Viscoelastic Kelvin-Helmholtz instability

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1. Introduction

The Kelvin-Helmholtz instability is integral to the transition to turbulence in Newtonian fluids. By finding a purely elastic version of this instability, we seek to probe a generic mechanism for the transition to purely elastic turbulence, one that is presumably present in other contexts, such as the oscillatory channel flow problem of Searle & Morozov (2016*a*). To explore this mechanism in the purely elastic regime we have constructed a simple model system of a purely elastic flow which generates large jumps in the first normal stress difference and so ought to lead to instability.

This flow has not been considered in the past because it does not occur in many of the natural contexts where we see the Newtonian Kelvin-Helmholtz. For example, the archetypical Kelvin-Helmholtz instability takes place for flow over a backwards facing step. In this geometry, the fluids inertia leads to an overshoot with vortex behind the step and a hyperbolic tangent streamwise flow velocity. In Purely elastic fluids the flow is Stokesian, and so the fluid velocity is much smoother with no overshoot.

The backwards facing step is not the flow we are thinking of when we examine this instability, however. We are thinking of flows where elastic effects lead to sharp changes in velocity and stress. Such jumps in first normal stress difference are thought to lead to an instability see **?**, and we hope to expose this as a generic mechanism by which a flow might become unstable.

1.1. *Exact coherent structures and Turbulence*

In 1997 Fabian Waleffe identified a Newtonian self sustaining process in plane Couette flow (Waleffe 1997). Streamwise rolls redistribute the streamwise velocity into a streaky flow. This streaky flow is unstable through a Kelvin-Helmholtz instability leading to a symmetry bifurcation to a three dimensional flow. Finally, the nonlinear effects due to this instability re-energises the original streamwise rolls. This exact solution to the Navier-Stokes equations is thought to be a component of the transition to turbulence. In 1998 Waleffe constructed a bifurcation diagram for this exact solution (Waleffe 1998)

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containing what looks like a bifurcation from infinity, just the behaviour expected of the transition to turbulence in plane Couette flow.

Using this work in Newtonian fluid dynamics as a template for the structure of purely elastic turbulence reveals a similar pattern of streaks and, a similar self-sustaining process (Searle & Morozov 2016*b*). An important step towards explaining this process in purely elastic flows is establishing the mechanism by which the streaks become unstable. It seems this is also a Kelvin-Helmholtz instability as the streaks shear with the rest of the fluid. The Kelvin-Helmholtz instability is integral to the transition to turbulence in Newtonian fluids. It is hoped that by finding a purely elastic version of this instability, we might be able to probe one of the important mechanisms for the transition to purely elastic turbulence, One that is presumably present in other contexts, such as the oscillatory channel flow problem of Searle & Morozov (2016*a*). To explore this mechanism in the purely elastic regime we have constructed a simple model system of a shearing viscoelastic fluid.

1.2. *Shear flow instabilities of Non-Newtonian fluids*

The shear flow instability we are interested in relates to many other instabilities in both Newtonian and Non-Newtonian fluids where the fluid inertia is not dominant. These instabilities mostly relate to a flow of two fluids with differing properties and an instability at the interface between them. I will break them up into 4 classes, Newtonian viscously stratified, shear thinning viscously stratified, Shear banded and Elastically stratified instabilities.

1.3. *Viscous two layer Newtonian shear flows*

In 1967 Yih (Yih 1967) performed a long wavelength analytic stability calculation of two Newtonian fluids in two-layer viscously stratified plane Couette and plane Poiseuille flows. This instability was shown to be an instability of the interface. Many subsequent authors copy Yih's technique and notation.

In 1983 the viscous stratification instability was studied in a flow which is a close relation to ours. Hooper and Boyd tackle the Newtonian two-layer viscous instability (Hooper & Boyd 1983) in a flow of two fluids both in unbounded Couette flow with a jump in the shear rate at the interface. They use both numerical and shortwave asymptotic techniques to obtain that the shortwave instability is always unstable without surface tension. Hinch (Hinch 1984) explained the mechanism of the Hooper and Boyd viscous stratification of above. Says that this instability ought to be unimportant since it will be damped by surface tension.

Renardy (Renardy 1985) repeats both the Yih and the Hooper Newtonian fluid problems but does a full numerical linear stability analysis using a Chebyshev-Tau method of Orszag. The problems with and without walls yield approximately the same critical Reynold's number. Above this the unstable eigenmodes for each problem are approximately the same. They include surface tension, density, viscosity, and volume fraction ratio. Hooper (1985) then looked at a thin film on top of another Newtonian fluid via a long wavelength linear stability analysis. Suggest a mechanism based on the lag between phases of velocity and phase of interface. Hooper (1985) then looked at the nonlinear stability of stratified Couette-Poiseuille flow. They performed a weakly nonlinear analysis to transform the problem to a Kuramoto-Sivishinsky equation and they don't find any travelling waves. Renardy (Renardy 1989) also looked at this nonlinear stability problem and found small amplitude travelling waves. Yiantsios & Higgins (1988) do a numerical linear stability of two superposed Newtonian fluids, looking at the short wave asymptotics like Hooper and Boyd but include surface tension and gravitational effects.

1.4. *Viscous two layer shear thinning Non-Newtonian shear flows*

Waters (Waters 1983) does a long wavelength linear stability analysis for shear thinning power law fluids in plane Couette flow and claims that shear thinning has a large effect. Wong and Jeng (Wong & Jeng 1987) do a long wave linear stability analysis for co-extruded pipe flow of Ellis model shear thinning non-Newtonian fluids. Find that if inner/outer fluid is more viscous shear thinning will stabilize/destabilize the flow.

Weinstein (Weinstein 1990) examined the flow of shear thinning fluids down and incline. Use the Carreau model and look at multilayered flow using analytic asymptotics as well as a finite difference shooting method for the eigenvalues. Find that effects of shear-thinning are complicated.

Pinarbasi (Pinarbasi & Liakopoulos 1995) in 1995 looked at two shear thinning models for the 2-layer Poiseuille flow, Carreau-Yasuda and the Bingham like model. They use a pseudospectral method eigensolver to show that shear thinning destabilizes.

1.5. *shear banded Non-Newtonian shear flows*

McLeish (Mcleish & Road 1987) look at the stability of the interface between two phases of linear polymer melts using the Doi-Edwards model. Show that a shear-banded state has an unstable interface between the fluids. Claims that the normal stress effect determines instability. Renardy (Renardy 1995*a*) returned to this problem to examine how a shear banding instability might be responsible for spurt in two layer Couette flow. Used the Johnson-Segalman model with a pseudospectral eigenvalue solver.

Mike Graham does a review of the sharkskin instability in (Graham 1999) which discusses a lot of previous work and which I found interesting as an insight into how the wall-slip and shear-banding instability factions think about melt fracture.

Fielding (Fielding 2005) looks at the linear stability of the shear-banded flow problem. Use the Johnson-Segalman model, finds that the shear banded state can be linearly unstable. The mechanism is not clear. Then Wilson (Wilson 2006) has a nice little article discussing how an instability can depend or not on changing the constitutive model. Using a small amount of diffusion in the diffusive Johnson-Segalman model doesn't effect the instability even though the limit of zero diffusion is singular.

Fielding and Wilson (Fielding & Wilson 2010) look at the shear banding interfacial instability in plane Poiseuille flow via the diffusive Johnson Segalman model. They do both a numerical linear stability and find nonlinear travelling waves.

Nicolas & Morozov (2012) look at the shear banding instability, this time in Taylor-Couette flow via the diffusive Johnson-Segalman model. Find a previously unnoticed azimuthally wavy bulk instability. I particularly enjoyed the comparison with the co-rotating problem where there is no shear band.

1.6. *Elastic multilayer Non-Newtonian shear flows*

The study of an elastically stratified shear flow began in 1969 when Li (Li 1969) examined two Oldroyd-B fluids with different viscosities and elasticities. This was a asymptotic expansion of the linear stability to long wave length disturbances along the same lines as Yih (1967). They found that elasticity can both stabilize and destabilise the flow. This study along with some subsequent ones (Waters & Keeley 1987; Anturkar *et al.* 1990), used an incorrect boundary condition for the traction at the interface between the fluids. Chen (Chen 1991*a*) later corrects this mistake, and subsequent studies including all the papers by Renardy contain the correct boundary condition.

Khan (1976) studied an experimental realisation of the plane Poiseuille flow of two viscoelastic fluids in a duct. They find that the influence of viscosity dominates over that of elasticity stratification in their flow. They attempt some analytic and numerical work on the Coleman-Noll second order fluid to back up their experiments, but they use the wrong interface traction boundary condition and their methods look dodgy. In Khan & Han (1977) the same authors publish a flawed theory paper using the same constitutive model.

Han (1985) perform a co-extrusion experiment on a pipe flow of two polymer melts. Mainly this is just cross-sections of the extrudate showing that lower viscosity component wraps higher viscosity component. They have no birefringence data but somehow they do get some normal stress data. They claim that the viscous stratification is more important than the elastic for their system.

Waters and Keeley (Waters & Keeley 1987) use an Newtonian fluid and power-law shear thinning Oldroyd-B fluid in stratified plane Couette flow. They do long wave asymptotics. Again, this paper uses the incorrect traction boundary condition at the interface. Renardy (Renardy 1988) looked at the stability of the interface of two layer Couette flow of UCM fluids via short wave asymptotics and a Chebyshev-tau method. They find elastic differences between the layers alone can stabilize or destabilize the flow. Claim that elasticity can enhance the lubrication effect.

Anturkar (Anturkar *et al.* 1990) do a problem involving multiple viscoelastic layers in planar co-extrusion. Do numerical and analytical linear stability analysis on the Oldroyd-B + Carreau shear thinning model. Again they use the incorrect traction boundary condition at the interfaces between the fluids.

Chen (Chen 1991*b*) looked at the elastic instability in the co-extrusion of UCM fluids in a pipe flow. He performed a long wavelength linear stability analysis and found an elastic instability. In (Chen 1991*a*) he then went on to look at the Couette flow of viscoelastic fluids and corrects the error in previous papers (Li 1969; Waters & Keeley 1987; Anturkar *et al.* 1990) for the interface traction boundary condition. Previous authors took the boundary condition on the unperturbed interface between the fluids, rather than the perturbed one. It is demonstrated that you can get a purely elastic linear instability via a long wave asymptotic analysis. The linear stability of a two layer film of UCM fluids down an inclined plane was examined by a long wave asymptotic analysis as well as using a pseudospectral method (Chen 1992). The result is that there is an instability when the more elastic component is next to the wall. The shortwave instability of the pipe co-extrusion flow is examined in (Chen & Joseph 1992) using the UCM model. Hypothesis formed about the origin of sharkskin being due to differing elastic properties of the fluid close to the wall. Find growth rates are in fact higher for the elastic instabilities than the viscous Newtonian ones in some settings. Many of the previous viscoelastic references are mentioned in (Larson 1992).

Su & Khomami (1992*b*) find a purely elastic interfacial instability in superposed Poiseuille flow of Oldroyd-B fluids. This is a Plane Poiseuille flow version of (Chen 1992). Long wave asymptotic and some numerical linear stability using a pseudospectral technique. Show that strength of elastic instability is comparable to that of the Newtonian viscous one. They have some low β low Re results. In Su & Khomami (1992*a*) they look at converging and parallel channel co-extrusion of two truncated power law Oldroyd-B fluids. Find that shear thinning has a bigger effect than elasticity. Many phase diagrams.

Hinch, Harris and Rallison (Hinch *et al.* 1992) provide a general understanding of the mechanism behind Chens pipe elastic stratification instability (Chen 1991*b*). They look for an arbitrary constitutive model and include the effects of jumps in the first and second normal stress differences in order to expose how the traction boundary condition can bring about instability.

Wilson and Khomami (Wilson & Khomami 1992, 1993*a*,*b*) performed a three part experimental investigation of the interfacial instabilities in multilayered flow of viscoelastic liquids. They can generate disturbances of particular wavelengths in their setup, to probe the dispersion relation. The compare with the Oldroyd-B with power law shear thinning fluid and find reasonable agreement. They also examine nonlinear effects (Wilson & Khomami 1993*a*) using the same apparatus and they find that there are shear stress imbalances in their flows supporting the proposed instability mechanism, and they show that differences in the normal stress difference enough to change the dispersion relation, lending support to the presence of an elastic instability. Find a supercritical bifurcation and it looks like the interface rolls up in a Kelvin-Helmholtz like way. They say they see things similar to predictions made in Renardy (1989). No subcritical bifurcation is observed. Complex 3D pattern occurs which made it difficult to look at the purely elastic effects divorced from the viscous stratification effects. For compatible polymers (Wilson & Khomami 1993*b*) they observe suppression of the growth rate due to diffusive and convective mixing.

Chen (Chen & Zhang 1993) looked at the stability of the interface in the pipe co-extrusion experiment via pseudospectral eigenvalue solver using the Oldroyd-B model. Discusses two kinds of interfacial instability, capillary instability due to surface tension and elastic due to first normal stress difference. Explains again the Hinch mechanism, but perhaps easier to understand. The Capillary instability can be stabilised by the elasticity stratification. In 1994 (Chen & Crighton 1994) Chen examines the instability of a large *Re* flow of a Newtonian fluid over an Oldroyd-B fluid. There is an asymptotic expansion for the linear stability in the short wave limit.

Azaiez & Homsy (1994) do the linear stability of a free shear flow of the Oldroyd-B, co-rotational Jefferys and Gisekus fluids. For large elasticity they find suppression of the Newtonian Kelvin-Helmholtz instability. They do asymptotic long wave linear stability such that the Reynold's number is small but finite and the Weissenberg number is large with a constant elasticity. They also do a numerical linear stability via an orthogonal shooting algorithm. In the appendix of this paper Hinch (Azaiez & Homsy 1994) takes the long wavelength limit of their problem in a far more elegant way and exposes a stabilisation mechanism based on elastic hoop stresses in the curved streamlines.

Renardy returned to the two-layer flow of the elastically stratified UCM model to look at the weakly nonlinear problem (Renardy 1995*b*). They redid the numerical linear stability analysis and then found the bifurcations were supercritical. Coward and Renardy (Coward & Renardy 1997) looked at a thin film core-annular pipe flow of UCM fluids via a weakly nonlinear analysis.

Wilson and Rallison (Wilson & Rallison 1997) looked at the short wave instability of the elastically stratified two layer plane Poiseuille flow of Oldroyd-B liquids. This was an asymptotic linear stability analysis for short waves. Possibly this is a different mechanism than the Hinch one because it happens for short wavelengths.

In 1997, Laure (Laure *et al.* 1997) looked at the linear stability of a multilayer plane Poiseuille flow of Oldroyd-B fluids. They did a long wave asymptotic analysis in the $Re \to 0$, $Wi \neq 0$ limit. They also did some numerical calculations. They assume that viscous and elastic effects are additive in the dispersion relation. Ganpule & Khomami (1998) thinks this is wrong. In the Ganpule & Khomami (1998) paper they also look at four models, UCM, multimode Gisekus, Gisekus and modified PTT. They repeat the linear stability analysis and do a perturbation energy analysis using the UCM model. They also compare all their results to experiments in (Khomami & Ranjbaran 1997). Those experiments are performed on the multilayer pressure driven flow of polymeric melts. In Ganpule & Khomami (1999) they published a similar theory paper but on just two layers in more detail rather than a multilayered flow. Show how the Oldroyd-B is qualitative match to experiments but gets the wrong growth rates. Attribute this to shear thinning of the experiment in (Khomami & Ranjbaran 1997). Again they use an energy disturbance analysis to study the mechanism and it seems they agree with the Hinch mechanism of (Hinch *et al.* 1992). It seems whatever the mechanism they settle on in this paper, Miller and Rallison disagree with it in (Miller & Rallison 2007*b*).

Scotto & Laure (1999) looks at the linear stability of a three layer Poiseuille flow of Oldroyd-B fluids using a Chebyshev-tau method. I think they do the dodgy thing of adding together affects from the viscous and elastic terms in the dispersion relation.

Renardy (Renardy & Renardy 1999) looks at the spanwise instability between two Gisekus Couette layers with only a second normal stress jump. They use a fully spectral Chebyshev-Tau method and find that short waves are always unstable and without surface tension whereas long waves depend on the depth ratio.

Wilson & Rallison (1999) looked at the instability of channel flows of elastic liquids having continuously stratified polymer concentration and Wi. They look at the long wavelength limit of the Oldroyd-B model using what I think is a similar method to to Hinch in Azaiez & Homsy (1994). Then they do linear stability using a shooting Newton-Rhaphson method. They compare the results of the Oldroyd-B model to a White-Mieztner fluid. Conclude that blurring of the interface between elasticities would suppress the instability, so it is not likely to be responsible for sharkskin. Their instability looks to be the closest to ours as far as I can gather.

Wilson *et al.* (1999) concerns the structure of the eigenvalue spectrum for the instability of UCM and Oldroyd-B fluids in the two-layer channel flow problem. Possibly a useful comparison there for our eigenvalue spectra.

Khomami et al performed linear stability analysis of a two layer Oldroyd-B and Newtonian pressure-driven channel flow and compared it to an experiment with an elastic Boger fluid and a Newtonian fluid in (Khomami & Su 2000; Khomami *et al.* 2000). They found that they could fit their experiments with the Oldroyd-B model and find that their experiments saturated to 2D structures unlike the 3D structures seen in (Khomami & Ranjbaran 1997) and they suggest that it is because this was a second normal stress effect a la (Renardy & Renardy 1999) which is not present in the current experimental setup. Their second paper, (Khomami *et al.* 2000) does the theoretical weakly nonlinear stability analysis and finds supercritical bifurcations as expected.

For some entertainment if you have read this far, (Renardy 2000) is a review article which I think states that my PhD is pointless, "Much early work on stability of shear flows was motivated by the desire to explain melt fracture (viscoelastic turbulence) as a result of an instability similar to the laminar/turbulent transition in Newtonian flows. This is no longer believed to be a realistic proposition."

Brady (Brady & Carpen 2002) claim to have a generic instability that explains Renardy's (Renardy & Renardy 1999) second normal stress jump result, look at a variety of flows including falling film and Newtonian fluid of non-Newtonian flow. They use their own viscous-suspension flow constitutive model and examine the effects of a jump in the second normal stress.

Meulenbroek (Meulenbroek *et al.* 2004) show that there is weakly nonlinear subcritical instability in Poiseuille flow of Oldroyd-B fluids. But seeing as you are an author I expect you know that better than me.

Miller looks at the interfacial instability between sheared UCM and Oldroyd-B liquids in channel flow (Miller & Rallison 2007*b*) and a pipe flow (Miller & Rallison 2007*a*). They don't like the distinction between short and long wave instabilities. They don't think that surface tension stabilises as claimed in (Chen & Joseph 1992). Look at a new limit, where $L < k^{-1} < U_0\lambda$ the wavelength is long compared to the channel width but short compared to the relaxation length scale. They use a combination of short and long wave techniques. They claim that their instability has a different origin to the Hinch one I think.

Bonhomme, O, Morozov (Bonhomme *et al.* 2011) find an elastic instability in the core annular flow of two different Oldroyd-B fluids and perform short wave linear stability asymptotics. Also some experiments, but I imagine you would know all about this.

Wongsomnuk, (Wongsomnuk *et al.* 2000) do a little experiment to show the sharkskin instability onset. I am sure there is probably a better experiment for this out there, but this is just the one that came up.

Yoo & Viñals (2013) does something that may be related to our instability, but it is a long shot. They look at the effect of viscoelasticity on an instability at the interface between lamellar phases with different orientations under oscillatory shear. They use a linear constitutive law which I don't recognise.

Dubief *et al.* (2013) looks at the mechanism of elasto-inertial turbulence in Samanta *et al.* (2013). Possibly related to our instability, but we have talked about this instability before and I don't think Yves has convinced you it exists yet?

Mahdaoui (Mahdaoui *et al.* 2013) do a numerical investigation of the co-extrusion instability. They use a complicated multiscale simulation technique on the Carreau-Yasuda multimode Johnson-Segalman model in 2D along with some possibly molecular dynamics techniques in areas which are hard to resolve.

Beaumont (Beaumont *et al.* 2013) do some experiments looking at turbulent flows in a Taylor-Couette flow of purely elastic wormlike micelles. Find inhomogeneous turbulence and suggest the existence of a bulk instability.

Fardin (Fardin *et al.* 2014) has a review article on the Taylor-Couette flow for soft matter. It references you a lot and discusses the possibility of a purely elastic KH instability. No doubt you are partially responsible for getting that line put in there!

McKinley (McKinley *et al.* 1996) gives us the geometrical Wi scaling. Zilz 2012 (Zilz *et al.* 2012) have nice serpentine channel purely elastic flow which supports the Pakdel-McKinley condition if we choose to use it.

2. Problem formulation

We propose that in fact, besides the shear banding and coextrusion instabilities, there is another mechanism by which a purely elastic fluid may become unstable. Using a linear stability analysis we examine a new Kelvin-Helmholtz like instability in both Oldroyd-B and FENE fluids. A purely elastic fluid is found to be unstable at sufficiently high Weissenberg number and this behaviour is consistent across both Oldroyd-B and FENE constituitive models.

There are a number of ways we might formulate a shear layer problem. We are interested in a generic instability for flows with a sharp change in the streamwise velocity, and so large normal stresses in the fluid. Precisely because we are searching for a generic mechanism, we will not attempt to justify our particular choice of a hyperbolic Cosine shear layer. As well examining the dependence on flow properties (β, Wi and Re) of the instability, we are also interested in the influence of boundaries so that we might assess the relevence of the instability to common experimental and real-life examples of flows of viscoelastic fluids - including both the plane Couette and Oscillatory flow problems of Searle & Morozov (2016*a*) and 2016*b*. As such we consider a nearly free shear layer (with a finite, but large, region of constant velocity between the shear layer and the walls) as well as a free shear layer.

The flow we consider is a hyperbolic tangent shaped laminar profile for the streamwise velocity across a channel, where $y = \pm\Delta$ is the region of shear. $y = 0$ is the centreline

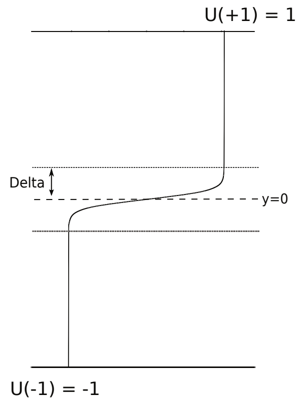

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Figure 1: Diagram of a shear layer in a channel.

of the channel, so that if $\Delta << 1$ the flow tends to a free shear instability. This gives a base flow profile,

$$
\begin{aligned}
U(y) &= \tanh(y/\Delta)\coth(1/\Delta) \\
V &= 0 \\
T_{xx}(y) &= 2Wi\left(\frac{\partial U}{\partial y}\right)^2 \\
T_{xy}(y) &= \frac{\partial U}{\partial y} \\
T_{yy}(y) &= 0
\end{aligned}
\tag{2.1}
$$

U and V are the base streamwise (x) and wall normal (y) velocities respectively. T is the base stress tensor. We use Gauss-Labatto points in the wall-normal (y) direction for the base profile, and decomposed the disturbances to this flow into Fourier modes.

$$g(y) = \sum_{n=-N}^{N} \tilde{g}(y)e^{ikx+\lambda t} \tag{2.2}$$

for all disturbance variables $g = u, v, p, \tau_{i,j}$. Where N is the number of Fourier modes, k is the streamwise wavenumber of the disturbance and λ is the growth rate of the mode. This program uses two domains of pseudo-spectral points to give increased resolution in the centre of the channel. The analysis was repeated in the limit of no boundaries on the flow by performing a passive transform of the coordinate systems such that,

$$z = \frac{y}{1-y}, \qquad z > 0 \tag{2.3}$$

$$z = \frac{y}{1+y}, \qquad z < 0 \tag{2.4}$$

$$U = \tanh z \tag{2.5}$$

This system ought to be completely insensitive to the boundary conditions at the walls so we refer to it as 'free shear' throughout this manuscript.

The dimensionless quantities used throughout are non-dimensionalised relative to the instability size, Δ, rather than the half channel width. This allows us to effectively examine the dependence of the instability on Δ as we approach the free shear limit of $\Delta = 0$.

We used an Oldroyd-B fluid at low Reynolds number (typically $Re < 10.0$) and a small ratio between the solvent and total viscosities $\beta = \frac{\mu_s}{\mu_s+\mu_p}$. This approximates a purely elastic flow, where the polymer contribution to the stress on the fluid is much larger than the solvent contribution. The Oldroyd-B Navier stokes and constitutive equations are then:

$$Re\left[\frac{d\mathbf{v}}{dt} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = -\nabla p + \beta\nabla^2\mathbf{v} + \frac{1-\beta}{Wi}\nabla\cdot\tau \tag{2.6}$$

$$\dot{\tau}/Wi + \overset{\nabla}{\tau} = (\nabla\mathbf{v})^T + \nabla\mathbf{v} \tag{2.7}$$

With $\mathbf{v}$ as the total (base flow and disturbance) velocity and τ as the total stress in the polymeric fluid. The FENE-CR viscoelastic fluid has the advantage of being similarly easy to analyse using our spectral method, but also including a finite extensibility for the polymers. This leads to a stress which depends on a conformation tensor ($\mathbf{C}$) for the polymer dumbbells via a non-linear spring force:

$$\tau = \frac{1-\frac{3}{L^2}}{1+\frac{L^2}{tr(\mathbf{C^2})}}\mathbf{C} \tag{2.8}$$

Again, we used Gauss-Labatto points in the wall normal direction and Fourier modes in the streamwise direction.

3. Purely elastic linear stability analysis

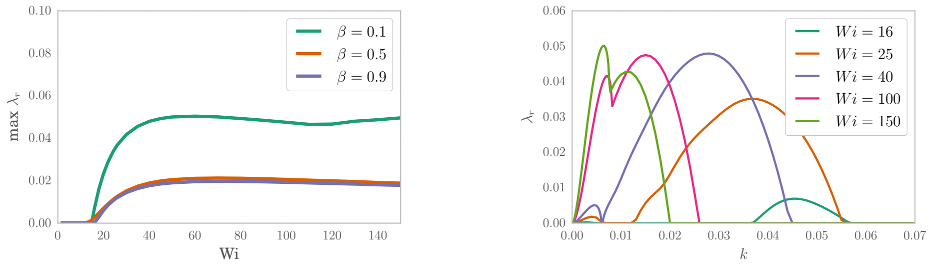
At low Reynolds number, low β and sufficiently large Weissenberg number we observe an instability across a range of streamwise disturbance wavenumbers. This instability persists for larger Δ, albeit with a fairly complex dependence on the boundaries. Before we examine the effect of the walls we will consider the instability in the free shear limit.

3.1. *Free shear layer*

For a free shear layer we observe an instability grow from around $k \sim 0.06$ and move to lower wavenumber as the Weissenberg number increases. The dispersion relations shown in figure 2b show this dependence of the unstable eigenvalues on Wi. The dispersion relation for the instability remains broad in k until $Wi \sim 100$ where a new eigenvector becomes dominant.

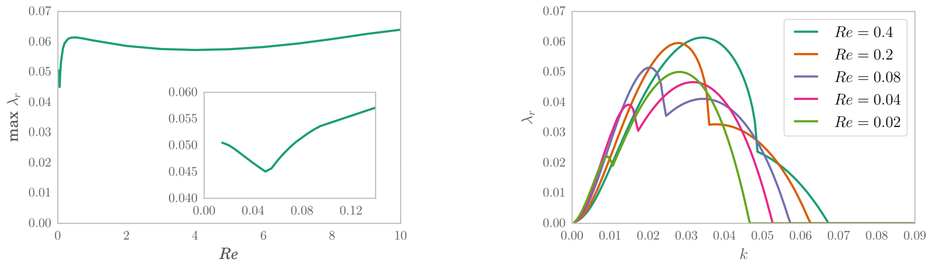
The height of the instability also increases with increasing Weissenberg number. Extracting the peak growth rate from the dispersion relations at a fixed Re for various Wi allows us to show how the instability depends on both β and Wi (figure 2a). The maximum growth rate saturates at $Wi \sim 50$ and remains approximately constant with increasing Weissenberg number. The slight nonmonotonicity in the curve is the result of the switchover in the dominant instability eigenmode.

As the Reynold's number is reduced the Newtonian Kelvin-Helmholtz instability disappears and an entirely separate, purely elastic instability arises. Although the dominant eigenvalue changes often as Re decreases, the flow is still unstable even down to $Re = 0.01$ (see figures 3a and 3b). Again, the complicated dependence on Re appears to be due to the appearance and disappearance of new dominant eigenmodes of the instability.



(a) (b)

Figure 2: a) Free shear version of the instability. Plot of the maximum growth rate against Weissenberg number at $Re = 0.01$. b) Dispersion relations at various Weissenberg numbers to accompany the trend in figure a).



(a) (b)

Figure 3: a) Free shear version of the instability. Plot of the maximum growth rate against Reynold's number with $Wi = 50$. b) Dispersion relations at various Reynold's numbers to accompany the trend in figure a).

3.2. *Shear instability with walls*

Now we use equations 2.1 for a two dimensional flow with boundaries. To validate our approach, we examine the high Reynolds number regime in order to recover a Newtonian Kelvin-Helmholtz instability. At high Reynolds number the instability is insensitive to the distance to the walls when $\Delta = 0.1$ (figure 4) and is consistent with the Kelvin-Helmholtz instability given in **?**.

We see the maximum growth rate of the instability decrease with increasing Weissenburg number, and this trend is stronger for smaller β (figure 5). This is the same behaviour reported in Azaiez & Homsy (1994) for the Kelvin-Helmholtz instability of a polymeric fluid.

At low Reynolds number, we can examine the purely elastic shear layer instability of above. In this case find that the width of the channel has a greater affect on the instablity of the flow. However, the flow is still unstable at very low Δ (figure 6). At $\Delta = 0.1$ the flow is unstable for $Re < 0.01$ far lower than the usual threshold of $Re \sim 1000$ for a Newtonian Kelvin-Helmholtz instability (figure 11b).

If we examine the eigenvectors of the instability, we see that there are very large polymer stresses in the $\pm\Delta$ region (figure 8b). This corresponds to the large shear rate brought about as the two flows pass each other. Examining the flow field, we see that

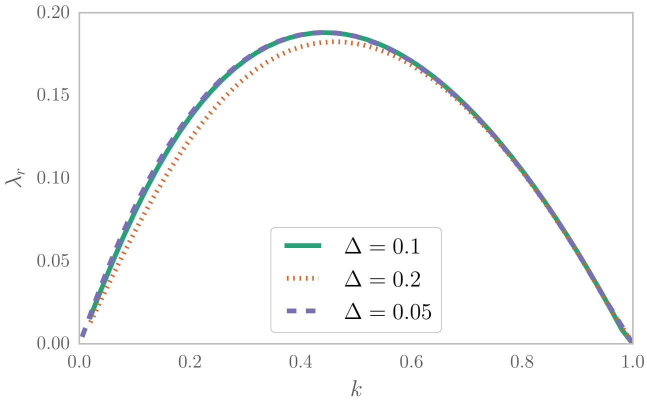


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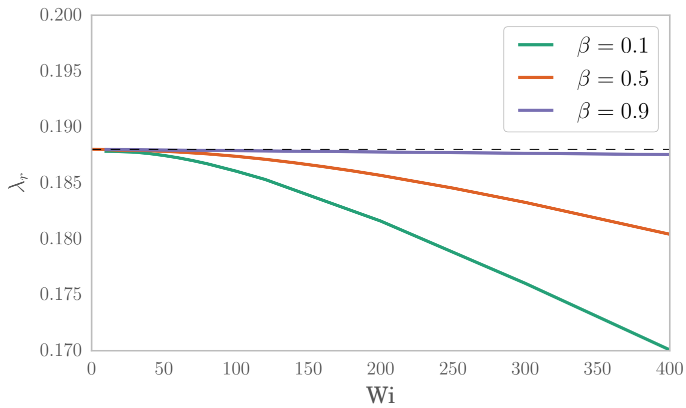
Figure 4: Dispersion relations for various values of Δ and $Re = 1000$, $Wi = 0.01$. When results are rescaled, the instability is found to be insensitive for $\Delta = 0.1$.



[h]

Figure 5: Peak growth rate against Weissenberg number at $Re = 1000$, $\Delta = 0.1$ The height of the dispersion relation is reduced for the Kelvin-Helmholtz instability as the Weissenberg number is increased. The grey dashed line is the height of the dispersion relation for the Newtonian fluid. The decrease in λ_r corresponds to increased stability of the base flow, this matches the observations made in Azaiez & Homsy (1994)

there are large vortices arranged with opposite rotational directions above and below the shear region which extend almost to the walls. This is a very different situation from that observed for the Newtonian Kelvin-Helmholtz instability, where there is an exponential decay in the velocity outside the region of shear (figure 7). The larger penetration depth out of the shear layer by the viscoelastic eigenvector is likely linked to the stronger dependence on Δ relative to the Newtonian instability.

As Δ decreases for a particular purely elastic instability, the width of the instability increases slightly, however in the limit of very small Δ the eigenvector appears independent of the walls (figures 8a and 9). The extended vortices in our purely elastic case seem to correspond to the eigenvectors in Azaiez & Homsy (1994) at larger elasticity ($G \sim Wi/Re$) . Their $G = 5$ case would correspond to only $Wi = 0.5$ for $Re = 0.1$ in our

[h]

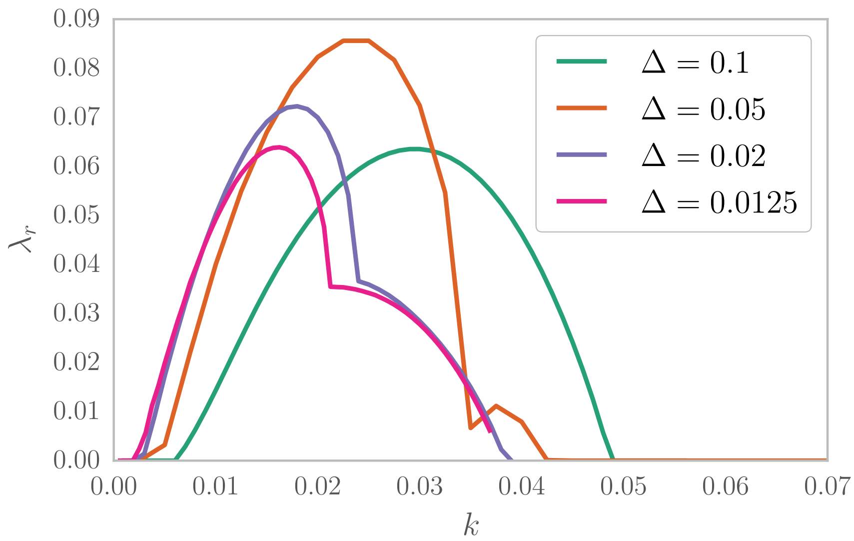


Figure 6: Dispersion relations for various values of Δ at $Re = 0.1$, $Wi = 100$. The sensitivity to the walls seems only to increase the strength of the instability. Even when the system is insensitive to the walls, at $\Delta = 0.0125$, there is still an instability present.

units, such that the instability they observe is still the inertial instability (see Azaiez & Homsy (1994) figure 8). Nonetheless at least one of their observations does apply. The fact that the velocity disturbance is essentially zero in the center of the channel, implies that the shear layer will be less likely to roll up. In the Newtonian instability, net velocities within the shear layer will distort its shape, however the purely elastic instability appears to have less of an effect on the shear layer.

As the Reynold's number is reduced the walls become more important to the instability, this makes a true $Re = 0$ instability difficult for us to observe. In order to remove the dependence on the walls and recover a free shear instability it is necessary to move them further away. This problem gets worse at lower Re. By introducing just a very small contribution from inertia, we can remove the dependence on the walls. This is partly why all of our results are taken for $Re \lesssim 0.1$ on the length scale of the instability.

The analysis for the system in the channel has implications for the plane Couette problem. The Kelvin-Helmholtz instability in the Newtonian problem takes place on a length scale of about 10% of the width of the channel, or $\Delta = 0.1$. If this length scale is similar in the purely elastic case, we can say that the viscoelastic Kelvin-Helmholtz instability is sensitive to the walls at this Δ and a $Re \lesssim 1$ (see figure 10). This means that on the scale of the channel, $Re \gtrsim 10$ for the walls not to affect the instability. So, the important results for the purely elastic instability from the point of view of the plane Couette problem are probably those for which the walls are relevant.

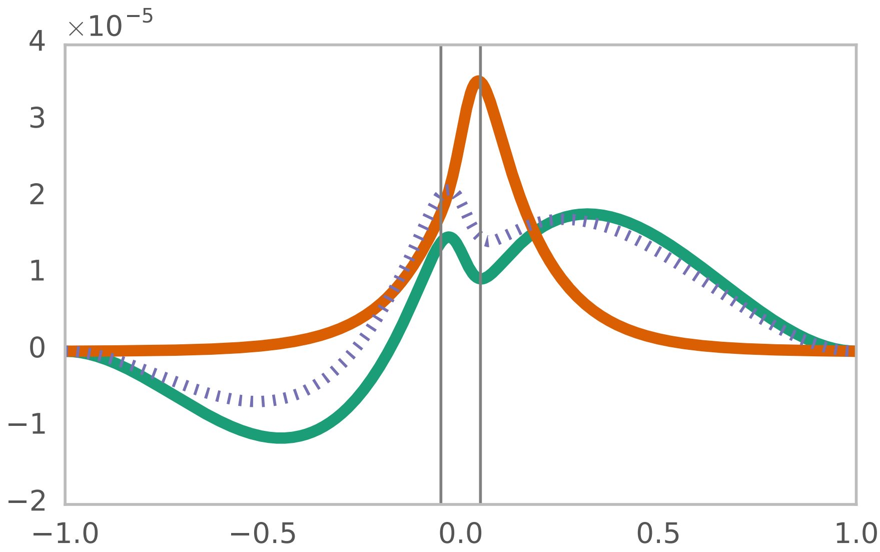


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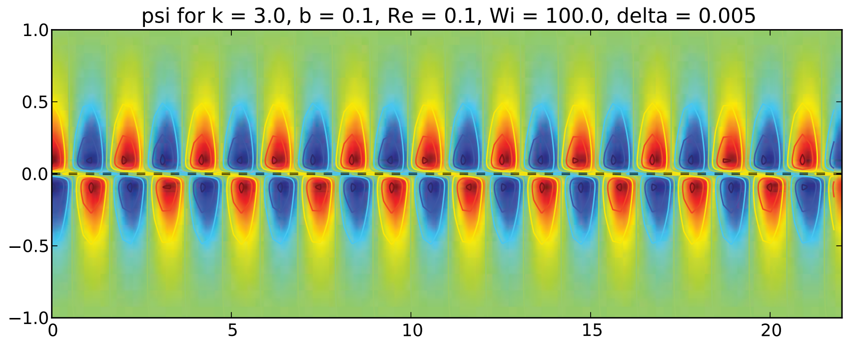
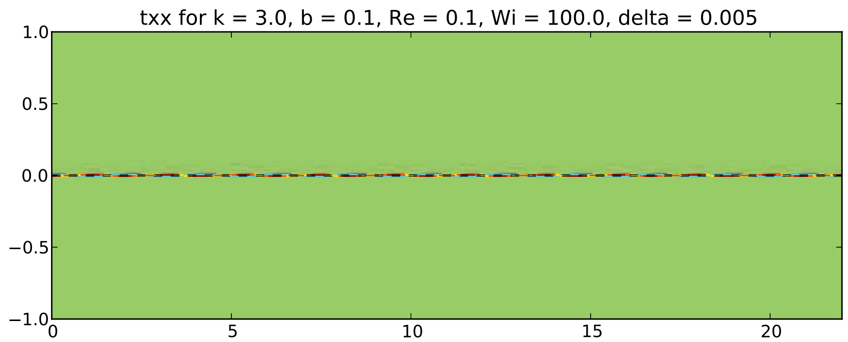
Figure 7: Streamfunction of the Newtonian Kelvin Helmholtz instability at $Re = 100$, $k = 0.722$ compared with the purely elastic Oldroyd-B and FENE-CR instability for $\beta = 0.1$, $Re = 0.0$, $k = 0.02$, $Wi = 50.0$, both at $\Delta = 0.05$. Newtonian eigenvector is in red, with the Oldroyd-B in green and the FENE model in dashed purple at $L = 100$. The Newtonian Kelvin Helmholtz instability exponentially decays towards the channel walls whilst large vortices are present outside the shear layer in the purely elastic case.



(a)



(b)

Figure 8: a) Streamfunction of the largest eigenfunction of the instability when $Re = 0.1$, $Wi = 100$, $\Delta = 0.005$. Red is positive (streamlines of the flow point clockwise around the contours), Blue is negative (streamlines of the flow point anticlockwise around the contours) b) xx stress for the same instability.

[h]

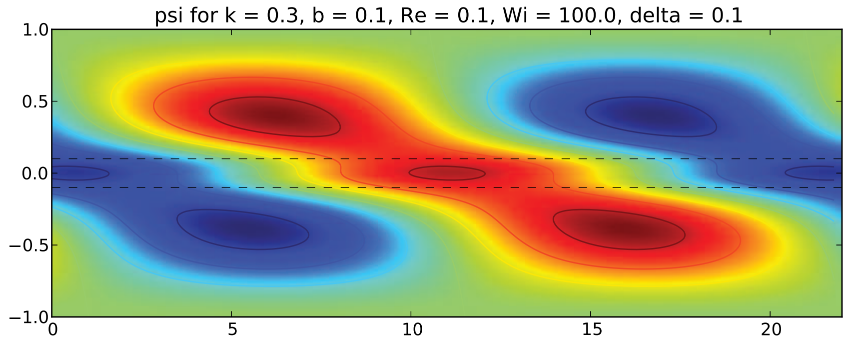
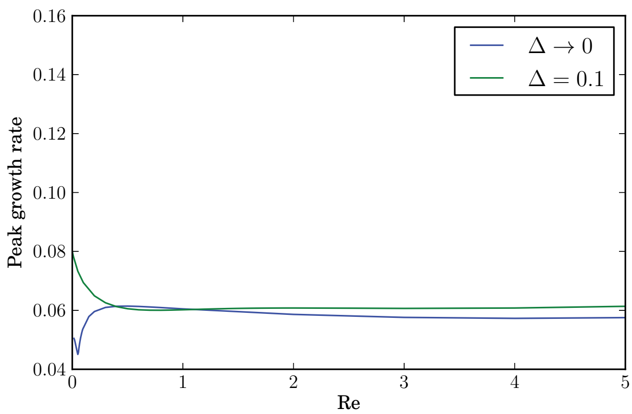
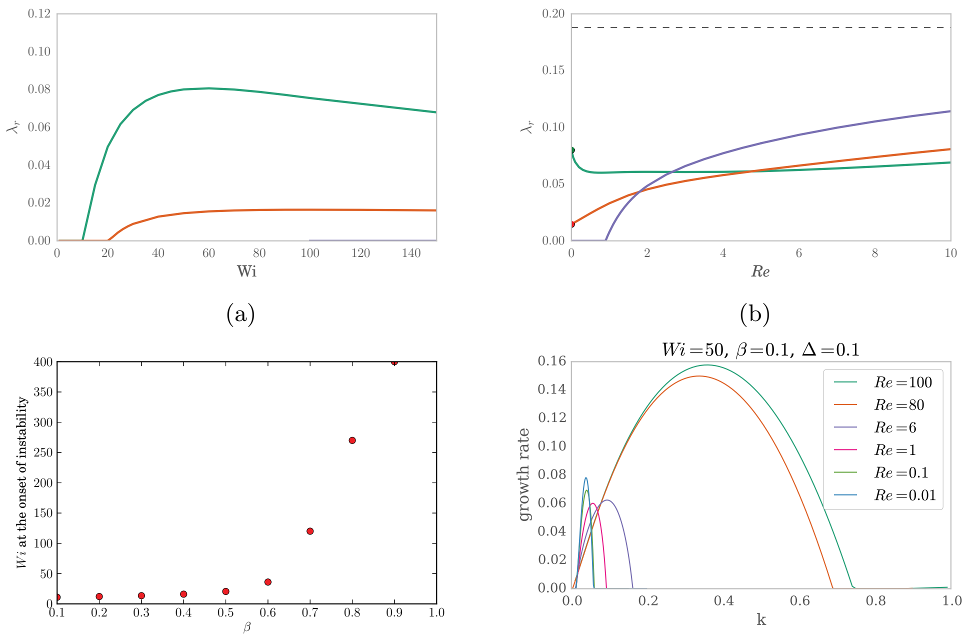


Figure 9: Streamfunction of the eigenfunction of the instability at $\Delta = 0.1$, $Wi = 100$, $Re = 0.1$. The magnitude is scaled by the xx stress in the centre of the channel.



[h]

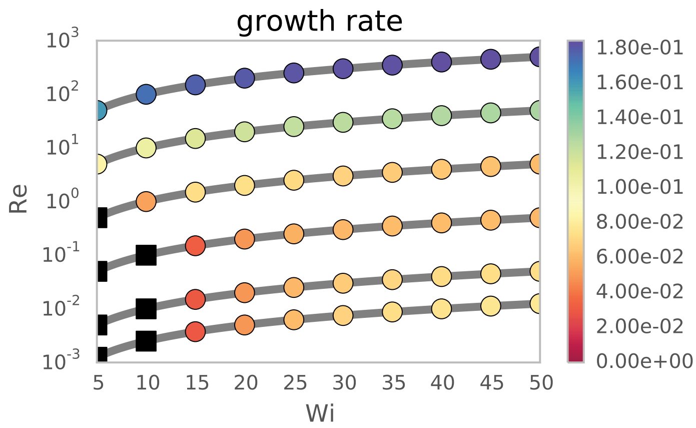
Figure 10: The data in red corresponds to the free shear system and the green gives the $\Delta = 0.1$ data. By $Re = 1$ it is clear that the Reynold's number is sufficient such that the walled system behaves as though it were insensitive to the walls.



(a) (b)

(c) (d)

Figure 11: a) The change in the height of the dispersion relation as Wi is increased at $Re = 0$ with $\Delta = 0.1$. The green and red lines gives the behaviour at $\beta = 0.1$, $\beta = 0.5$ respectively. b) The change in the height of the dispersion relation as the Reynolds number is reduced at $Wi = 5.0$. The colours give the values of β as above, with blue being the behaviour for $\beta = 0.9$. At low Re the instability is clearly still present for large polymer concentrations, consistent with purely elastic turbulence with $\Delta = 0.1$. c) The value of the Weissenberg number at which the velocity profile becomes unstable against β at $\Delta = 0.1$. The onset of the purely elastic instability moves to higher Wi as β increases. d) Dispersion relations at Wi =50 for $\Delta = 0.1$.



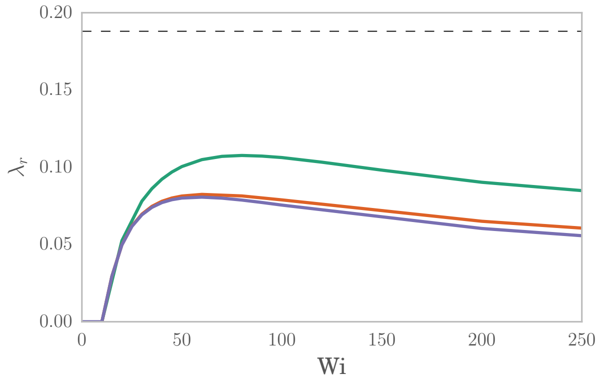
[h]

Figure 12: Phase diagram showing the growth rate for various values of the Reynold's and Weissenberg numbers at $\beta = 0.1$. Grey lines represent lines of constant elasticity. Black squares are data points for which the flow is linearly stable.

3.3. *Consistency of results with the FENE model*

The finite extensibility version of the model gives good agreement with the Oldroyd-B version. Shorter lengths bring about a stronger instability and do not delay it to higher Weissenberg number. Otherwise similar dispersion relations are observed with a similar dependence on Δ. (figure 13). The dependence of the instability on wavenumber is not shown, but is exactly the same as that seen for the Oldroyd-B model.

A shorter length scale for the polymers results in an amplification of the instability. This suggests that perhaps the instability mechanism involves the shear stress on the polymers.



[h]

Figure 13: The FENE-CR model fluid at $Re = 0$ when $\Delta = 0.1$, ranging from $L = 100$ (top curve), $L = 500$ and the Oldroyd-B model in the bottom curve. Results are consistent with figure 11a.

4. Conclusions

In conclusion, we can say that there is a Kelvin-Helmholtz like instability for purely elastic shear flows. The unstable region is much larger than that seen in Newtonian fluids relative to the region of shear. The wavenumber of the instability is much smaller in the viscoelastic case ($k < 0.06$), suggesting a larger minimal flow unit is required in viscoelastic fluids. This instability is enhanced slightly by introducing a finite extensibility to the polymers but remains of essentially the same form. The instability has grown to its maximum by the time it reaches an effective Weissenberg number $Wi \sim 50$ after which it is saturated. The instability is also strongly dependent on the walls at low Reynold's number.

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