops, but it requires the solution of a larger system of equations (number of loops + number of nodes) although reducing the risk of failures.

The optimization methods (Collins et al., 1978, Contro and Franzetti, 1982), minimize a non-linear convex objective function subject to linear equality and inequality constraints using mathematical optimization techniques. The advantage is obvious since the convexity of the objective function combined to the linear constraints guarantees the existence and uniqueness of the solution. Unfortunately the numerical solution of the problem requires efficient non-linear programming algorithms, thus reducing the practical utilization of these methods, particularly when dealing with large complex networks.

The proposed method (originally developed by Todini, 1979) may be regarded as a bridge between the optimization based and the Newton-Raphson based techniques in that it starts from the minimization of a slightly modified Content Model (Collins et al. 1978) in order to prove the existence and uniqueness of the solution of the partly linear and partly non-linear system to be simultaneously solved in terms of unknown nodal heads and unknown flows in pipes. The Newton-Raphson technique is then applied in this enlarged space of flows and heads where the proof of the existence and unicity of the solution holds, which is the key to the unconditional convergence of the proposed method, and finally the problem is algebraically reconducted to the recursive solution of a linear system of size equal to the number of unknown nodal heads and a matrix projection of the results over the unknown pipe flows. The special structure of the resulting system matrix, a sparse Stieltjes matrix, symmetrical, positive definite which non zero elements can be stored in number of nodes + number of pipes locations, allows for an efficient solution by using the Incomplete Choleski Factorization/Modified Conjugate Gradient algorithm (ICF/MCG) due to D. Kershaw, 1978.

In this paper the authors will develop the basic equations for a simple pipe network, the extension to systems including pumps and valves is given in Salgado et al. 1987.

2. THE DERIVATION OF THE PROPOSED NEWTON RAPHSON ALGORITHM
Given a pipe network defined by its topology, pipe characteristics and system constraints (water demands q or fixed heads H_0 for each node and head losses laws for each pipe $f_i(Q_i)$) the problem consists in determining all the flowrates Q in the pipes as well as all the unknown heads H at the nodes on the assumption of steady state conditions.

In matrix form the problem can be formulated as follows:

$$A12 \quad H + F(Q) = -A10 \quad H_0 \quad (1)$$

$$\begin{aligned} A21 \quad Q &= q \\ A12 &= A21^T \quad (\text{np,nn}) \text{ unknown head nodes} \\ &\quad \text{incidence matrix} \\ A10 &= A01^T \quad (\text{np,no}) \text{ fixed head nodes} \\ &\quad \text{incidence matrix} \\ Q^T &= [Q_1, Q_2, \dots, Q_{np}]^T (1, np) \text{ flowrates in each pipe} \\ q^T &= [q_1, q_2, \dots, q_{nn}]^T (1, nn) \text{ nodal demands} \\ H^T &= [H_1, H_2, \dots, H_{nn}]^T (1, nn) \text{ unknown nodal heads} \\ H_0^T &= [H_0_1, H_0_2, \dots, H_0_no]^T (1, no) \text{ fixed nodal heads} \\ F^T(Q) &= [f_1, f_2, \dots, f_{np}]^T (1, np) \text{ laws expressing head losses in pipes } (f_i = f_i(Q_i)) \end{aligned}$$

with

$$\begin{aligned} nn &\text{ number of nodes with unknown head} \\ no &\text{ number of nodes with fixed head} \\ np &\text{ number of pipes with unknown flow} \end{aligned}$$

$$A12(i,j) = \begin{cases} 1 & \text{if flow of pipe } i \text{ enters node } j \\ 0 & \text{if pipe } i \text{ and node } j \text{ are not connected} \\ -1 & \text{if flow of pipe } i \text{ leaves node } j \end{cases}$$

and with $A10$ defined similarly to $A12$ for fixed head nodes.

The system represented by equation (1) may have more than one solution depending upon the shape of $f_i(Q_i)$. If all $f_i(Q_i)$ are monotonically increasing functions, it can be proved that the solution of system (1) exists and is unique (Pilati and Todini, 1984). A more general approach to the proof of the existence and uniqueness of the solution can be derived with reference to the Content Model (Collins et al., 1978). Assuming the Hazen-Williams head losses function, one can write for each pipe:

$$f_i(Q_i) = R_i |Q_i|^{n_i-1} Q_i \quad (2)$$

with R_i a constant. After integration of f_i from 0 to Q_i the Content Model becomes:

$$\begin{aligned} \text{min. } C(Q) &= \sum_{i=1}^{\text{np}} R_i |Q_i|^{n_i+1} / (n_i+1) + \\ &+ \sum_{j=1}^{\text{no}} H_0_j \sum_{i=1}^{\text{np}} A01(j,i) Q_i \quad (3) \\ \text{s.t. } \sum_{i=1}^{\text{np}} A21(j,i) Q_i - q_j &= 0 \quad j = 1, \text{nn} \end{aligned}$$

It can be noticed that due to the definition of $f_i(Q_i)$ there is no need for the additional constraints $Q_i \geq 0$ which were introduced in the original Content Model formulation. This allows to transform the constrained minimization into an unconstrained one by means of Lagrange multipliers, i.e.:

$$\begin{aligned} \text{min. } L(Q, \lambda) &= \sum_{i=1}^{\text{np}} R_i |Q_i|^{n_i+1} / (n_i+1) + \\ &+ \sum_{j=1}^{\text{no}} H_0_j \sum_{i=1}^{\text{np}} A01(j,i) Q_i \quad (4) \\ &+ \sum_{j=1}^{\text{nn}} \lambda_j \sum_{i=1}^{\text{np}} (A21(j,i) Q_i - q_j) \end{aligned}$$

Since all R_i are positive when all $n_i > 0$, L is convex and the solution of this problem exists and is unique. This coincides with the sufficient conditions for a minimum.

The solution can thus be found by imposing all the necessary conditions for an extreme:

$$\begin{aligned} \frac{\partial L}{\partial Q_i} &= 0 & i=1,np \\ \frac{\partial L}{\partial \lambda_j} &= 0 & j=1,nn \end{aligned} \quad (5)$$

to get, in matrix form:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} Q \\ \lambda \end{bmatrix} = \begin{bmatrix} -A_{10} H_0 \\ q \end{bmatrix} \quad (6)$$

where

$$A_{11} = \begin{bmatrix} R_1 |Q_1|^{n_1-1} \\ R_2 |Q_2|^{n_2-1} \\ . \\ . \\ R_{np} |Q_{np}|^{n_{np}-1} \end{bmatrix} \quad (7)$$

is an (np,np) diagonal matrix.

By comparison with equation (1) it is immediate to assign a physical meaning to the Lagrange multipliers; they represent in fact the unknown nodal heads. Finally after substitution one gets:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} Q \\ H \end{bmatrix} = \begin{bmatrix} -A_{10} H_0 \\ q \end{bmatrix} \quad (8)$$

Equation (8) equals equation (1) when the head losses functions are given by equations (2).

It is essential to point out that this proves that the solution of the system of non-linear equations (8), as well as the solution of the Content Model, exists and is unique under the given assumptions, only in the space of all the unknown Q and H . It will be shown later, that in reduced dimension spaces, such as the space of all pipes

or the space of all loops, the solution is not unique but can be reconduted to the unique solution by providing additional information.

In order to solve the system of non-linear equations (8), the Newton-Raphson technique can be used, provided that matrix A_{11} does not become singular, which happens when the heads at the extremes of a pipe are identical and consequently the flow in the pipe vanishes. This problem can be avoided by defining a lower bound for the elements of matrix A_{11} .

The Newton-Raphson iterative scheme can thus be obtained by differentiating both sides of equation (8) with respect to Q and H to get:

$$\begin{bmatrix} N A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} dQ \\ dH \end{bmatrix} = \begin{bmatrix} dE \\ dq \end{bmatrix} \quad (9)$$

with

$$N = \begin{bmatrix} n_1 & & & \\ & n_2 & & \\ & . & . & \\ & & & n_{np} \end{bmatrix} \quad (np,np) \text{ diagonal matrix}$$

and where

$$\begin{aligned} dE &= A_{11} Q^k + A_{12} H^k + A_{10} H_0 \\ dq &= A_{21} Q^k - q \end{aligned} \quad (10)$$

are the residuals to be iteratively reduced to zero and Q^k and H^k the flows and heads at iteration k .

Assuming

$$N A_{11} = D^{-1} \quad (11)$$

(and therefore $D A_{11} = N^{-1}$, N , A_{11} and D being diagonal), the inverse of the system matrix can be obtained analytically (F. Ayres, 1962) by partitioning:

$$\begin{bmatrix} D^{-1} & A_{12} \\ A_{21} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad (12)$$

with

$$\begin{aligned} B_{11} &= D - D A_{12} (A_{21} D A_{12})^{-1} A_{21} D \\ B_{12} &= D A_{12} (A_{21} D A_{12})^{-1} \\ B_{21} &= (A_{21} D A_{12})^{-1} A_{21} D \\ B_{22} &= -(A_{21} D A_{12})^{-1} \end{aligned} \quad (13)$$

The solution of equation (9) can be found bearing in mind that:

$$\begin{aligned} dQ &= B_{11} dE + B_{12} dq \\ dH &= B_{21} dE + B_{22} dq \end{aligned} \quad (14)$$

by substituting for equations (13) into (14) to give:

$$\begin{aligned} dH &= (A_{21} D A_{12})^{-1} A_{21} D (A_{11} Q^k + A_{12} H^k + A_{10} H_0) + \\ &\quad - (A_{21} D A_{12})^{-1} (A_{21} Q^k - q) = \\ &= H^k + (A_{21} D A_{12})^{-1} A_{21} D (A_{11} Q^k + A_{10} H_0) + \\ &\quad + (q - A_{21} Q^k) \end{aligned} \quad (15)$$

$$\begin{aligned} dQ &= \{ D - D A_{12} (A_{21} D A_{12})^{-1} A_{21} D \} (A_{11} Q^k + A_{12} H^k + \\ &\quad + A_{10} H_0) + D A_{12} (A_{21} D A_{12})^{-1} (A_{21} Q^k - q) = \\ &= D (A_{11} Q^k + A_{10} H_0) - D A_{12} \left[(A_{21} D A_{12})^{-1} (A_{21} D \right. \\ &\quad \left. (A_{11} Q^k + A_{10} H_0) + (q - A_{21} Q^k) \right] \end{aligned} \quad (16)$$

Substituting for equation (15) into equation (16), bearing in mind the definition of D and that:

$$\begin{aligned} dQ &= Q^k - Q^{k+1} \\ dH &= H^k - H^{k+1} \end{aligned} \quad (17)$$

one finally obtains the recursive Newton-Raphson algorithm:

$$\begin{aligned} H^{k+1} &= -(A_{21} N^{-1} A_{11}^{-1} A_{12})^{-1} (A_{21} N^{-1} (Q^k + A_{11}^{-1} A_{10} H_0) + \\ &\quad + (q - A_{21} Q^k)) \end{aligned} \quad (18)$$

$$Q^{k+1} = (I - N^{-1}) Q^k - N^{-1} A_{11}^{-1} (A_{12} H^{k+1} + A_{10} H_0) \quad (19)$$

where A_{11} is computed using Q^k .

Summarizing, it was shown that the minimization of the Content Model under equality constraints can be reconducted to the iterative solution of a system of linear equations of size equal to the number of nodes (nn) plus a scalar projection and a linear combination of the results, of size equal to the number of pipes (np). This algorithm can be regarded as the correct formulation of the so called nodal Newton-Raphson.

In the next paragraph the linear theory and the loop Newton-Raphson methods will be derived from equation (9) allowing for a unified view of the Newton-Raphson algorithms commonly used for flow network analysis.

