

LeptonWeak

January 29, 2020

1) Muon-neutrino electron scattering

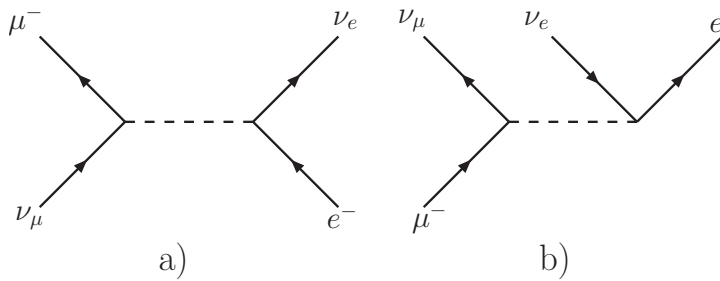


Fig 1: Feynman diagrams, time axis bottom to top. a) $\nu_\mu e^- \rightarrow \mu^- \nu_e$ b) $\mu^- \rightarrow e^- \nu_\mu \bar{\nu}_e$

The amplitudes for the process $\nu_\mu + e^- \rightarrow \mu^- + \nu_e$ (fig. 1a) with four momenta $k_i + p_i = k_f + p_f$ are calculated from

$$T_{fi} = -\frac{g^2}{2} \bar{u}(k_f) \gamma^\mu \frac{1 - \gamma^5}{2} u(k_i) \frac{g_{\mu\nu} - q_\mu q_\nu / M_W^2}{t - M_W^2} \bar{u}(p_f) \gamma^\nu \frac{1 - \gamma^5}{2} u(p_i) \quad (1)$$

For squared momentum transfers t small compared to M_W^2 this reduces with $g^2 = 8G_F M_W^2 / \sqrt{2}$ to

$$T_{fi} = \frac{4G_F}{\sqrt{2}} \bar{u}(k_f) \gamma^\mu \frac{1 - \gamma^5}{2} u(k_i) \bar{u}(p_f) \gamma_\mu \frac{1 - \gamma^5}{2} u(p_i) ,$$

where $G_F = 1,16638 \cdot 10^{-5} \text{ GeV}^{-2}$ is the Fermi constant.

```
[1]: from sympy import *
```

```
[2]: import heppackv0 as hep
```

Reading heppackv0.py

Done

```
[3]: print('The CM system is used')
E0,E1,E2,E3,p1,p3,m,M,s,x,y,z,a=symbols('E0 E1 E2 E3 p1 p3 m M s x y z_
↪a',positive=True)
theta, theta2,theta3=symbols('theta theta2 theta3')
```

The CM system is used

```
[4]: ki=[E0,0,0,0]
pin=[E1,m,pi,pi]
kf=[E3,M,theta,0]
pf=[E2,0,pi-theta,pi]
```

The current products $\bar{u}(k_f)\gamma^\mu\frac{1-\gamma^5}{2}u(k_i)\bar{u}(p_f)\gamma^\mu\frac{1-\gamma^5}{2}u(p_i)$ are contained in the package hep-packv0. Only the amplitude with all $h_i = -1/2$ survives. Clearly here the direct evaluation of the helicity amplitudes is much easier than the standard method as e.g. used in the book “Gauge theories in particle physics” by Aitchison & Hey

```
[5]: t1=simplify(hep.dotprod4(hep.ubuw(pf,-1,pin,-1),hep.ubuw(kf,-1,ki,-1)));t1
```

[5]:

$$2\sqrt{E_0}\sqrt{E_2}\left(\sqrt{E_1-m}+\sqrt{E_1+m}\right)\left(\sqrt{E_3-M}+\sqrt{E_3+M}\right)$$

```
[6]: simplify(expand(t1*t1))
```

[6]:

$$16E_0E_2\left(E_1E_3+E_1\sqrt{E_3^2-M^2}+E_3\sqrt{E_1^2-m^2}+\sqrt{E_1-m}\sqrt{E_1+m}\sqrt{E_3-M}\sqrt{E_3+M}\right)$$

As usual the square roots are converted by hand, e.g. $\sqrt{E_3^2-M^2} = p_3 = E_2$. One line further the energies are expressed as functions of s , the CM energy squared.

```
[7]: t1sqv1=16*E0*E2*(E1*E3+E1*E2+E3*E0+E0*E2);t1sqv1
```

[7]:

$$16E_0E_2(E_0E_2+E_0E_3+E_1E_2+E_1E_3)$$

```
[8]: t1sq=simplify(t1sqv1.subs(E0,(s-m*m)/2/sqrt(s)).subs(E1,(s+m*m)/2/sqrt(s)).
↪subs(E2,(s-M*M)/2/sqrt(s))
.subs(E3,(s+M*M)/2/sqrt(s)));t1sq
```

[8]:

$$4M^2m^2-4M^2s-4m^2s+4s^2$$

or $t1sq = 4(s-m^2)(s-M^2)$ which after including the coupling $(4G_F/\sqrt{2})^2$ yields

$$|T_1|^2 = \sum_i |T_i|^2 = 32G_F^2(s-M^2)(s-m^2)$$

and for neutrino scattering off unpolarized electrons

$$\overline{\sum_i} |T_i|^2 = 16G_F^2(s-M^2)(s-m^2) .$$

Finally the cross section is obtained by dividing the result by $16\pi(s - m^2)^2$, that is

$$\frac{d\sigma}{dt}(\nu_\mu + e^- \rightarrow \mu^- + \nu_e) = \frac{G_F^2}{\pi(s - m^2)}(s - M^2) .$$

Experimentally neutrinos are scattered off electrons at rest with $s = 2E_0m + m^2$.

Next $d\sigma/d\Omega$ is calculated by multiplying $\sum_i |T_i|^2$ with the usual kinematical factors

[9]: `simplify(t1sq/2*(s-M**2)/(s-m**2)/s/64/pi/pi)`

[9]:

$$\frac{M^4 - 2M^2s + s^2}{32\pi^2s}$$

i.e. after including the coupling

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{4\pi^2s}(s - M^2)^2$$

with the total cross section

$$\sigma(\nu_\mu + e^- \rightarrow \mu^- + \nu_e) = \frac{G_F^2}{\pi s}(s - M^2)^2$$

which simplifies at high energies to

$$\sigma = \frac{G_F^2}{\pi}s .$$

The unlimited increase with s violates unitarity.

In order to find the unitarity limit we invoke the partial wave expansion of the scattering amplitude, which reads in the high energy limit (eq. 2.236 and 2.267 of the book, explanations in the text and table of the $d_{\lambda,\mu}^J$)

$$T_{\lambda_3\lambda_4,\lambda_1\lambda_2} = 16\pi \sum_J (2J+1) t_{\lambda_3\lambda_4,\lambda_1\lambda_2}^J (\sqrt{s}) d_{\lambda,\mu}^J(\Theta) e^{i(\lambda-\mu)\phi}$$

For an isotropic amplitude the expansion reduces to one term $T_1 = 16\pi t^0$, where we have omitted the helicity indices of t^0 . Like in the spinless case the representation $t^J = e^{i\delta_J} \sin \delta_J$ restricts $|t^J| < 1$ and $\Re(t^J) < 1/2$. The first order born approximation is ok for calculating $|T_{fi}|$ without a possible phase Φ . In order to ensure $|T_{fi}| \cos \Phi < 1/2$ for all values of Φ the condition $|T_1| < 8\pi$ must therefor hold resulting in

$$\frac{4G_F}{\sqrt{2}}2s < 8\pi$$

or

$$s < \frac{\sqrt{2}\pi}{G_F}$$

corresponding to a maximum neutrino energy of 310 GeV in the CM system. The violation of the unitarity limit is very much softened by including the W boson propagator of fig.1. The propagator term in (1) contributes only for $t > 8 \text{ GeV}^2$ (i.e. for CM system energies $> 10 \text{ GeV}$) more than 1% to the cross section. At these energies the difference between muon momenta and energies is $\mathcal{O}(10^{-5})$ and therefor t can be safely approximated by $-s(1 - \cos \Theta)/2$. In addition the $q_\mu q_\nu / M_W^2$

term in the nominator can be neglected because with the help of the Dirac equation $q_\nu \gamma^\nu u$ reduces to Mu . At these energies the amplitude T_1 is thus replaced by

$$T_1 = 4\sqrt{2}G_F \frac{-sM_W^2}{M_W^2 + s(1 - \cos \Theta)/2}$$

leading to

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{4\pi^2} \frac{sM_W^4}{(M_W^2 + s(1 - \cos \Theta)/2)^2} .$$

From this the total cross section is easily derived with the help of

[10]: `simplify(integrate(1/(a**2+s/2*(1-z))**2,(z,1,-1)))`

[10]:

$$-\frac{2}{a^2(a^2 + s)}$$

(here $z=\cos(\Theta), a = M_W$)

$$\sigma = \frac{G_F^2}{\pi} \frac{M_W^2 s}{(M_W^2 + s)}$$

leading to the constant high energy limit

$$\sigma = \frac{G_F^2 M_W^2}{\pi} .$$

The dependence of T_1 on Θ obviously requires a superposition of many partial waves. Using $\int d_{00}^J d \cos \Theta = 0$ for $J > 1$ the s wave can be easily calculated via integrating $T_{\lambda_3 \lambda_4, \lambda_1 \lambda_2} = T_1$:

[11]: `simplify(integrate(1/(a**2+s/2*(1-z)),(z,-1,1)))`

[11]:

$$-\log \left(\left(\frac{a^2}{a^2 + s} \right)^{\frac{2}{s}} \right)$$

leading to

$$t^0 = \frac{G_F M_W^2}{2\sqrt{2}\pi} \ln \left(1 + \frac{s}{M_W^2} \right)$$

Surprisingly the amplitude again rises with s but only logarithmically. With a W mass of 80.4 GeV the limit $t_0 = 1/2$ will only be reached at very high energies.

[12]: `r=1.116e-5*80.4**2/sqrt(2)/pi;r`

[12]:

$$\frac{0.0360700128\sqrt{2}}{\pi}$$

[13]: `sqrtofs=80.4*sqrt((exp(1/r)).evalf());sqrtofs`

[13]:

$$1.89971596831086 \cdot 10^{15}$$

that is, CM beam energies of $\approx 10^{15}$ GeV or laboratory energies close to the Planck mass. This extremely weak violation of unitarity needs not concern us.

Historically the unitarity argument was a major motivation in the search for an “intermediate vector boson”.

2) Muon decay rate

$\overline{\sum_i |T_i|^2}$ for muon decay can be directly taken from the scattering formula invoking crossing symmetry. Consider fig. 1a with the time arrow pointing from top to bottom and in addition bending the ν_e line downward. Then the diagram describes the decay $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$, which is equivalent to the diagram in fig. 1b describing $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$ with the usual direction of time. The variable $s = (k_i + p_i)^2 = (k_f + p_f)^2$ is with $p_f \rightarrow -p_f$ transformed to $s' = (k_f - p_f)^2$. For a muon at rest $k_f = (M, 0)$ and denoting the ν_e energy again by E_2 it follows $s' = M^2 - 2ME_2$ resulting in

$$\overline{\sum_i |T_i|^2} = 16G_F^2(M^2 - s')(s' - m^2) = 32G_F^2ME_2(M^2 - m^2 - 2E_2M) = 32G_F^2M^3E_2 \left(1 - x - 2\frac{E_2}{M}\right)$$

with $x = m^2/M^2$. The extra $(-)$ sign is according to the crossing symmetry rules due to the bending of one fermion line.

The differential decay rate for the three body decay is given by

$$\frac{d^2\Gamma(E_1, E_2)}{dE_1 dE_2} = \frac{1}{64\pi^3 M} \overline{\sum_i |T_{fi}(E_1, E_2)|^2}$$

i.e.

$$\frac{d^2\Gamma(E_1, E_2)}{dE_1 dE_2} = \frac{G_F^2 M^2}{2\pi^3} E_2 \left(1 - x - 2\frac{E_2}{M}\right)$$

In order to calculate the total decay rate, $dG = E_2 \left(1 - x - 2\frac{E_2}{M}\right)$ has to be integrated over E_2 in the limits of the contour $E_2 = f(E_1)$ in the E_2, E_1 plane, where E_1 is the energy of the electron.

[14]: `dG=(E2*(1-2*E2/M-x));dG`

[14]:

$$E_2 \left(-\frac{2E_2}{M} - x + 1\right)$$

The possible values of E_1, E_2 are restricted by 3-momentum conservation $\vec{p}_3 = -\vec{p}_1 - \vec{p}_2$. Choosing the direction of \vec{p}_1 as the x -axis of the coordinate system results in $p_3^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos \theta_2$. We express $\cos \theta_2$ as function of E_1, E_2 and solve the equation for $\cos \theta_2 = \pm 1$

[15]: `costheta2=simplify(expand((M-E1-E2)**2-(E1**2-M**2*x)-E2**2)/2/E2/
↪sqrt(E1**2-M**2*x));costheta2`

[15]:

$$\frac{E_1 E_2 - E_1 M - E_2 M + \frac{M^2 x}{2} + \frac{M^2}{2}}{E_2 \sqrt{E_1^2 - M^2 x}}$$

```
[16]: h1=solve(costheta2-1,E2)
      highlim=h1[0];highlim
```

[16]:

$$\frac{M(-2E_1 + Mx + M)}{2(-E_1 + M + \sqrt{E_1^2 - M^2x})}$$

```
[17]: h2=solve(costheta2+1,E2)
      lowlim=h2[0];lowlim
```

[17]:

$$\frac{M(2E_1 - Mx - M)}{2(E_1 - M + \sqrt{E_1^2 - M^2x})}$$

```
[18]: simplify(costheta2.subs(E1,M/2*(1+x)))
```

[18]:

$$\frac{x-1}{\sqrt{x^2-2x+1}}$$

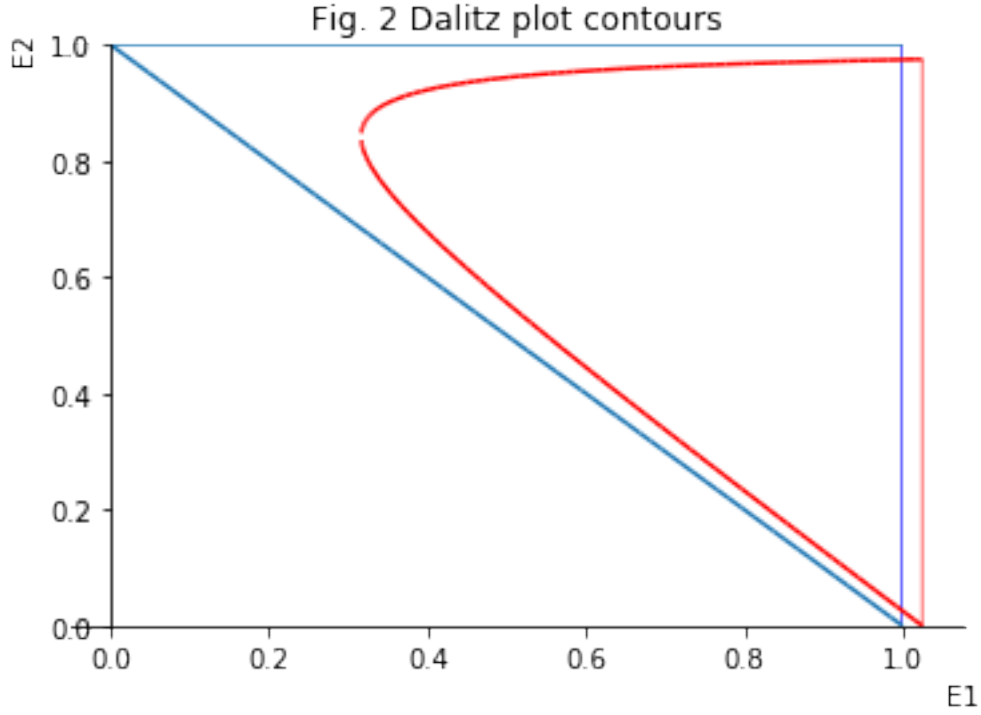
```
[19]: simplify(costheta2.subs(x,0).subs(E1,M/2))
```

[19]:

$$-1$$

The vertical straight line at the upper limit of $E_1 = M(1+x)/2$ satisfies $\cos \theta_2 = -1$. The negative sign is proven in the limit $m = 0$. Fig. 2 shows the resulting contour in red for fictitious masses $M = 2$, $m = \sqrt{0.1}$. The blue triangle represents the the case $m = 0$.

```
[45]: # Start with the triangle representing the decay of a particle with
      #M=2 GeV into 3 massless particles.
      pl1=plot(1,1-E1,(E1,0,1),ylim=[0,1],ylabel=E2,title='Fig. 2 Dalitz plot_
      ↪contours',show=False)
      pl11=plot_implicit(Eq(x,1.0),(x,0,2),(y,0,1),show=False)
      # Now m=0.316
      pl2=plot(highlim.subs(M,2).subs(x,0.025),lowlim.subs(M,2).subs(x,0.025),
      (E1,0.3165,1.025),show=False)
      pl2[0].line_color='r'
      pl2[1].line_color='r'
      pl21=plot_implicit(Eq(x,1.025),(x,0,2),(y,0,0.975),show=False)
      pl21[0].line_color='r'
      pl2.append(pl21[0])
      pl1.append(pl11[0])
      pl1.extend(pl2)
      pl1.show()
```



[21]: `dGdE=simplify(integrate(dG,(E2,highlim,lowlim)));dGdE`

[21]:

$$\frac{\sqrt{E_1^2 - M^2 x} (-4E_1^2 + 3E_1 M x + 3E_1 M - 2M^2 x)}{6M}$$

Multiplying with $G_F^2 M^2 / 2\pi^3$ leads to

$$\frac{d\Gamma}{dE_1} = \frac{G_F^2 M}{12\pi^3} p_1 (3E_1 M (1+x) - 4E_1^2 - 2M^2 x)$$

For muon decay $x = 2.3 \cdot 10^{-5}$ and can be set to 0 in most cases.

