Rosenbluthandmore

December 18, 2019

[1]:
```
from sympy import *
```

[2]:
```
import heppackv0 as hep
```

```
Reading heppackv0.py

Done
```

1) Kinematics

[3]:
```
theta,theta_L,F1,F2,G_E,G_M,tau,alpha=symbols('theta theta_B F1 F2 G_E G_M tau␣
↪alpha',real=True)
t,t0,s,s0=symbols('t t0,s,s0',real=True)
E,M,omega,E_i,E_f=symbols('E M omega E_i E_f',positive=True)
```

For reasons which become clear at the end of the notebook we calculate ep scattering in the Breit frame, which is defined by a purely space like virtual photon $q^\mu = (0, \vec{q})$. Let the photon run along the positive x-axis, $q^\mu = (0, Q, 0, 0)$. We take the incoming proton along the negativ x-axis. To ensure $q^0 = 0$ the asignment $k^\mu = (\sqrt{Q^2/4 + M^2}, Q/2, 0, 0)$ or conveniently $k^\mu = (\omega, \sqrt{\omega^2 - M^2}, 0, 0)$ is necessary. At the electron photon vertex the same line of arguments also only allows a symmetric arrangement. A massless electron with energy E and angle $\theta/2$ with respect to the z-axis is reflected at the z-axis. Therefore $p_i^\mu = (E, E\sin\theta/2, 0, 0)$, $p_f^\mu = (E, -E\sin\theta/2, 0, 0)$, i.e.

[4]:
```
pin=[E,0,theta/2,0]
ki=[omega,M,pi/2,pi]
pf=[E,0,theta/2,pi]
kf=[omega,M,pi/2,0]
```

resulting in the four-vectors

[5]:
```
pin4=hep.fourvec(pin);pin4
```

[5]:

$$\left[E,\quad 2E\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right),\quad 0,\quad E\left(-\sin^2\left(\frac{\theta}{4}\right)+\cos^2\left(\frac{\theta}{4}\right)\right)\right]$$

[6]:
```
pf4=hep.fourvec(pf);pf4
```

[6]:

$$\left[E,\quad -2E\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right),\quad 0,\quad E\left(-\sin^2\left(\frac{\theta}{4}\right)+\cos^2\left(\frac{\theta}{4}\right)\right)\right]$$

[7]:
```
ki4=hep.fourvec(ki);ki4
```

[7]:

$$\left[\omega,\quad -\sqrt{-M^2+\omega^2},\quad 0,\quad 0\right]$$

[8]:
```
kf4=hep.fourvec(kf);kf4
```

[8]:

$$\left[\omega,\quad \sqrt{-M^2+\omega^2},\quad 0,\quad 0\right]$$

The four momentum transfer squared t can be calculated at the electron and the proton vertex

[9]:
```
qvB1=[0,2*E*sin(theta/2),0,0];qvB1
```

[9]:

$$\left[0,\quad 2E\sin\left(\frac{\theta}{2}\right),\quad 0,\quad 0\right]$$

[10]:
```
tBv1=-qvB1[1]**2;tBv1
```

[10]:

$$-4E^2\sin^2\left(\frac{\theta}{2}\right)$$

[11]:
```
qvB2=[0,-2*sqrt(omega**2-M**2),0,0];qvB2
```

[11]:

$$\left[0,\quad -2\sqrt{-M^2+\omega^2},\quad 0,\quad 0\right]$$

[12]:
```
tBv2=-qvB2[1]**2;tBv2
```

[12]:

$$4M^2-4\omega^2$$

From $t=(p_f-p_i)^2=(k_i-k_f)^2$ we determine $\sin\theta/2$:

[13]:
```
sinthetahalf=sqrt(omega**2-M**2)/E;sinthetahalf
```

[13]:

$$\frac{\sqrt{-M^2+\omega^2}}{E}$$

[14]:
```
costhetahalf=simplify(sqrt(1-sinthetahalf**2));costhetahalf
```

[14]:

$$\frac{\sqrt{E^2 + M^2 - \omega^2}}{E}$$

The CM energy squared is calculated from $s = (p_i + k_i)^2 = M^2 + 2p_i \cdot k_i$. Substituting $\sqrt{\omega^2 - M^2}/E$ for $\sin\theta/2$ we obtain s and $s_0 = s - M^2$

[15]:
```
sB=simplify((M**2+2*hep.dotprod4(pin4,ki4)).subs(sin(theta/2),sinthetahalf));
↪print('s=');sB
```

s=

[15]:

$$2E\omega - M^2 + 2\omega^2$$

[16]:
```
s0B=sB-M**2;print('s_0=');s0B
```

s_0=

[16]:

$$2E\omega - 2M^2 + 2\omega^2$$

2) Proton as Dirac particle

Treating the proton as a pointlike fermion like the μ there are 4 independent amplitudes for $m = 0$. They are called Dirac amplitudes. We also calculate the squares subsequently.

[17]:
```
tmp1D=hep.dotprod4(hep.ubu(pf,-1,pin,-1),hep.ubu(kf,-1,ki,-1));tmp1D
```

[17]:

$$-4E\sqrt{-M^2 + \omega^2}\left(\sin\left(\frac{\theta}{2}\right) + 1\right)$$

[18]:
```
simplify(tmp1D.subs(sin(theta/2),sinthetahalf))
```

[18]:

$$-4E\sqrt{-M^2 + \omega^2} + 4M^2 - 4\omega^2$$

[19]:
```
t1D=-4*E*sqrt(omega**2-M**2)-4*(omega**2-M**2);t1D
```

[19]:

$$-4E\sqrt{-M^2 + \omega^2} + 4M^2 - 4\omega^2$$

[20]:
```
t1Dsq=t1D**2;t1Dsq
```

[20]:

$$\left(-4E\sqrt{-M^2+\omega^2}+4M^2-4\omega^2\right)^2$$

[21]:
```
tmp2D=hep.dotprod4(hep.ubu(pf,-1,pin,-1),hep.ubu(kf,-1,ki,1));tmp2D
```

[21]:

$$-4EM\cos\left(\frac{\theta}{2}\right)$$

[22]:
```
t2D=tmp2D.subs(cos(theta/2),costhetahalf);t2D
```

[22]:

$$-4M\sqrt{E^2+M^2-\omega^2}$$

[23]:
```
t2Dsq=t2D**2;t2Dsq
```

[23]:

$$16M^2\left(E^2+M^2-\omega^2\right)$$

[24]:
```
tmp3D=hep.dotprod4(hep.ubu(pf,-1,pin,-1),hep.ubu(kf,1,ki,1));tmp3D
```

[24]:

$$4E\sqrt{-M^2+\omega^2}\left(\sin\left(\frac{\theta}{2}\right)-1\right)$$

[25]:
```
simplify(tmp3D.subs(sin(theta/2),sinthetahalf))
```

[25]:

$$-4E\sqrt{-M^2+\omega^2}-4M^2+4\omega^2$$

[26]:
```
t3D=-4*E*sqrt(omega**2-M**2)+4*(omega**2-M**2);t3D
```

[26]:

$$-4E\sqrt{-M^2+\omega^2}-4M^2+4\omega^2$$

[27]:
```
t3Dsq=t3D**2;t3Dsq
```

[27]:

$$\left(-4E\sqrt{-M^2+\omega^2}-4M^2+4\omega^2\right)^2$$

[28]:
```
tmp4D=hep.dotprod4(hep.ubu(pf,-1,pin,-1),hep.ubu(kf,1,ki,-1));tmp4D
```

[28]:

$$4EM\cos\left(\frac{\theta}{2}\right)$$

[29]:
```
t4D=-t2D;t4D
```

[29]:

$$4M\sqrt{E^2+M^2-\omega^2}$$

[30]:
```
t4Dsq=t4D**2;t4Dsq
```

[30]:

$$16M^2\left(E^2+M^2-\omega^2\right)$$

The cross section is determined by $\overline{\sum}|T_{fi}|_D^2$, the average (factor 1/4) of the sum (factor 2) of all squared amplitudes

[31]:
```
tsqavD=simplify(t1Dsq+t2Dsq+t3Dsq+t4Dsq)/2;tsqavD
```

[31]:

$$16E^2\omega^2+32M^4-48M^2\omega^2+16\omega^4$$

which can be easily expressed in invariant form:

[32]:
```
tsqavD_inv=4*s0**2+2*t**2+4*t*s;print('tsqav_inv=');tsqavD_inv
```

tsqav_inv=

[32]:

$$4st+4s_0^2+2t^2$$

[33]:
```
simplify(4*sB*tBv2+4*s0B**2+2*tBv2**2-tsqavD)
```

[33]:

$$0$$

The invariant expression for $\overline{\sum}|T_{fi}|_D^2$ was derived in the eemumu notebook using only the crossing symmetry. This method is here verified, at least for $m=0$.

3) Pauli terms

The proton is no Dirac particle. First of all it has an anomalous magnetic moment. This can be accomodated by including a Pauli Term $\bar{u}(k_f)\frac{i\sigma^{\mu\nu}q_\nu}{2M}u(k_i)$ in the interaction. Consult the textbooks (e.g. Bjorken Drell) for more details.

[34]:
```
tmp1P=hep.dotprod4(hep.ubu(pf,-1,pin,-1),hep.ubuP(kf,-1,ki,-1));tmp1P
```

[34]:

$$-4E\sqrt{-M^2+\omega^2}\left(\sin\left(\frac{\theta}{2}\right)+1\right)$$

[35]:
```
t1P=t1D;t1P
```

[35]:

$$-4E\sqrt{-M^2+\omega^2}+4M^2-4\omega^2$$

[36]:
```
t1Psq=t1P**2;t1Psq
```

[36]:

$$\left(-4E\sqrt{-M^2+\omega^2}+4M^2-4\omega^2\right)^2$$

[37]:
```
tmp2P=hep.dotprod4(hep.ubu(pf,-1,pin,-1),hep.ubuP(kf,-1,ki,1));tmp2P
```

[37]:

$$\frac{4E\sqrt{-M+\omega}\sqrt{M+\omega}\sqrt{-M^2+\omega^2}\cos\left(\frac{\theta}{2}\right)}{M}$$

[38]:
```
tmp2P=4*E*(omega**2-M**2)*cos(theta/2)/M;tmp2P
```

[38]:

$$\frac{4E\left(-M^2+\omega^2\right)\cos\left(\frac{\theta}{2}\right)}{M}$$

[39]:
```
t2P=simplify(tmp2P.subs(cos(theta/2),costhetahalf));t2P
```

[39]:

$$\frac{4\left(-M^2+\omega^2\right)\sqrt{E^2+M^2-\omega^2}}{M}$$

[40]:
```
t2Psq=t2P**2;t2Psq
```

[40]:

$$\frac{16\left(-M^2+\omega^2\right)^2\left(E^2+M^2-\omega^2\right)}{M^2}$$

[41]:
```
tmp3P=hep.dotprod4(hep.ubu(pf,-1,pin,-1),hep.ubuP(kf,1,ki,1));tmp3P
```

[41]:

$$4E\sqrt{-M^2+\omega^2}\left(\sin\left(\frac{\theta}{2}\right)-1\right)$$

[42]:
```
t3P=t3D;t3P
```

[42]:

$$-4E\sqrt{-M^2+\omega^2}-4M^2+4\omega^2$$

[43]:
```
t3Psq=t3P**2;t3Psq
```

[43]:

$$\left(-4E\sqrt{-M^2+\omega^2}-4M^2+4\omega^2\right)^2$$

[44]:
```
tmp4P=hep.dotprod4(hep.ubu(pf,-1,pin,-1),hep.ubuP(kf,1,ki,-1));tmp4P
```

[44]:

$$-\frac{4E\sqrt{-M+\omega}\sqrt{M+\omega}\sqrt{-M^2+\omega^2}\cos\left(\frac{\theta}{2}\right)}{M}$$

[45]:
```
tmp4P=-4*E*(omega**2-M**2)*cos(theta/2)/M;tmp4P
```

[45]:

$$-\frac{4E\left(-M^2+\omega^2\right)\cos\left(\frac{\theta}{2}\right)}{M}$$

[46]:
```
t4P=simplify(tmp4P.subs(cos(theta/2),costhetahalf));t4P
```

[46]:

$$\frac{4\left(M^2-\omega^2\right)\sqrt{E^2+M^2-\omega^2}}{M}$$

[47]:
```
t4Psq=t4P**2;t4Psq
```

[47]:

$$\frac{16\left(M^2-\omega^2\right)^2\left(E^2+M^2-\omega^2\right)}{M^2}$$

The average of the squared Pauli terms $\overline{\sum}|T_{fi}|^2_P$ is given by

[48]:
```
tsqavP=simplify(t1Psq+t2Psq+t3Psq+t4Psq)/2;tsqavP
```

[48]:

$$-16E^2\omega^2+\frac{16E^2\omega^4}{M^2}+32M^4-80M^2\omega^2+64\omega^4-\frac{16\omega^6}{M^2}$$

[49]:
```
tsqavP_inv=2*t**2-(s*t**2+s0**2*t)/M**2;print('tsqavP_inv=');tsqavP_inv
```

tsqavP_inv=

[49]:

$$2t^2-\frac{st^2+s_0^2t}{M^2}$$

[50]:
```
simplify(2*tBv2**2-(sB*tBv2**2+s0B**2*tBv2)/M**2-tsqavP)
```
[50]:

$$0$$

In addition we need the interference terms $T_{fi\,D}T_{fi\,P}$

[51]:
```
t1DP=simplify(t1D*t1P);t1DP
```
[51]:

$$16\left(E\sqrt{-M^2+\omega^2}-M^2+\omega^2\right)^2$$

[52]:
```
t2DP=simplify(t2D*t2P);t2DP
```
[52]:

$$16\left(M^2-\omega^2\right)\left(E^2+M^2-\omega^2\right)$$

[53]:
```
t3DP=simplify(t3D*t3P);t3DP
```
[53]:

$$16\left(E\sqrt{-M^2+\omega^2}+M^2-\omega^2\right)^2$$

[54]:
```
t4DP=simplify(t4D*t4P);t4DP
```
[54]:

$$16\left(M^2-\omega^2\right)\left(E^2+M^2-\omega^2\right)$$

The average of $2T_{fi\,D}T_{fi\,P}$ is given by

[55]:
```
tsqavDP=simplify(t1DP+t2DP+t3DP+t4DP);tsqavDP
```
[55]:

$$64M^4-128M^2\omega^2+64\omega^4$$

[56]:
```
tsqavDP_inv=4*t**2;print('tsqavDP_inv=');tsqavDP_inv
```
```
tsqavDP_inv=
```
[56]:

$$4t^2$$

[57]:
```
simplify(4*tBv2**2-tsqavDP)
```
[57]:

$$0$$

4) Rosenbluth formula

We now account for the fact that the proton is not pointlike but a hadron with finite radius by including formfactors F_1 and F_2. They may depend on t, the only invariant available at the photon proton vertex. The normalization is $F_1(0) = 1$, i.e. the proton charge in units of e, and $F_2(0) = \kappa$, with $\kappa = 1.71$ the anomalous magnetic momentt of the proton in units of nuclear magnetons. In summary

[58]:
```
tsqavRB=tsqavD_inv*F1**2+tsqavP_inv*F2**2+F1*F2*tsqavDP_inv;tsqavRB
```

[58]:

$$F_1^2\left(4st+4s_0^2+2t^2\right)+4F_1F_2t^2+F_2^2\left(2t^2-\frac{st^2+s_0^2t}{M^2}\right)$$

Traditionally this result for the average of the squared ep scattering amplitudes is presented in a form known as Rosenbluth formula

[59]:
```
tsqavRB_std=(F1**2-t/4/M**2*F2**2)*(4*s*t+4*s0**2)+2*t**2*(F1+F2)**2;tsqavRB_std
```

[59]:

$$2t^2\left(F_1+F_2\right)^2+\left(F_1^2-\frac{F_2^2t}{4M^2}\right)\left(4st+4s_0^2\right)$$

[60]:
```
simplify(tsqavRB-tsqavRB_std)
```

[60]:

$$0$$

Before calculating the cross section we replace F_1 and F_2 by the electric and magnetic form factors $G_E = F_1 + \frac{t}{4M^2}F_2$, $G_M = F_1 + F_2$ with $G_E(0) = 1$ and $G_M(0) = 2.71$ (the magnetic moment of the proton in nuclear magnetons units).

[61]:
```
tmp1=(F1+F2).subs(F2,(G_M-G_E)/(1-t/4/M**2))
```

[62]:
```
simplify(tmp1.subs(F1,(G_E-G_M*t/4/M**2)/(1-t/4/M**2)))
```

[62]:

$$G_M$$

for $F_1 + F_2$ and

[63]:
```
tmp2=simplify((F1**2).subs(F1,(G_E-G_M*t/4/M**2)/(1-t/4/M**2))+(-F2**2*t/4/M**2).
↪subs(F2,(G_M-G_E)/(1-t/4/M**2)));tmp2
```

[63]:

$$\frac{-4G_E^2M^2+G_M^2t}{-4M^2+t}$$

for $F_1^2 - F_2^2t/4M^2$ which can be simplified using the definition $\tau = -t/4M^2$. The result is then used in the alternative expression for $\overline{\sum}|T_{fi}|^2_{RB}$

[64]:
```
tmp3=simplify(tmp2.subs(t,-4*M**2*tau));tmp3
```

[64]:

$$\frac{G_E^2 + G_M^2\tau}{\tau + 1}$$

[65]:
```
tsqavRB_alt=tmp3*(4*s*t+4*s0**2)+2*t**2*G_M**2;tsqavRB_alt
```

[65]:

$$2G_M^2 t^2 + \frac{\left(G_E^2 + G_M^2\tau\right)\left(4st + 4s_0^2\right)}{\tau + 1}$$

The cross section is obtained by multiplication with the usual kinematic and charge factors

[66]:
```
fac=pi*alpha**2/s0**2/t**2;fac
```

[66]:

$$\frac{\pi\alpha^2}{s_0^2 t^2}$$

[67]:
```
dsigdtRB=fac*tsqavRB_alt;print('dsigma_RB/dt=');dsigdtRB
```

dsigma_RB/dt=

[67]:

$$\frac{\pi\alpha^2\left(2G_M^2 t^2 + \frac{\left(G_E^2+G_M^2\tau\right)\left(4st+4s_0^2\right)}{\tau+1}\right)}{s_0^2 t^2}$$

which can be evaluated in any frame, e.g. in the rest system of the proton (laboratory system). We quote the standard formula with a rather intransparent mixture of invariants and lab system variables on the right hand side

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4\alpha^2 E'^2\cos^2(\theta/2)}{t^2[1+(2E/M)\sin^2(\theta/2)]}\left(\frac{G_\mathrm{E}^2 + \tau G_\mathrm{M}^2}{1+\tau} + 2\tau G_\mathrm{M}^2\tan^2(\theta/2)\right)$$

with E denoting the energy of the incomig electron and E', θ the energy and angle of the sattered electron.

5) Polarization of the recoil proton

For the calculation of polarization phenomena it is useful to have the squared Rosenbluth amplitudes at hand:

[68]:
```
t1sqRB=t1Dsq*F1**2+t1Psq*F2**2+2*F1*F2*t1D*t1P;t1sqRB
```

[68]:

$$F_1^2\left(-4E\sqrt{-M^2+\omega^2} + 4M^2 - 4\omega^2\right)^2 + 2F_1F_2\left(-4E\sqrt{-M^2+\omega^2} + 4M^2 - 4\omega^2\right)^2$$

$$+F_2^2\left(-4E\sqrt{-M^2+\omega^2}+4M^2-4\omega^2\right)^2$$

[69]:
```
t2sqRB=t2Dsq*F1**2+t2Psq*F2**2+2*F1*F2*t2D*t2P;t2sqRB
```

[69]:

$$16F_1^2M^2\left(E^2+M^2-\omega^2\right)-32F_1F_2\left(-M^2+\omega^2\right)\left(E^2+M^2-\omega^2\right)+\frac{16F_2^2\left(-M^2+\omega^2\right)^2\left(E^2+M^2-\omega^2\right)}{M^2}$$

[70]:
```
t3sqRB=t3Dsq*F1**2+t3Psq*F2**2+2*F1*F2*t3D*t3P;t3sqRB
```

[70]:

$$F_1^2\left(-4E\sqrt{-M^2+\omega^2}-4M^2+4\omega^2\right)^2+2F_1F_2\left(-4E\sqrt{-M^2+\omega^2}-4M^2+4\omega^2\right)^2+F_2^2\left(-4E\sqrt{-M^2+\omega^2}-4M^2\right.$$

[71]:
```
t4sqRB=t4Dsq*F1**2+t4Psq*F2**2+2*F1*F2*t4D*t4P;t4sqRB
```

[71]:

$$16F_1^2M^2\left(E^2+M^2-\omega^2\right)+32F_1F_2\left(M^2-\omega^2\right)\left(E^2+M^2-\omega^2\right)+\frac{16F_2^2\left(M^2-\omega^2\right)^2\left(E^2+M^2-\omega^2\right)}{M^2}$$

We study the polarization transfer of an incoming electron with negative helicity to the recoil proton. From the definition of the amplitudes given above it follows that up to kinematical factors the polarization, i.e. number of protons with helicity $-1/2$ minus the number of proton with helicity $+1/2$ is given by $(t_1^2+t_2^2-t_3^2-t_4^2)_{RB}$

[72]:
```
pol=simplify(t1sqRB+t2sqRB-t3sqRB-t4sqRB);pol
```

[72]:

$$64E\sqrt{-M^2+\omega^2}\left(-F_1^2M^2+F_1^2\omega^2-2F_1F_2M^2+2F_1F_2\omega^2-F_2^2M^2+F_2^2\omega^2\right)$$

which can be further simplified by using the electric and magnetic form factors and expressing τ by Breit system variables

[73]:
```
tauB=-tBv2/4/M**2;tauB
```

[73]:

$$\frac{-M^2+\omega^2}{M^2}$$

[74]:
```
F1B=simplify((G_E+tauB*G_M)/(1+tauB));F1B
```

[74]:

$$\frac{G_E M^2 - G_M \left(M^2 - \omega^2\right)}{\omega^2}$$

[75]:
```
F2B=simplify((G_M-G_E)/(1+tauB));F2B
```

[75]:

$$\frac{M^2 \left(-G_E + G_M\right)}{\omega^2}$$

[76]:
```
pol1=pol.subs(F1,F1B)
longpol=simplify(pol1.subs(F2,F2B));print('longpol=');longpol
```

```
longpol=
```

[76]:

$$64EG_M^2 \left(-M^2 + \omega^2\right)^{\frac{3}{2}}$$

Next the transverse polarization i.e proton spin along the positive or negative x axis is required. The proton states with spin up and down along the x-axis are calculated via

$$\begin{aligned} |\uparrow\rangle &= \tfrac{1}{\sqrt{2}}\left(|R\rangle + |L\rangle\right) \\ |\downarrow\rangle &= \tfrac{1}{\sqrt{2}}\left(-|L\rangle + |R\rangle\right) \ . \end{aligned}$$

where $|R\rangle$ or $|L\rangle$ denotes a proton with positive or negative helicity. For more explanations study the notebook eemumu. The ampltudes t_i have been defined as

$$t_1 = \langle e_L, p_L|T|e_L, p_L\rangle t_4 = \langle e_L, p_R|T|e_L, p_L\rangle t_2 = \langle e_L, p_L|T|e_L, p_R\rangle t_3 = \langle e_L, p_R|T|e_L, p_R\rangle$$

For incoming protons with negative helicity we have therefore to calculate

$$\frac{1}{2}((t_1 + t_4)^2 - (t_4 - t_1)^2) = 2t_1 t_4$$

and similarly $2t_2t_3$ for incoming protons with positive helicity. We calculate the resulting expressions for the Dirac, Pauli and interference terms separately.

[77]:
```
Dterm=simplify(t1D*t4D+t2D*t3D)*2*F1**2;Dterm
```

[77]:

$$64F_1^2 M \left(M^2 - \omega^2\right) \sqrt{E^2 + M^2 - \omega^2}$$

[78]:
```
Pterm=simplify(t1P*t4P+t2P*t3P)*2*F2**2;Pterm
```

[78]:

$$\frac{64F_2^2 \left(M^2 - \omega^2\right)^2 \sqrt{E^2 + M^2 - \omega^2}}{M}$$

[79]:
```
DPterm=simplify(t1D*t4P+t1P*t4D+t2D*t3P+t2P*t3D)*2*F1*F2;DPterm
```

[79]:

$$\frac{64F_1F_2\left(M^2-\omega^2\right)\left(2M^2-\omega^2\right)\sqrt{E^2+M^2-\omega^2}}{M}$$

[80]:
```
tpol=simplify(Dterm+Pterm+DPterm);tpol
```

[80]:

$$\frac{64\left(M^2-\omega^2\right)\sqrt{E^2+M^2-\omega^2}\left(F_1^2M^2+F_1F_2\left(2M^2-\omega^2\right)+F_2^2\left(M^2-\omega^2\right)\right)}{M}$$

Switching to the electromagnetic form factors yields

[81]:
```
tpol1=simplify(tpol.subs(F1,F1B))
transpol=simplify(tpol1.subs(F2,F2B));print('transpol=');transpol
```

transpol=

[81]:

$$64G_EG_MM\left(M^2-\omega^2\right)\sqrt{E^2+M^2-\omega^2}$$

The kinematical factors drop out for the ratio and therefore the real ratio of transverse and longitudinal polarization of the recoil protons is given by

[82]:
```
ToverL=simplify(transpol/longpol);ToverL
```

[82]:

$$-\frac{G_EM\sqrt{E^2+M^2-\omega^2}}{EG_M\sqrt{-M^2+\omega^2}}$$

or after inserting the Breit system expressions for $\sin\theta/2$ and $\cos\theta/2$ by hand

[83]:
```
TL_alt=-G_E*M/G_M/E/tan(theta/2);print('ToverL=');TL_alt
```

ToverL=

[83]:

$$-\frac{G_EM}{EG_M\tan\left(\frac{\theta}{2}\right)}$$

This beautiful result was derived by A.I.Akhiezer and M.P. Rekalo (Soviet Journal of Particle and Nuclear Physics 4,277,1974) with standard methods. At first sight it seems to be not very useful because it is derived in the Breit system and the orientation of spins is generally not an invariant. However the transformation from the Breit system to the labsystem is along the direction of the incoming proton, therefore the longitudinal polarization is invariant. The invariant total cross section for incoming lefthanded electrons is given by $t_1^2 + t_2^2$. Therefore the transverse polarization is also invariant. It only remains to express the Breit-system variables E,θ by invariants which are then calculated in the lab system. Rename E by E_B and use the Breit-system expression for s given above:

[84]:
```
E_B=simplify(((s+M**2-2*omega**2)/2/omega).subs(omega,sqrt(M**2-t/4)));E_B
```

[84]:

$$\frac{-M^2 + s + \frac{t}{2}}{\sqrt{4M^2 - t}}$$

In the lab (rest system of the incoming proton) the enegy E_i of the incoming electron is given by $s_0/2M$.

[85]:
```
E_i=(s-M**2)/2/M;E_i
```

[85]:

$$\frac{-\frac{M^2}{2} + \frac{s}{2}}{M}$$

The energy E_f of the outgoing electron is easily calculated from four-momentum conservation, $p_i + k_i = p_f + k_f$, i.e. $(q + k_i)^2 = M^2$ or $q^2 + 2q \cdot k_i = 0$ yielding $E_f = (2E_i M + t)/2M$

[86]:
```
E_f=(s-M**2+t)/2/M;E_f
```

[86]:

$$\frac{-\frac{M^2}{2} + \frac{s}{2} + \frac{t}{2}}{M}$$

[87]:
```
simplify(2*E_B*sqrt(4*M**2-t)-2*M*(E_i+E_f))
```

[87]:

$$0$$

i.e. using $\tau = -t/4M^2$

$$E = \frac{E_i + E_f}{2\sqrt{1+\tau}}$$

On the other hand we have $t = -4E_i E_f \sin^2\theta_L/2$ in the lab and $t = -4E^2 \sin^2\theta/2$ in the Breit-system, i.e. $\tan^2\theta/2\,(4E^2 + t) = \tan^2\theta_L/2\,(4E_i E_f + t)$. Using

[88]:
```
simplify((4*E_i*E_f+t)/(4*E_B**2+t))
```

[88]:

$$1 - \frac{t}{4M^2}$$

we obtain $\tan(\theta/2) = \sqrt{1+\tau}\tan(\theta_L/2)$ and finally for the ratio of transverse and longitudinal polarizations of the recoil proton in the laboratory system

$$\frac{T}{L} = \frac{-2G_E M}{G_M(E_i + E_f)\tan\frac{\theta_L}{2}}$$

Here G_E and G_M come in with equal weight, whereas ep scattering (traditionally used for form factor measurements) is according to the Rosenbluth formula already for moderate t dominated by G_M. This makes the evaluation of $G_E(t)$ very difficult. The realization of experiments using a single set up for determining the transverse and longitudinal polarization of the recoil protons in a longitudinally polarized electron beam is a major achievement of experimental physics (see e..g. A.J.R. Puckett et al, arXiv:1707.08587v3).

[ ]: