Surface Wave Height Exceedance Probabilities on Two-Layer Fluids

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Background

Exceedance probabilities:

- The probability that a given wave will exceed some value, often expressed as multiples of the significant wave height, Hs.

Significant wave height:

- Average wave height of the largest third of waves in the wave field [5]. Also calculated as $4 \cdot \mathrm{std}(\eta)$.

Wave height:

- The distance between the crest (maximum) and the trough (minimum) of a wave, where a wave is the surface profile between two consecutive downward zero crossings [5].

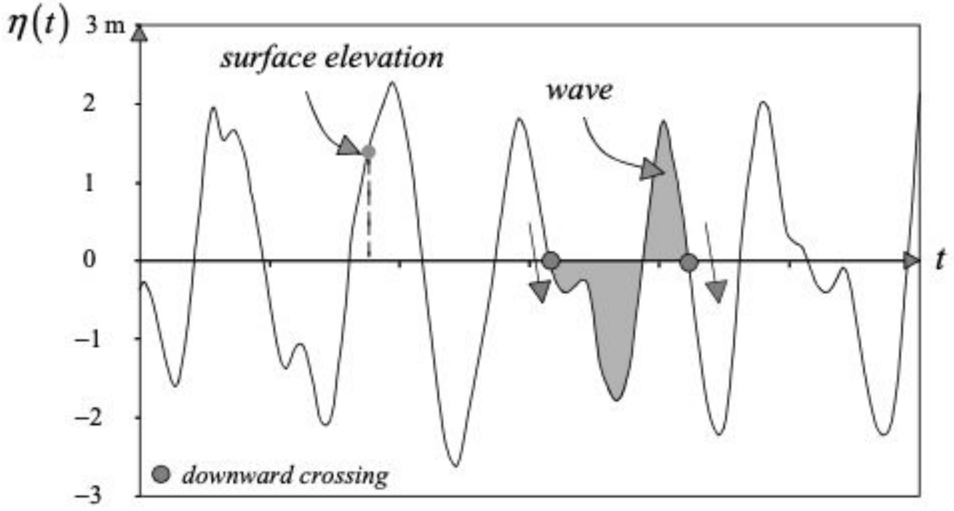
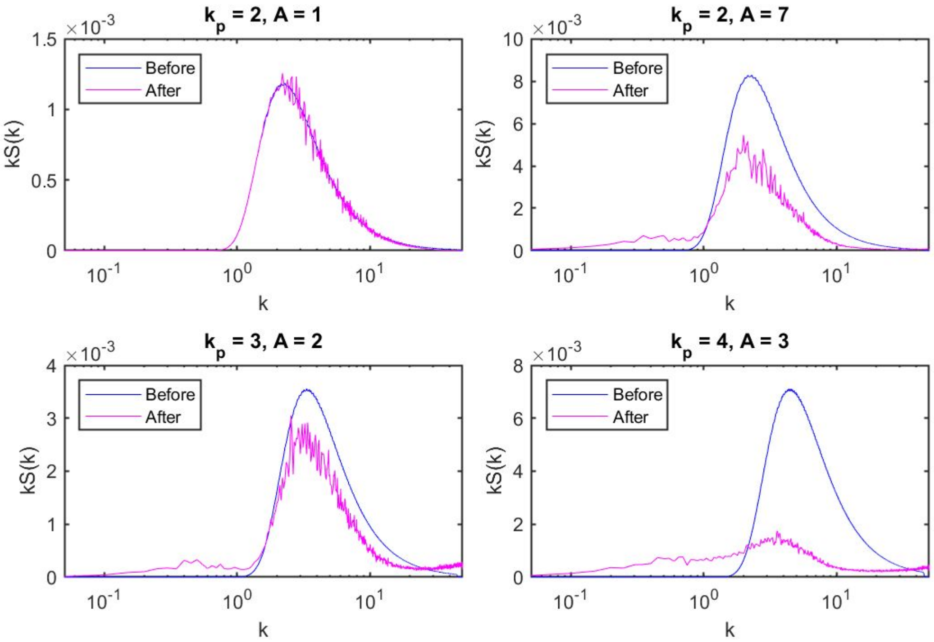


Figure 1: Example of a surface profile with the grey area showing an individual wave (Holthuijsen, 2010).

The wave height spectrum:

- Contains all the statistics of the surface elevation, assuming that the surface elevation is a stationary, Gaussian process [5].
- k_p is the peak wave number and A is a parameter that determines the significant wave height.
- Steepness is the significant wave height multiplied by the average wavenumber of the spectrum [5].
  - Increasing A and/or k_p increases steepness [3].
- The occurence rate of large waves depends on the shape of the wave spectrum [2].
- On a two-layer fluid, interactions between the surface and the interface between the two layers can filter out high frequency waves [3].



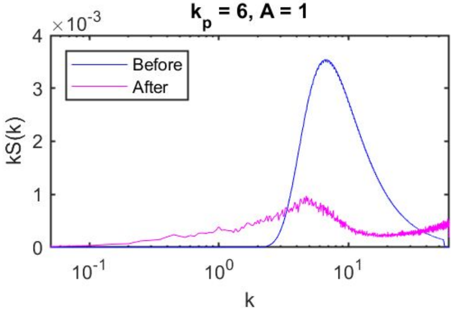


Figure 2: Non-dimensional wave height spectra used in the simulation from before (blue) and after (pink) spectral evolution. Spectra from Gemmrich & Monahan (2021).

Class 3 Resonant Triad

In two-layer fluids, a class 3 resonant triad can allow energy to move from the surface to the interface between the two layers in the generation and maintenance of an interfacial wave [1].

- Two surface waves and an interfacial wave all propagating in the same direction.
- The wavenumbers of the two surface waves must be very close to each other in value. The wavenumber of the interfacial wave must be the difference of wavenumbers of the two surface waves.
- The frequencies of the waves must also balance.

In this case $\frac{\rho_1}{\rho_2} = R = 0.97$ which is a realistic lower bound on density ratios in the ocean [3].

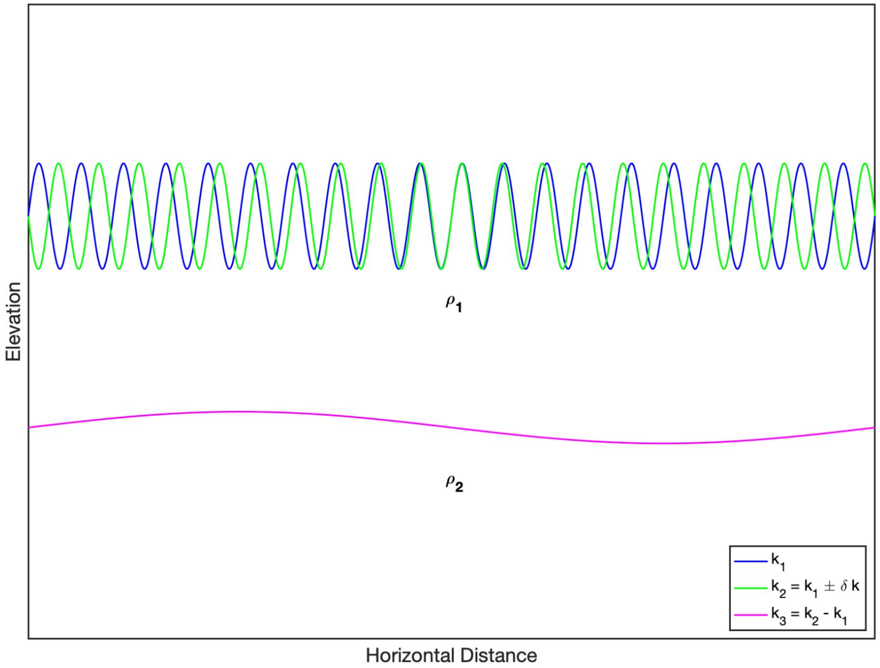


Figure 3: Sketch of a class 3 triad where each wave is propagating in the same direction.

Rayleigh distributed wave heights

If the bandwidth of the spectrum is sufficiently narrow, the wave heights will be distributed according to a Rayleigh distribution [6].

$$P\left(H/H_s > \alpha\right) = \exp\left(-2\alpha\right) \quad (1)$$

Taking $\ln\left(-\ln P\right)$ of equation 1 results in

$$\ln\left(-\ln P\right) = 2\ln\left(\alpha\right) + \ln\left(2\right) \quad (2)$$

If the left hand side is plotted against $\ln\left(\alpha\right)$, then the wave height exceedance probability curve will be a straight line with a slope of 2, and an intercept of $\ln\left(2\right)$

Plotting the exceedance curves in this way shows much more detail in the tail end of the exceedance curves, and provides an easy comparison to Rayleigh distributed wave heights and crest heights.

Motivation

Extreme waves ($H > 2.2\, H_s$) can do serious damage to ships, offshore platforms, and smaller vessels.

Two-layer fluids are common

- In river estuaries due to fresh river water overlying salty ocean water.
- In polar regions due to seasonal ice melt.

In the Arctic, climate change is reducing the amount of sea ice present throughout the year.

- Increased shipping through the region.
- Larger significant wave heights.

Figure 4: Damage to a ship from a rogue wave. Picture taken by H. Gunther and W. Rosenthal. Accessed from: https://www.esa.int/ESA_Multimedia/Images/2004/06/Damage_done_by_a_rogue_wave

Simulation

Monte Carlo simulation was the method used to determine if there are differences in wave height exceedance probabilities after spectral evolution on a two-layer fluid.

- Random component model

$$\eta(x) = \sum_{n=1}^{N} a_n \cos(k_n x) + b_n \sin(k_n x) \qquad (3)$$

- a_n and b_n are random coefficients generated from a normal distribution with zero mean, and variance equal to the value of the spectrum, S_n, at k_n.

$$a_n, b_n = \mathcal{N}\left(0, \sqrt{S_n}\right) \qquad (4)$$

- Each spectrum was used to simulate surface profiles.

- Wave heights, crest heights, and the significant wave height were stored from each run.

Results

Wave height exceedance probabilities are reduced after spectral evolution on a two-layer fluid in 4 of 5 cases simulated.

- Case [k_p,A] = [2,1] had no significant changes in the large wave regime.

- All cases were less likely to exceed a given wave height than Rayleigh distributed wave heights.

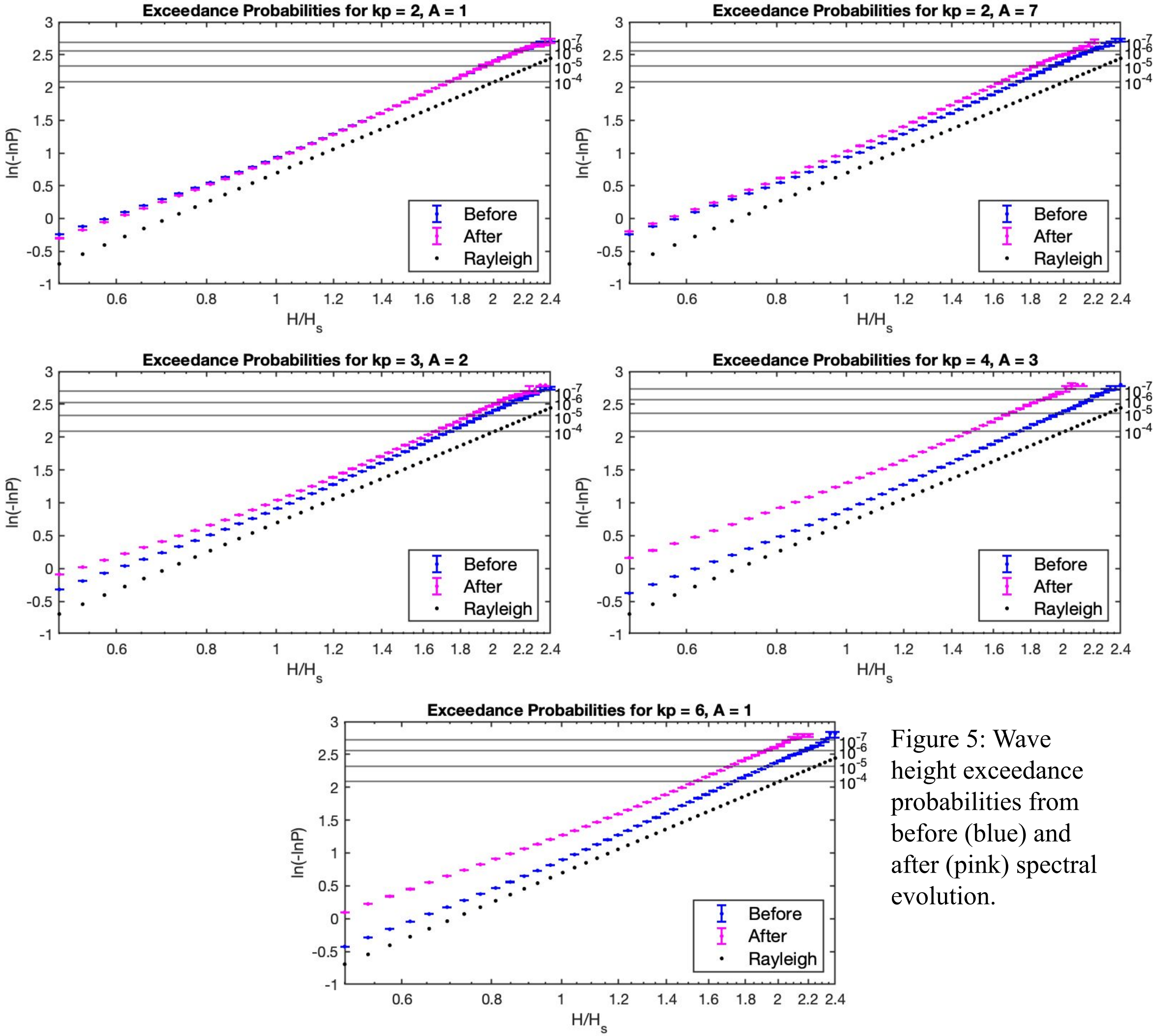


Figure 5: Wave height exceedance probabilities from before (blue) and after (pink) spectral evolution.

Relative change curves:

- The magnitude of the relative change in wave height exceedance probabilities was fairly well correlated with the initial steepness of the wave height spectrum.

and

- The magnitude of the relative change in wave height exceedance probabilities was well correlated with energy lost relative to the initial spectrum.

- In the most extreme case, the probability of exceeding a wave larger than 2Hs became less common by a factor of approximately 1/100.

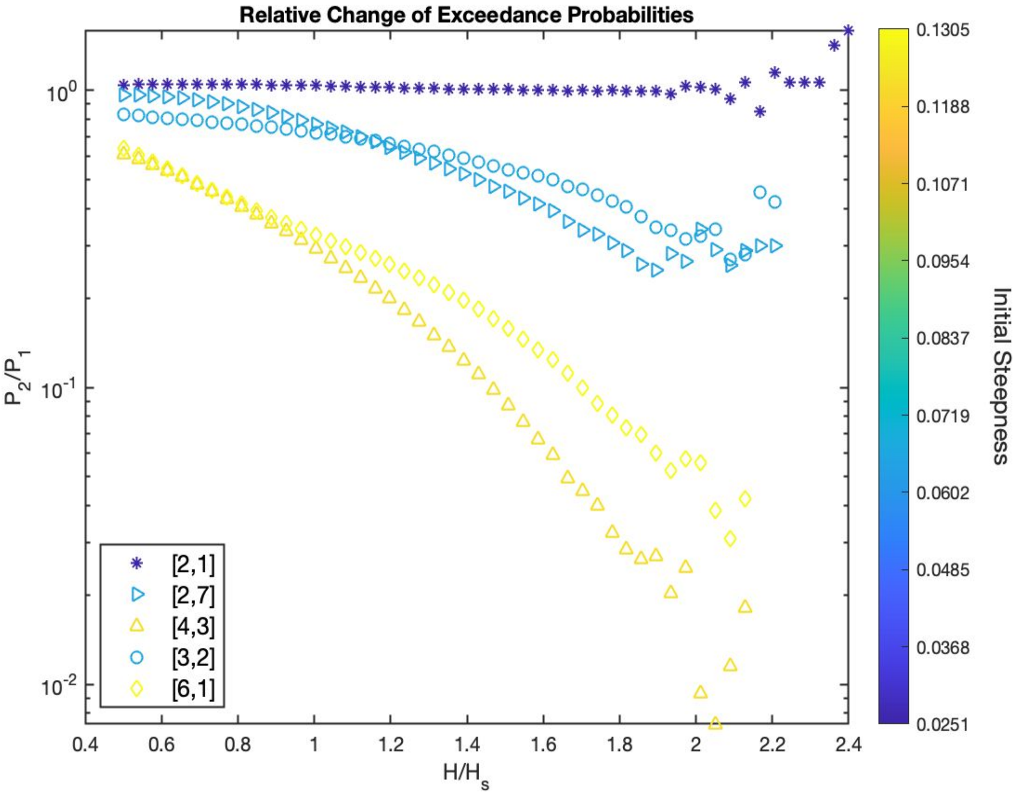


Figure 6: Relative change in exceedance probabilities from before (P_1) and after (P_2) two-layer spectral evolution. The legend shows the case ($[k_p,A]$) and the colour indicates the initial steepness of the spectrum.

Crest heights

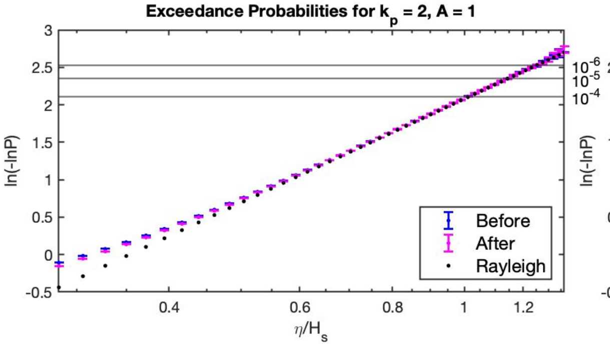
Crest height exceedance probabilities had different behaviour from wave height exceedance probabilities.

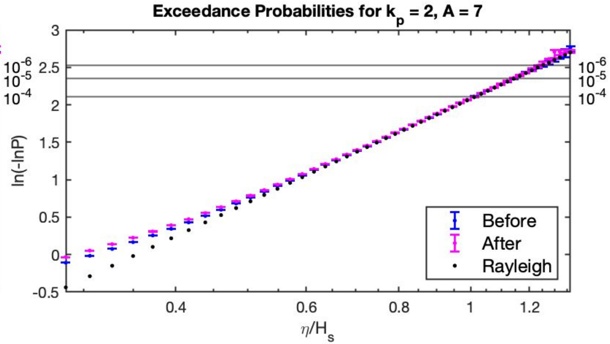
- In the small wave regime crest heights also have lower exceedance probabilities.

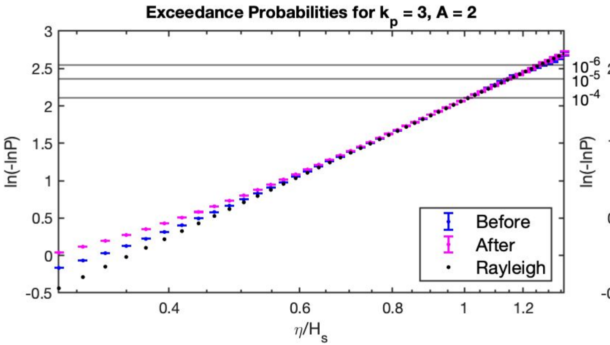
but

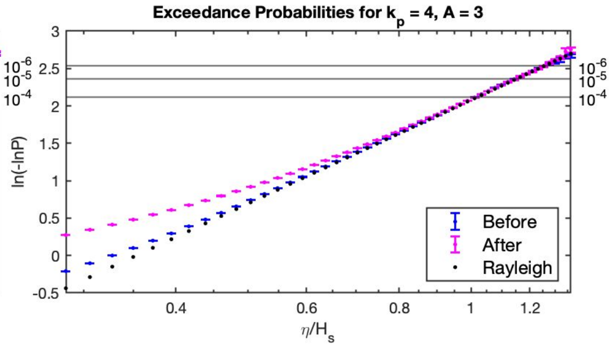
- In the large wave regime crest height exceedance from before and after spectral evolution converge to the Rayleigh exceedance curve

- Crest heights and trough depths must become less correlated.









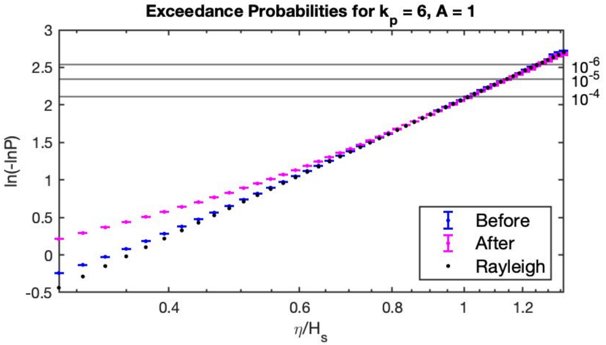


Figure 7: Crest height exceedance probabilities from before (blue) and after (pink) spectral evolution.

Crest-trough correlations

Prior to spectral evolution the crest trough correlation coefficient was approximately 0.33 in each case.

- After spectral evolution crest-trough correlations decreased in each statistically significant case.

- The magnitude of decrease was
  - Well correlated with energy lost relative to the initial spectrum.
  - Fairly well correlated with initial steepness.

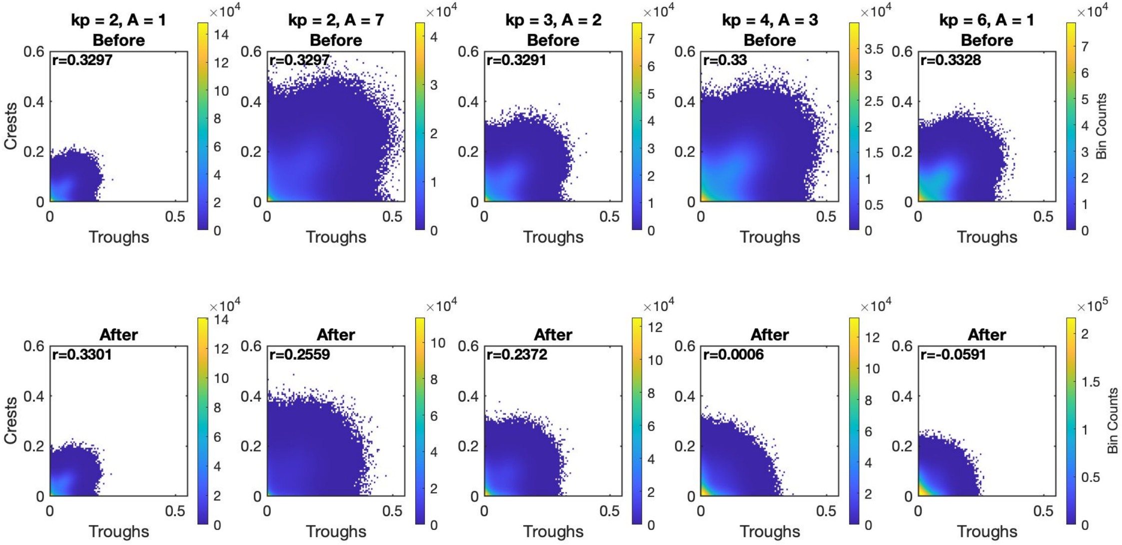


Figure 8: binned scatter plots of crest heights and trough depths from before and after spectral evolution in each case. The crest-trough correlation coefficient, r, is also quoted.

Lower crest-trough correlations corresponding to reduced exceedance wave height probabilities is qualitatively consistent with a correction to Rayleigh distributed wave height exceedance probabilities derived by Naess (1985).

$$P\left(H/H_s > \alpha\right) = \exp\left(-\frac{2\alpha}{\beta}\right) \qquad (5)$$

$$\beta = \frac{1}{2}\left(1 - \tilde{r}\right), \;\; \tilde{r}\epsilon\left[-1, 0\right] \qquad (6)$$

- The slope of 2 remains the same, but the intercept becomes $\ln\left(\frac{2}{\beta}\right)$ [7].
- Decreased crest-trough correlations result in larger intercept values.

The reduction in crest-trough correlations and subsequent reduced wave height exceedance probabilities is also consistent with new research (in prep) that finds crest-trough correlations to be the best predictor of extreme waves [4].

Conclusions

- Wave height exceedance probabilities were reduced in four of the five cases simulated.
- The magnitude of the reductions in wave height exceedance probabilities were correlated with
  - Initial steepness of the wave height spectrum.
  - Energy lost relative to the initial wave height spectrum.

- Crest heights were the unchanged from Rayleigh distributed crest heights in the limit of large crest heights.
- Crest-trough correlations decreased in each significant case.
  - The magnitude of the decrease of crest-trough correlations were correlated with
    - The initial steepness of the spectrum.
    - The energy lost relative to the initial spectrum.

References

[1] Alam, M.-R. (2012). A new triad resonance between co-propagating surface and interfacial waves. *Journal of Fluid Mechanics*, *691* , 267.

[2] Gemmrich, J., Garrett, C. (2011). Dynamical and statistical explanations of observed occurrence rates of rogue waves. *Natural Hazards and Earth System Sciences, 11*(5), 1437-1446

[3] Gemmrich, J., & Monahan, A. (2021). Surface and interfacial waves in a strongly stratified upper ocean. *Journal of Physical Oceanography, 51*(2), 269-278

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[6] Longuet-Higgins, M. S. (1952). On the statistical distribution of the height of sea waves. *Journal of Marine Research*, *11* , 245-266.

[7] Naess, A. (1985). On the distribution of crest to trough wave heights. *Ocean Engineering, 12* (3), 221-234.