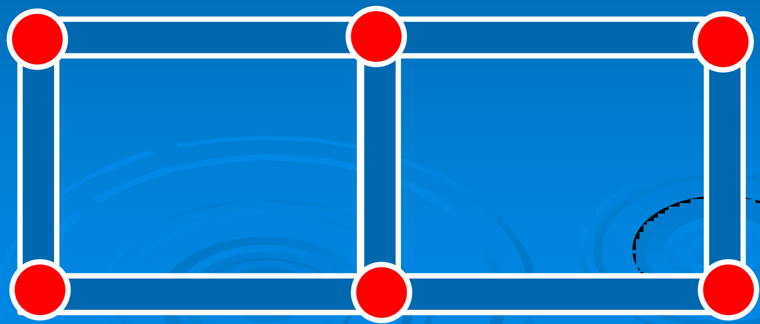
Water Pipe Networks

Water Distribution Analysis Via Excel

Lecture 2



Pipe Network Definition

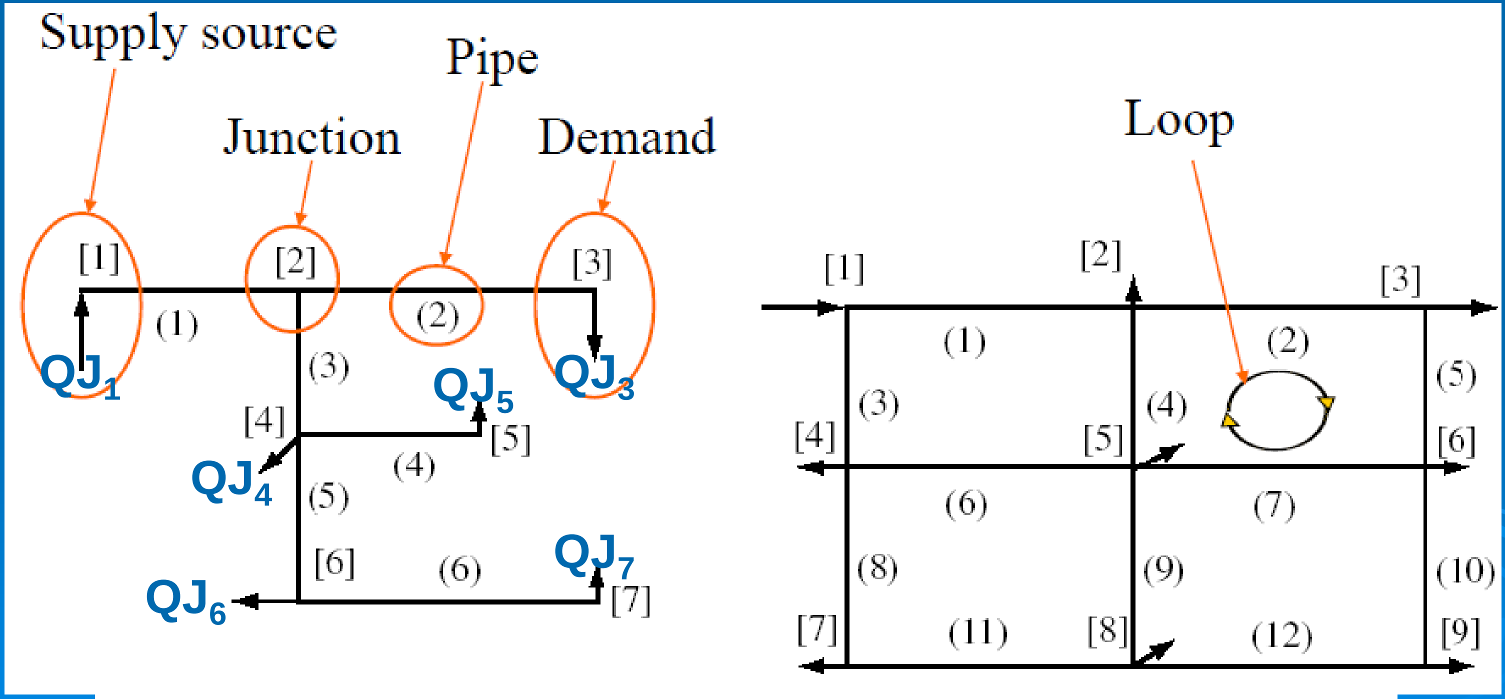
Before Discussing How to Solve the Flow in Pipe Network, Let's Define the Following:

-Pipe;

-Node;

-Loop;

-Demand and Supply (QJ).



(a) A small branched system. 6 pipes, 7 nodes

(b) A small looped system. 12 pipes, 9 nodes

Objective of this Session

•The objective of this session is to examine the use of Excel to analyze a water distribution network.
•Excel is a commonly available spreadsheet package that has been widely used as a computational tool in almost all engineering applications.
Despite the demonstrated examples are simple, they enables trainees to analyze realistic applications while still requiring manual development of the governing equations to reinforce the underlying engineering principles.
•It is also believed that such full understanding should come first and before getting first hand on the application of commercial pipe network package.

Progress in Network Analysis

- Pre Computer Age:

- Graphical approaches:
- Hardy Cross Developed his famous method of solving pipe networks in 1936.

- The Dawn of the Computer Age:

- In 1957, Hoag and Weinberg adapted Hardy Cross approach in digital computers.

- Advanced Computer Methods

Steps of Network Analysis

In order to hydraulically analyze a given network, two steps should be conducted:

-Step 1: Formulation of the governing equations;
-Step 2: Numerical solution of the obtained equations.

Formulation of Network Equations

Different approaches are found in the literature for the formulation of the network equations. Examples of these approaches are:

- Using Junction equations;
- Using loop equations;
- Using pipe equations;
- Using a mix.

Based on the above approaches, the following methods have been developed:

- Q-Method: Solving for Pipe flows as unknowns (Q_p);
- H-Method: Solving for Heads at junctions as unknowns (H_J);
- ΔQ-Method: Solving for Corrective flow rates as unknowns (ΔQ_p) where: ($Q_p = Q_o \pm \Delta Q_p$), Q_o is a previous pipe flow guess;
- ΔH-Method: Corrective heads at nodes as unknowns (ΔH_j) where: ($H_j = H_o \pm \Delta H_j$), H_o is a previous nodal head guess.

Solution of Network Equations

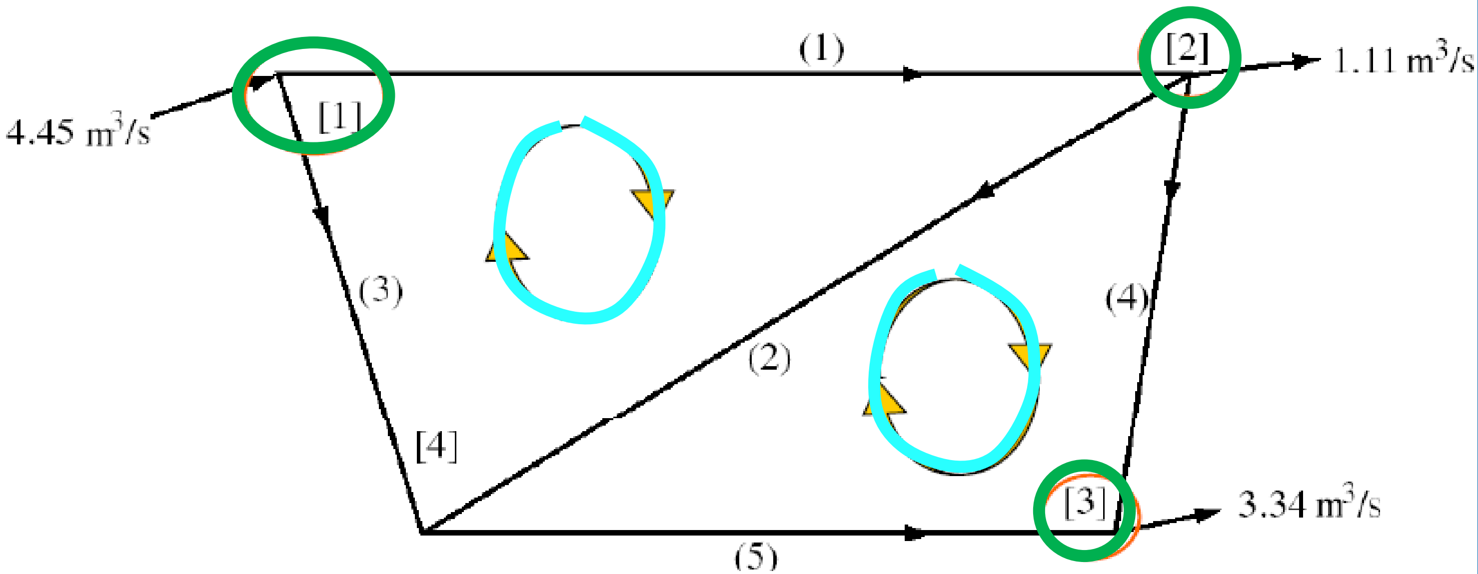
The resulted governing equations are nonlinear (continuity equations are linear whereas energy equations or loop equations are non-linear) thus we need methods to solve non-linear equations. Such methods include:

- By iteration using some correction formula (such as HCM);
- By minimization of a target function (we could use Excel solver);
- By linearization to convert equations into linear equations then use matrix manipulation for this regard with some iterations;
- Direct solution of non-linear equations by using Newton-Raphson method.

Formulation of Network Equations

Example of Q-Method

List all the governing equations to solve the below network using the Q-method.



$$\sum Q_{in} - \sum Q_{out} = 0$$

Node [1]: $Q_1 + Q_3 - 4.45 = 0$

Node [2]: $-Q_1 + Q_2 + Q_4 + 1.11 = 0$

Node [3]: $-Q_4 - Q_5 + 3.34 = 0$

Loop 1-2-3: $K_1 Q_1^n + K_2 Q_2^n - K_3 Q_3^n = 0$

Loop 4-5-2: $K_4 Q_4^n - K_5 Q_5^n - K_2 Q_2^n = 0$

Five Equations in Five Unknowns

H-Method

- *H*-equations (assume head as unknowns)
  - Solve the exponential equation for the flow

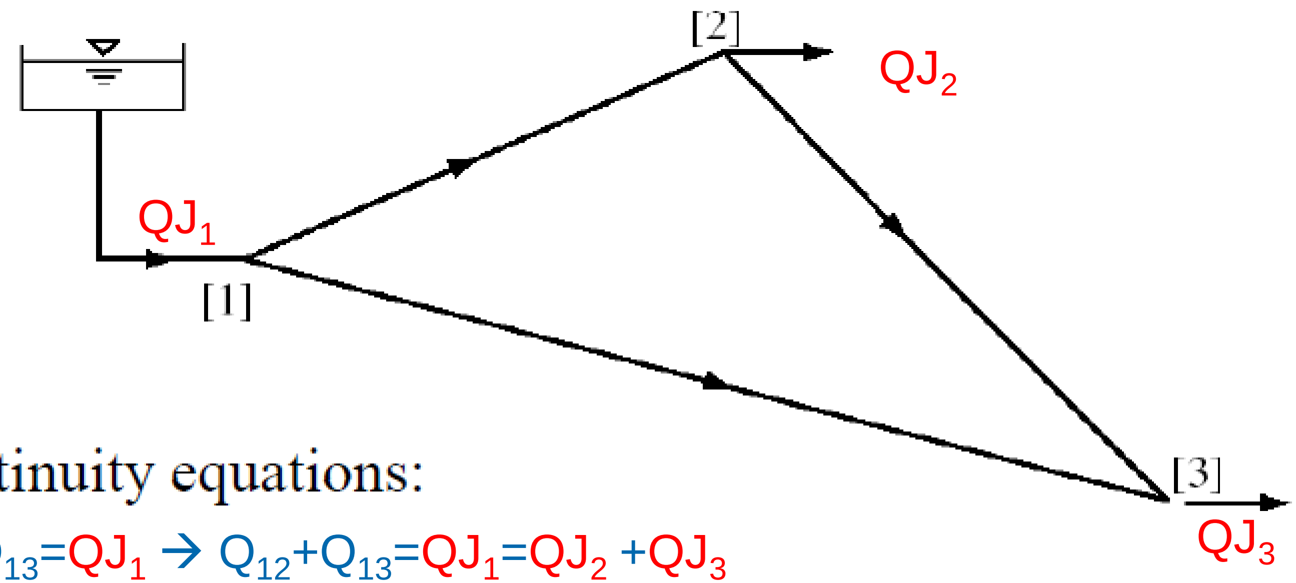
$$Q_{ij} = (h_{f\,ij} / K_{ij})^{1/n_{ij}} = [(H_i - H_j)/K_{ij})]^{1/n_{ij}}$$

    - Subscript ij = for the pipe from node i to node j
  - Substitute the above into junction continuity equ.

$$QJ_j - \sum\{[(H_i - H_j)/K_{ij})]^{1/n_{ij}}\}_{in}$$

$$+\sum\{[(H_i - H_j)/K_{ij})]^{1/n_{ij}}\}_{out} = 0$$

Example of H-Method



Continuity equations:

$Q_{12}+Q_{13}=QJ_1 \rightarrow Q_{12}+Q_{13}=QJ_1=QJ_2+QJ_3$

$Q_{12}=Q_{23}+QJ_2 \rightarrow Q_{12}-Q_{23}=QJ_2$

$Q_{13}+Q_{23}=QJ_3 \rightarrow Q_{13}+Q_{23}=QJ_3$

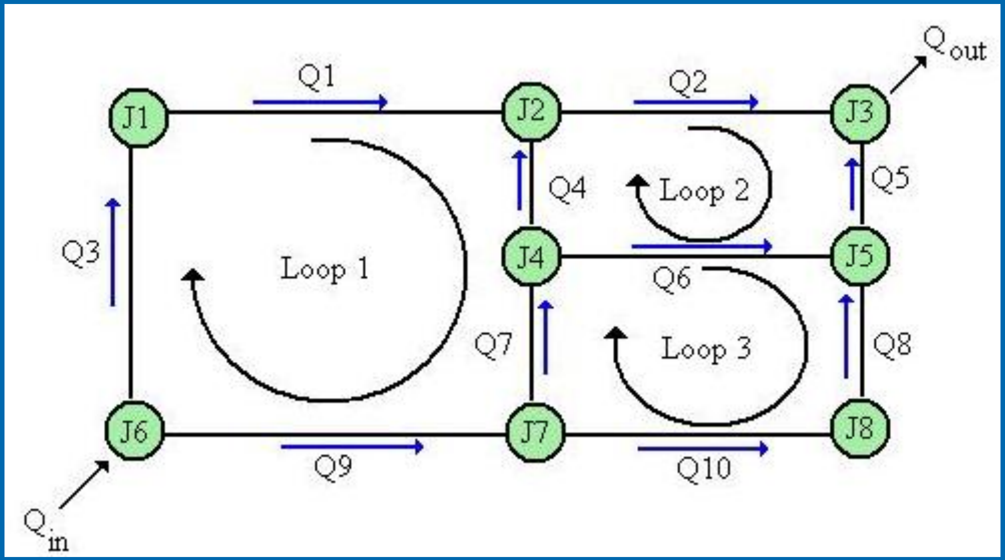
By using the Head expression and substitute back, the above equations reduce to:

$$\left.\begin{aligned}
\left[\frac{H_1-H_2}{K_{12}}\right]^{\frac{1}{2}}+\left[\frac{H_1-H_3}{K_{13}}\right]^{\frac{1}{2}}&=QJ_2+QJ_3\\
\left[\frac{H_1-H_2}{K_{12}}\right]^{\frac{1}{2}}-\left[\frac{H_2-H_3}{K_{23}}\right]^{\frac{1}{2}}&=QJ_2\\
\left[\frac{H_1-H_3}{K_{13}}\right]^{\frac{1}{2}}+\left[\frac{H_2-H_3}{K_{23}}\right]^{\frac{1}{2}}&=QJ_3
\end{aligned}\right\}\ \text{Three Equations in Three Unknowns}$$

Hardy Cross Method (HCM)

Example of ΔQ method

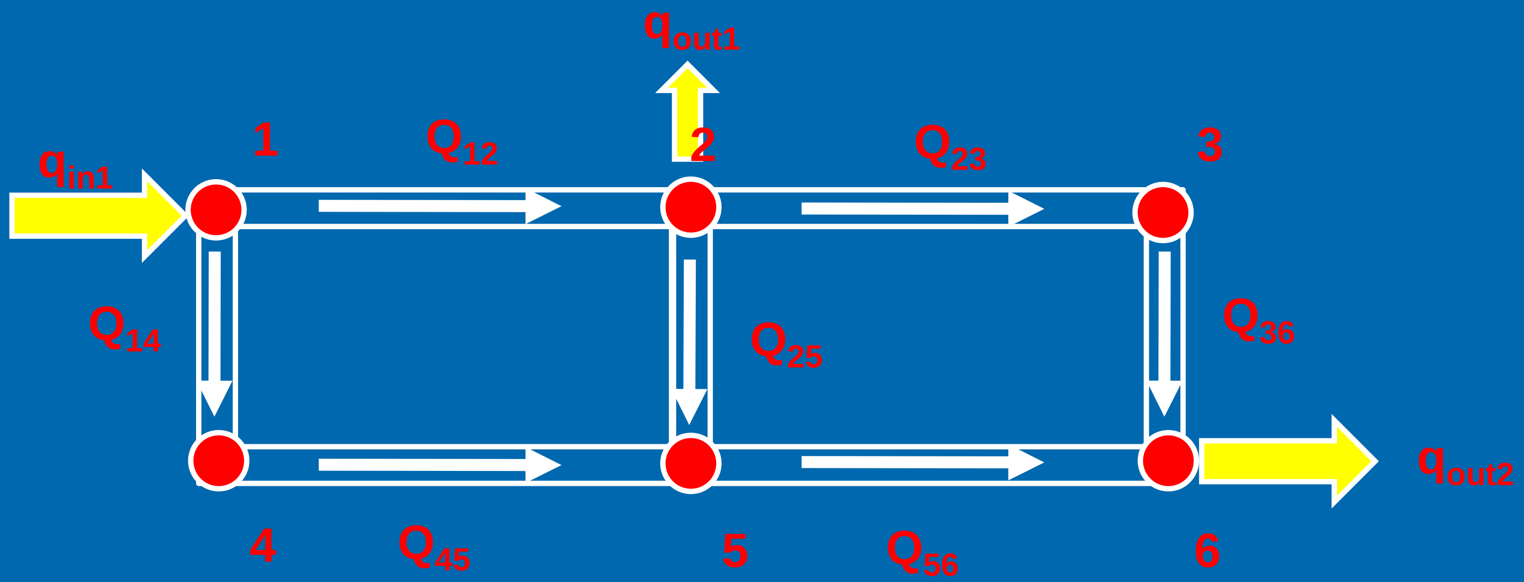
Hardy Cross 1936



The method was first published in November 1936 by Hardy Cross, a structural engineering professor at the University of Illinois at Urbana–Champaign.The Hardy Cross method is an adaptation of the Moment distribution method, which was also developed by Hardy Cross as a way to determine the moments in indeterminate structures.