3. A UNIFYING ASSESSMENT OF THE LOOP GRADIENT PROCEDURES

If one defines
 $M_{13} = M_{31}^T$ (np,nl) loop incidence matrix
as:

$$\begin{aligned} M_{13}(i,k) &= \begin{cases} 1 & \text{if flow of pipe } i \text{ is in the positive} \\ & \text{direction of loop } k, \text{ to which it belongs} \\ & 0 & \text{if pipe } i \text{ does not belong to loop } k \end{cases} \\ M_{13}(i,k) &= \begin{cases} -1 & \text{if flow of pipe } i \text{ is in the negative} \\ & \text{direction of loop } k \text{ to which it belongs} \end{cases} \end{aligned}$$

where $nl = np - nn$ is the number of independent loops, (which is always possible provided that no-1 extra dummy pipes are added when $no > 1$), it is very easy to verify the following topological properties:

$$\begin{aligned} M_{31} A_{12} &= 0 & A_{21} M_{13} &= 0 \\ M_{31} A_{10} &= 0 & A_{01} M_{13} &= 0 \end{aligned} \quad (20)$$

These properties allow to modify the original Newton-Raphson procedure. In fact if one premultiplies equation (9) by the following (np,np+nn) matrix:

$$\begin{bmatrix} M_{31} & 0 \\ 0 & I \end{bmatrix}$$

where I is an (nn, nn) identity matrix, one obtains:

$$\begin{bmatrix} M_{31} & A_{11} \\ A_{21} \end{bmatrix} \begin{bmatrix} dQ \\ dq \end{bmatrix} = \begin{bmatrix} dF \\ dq \end{bmatrix} \quad (21)$$

with $dF = M_{31} dE = M_{31} (A_{11} Q^k + A_{12} H^k + A_{10} H_0)$
 $= M_{31} A_{11} Q^k$

and

$$dq = A_{21} Q^k - q$$

which gives rise to the following recursive algorithm:

$$Q^{k+1} = Q^k - \begin{bmatrix} M_{31} & A_{11} \\ A_{21} \end{bmatrix}^{-1} \begin{bmatrix} M_{31} A_{11} Q^k \\ A_{21} Q^k - q \end{bmatrix} \quad (22)$$

This is nothing else than the linear theory method, where the matrix to be inverted at each step is of size (np, np) and the unknown variables have been reduced from the original $np+nn$ unknowns of equation (9) to the np dQ , by eliminating nn equations together with the dependence upon the nn dH .

The solution of this problem in terms of flows is unique, but the solution provided for the original problem is not, since by reducing the space of unknowns, one loses the information on the externally fixed heads H_0 . This additional information, though, can be easily added, a posteriori, to the solution of the mathematical problem defined by equation (22).

Following the same logics one can proceed further on in the reduction of the solution space by changing the unknown variables. If one defines dQ_l , the loop discharge corrections, as:

$$dQ = M_{13} dQ_l \quad (23)$$

the system represented by equation (22) can be reduced to:

$$(M_{31} A_{11} M_{13}) dQ_l = M_{31} A_{11} Q^k \quad (24)$$

where the symmetrical positive definite matrix $M_{31} A_{11} M_{13}$ is of size (nl, nl) . This formulation gives rise to the following recursive algorithm:

$$Q^{k+1} = Q^k - M_{13} (M_{31} A_{11} M_{13})^{-1} M_{31} A_{11} Q^k \quad (25)$$

which is known as the loop Newton-Raphson. The solution of this problem, although unique, does not provide a unique solution in terms of Q for the original problem. As one can see, additional nn equations have been eliminated, together with all the information on the outflow demands q . This can be overcome by providing an initial solution Q_0 that satisfies the condition:

$$A_{21} Q_0 = q \quad (26)$$

since due to the property $A_{21} M_{13} = 0$ any other solution Q^k will also satisfy the same condition. In addition, as for the linear theory, one has also to introduce the information on the fixed heads H_0 in order to find the unique solution of the original problem.

Summarizing, it was shown that the basic application of the Newton-Raphson procedure upon the space of all pipes and of all independent nodes, leads, via simple algebraic manipulations, to three basic algorithms. Each algorithm requires the recursive solution of a system of linear equations of different size (nn, np, nl) , but they are all equivalent in terms of convergence properties, i.e. the gain at each gradient iteration is the same, since one can project the results obtained in one problem space into the space of the remaining two by simple linear algebra manipulations. In the next paragraph a comparison of the three algorithms will be presented in order to identify the most convenient formulation for the solution of large pipe networks.

4. PROS AND CONS OF THE DIFFERENT FORMULATIONS

As mentioned earlier, the three different formulations of the Newton-Raphson algorithm, although equivalent in terms of convergence, lead to the solution of systems of linear equations of different size.

In order to assess the relative merits of the different

formulations for solving large pipe network analysis problems, the comparison will be made in terms:

- simplicity of input
- initial solution
- size of the system of linear equations
- efficient solution of the system of equations

Due to the large problems to be solved one should minimize the amount of information to be provided, avoiding duplication. The typical input for the nodal gradient consists of a table where the initial and ending nodes, the length, the diameter and the other elements that characterize the pipes are provided. In addition, a table providing all the information related to the nodes, must be given. On top of that, the loop based algorithms (linear theory and loop gradient) require the definition of the loop incidence matrix: this must be provided either as part of the input data or it can be computed from the node incidence matrix A_{12} by partitioning it into two portions, A_1 of size (nn, nn) and A_2 of size (nl, nn)

$$A_{12} = \begin{bmatrix} A_1 \\ \cdot \\ A_2 \end{bmatrix} \quad (27)$$

and imposing $M_{31} A_{12} = 0$ to get

$$M_{31} = [-A_2 A_1^{-1}; I] \quad (28)$$

with I an (nl, nl) identity matrix.

Therefore, from the point of view of the input requirements one can conclude that the nodal formulation is simpler, also considering that there is no need of adding dummy pipes to the topology every time that a new fixed head node is introduced.

In terms of initial solution, both the nodal formulation and the linear theory do not require a particular starting solution, while the loop based gradient needs an initial solution Q^0 satisfying equation (26). This can be obtained by partitioning Q^0 into two portions Q_1 of size nn , and Q_2 of size nl and solving for Q_1 :

$$Q_1 = A_1^{-1} (q - A_2 Q_2) \quad (29)$$

once an initial value is assigned to Q_2 .

From the point of view of the initial solution, the nodal gradient as well as the linear theory are thus preferable to the loop gradient.

In terms of matrix size, the three methods have the following requirements:

nodal gradient (nn, nn) symmetrical

linear theory	(np,np) non symmetrical
loop gradient	(nl,nl) symmetrical

These requirements give an idea of the size of the problem to be solved, but do not account for the matrix sparsity which involves very small storage requirements.

In practice, allowing for symmetry, the number of non zero elements for the nodal gradient is equivalent to the number of pipes plus the number of nodes, while for the loop gradient it equals the number of loops plus the number of pipes shared by different loops. The linear theory is heavily penalized in terms of storage occupation. Its non symmetrical matrix will contain a number of non zero elements equal to three times the number of pipes plus one time the number of pipes shared by different loops.

This allows to select the loop gradient as the procedure with smaller storage requirements, followed by the nodal gradient, the storage requirements of the linear theory being much higher.

Although the previous items give some indication for choosing the appropriate procedure, the last one, the efficiency of the linear system solver, is definitely the most important. The solution of the linear system must in fact be found more than once (in general from 5 to 10 times) and the method that allows for the most efficient solution will obviously be the most appropriate.

As well as for the storage requirements, the linear theory, is far below the capabilities of the other two formulations: the non symmetrical matrix must be inverted using direct elimination methods that, in the case of sparse non symmetrical matrices, will require a number of operations which increases with the square of the number of pipes.

The two other methods result into very similar matrices and, at a first glance one could conclude that the most suitable method is the one that has a smaller matrix size. This is not particularly true. Let first assume that the solution will be found by an iterative method: the number of operations for one iteration depends almost linearly upon the size of the matrix, but the number of iterations depends upon the spectral radius of the matrix. If, on the other hand, one uses a direct method which takes advantage of symmetry, (such as a factorization or a symmetric matrices Gauss elimination), the number of operations depends not only upon the size, but also upon the density,

due to the filling in. And in fact the results found by many authors (Hamam and Brameller, 1971, Carpentier, Cohen and Hamam, 1987, Salgado, Todini and O'Connell, 1987), although not conclusive, indicate that solving the system in the space of nodes requires consistently less time than solving it in the space of loops.

All these considerations can be summarized in the following table:

	NODAL	LINEAR T.	LOOP
INPUT requirements	1	2	2
INITIAL SOLUTION requirements	1	1	2
SIZE of system of linear equations	2	3	1
EFFICIENT SOLUTION of system of equat.	1	3	2

which leads to the obvious conclusion that the proposed nodal version of the Newton-Raphson method, which includes the Hybrid method proposed by Hamam and Brameller as a particular case when all the exponents n are constant and equal to 2, and which does not require the definition of loops, is the most suitable procedure for solving large network systems, particularly when using microcomputers or PC's.

5. THE ICF/MCG SOLUTION FOR THE SYSTEM OF LINEAR EQUATIONS A method for solving the system of linear equations represented by the first of equations (18-19), is here presented, which seems convenient for the solution of the system matrix when using microcomputers or PC's.

Bearing in mind the structure of the (nn, nn) system matrix A :

$$\begin{aligned} A &= A_{21} (N A_{11})^{-1} A_{12} = \\ &= A_{21} D A_{12} \end{aligned} \quad (30)$$

The special nature of the topological matrices A_{12} and A_{21} produce a matrix A with the following characteristics:

- Symmetric
- Positive definite
- Stieltjes type

which elements are defined as:

$$\begin{aligned} A(j,j) &= \text{the sum of the } D(i,i) \text{ relevant to} \\ &\quad \text{all pipes } i \text{ connected to node } j \\ A(j,jj) &= 0 \text{ if node } j \text{ is not connected to} \\ &\quad \text{node } jj \\ A(j,jj) &= -D(i,i) \text{ if pipe } i \text{ connects node } j \\ &\quad \text{with an unknown head node } jj \\ &= 0 \text{ if pipe } i \text{ connects node } j \text{ with} \\ &\quad \text{a fixed head node } jj \end{aligned}$$

Due to symmetry the number of non zero elements to be retained in the matrix equals at most the number of nodes (the nn diagonal elements) plus the number of pipes (the np extra-diagonal elements).

The low density and sparsity of the matrix suggests that an iterative scheme should be preferred to a direct inversion to avoid filling in. Unfortunately, only few elements of the principal diagonal, relevant to nodes connected to some fixed head node, are strictly dominant i.e. greater than the sum of the absolute value of the extra-diagonal terms. This only defines a weak convergence criterium (Stoer and Burlish, 1975), and the use of iterative schemes such as Jacobi or Gauss-Seidel generally results in tiresome computations. Higher convergence may be obtained with over-relaxation methods, although increasing the computational effort in order to find optimal over-relaxation parameters.

An alternative method, which in theory converges to the exact solution in a limited number of steps, being insensitive to the weak convergence criterium, is the Conjugate Gradient method (Hestenes and Stiefel, 1952). If one rewrites the first of equations (18-19) as:

$$A x = b \quad (31)$$

with

$$\begin{aligned} x &= H^{k+1} \\ b &= -\{A_{21} N^{-1}(Q^k + A_{11}^{-1} A_{10} H_0) + (q - A_{21} Q^k)\} \end{aligned}$$

the matrix A being symmetrical positive definite, the solution of (31) is reconduted to the minimization of the quadratic convex functional :

$$J(x) = 1/2 x^T A x - b^T x \quad (32)$$

which can be solved using the following recursive conjugate gradient scheme:

$$\begin{aligned} r_0 &= Ax_0 - b & k = 0 \\ p_0 &= r_0 \\ \alpha_k &= -r_k^T p_k / p_k^T A p_k \\ x_{k+1} &= x_k + \alpha_k p_k & k > 0 \\ r_{k+1} &= r_k + \alpha_k A p_k \\ \beta_{k+1} &= -r_{k+1}^T A p_k / p_k^T A p_k \\ p_{k+1} &= r_{k+1} + \beta_{k+1} p_k \end{aligned} \quad (33)$$

In exact algebra the scheme converges at most in nn steps, nn being the rank of the matrix A , but due to truncation errors the number of steps required is much larger (circa 2-2.5 nn). Although the number of operations required at each step is small, due to the low density of A , the scheme is not adequate for large problems ($nn \sim 1000$).

Kershaw, 1978, and more recently Ajiz and Jennings, 1984, have proposed modified versions of the conjugate gradient which are particularly suited for the solution of large problems. As previously mentioned due to the sparsity of A , when dealing with large networks direct methods are not appropriate for solving this problem, due to the increased memory occupation and computational effort they require. To benefit of the symmetry and positive definition of matrix A one could actually perform a Choleski factorization to give:

$$A = L L^T \quad (34)$$

and the system (31) could then be solved in two steps :

$$\begin{aligned} y &= L^{-1} b \\ x &= (L^T)^{-1} y \end{aligned} \quad (35)$$

Unfortunately if one performs the complete Choleski factorization the number of non zero elements in the original matrix increases in a more or less unpredictable manner, which again prevents its use for large systems. Kershaw proposed the use of the so called incomplete Choleski factorization in order to provide an approximate inverse of A to be used as a conditioning factor in the conjugate gradient algorithm. The Choleski factorization is carried out only for the non-zero elements of the matrix A : this is always possible, A being a Stieltjes matrix. Matrix A is thus decomposed as :

$$A \approx M M^T \quad (36)$$

where M is a lower triangular matrix which non-zero elements correspond to the non-zero elements of A (at most $nn + np$). Using M the recursive modified conjugate gradient becomes:

$$\begin{aligned} r_0 &= Ax_0 - b & k = 0 \\ p_0 &= r_0 \\ \alpha_k &= -r_k^T p_k / p_k^T A p_k \\ x_{k+1} &= x_k + \alpha_k p_k & k > 0 \\ r_{k+1} &= r_k + \alpha_k A p_k \\ s_{k+1} &= (M^T)^{-1} M^{-1} r_{k+1} \\ \beta_{k+1} &= -s_{k+1}^T A p_k / p_k^T A p_k \\ p_{k+1} &= s_k + \beta_{k+1} p_k \end{aligned} \quad (37)$$

The convergence properties of this or similar schemes (convergence is generally reached in 30-50 iterations even

for extremely large systems), discussed in Kershaw, 1978, and in Ajiz and Jennings, 1984, combined to the limited number of operations required (linearly proportional to nn) and to the limited amount of computer memory required (512 kb for nn = 1000, np = 2000) make this scheme particularly useful for the solution of large and complex networks on small machines such as the increasingly available Personal Computers.

4. CONCLUSIONS

It has been shown that in order to find the solution of the system of partly linear and partly non-linear equations describing the network flow problem, the application of the Newton-Raphson technique in the space of both unknown pipe flows and unknown nodal heads, where the existence and unicity of the solution can be proved, leads to an extremely convergent scheme. The scheme which requires the iterative solution of a system of linear equations compares favourably towards the linear theory and the loop Newton-Raphson procedures. Practical applications of this scheme, which can handle at the same time looped and open networks, enlightened its convergence properties (Salgado et al., 1987), which are not affected neither by the choice of an initial guess solution nor by the complexity of the network. In addition the special characteristics of the resulting system of linear equations, which can be solved at each gradient step by means of the Modified Conjugate Gradient/Incomplete Choleski Factorization algorithm, allows for its use on Personal Computers even in the case of extremely large and complex networks.

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