Time iteration method for linear stability analysis

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1 Outline

- Use streamfunction equations to make $2N + 1$, $(2N + 1)M$ matrix equations for the operators.
  - split the 6th order operator into three steps, solve these 3 equations simultaneously. (2 steps for 4th order)
  - apply the boundary conditions to this matrix.
  - invert this matrix
  - save only the part of the new matrix which multiplies the function of stresses, this will be the inverse of the operator.
  - repeat this for all z Fourier modes.
- Make matrices for the operators from the zeroth mode equations
- LOOP
- make vectors from stresses for the functions of stress of the RHS of equations 5 and 6
- put boundary conditions in appropriate elements of these vectors
- Use the inverted operator matrices made before the loop on these vectors to find the stream function modes.
- Stitch the answers together to give the full stream function vectors
- Uses the stream functions to solve for the velocities

- solve the zeroth mode equations to find δw_0, δv_0 by making vectors for the terms constant in the velocity disturbances and then left multiplying these by the appropriate precalculated matrices.
- use the incompressibility equation to find δu_0.
- use the zeroth mode of the x component of the Navier-Stokes equation to find δp_0.
- Use equation 4 to calculate the rest of the components of the velocities.

- Use the x-component of the Navier-Stokes equation to solve for the pressure.
- Calculate magnitude of the disturbance
- Calculate the time derivative of the vector of stress disturbance variables
- Calculate new vector of stress disturbances from old using an approximation of the derivative
- END LOOP
- use a plot of the log of the magnitudes against time to find the growth rate λ

2 Description

We will do a linear stability analysis using a disturbance of the form,

$$(v_x, v_y, v_z) = (U, V, W) + (\delta u(y, z, t), \delta v(y, z, t), \delta w(y, z, t))e^{ikx} \tag{1}$$

$$T_{ij} = T_{ij} + \delta\tau_{ij}(t)e^{ikx} \tag{2}$$

The stress equations will be solved for the time derivatives and then these derivatives will be used to calculate the stresses at the next time step. However, before they can be solved, we must first find $\delta u, \delta v, \delta w, \delta p$. We do this using a stream function representation of the Navier-Stokes equations.

2.1 Calculating the stream functions

The stream functions are given by

$$\mathbf{v} = \nabla \times \nabla \times \phi \hat{\mathbf{j}} + \nabla \times \psi \hat{\mathbf{j}} \tag{3}$$

so that the disturbance velocities are

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \partial_{xy}\phi - \partial_z \psi \\ -\partial_{xx}\phi - \partial_{zz}\phi \\ \partial_{yz}\phi + \partial_x \psi \end{pmatrix} \tag{4}$$

By taking $\hat{\mathbf{j}} \cdot \nabla \times \nabla \times (\ \cdot \ )$ and $\hat{\mathbf{j}} \cdot \nabla \times (\ \cdot \ )$ of the Navier-stokes equation, we find

$$\begin{aligned} \nabla^4 \Delta_2 \phi = -\frac{1-\beta}{\beta Wi} \big(& -k^2 \partial_y \delta t_{xx} + (k^2 \partial_y - \partial_{yzz}) \delta t_{yy} + \partial_{yzz} \delta t_{zz} \\ & + (ik^3 - ik\partial_{zz} + ik\partial_{yy}) \delta t_{xy} + 2ik\partial_{yz} \delta t_{xz} \\ & + (k^2 \partial_z + \partial_{yyz} - \partial_{zzz}) \delta t_{yz} \big) \end{aligned} \tag{5}$$

$$\begin{aligned} \nabla^2 \Delta_2 = \frac{1-\beta}{\beta Wi} \big(& ik\partial_z \delta t_{xx} + \partial_{yz} \delta t_{xy} + (\partial_{zz} + k^2) \delta t_{xz} \\ & - ik\partial_y \delta t_{yz} - ik\partial_z \delta t_{zz}) \end{aligned} \tag{6}$$

where

$$\Delta_2 = \partial_{xx} + \partial_{zz} \tag{7}$$

The right hand sides of equations 5 and 6 are used to generate vectors $a(\mathbf{T})$ and $b(\mathbf{T})$, so matrix methods can be used to solve equations for the stream functions.

$$\left(-k^2 + \partial_{yy} - (n\gamma)^2\right)^2 (-k^2 - (n\gamma)^2)\phi_n = a \tag{8}$$

$$\left(-k^2 + \partial_{yy} - (n\gamma)^2\right) (-k^2 - (n\gamma)^2)\psi_n = b \tag{9}$$

Where n is the z Fourier mode. The boundary conditions will be applied to matrices for the Left-handside operators. These matrices will then be inverted to find the stream functions. This can be performed once at the beginning of the program for each z mode, giving $2N+1$ $(2N+1)M$ matrices. The matrices will be inverted in pieces, so that at most a second order derivative is inverted. For example, for equation 5 we define extra variables (for the purpose of illustration)

$$
\begin{aligned}
m &= \Delta_2 \phi_n && (10)\\
p &= \nabla^2 \Delta_2 \phi_n && (11)\\
a &= \nabla^4 \Delta_2 \phi_n && (12)\\
& && (13)
\end{aligned}
$$

These then allow us to form a matrix of simulutaneous equations and form an equation.

$$
\begin{bmatrix} -\Delta_2 & \mathbf{I} & 0 \\ 0 & -\nabla^2 & \mathbf{I} \\ 0 & 0 & \nabla^2 \end{bmatrix} \begin{pmatrix} \phi_n \\ m \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \tag{14}
$$

This matrix is then inverted. The top right block in the inverted matrix corresponding to the vector a and the stream function ϕ_n (the inverse of the operator) is then stored for each z Fourier mode.

2.1.1 Boundary conditions

The boundary conditions derived from equation 4 are,

$$
\begin{aligned}
\phi_n(\pm 1) &= 0 && (15)\\
\phi_n'(\pm 1) &= 0 && (16)\\
\psi_n(\pm 1) &= 0 && (17)\\
& && (18)
\end{aligned}
$$

where $\phi' \equiv \frac{\partial \phi}{\partial y}$. To impose the boundary conditions on ϕ and ψ equations must be added to the matrices for the calculation of the ϕ and ψ z-modes. The upper and lower boundary arrays are given by the coefficients of the stream function components in,

$$
\phi_n(\pm 1) = \sum_{m=0}^{M-1} (\pm 1)^m \widetilde{\phi}_{n,m} = 0 \tag{19}
$$

and similarly for ψ. Writing a similar equation for the derivative boundary condition gives,

$$
\phi_n(\pm 1) = \partial_y \sum_{m=0}^{M-1} \cos\left(m \arccos \pm 1\right) \widetilde{\phi}_m = 0 \tag{20}
$$

so that

$$\phi_n(\pm 1) = \sum_{m=0}^{M-1} \frac{-m \sin\left(m \arccos \pm 1\right)}{\sqrt{1 \mp 1}} \tilde{\phi}_m = 0 \tag{21}$$

Which at all m gives a coefficient of zero when $y = -1$ and is undefined for $y = 1$. I am not sure how to account for this boundary condition, but this does not seem right!

2.2 Velocities and pressure

Next I will need to calculate all components except the zeroth component of the velocities using equation 4. This just uses straight forward matrix products.

To calculate the zeroth component of the velocities, we will need to use the Navier-Stokes equation with the incompressibility equation for the zeroth mode. For the zeroth z mode all z deriviatives are zero. Firstly, we can calculate δw_0 component by solving,

$$0 = \beta \nabla_0^2 \delta w_0 + \frac{1-\beta}{Wi} \left(\nabla \cdot \delta\tau\right)_z \tag{22}$$

for δw_0, where $\nabla_0^2 = -k^2 + \partial_{yy}$. First we form a matrix for the operator ∇_0^2 and a vector containing the stress component. At this point we can impose boundary conditions on δw_0 such that it is zero at the walls. We can do this by adding rows of the boundary condition arrays to the operator matrix and inserting zeros into the vector on the RHS. Then we invert this operator and multiply by the vector to find δw_0.

To calculate the other velocities, we will use the Navier-Stokes equations and incompressibility to solve for δv_0 and then δp and δu_0. Boundary condition rows can also be inserted into the equations for δv_0 and δu_0. For δv we have,

$$\begin{aligned} &\frac{1}{ik}\left(-\beta \nabla_0^2 \frac{\partial_{yy}\delta v}{ik} + \frac{1-\beta}{Wi}\partial_y \left(\nabla \cdot \delta\tau\right)_{x0}\right) \\ =&\beta \nabla_0^2 \delta v + \frac{1-\beta}{Wi}\left(\nabla \cdot \delta\tau\right)_{y0} \end{aligned} \tag{23}$$

where $\left(\nabla \cdot \delta\tau\right)_{x0}$ is the x component of the divergence of the stress for the 0th z-mode. This simplifies to:

$$\beta \nabla_0^2 \left(1 + \frac{\partial_{yy}}{k^2}\right) \delta v_0 = \frac{1-\beta}{Wi}\left(\frac{\partial_y}{ik}\left(\nabla \cdot \delta\tau\right)_{x0} - ik\delta t_{xy} - \partial_y \delta t_{yy}\right) \tag{24}$$

Then using the incompressibility equation, the streamwise disturbance velocity is:

$$\delta u_0 = \frac{\partial_y \delta v_0}{ik} \tag{25}$$

To calculate the pressure we can used the x component of the Navier-stokes equation,

$$p_0 = \frac{\beta}{ik}\nabla^2 \delta u_0 + \frac{1-\beta}{ikWi}\left(\nabla \cdot \delta\tau\right)_{x0} \tag{26}$$

The operators for δw_0 and δv_0 equations should be created before the start of the main time iteration loop. The operator on the left-hand-side of equation 24 is fourth order, so to invert it I will again use the inversion method discussed above.

2.3 Time iteration

XX EQUATION:

$$\begin{aligned}\frac{\partial \delta c_{xx}}{\partial t} = -\frac{1}{W_i}\delta c_{xx} - \left[ikU + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}\right]\delta c_{xx} - \left[\delta v\frac{\partial}{\partial y} + \delta w\frac{\partial}{\partial z}\right]C_{xx} \\ +2ik\delta u C_{xx} + 2C_{xy}\frac{\partial \delta u}{\partial y} + 2C_{xz}\frac{\partial \delta u}{\partial z} + 2\delta c_{xy}\frac{\partial U}{\partial y} + 2\delta c_{xz}\frac{\partial U}{\partial z}\end{aligned} \tag{27}$$

YY EQUATION:

$$\begin{aligned}\frac{\partial \delta c_{yy}}{\partial t} = -\frac{1}{W_i}\delta c_{yy} - \left[ikU + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}\right]\delta c_{yy} - \left[\delta v\frac{\partial}{\partial y} + \delta w\frac{\partial}{\partial z}\right]C_{yy} \\ +2ik\delta v C_{xy} + 2C_{yy}\frac{\partial \delta v}{\partial y} + 2C_{yz}\frac{\partial \delta v}{\partial z} + 2\delta c_{yy}\frac{\partial V}{\partial y} + 2\delta c_{yz}\frac{\partial V}{\partial z}\end{aligned} \tag{28}$$

ZZ EQUATION:

$$\begin{aligned}\frac{\partial \delta c_{zz}}{\partial t} = -\frac{1}{W_i}\delta c_{zz} - \left[ikU + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}\right]\delta c_{zz} - \left[\delta v\frac{\partial}{\partial y} + \delta w\frac{\partial}{\partial z}\right]C_{zz} \\ +2ik\delta w C_{xz} + 2C_{yz}\frac{\partial \delta w}{\partial y} + 2C_{zz}\frac{\partial \delta w}{\partial z} + 2\delta c_{yz}\frac{\partial W}{\partial y} + 2\delta c_{zz}\frac{\partial W}{\partial z}\end{aligned} \tag{29}$$

XY EQUATION:

$$\frac{\partial \delta c_{xy}}{\partial t} = -\frac{1}{W_i}\delta c_{xy} - \left[ikU + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}\right]\delta c_{xy} - \left[\delta v\frac{\partial}{\partial y} + \delta w\frac{\partial}{\partial z}\right]C_{xy}$$
$$+C_{yy}\frac{\partial \delta u}{\partial y} + C_{yz}\frac{\partial \delta u}{\partial z} + \delta c_{yy}\frac{\partial U}{\partial y} + \delta c_{yz}\frac{\partial U}{\partial z}$$
$$+ik\delta v C_{xx} + C_{xz}\frac{\partial \delta v}{\partial z} + \delta c_{xy}\frac{\partial V}{\partial y} + \delta c_{xz}\frac{\partial V}{\partial z} - C_{xy}\frac{\partial \delta w}{\partial z} \tag{30}$$

XZ EQUATION:

$$\frac{\partial \delta c_{xz}}{\partial t} = -\frac{1}{W_i}\delta c_{xz} - \left[ikU + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}\right]\delta c_{xz} - \left[\delta v\frac{\partial}{\partial y} + \delta w\frac{\partial}{\partial z}\right]C_{xz}$$
$$+C_{yz}\frac{\partial \delta u}{\partial y} + C_{zz}\frac{\partial \delta u}{\partial z} + \delta c_{yz}\frac{\partial U}{\partial y} + \delta c_{zz}\frac{\partial U}{\partial z}$$
$$+ik\delta w C_{xx} + C_{xy}\frac{\partial \delta w}{\partial y} + \delta c_{xy}\frac{\partial W}{\partial y} + \delta c_{xz}\frac{\partial W}{\partial z} - C_{xz}\frac{\partial \delta v}{\partial y} \tag{31}$$

YZ EQUATION:

$$\frac{\partial \delta c_{yz}}{\partial t} = -\frac{1}{W_i}\delta c_{yz} - \left[ikU + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}\right]\delta c_{yz} - \left[\delta v\frac{\partial}{\partial y} + \delta w\frac{\partial}{\partial z}\right]C_{yz}$$
$$+ik\delta v C_{xz} + C_{zz}\frac{\partial \delta v}{\partial z} + \delta c_{zz}\frac{\partial V}{\partial z}$$
$$+ik\delta w C_{xy} + C_{yy}\frac{\partial \delta w}{\partial y} + \delta c_{yy}\frac{\partial W}{\partial y} - ik\delta u C_{yz} \tag{32}$$

Using the equations above we can now calculate the time derivatives of the stresses. Using a numerical iteration technique, we can find the new stress values after an appropriate time interval.

Before each iteration, we will calculate the magnitude of the instability (the 2-norm of the disturbance velocities and stresses vector). Using this we will build up a list of times and magnitudes. The gradient of the graph of the log of the magnitudes against time is the growth rate.

3 Variable names

3.1 parameters

- number of time steps NSTEPS

- width of a time step DT
- number of Fourier modes N
- number of Chebychev modes M
- gamma, the spanwise wavenumber of the base flow
- kx, the streamwise wavenumber of the disturbance
- Wi, the Weissenberg number
- Re, the Reynolds number
- L, the height of the system

3.2 operators

- derivative and double derivative operator arrays MDY, MDYY, MDZ, MDZZ
- List of $2 * N + 1$ arrays of $(2 * N + 1)M$ phi_operator
- List of $2 * N + 1$ arrays of $(2 * N + 1)M$ psi_operator
- Upper and lower boundary arrays BTOP, BBOT
- Upper and lower derivative boundary arrays DBTOP DBBOT
- Zeroth mode array dwZerothOp
- Zeroth mode array dvZerothOp

3.3 variables

- Arrays for U, V, W and Cij
- Arrays for du, dv, dw, dp, and $dcij$
- Small arrays for zeroth modes $du0$, $dv0$, $dw0$
- Arrays for ϕ and ψ phi and psi