The purely elastic self sustaining process in plane couette flow

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1. Introduction

We present a study of a purely elastic analogue of the Newtonian self-sustaining process presented by Fabian Waleffe **?**.

In 1997 Waleffe published a seminal article that demonstrated the existence of an exact coherent state that sustains turbulence in Newtonian plane Couette fluid turbulence. This solution consists of a self-sustaining process (SSP) of three phases. First the lift-up mechanism (sometimes called shear tilting), first studied by Ellingson, Palm and Landahl **??** induced by streamwise rolls moves fluid between the plates. Due to conservation of momentum, this creates spanwise streaks in the streamwise velocity. Second, these streaks become wavy in the streamwise direction. This is due to a linear instability in the streaky profile from a Kelvin-Helmholtz like instability as the streaks shear with the rest of the fluid. Finally, the nonlinear self-interaction terms from the disturbance which produces these wavy streaks regenerate the original streamwise rolls.

This mechanism has proven very important. Since then, similar structures have been found in Newtonian pipe flow, both in numerical simulations **?** and in experiments **?**. The success of these exact solutions has fuelled the move towards an understanding of Newtonian turbulence in terms of these exact solutions, with the turbulent attactor constructed out of many exact solutions which become more densely packed as the Reynold's number increases. According to this picture, a turbulent fluid trajectory in phase space 'pinballs' between these states, spending most of its time very close to one of these solutions **?**.

In 1990 Larson, Shaqfeh and Muller provided theoretical and experimental evidence of purely elastic instability **?**. The mechanism behind this instability is the Weissenberg effect**?**. Later, Byars et al. **?** found spiral patterns in a low Reynold's number plate-plate flow. These instabilities are part of a body of evidence which supports the existence of a different kind of turbulence, purely elastic turbulence, where the instabilities are generated and sustained by this Normal stress mechanism. Curved streamlines in the fluid lead to large normal stresses, which create more curved streamlines. It is this feedback

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which it is supposed will lead to a state of continuous instability. We think that this mechanism can be carried over to provide a viscoelastic analogue of the Newtonian SSP.

Studies have already been carried out on a viscoelastic self-sustaining process **???**. However, these studies were concerned with high Reynold's number viscoelastic flows. Primarly they were interested in discovering how a small concentration of polymeric fluid can bring about a reduction in viscous drag in a fluid, a phenomenon known as drag reduction. They sought to explain drag reduction by an appeal to the effect of the polymers on Waleffe's Newtonian self-sustaining process. These studies found that there is a minimum Reynold's number below which the SSP solution ceases to exist. We are find results which are consistent with this, however, the streak instabilty reappears at much lower Reynold's number, in the purely elastic regime. Neither the simulations by Sureshkumar **?** nor of Stone **?** give any results for Re 1. They also introduce an unphysical approximation to the equations via a stress diffusion term. It is not clear what effect this term will have in the purely elastic limit (or is it? I just don't know). The mechanism these papers suggest for the effect of the polymeric fluid on the SSP is that of vortex unwinding. The polymeric stress opposes the nonlinearities that produce the vortices in the third phase of the process. In our study, the polymeric stresses are responsible for the production of the voritices. But this mechanism cannot take place in the Navier-stokes equation, and must take place in the constituitive equation because we are at low Reynold's number.

In the absence of any clear notions of what might structure purely elastic turbulence, we have attempted to construct a viscoelastic analogue of the SSP. By analogy of the Weissenberg number with the Reynold's number, purely elastic fluids can be expected to undergo a transition to turbulence at high Weissenberg number. This is strongly suggested by previous nonlinear stability analyses e.g **?**.

We use the Oldroyd-B model for the constituitive equation for the polymeric component of the fluid and the full Navier-Stokes equations,

$$Re\left[\frac{d\mathbf{v}}{dt} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = -\nabla p + \beta\nabla^2\mathbf{v} + \frac{1-\beta}{Wi}\nabla\cdot\tau \tag{1.1}$$

$$\nabla\cdot\mathbf{v} = 0 \tag{1.2}$$

$$\dot{\tau}/Wi + \overset{\nabla}{\tau} = (\nabla\mathbf{v})^T + \nabla\mathbf{v} \tag{1.3}$$

where Re is the Reynold's number, Wi is the Weissenberg number, ∇ is the gradient operator, ∇^2 is the Laplacian and $\overset{\nabla}{\tau}$ is the upper convected derivative of the stress tensor. By direct numerical simulation, we investigate the three phases of the process. First we use a Newton-Raphson method to show that streaky flow is a solution at low Re to the these equations when forced by streamwise independent rolls. Then we do a linear stability analysis of the streaky flow to show that these streaks are unstable to perturbations both dependent and independent of the streamwise wavenumber k_x. Finally we investigate how the eigenvectors which correspond to these instabilities will affect the base flow, completing the SSP.

At this point it is important to note that we are using the Oldroyd-B fluid and so our results have limitations. They include all of the ingredients for the normal stress effect believed to be responsible for viscoelastic turbulence. However, they do not capture the shear thinning behaviour. It is also well known that under extensional flows (when $\lambda\dot{\epsilon} > 1/2$) the Oldroyd-B model breaks down. In the self-sustaining process as we have outlined it, there ought to be no extensional flows, so this danger turns out to be irrelevent (is this actually true?). Support for this point of view can be found in the studies mentioned earlier Stone **?** finds that in their simulations the polymer extension never exceeds 10% of the contour length (check understanding of this fact).

The viscoelastic streamwise vortex system we observe does produce streaks and these streaks do become unstable. The instabilities we observe can be wavy in the streamwise direction, but tend to have very long wavelengths. The instability can only be tracked for Free slip Boundary conditions, suggesting that polymer slip at the walls might be important in this system.

2. Streaky profile

We use a Chebyshev-Fourier decomposition, with Chebyshev polynomials in the wall-normal (y) direction and Fourier modes in the spanwise (z) direction. We then solve for the velocity in the streamwise (x) direction, subject to forcing in the y and z directions, via a Newton-Raphson iteration of the governing Navier-Stokes and Oldroyd-B equations. In order to decompose the system onto the computational grid, we take a Fourier and Chebyshev transform of the variables,

$$\check{G}(y,z) = \sum_{n=-N}^{N} \sum_{m=0}^{M+1} G_{m,n} \mathrm{e}^{i\gamma z} T_m(y) \tag{2.1}$$

where g stands for any of the base profile variables (U, V, W, τ_{ij}) in the problem and $T_m(y)$ is the mth Chebyshev polynomial of the first kind.

Given that the base profile is independent of the streamise direction and time, we reduce the Navier-Stokes and Oldroyd-B equations and then solve them using a simple Newton-Rhaphson iteration for the base Streamwise velocity U. The system is driven by the standard no-slip boundary conditions on the streamwise velocity at the walls,

$$U(\pm 1) = 1 \tag{2.2}$$

$$\tag{2.3}$$

and forcing terms introduced via fixing of the wall-normal and spanwise base profile velocities,

$$V(y,z) = V_0 \hat{v}(y) cos(\gamma z) \tag{2.4}$$

$$W(y,z) = \frac{V_0}{\gamma} \frac{\partial \hat{v}}{\partial y} sin(\gamma z) \tag{2.5}$$

Where,

$$\hat{v}(y) = \frac{\cos(py)}{\cos(p)} - \frac{\cosh(\gamma y)}{cosh(\gamma)} \tag{2.6}$$

p is given by solutions to $p \tan p + \gamma \tanh \gamma = 0$ and governs the number of rolls in the wall-normal direction. These velocities are a guess for the rolls derived from the lowest order eigenmode of the operator ∇^4, precisely the same rolls used in **?**. This provides us with the streaky profile shown below.

After using the above Newton-Rhaphson method we obtain a streaky profile in the fluid. At high Re we find streaks in the streamwise velocity similar to those in **?**. However, as we decrease the Reynold's number for constant and large Weissenberg number, we find that the streaks in the streamwise velocity become less pronounced. This can be explained by considering that at low Reynold's number the fluid has a higher inertia, so it is more difficult for the lift up effect to produce streaks in the streamwise velocity.

Although we do not see streaks in the streamwise velocity, we do see them in the first

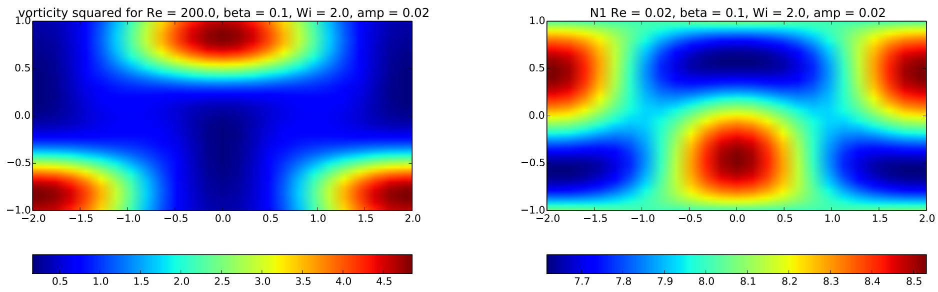

Figure 1: a) The magnitude of the vorticity of the fluid at $Re = 200$, $Wi = 2.0$, amp=0.02. b) The first normal stress difference in the polymeric fluid at $Re = 0.02$, $Wi = 2$, amp=0.02. The Newtonian vorticity and purely elastic first normal stress difference have large gradients in similar regions. However, the first normal stress difference also shows large gradients next to the wall.

normal stress difference, $N_1 = T_{xx} - T_{yy}$ (figure **??**). These streaks appear in much the same place as the streamwise velocity streaks appear in the Newtonian self-sustaining process. As noted earlier, instabilities in viscoelastic fluids are brought about by large changes in the first normal stress difference, since this brings about polymer stretching of the kind seen in the Weissenberg effect. It is important to note that the purely elastic 1st normal stress difference shows very large gradients near the wall, suggesting that resolving the instability in this base profile will be more difficult than in the Newtonian case.

3. Linear Stability Analysis

Having obtained the full base profile of the problem, we then performed a linear stability analysis, to look for instabilities that might produce waviness in the streaks. These wavy instabilities are responsible for sustaining the exact coherent state in the Newtonian version of the process.

We decompose the disturbance velocities in a similar way to above (equation 2.4), but include streamwise dependence

$$\breve{g}(x, y, z, t) = \sum_{n=-N}^{N} \sum_{m=0}^{M+1} g_{m,n}(k_x) \mathrm{e}^{i(k_x x + \gamma z)} T_m(y) \mathrm{e} i \lambda t \tag{3.1}$$

where g can be any of the disturbance variables (u, v, w, p, τ_{ij}). To increase the numerical stability of the problem, we use free slip boundary conditions on the disturbance velocities, $\frac{\partial w}{\partial y}(\pm 1) = v(\pm 1) = \frac{\partial w}{\partial y}(\pm 1) = 0$.

The linearised system of equations now gives an eigenvalue problem for the growth rate of the instability λ at every streamwise wavenumber of the disturbance.

The solution to the eigenvalue problem provides spectra for which eigenmodes with positive growth rates are unstable.

As the Reynold's number is decreased, we find that the base profile becomes more stable. The dispersion relation is reduced in height and moves to lower streamwise wavenumbers. By about $Re = 100$ the base profile has become completely stable. The Newtonian

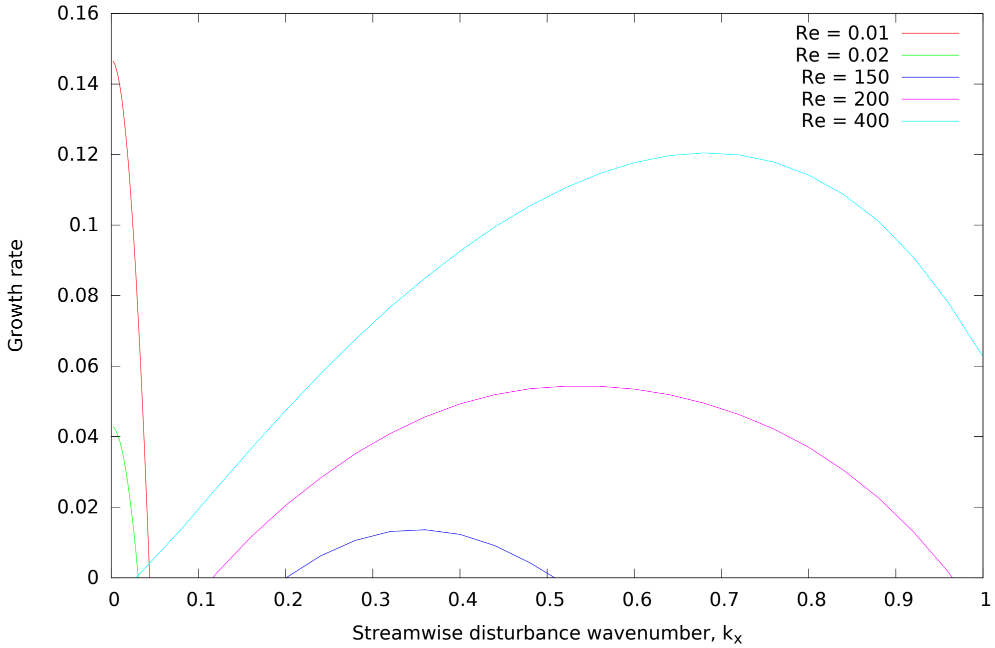

Figure 2: Dispersion relations as the Reynold's number is decreased at $Wi = 2$, $\beta = 0.1$ and amp $= 0.02$. A clear instability can be seen at low k_x in the purely elastic regime.

instability is no longer present at this Reynold's number. However, once the Reynolds number becomes negligible in comparison to the Weissenberg number, we begin to see a purely elastic instability arise at very low streamwise wavenumber 2. This purely elastic instability is hugely amplified by further reductions in the Reynold's number.

We can further examine how the purely elastic instability changes with changing Weissenberg number. We find that it grows as the Weissenberg number increases, and saturates by around Wi 20. Doubling the amplitude of the rolls increases the width of the instablity by about a third and the height, 3 fold.

4. Nonlinear feedback on the rolls

In Fabian Waleffe's treatment of the Newtonian version of this self-sustaining process **?**, he found that the feedback on the original rolls was provided by the eigenmode of the instability. The eigenmodes of the viscoelastic instability is quite different to that of the Newtonian instability. Although the components of the instability are large at the walls, the velocity is less important for the instability of the purely elastic fluid than the first normal stress difference. We find that N_1 is large away from the walls, another encouraging sign for a viscoelastic self-sustaining process (4).

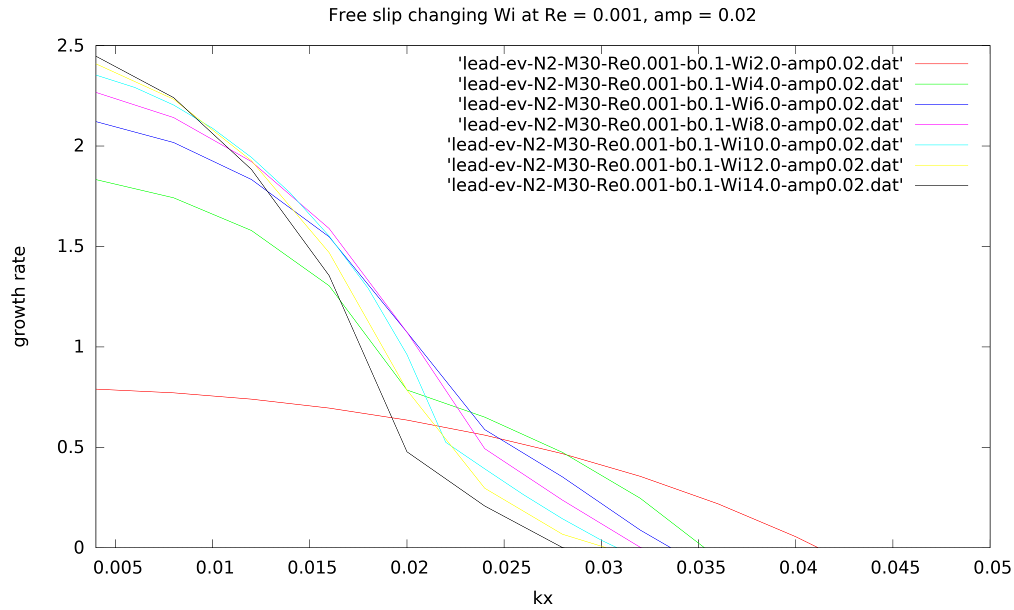

Figure 3: Dispersion relations as the Wi is decreased at $Re = 0.001$, $\beta = 0.1$ and amp $=$ 0.02. The instability grows and then saturates.

4.1. *Cauchy boundary conditions*

To investigate the affect of free slip boundary conditions on linear stability analysis, we implemented Cauchy boundary conditions,

$$\alpha \left.\frac{\partial u}{\partial y}\right|_{y=\pm1} + (1-\alpha)u(y=\pm1) = 0 \tag{4.1}$$

$$v(y=\pm1) = 0 \tag{4.2}$$

$$\alpha \left.\frac{\partial w}{\partial y}\right|_{y=\pm1} + (1-\alpha)w(y=\pm1) = 0 \tag{4.3}$$

Where α is a control parameter which controls the relative strength of the free slip to no-slip conditions. (Slip length?)

We found that on decreasing α the instability grows in strength and broadens. It appears that the no-slip system is infinitely unstable (figure 5).

5. No slip case

When we completely remove slip at the walls the instability appears infinitely amplified (figure 6). A possible explanation for this is that the instability moves into a region very close to the boundaries as we introduce free slip and can no longer be resolved. It is clear from figure 7 that the first normal stress difference introduced via the instability moves towards the walls as α tends to zero.

6. Discussion

I do not understand at all how to interpret these results.

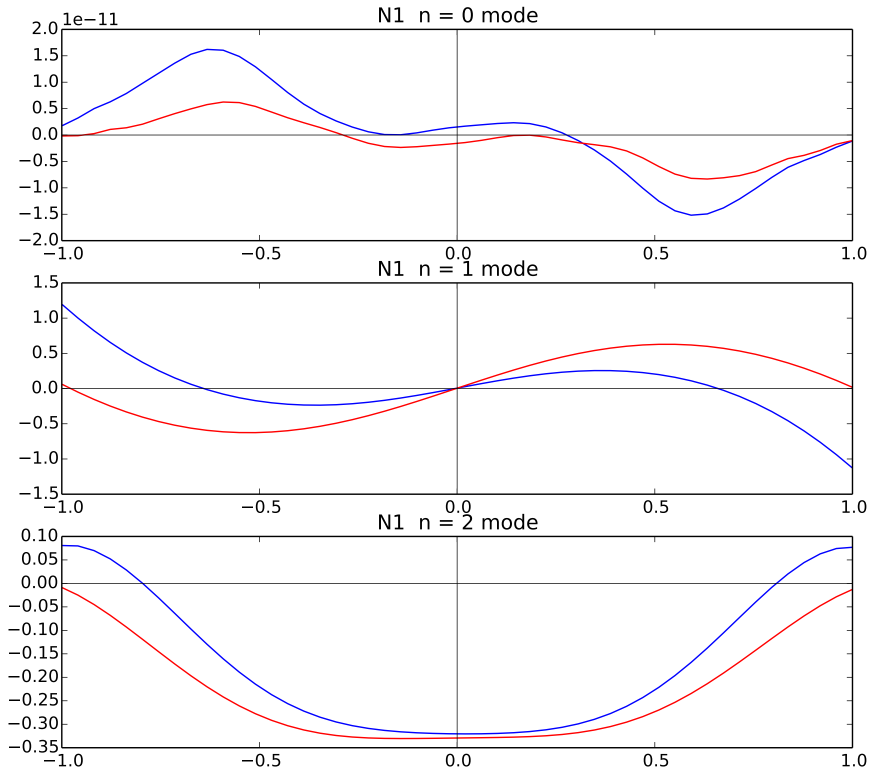

Figure 4: Eigenmodes of the first normal stress difference of the viscoelastic instability at $Wi = 2.0$, $Re = 0.02$, $\beta = 0.1$, amp=0.02 and $k_x = 0.01$.

7. Conclusion

We have shown that the velocity profile for the plane Couette viscoelastic self-sustaining process is susceptible to a viscoelastic lift up effect and does become streaky in the purely elastic regime. This streaky base profile is linearly unstable, with a range of wave numbers between about $k_x = 0$ and $k_x = 0.04$. The eigenmodes of this instability are localised in the centre of the channel.

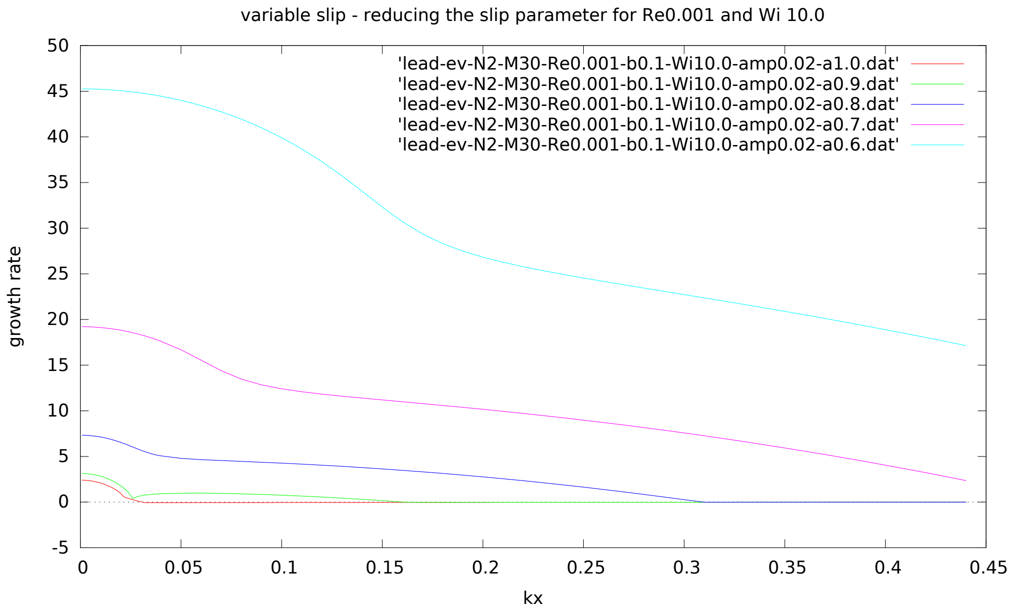

Figure 5: Dispersion relations as α the slip parameter, is reduced. As the system becomes closer to no-slip at the walls the instability grows.

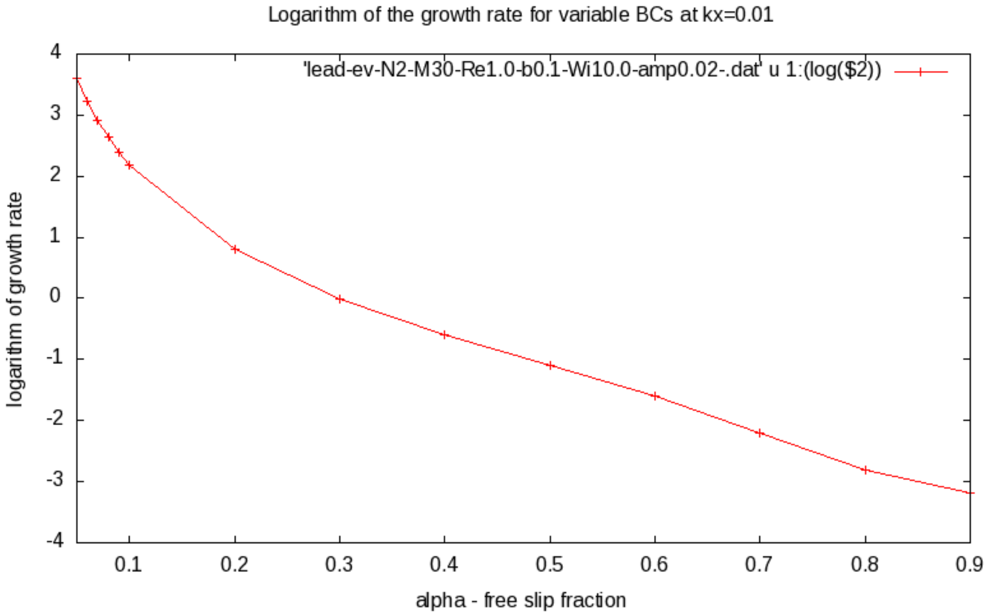

Figure 6: Growth rate of an eigenvalue at $k_x = 0.01$ as α the slip parameter, is reduced. The growth rate assymtotes to the no-slip condition.

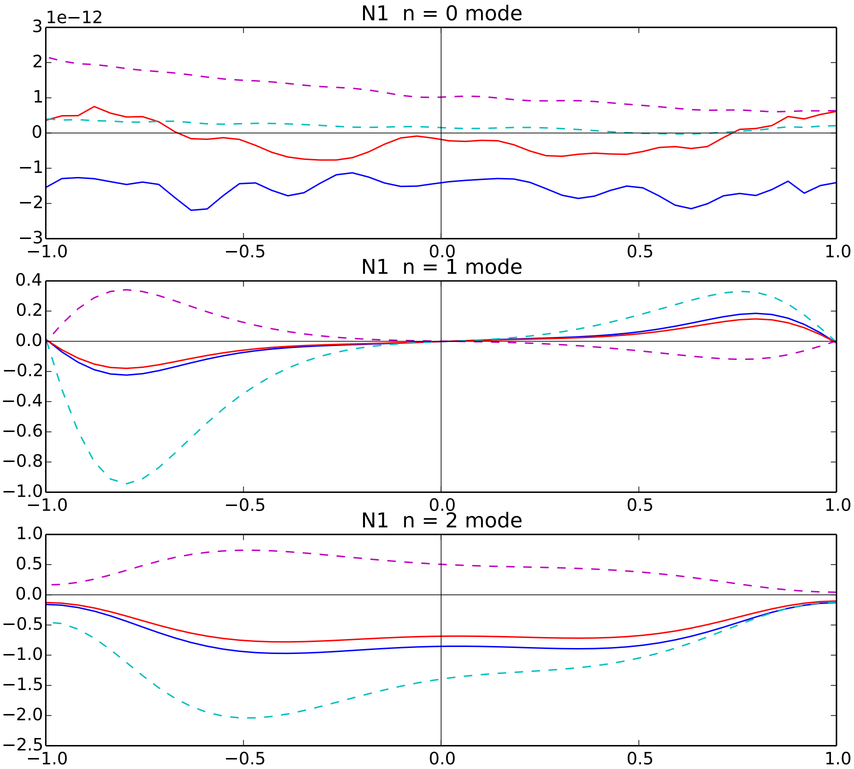

Figure 7: Eigenmodes of the first normal stress difference at $k_x = 0.01$ as α the slip parameter, is reduced. As the slip at the walls is reduced, the first normal stress difference of the instablity becomes more and more localised at the walls.