eeee

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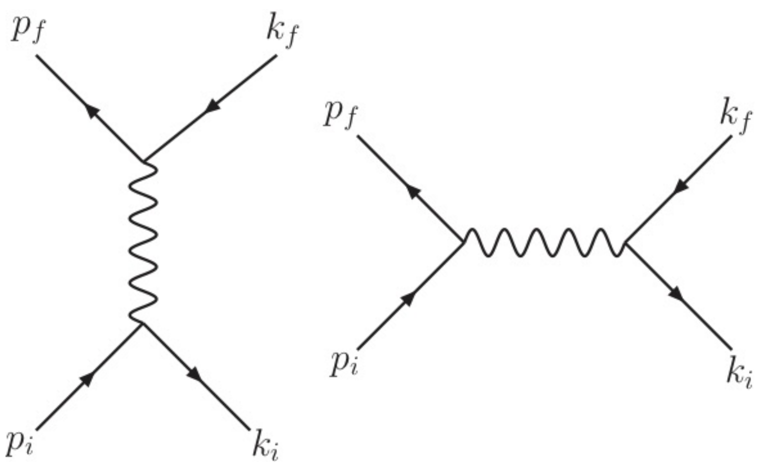
Bhabha and Moller scattering

1 Introduction



In calculating the cross section $e^-e^+ \to e^-e^+$ (Bhabha scattering) the annihilation or s channel amplitude (left figure) and the scattering or t channel amplitude (right figure) both contribute. (Note that the time arrow is here pointing upwards.) Neglecting the charge factor $(-\imath e^2)$ one has

$$T_{fi} = \bar{v}(k_i)\gamma^\mu u(p_i)\bar{u}(p_f)\gamma_\mu v(k_f)\frac{1}{s} - \bar{u}(p_f)\gamma^\mu u(p_i)\bar{v}(k_i)\gamma_\mu v(k_f)\frac{1}{t}$$

The minus sign is requested by the Pauli principle. Its application is easier understood in the case of electron electron scattering. The Feynman rules interpret positrons as electrons of negative energy moving backward in time. This explains the use of the Pauli principle also here. Another approach to understand the minus sign is to calculate the scattering diagramm as scattering of an electron off a positively charged fermion, i.e $T_S = \bar{u}(p_f)\gamma^\mu u(p_i)\bar{u}(k_f)\gamma_\mu u(k_i)/t$ leading to charge factor which is the negative of the one used in the annihilation diagram.

2 Kinematics

The CM system is used

[1]:
```
from sympy import *
```

[2]:
```
import heppackv0 as hep
```

```
Reading heppackv0.py

Done
```

[3]:
```
E,m,theta,pin,ki,pf,kf,alpha=symbols('E m theta pin ki pf kf alpha',real=True)
u,t,t0,s,s0=symbols('u t t0,s,s0',real=True)
```

[4]:
```
pin=[E,m,0,0]
ki=[E,m,pi,pi]
pf=[E,m,theta,0]
kf=[E,m,pi-theta,pi]
```

From $t = (p_f - p_i)^2 = -4p^2 \sin^2(\theta/2) = (s - 4m^2)\sin^2(\theta/2)$ we get

[5]:
```
sinthetahalf=sqrt(t/(4*m**2-s))
```

[6]:
```
costhetahalf=sqrt(u/(4*m**2-s))
```

3 The amplitudes

With $|T|^2 = |T_s|^2 + |T_t|^2 - 2T_sT_t$ we need explicit expressions for all amplitudes, although the average of the squared annihilation and scattering amplitudes could be taken from the eemumu notebook. There are 8 independent helicity amplitudes $T_{fi} = T(\lambda_3\lambda_4;\lambda_1\lambda_2)$ for incoming positrons with positive helicity λ_2. $T_{k,s}$ and $T_{k,t}$ are calculated separetely ($k = 1..8$).

1) $T(-+;-+)$:

[7]:
```
T1s=hep.dotprod4(hep.vbu(ki,1,pin,-1),hep.ubv(pf,-1,kf,1))/s
T1s
```

[7]:
$$-\frac{4E^2(\cos(\theta)+1)}{s}$$

[8]:
```
T1sv1=-2*cos(theta/2)**2;T1sv1
```

[8]:
$$-2\cos^2\left(\frac{\theta}{2}\right)$$

[9]:
```
T1s=T1sv1.subs(cos(theta/2),costhetahalf);T1s
```

[9]:
$$-\frac{2u}{4m^2-s}$$

[10]:
```
T1tv1=hep.dotprod4(hep.ubu(pf,-1,pin,-1),hep.vbv(ki,1,kf,1))/t;T1tv1
```

[10]:
$$\frac{(8E^2-4m^2)\cos^2\left(\frac{\theta}{2}\right)}{t}$$

[11]:
```
T1t=simplify((2*s-4*m**2)*costhetahalf**2/t);T1t
```

[11]:

$$-\frac{2u\left(2m^2-s\right)}{t\left(4m^2-s\right)}$$

2) $T(-,-;-,+)$:

[12]:
```
T2sv1=hep.dotprod4(hep.vbu(ki,1,pin,-1),hep.ubv(pf,-1,kf,-1))/s;T2sv1
```

[12]:

$$-\frac{4Em\sin\left(\theta\right)}{s}$$

[13]:
```
T2s=-8*m*sinthetahalf*costhetahalf/2/sqrt(s);T2s
```

[13]:

$$-\frac{4m\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{\sqrt{s}}$$

[14]:
```
T2tv1=hep.dotprod4(hep.ubu(pf,-1,pin,-1),hep.vbv(ki,1,kf,-1))/t;T2tv1
```

[14]:

$$\frac{2Em\sin\left(\theta\right)}{t}$$

[15]:
```
T2t=4*m*sinthetahalf*costhetahalf*sqrt(s)/2/t;T2t
```

[15]:

$$\frac{2m\sqrt{s}\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{t}$$

3) $T(+,+;-,+)$:

[16]:
```
T3s=hep.dotprod4(hep.vbu(ki,1,pin,-1),hep.ubv(pf,1,kf,1))/s;T3s
```

[16]:

$$-\frac{4Em\sin\left(\theta\right)}{s}$$

[17]:
```
T3s=T2s;T3s
```

[17]:

$$-\frac{4m\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{\sqrt{s}}$$

[18]:
```
T3t=hep.dotprod4(hep.ubu(pf,1,pin,-1),hep.vbv(ki,1,kf,1))/t;T3t
```

[18]:

$$\frac{2Em\sin\left(\theta\right)}{t}$$

[19]:
```
T3t=T2t;T3t
```

[19]:

$$\frac{2m\sqrt{s}\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{t}$$

4) $T(+,-;-,+)$:

[20]:
```
T4sv1=hep.dotprod4(hep.vbu(ki,1,pin,-1),hep.ubv(pf,1,kf,-1))/s;T4sv1
```

[20]:

$$\frac{4E^2\left(\cos\left(\theta\right)-1\right)}{s}$$

[21]:
```
T4s=-2*sinthetahalf**2;T4s
```

[21]:

$$-\frac{2t}{4m^2-s}$$

[22]:
```
T4tv1=hep.dotprod4(hep.ubu(pf,1,pin,-1),hep.vbv(ki,1,kf,-1))/t;T4tv1
```

[22]:

$$\frac{4m^2\sin^2\left(\frac{\theta}{2}\right)}{t}$$

[23]:
```
T4t=4*m**2*sinthetahalf**2/t;T4t
```

[23]:

$$\frac{4m^2}{4m^2-s}$$

5) $T(-,+;+,+)$:

[24]:
```
T5s=hep.dotprod4(hep.vbu(ki,1,pin,1),hep.ubv(pf,-1,kf,1))/s;T5s
```

[24]:

$$\frac{4Em\sin\left(\theta\right)}{s}$$

[25]:
```
T5s=-T2s;T5s
```

[25]:

$$\frac{4m\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{\sqrt{s}}$$

[26]:
```
T5t=hep.dotprod4(hep.ubu(pf,-1,pin,1),hep.vbv(ki,1,kf,1))/t;T5t
```

[26]:

$$-\frac{2Em\sin\left(\theta\right)}{t}$$

[27]:
```
T5t=-T2t;T5t
```

[27]:

$$-\frac{2m\sqrt{s}\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{t}$$

6) $T(-,-;+,+)$:

[28]:
```
T6sv1=hep.dotprod4(hep.vbu(ki,1,pin,1),hep.ubv(pf,-1,kf,-1))/s
T6sv1
```

[28]:

$$-\frac{4m^2\cos(\theta)}{s}$$

[29]:
```
T6s=simplify(-4*m**2*(costhetahalf**2-sinthetahalf**2)/s)
T6s
```

[29]:

$$\frac{4m^2(t-u)}{s(4m^2-s)}$$

[30]:
```
T6t=hep.dotprod4(hep.ubu(pf,-1,pin,1),hep.vbv(ki,1,kf,-1))/t;T6t
```

[30]:

$$-\frac{4m^2\sin^2\left(\frac{\theta}{2}\right)}{t}$$

[31]:
```
T6t=-T4t;T6t
```

[31]:

$$-\frac{4m^2}{4m^2-s}$$

7) $T(+,+;+,+):$

[32]:
```
T7s=hep.dotprod4(hep.vbu(ki,1,pin,1),hep.ubv(pf,1,kf,1))/s;T7s
```

[32]:

$$-\frac{4m^2\cos(\theta)}{s}$$

[33]:
```
T7s=T6s;T7s
```

[33]:

$$\frac{4m^2(t-u)}{s(4m^2-s)}$$

[34]:
```
T7tv1=hep.dotprod4(hep.ubu(pf,1,pin,1),hep.vbv(ki,1,kf,1))/t;T7tv1
```

[34]:

$$\frac{8E^2-4m^2\sin^2\left(\frac{\theta}{2}\right)-4m^2}{t}$$

[35]:
```
T7t=simplify((2*s-4*m**2*sinthetahalf**2-4*m**2)/t);T7t
```

[35]:

$$-\frac{4m^2t+2(2m^2-s)(4m^2-s)}{t(4m^2-s)}$$

8) $T(+,-;+,+):$

[36]:
```
T8sv1=hep.dotprod4(hep.vbu(ki,1,pin,1),hep.ubv(pf,1,kf,-1))/s;T8sv1
```

[36]:

$$-\frac{4Em\sin(\theta)}{s}$$

[37]:
```
T8s=T2s;T8s
```

[37]:

$$-\frac{4m\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{\sqrt{s}}$$

[38]:
```
T8tv1=hep.dotprod4(hep.ubu(pin,1,pf,1),hep.vbv(ki,1,kf,-1))/t;T8tv1
```

[38]:

$$\frac{2Em\sin(\theta)}{t}$$

[39]:
```
T8t=T2t;T8t
```

[39]:

$$\frac{2m\sqrt{s}\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{t}$$

4 Cross sections

Start with $\frac{1}{2}\sum_k |T_{s,k}|^2$

[40]:
```
tsqavsv1=simplify(T1s**2+T2s**2+T3s**2+T4s**2+T5s**2+T6s**2+T7s**2+T8s**2)/2;
↪tsqavsv1
```

[40]:

$$\frac{2\left(8m^4(t-u)^2+16m^2stu+s^2\left(t^2+u^2\right)\right)}{s^2\left(4m^2-s\right)^2}$$

[41]:
```
tsqavsv2=simplify(tsqavsv1.subs(u,4*m**2-s-t));tsqavsv2
```

[41]:

$$\frac{2\left(8m^4-8m^2t+s^2+2st+2t^2\right)}{s^2}$$

[42]:
```
tsqavs=2*(2*t0**2+s**2+2*s*t)/s**2;tsqavs
```

[42]:

$$\frac{2s^2+4st+4t_0^2}{s^2}$$

This is the result of the eemumu notebook. An alternative form can be found in the textbook of Landau and Lifshitz:

[43]:
```
tsqavsalt=2*(u**2+t**2+8*m**2*(s-m**2))/s**2;tsqavsalt
```

[43]:

$$\frac{16m^2\left(-m^2+s\right)+2t^2+2u^2}{s^2}$$

[44]:
```
check1=simplify((tsqavsv2-tsqavsalt).subs(u,4*m**2-s-t));check1
```

[44]:

$$0$$

Next step $\frac{1}{2}\sum_k |T_{t,k}|^2$:

[45]:
```
tsqavtv1=simplify(T1t**2+T2t**2+T3t**2+T4t**2+T5t**2+T6t**2+T7t**2+T8t**2)/2;
↪tsqavtv1
```

[45]:

$$\frac{2\left(8m^4t^2+4m^2stu+u^2\left(2m^2-s\right)^2+\left(2m^2t+\left(2m^2-s\right)\left(4m^2-s\right)\right)^2\right)}{t^2\left(4m^2-s\right)^2}$$

[46]:
```
tsqavtv2=simplify(tsqavtv1.subs(u,4*m**2-s-t));tsqavtv2
```

[46]:

$$\frac{2\left(8m^4-8m^2s+2s^2+2st+t^2\right)}{t^2}$$

This is tsqavsv2 (line 43) with $s \leftrightarrow t$ as it should be. The alternative expression of Landau Lifshitz is:

[47]:
```
tsqavtalt=2*(u**2+s**2+8*m**2*(t-m**2))/t**2;tsqavtalt
```

[47]:

$$\frac{16m^2\left(-m^2+t\right)+2s^2+2u^2}{t^2}$$

[48]:
```
check2=simplify((tsqavtv2-tsqavtalt).subs(u,4*m**2-s-t));check2
```

[48]:

$$0$$

Finally the interference term -$\sum_k T_{s,k}T_{t,k}$

[49]:
```
Tstv1=-simplify(T1s*T1t+T2s*T2t+T3s*T3t+T4s*T4t+T5s*T5t+T6s*T6t+T7s*T7t+T8s*T8t);
↪Tstv1
```

[49]:

$$-\frac{4\left(-4m^4t\left(t-u\right)-2m^2st\left(t+4u\right)-2m^2\left(t-u\right)\left(2m^2t+\left(2m^2-s\right)\left(4m^2-s\right)\right)+su^2\left(2m^2-s\right)\right)}{st\left(4m^2-s\right)^2}$$

[50]:
```
Tst=simplify(Tstv1.subs(u,4*m**2-s-t));Tst
```

[50]:

$$-\frac{16m^4}{st}+\frac{4s}{t}+8+\frac{4t}{s}$$

[51]:
```
print ('or'); Tstv2=2/s/t*(-8*m**4+2*(4*m**2-u)**2);Tstv2
```

```
or
```

[51]:

$$\frac{2\left(-8m^4+2\left(4m^2-u\right)^2\right)}{st}$$

[52]:
```
print ('or a la Landau Lifshitz '); Tstalt=4/t/s*(u-2*m**2)*(u-6*m**2);Tstalt
```

```
or a la Landau Lifshitz
```

[52]:

$$\frac{4\left(-6m^2+u\right)\left(-2m^2+u\right)}{st}$$

[53]:
```
check3=simplify((Tstv2-Tstalt));check3
```

[53]:

$$0$$

Multiplying with the charge factor $16\pi^2\alpha^2$ and dividing by the kinematic factor $16\pi s(s-4m^2)$ we obtain

$$\frac{d\sigma}{dt}(e^-e^+ \to e^-e^+) = \frac{2\pi\alpha^2}{s(s-4m^2)}(A+B+C)$$

with

$$A = 1+2\frac{t}{s}+2\frac{t_0^2}{s^2}B = 1+2\frac{s}{t}+2\frac{s_0^2}{t^2}C = 2\left(2+\frac{s}{t}+\frac{t}{s}-\frac{4m^4}{st}\right)$$

using the abbreviatons $s_0 = s-m^2$ and $t_0 = t-m^2$. If you prefer the Landau Lifshitz version (Note their A is our B and vice versa), here it is:

$$A = \frac{1}{s^2}(t^2+u^2+8m^2s_0)B = \frac{1}{t^2}(s^2+u^2+8m^2t_0)C = \frac{2}{st}(u-2m^2)(u-6m^2)$$

In calculating the cross section for electron electron scatterin $e^-e^- \to e^-e^-$ (Moller scattering) the annihilation diagramm (s channel) has to be replaced by the crossed exchange diagram with the photon propagator given by $1/u$ (so called u channel).

If one is only interested in the cross section it suffices to replace $s \leftrightarrow u$ in the formula for the average squared amplitudes, i.e.

$$A' = 1+2\frac{t}{u}+2\frac{t_0^2}{u^2} = \frac{1}{u^2}(t^2+s^2+8m^2u_0^2)C' = 2\left(2+\frac{u}{t}+\frac{t}{u}-\frac{4m^4}{ut}\right) = \frac{2}{ut}(s-2m^2)(s-6m^2)\ .$$

Obviously B does not change. For later use the u channel amplitudes are also explicitly given.

1) $T_u(-,+;-,+)$:

[54]:
```
T1uv1=hep.dotprod4(hep.ubu(kf,1,pin,-1),hep.ubu(pf,-1,ki,1))/u;T1uv1
```

[54]:

$$\frac{4m^2\cos^2\left(\frac{\theta}{2}\right)}{u}$$

[55]:
```
T1u=simplify(4*m**2*costhetahalf**2/u);T1u
```

[55]:

$$\frac{4m^2}{4m^2-s}$$

2) $T_u(-,-;-,+)$:

[56]:
```
T2uv1=hep.dotprod4(hep.ubu(kf,-1,pin,-1),hep.ubu(pf,-1,ki,1))/u;T2uv1
```

[56]:

$$-\frac{2Em\sin(\theta)}{u}$$

[57]:
```
T2u=-4*m*sinthetahalf*costhetahalf*sqrt(s)/2/u;T2u
```

[57]:

$$-\frac{2m\sqrt{s}\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{u}$$

3) $T_u(+,+;-,+)$:

[58]:
```
T3u=hep.dotprod4(hep.ubu(kf,1,pin,-1),hep.ubu(pf,1,ki,1))/u;T3u
```

[58]:

$$\frac{2Em\sin(\theta)}{u}$$

[59]:
```
T3u=-T2u;T3u
```

[59]:

$$\frac{2m\sqrt{s}\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{u}$$

4) $T_u(+,-;-,+)$:

[60]:
```
T4uv1=hep.dotprod4(hep.ubu(kf,-1,pin,-1),hep.ubu(pf,1,ki,1))/u;T4uv1
```

[60]:

$$\frac{\left(-8E^2+4m^2\right)\sin^2\left(\frac{\theta}{2}\right)}{u}$$

[61]:
```
T4u=(2*s-4*m**2)*sinthetahalf**2/u;T4u
```

[61]:

$$\frac{t\left(-4m^2+2s\right)}{u\left(4m^2-s\right)}$$

[62]:
```
T4u.subs(m,0)
```

[62]:

$$-\frac{2t}{u}$$

5) $T_u(-,+;+,+)$:

[63]:
```
T5u=hep.dotprod4(hep.ubu(kf,1,pin,1),hep.ubu(pf,-1,ki,1))/u;T5u
```

[63]:

$$-\frac{2Em\sin(\theta)}{u}$$

[64]:
```
T5u=T2u;T5u
```

[64]:

$$-\frac{2m\sqrt{s}\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{u}$$

6) $T_u(-,-;+,+):$

[65]:
```
T6u=hep.dotprod4(hep.ubu(kf,-1,pin,1),hep.ubu(pf,-1,ki,1))/u;T6u
```

[65]:

$$-\frac{4m^2\cos^2\left(\frac{\theta}{2}\right)}{u}$$

[66]:
```
T6u=-T1u;T6u
```

[66]:

$$-\frac{4m^2}{4m^2-s}$$

7) $T_u(+,+;+,+):$

[67]:
```
T7uv1=hep.dotprod4(hep.ubu(kf,1,pin,1),hep.ubu(pf,1,ki,1))/u;T7uv1
```

[67]:

$$\frac{-8E^2+4m^2\cos^2\left(\frac{\theta}{2}\right)+4m^2}{u}$$

[68]:
```
T7u=simplify((-2*s+4*m**2*costhetahalf**2+4*m**2)/u);T7u
```

[68]:

$$\frac{2\left(2m^2u+\left(2m^2-s\right)\left(4m^2-s\right)\right)}{u\left(4m^2-s\right)}$$

[69]:
```
T7u.subs(m,0)
```

[69]:

$$-\frac{2s}{u}$$

8) $T_u(+,-;+,+):$

[70]:
```
T8uv1=hep.dotprod4(hep.ubu(kf,-1,pin,1),hep.ubu(pf,1,ki,1))/u;T8uv1
```

[70]:

$$-\frac{2Em\sin\left(\theta\right)}{u}$$

[71]:
```
T8u=T2u;T8u
```

[71]:

$$-\frac{2m\sqrt{s}\sqrt{\frac{t}{4m^2-s}}\sqrt{\frac{u}{4m^2-s}}}{u}$$

[ ]: