Supplementary File S2: Assessing the modified ABC and regression adjustment using a numerical experiment

1 Introduction

This supplementary file presents a simple numerical experiment with multivariate normal likelihood function (where the presumed true model parameter values and the exact form of the true posterior are known) to assess the modified ABC approximations (using Algorithm 4) and the proposed ABC posterior adjustment methodology (described in the main paper) at a different number of proposal draws (i.e., $N = 500$, 1000, 2000, 3000, 4000, and 5000 samples); where we further explore whether the resulting approximated posterior is independent of N, and mutually compatible ABC approximations are achieved at the different values of N). Here, the modified regression adjustment is also compared with the standard local-linear regression adjustment with heteroscedastic errors proposed by Beaumont et al. [1]. Findings from the numerical experiments are also used to determine the minimal number of proposal draws needed when fitting the complex stochastic simulation model (formally described in main paper) due to the high computational costs involved in i) model simulation, ii) estimation of the multidimensional summary statistics for the entire host population (especially the summary component which estimates the B-D-C model parameters during realisations of parasite population explosion as discovered in the Supplementary File S1), and iii) implementing sequential Monte Carlo ABC methods (whose computational cost increases quadratically as a function of N).

2 Description of the toy model and modelling problem

For the numerical experiment based a toy model defined below, artificial multivariate data $\mathbf{X} \sim \mathcal{N}_k(\boldsymbol{\theta}, \Sigma)$ was simulated from a multivariate normal (MVN) distribution for a k-dimensional random variables $\mathbf{X} = (X_1, X_2, \cdots, X_k)^\top$ with k-dimensional mean vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_k)^\top$ and $k \times k$ covariance matrix Σ. For simplicity, we set $k = 6$ and assume that the true mean vector (or population mean) of $\mathbf{X}$ (in the toy model) is $\boldsymbol{\theta} = (0.5, 1.0, 1.5, 2.0, 2.5, 3.0)^\top$ with a positive-definite symmetric covariance matrix $\Sigma = \mathrm{Var}(\mathbf{X})$ (which was randomly generated in R for the toy model) also known. Specifically, we randomly assumed that

$$\mathbf{X}=\begin{pmatrix}X_1\\X_2\\X_3\\X_4\\X_5\\X_6\end{pmatrix}\sim\mathcal{N}_6\left[\begin{pmatrix}0.5\\1.0\\1.5\\2.0\\2.5\\3.0\end{pmatrix},\quad\begin{pmatrix}24.5134 & 11.6042 & 8.2851 & 15.6787 & 19.6029 & 18.4657\\11.6042 & 36.1535 & 16.9813 & 9.1931 & 12.6557 & 33.0837\\8.2851 & 16.9813 & 24.7937 & 5.0379 & 18.2924 & 16.5758\\15.6787 & 9.1931 & 5.0379 & 16.1338 & 11.7926 & 9.8223\\19.6029 & 12.6557 & 18.2924 & 11.7926 & 35.7758 & 18.0006\\18.4657 & 33.0837 & 16.5758 & 9.8223 & 18.0006 & 35.2091\end{pmatrix}\right]. \tag{1}$$

Toy model:

Let suppose $\mathbf{X}$ (with n number of observations) is a multivariate data randomly generated from a 6-dimensional MVN distribution with density function $f_{\mathbf{X}}(\cdot \mid \boldsymbol{\theta},\Sigma)$ given by equation 2, where the population mean vector $\boldsymbol{\theta}=(\theta_1,\theta_2,\cdots,\theta_6)^\top$ (with 6 unknown model parameters) and known covariance matrix Σ (as specified in equation 1); such that

$$f_{\mathbf{X}}\left(X_1,X_2,\cdots,X_6;\boldsymbol{\theta},\Sigma\right)=\frac{1}{\sqrt{(2\pi)^6\left(\det\Sigma\right)}}\exp\left\{-\frac{1}{2}\left(\mathbf{X}-\boldsymbol{\theta}\right)^\top\Sigma^{-1}\left(\mathbf{X}-\boldsymbol{\theta}\right)\right\}. \tag{2}$$

Assuming that $\boldsymbol{\theta}\in\mathbb{R}^6$ is also a random variable, let suppose the prior distribution of $\boldsymbol{\theta}$, $\pi(\boldsymbol{\theta})\propto\mathcal{N}_6\left(\boldsymbol{\mu}_0,\Sigma_0\right)$ is MVN with mean $\boldsymbol{\mu}_0$ and covariance matrix Σ_0.

Modelling problem:

Given the simulated MVN data $\mathbf{X}$ (whose population mean vector is assumed to be unknown), and the MVN prior density $\pi(\boldsymbol{\theta})\propto\mathcal{N}_6\left(\boldsymbol{\mu}_0,\Sigma_0\right)$; we want to estimate the posterior predictive distribution $p(\boldsymbol{\theta}\mid\mathbf{X},\Sigma)$ using the weighted-iterative ABC algorithm (outlined in the main paper) as well as perform regression adjustment using both the proposed posterior correction method (defined in the main paper) and standard local-linear regression adjustment with heteroscedastic errors proposed by Beaumont et al. [1]. Here, we assume that the true likelihood $f(\mathbf{X}\mid\boldsymbol{\theta},\Sigma)$ is unknown (for the sake of ABC fitting and assessment). The accuracy of the posterior estimates from the ABC fitting and ABC post-processing analyses (based on the toy model) at different values of N (where N is the number of proposal draws) are also compared to the true hyperparameter posterior mean estimator of $\boldsymbol{\theta}\in\mathbb{R}^6$ defined in accordance with Lemma 1 and the true parameter values using some standard accuracy measures (i.e., the bias, variance and mean square error of the posterior estimates as well as their corresponding 95% credible intervals).

Lemma 1. Suppose that $\mathbf{X}\sim\mathcal{N}_k\left(\boldsymbol{\theta},\Sigma\right)$ is a multivariate data (with sample size of n) generated from a MVN distribution with unknown mean vector $\boldsymbol{\theta}\in\mathbb{R}^k$ and known covariance matrix Σ. Let assume that the prior distribution of $\boldsymbol{\theta}$, $\pi(\boldsymbol{\theta}) \propto \mathcal{N}_k(\boldsymbol{\mu}_0, \Sigma_0)$ is multivariate normal with mean $\boldsymbol{\mu}_0 \in \mathbb{R}^k$ and covariance matrix Σ_0. Then, given the MVN data $\mathbf{X}$, the resulting posterior distribution $p(\boldsymbol{\theta} \mid \mathbf{X}, \Sigma)$ and prior $\pi(\boldsymbol{\theta})$ are conjugate distributions; such that $p(\boldsymbol{\theta} \mid \mathbf{X}, \Sigma) \propto \mathcal{N}_k\left(\hat{\boldsymbol{\theta}}_n, \Sigma_n\right)$ is MVN with the exact posterior hyperparameter mean estimator given as

$$\hat{\boldsymbol{\theta}}_n = \Sigma_n\left(\Sigma_0^{-1}\boldsymbol{\mu}_0 + n\Sigma^{-1}\bar{\mathbf{X}}\right), \tag{3}$$

where the covariance matrix $\Sigma_n = (\Sigma_0^{-1} + n\Sigma^{-1})^{-1}$, $\bar{\mathbf{X}} \in \mathbb{R}^k$ is the sample mean vector, and n is the sample size of the observed data [2].