Governing Equations

a. Conservation of Mass (Nodal Equations)



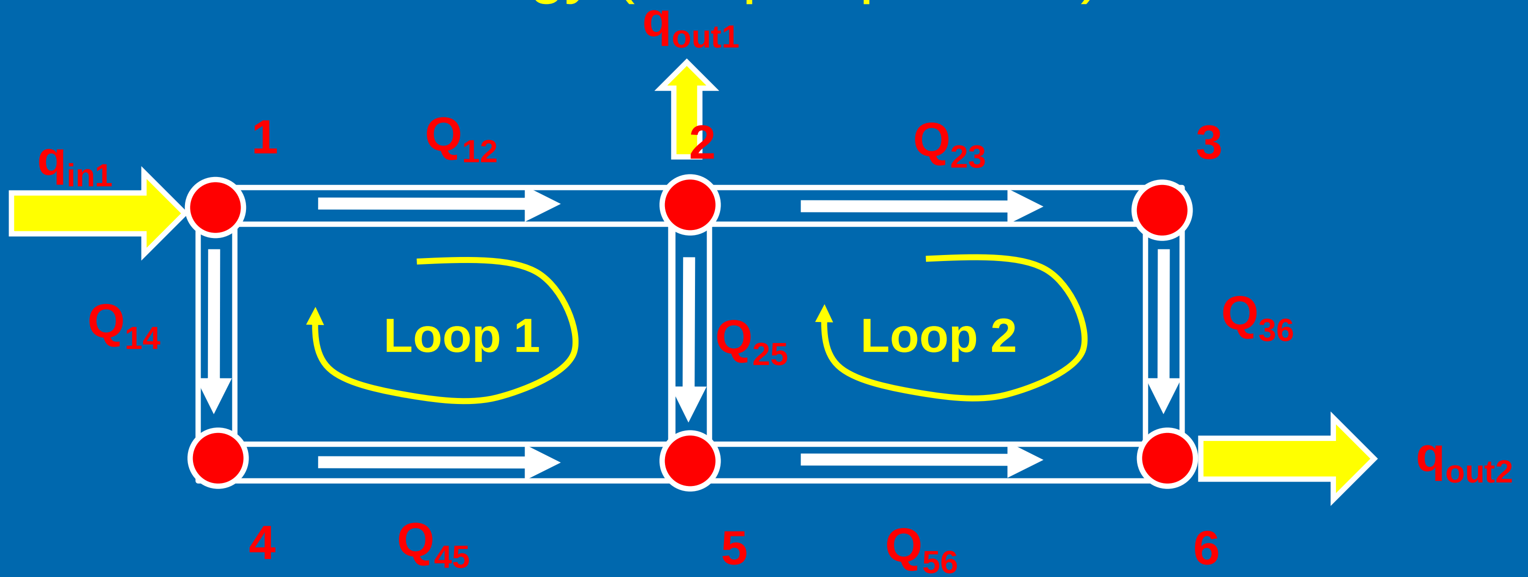
Hardy Cross Method requires an initial guesses of all pipe flows under the condition that such guesses satisfy the conservation of mass at each node.

For example: Each **Node** We Could Write a Mass Conservation Equation:

Example @ Node 2: $Q_{12}=q_{out1}+Q_{23}+Q_{25}$

Governing Equations

b. Conservation of Energy (Loop Equations)



For Each closed **Loop,** the summation of head lost should vanish:

Example @ Loop 1: $H_{L12}+H_{L25}+H_{L54}+H_{L41}$

For Each **Link** We Could Write the head lost:

Example @ Link1-2: $H_{L12}= K_{12}.Q_{12}|Q_{12}|$ Where: $K = \left(\dfrac{8\,fL}{gD^5\pi^2} \right)$

Thus, For Loop 1: $K_{12}.Q_{12}|Q_{12}|+ K_{25}.Q_{25}|Q_{25}|+ K_{54}.Q_{54}|Q_{54}|+ K_{41}.Q_{41}|Q_{41}|=0$

Solution Steps Using HCM

Solution of Pipe Network via HCM is iterative as follow:

1. Consider a positive flow direction for all loops (say clock wise direction is positive;
2. Assume flow discharges for all pipes satisfying the mass conservation at each node;
3. Calculate a first approximation of the flow correction for each loop using the following equation given by Hardy Cross:

$$\Delta Q = -\frac{\sum K_i\, Q_i^2}{\sum |2K_i Q_i|}$$

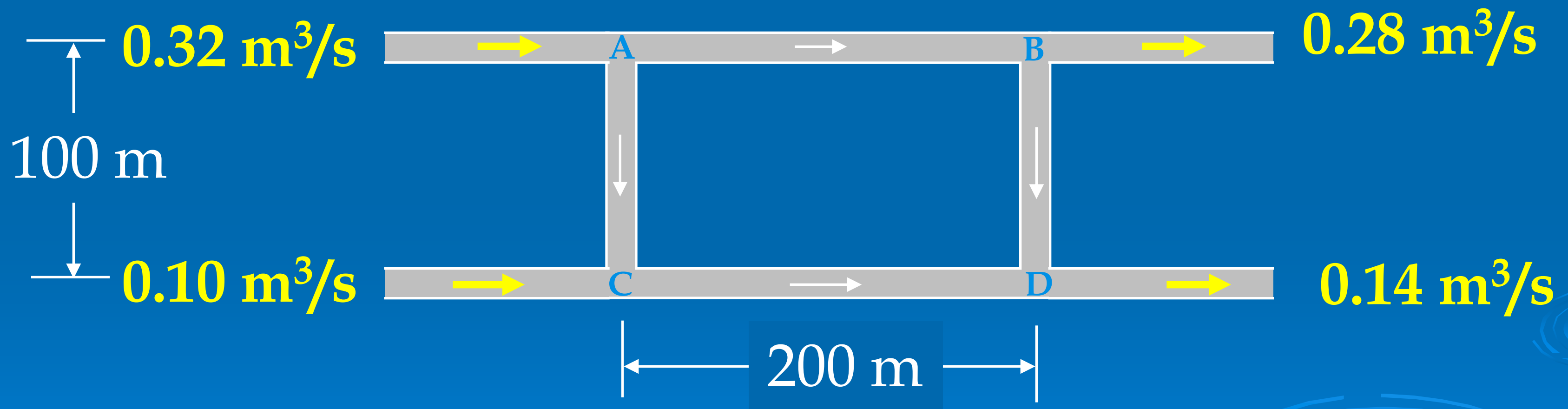
4. Calculate the corrected Q and iterate till corrections vanish.

An Alternative Approach…
Using Excel Solver

An alternative approach to avoid using the flow correction equation given by Hardy Cross is to directly use the Excel Solver to obtain the suitable loop flow correction DQ that is required to make the summation of the head losses across any loop equal zero.

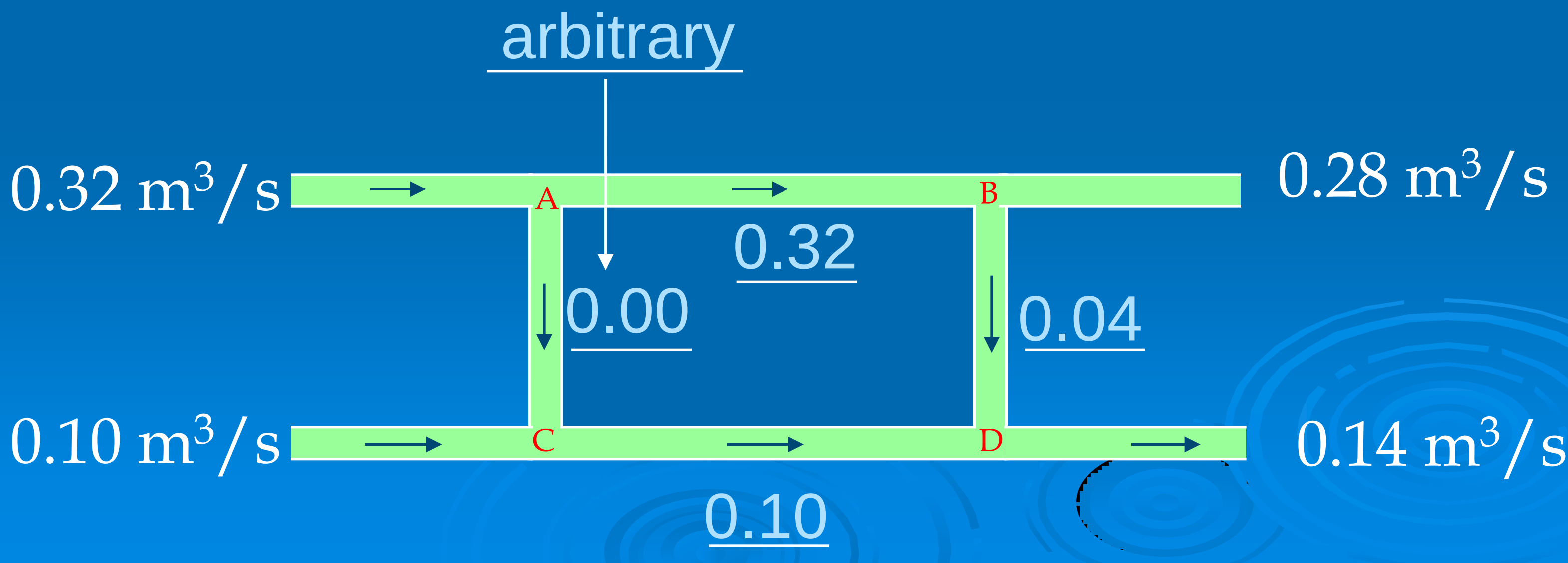
Example of Network Analysis Using HCM via Excel

Find the flows in the loop given the inflows and outflows. The pipes are all 250 mm cast iron (ε=0.26 mm).



Example of Network Analysis Using HCM via Excel (Cont.)

- Assign a flow to each pipe link
- Flow into each junction must equal flow out of the junction



Example of Network Analysis Using Hardy Cross Method

- Calculate the head loss in each pipe

$$h_f = \left(\frac{8fL}{gD^5\pi^2}\right)Q^2$$

f=0.02 for Re>200000

$$h_f = kQ|Q|$$

Sign convention +CW

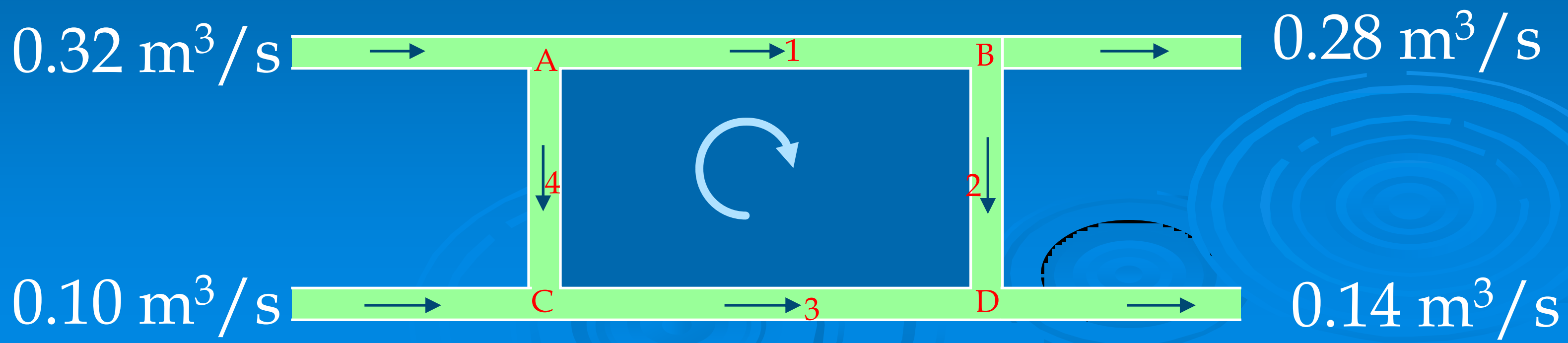
$$k_1 = \left(\frac{8(0.02)(200)}{(9.8)(0.25)^5\pi^2}\right) = 339\,\frac{s^2}{m^5}$$

$k_1, k_3 = 339$

$k_2, k_4 = 169$

$$h_{f_1} = 34.7m$$
$$h_{f_2} = 0.222m$$
$$h_{f_3} = -3.39m$$
$$h_{f_4} = -0.00m$$

$$\sum_{i=1}^{4} h_{f_i} = 31.53m$$



Example of Network Analysis Using Hardy Cross Method

- The head loss around the loop isn't zero
- Need to change the flow around the loop
  - the clockwise flow is too great (head loss is positive)
  - reduce the clockwise flow to reduce the head loss
- Solution techniques
  - Hardy Cross loop-balancing (Optimizes Correction)
  - Use a numeric solver (Solver in Excel) to find a change in flow that will give zero head loss around the loop
  - Use Network Analysis software (EPANET / WATERCAD)

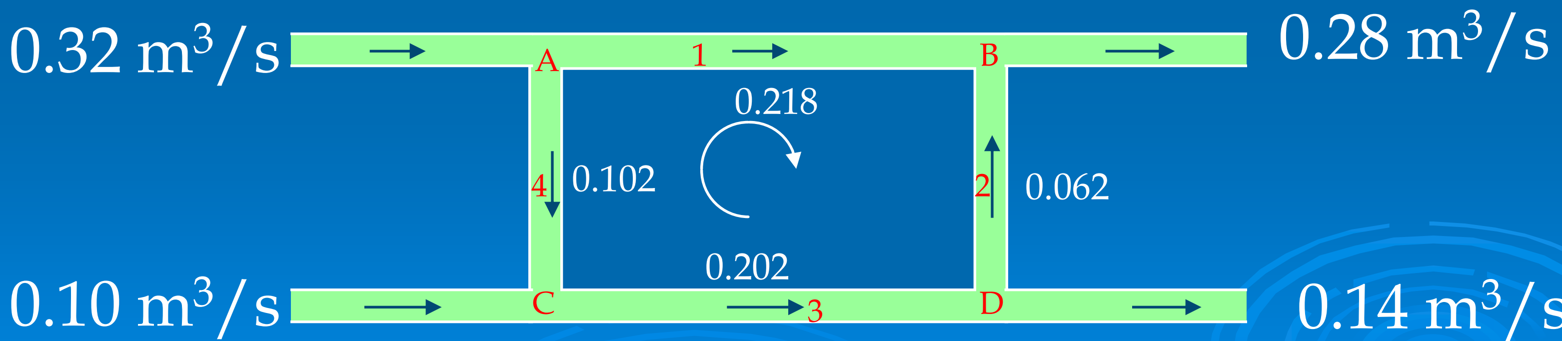
Hardy Cross Using Excel Numeric Solver

- Set up a spreadsheet as shown below.
- the numbers in **bold** were entered, the other cells are calculations
- initially ΔQ is 0
- use "solver" to set the sum of the head loss to 0 by changing ΔQ
- the column $Q_0 + \Delta Q$ contains the correct flows

ΔQ	0.000						
pipe	f	L	D	k	Q_0	$Q_0+\Delta Q$	h_f
P1	**0.02**	**200**	**0.25**	339	**0.32**	0.320	34.69
P2	**0.02**	**100**	**0.25**	169	**0.04**	0.040	0.27
P3	**0.02**	**200**	**0.25**	339	**-0.1**	-0.100	-3.39
P4	**0.02**	**100**	**0.25**	169	**0**	0.000	0.00
						Sum Head Loss	31.575

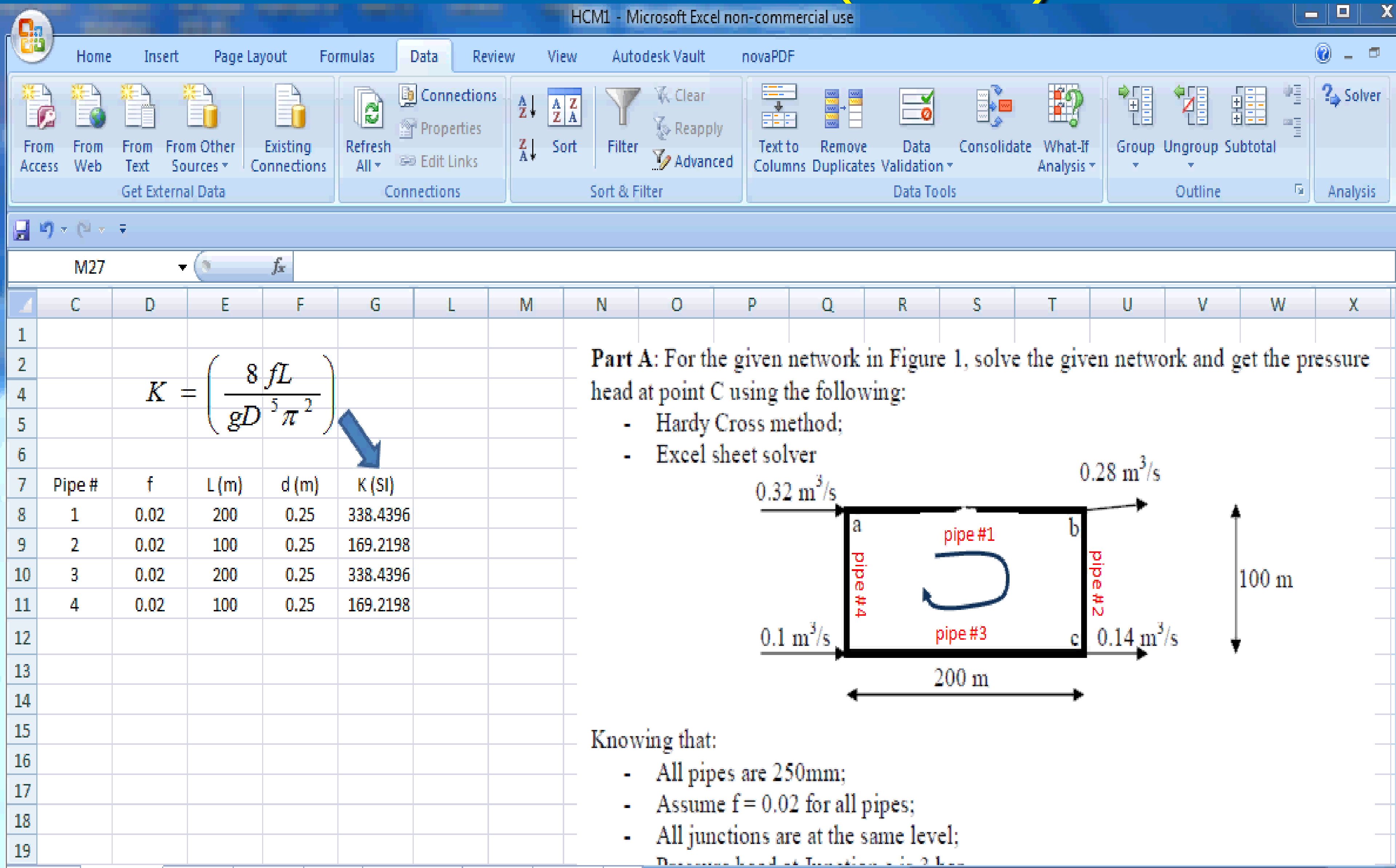
Solution to Loop Problem

$Q_0 + \Delta Q$
0.218
–0.062
–0.202
–0.102



Better solution is software with a GUI showing the pipe network.

Solution of A Single Loop Problem Using HCM Via Excel (Video)



$$K = \left(\frac{8\, fL}{gD^5 \pi^2} \right)$$

Pipe #	f	L (m)	d (m)	K (SI)
1	0.02	200	0.25	338.4396
2	0.02	100	0.25	169.2198
3	0.02	200	0.25	338.4396
4	0.02	100	0.25	169.2198

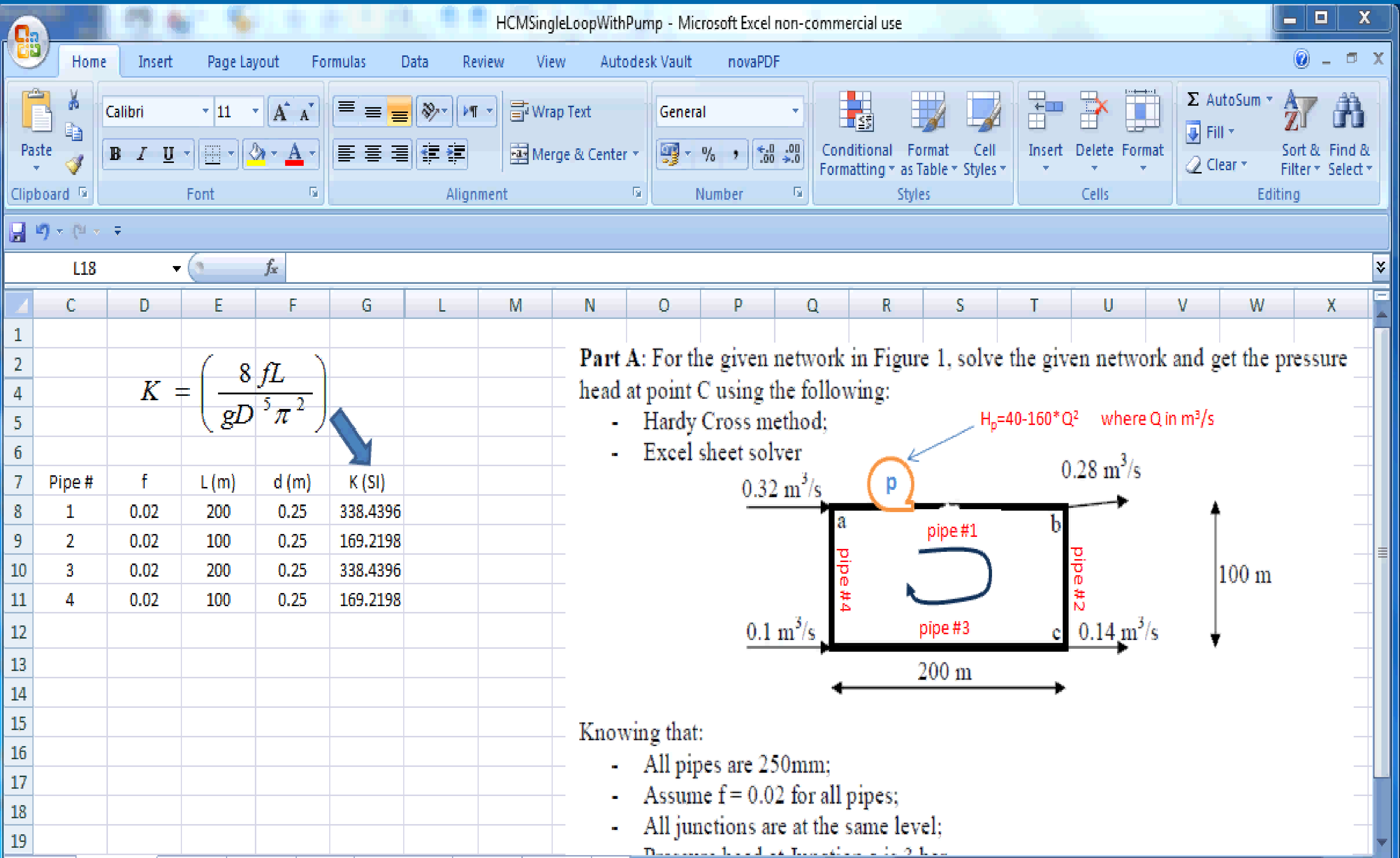
Part A: For the given network in Figure 1, solve the given network and get the pressure head at point C using the following:

- Hardy Cross method;
- Excel sheet solver

Knowing that:

- All pipes are 250mm;
- Assume f = 0.02 for all pipes;
- All junctions are at the same level;

Solution of a Single Loop Problem with a Pump Using HCM Via Excel (Video)



Solution of a Multiple Loops Problem Using HCM Via Excel (Video)

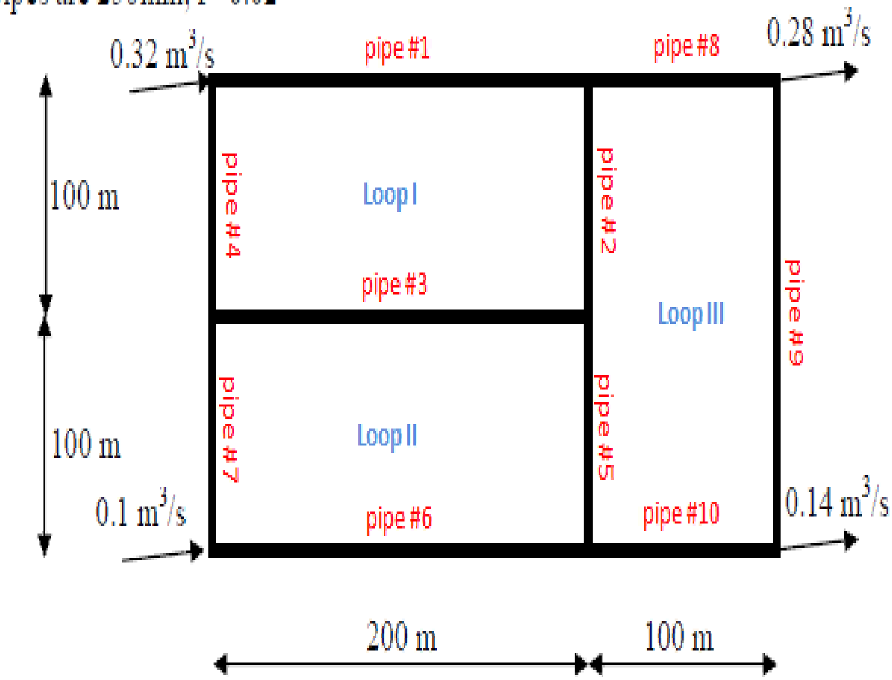
$$K = \left(\frac{8 fL}{g D^5 \pi^2} \right)$$

Loop	Pipe #	L(m)	D(m)	f	K
I	1	200	0.25	0.02	338.4396
	2	100	0.25	0.02	169.2198
	3	200	0.25	0.02	338.4396
	4	100	0.25	0.02	169.2198
II	3	200	0.25	0.02	338.4396
	5	100	0.25	0.02	169.2198
	6	200	0.25	0.02	338.4396
	7	100	0.25	0.02	169.2198
III	8	100	0.25	0.02	169.2198
	9	200	0.25	0.02	338.4396
	10	100	0.25	0.02	169.2198
	5	100	0.25	0.02	169.2198
	2	100	0.25	0.02	169.2198

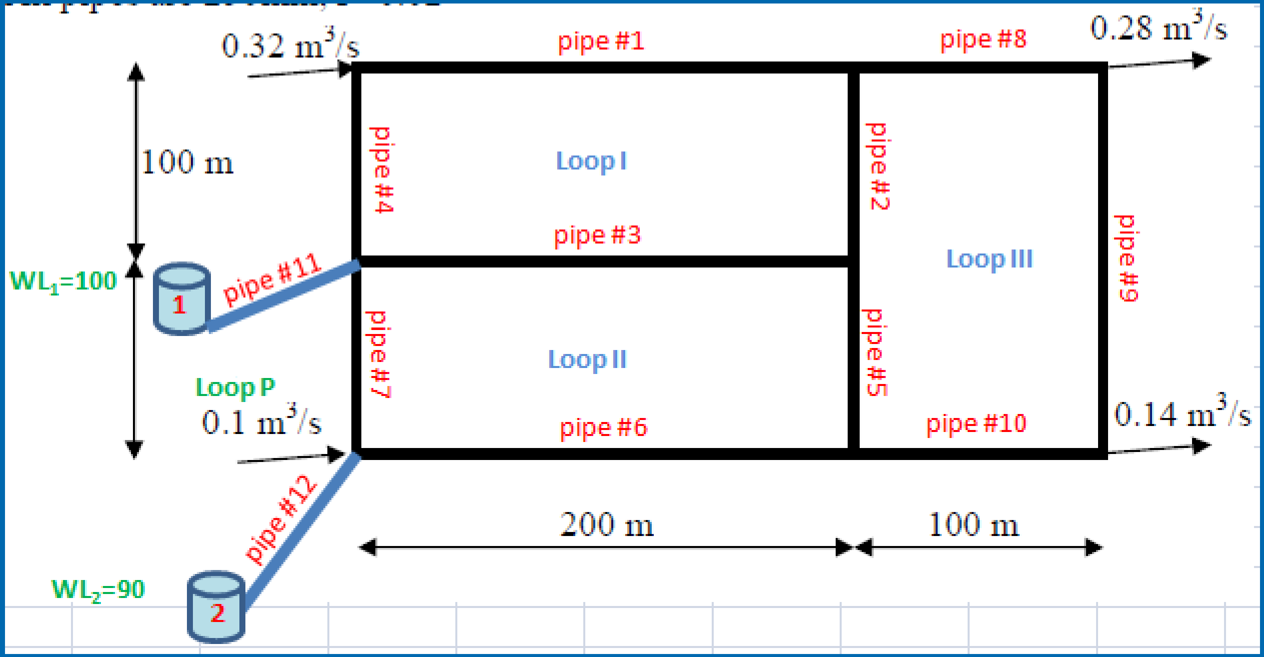
Solve the following network using:

- Excel solver;
- Modified Hardy-Cross Jacobian Matrix Approach taken in class

All pipes are 250mm, f= 0.02

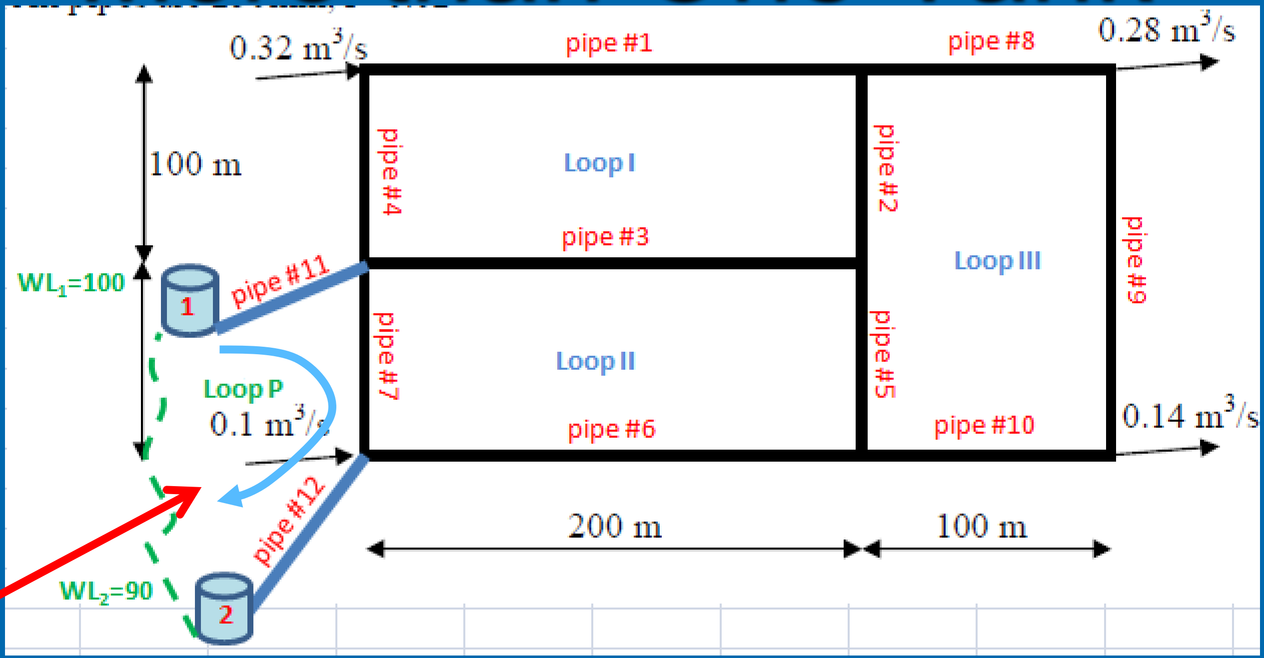


Solution of a Network Including More than One Tank



The common practice is to form a pseudo loop that includes the given tanks as follow:

Solution of a Network Including More than One Tank



Pseudo Loop

The total head lost equation in such pseudo loop could be written as:

The water level between the two tanks should balance the summation of head lost through the pseudo loop.

$WL_1 - WL_2 = H_{L11} + H_{L7} + H_{L12}$ ➡ $WL_1 - WL_2 - [H_{L11} + H_{L7} + H_{L12}] = 0$

Solution of a Network with Tanks- Pseudo Loop Approach via HCM (Video)

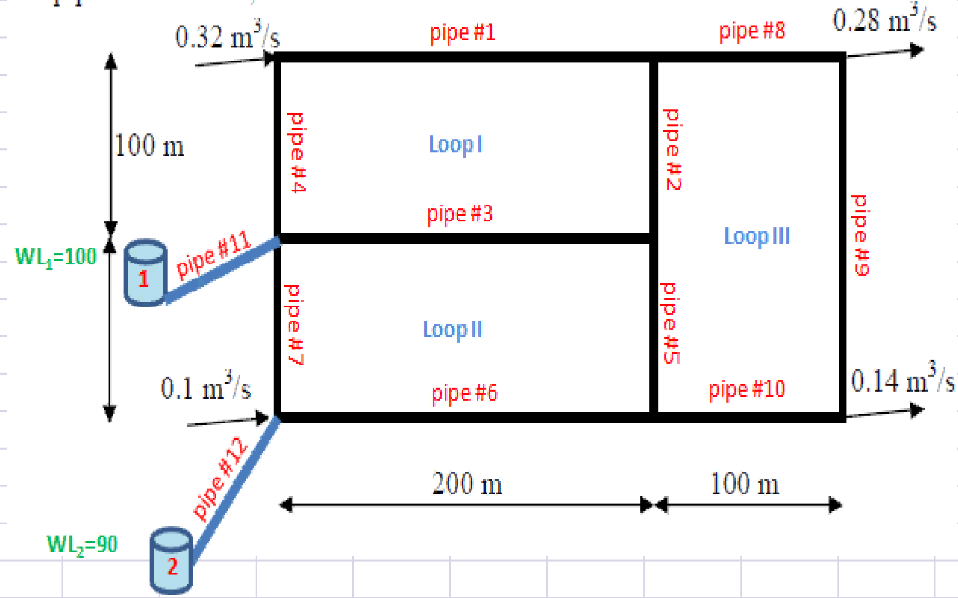
$$K = \left(\frac{8\,fL}{gD^5\pi^2} \right)$$

Loop	Pipe #	L(m)	D(m)	f	K
I	1	200	0.25	0.02	338.4396
	2	100	0.25	0.02	169.2198
	3	200	0.25	0.02	338.4396
	4	100	0.25	0.02	169.2198
II	3	200	0.25	0.02	338.4396
	5	100	0.25	0.02	169.2198
	6	200	0.25	0.02	338.4396
	7	100	0.25	0.02	169.2198
III	8	100	0.25	0.02	169.2198
	9	200	0.25	0.02	338.4396
	10	100	0.25	0.02	169.2198
	5	100	0.25	0.02	169.2198
	2	100	0.25	0.02	169.2198

Solve the following network using:

- Excel solver;
- Modified Hardy-Cross Jacobian Matrix Approach taken in class

All pipes are 250mm, f= 0.02



Solving HCM Using Matrices and Network Jacobian

- This method is also called as simultaneous loops equations in terms of ΔQ_i;
- The loops equations could be written as:

$$[J].\{\Delta Q\} = \{-\Sigma H_L\}$$

Jacobian of Loops Network

Unknown Corrections of Flow in Each Loop (including Pseudo loops)

Negative Sum of Head Lost in Each Loop

N_{loops} xN_{loops}

N_{loops} x1

N_{loops} x1

Solving HCM Using Matrices and Network Jacobian

What is the Jacobian Matrix [J] ?

- [J] is a square and symmetric matrix
- [J] has dimensions of $N_{loops} x N_{loops}$

  Where: N_{loops} = No. of actual Loops +No. of pseudo loops

- [J] represents the first derivatives of the loops head lost functions evaluated at known Q values, i.e. $\delta(H_L)/\delta(Q)$ evaluated at Q_{old} , where H_L is the head lost function.
- Using Darcy, the head lost function is .$H_L = KQ^2$
- Then, $|\delta(H_L)/\delta(Q)| = 2KQ = 2KQ^2/Q = |2H_L/Q|$

Solving HCM Using Matrices and Network Jacobian

What is the Jacobian Matrix [J] ?

- Accordingly, [J] could be written as:

$$[J]=\begin{bmatrix} |\partial H_L/\partial Q|_1 & -|2H_L/Q|_{1-2} & -|2H_L/Q|_{1-3} \\ \square & |\partial H_L/\partial Q|_2 & -|2H_L/Q|_{2-3} \\ \square & \square & |\partial H_L/\partial Q|_3 \end{bmatrix}$$

- Or,

$$[J]=\begin{bmatrix} \sum_{Loop1}|2H_L/Q| & -|2H_L/Q|_{1-2} & -|2H_L/Q|_{1-3} \\ \square & \sum_{Loop2}|2H_L/Q| & -|2H_L/Q|_{2-3} \\ \square & \square & \sum_{Loop3}|2H_L/Q| \end{bmatrix}$$

Symmetric Non Zero Elements

- Remember, [J] matrix is a symmetric matrix

Solving HCM Using Matrices and Network Jacobian

What is the Jacobian Matrix [J] ?

$$[J]=\begin{bmatrix} \sum_{Loop1}|2H_L/Q| & -|2H_L/Q|_{1-2} & -|2H_L/Q|_{1-3} \\ \square & \sum_{Loop2}|2H_L/Q| & -|2H_L/Q|_{2-3} \\ \square & \square & \sum_{Loop3}|2H_L/Q| \end{bmatrix}$$

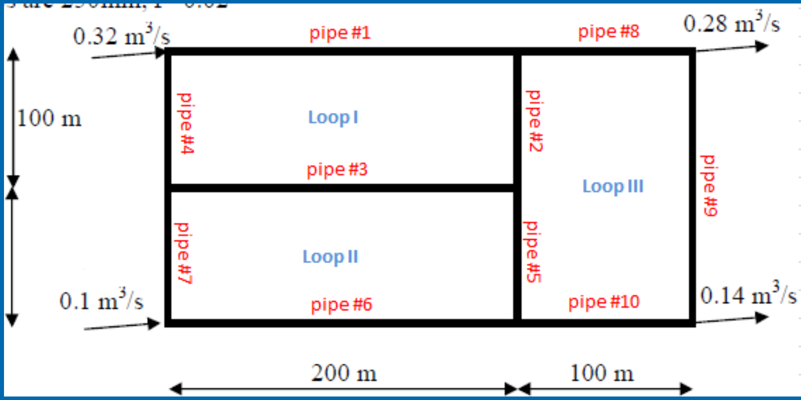
-The off-diagonal elements of the [J] matrix represent the negative gradient of the head lost function for the pipes in common between different loops?

- For instance: the term $-|2H_L/Q|_{1-2}$ represents the negative head-lost function for the pipe in common between loop 1 and loop 2.

Solving HCM Using Matrices and Network Jacobian

Example of Matrix [J] ?

$$[J]=\begin{bmatrix} \sum_{Loop1}|2H_L/Q| & -|2H_L/Q|_{1-2} & -|2H_L/Q|_{1-3} \\ & \sum_{Loop2}|2H_L/Q| & -|2H_L/Q|_{2-3} \\ & & \sum_{Loop3}|2H_L/Q| \end{bmatrix}$$



-For example, as per the shown pipe network loops:

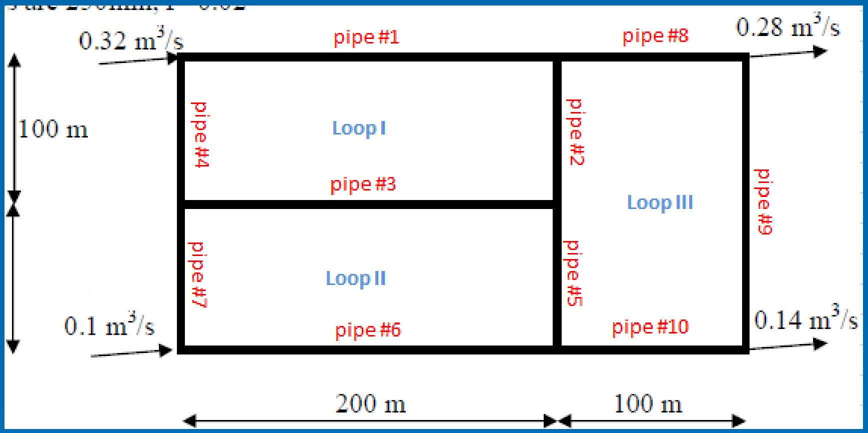
$\sum_{Loop2}|2H_L/Q|$ $= 2|H_{L3}/Q_3|+2|H_{L5}/Q_5|+2|H_{L6}/Q_6|+2|H_{L7}/Q_7|$

$-|2H_L/Q|_{1-2}$ $=-2|H_{L3}/Q_3|$ $-|2H_L/Q|_{2-3}$ $=-2|H_{L5}/Q_5|$

$-|2H_L/Q|_{1-3}$ $=-2|H_{L2}/Q_2|$

Solving HCM Using Matrices and Network Jacobian

Creation of the Negative Sum of Head Lost Vector $\{-\Sigma H_L\}$:



For the shown three loops:

$$\{-\Sigma H_L\} = \begin{Bmatrix} -\sum H_{L\ loop1} \\ -\sum H_{L\ loop2} \\ -\sum H_{L\ loop3} \end{Bmatrix}$$

Solving HCM Using Matrices and Network Jacobian

- The loops equations could be solved to get the flow corrections DQ for each loop as follow:

$$[J].\{\Delta Q\} = \{-\Sigma H_L\}$$

Using Matrix Inverse:

$$\{\Delta Q\} = [J]^{-1} \{-\Sigma H_L\}$$

Using Excel for Matrix Manipulation

- Example 1: Matrix Multiplications:

If $A = \begin{bmatrix} -2 & 1 & 3 \\ -4 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 3 & -1 \\ 4 & -3 \end{bmatrix}$; Find $A.B$ and name the resulting matrix as E

Enter the matrices A and B anywhere into the Excel sheet as:

I15 =

	A	B	C	D	E	F	G	H
1			Matrix *A*				Matrix *B*	
2		-2	1	3			2	0
3		-4	0	5			3	-1
4							4	-3
5								

Notice that Matrix A is in cells **B2:D3,** and Matrix B in cells **G2:H4**

Using Excel for Matrix Manipulation

➢ Example 1: Matrix Multiplications (Cont.):

We multiply Row by Column and the first matrix has 2 rows and the second has 2 columns, so the resulting matrix will have 2 rows by 2 columns.. **Highlight** the cells where you want to place the resulting matrix *E*:

D7 =

	A	B	C	D	E	F	G	H
1			Matrix *A*				Matrix *B*	
2		-2	1	3			2	0
3		-4	0	5			3	-1
4							4	-3
5								
6				Matrix *E* = *A*.*B*				
7								
8								

Once you have highlighted the resulting matrix, and <u>while it is still highlighted</u>, enter the following formula:

=MMULT(B2:D3,G2:H4)

Using Excel for Matrix Manipulation

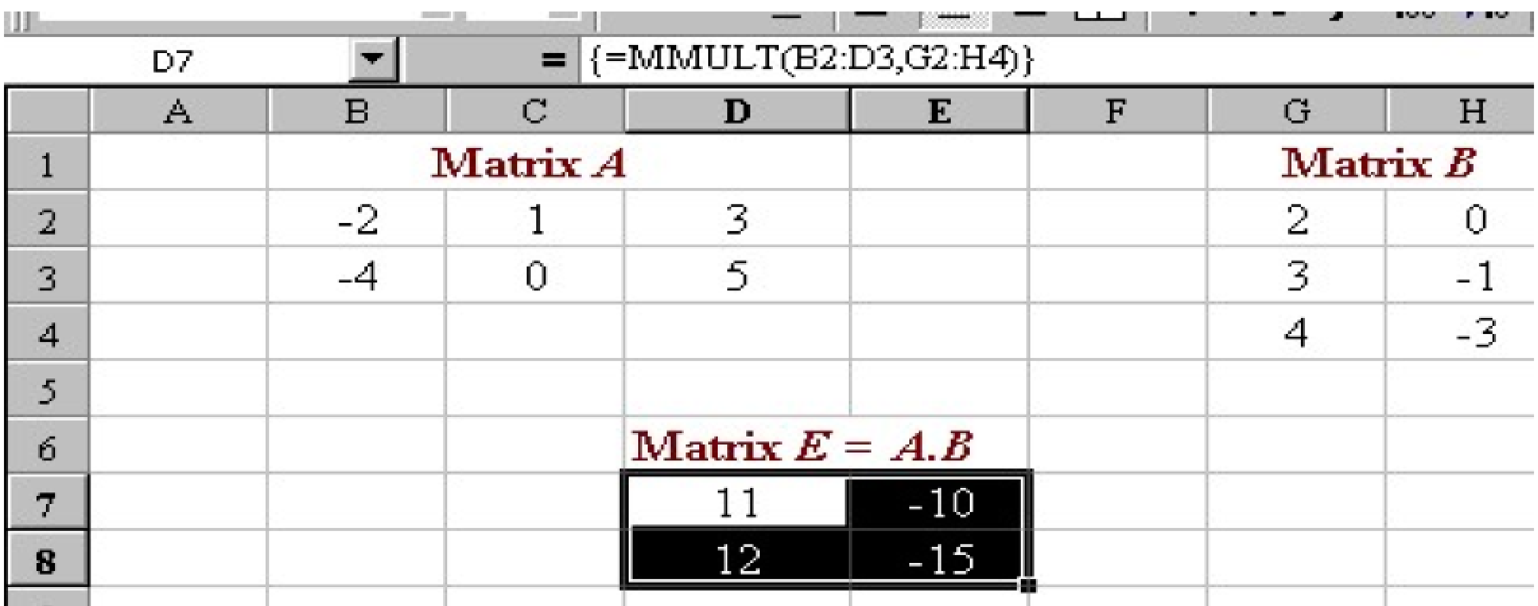
- Example 1: Matrix Multiplications (Cont.):

When the formula is entered, press the **Ctrl** key and the **Shift** key simultaneously, then press the **Enter** key. This will change the formula you just wrote to:

{=MMULT(B2:D3,G2:H4)}

If you don't press these keys simultaneously (holding down Shift and Ctrl then press Return), the result will appear only in one cell or, you will get some error message).

The resulting matrix will be:



D7 = {=MMULT(B2:D3,G2:H4)}

	A	B	C	D	E	F	G	H
1			Matrix *A*				Matrix *B*	
2		-2	1	3			2	0
3		-4	0	5			3	-1
4							4	-3
5								
6				Matrix *E* = *A.B*				
7				11	-10			
8				12	-15			

Using Excel for Matrix Manipulation

- Example 2: Matrix Inversion:

If $A = \begin{bmatrix} -2 & 1 & 3 \\ -4 & 0 & 5 \\ 3 & 5 & 2 \end{bmatrix}$; Find the inverse or A^{-1}

Enter the matrices A into the Excel sheet as:

	A	B	C	D
1			Matrix *A*	
2		-2	1	3
3		-4	0	5
4		3	5	2

Notice that Matrix A is in cells **B2:D4**

Using Excel for Matrix Manipulation

➢ Example 2: Matrix Inversion (Cont.):

We find the inverse of matrix *A* by **Highlighting** the cells where you want to place the resulting matrix A^{-1}

	A	B	C	D	E	F	G	H
1			**Matrix *A***			**Inverse Matrix A^{-1}**		
2		-2	1	3				
3		-4	0	5				
4		3	5	2				

Once you have highlighted the resulting matrix, and while it is still highlighted, enter the following formula:

= MINVERSE(B2:D4)

When the formula is entered, press the **Ctrl** key and the **Shift** key simultaneously, then press the **Enter** key. This will change the formula you just wrote to:

{= MINVERSE(B2:D4)}

If you don't press these keys simultaneously (holding down Shift and Ctrl then press Return), the result will appear only in one cell or, you will get some error message).

Using Excel for Matrix Manipulation

- Example 2: Matrix Inversion (Cont.):

The resulting matrix will be:

F2 = {=MINVERSE(B2:D4)}

	A	B	C	D	E	F	G	H
1			Matrix *A*				Inverse Matrix A^{-1}	
2		-2	1	3		-1.923	1	0.385
3		-4	0	5		1.769	-1	-0.15
4		3	5	2		-1.54	1	0.308

Limitations of Hardy Cross Method (HCM)

- You need first to guess the initial flow in all pipes and the initial pipe flow should satisfy the continuity equations at each node;
- It could take long period to converge especially for big systems;

- Some times it fails to converge;
- Original method was restricted to closed looped systems;
- Original method did not simulate pumps and valves;
- Its coding manipulation is not in the matrices form

Further Readings

Next Slides are out of scope of Final Exam

Are there any other method that does not require initial flow guess based on continuity and can be casted in a matrix form?

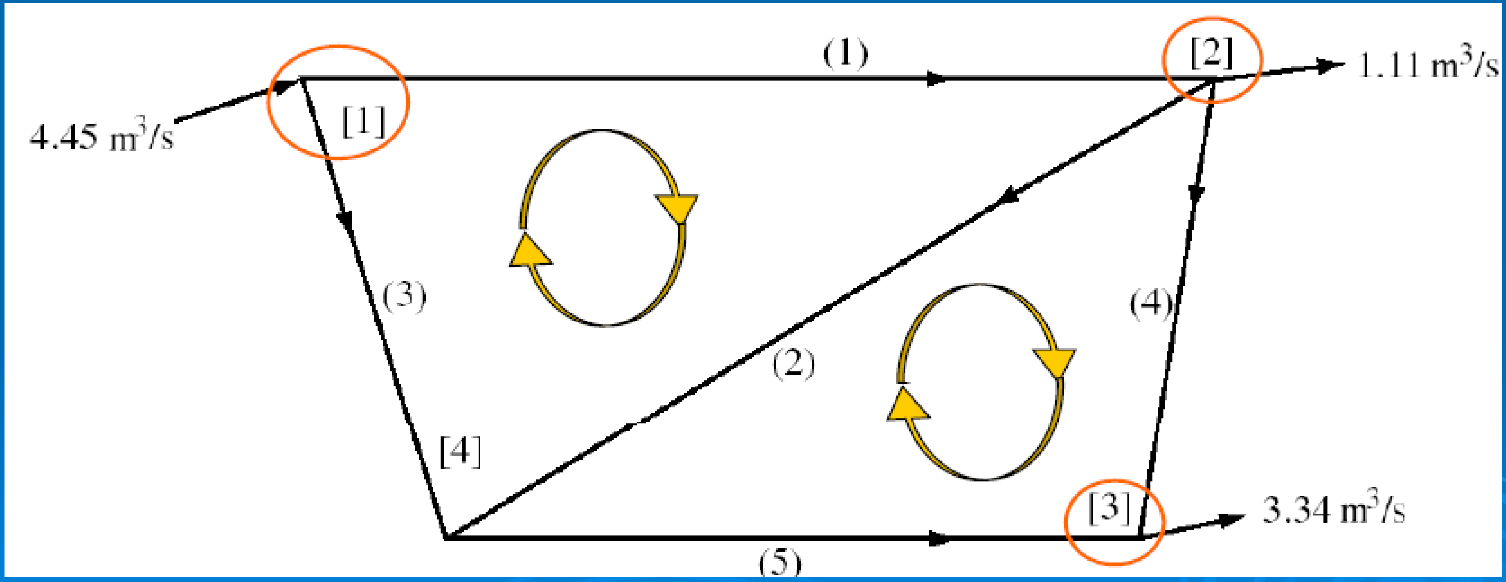
Progress in Solution Techniques of Pipe Network Equations

- Hardy Cross Method (Example of ΔQ method);
- The Simultaneous Node Method;
- The Simultaneous Loop Method;
- The Linear Method (Simultaneous Pipe Method, 1972);
- The Gradient Method (Simultaneous Network Method, 1987);

Using Optimization Algorithms.

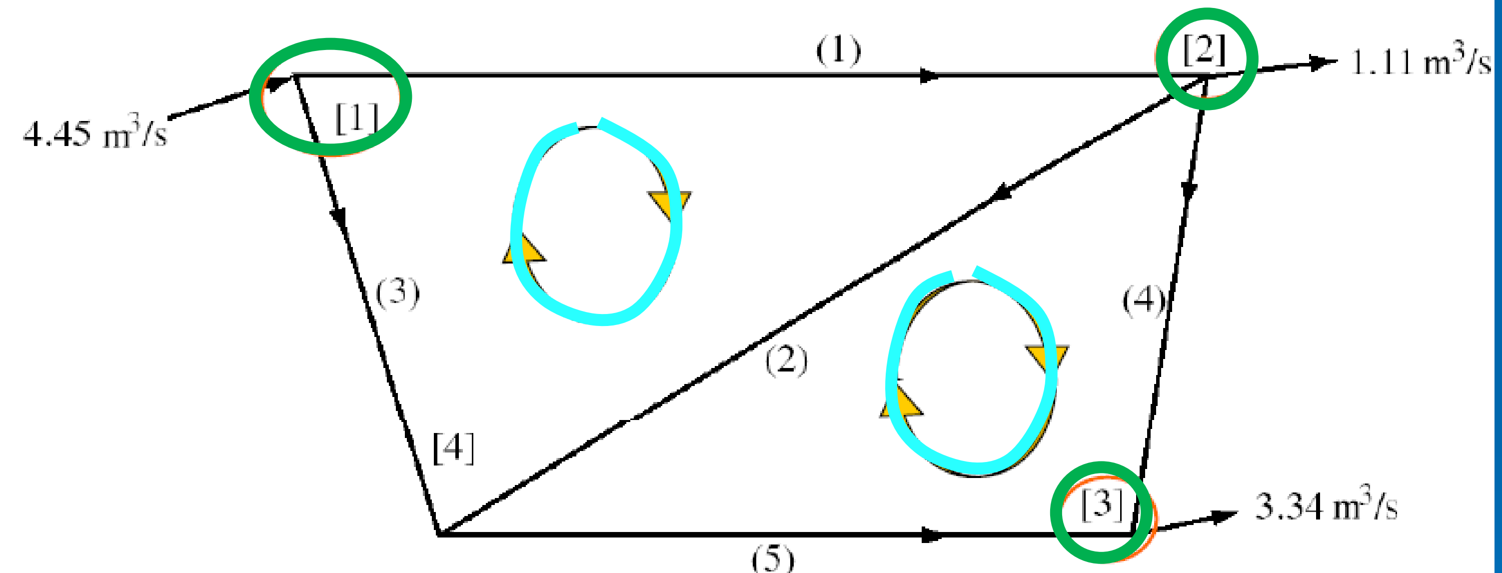
Solution of the Q-Method using Linearization

Let's formulate the governing equations of the given network using the Q-method then let's solve the obtained equations using linearization.



Solution of the Q-Method using Linearization (Cont.)

List all the governing equations to solve the below network using the Q-method.



$$\sum Q_{in} - \sum Q_{out} = 0$$

Node [1]: $Q_1 + Q_3 - 4.45 = 0$

Node [2]: $-Q_1 + Q_2 + Q_4 + 1.11 = 0$

Node [3]: $-Q_4 - Q_5 + 3.34 = 0$

Loop 1-2-3: $K_1 Q_1^n + K_2 Q_2^n - K_3 Q_3^n = 0$

Loop 4-5-2: $K_4 Q_4^n - K_5 Q_5^n - K_2 Q_2^n = 0$

Five Equations in Five Unknowns

Let us Look Carefully and Examine the Produced Equations…

Solution of the Q-Method using Linearization (Cont.)

Some of the governing equations are linear and others are non-linear.

Linear Equations

$$\left\{\begin{array}{l} \text{Node [1]: } Q_1 + Q_3 - 4.45 = 0 \\ \text{Node [2]: } -Q_1 + Q_2 + Q_4 + 1.11 = 0 \\ \text{Node [3]: } -Q_4 - Q_5 + 3.34 = 0 \end{array}\right.$$

$$\left.\begin{array}{l} \text{Loop 1-2-3: } K_1 Q_1^n + K_2 Q_2^n - K_3 Q_3^n = 0 \\ \text{Loop 4-5-2: } K_4 Q_4^n - K_5 Q_5^n - K_2 Q_2^n = 0 \end{array}\right\}$$

Nonlinear Equations

We have two avenues for the solution,:

-To deal with the set of equations as non-linear and to use Newton-Raphson or other iterative solvers to solve the equations, or,

-To linearize the equations and to make use of the efficient linear solvers.

Let us Go with Linearization

Solution of the Q-Method using Linearization (Cont.)

Let us carry out linearization of the following set of equations:

$$\text{Linear Equations}\left\{\begin{array}{l}\text{Node [1]: } Q_1 + Q_3 - 4.45 = 0\\ \text{Node [2]: } -Q_1 + Q_2 + Q_4 + 1.11 = 0\\ \text{Node [3]: } -Q_4 - Q_5 + 3.34 = 0\end{array}\right.$$

$$\left.\begin{array}{l}\text{Loop 1-2-3: } K_1Q_1^n + K_2Q_2^n - K_3Q_3^n = 0\\ \text{Loop 4-5-2: } K_4Q_4^n - K_5Q_5^n - K_2Q_2^n = 0\end{array}\right\}\text{Nonlinear Equations}$$

Note that: n = 2 in case of using Darcy equation

Linear Equations

$$\begin{array}{l}\text{Node [1]: } Q_1 + Q_3 - 4.45 = 0\\ \text{Node [2]: } -Q_1 + Q_2 + Q_4 + 1.11 = 0\\ \text{Node [3]: } -Q_4 - Q_5 + 3.34 = 0\end{array}$$

Loop 1-2-3: $K_1|Q_1|Q_1+K_2|Q_2|Q_2-K_3|Q_3|Q_3=0$

Loop 4-5-2: $K_4|Q_4|Q_4-K_5|Q_5|Q_5-K_2|Q_2|Q_2=0$

Loop 1-2-3: $K_1|Q^o_1|Q_1+K_2|Q^o_2|Q_2-K_3|Q^o_3|Q_3=0$

Loop 4-5-2: $K_4|Q^o_4|Q_4- K_5|Q^o_5|Q_5-K_2|Q^o_2|Q_2=0$

Solution of the Q-Method using Linearization (Cont.)

The set of equations eventually reduces to the following linear system:

Node [1]: $Q_1 + Q_3 - 4.45 = 0$
Node [2]: $-Q_1 + Q_2 + Q_4 + 1.11 = 0$
Node [3]: $-Q_4 - Q_5 + 3.34 = 0$

Loop 1-2-3: $K_1|Q^o_1|Q_1 + K_2|Q^o_2|Q_2 - K_3|Q^o_3|Q_3 = 0$
Loop 4-5-2: $K_4|Q^o_4|Q_4 - K_5|Q^o_5|Q_5 - K_2|Q^o_2|Q_2 = 0$

The above linear set of equations can be written in a matrix format as:

[A]{Q} ={F}

Where:

$$[A] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ K_1|Q_1^o| & K_2|Q_2^o| & -K_3|Q_3^o| & 0 & 0 \\ 0 & -K_2|Q_2^o| & 0 & K_4|Q_4^o| & -K_5|Q_5^o| \end{bmatrix} \qquad \{F\} = \begin{bmatrix} 4.45 \\ -1.11 \\ -3.34 \\ 0 \\ 0 \end{bmatrix}$$

Solution of the Q-Method using Linearization (Cont.)

The value of the unknown $\{Q_{new}\}$ can be obtained using iteration as follow:

$$\{Q_{new}\} = [A]^{-1}\{F\} \ \& \ Q^o = (Q^o_{old} + Q_{new})/2$$

Where:

$$[A] = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ K_1|Q_1^o| & K_2|Q_2^o| & -K_3|Q_3^o| & 0 & 0 \\ 0 & -K_2|Q_2^o| & 0 & K_4|Q_4^o| & -K_5|Q_5^o| \end{bmatrix}}_{\text{Transformation Matrix}} \qquad \{F\} = \underbrace{\begin{bmatrix} 4.45 \\ -1.11 \\ -3.34 \\ 0 \\ 0 \end{bmatrix}}_{\text{Force Vector Or Demand Vector}}$$

Linearization via Excel (Video)