eemumu

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1) Introduction

A jupyter notebook to calculate basic QED cross sections: electron positron annihilation into muon pairs and electron muon scattering. In the textbooks these processes are treated using the trace technique for obtaining the spin sums. We simply calculate each amplitude explicitly.

As a first step the package sympy and the private package heppackv0 have to be imported.

[1]:
```
from sympy import *
```

[2]:
```
import heppackv0 as hep
```

```
Reading heppackv0.py

Done
```

Definition of the variables used: p_1=incoming electron, p_2 incoming positron, p_3 outgoing muon, p_4 outgoing antimuon. Each 4 momentum is characterized by energy, mass, polar angle azimuth angle. We use the CM system. Python requires definition of the symbols.

[46]:
```
E,M,m,theta,alpha=symbols('E M m theta alpha',real=True)
t,t0,s,s0,phi,P,beta,p=symbols('t t0,s,s0,phi,P,beta,p',real=True)
```

[4]:
```
p1=[E,m,0,0]
p2=[E,m,pi,pi]
p3=[E,M,theta,0]
p4=[E,M,pi-theta,pi]
```

2) The amplitudes

The amplitudes are products of currents divided by the propagator. The products of currents are provided by heppackv0. There are 8 independent helicity amplitudes $T_{fi} = T(\lambda_3\lambda_4;\lambda_1\lambda_2)$ for incoming positrons with positive helicity λ_2. The 8 amplitudes with negative helicity λ_2 are not independent but are up to a possible phase factor given by $T(-\lambda_3 - \lambda_4; -\lambda_1 - \lambda_2)$ due to the parity invariance of the electromagnetic interaction.

1) $T(-+;-+)$:

The amplitude

$$t_1 = -e^2\bar{v}_+(p_2)\gamma^\mu u_-(p_1)\bar{u}_-(p_3)\gamma_\mu v_+(p_4)\frac{1}{q^2}$$

is without the charge factor $e^2 = 4\pi\alpha$ given by

[5]:
```
t1=-hep.dotprod4(hep.vbu(p2,1,p1,-1),hep.ubv(p3,-1,p4,1))/4/E**2
t1
```

[5]:

$$\cos(\theta) + 1$$

and for the other combinations of helicities

2) $T(--;-+)$:

[6]:
```
t2=-hep.dotprod4(hep.vbu(p2,1,p1,-1),hep.ubv(p3,-1,p4,-1))/4/E**2
t2
```

[6]:

$$\frac{M \sin(\theta)}{E}$$

3) $T(++;-+)$:

[7]:
```
t3=-hep.dotprod4(hep.vbu(p2,1,p1,-1),hep.ubv(p3,1,p4,1))/4/E**2
t3
```

[7]:

$$\frac{M \sin(\theta)}{E}$$

4) $T(+-;-+)$:

[8]:
```
t4=-hep.dotprod4(hep.vbu(p2,1,p1,-1),hep.ubv(p3,1,p4,-1))/4/E**2
t4
```

[8]:

$$-\cos(\theta) + 1$$

5) $T(-+;++)$:

[9]:
```
t5=-hep.dotprod4(hep.vbu(p2,1,p1,1),hep.ubv(p3,-1,p4,1))/4/E**2
t5
```

[9]:

$$-\frac{m \sin(\theta)}{E}$$

6) $T(--;++)$:

[10]:
```
t6=-hep.dotprod4(hep.vbu(p2,1,p1,1),hep.ubv(p3,-1,p4,-1))/4/E**2
t6
```

[10]:

$$\frac{Mm \cos(\theta)}{E^2}$$

7) $T(++;++)$:

[11]:
```
t7=-hep.dotprod4(hep.vbu(p2,1,p1,1),hep.ubv(p3,1,p4,1))/4/E**2
t7
```

[11]:

$$\frac{Mm\cos(\theta)}{E^2}$$

8) $T(+-;++)$:

[12]:
```
t8=-hep.dotprod4(hep.vbu(p2,1,p1,1),hep.ubv(p3,1,p4,-1))/4/E**2
t8
```

[12]:

$$\frac{m\sin(\theta)}{E}$$

The following two lines are used for tests:

[13]:
```
test1=-hep.dotprod4(hep.vbu(p2,-1,p1,-1),hep.ubv(p3,-1,p4,1))/4/E**2;test1
```

[13]:

$$-\frac{m\sin(\theta)}{E}$$

[14]:
```
test2=-hep.dotprod4(hep.vbu(p2,-1,p1,1),hep.ubv(p3,1,p4,1))/4/E**2;test2
```

[14]:

$$-\frac{M\sin(\theta)}{E}$$

3) Cross sections

Because of parity invariance the 8 amplitudes containing $\bar{v}_-$ need not to be calculated explicitly. Summing over outgoing and averaging over incoming helicities we thus get for the average of squared amplitudes

[15]:
```
tsqav=simplify((t1**2+t2**2+t3**2+t4**2+t5**2+t6**2+t7**2+t8**2)/2)
print('tsqav=')
tsqav
```

```
tsqav=
```

[15]:

$$\cos^2(\theta)+1+\frac{M^2\sin^2(\theta)}{E^2}+\frac{m^2\sin^2(\theta)}{E^2}+\frac{M^2m^2\cos^2(\theta)}{E^4}$$

This can be expressed by the invariants s, t, t_0 (remember s= CM energy squared, t=four momentum transfer squared, $t_0 = t - m^2 - M^2$):

[16]:
```
tsqav_inv=2*(2*t0**2+s**2+2*t*s)/s**2
print('tsqav_inv='); tsqav_inv
```

```
tsqav_inv=
```

[16]:

$$\frac{2s^2 + 4st + 4t_0^2}{s^2}$$

Proof: Calulate s, t, t_0 by their CM values

[17]:
```
tCM=m**2+M**2-2*E**2+2*sqrt(E**2-m**2)*sqrt(E**2-M**2)*cos(theta)
tCM
```

[17]:

$$-2E^2 + M^2 + m^2 + 2\sqrt{E^2 - M^2}\sqrt{E^2 - m^2}\cos(\theta)$$

[18]:
```
t0CM=tCM-m**2-M**2
t0CM
```

[18]:

$$-2E^2 + 2\sqrt{E^2 - M^2}\sqrt{E^2 - m^2}\cos(\theta)$$

[19]:
```
sCM=4*E**2
sCM
```

[19]:

$$4E^2$$

and then:

[20]:
```
tsqav_invv1=tsqav_inv.subs(s,sCM)
tsqav_invv2=tsqav_invv1.subs(t,tCM)
tsqav_invv3=tsqav_invv2.subs(t0,t0CM)
proof=simplify(tsqav_invv3-tsqav)
proof
```

[20]:

$$0$$

dsigma/dt is obtained by multipliying with the charge factor $e^4 = 16\pi^2\alpha^2$ and dividing by

[21]:
```
factor=16*pi*s**2*(1-4*m**2/s)
factor
```

[21]:

$$16\pi s^2 \left(-\frac{4m^2}{s} + 1\right)$$

resulting in

[22]:
```
dsigdt=2*pi*alpha**2/s**2*1/(1-4*m**2/s)*(1+2*t0**2/s**2+2*t/s)
dsigdt
```

[22]:

$$\frac{2\pi\alpha^2 \left(1 + \frac{2t}{s} + \frac{2t_0^2}{s^2}\right)}{s^2 \left(-\frac{4m^2}{s} + 1\right)}$$

[23]:
```
proof=simplify(dsigdt-16*pi**2*alpha**2/factor*tsqav_inv)
proof
```

[23]:

$$0$$

That is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{2\pi\alpha^2}{s^2}\frac{1}{1-4m^2/s}\left(1+\frac{2t_0^2}{s^2}+\frac{2t}{s}\right) \quad .$$

Expressed in CM variables we get for $m = 0$:

[24]:
```
tsqav_invv4=simplify(tsqav_invv3.subs(m,0))
tsqav_invv4
```

[24]:

$$-\sin^2(\theta)+2+\frac{M^2\sin^2(\theta)}{E^2}$$

$d\sigma/d\Omega$ is easily calculated:

[25]:
```
factor1=factor.subs(m,0)
factor2=factor1.subs(s,sCM)
dsigdtv1=16*pi**2*alpha**2/factor2*tsqav_invv4
print('dsigdtv1='); dsigdtv1
```

```
dsigdtv1=
```

[25]:

$$\frac{\pi\alpha^2\left(-\sin^2(\theta)+2+\frac{M^2\sin^2(\theta)}{E^2}\right)}{16E^4}$$

[26]:
```
dsigdOm=dsigdtv1*E**2*sqrt(1-M**2/E**2)/pi
dsigdOm
```

[26]:

$$\frac{\alpha^2\sqrt{1-\frac{M^2}{E^2}}\left(-\sin^2(\theta)+2+\frac{M^2\sin^2(\theta)}{E^2}\right)}{16E^2}$$

or using $\beta_\mu = \sqrt{1-M^2/E^2}$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2\beta_\mu}{16E^2}\left(1+\frac{M^2}{E^2}+\beta_\mu^2\cos^2\Theta\right) \quad .$$

For $m = M = 0$ we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{2\pi\alpha^2}{s^2}\left(\frac{t^2+u^2}{s^2}\right)$$

or

[27]:
```
dsigdOmv1=dsigdOm.subs(M,0)
dsigdOmv1
```

[27]:

$$\frac{\alpha^2 \left(-\sin^2\left(\theta\right)+2\right)}{16E^2}$$

that is

$$\frac{d\sigma}{d\Omega}(e^-e^+ \to \mu^-\mu^+) = \frac{\alpha^2}{16E^2}(1+\cos^2\Theta)$$

with the total cross section

[28]:
```
sigQED=2*pi*integrate(sin(theta)*dsigdOmv1,(theta,0,pi))
print('sigQED=')
sigQED
```

```
sigQED=
```

[28]:

$$\frac{\pi\alpha^2}{3E^2}$$

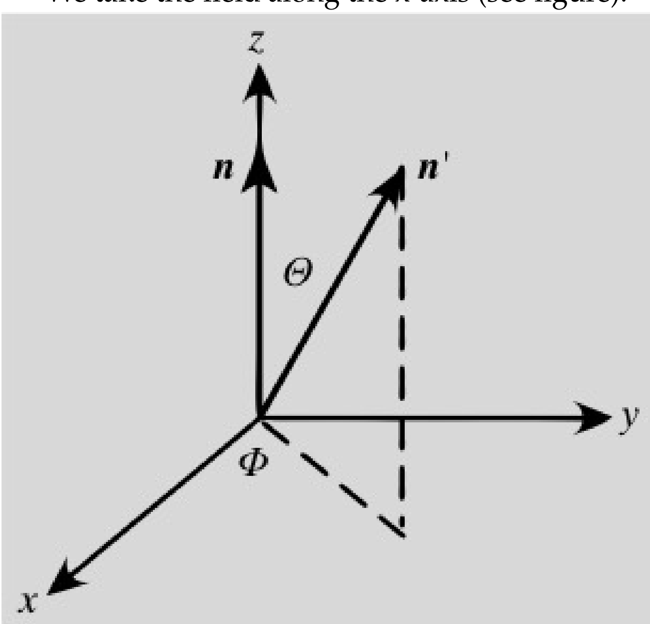
or

$$\sigma_{QED} = \frac{4\pi\alpha^2}{3s} \quad .$$

4) Polarized beams

It is also very interesting to study myon production by polarized electrons and positrons. In electron positron storage rings the beams become polarized after a while with the positron spin along the field (upward) and the electron spin downward. The standard method is again using traces and/or discussing general properties of products of currents. Having all amplitudes available the direct calculation is easy after understanding the kinematics.

We take the field along the x-axis (see figure).



Instead of calculating the production of myons in the direction ϕ in the xy plane we consider the equivalent kinematical situation of e^+ incoming along ϕ, e^- in the opposite direction, production of myons (antimyons) along the z (-z) axis. Thus we don't need to set up new ϕ dependent vectors p_3, p_4.

x-axis upward spin states of positrons travelling in direction ϕ in the xy plane are calculated from the first column of the transformation matrix

$$D^{1/2} = \begin{pmatrix} \cos(\theta/2) & -\mathrm{e}^{-\mathrm{i}\phi}\sin(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} .$$

i.e. with $\theta = \pi/2$

$$|{\uparrow}\rangle = \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\rangle + \mathrm{e}^{\mathrm{i}\phi}|\tfrac{-1}{2}\rangle\right)$$

Likewise from the second column for electrons with spin down

$$|{\downarrow}\rangle = \tfrac{1}{\sqrt{2}}\left(-\mathrm{e}^{-\mathrm{i}\phi}|\tfrac{1}{2}\rangle + |\tfrac{-1}{2}\rangle\right)$$

The translation into helicity states on the right hand side results in (watch the flip for electrons travelling in the opposite direction):

$$\begin{aligned} |e^+\uparrow\rangle &= \tfrac{1}{\sqrt{2}}\left(|e^+_+\rangle + \mathrm{e}^{\mathrm{i}\phi}|e^+_-\rangle\right) \\ |e^-\downarrow\rangle &= \tfrac{1}{\sqrt{2}}\left(-\mathrm{e}^{-\mathrm{i}\phi}|e^-_-\rangle + |e^-_+\rangle\right) . \end{aligned}$$

The initial state is given by the product of these states, for $m = 0$ only the mixed states contribute to the current.

$$|e^+\uparrow\rangle\,|e^-\downarrow\rangle = \tfrac{1}{2}\left(-\mathrm{e}^{-\mathrm{i}\phi}|e^-_-\rangle|e^+_+\rangle + \mathrm{e}^{\mathrm{i}\phi}|e^-_+\rangle|e^+_-\rangle\right) .$$

Because of parity invariance the amplitude

$$\bar{v}_-(p_2)\gamma^\mu u_+(p_1)\bar{u}_-(p_3)\gamma_\mu v_+(p_4)\frac{1}{q^2}$$

equals t_4 and therefore the amplitude pol1 for $\mu^-_-\mu^+_+$ production is

[29]:
```
pol1tmp=(-exp(-I*phi)*t1+exp(I*phi)*t4)/2
pol1tmp
```

[29]:

$$\frac{(-\cos(\theta)+1)\,e^{i\phi}}{2} - \frac{(\cos(\theta)+1)\,e^{-i\phi}}{2}$$

[30]:
```
pol1=-cos(phi)*cos(theta)+I*sin(phi);print('pol1='); pol1
```

```
pol1=
```

[30]:

$$i\sin(\phi) - \cos(\phi)\cos(\theta)$$

Similarly for $\mu^-_+\mu^+_-$

[31]:
```
pol2tmp=(-exp(-I*phi)*t4+exp(I*phi)*t1)/2
pol2tmp
```

[31]:

$$-\frac{(-\cos(\theta)+1)\,e^{-i\phi}}{2} + \frac{(\cos(\theta)+1)\,e^{i\phi}}{2}$$

[32]:
```
pol2=cos(phi)*cos(theta)+I*sin(phi);pol2
```
[32]:

$$i \sin(\phi) + \cos(\phi)\cos(\theta)$$

[33]:
```
polsqP=2*simplify(pol1*conjugate(pol1))
polsqP
```
[33]:

$$2\sin^2(\phi) + 2\cos^2(\phi)\cos^2(\theta)$$

[34]:
```
polsqP=2-2*sin(theta)**2*cos(phi)**2;polsqP
```
[34]:

$$-2\sin^2(\theta)\cos^2(\phi) + 2$$

resulting in

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e^-_\downarrow e^+_\uparrow \to \mu^-\mu^+) = \frac{\alpha^2}{8E^2}(1 - \sin^2\theta\cos^2\phi)$$

as in the standard reference Y.S.Tsai, PRD12, 3533 (1975), which also only treats massless electrons ($m = 0$).

The polarization of the beams is never complete. Designing the product of polarizations by P^2 with $0 \leq P^2 \leq 1$ we must add polsqP and polsqU in an obvious manner.

[35]:
```
polsqU=1+cos(theta)**2;polsqU
```
[35]:

$$\cos^2(\theta) + 1$$

[36]:
```
sig=P**2*polsqP+(1-P**2)*polsqU;sig
```
[36]:

$$P^2\left(-2\sin^2(\theta)\cos^2(\phi) + 2\right) + \left(-P^2 + 1\right)\left(\cos^2(\theta) + 1\right)$$

[37]:
```
simplify(sig)
```
[37]:

$$2P^2\left(-\sin^2(\theta)\cos^2(\phi) + 1\right) + \left(-P^2 + 1\right)\left(\cos^2(\theta) + 1\right)$$

or alternatively

[38]:
```
sigalt=polsqU-P**2*sin(theta)**2*cos(2*phi);sigalt
```
[38]:

$$-P^2\sin^2(\theta)\cos(2\phi) + \cos^2(\theta) + 1$$

[39]:
```
simplify(sig-sigalt)
```
[39]:

$$0$$

Finally we follow the custom of the experimentalists, to take the storage ring plane spanned by the x and z axis as scattering plane and the field along the y axis. With $\phi \to \pi/2 - \phi$ we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e^-_\downarrow e^+_\uparrow \to \mu^-\mu^+) = \frac{\alpha^2}{16E^2}(1 + \cos^2\theta + P^2\sin^2\theta\cos 2\phi)$$

5) Mott and Rutherford scattering

The cross section for electron myon scattering can be immediately obtained from tsqav_inv with the substitution $s \leftrightarrow t$ following from crossing symmetry.

[40]:
```
tsqav_sc=2*(2*s0**2+t**2+2*t*s)/t**2
print('tsqav_sc='); tsqav_sc
```

```
tsqav_sc=
```

[40]:

$$\frac{4st + 4s_0^2 + 2t^2}{t^2}$$

[41]:
```
16*pi**2*alpha**2/(16*pi*s0**2*(1-4*m**2*M**2/s0**2))*tsqav_sc
```

[41]:

$$\frac{\pi\alpha^2\left(4st + 4s_0^2 + 2t^2\right)}{s_0^2 t^2\left(-\frac{4M^2m^2}{s_0^2} + 1\right)}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(e\mu \to e\mu) = \frac{4\pi\alpha^2}{t^2}\frac{1}{1 - 4m^2M^2/s_0^2}\left(1 + \frac{t^2}{2s_0^2} + \frac{ts}{s_0^2}\right) \quad .$$

Scattering on muons at rest (Lab system) reads with $s_0 = s - m^2 - M^2 = 2EM$, $p = \sqrt{E^2 - m^2}$, $\beta = p/E$

[42]:
```
dsigdtLab=4*pi*alpha**2/t**2/beta**2*(1+t**2/8/E**2/M**2+t*s/4/E**2/M**2);
↪dsigdtLab
```

[42]:

$$\frac{4\pi\alpha^2\left(1 + \frac{st}{4E^2M^2} + \frac{t^2}{8E^2M^2}\right)}{\beta^2 t^2}$$

An important application is $M \to \infty$ yielding the Mott cross section for electron scattering in a Coulomb field. With $s = M^2$ it reads

[43]:
```
dsigdtMott=4*pi*alpha**2/t**2/beta**2*(1+t/4/E**2);dsigdtMott
```

[43]:

$$\frac{4\pi\alpha^2\left(1 + \frac{t}{4E^2}\right)}{\beta^2 t^2}$$

[44]:
```
dsigdtMottv1=dsigdtMott.subs(t,-4*beta**2*E**2*sin(theta/2)**2);dsigdtMottv1
```

[44]:

$$\frac{\pi\alpha^2\left(-\beta^2\sin^2\left(\frac{\theta}{2}\right) + 1\right)}{4E^4\beta^6\sin^4\left(\frac{\theta}{2}\right)}$$

[45]:
```
dsigdtMottv1.subs(E,p/beta)
```

[45]:

$$\frac{\pi\alpha^2\left(-\beta^2\sin^2\left(\frac{\theta}{2}\right)+1\right)}{4\beta^2p^4\sin^4\left(\frac{\theta}{2}\right)}$$

or finally with $dt = p^2 d\Omega/\pi$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} = \frac{\alpha^2}{4p^2\beta^2\sin^4(\theta/2)}\left(1-\beta^2\sin^2(\theta/2)\right)$$

For $\beta \to 0$ one has $p \to m\beta$ and only the Rutherford cross section survives:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Ruth}} = \frac{\alpha^2}{4m^2\beta^4\sin^4(\theta/2)}$$

Another important application of line 39 is myon scattering of electrons at rest. Using the second equality in $t = (p_a - p_e)^2 = (k_a - k_e)^2$ we get $t = 2(E - E')M$, withe E' the energy of the outgoing myon. Defining the relative energy loss $y = -t/s_0 = (E - E')/E$ we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y}(\mu e \to \mu e) = \frac{2\pi\alpha^2}{Em\beta_\mu^2 y^2}\left(1 - y\frac{s}{s_0} + \frac{y^2}{2}\right)$$

the starting point for calculating the energy loss of myons in matter.