Remark. For the numerical experiment, pseudo-observed data with a sample size of $n = 1000$ was simulated (in R software) from MVN with true mean and covariance matrix specified per equation 1 (where the true population mean was already known). This pseudo-observed data was considered as the observed data to be used for ABC fitting, where we assumed the true parameter mean vector $\boldsymbol{\theta} \in \mathbb{R}^6$ of the pseudo-observed data is unknown with 6 model parameters (as a form of an inverse problem), but the covariance matrix Σ is known. For simplicity, we also set the covariance matrix of the prior $\Sigma_0 = \Sigma$ (since Σ is not to be estimated, and thus, not of interest). We further assume that the prior mean vector $\boldsymbol{\mu}_0 \in \mathbb{R}^6$ is the only hyperparameter to consider in the prior distribution $\pi(\boldsymbol{\theta})$ during the ABC analysis and evaluation of the exact posterior hyperparameter mean estimator given by equation 3 under Lemma 1. Additionally, since the sample mean vector $\bar{\mathbf{X}}$ is known to be a sufficient summary statistics for the population mean of MVN distribution (with known covariance matrix), the sample mean vector was considered as summary statistics for both pseudo-observed and simulated data from the toy model during ABC fitting (based on the prior or proposal samples). The weighted Euclidean distance metric (given by equation 4) was used as the discrepancy measure in the proposed weighted-iterative ABC algorithm to compare between the pseudo-observed and simulated data; where

$$\rho(y_{\text{sim}}, y_{\text{obs}}) = \left[\sum_{j=1}^{m} \omega_j \left(s_{\text{sim},j} - s_{\text{obs},j}\right)^2\right]^{1/2}. \tag{4}$$

The main results from the numerical experiment (based on the toy model) are presented in section 3.

3 Summary of results from the numerical experiment

The MVN model (with 6 model parameters described in section 2) was fitted using the modified weighted-iterative ABC with sequential Monte Carlo and adaptive importance sampling (given by Algorithm 4 in the main paper). The modified ABC algorithm was set-up to have a fixed number of iterations or time steps at $T = 10$ (i.e., a total of 10 time steps), and a set of monotonically decreasing tolerances (ϵ_t, $t = 1, 2, \cdots, 10$) at each ABC time step t was carefully pre-specified based on the total number of proposal draws or prior samples (N) according to the following: if $N < 1000$, $\epsilon_t = 0.5$, 0.43, 0.4, 0.35, 0.3, 0.2, 0.1, 0.08, 0.06, 0.02; whereas if $N \geq 1000$, $\epsilon_t = 0.5$, 0.3, 0.2, 0.1, 0.08, 0.07, 0.06, 0.03, 0.02, 0.01. To examine the robustness of the modified weighted-iterative ABC based on the choice of N and pre-specified tolerances, the ABC fitting of the toy model was done at different values of N: $N = 500$, 1000, 2000, 3000, 4000, and 5000, respectively; and the resulting posterior distributions were further adjusted to estimate the posterior mean of the model parameter $\boldsymbol{\theta}$ using the proposed posterior correction method (defined in the main paper) and standard local-linear regression adjustment with heteroscedastic errors proposed by Beaumont et al. [1] (for comparison purposes) across the 6 model parameters.

Figure 1 is a comparative plot showing the variability in the unadjusted posterior mean estimates of the model parameters with their respective 95% credible intervals at the different values of N. Figure 1 shows that the ABC approximations from the weighted-iterative ABC with SMC and SIS (Algorithm 4) resulted in mutually compatible approximations at the different values of N based on the pre-specified tolerance and ABC time steps. Thus, it can be inferred (from Figure 1) that the resulting posterior from the modified ABC-SMC algorithm is independent of N (for $500 \leq N \leq 5000$), and the degree of variability in the posterior distributions are not significantly different irrespective of the number of proposal draws from the importance distribution with density g defined by equation 5 (from $N = 500$ to $N = 5000$);where,

$$g_t(\theta) = \sum_{i=1}^{N} W_i^{(t-1)} K_t\left(\theta \mid \theta_i^{(t-1)}\right) / \sum_{i=1}^{N} W_i^{(t-1)}. \tag{5}$$

Nonetheless, Figure 2 indicates that the computational cost (or cost in time) increases quadratically as the number of proposal draws increases from $N = 500$ to $N = 5000$ during ABC fitting of the toy simulation model (run in parallel using over 20 CPU cores of a multi-core processor). Based on Figures 1 and 2, it will be cost-effective to fit the complex stochastic simulation (described in the main paper) at $N = 500$ due to the high computational cost associated with model simulation from the complex model (especially during realisations of parasite population explosion), estimation of the multidimensional summary statistics (across the entire host population) and the potential cost of implementing the modified ABC algorithm at higher values of $N \gg 500$.

After ABC fitting of the toy model, the approximate posterior mean was estimated and its posterior distribution adjusted using the modified regression adjustment with $L2$ regularisation and the standard local-linear regression adjustment (with heteroscedastic errors) at the different values of N. Figures 3–8 are goodness-of-fit density plots at the different values of N, which graphically show the unadjusted and adjusted posterior distributions against the prior distributions (of the fitted toy simulation model). It can be seen from the goodness-of-fit density plots that the unadjusted posterior based on the modified ABC algorithm as well as the adjusted posterior using the modified regression adjustment (with heteroscedastic errors and $L2$ regularisation) performed well when compared to the prior distribution across all model parameters at the different values of N. However, at $1000 \leq N \leq 4000$, the posterior adjustments from Beaumont et al. [1] standard local-linear regression (with heteroscedastic errors) resulted in very flat and poorly adjusted for a few model parameters (relative to the prior distribution). Hence, the modified regression adjustment, which can deal with potential multicollinearity in the regression predictors (in the neighbourhood of the observed summaries), supercollinearity and shrink regression coefficients of predictors with less contribution, appears to be more robust in adjusting the posterior distribution than the standard correction method. At certain simulation realisations with very high multicollinearity, the standard local-linear regression could not be implemented at all or performed poorly since the design matrix X or the term $X^\top W X$ (where W was the diagonal weighting matrix) was either singular or close to being singular. The unadjusted and adjusted posterior means were computed and compared with the true parameter values and the true posterior mean estimates at the different values of N.

Additionally, the bias, variance, and mean square error (MSE) of the posterior mean estimates, as well as their corresponding 95% credible intervals, were estimated to compare the performance of approximations from the weighted-iterative ABC and the two regression adjustments numerically (Tables 1–3). Generally, there was no significant difference in the degree of accuracy between the unadjusted and adjusted posterior mean estimates at the different values of N; and the true posterior mean estimates (based on equation 3) was found in their respective estimated credible intervals. However, the MSE of the adjusted posterior mean based on the proposed regression correction (with $L2$ regularisation) resulted in relatively smaller MSE and credible interval width most of the time (especially at $N \leq 1000$). Hence, it can be inferred from Tables 1–3 that the proposed regression adjustment is relatively robust in estimating the posterior mean compared to the standard local-linear regression of Beaumont et al. [1]. In addition, it can further be adapted to estimate the posterior mean after ABC fitting of the complex stochastic simulation model in the presence of high multicollinearity (since the multidimensional summary statistics to be used for calibrating the complex model appears to be highly correlated). Also, since the number of predictor variables exceed the number of posterior samples at $N = 500$ (a condition which results in suppercollinearity), the proposed regression adjustment will be more suitable for posterior correction. Hence, the standard local-linear regression may not be possible to be implemented or result in incorrect adjustments in these aforementioned instances.

Remark. Based on findings from the numerical experiment, it is recommended to fit the complex simulation model based on $N = 500$ proposal draws from the importance density, using the weighted-iterative ABC algorithm since it will be cost-effective to calibrate the complex model at $N = 500$; whereas the adjusted posterior of the resulting ABC posterior (based on the modified ABC regression correction) is considered for subsequent analyses including hypotheses testing.

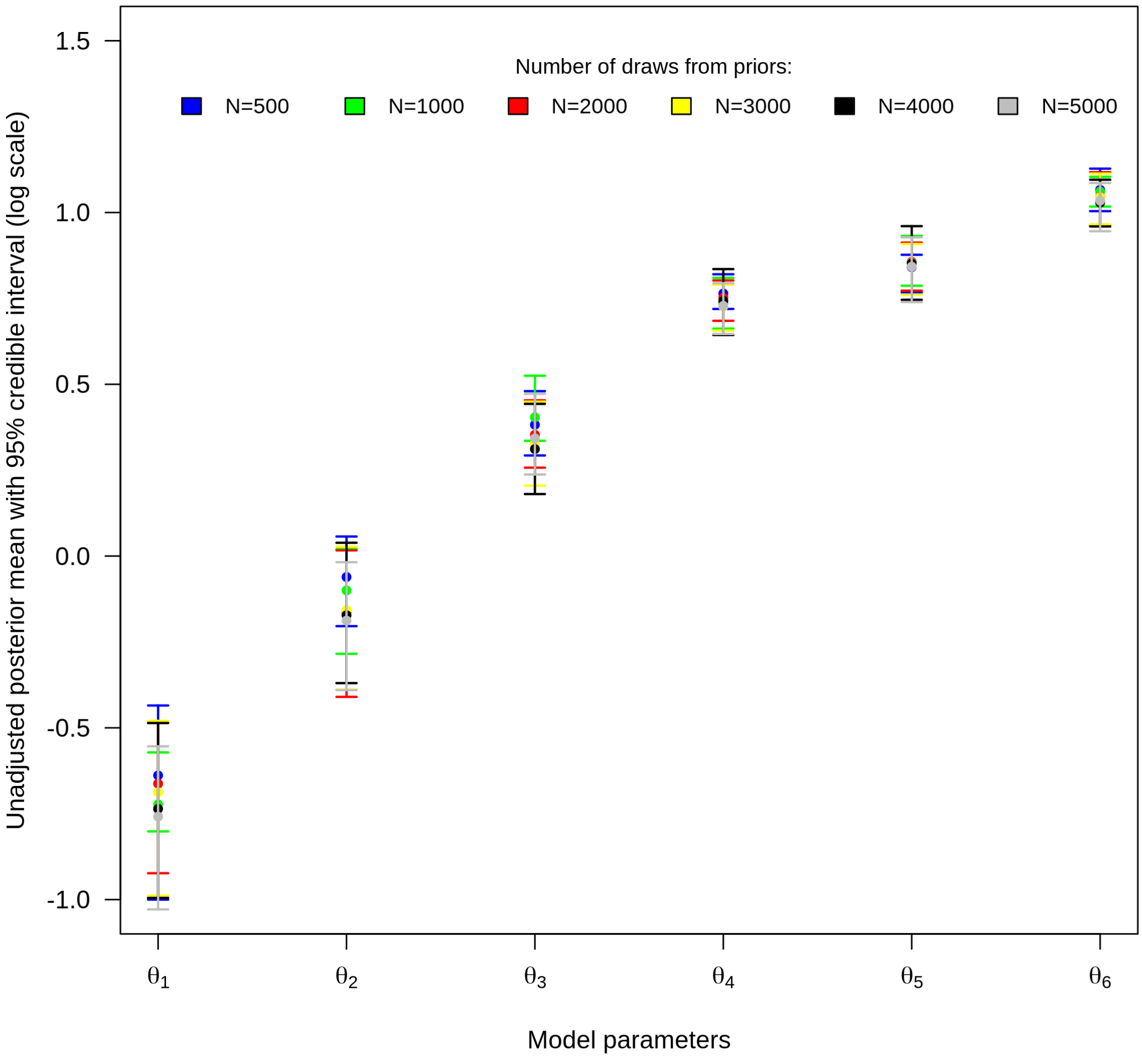

Figure 1: Comparative plot of the unadjusted posterior mean estimates of the toy model parameters with their respective 95% credible intervals (on logarithmic scale) at different values of N ($N = 500$, 1000, 2000, 3000, 4000, and 5000 proposal samples).

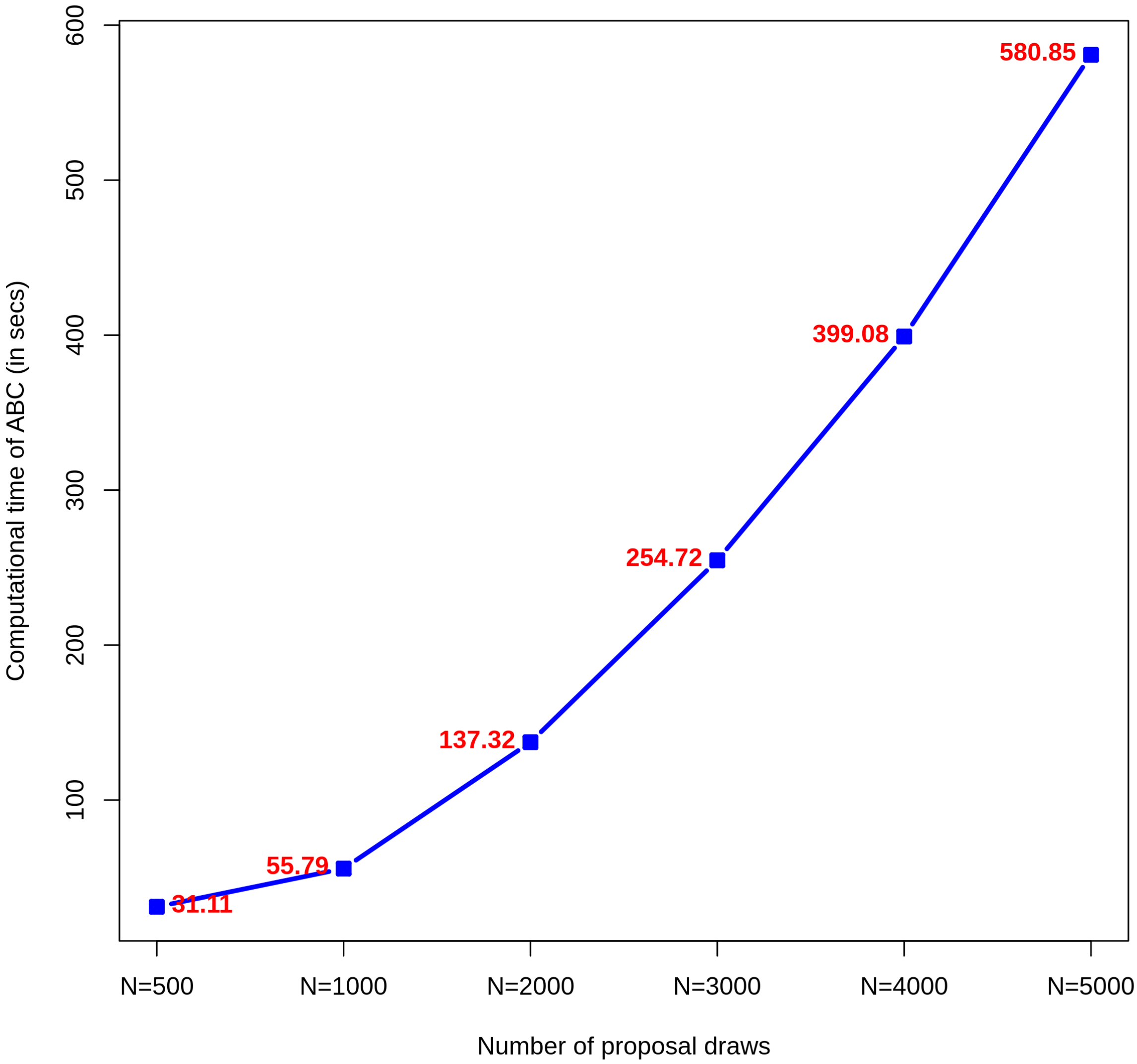

Figure 2: Computational times (in secs) of ABC fitting of the toy model at different values of N ($N = 500$, 1000, 2000, 3000, 4000, and 5000 proposal samples).

Table 1: Comparison between unadjusted ABC posterior mean estimates ($\hat{\boldsymbol{\theta}}_{\text{unadj}}$), true parameter values ($\hat{\boldsymbol{\theta}}_0$) and conjugate posterior mean estimates ($\hat{\boldsymbol{\theta}}_{\text{conjugate}}$) of the 6 parameters of the toy model across different values of N (from $N = 500$ to $N = 5000$).

Parameters	$\hat{\boldsymbol{\theta}}_{\text{unadj}}$	$\hat{\boldsymbol{\theta}}_0$	$\hat{\boldsymbol{\theta}}_{\text{conjugate}}$	bias($\hat{\boldsymbol{\theta}}_{\text{unadj}}$)	Var($\hat{\boldsymbol{\theta}}_{\text{unadj}}$)	MSE($\hat{\boldsymbol{\theta}}_{\text{unadj}}$)	95% Cred. Int.
				N=500			
θ_1	0.5280	0.5	0.5063	0.02802	0.0086	0.0094	0.3690—0.6477
θ_2	0.9408	1.0	0.8980	-0.0592	0.0065	0.0100	0.8162—1.0587
θ_3	1.4656	1.5	1.4664	-0.0344	0.0099	0.0111	1.3403—1.6164
θ_4	2.1475	2.0	2.1252	0.1475	0.0062	0.0279	2.0530—2.2712
θ_5	2.3186	2.5	2.4327	-0.1814	0.0083	0.0412	2.1536—2.4037
θ_6	2.9038	3.0	2.9467	-0.0962	0.0162	0.0255	2.7292—3.0889
				N=1000			
θ_1	0.4853	0.5	0.5058	-0.0147	0.0015	0.0017	0.4487—0.5657
θ_2	0.9047	1.0	0.8975	-0.0953	0.0076	0.0166	0.7542—1.0208
θ_3	1.4979	1.5	1.4629	-0.0021	0.0097	0.0097	1.3987—1.6916
θ_4	2.0873	2.0	2.1241	0.0874	0.0103	0.0179	1.939—2.2482
θ_5	2.3485	2.5	2.4320	-0.1515	0.0108	0.0338	2.1973—2.5405
θ_6	2.8892	3.0	2.9465	-0.1108	0.0083	0.0206	2.7659—3.0162
				N=2000			
θ_1	0.5155	0.5	0.5074	0.0155	0.0045	0.0047	0.3977—0.6163
θ_2	0.8487	1.0	0.8964	-0.1513	0.0102	0.0331	0.6647—1.0167
θ_3	1.4233	1.5	1.4584	-0.0767	0.0068	0.0127	1.2936—1.5739
θ_4	2.1219	2.0	2.1216	0.1219	0.0056	0.0205	1.9836—2.2309
θ_5	2.355	2.5	2.4305	-0.1450	0.0087	0.0297	2.1661—2.4906
θ_6	2.8469	3.0	2.9465	-0.1531	0.0191	0.0425	2.6208—3.0568
				N=3000			
θ_1	0.5028	0.5	0.5070	0.0028	0.0055	0.0055	0.3732—0.6188
θ_2	0.8546	1.0	0.8971	-0.1454	0.0113	0.0325	0.6780—1.0287
θ_3	1.3874	1.5	1.4592	-0.1126	0.0099	0.0225	1.2278—1.5689
θ_4	2.0671	2.0	2.1228	0.0671	0.0072	0.0117	1.9282—2.2036
θ_5	2.3484	2.5	2.4321	-0.1516	0.0113	0.0342	2.1416—2.4808
θ_6	2.8438	3.0	2.9476	-0.1562	0.0160	0.0404	2.6251—3.0456
				N=4000			
θ_1	0.4791	0.5	0.5070	-0.0209	0.0046	0.0051	0.3697—0.6150
θ_2	0.8417	1.0	0.8968	-0.1583	0.0077	0.0328	0.6909—1.0394
θ_3	1.3654	1.5	1.4595	-0.1345	0.0119	0.0300	1.1978—1.5574
θ_4	2.1020	2.0	2.1231	0.1020	0.0132	0.0236	1.9029—2.3057
θ_5	2.3448	2.5	2.4320	-0.1552	0.0191	0.0431	2.1084—2.6126
θ_6	2.7976	3.0	2.9473	-0.2024	0.0103	0.0513	2.6104—2.9908
				N=5000			
θ_1	0.4683	0.5	0.5072	-0.0317	0.0039	0.0049	0.3577—0.5750
θ_2	0.8288	1.0	0.8965	-0.1712	0.0073	0.0366	0.6769—0.9823
θ_3	1.4084	1.5	1.4590	-0.0915	0.0109	0.0193	1.2680—1.6036
θ_4	2.0716	2.0	2.1221	0.0716	0.0081	0.0132	1.9085—2.2131
θ_5	2.3222	2.5	2.4319	-0.1778	0.01541	0.0470	2.0948—2.5289
θ_6	2.8138	3.0	2.9468	-0.1862	0.0135	0.0482	2.5742—2.9624

Table 2: Comparison between the standard adjusted posterior mean estimates ($\hat{\boldsymbol{\theta}}_{\text{adj}}$), true parameter values ($\hat{\boldsymbol{\theta}}_0$) and conjugate posterior mean estimates ($\hat{\boldsymbol{\theta}}_{\text{conjugate}}$) of the 6 parameters of the toy model across different values of N (from $N = 500$ to $N = 5000$).

Parameters	$\hat{\boldsymbol{\theta}}_{\text{adj}}$	$\hat{\boldsymbol{\theta}}_0$	$\hat{\boldsymbol{\theta}}_{\text{conjugate}}$	bias($\hat{\boldsymbol{\theta}}_{\text{adj}}$)	Var($\hat{\boldsymbol{\theta}}_{\text{adj}}$)	MSE($\hat{\boldsymbol{\theta}}_{\text{adj}}$)	95% Cred. Int.
				N=500			
θ_1	0.5502	0.5	0.5063	0.0502	0.0071	0.0097	0.4418—0.7090
θ_2	0.9568	1.0	0.8980	-0.0432	0.0336	0.0354	0.6571—1.1809
θ_3	1.4172	1.5	1.4664	-0.0828	0.0141	0.0210	1.1992—1.5399
θ_4	2.1245	2.0	2.1252	0.12447	0.0017	0.0172	2.0658—2.1730
θ_5	2.3436	2.5	2.4327	-0.1564	0.0079	0.0324	2.2000—2.4866
θ_6	2.9435	3.0	2.9467	-0.0565	0.0178	0.0210	2.7030—3.0890
				N=1000			
θ_1	0.6350	0.5	0.5058	0.1350	0.1554	0.1736	0.1522—1.3605
θ_2	1.066	1.0	0.8976	0.0660	0.0930	0.0973	0.5970—1.5427
θ_3	1.6321	1.5	1.4629	0.13205	0.0629	0.0803	1.1401—1.8758
θ_4	2.0795	2.0	2.1241	0.0795	0.0129	0.0193	1.9184—2.2594
θ_5	2.4111	2.5	2.4320	-0.0889	0.0667	0.0745	2.0394—2.6897
θ_6	3.1265	3.0	2.9465	0.1266	0.2522	0.2683	2.1989—3.7414
				N=2000			
θ_1	0.5200	0.5	0.5074	0.0200	0.0050	0.0054	0.3874—0.6491
θ_2	0.8283	1.0	0.8964	-0.1717	0.0076	0.0371	0.6665—0.9776
θ_3	1.4294	1.5	1.4584	-0.0706	0.0202	0.0252	1.2091—1.6672
θ_4	2.0865	2.0	2.1216	0.0866	0.0091	0.0166	1.9378—2.2277
θ_5	2.3017	2.5	2.4305	-0.1983	0.1059	0.1452	1.8252—2.8712
θ_6	2.8308	3.0	2.9464	-0.1692	0.0174	0.0460	2.5979—3.0677
				N=3000			
θ_1	0.5561	0.5	0.5070	0.0561	0.0099	0.0131	0.3366—0.6926
θ_2	1.0135	1.0	0.8971	0.0136	0.0302	0.0304	0.6207—1.2354
θ_3	1.3542	1.5	1.4592	-0.1458	0.4966	0.5181	0.7311—3.3913
θ_4	2.0331	2.0	2.1228	0.0331	0.1064	0.1075	1.6225—2.7373
θ_5	2.3965	2.5	2.4321	-0.1035	0.0295	0.0402	1.9518—2.6140
θ_6	3.0575	3.0	2.9476	0.0575	0.1713	0.1746	2.1120—3.6480
				N=4000			
θ_1	0.4122	0.5	0.5070	-0.0878	0.1041	0.1118	0.2259—0.9687
θ_2	0.9166	1.0	0.8968	-0.0834	0.0295	0.0365	0.5314—1.1814
θ_3	1.4103	1.5	1.4595	-0.0896	0.0125	0.02061	1.1861—1.6170
θ_4	2.0267	2.0	2.1231	0.0268	0.0497	0.0504	1.7632—2.4752
θ_5	2.5443	2.5	2.4320	0.0443	0.1350	0.1370	1.6821—3.1107
θ_6	2.8656	3.0	2.9473	-0.1344	0.0204	0.0385	2.5604—3.0843
				N=5000			
θ_1	0.4411	0.5	0.5072	-0.0589	0.0067	0.0103	0.3259—0.5829
θ_2	0.7651	1.0	0.8965	-0.2349	0.0113	0.0665	0.6276—1.0433
θ_3	1.4439	1.5	1.4590	-0.0561	0.0095	0.0126	1.2722—1.6720
θ_4	2.0309	2.0	2.1221	0.0310	0.0115	0.0124	1.8560—2.2761
θ_5	2.3640	2.5	2.4319	-0.1360	0.0124	0.0309	2.1593—2.5994
θ_6	2.7485	3.0	2.9468	-0.2515	0.0036	0.0669	2.6816—2.8948

Table 3: Comparison between the proposed adjusted posterior mean estimates ($\hat{\boldsymbol{\theta}}_{\text{adj}}$), true parameter values ($\hat{\boldsymbol{\theta}}_0$) and conjugate posterior mean estimates ($\hat{\boldsymbol{\theta}}_{\text{conjugate}}$) of the 6 parameters of the toy model across different values of N (from $N = 500$ to $N = 5000$).

Parameters	$\hat{\boldsymbol{\theta}}_{\text{adj}}$	$\hat{\boldsymbol{\theta}}_0$	$\hat{\boldsymbol{\theta}}_{\text{conjugate}}$	bias($\hat{\boldsymbol{\theta}}_{\text{adj}}$)	Var($\hat{\boldsymbol{\theta}}_{\text{adj}}$)	MSE($\hat{\boldsymbol{\theta}}_{\text{adj}}$)	95% Cred. Int.
				N=500			
θ_1	0.5092	0.5	0.5063	0.0092	0.0082	0.0083	0.3690—0.6381
θ_2	0.9691	1.0	0.8980	-0.0309	0.0064	0.0074	0.8160—1.0586
θ_3	1.4422	1.5	1.4664	-0.0578	0.0099	0.0132	1.3403—1.6164
θ_4	2.1352	2.0	2.1252	0.1352	0.0031	0.0214	2.0358—2.1947
θ_5	2.2658	2.5	2.4327	-0.2342	0.0057	0.0605	2.1927—2.4078
θ_6	2.9442	3.0	2.9467	-0.0557	0.0174	0.0204	2.6854—3.1162
				N=1000			
θ_1	0.5059	0.5	0.5058	0.0059	0.0011	0.0011	0.4460—0.5507
θ_2	0.8756	1.0	0.8976	-0.1244	0.0076	0.0230	0.7542—1.0208
θ_3	1.4595	1.5	1.4629	-0.0405	0.0097	0.0114	1.3988—1.6918
θ_4	2.0740	2.0	2.1241	0.0739	0.0087	0.0142	1.9511—2.2306
θ_5	2.3327	2.5	2.4320	-0.1673	0.0106	0.0386	2.2014—2.5397
θ_6	2.8354	3.0	2.9464	-0.1645	0.0083	0.0354	2.7659—3.0162
				N=2000			
θ_1	0.5209	0.5	0.5074	0.0209	0.0036	0.0041	0.4083—0.5990
θ_2	0.8851	1.0	0.8964	-0.1149	0.0102	0.0234	0.6648—1.0167
θ_3	1.4342	1.5	1.4584	-0.0657	0.0059	0.0102	1.3191—1.5620
θ_4	2.1169	2.0	2.1216	0.1169	0.0046	0.0183	2.0053—2.2160
θ_5	2.3697	2.5	2.4304	-0.1303	0.0071	0.0241	2.1784—2.4740
θ_6	2.8744	3.0	2.9465	-0.1256	0.0176	0.0334	2.5819—3.0153
				N=3000			
θ_1	0.50178	0.5	0.5070	0.0017	0.0052	0.0052	0.3784—0.6144
θ_2	0.8439	1.0	0.8971	-0.1561	0.0108	0.0351	0.6908—1.0490
θ_3	1.4079	1.5	1.4592	-0.0921	0.0082	0.0167	1.2420—1.5525
θ_4	2.0646	2.0	2.1228	0.0646	0.0043	0.0085	1.9617—2.1635
θ_5	2.3612	2.5	2.4321	-0.1388	0.0084	0.0277	2.1541—2.4752
θ_6	2.8406	3.0	2.9476	-0.1594	0.0132	0.0387	2.6553—3.0584
				N=4000			
θ_1	0.4830	0.5	0.5070	-0.0170	0.0038	0.0041	0.3742—0.5953
θ_2	0.8494	1.0	0.8968	-0.1506	0.0077	0.0304	0.6909—1.0394
θ_3	1.3655	1.5	1.4595	-0.1344	0.0119	0.0299	1.1978—1.5574
θ_4	2.1047	2.0	2.1231	0.1047	0.0106	0.0216	1.9211—2.2789
θ_5	2.3459	2.5	2.4320	-0.1541	0.0129	0.0366	2.1378—2.5463
θ_6	2.7938	3.0	2.9473	-0.2062	0.0088	0.0513	2.6270—2.9766
				N=5000			
θ_1	0.4757	0.5	0.5072	-0.0243	0.0030	0.0035	0.3735—0.5520
θ_2	0.8180	1.0	0.8965	-0.1810	0.0056	0.0384	0.6950—0.9689
θ_3	1.3951	1.5	1.4589	-0.1049	0.0098	0.02078	1.2522—1.6369
θ_4	2.0777	2.0	2.1221	0.0777	0.0052	0.0112	1.9547—2.2093
θ_5	2.3147	2.5	2.4319	-0.1853	0.0091	0.0435	2.1605—2.5080
θ_6	2.8034	3.0	2.9468	-0.1966	0.0094	0.0480	2.6093—2.9506

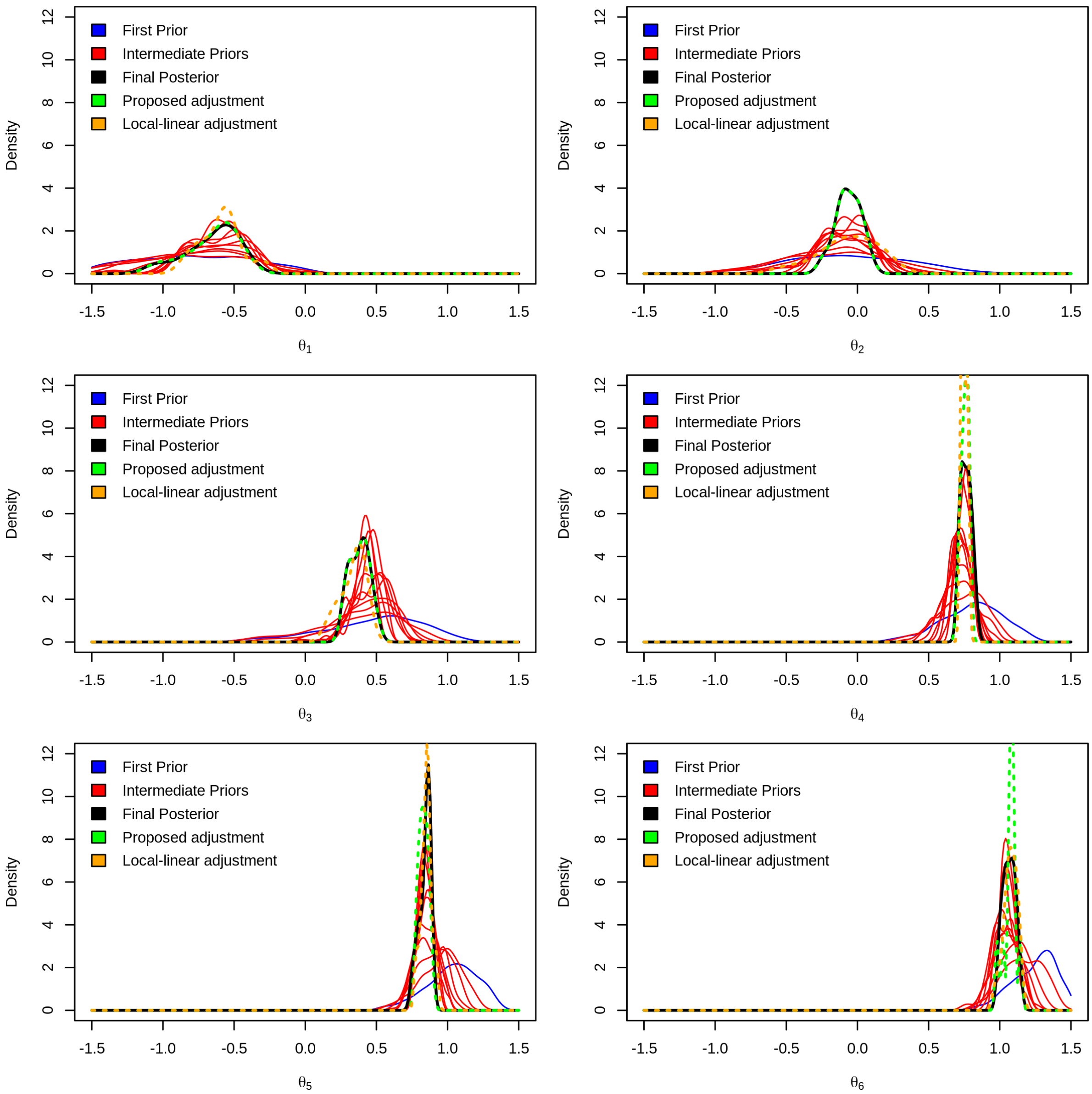

Figure 3: Goodness-of-fit density plots of the (unadjusted) approximate posterior distribution (in black) for the 6 parameters of the toy model against the sequentially improving prior distributions and the adjusted posterior from the two regression adjustments at $N = 500$ (on logarithmic scale).

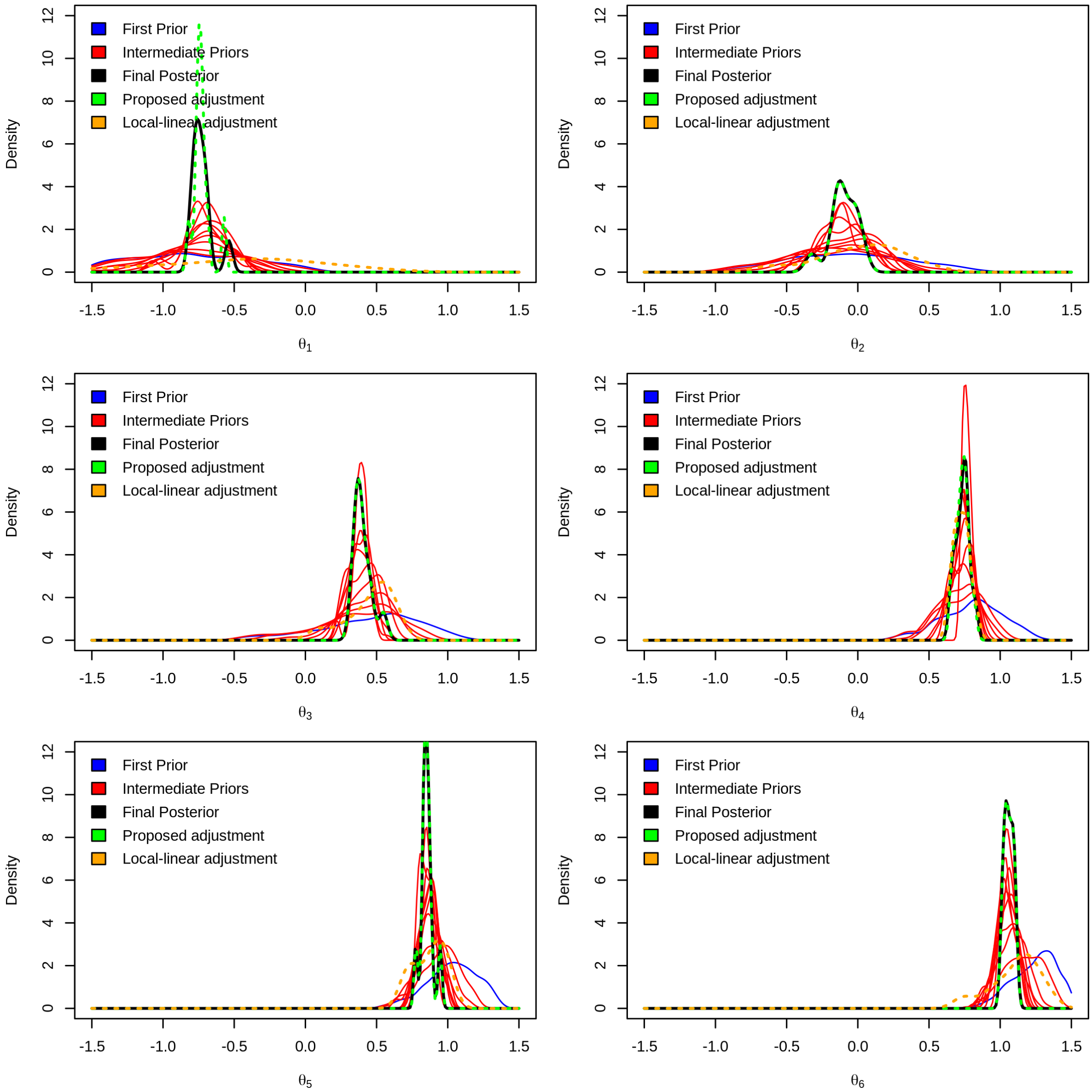

Figure 4: Goodness-of-fit density plots of the (unadjusted) approximate posterior distribution (in black) for the 6 parameters of the toy model against the sequentially improving prior distributions and the adjusted posterior from the two regression adjustments at $N = 1000$ (on logarithmic scale).

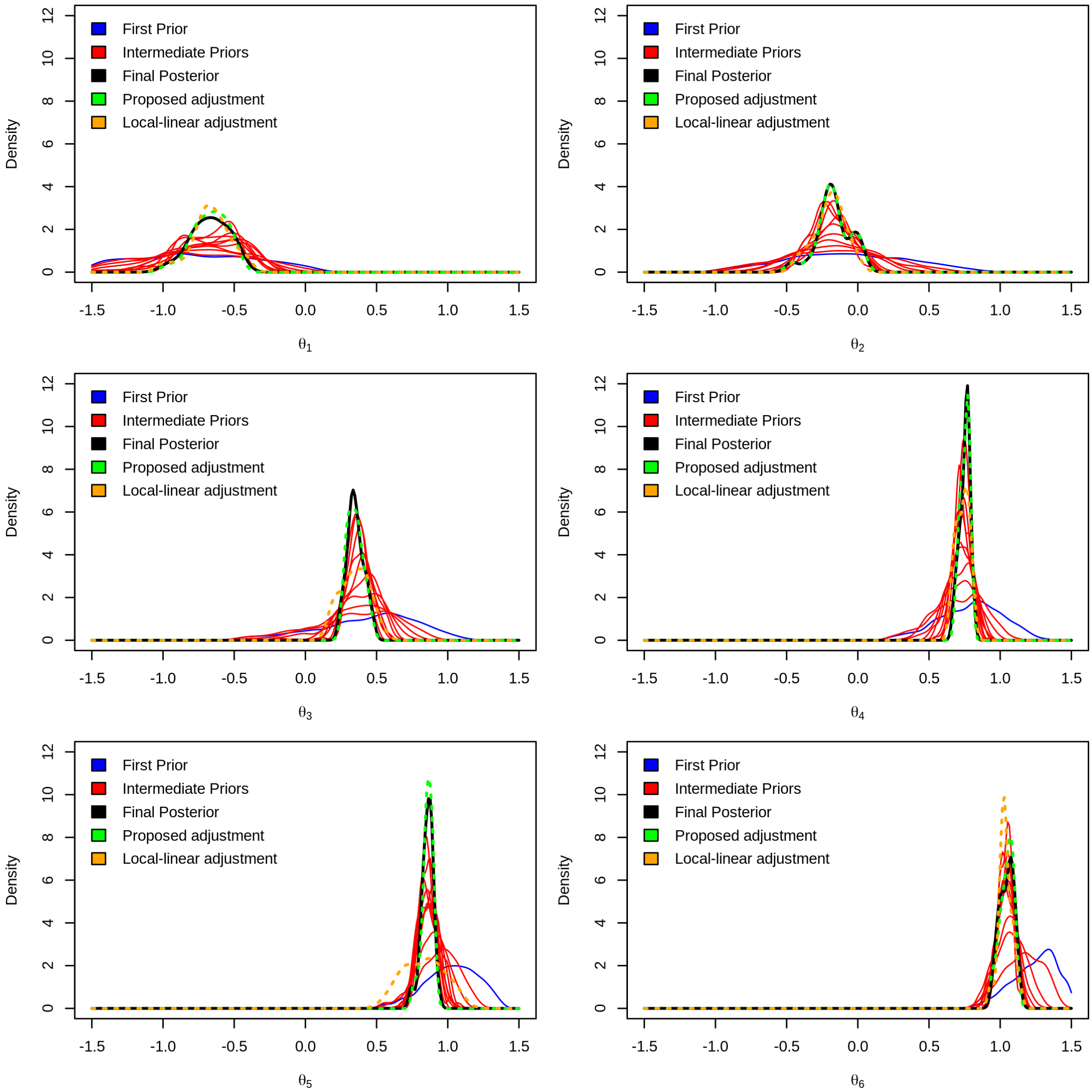

Figure 5: Goodness-of-fit density plots of the (unadjusted) approximate posterior distribution (in black) for the 6 parameters of the toy model against the sequentially improving prior distributions and the adjusted posterior from the two regression adjustments at $N = 2000$ (on logarithmic scale).

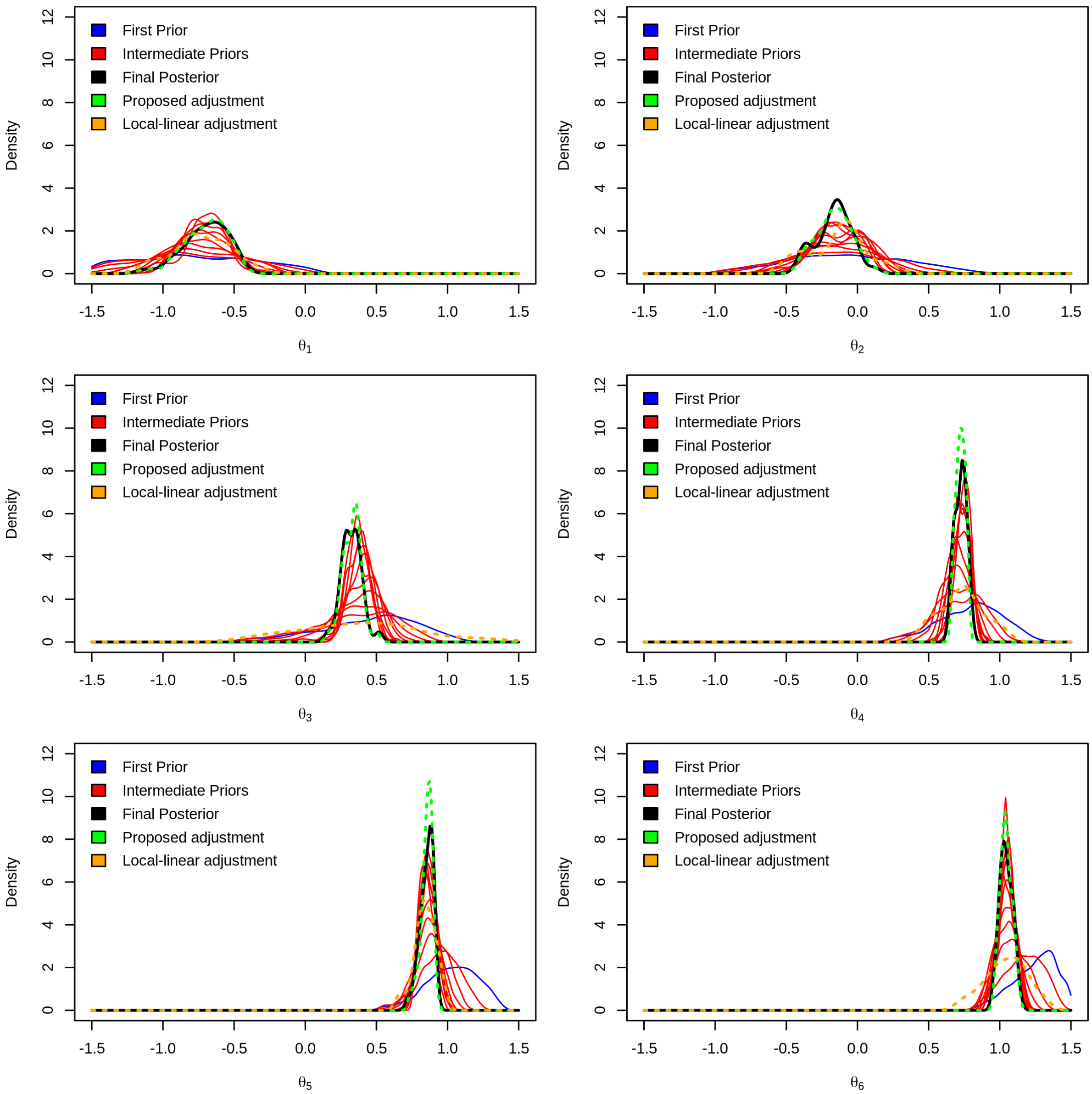

Figure 6: Goodness-of-fit density plots of the (unadjusted) approximate posterior distribution (in black) for the 6 parameters of the toy model against the sequentially improving prior distributions and the adjusted posterior from the two regression adjustments at $N = 3000$ (on logarithmic scale).

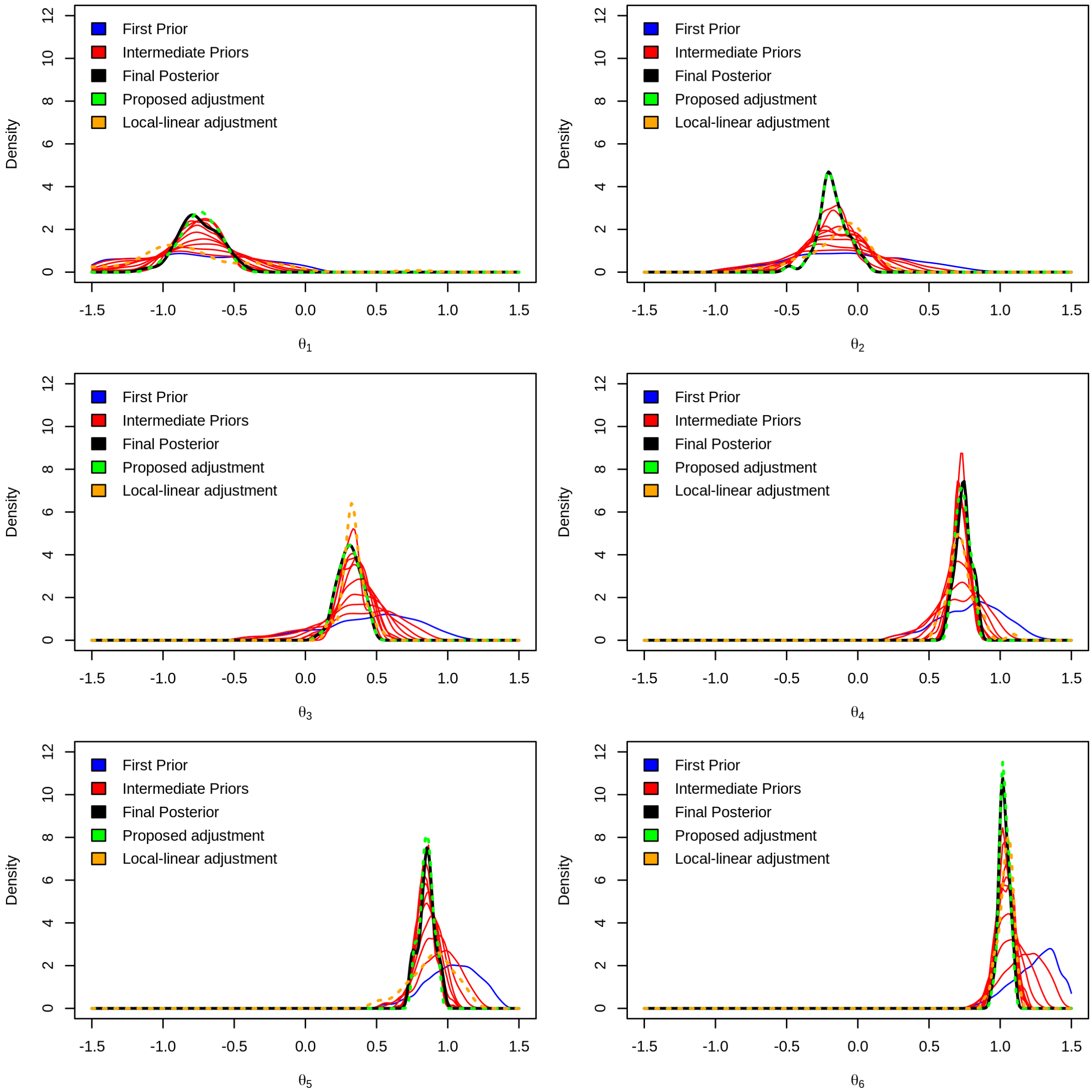

Figure 7: Goodness-of-fit density plots of the (unadjusted) approximate posterior distribution (in black) for the 6 parameters of the toy model against the sequentially improving prior distributions and the adjusted posterior from the two regression adjustments at $N = 4000$ (on logarithmic scale).

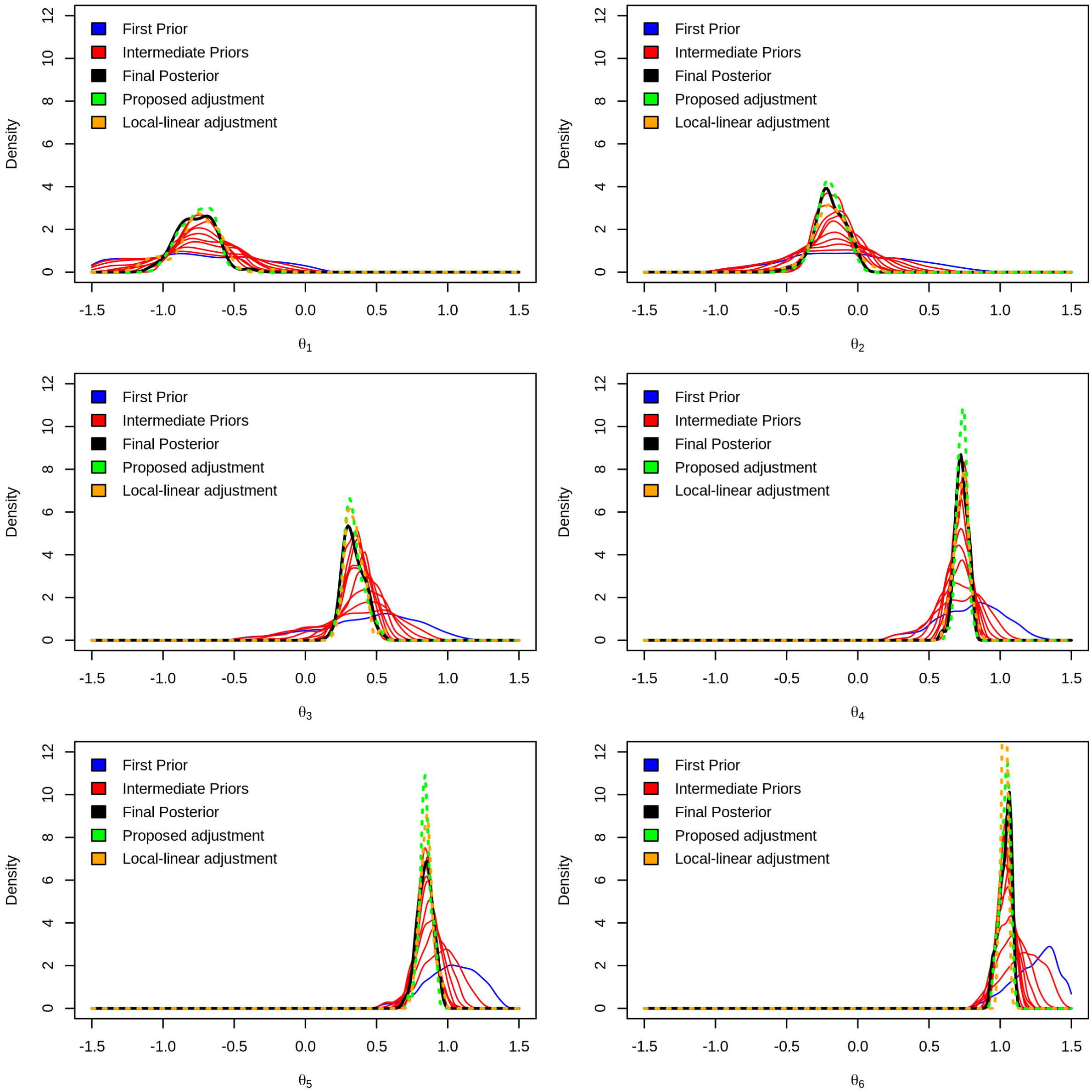

Figure 8: Goodness-of-fit density plots of the (unadjusted) approximate posterior distribution (in black) for the 6 parameters of the toy model against the sequentially improving prior distributions and the adjusted posterior from the two regression adjustments at $N = 5000$ (on logarithmic scale).

Bibliography

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