2D Poiseuille Exact Coherent State equations (the other streamfunction)

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1 The full equations

Equations given are the components of the Navier-Stokes equation for the streamfunction and Oldroyd-B equation for the stress.

$$\mathrm{Re}\ \left[\partial_t \nabla^2 \psi + (\mathbf{u} \cdot \nabla)\, \nabla^2 \psi\right] = \nabla p + \beta \nabla^4 \psi - (1-\beta) \nabla \times \nabla \cdot \boldsymbol{\tau} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\partial_t \mathbf{c}\, (\mathbf{u} \cdot \nabla)\, \mathbf{c} = (\nabla \mathbf{u})^T c + c\, (\nabla \mathbf{u}) - \boldsymbol{\tau} \tag{3}$$

$$\begin{aligned} (\nabla \mathbf{u})_{i,j} &= \partial_i u_j \\ T_{ij} &= \frac{1}{\mathrm{Wi}} (C_{ij} - \mathbf{I}_{ij}) \\ u &= \partial_y \psi \\ v &= -\partial_x \psi \end{aligned}$$

1.1 ψ

$$\begin{aligned} 0 = &-\mathrm{Re}\ \nu \partial_x \nabla^2 \psi + \mathrm{Re}\ u \partial_x \nabla^2 \psi + \mathrm{Re}\ v \partial_y \nabla^2 \psi \\ &- \beta \nabla^4 \psi + (1-\beta) \left(\partial_{xx} T_{xy} + \partial_{xy}(T_{yy} - T_{xx}) - \partial_{yy} T_{xy}\right) \end{aligned} \tag{4}$$

1.2 C_{xx}

$$\begin{aligned} 0 = &\ \nu \partial_x C_{xx} - (u \partial_x + v \partial_y) C_{xx} \\ &+ 2 C_{xx} \partial_x u + 2 C_{xy} \partial_y u - T_{xx} \end{aligned} \tag{5}$$

1.3 C_{yy}

$$\begin{aligned} 0 = &\ \nu \partial_x C_{yy} - (u \partial_x + v \partial_y) C_{yy} \\ &+ 2 C_{xy} \partial_x v + 2 C_{yy} \partial_y v - T_{yy} \end{aligned} \tag{6}$$

1.4 C_{xy}

$$
\begin{aligned}
0 = {} & \nu\partial_x C_{xy} - (u\partial_x + v\partial_y)C_{xy} \\
& + C_{xx}\partial_x v + C_{yy}\partial_y u - T_{xy}
\end{aligned} \tag{7}
$$

1.5 ψ_0

$$
\begin{aligned}
0 = {} & -\mathrm{Re}\ \overline{v\partial_y u} \\
& + \beta\overline{\partial_{yyy}\psi} + (1-\beta)\overline{\partial_y T_{xy}} + 2
\end{aligned} \tag{8}
$$

2 The 1st order terms in a Taylor expansion

These are the equations as they will need to be entered into the Jacobian matrix.

$$
\psi = \Psi + \delta\psi \tag{9}
$$

$$
C_{ij} = C_{ij} + \delta c_{ij} \tag{10}
$$

2.1 ψ

$$
\begin{aligned}
0 = {} & -\nu\mathrm{Re}\ \partial_x\nabla^2\delta\psi + \mathrm{Re}\ (\partial_x\nabla^2\Psi)\partial_y\delta\psi + \mathrm{Re}\ U\partial_x\nabla^2\delta\psi \\
& - \mathrm{Re}\ (\partial_y\nabla^2\Psi)\partial_x\delta\psi + \mathrm{Re}\ V\partial_y\nabla^2\delta\psi - \beta\nabla^4\delta\psi \\
& + \frac{(1-\beta)}{\mathrm{Wi}}\left(\partial_{xx}\delta c_{xy} + \partial_{xy}(\delta c_{yy} - \delta c_{xx}) - \partial_{yy}\delta c_{xy}\right)
\end{aligned} \tag{11}
$$

2.2 C_{xx}

$$
\begin{aligned}
0 = {} & \nu\partial_x\delta c_{xx} - (\mathbf{V}\cdot\nabla)\delta c_{xx} \\
& - (\partial_x C_{xx})\partial_y\delta\psi + (\partial_y C_{xx})\partial_x\delta\psi \\
& + 2(\partial_x U)\delta c_{xx} + 2(\partial_y U)\delta c_{xy} \\
& + 2C_{xx}\partial_{xy}\delta\psi + 2C_{xy}\partial_{yy}\delta\psi - \frac{1}{\mathrm{Wi}}\delta c_{xx}
\end{aligned} \tag{12}
$$

2.3 C_{yy}

$$
\begin{aligned}
0 = {} & \nu\partial_x\delta c_{yy} - (\mathbf{V}\cdot\nabla)\delta c_{yy} \\
& - (\partial_x C_{yy})\partial_y\delta\psi + (\partial_y C_{yy})\partial_x\delta\psi \\
& + 2(\partial_x V)\delta c_{xy} + 2(\partial_y V)\delta c_{yy} \\
& - 2C_{xy}\partial_{xx}\delta\psi - 2C_{yy}\partial_{xy}\delta\psi - \frac{\delta c_{yy}}{\mathrm{Wi}}
\end{aligned} \tag{13}
$$

2.4 C_{xy}

$$
\begin{aligned}
0 = &\nu\partial_x\delta c_{xy} - (\mathbf{V}\cdot\nabla)\delta c_{xy} \\
&- (\partial_x C_{xy})\partial_y\delta\psi + (\partial_y C_{xy})\partial_x\delta\psi \\
&+ (\partial_y U)\delta c_{yy} + (\partial_x V)\delta c_{xx} \\
&+ C_{yy}\partial_{yy}\delta\psi - C_{xx}\partial_{xx}\delta\psi - \frac{\delta c_{xy}}{\mathrm{Wi}}
\end{aligned} \tag{14}
$$

2.5 ψ_0

$$
\begin{aligned}
0 = &+\mathrm{Re}\ \overline{(\partial_x\Psi)\partial_{yy}\delta\psi} + \mathrm{Re}\ \overline{(\partial_{yy}\Psi)\partial_x\delta\psi} \\
&+ \beta\overline{\partial_{yyy}\delta\psi} + \frac{1-\beta}{\mathrm{Wi}}\overline{\partial_y\delta c_{xy}}
\end{aligned} \tag{15}
$$