[22]: `dGdE0=simplify(dGdE.subs(x,0));dGdE0`

[22]:

$$\frac{E_1^2 \left(-\frac{2E_1}{3} + \frac{M}{2} \right)}{M}$$

[23]: `integrate(dGdE0,(E1,0,M/2))`

[23]:

$$\frac{M^3}{96}$$

Thus, neglecting the electron mass the electron energy spectrum is given by

$$\frac{d\Gamma_0}{dE_1} = \frac{G_F^2 M}{12\pi^3} (3E_1^2 M - 4E_1^3)$$

with the total decay rate

$$\Gamma_0 = \frac{G_F^2 M^5}{192\pi^3}$$

The relative differential decay rates are defined by

$$\frac{d\Gamma_0}{\Gamma_0 dE_1} = \frac{16}{M^4} (3E_1^2 M - 4E_1^3)$$

and

$$\frac{d\Gamma}{\Gamma_0 dE_1} = \frac{16}{M^4} p_1 (3E_1 M (1+x) - 4E_1^2 - 2M^2 x)$$

[24]: `G0plot=16/M**4*simplify(dGdE0*6*M);G0plot`

[24]:

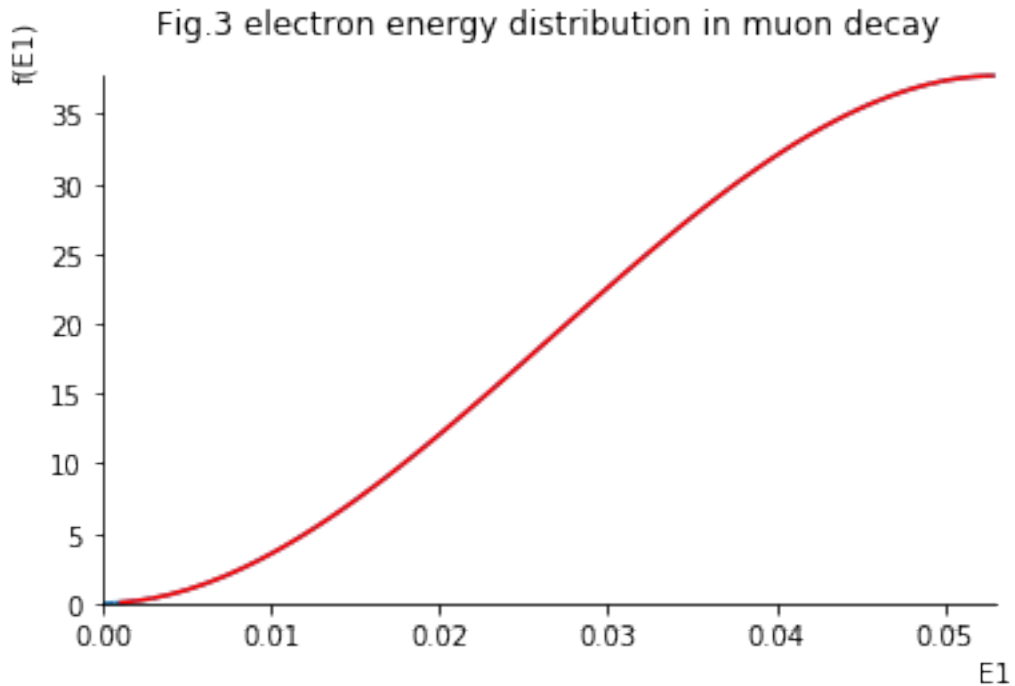
$$\frac{16E_1^2 (-4E_1 + 3M)}{M^4}$$

[25]: `Gplot=16/M**4*dGdE*6*M;Gplot`

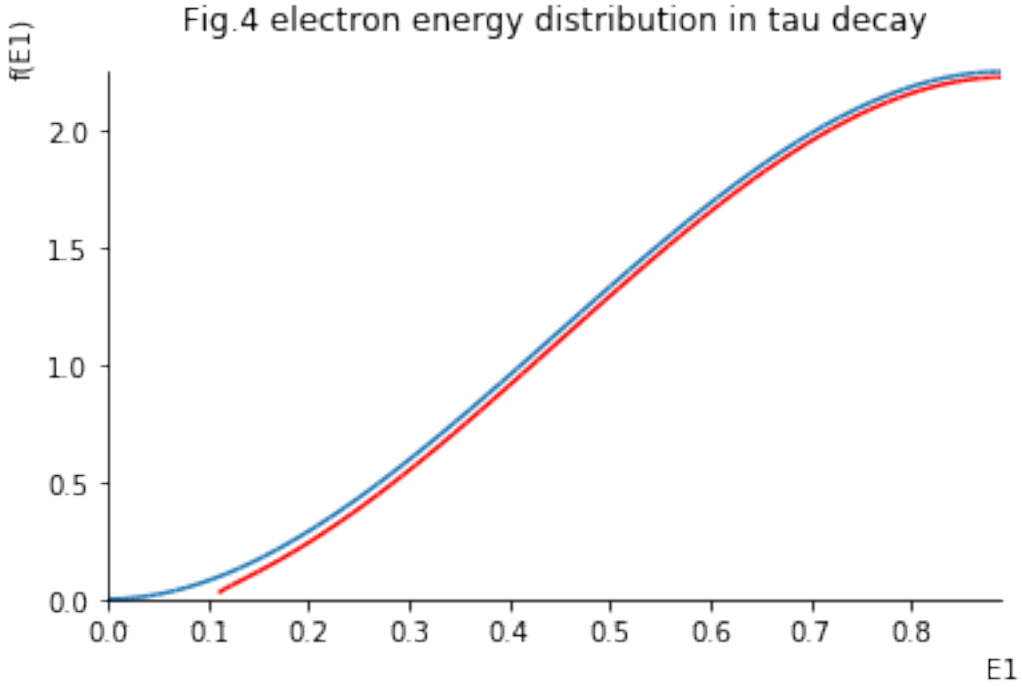
[25]:

$$\frac{16\sqrt{E_1^2 - M^2 x} (-4E_1^2 + 3E_1 M x + 3E_1 M - 2M^2 x)}{M^4}$$

[26]: `p2=plot(G0plot.subs(M,0.106),(E1,0,0.053),
title='Fig.3 electron energy distribution in muon decay',
show=False)
p21=plot(Gplot.subs(M,0.106).subs(x,2.3E-5),(E1,0.001,0.053),show=False)
p21[0].line_color='r'
p2.append(p21[0])
p2.show()`



```
[27]: p3=plot(G0plot.subs(M,1.78),(E1,0,0.89),
title='Fig.4 electron energy distribution in tau decay',
show=False)
p31=plot(Gplot.subs(M,1.78).subs(x,3.5E-3),(E1,0.11,0.89),show=False)
p31[0].line_color='r'
p3.append(p31[0])
p3.show()
```



For muon decay the curves for $m = 0$ and $m = 0.51$ MeV are practically identical except for very small E_1 (test it!), whereas for $\tau \rightarrow \mu\nu\bar{\nu}$ there is a marked difference. The integral $\int \frac{d\Gamma}{dE_1} dE_1$ looks very ugly so we calculate Γ via $\int \frac{d\Gamma}{dp_1} dp_1 = \int \frac{d\Gamma}{dE_1} \frac{p_1}{E_1} dp_1$

```
[28]: dgamdp=p1**2*(3*M*(1+x)-4*E1-2*M**2*x/E1);dgamdp
```

[28]:

$$p_1^2 \left(-4E_1 + 3M(x+1) - \frac{2M^2x}{E_1} \right)$$

```
[29]: dgamdp1=dgamdp.subs(E1,sqrt(p1**2+m**2));dgamdp1
```

[29]:

$$p_1^2 \left(-\frac{2M^2x}{\sqrt{m^2 + p_1^2}} + 3M(x+1) - 4\sqrt{m^2 + p_1^2} \right)$$

```
[30]: Gam1=simplify(integrate(dgamdp1,(p1,0,M*(1-x)/2)));Gam1
```

[30]:

$$-\frac{M^4 x (x-1)^3}{8} - \frac{M^4 (x-1)^3}{8} + \frac{M^3 x (x-1) \sqrt{M^2 (x-1)^2 + 4m^2}}{4} + \frac{M^3 (x-1)^3 \sqrt{M^2 (x-1)^2 + 4m^2}}{16} - M^2 m^2 x$$

```
[31]: h1=simplify(M**2*(x-1)**2+4*m**2).subs(x,m**2/M**2);h1
simplify(h1-M**2*(1+x)**2).subs(x,m**2/M**2))
```

[31]:

$$0$$

The complicated expression for Γ_1 can be greatly simplified by realizing that the square-roots are given by $M(1+x)/2$, as proven in the last cell.

```
[32]: simplify(Gam1.subs(sqrt(M**2*(x-1)**2+4*m**2),M*(1+x)).subs(m**2,x*M**2))/6/M
```

[32]:

$$\frac{M^3 \left(-x^4 + 8x^3 - 24x^2 \operatorname{asinh}\left(\frac{x-1}{2\sqrt{x}}\right) - 8x + 1 \right)}{96}$$

Setting $h_2 = \frac{x-1}{2\sqrt{x}}$ we replace the $\operatorname{asinh}(h_2)$ by $\ln\left(h_2 + \sqrt{1+h_2^2}\right)$

```
[33]: h2=(x-1)/2/sqrt(x)
h3=simplify(h2+sqrt(h2**2+1));h3
```

[33]:

$$\frac{x + \sqrt{4x + (x-1)^2} - 1}{2\sqrt{x}}$$

I cannot understand why sympy does not simplify the square-root to $1+x$ and thus the whole expression to \sqrt{x} . Using $\sinh^{-1}(h_2) = \ln \sqrt{x}$ we finally obtain the total decay rate Γ :

$$\Gamma = \frac{G_F^2 M^5}{192\pi^3} \left(1 - 8x + 8x^3 - x^4 - 12x^2 \ln x \right)$$

which is mostly abbreviated by

$$\Gamma = \frac{G_F^2 M^5}{192\pi^3} (1 - 8x) .$$

3) Direct calculation of crossed amplitudes

a) Muon decay

For those who don't appreciate the utilization of crossing symmetry we add a direct calculation of the muon decay matrix elements, $\mu^-(k_i) \rightarrow e^-(p_{\text{out}}) + \bar{\nu}_e(p_f) + \nu_\mu(k_f)$:

```
[34]: ki=[M,M,0,0]
pout=[E1,m,0,0]
kf=[E3,0,theta3,0]
```

```
pf=[E2,0,theta2,0]
```

```
[35]: dec1=simplify(hep.dotprod4(hep.ubvw(pout,-1,pf,1),hep.ubuw(kf,-1,ki,-1)));dec1
```

[35]:

$$2\sqrt{2}\sqrt{E_2}\sqrt{E_3}\sqrt{M}\left(\sqrt{E_1-m}+\sqrt{E_1+m}\right)\sin\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_3}{2}\right)$$

```
[36]: dec2=simplify(hep.dotprod4(hep.ubvw(pout,1,pf,1),hep.ubuw(kf,-1,ki,-1)));dec2
```

[36]:

$$2\sqrt{2}\sqrt{E_2}\sqrt{E_3}\sqrt{M}\left(-\sqrt{E_1-m}+\sqrt{E_1+m}\right)\sin\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_3}{2}\right)$$

```
[37]: dec3=simplify(hep.dotprod4(hep.ubvw(pout,-1,pf,1),hep.ubuw(kf,-1,ki,1)));dec3
```

[37]:

$$2\sqrt{2}\sqrt{E_2}\sqrt{E_3}\sqrt{M}\left(\sqrt{E_1-m}+\sqrt{E_1+m}\right)\sin\left(\frac{\theta_3}{2}\right)\cos\left(\frac{\theta_2}{2}\right)$$

```
[38]: dec4=simplify(hep.dotprod4(hep.ubvw(pout,1,pf,1),hep.ubuw(kf,-1,ki,1)));dec4
```

[38]:

$$2\sqrt{2}\sqrt{E_2}\sqrt{E_3}\sqrt{M}\left(-\sqrt{E_1-m}+\sqrt{E_1+m}\right)\cos\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_3}{2}\right)$$

```
[39]: simplify(dec1**2+dec2**2+dec3**2+dec4**2)
```

[39]:

$$16E_2E_3M\left(E_1-2\sqrt{E_1^2-m^2}\cos^2\left(\frac{\theta_3}{2}\right)+\sqrt{E_1^2-m^2}\right)$$

$\sum_i |\text{dec}_i|^2$ is proportional to the invariant $p_{\text{out}} \cdot k_f$.

```
[40]: h4=hep.dotprod4(hep.fourvec(kf),hep.fourvec(pout));h4
```

[40]:

$$E_3\left(E_1-\sqrt{E_1^2-m^2}\cos(\theta_3)\right)$$

```
[41]: h4v1=E3*E1-2*E3*sqrt(E1**2-m**2)*cos(theta3/2)**2+E3*sqrt(E1**2-m**2);h4v1
```

[41]:

$$E_1E_3-2E_3\sqrt{E_1^2-m^2}\cos^2\left(\frac{\theta_3}{2}\right)+E_3\sqrt{E_1^2-m^2}$$

```
[42]: simplify(h4v1-h4)
```

[42]:

$$0$$

With $\sum_i |\text{dec}_i|^2 = 16E_2M(p_{\text{out}} \cdot k_f)$ one obtains

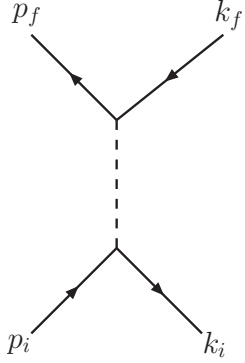
$$\overline{\sum_i |T_i|^2} = 8G_F^2 \frac{1}{2} \sum_i |\text{dec}_i|^2 = 64G_F^2 E_2 M(p_{\text{out}} \cdot k_f)$$

and from $k_i = p_{\text{out}} + k_f + p_f$ follows $(p_{\text{out}} + k_f)^2 = (k_i - p_f)^2$ or $m^2 + 2(p_{\text{out}} \cdot k_f) = M^2 - 2E_2 M$, that is $p_{\text{out}} \cdot k_f = (M^2 - m^2 - 2E_2 M)/2$ leading to

$$\sum_i |T_i|^2 = 32G_F^2 E_2 M (M^2 - m^2 - 2E_2 M) .$$

This confirms the result of the crossing method.

b) Anti-neutrino electron scattering



The amplitudes for the process $\bar{\nu}_\mu + e^- \rightarrow \mu^- + \bar{\nu}_e$ with four momenta $p_i + k_i = p_f + k_f$ are for energies much smaller than M_W calculated from

$$T_{fi} = \frac{4G_F}{\sqrt{2}} \bar{v}(k_i) \gamma^\mu \frac{1 - \gamma^5}{2} u(p_i) \bar{u}(p_f) \gamma_\mu \frac{1 - \gamma^5}{2} v(k_f) ,$$

[43]: `ki=[E0,0,pi,pi]
pin=[E1,m,0,0]
kf=[E2,0,pi-theta,pi]
pf=[E3,M,theta,0]`

[44]: `t11=simplify(hep.dotprod4(hep.vbuw(ki,1,pin,-1),hep.ubvw(pf,-1,kf,1)));t11`

[44]:

$$-\sqrt{E_0}\sqrt{E_2} \left(\sqrt{E_1 - m} + \sqrt{E_1 + m} \right) \left(\sqrt{E_3 - M} + \sqrt{E_3 + M} \right) (\cos(\theta) + 1)$$

All other amplitudes are 0. From comparing with line (5) it follows immediately

$$T_1 = \frac{4G_F}{\sqrt{2}} \sqrt{(s - m^2)(s - M^2)} (1 + \cos \Theta)$$

and

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{16\pi^2 s} (s - M^2)^2 (1 + \cos \Theta)^2$$

with the total cross section

$$\sigma(\bar{\nu}_\mu + e^- \rightarrow \mu^- + \bar{\nu}_e) = \frac{G_F^2}{3\pi s} (s - M^2)^2$$

which simplifies at high energies to

$$\sigma = \frac{G_F^2}{3\pi} s .$$

The unlimited increase with s again violates unitarity. The amplitude is pure p wave and reads at high energies

$$T_1 = \frac{4G_F}{\sqrt{2}} 2s d_{11}^1$$

leading to the condition

$$\frac{4G_F}{\sqrt{2}} 2s < 24\pi$$

or

$$s < \frac{3\sqrt{2}\pi}{G_F}$$

in Fermi's theory. For the s -channel the W -boson propagator has to be modified to

$$\frac{g_{\mu\nu} - q_\mu q_\nu / M_W^2}{s - M_W^2 + iM_W \Gamma_W}$$

in order to avoid the pole at $s = M_W^2$.

As a result the differential and total cross section have to be multiplied by $M_W^4 f_{BW}^r$ where f_{BW}^r is the relativistic Breit-Wigner function

$$f_{BW}^r = \frac{1}{(s - M_W^2)^2 + M_W^2 \Gamma_W^2}$$

e.g.

$$\sigma(\bar{\nu}_\mu + e^- \rightarrow \mu^- + \bar{\nu}_e) = \frac{G_F^2}{3\pi s} (s - M^2)^2 \frac{M_W^4}{(s - M_W^2)^2 + M_W^2 \Gamma_W^2}$$

with the high energy limit

$$\sigma = \frac{G_F^2 M_W^4}{3\pi s} .$$

[]: