Compton-scattering

January 5, 2021

[1]:
```
from sympy import *
import heppackv0 as hep
```

```
Reading heppackv0.py

Done
```

1 Kinematics

Amplitudes and cross section for Compton scattering, $\gamma + e \to \gamma + e$. Energy momentum conservation reads $k_i + p_i = k_f + p_f$. The CM system is used where p, θ denote the photon energy and scattering angle, E, m electron energy and mass.

[2]:
```
theta,t,t0,u,u0,x,y,z=symbols('theta t t0 u u0 x y z',real=True)
s,s0,E,p,m =symbols('s s0 E p m',positive=True)
```

[3]:
```
ki=[p,0,0,0]
pin=[E,m,pi,pi]
kf=[p,0,theta,0]
pf=[E,m,pi-theta,pi]
```

CM energy squared $s = (k_i + p_i)^2$ and $s_0 = s - m^2$ are expressed by CM variables:

[4]:
```
sCM=m**2+2*hep.dotprod4(hep.fourvec(ki),hep.fourvec(pin));sCM
```

[4]:

$$m^2 + 2p\left(E + \sqrt{E^2 - m^2}\right)$$

[5]:
```
sCM=m**2+2*p*(E+p);print('sCM= ');sCM
```

```
sCM=
```

[5]:

$$m^2 + 2p\,(E + p)$$

[6]:
```
s0CM=2*p*(E+p);s0CM
```

[6]:

$$2p\,(E + p)$$

It is not easy to express the scattering amplitudes by invariants. I found it best to express p and E by s and s_0 and write all amplitudes as function of s_0, $u_0 = (k_i - p_f)^2 - m^2$ and $t = (k_i - k_f)^2$ where the latter is mainly used as abbreviation $t = -u_0 - s_0$. One thus obtains rather simple expressions for $\sin(\theta/2)$ and $\cos(\theta/2)$

[7]:
```
pCM=s0/2/sqrt(s);pCM
```

[7]:

$$\frac{s_0}{2\sqrt{s}}$$

[8]:
```
ECM=(s+m**2)/2/sqrt(s);ECM
```

[8]:

$$\frac{\frac{m^2}{2} + \frac{s}{2}}{\sqrt{s}}$$

[9]:
```
tCM=-4*pCM**2*sin(theta/2)**2;tCM
```

[9]:

$$-\frac{s_0^2 \sin^2\left(\frac{\theta}{2}\right)}{s}$$

[10]:
```
solve(t+4*pCM**2*sin(theta/2)**2,sin(theta/2))
```

[10]:

$$\left[-\frac{\sqrt{s}\sqrt{-t}}{s_0}, \quad \frac{\sqrt{s}\sqrt{-t}}{s_0} \right]$$

[11]:
```
sinthetahalf=sqrt(-t*s)/s0;print('sinthetahalf=');sinthetahalf
```

```
sinthetahalf=
```

[11]:

$$\frac{\sqrt{s}\sqrt{-t}}{s_0}$$

[12]:
```
costhetahalf=sqrt(1-sinthetahalf**2).subs(s,s0+m**2);costhetahalf
```

[12]:

$$\sqrt{1 + \frac{t\left(m^2 + s_0\right)}{s_0^2}}$$

[13]:
```
costhetahalf=sqrt(m**2*t-u0*s0)/s0;print('costhetahalf=');costhetahalf
```

```
costhetahalf=
```

[13]:

$$\frac{\sqrt{m^2 t - s_0 u_0}}{s_0}$$

2 The amplitudes

The scattering amplitude is determined by the sum of the s channel and u channel Feynman graph:

$$T_{fi} = e^2 \bar{u}(p_f, s_f) \left(\not{\epsilon}_f^* \frac{\not{p}_i + \not{k}_i + m}{(p_i + k_i)^2 - m^2} \not{\epsilon}_i + \not{\epsilon}_i \frac{\not{p}_i - \not{k}_f + m}{(p_i - k_f)^2 - m^2} \not{\epsilon}_f^* \right) u(p, s_i)$$

The direct evaluation in the notebook is cumbersome. heppackv0.py contains the necessary procedure for calculating the 16 helicity amplitudes $T_{fi} = T_{\lambda_3\lambda_4;\lambda_1\lambda_2}$. 8 of these amplitudes are independent, here the 8 amplitudes for incoming photons with helicity $\lambda_1 = -1$. The helicity of the incoming and outgoing electrons are λ_2 and λ_4 respectively-

We evaluate generally the s channel helicity amplitude $T^s_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ and u channel helicity amplitude $T^u_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ separately by first calculating only the numerator and then dividing respectively by s_0 and u_0. In most cases $T^s_{\lambda_1\lambda_2;\lambda_3\lambda_4} = 0$.

2.1 $T1 = T_{--;--}$

[14]:
```
T1Sv1=simplify(hep.Ncompts(ki, -1, pin, -1, kf, -1, pf, -1));T1Sv1
```

[14]:

$$4\left(E^2 + Ep - m^2 + p\sqrt{E^2 - m^2} - \sqrt{E - m}\sqrt{E + m}\sqrt{E^2 - m^2}\right)\cos\left(\frac{\theta}{2}\right)$$

[15]:
```
T1Sv1=4*cos(theta/2)*(p**2+E*p);T1Sv1
```

[15]:

$$4\left(Ep + p^2\right)\cos\left(\frac{\theta}{2}\right)$$

which is obviously $2s_0 \cos(\theta/2)$. Dividing by s_0 yields

[16]:
```
T1S=2*cos(theta/2);T1S
```

[16]:

$$2\cos\left(\frac{\theta}{2}\right)$$

[17]:
```
T1Uv1=simplify(hep.Ncomptu(ki, -1, pin, -1, kf, -1, pf, -1));T1Uv1
```

[17]:

$$4\left(-E^2 + Ep + m^2 - p\sqrt{E^2 - m^2} - \sqrt{E - m}\sqrt{E + m}\sqrt{E^2 - m^2}\right)\sin^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

[18]:
```
T1Uv1=4*(E*p-3*p**2)*sin(theta/2)**2*cos(theta/2);T1Uv1
```

[18]:

$$\left(4Ep - 12p^2\right)\sin^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

[19]:
```
simplify(T1Uv1.subs(p,pCM).subs(E,ECM).subs(sin(theta/2),sinthetahalf))
```

[19]:

$$\frac{t\left(-m^2 - s + 3s_0\right)\cos\left(\frac{\theta}{2}\right)}{s_0}$$

[20]:
```
T1U=2*t*(s0-m**2)*cos(theta/2)/s0/u0;T1U
```

[20]:

$$\frac{2t\left(-m^{2}+s_{0}\right)\cos\left(\frac{\theta}{2}\right)}{s_{0}u_{0}}$$

[21]:
```
simplify((T1S+T1U).subs(cos(theta/2),costhetahalf))
```

[21]:

$$\frac{2\sqrt{m^{2}t-s_{0}u_{0}}\left(s_{0}u_{0}-t\left(m^{2}-s_{0}\right)\right)}{s_{0}^{2}u_{0}}$$

[22]:
```
T1=-2*sqrt(m**2*t-s0*u0)*(m**2*t+s0**2)/s0**2/u0;print('T1= ');T1
```

```
T1=
```

[22]:

$$-\frac{2\left(m^{2}t+s_{0}^{2}\right)\sqrt{m^{2}t-s_{0}u_{0}}}{s_{0}^{2}u_{0}}$$

[23]:
```
T1sq=simplify(T1**2);T1sq
```

[23]:

$$\frac{4\left(m^{2}t+s_{0}^{2}\right)^{2}\left(m^{2}t-s_{0}u_{0}\right)}{s_{0}^{4}u_{0}^{2}}$$

The ampltude $T(--;--)$ can be written as $T(--;--) = t_1 + t_2$ where t_1 depends only implicitly on m and t_2 is proportional to m^2, i.e. $t_2 = 0$ in the high energy approximation or for $m = 0$.

[24]:
```
t_1=T1.subs(m,0);t_1
```

[24]:

$$-\frac{2\sqrt{s_{0}}\sqrt{-u_{0}}}{u_{0}}$$

[25]:
```
t_1sq=t_1**2;t_1sq
```

[25]:

$$-\frac{4s_{0}}{u_{0}}$$

Next we calculate $T1sq - t_1^2$ and name it for notational simplicity t_2sq although it is not a real square and can be negative.

[26]:
```
t_2sq=simplify(T1sq+4*s0/u0);t_2sq
```

[26]:

$$\frac{4\left(s_{0}^{5}u_{0}+\left(m^{2}t+s_{0}^{2}\right)^{2}\left(m^{2}t-s_{0}u_{0}\right)\right)}{s_{0}^{4}u_{0}^{2}}$$

[27]:
```
t_2sq.subs(m,0)
```

[27]:

$$0$$

2.2 $T2 = T_{-+;--}$:

[28]:
```
T2S=simplify(hep.Ncompts(ki, -1, pin, -1, kf, -1, pf, 1));T2S
```

[28]:

$$0$$

[29]:
```
T2U=simplify(hep.Ncomptu(ki, -1, pin, -1, kf, -1, pf, 1));T2U
```

[29]:

$$-4mp\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)$$

[30]:
```
T2v1=T2U.subs(p,pCM).subs(sin(theta/2),sinthetahalf).subs(cos(theta/
↪2),costhetahalf);T2v1
```

[30]:

$$-\frac{2m\sqrt{-t}\left(m^2t - s_0u_0\right)}{s_0^2}$$

[31]:
```
T2=T2v1/u0;T2
```

[31]:

$$-\frac{2m\sqrt{-t}\left(m^2t - s_0u_0\right)}{s_0^2u_0}$$

[32]:
```
T2sq=T2**2;T2sq
```

[32]:

$$-\frac{4m^2t\left(m^2t - s_0u_0\right)^2}{s_0^4u_0^2}$$

2.3 $T3 = T_{++;--}$:

[33]:
```
T3Sv1=simplify(hep.Ncompts(ki, -1, pin, -1, kf, 1, pf, 1));T3Sv1
```

[33]:

$$-4mp\sin\left(\frac{\theta}{2}\right)$$

[34]:
```
tmp=simplify(T3Sv1.subs(p,pCM).subs(sin(theta/2),sinthetahalf));tmp
```

[34]:

$$-2m\sqrt{-t}$$

[35]:
```
T3S=tmp/s0;T3S
```

[35]:

$$-\frac{2m\sqrt{-t}}{s_0}$$

[36]:
```
T3Uv1=simplify(hep.Ncomptu(ki, -1, pin, -1, kf, 1, pf, 1));T3Uv1
```

[36]:

$$-4mp\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)$$

[37]:
```
tmp=simplify(T3Uv1.subs(p,pCM));tmp
```
[37]:

$$-\frac{2ms_0\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)}{\sqrt{s}}$$

[38]:
```
T3U=tmp.subs(sin(theta/2),sinthetahalf).subs(cos(theta/2),costhetahalf)/u0;T3U
```
[38]:

$$-\frac{2m\sqrt{-t}\left(m^2t-s_0u_0\right)}{s_0^2u_0}$$

[39]:
```
T3=simplify(T3S+T3U);T3
```
[39]:

$$-\frac{2m^3t\sqrt{-t}}{s_0^2u_0}$$

[40]:
```
T3sq=T3**2;T3sq
```
[40]:

$$-\frac{4m^6t^3}{s_0^4u_0^2}$$

2.4 $T4 = T_{+-;--}$:

[41]:
```
T4S=simplify(hep.Ncompts(ki, -1, pin, -1, kf, 1, pf, -1));T4S
```
[41]:

$$0$$

[42]:
```
T4U=simplify(hep.Ncomptu(ki, -1, pin, -1, kf, 1, pf, -1));T4U
```
[42]:

$$4\left(-E^2+Ep+m^2+p\sqrt{E^2-m^2}-\sqrt{E-m}\sqrt{E+m}\sqrt{E^2-m^2}\right)\sin^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

The factot inside the brackets is easiest reduced by hand to $Ep-p^2=\frac{m^2s_0}{2s}$ leading to

[43]:
```
T4=2*m**2*s0/s*sinthetahalf**2*costhetahalf/u0;T4
```
[43]:

$$-\frac{2m^2t\sqrt{m^2t-s_0u_0}}{s_0^2u_0}$$

[44]:
```
T4sq=T4**2;T4sq
```
[44]:

$$\frac{4m^4t^2\left(m^2t-s_0u_0\right)}{s_0^4u_0^2}$$

Finally the 4 squared small terms for incoming lefthanded photons are collected:

[45]:
```
small1=simplify(t_2sq+T2sq+T3sq+T4sq);small1
```

[45]:

$$\frac{4m^2t\left(2m^2t+s_0^2-2s_0u_0-u_0^2\right)}{s_0^2u_0^2}$$

2.5 $T5 = T_{--;-+}$:

Now the 4 amplitudes for incoming electrons with helicity $+1/2$ are calculated:

[46]:
```
T5S=simplify(hep.Ncompts(ki, -1, pin, 1, kf, -1, pf, -1));T5S
```

[46]:

$$0$$

[47]:
```
T5U=simplify(hep.Ncomptu(ki, -1, pin, 1, kf, -1, pf, -1));T5U
```

[47]:

$$2mp\left(\cos\left(\theta\right)+1\right)\sin\left(\frac{\theta}{2}\right)$$

[48]:
```
T2U
```

[48]:

$$-4mp\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)$$

[49]:
```
T5=-T2;T5
```

[49]:

$$\frac{2m\sqrt{-t}\left(m^2t-s_0u_0\right)}{s_0^2u_0}$$

[50]:
```
T5sq=T5**2;T5sq
```

[50]:

$$-\frac{4m^2t\left(m^2t-s_0u_0\right)^2}{s_0^4u_0^2}$$

2.6 $T6 = T_{-+;-+}$:

[51]:
```
T6S=hep.Ncompts(ki, -1, pin, 1, kf, -1, pf, 1);T6S
```

[51]:

$$0$$

[52]:
```
T6Uv1=hep.Ncomptu(ki, -1, pin, 1, kf, -1, pf, 1);T6Uv1
```

[52]:

$$2\Big(E^2\cos\left(\theta\right)+E^2-Ep\cos\left(\theta\right)-Ep-m^2\cos\left(\theta\right)-m^2-p\sqrt{E^2-m^2}\cos\left(\theta\right)-p\sqrt{E^2-m^2}-\sqrt{E-m}\sqrt{E+m}\sqrt{E$$

Sympy insists of converting expressions containing trigonometric functions of $\theta/2$ into functions of θ. I dont know how to avoid this. Therefore use brute force:

[53]:
```
T6Uv2=simplify(T6Uv1.subs(cos(theta),2*cos(theta/2)**2-1));T6Uv2
```

[53]:

$$4\left(E^{2}-Ep-m^{2}-p\sqrt{E^{2}-m^{2}}-\sqrt{E-m}\sqrt{E+m}\sqrt{E^{2}-m^{2}}\right)\cos^{3}\left(\frac{\theta}{2}\right)$$

[54]:
```
T6U=-4*(p**2+E*p)*cos(theta/2)**3;T6U
```

[54]:

$$\left(-4Ep-4p^{2}\right)\cos^{3}\left(\frac{\theta}{2}\right)$$

[55]:
```
tmp=simplify(T6U.subs(p,pCM).subs(E,ECM).subs(cos(theta/2),costhetahalf));tmp
```

[55]:

$$-\frac{\left(m^{2}t-s_{0}u_{0}\right)^{\frac{3}{2}}\left(m^{2}+s+s_{0}\right)}{ss_{0}^{2}}$$

[56]:
```
T6=-(sqrt(m**2*t-s0*u0))**3*2/s0**2/u0;T6
```

[56]:

$$-\frac{2\left(m^{2}t-s_{0}u_{0}\right)^{\frac{3}{2}}}{s_{0}^{2}u_{0}}$$

[57]:
```
T6sq=T6**2;T6sq
```

[57]:

$$\frac{4\left(m^{2}t-s_{0}u_{0}\right)^{3}}{s_{0}^{4}u_{0}^{2}}$$

Like T1 this amplitude consists of 2 terms, which can be separated by setting $m=0$

[58]:
```
t_3=T6.subs(m,0);t_3
```

[58]:

$$-\frac{2\left(-u_{0}\right)^{\frac{3}{2}}}{\sqrt{s_{0}}u_{0}}$$

[59]:
```
t_3sq=t_3**2;t_3sq
```

[59]:

$$-\frac{4u_{0}}{s_{0}}$$

[60]:
```
t_4sq=expand(T6**2+4*u0/s0);print('t_4sq is not a real square');t_4sq
```

```
t_4sq is not a real square
```

[60]:

$$\frac{4m^{6}t^{3}}{s_{0}^{4}u_{0}^{2}}-\frac{12m^{4}t^{2}}{s_{0}^{3}u_{0}}+\frac{12m^{2}t}{s_{0}^{2}}$$

2.7 $T7 = T_{++;-+}$:

[61]:
```
T7S=expand_trig(hep.Ncompts(ki, -1, pin, 1, kf, 1, pf, 1));T7S
```

[61]:

$$0$$

[62]:
```
T7Uv1=expand_trig(hep.Ncomptu(ki, -1, pin, 1, kf, 1, pf, 1));T7Uv1
```

[62]:

$$2\left(E^2\cos(\theta) - E^2 - Ep\cos(\theta) + Ep - m^2\cos(\theta) + m^2 + p\sqrt{E^2-m^2}\cos(\theta) - p\sqrt{E^2-m^2} - \sqrt{E-m}\sqrt{E+m}\sqrt{E$$

[63]:
```
T7U=simplify(T7Uv1.subs(cos(theta),2*cos(theta/2)**2-1));T7U
```

[63]:

$$4\left(-E^2 + Ep + m^2 - p\sqrt{E^2-m^2} + \sqrt{E-m}\sqrt{E+m}\sqrt{E^2-m^2}\right)\sin^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

This looks like T4U

[64]:
```
T7=T4;T7
```

[64]:

$$-\frac{2m^2t\sqrt{m^2t - s_0u_0}}{s_0^2u_0}$$

[65]:
```
T7sq=T7**2;T7sq
```

[65]:

$$\frac{4m^4t^2\left(m^2t - s_0u_0\right)}{s_0^4u_0^2}$$

2.8 $T8 = T_{+-;-+}$:

[66]:
```
T8S=simplify(hep.Ncompts(ki, -1, pin, 1, kf, 1, pf, -1));T8S
```

[66]:

$$0$$

[67]:
```
T8Uv1=simplify(hep.Ncomptu(ki, -1, pin, 1, kf, 1, pf, -1));T8Uv1
```

[67]:

$$2mp\left(\cos(\theta) - 1\right)\sin\left(\frac{\theta}{2}\right)$$

[68]:
```
T8=simplify((4*m*p*sinthetahalf**3/u0).subs(p,pCM));T8
```

[68]:

$$\frac{2ms\left(-t\right)^{\frac{3}{2}}}{s_0^2u_0}$$

[69]:
```
T8sq=(T8.subs(s,s0+m**2))**2;T8sq
```

[69]:

$$-\frac{4m^2t^3\left(m^2+s_0\right)^2}{s_0^4u_0^2}$$

The second set of small terms is given by the sum of the squared $T5$ to $T8$. Finally the small terms are collected.

[70]:
```
small2=simplify(T5sq+t_4sq+T7sq+T8sq);small2
```

[70]:

$$-\frac{4m^2t\left(2m^2t^2+2m^2tu_0+s_0t^2-2s_0u_0^2\right)}{s_0^3u_0^2}$$

[71]:
```
tmp1=simplify(small1+small2)
small=simplify(tmp1.subs(t,-s0-u0));small
```

[71]:

$$\frac{16m^4}{u_0^2}+\frac{32m^4}{s_0u_0}+\frac{16m^4}{s_0^2}+\frac{16m^2}{u_0}+\frac{16m^2}{s_0}$$

3 Cross section

The small terms have a clear structure:

[72]:
```
R=m**2/u0/s0*(s0+u0);print('Define R');R
```

```
Define R
```

[72]:

$$\frac{m^2\left(s_0+u_0\right)}{s_0u_0}$$

[73]:
```
simplify(small-16*(R+R**2))
```

[73]:

$$0$$

[74]:
```
R=-m**2*t/s0/u0;print('simpler alternative expression for R');R
```

```
simpler alternative expression for R
```

[74]:

$$-\frac{m^2t}{s_0u_0}$$

Putting all pieces together the sqared average of the amplitudes is given by

[75]:
```
tsqav=-2*u0/s0-2*s0/u0+8*(R+R**2);print('tsqav= ');tsqav
```

```
tsqav=
```

[75]:

$$\frac{8m^4t^2}{s_0^2u_0^2} - \frac{8m^2t}{s_0u_0} - \frac{2s_0}{u_0} - \frac{2u_0}{s_0}$$

i.e.

$$\overline{\sum} T_{fi}^2 = \left(\frac{-2u_0}{s_0} + \frac{2s_0}{-u_0} + \frac{-8m^2t}{s_0u_0} + \frac{8m^4t^2}{s_0^2u_0^2}\right) \quad .$$

The cross section $d\sigma/dt$ is obtained by multiplication with the kinematic factor $1/16\pi s_0^2$ and the charge factor $e^4 = 16\pi^2\alpha^2$ with the final result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(\gamma e^- \to \gamma e^-) = \frac{2\pi\alpha^2}{s_0^2}\left(\frac{-u_0}{s_0} + \frac{s_0}{-u_0} + \frac{-4m^2t}{s_0u_0} + \frac{4m^4t^2}{s_0^2u_0^2}\right) \quad .$$

or

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(\gamma e^- \to \gamma e^-) = \frac{2\pi\alpha^2}{s_0^2}\left(\frac{-u_0}{s_0} + \frac{s_0}{-u_0} - \frac{1}{2}g(s,t)\right)$$

with

$$g(s,t) = -8(R+R^2) = \frac{8m^2t}{s_0u_0} - \frac{8m^4t^2}{s_0^2u_0^2} \quad .$$

$g(s,t)/2$ is sometimes (see e.g. S. Gasiorowicz, Elementary Particle Physics) quoted as

$$g(s,t) = 1 - \left(\frac{s+m^2}{s-m^2} - \frac{2m^2}{s+t-m^2}\right)^2 \quad ,$$

which is of course equivalent to my definition.

Proof:

[76]:
```
simplify((1-((s0+2*m**2)/s0+2*m**2/u0)**2+4*(R+R**2)).subs(t,-s0-u0))
```

[76]:

$$0$$

For photon scattering off an electron at rest (so called laboratory system) the cross section reads particullary simple. Let ω, ω' denote the energy of the incoming and outgoing photon respectively and θ its scattering angle. Then $s_0 = 2m\omega$, $t = -4\omega\omega'\sin^2(\theta/2)$ and using the definition $u = (k_f - p_i)^2$ one gets $u_0 = -2m\omega'$. Without invoking computer algebra this leads to $R = -\sin^2(\theta/2)$ and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\pi\alpha^2}{2m^2\omega^2}\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right)$$

or

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right) \quad .$$

At low energies $\omega \to \omega'$ and therefore

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{2m^2}\left(2 - \sin^2\theta\right) \quad ,$$

which is the result of classical physics (Thomson scattering).

Sometimes it is usefull to express cross sections and amplitudes in terms of dimensionless invariants. Taking $s_0 = xm^2$ and $y = 1 + u_0/s_0 = -t/s_0$ yields

$$y = \frac{x\sin^2(\theta/2)}{1+x}$$

and

$$\frac{u_0}{s_0} = \frac{-(1 + x\cos^2(\theta/2)}{1+x}$$

in the CM system.

The range of x and y is given by $1 \leq x < \infty$ and $0 \leq y \leq x/(1+x)$, where the second relation is read off the CM expression for y. Using these new variables we get nice alternative expressions for R and $\overline{\sum} T_{fi}^2$.

[77]:
```
altR=R.subs(u0,(y-1)*s0).subs(s0,x*m**2).subs(t,-m**2*x*y);altR
```

[77]:

$$\frac{y}{x\,(y-1)}$$

[78]:
```
simplify(altR+altR**2)
```

[78]:

$$\frac{y\,(x\,(y-1)+y)}{x^2\,(y-1)^2}$$

[79]:
```
tsqavV1=2*(1-y)+2/(1-y)+8*altR+8*altR**2;tsqavV1
```

[79]:

$$-2y + 2 + \frac{2}{-y+1} + \frac{8y}{x\,(y-1)} + \frac{8y^2}{x^2\,(y-1)^2}$$

or even nicer

$$\overline{\sum} T_{fi}^2 = 2(1-y) + \frac{2}{1-y} - 8\frac{y}{x(1-y)} + 8\frac{y^2}{x^2(1-y)^2}$$

4 Discussion

The dependence of the cross section on s, θ is not obvious. For a better understanding we plot $\overline{\sum} T_{fi}^2$ as function of x, y where y is expressed by the CM scattering angle θ.

[80]:
```
yCM=simplify((-tCM/s0).subs(s,s0+m**2).subs(s0,x*m**2));yCM
```

[80]:

$$\frac{x\sin^2\left(\frac{\theta}{2}\right)}{x+1}$$

As said above $T1 = t_1 + t_2$ with $t_1 = 2/\sqrt{1-y}$ (which does not depend on m^2 explicitly), $t_2 \sim m^2$. Similarly $T6 = t_3 + t_4$ with $t_3 = 2\sqrt{1-y}$ and $t_4 \sim m^2$. We might call t_1, t_3 the big amplitudes because the dont vanish in the limit $m^2 \to 0$. The contribution of the big and small parts to $\overline{\sum} T_{fi}^2$ is written as function of x, θ in the following cells:

[81]:
```
bigplot=simplify((2*(1-yCM)**2+2)/(1-yCM));bigplot
```

[81]:

$$\frac{2\left(x+1\right)^{2}+2\left(x\cos^{2}\left(\frac{\theta}{2}\right)+1\right)^{2}}{\left(x+1\right)\left(x\cos^{2}\left(\frac{\theta}{2}\right)+1\right)}$$

[82]:
```
Rplot=simplify(altR.subs(y,yCM));Rplot
```

[82]:

$$-\frac{\sin^{2}\left(\frac{\theta}{2}\right)}{x\cos^{2}\left(\frac{\theta}{2}\right)+1}$$

[83]:
```
gplot=-8*simplify(Rplot+Rplot**2);gplot
```

[83]:

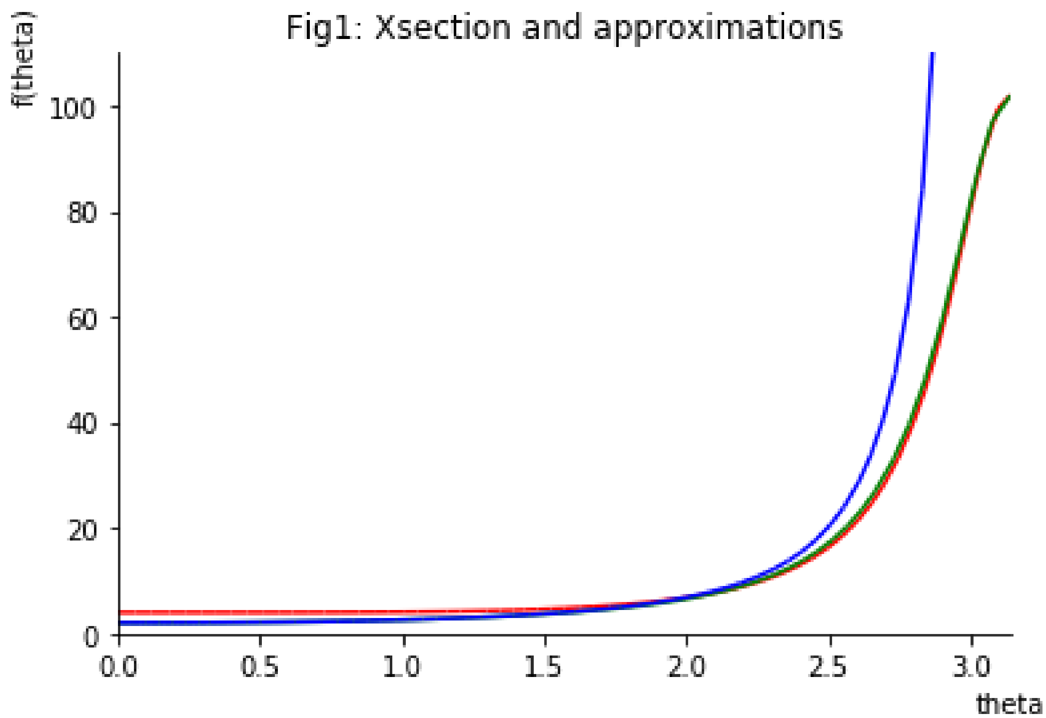
$$-\frac{4\left(-x-1\right)\left(-\cos\left(2\theta\right)+1\right)}{\left(x\cos\left(\theta\right)+x+2\right)^{2}}$$

[84]:
```
sigplot=bigplot-gplot
```

The cross section peaks in backward direction (red curve in fig. 1). The fig also demonstrates that already for $\theta > 150^\circ$ $\overline{\sum}T_{fi}^2$ is very well approximated by $t_1^2/2 = -2/(1-y) = -2s_0/u_0$ (green curve) alone. y depends implicitly on m^2 (cell 80). In the limit $m^2 \to 0$ all small terms vanish and with $x \to \infty$ we get $t_1^2 = 4/\cos^2(\theta/2)$. The last equality leads, however, to a very bad approximation for $\overline{\sum}T_{fi}^2$ (blue curve).

$x = 50$ corresponds to $\sqrt{s} \approx 4$ MeV, which is far in the relativistic regime. The reader is invited to vary the parameters.

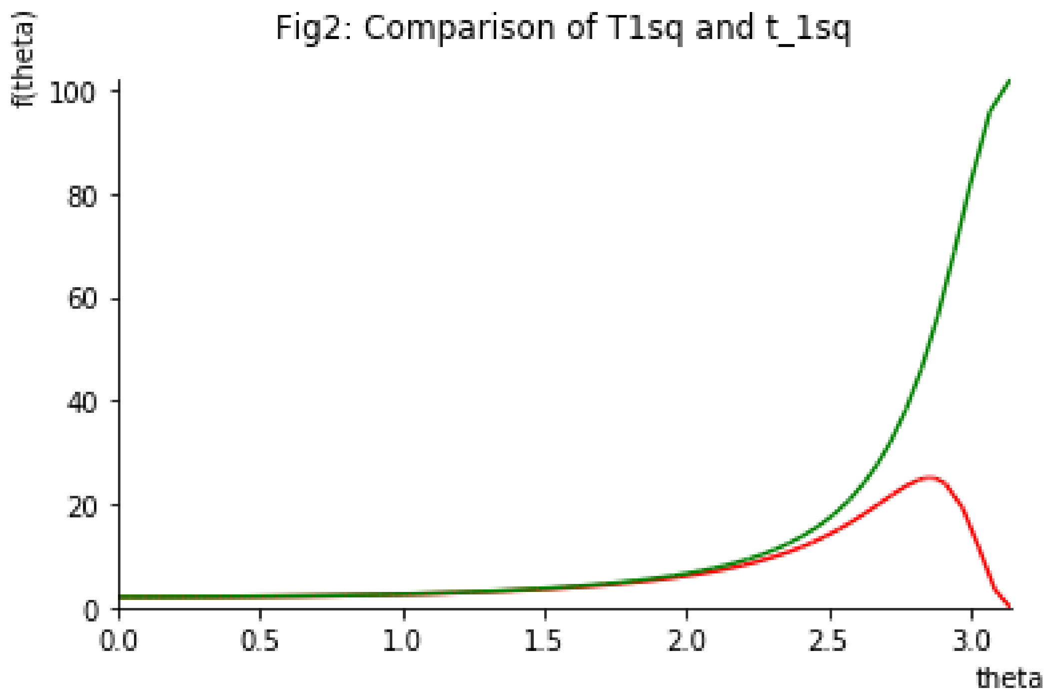
[105]:
```
p1=plot(sigplot.subs(x,50),2/(1-yCM).subs(x,50),2/cos(theta/2)**2,(theta,0,pi),
title='Fig1: Xsection and approximations',ylim=(0,110),show=False)
p1[0].line_color='r'
p1[1].line_color='g'
p1[2].line_color='b'
p1.show()
```



We get more insight into the subtle cancellations taking place by inspecting $T1^2 = T^2_{--;--}$:

[86]:
```
T1sqplot=simplify(T1sq.subs(t,-yCM*s0).subs(u0,(yCM-1)*s0).subs(s0,x*m**2))
```

[87]:
```
p2=plot(T1sqplot.subs(x,50)/2,2/(1-yCM).subs(x,50),
(theta,0,pi),title='Fig2: Comparison of T1sq and t_1sq',show=False)
p2[0].line_color='r'
p2[1].line_color='g'
p2.show()
```



Angular momentum conservation $\Delta J_3 = 0$ requires $T_{--;--} = 0$ at $\theta = \pi$. This is indeed the case, as is shown by the red curve in fig. 2 which bends over due to the large negative t_2sq. On the other hands side $t_1^2/2$ which yields such a good approximation to $\overline{\sum} T_{fi}^2$ badly violates the request $T_{--;--} = 0$ in the backward direction. The gap between the green and the red curve is miraculously filled by the angular momentum conserving helicity flip amplitudes $T_{++;--} = T3$ and $T_{+-;-+} = T8$.

[88]:
```
Tflipsq=simplify((T3sq+T8sq).subs(t,-y*s0).subs(u0,(y-1)*s0).subs(s0,x*m**2));
↪Tflipsq
```

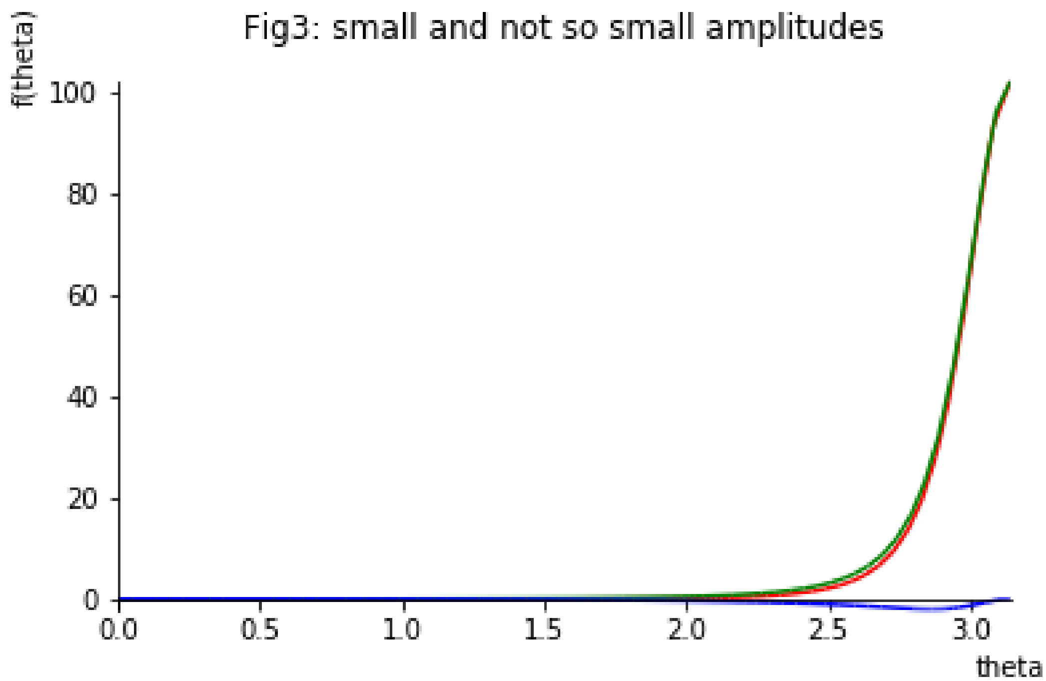
[88]:

$$\frac{4y^3\left((x+1)^2+1\right)}{x^3\,(y-1)^2}$$

[89]:
```
Tflipsqplot=Tflipsq.subs(y,yCM)
```

[90]:
```
t_2sqplot=simplify(t_2sq.subs(t,-yCM*s0).subs(u0,(yCM-1)*s0).subs(s0,x*m**2))
```

[91]:
```
p3=plot(Tflipsqplot.subs(x,50)/2,-t_2sqplot.subs(x,50)/2,
-gplot.subs(x,50),(theta,0,pi),title='Fig3: small and not so small␣
↪amplitudes',show=False)
p3[0].line_color='r'
p3[1].line_color='g'
p3[2].line_color='b'
p3.show()
```



Looking at the green and red curve one finds indeed t_2sq is approximately given by $T^2_{++;--} + T^2_{+-;-+}$. The lesson to be learned is that some of the so called small amplitudes ($\sim m^2$) are not at all small in the limit $\theta \to \pi$ but of the order $2s_0/m^2$. The takeover of the "small" amplitudes (peak of the red curve in fig.2) occurs at $\cos(\theta/2) \approx \sqrt{m^2/s_0} = \sqrt{1/x}$ for $x > 5$. Compare the really small $g(s_0, \theta)$, plotted in blue.

5 Polarization

To begin with the transfer of circular polarization from the incoming to the outgoing photon is investgated in the CM system. It is given by $T1^2 + T2^2 + T5^2 + T6^2 - T3^2 - T4^2 - T7^2 - T8^2$. We start with the small terms.

[92]:
```
lpsmall=simplify((t_2sq+T2sq+T5sq+t_4sq
-T3sq-T4sq-T7sq-T8sq).subs(t,-y*s0).subs(u0,(y-1)*s0).subs(s0,x*m**2));lpsmall
```

[92]:

$$-\frac{8y\left(y^2-2y+2\right)}{x\left(y^2-2y+1\right)}$$

[93]:
```
lp=lpsmall+4*(1-y)+4/(1-y);lp
```

[93]:

$$-4y+4+\frac{4}{-y+1}-\frac{8y\left(y^2-2y+2\right)}{x\left(y^2-2y+1\right)}$$

The polarization degree of the outgoing photon is obtained by deviding lp by $2\overline{\sum} T^2_{fi}$

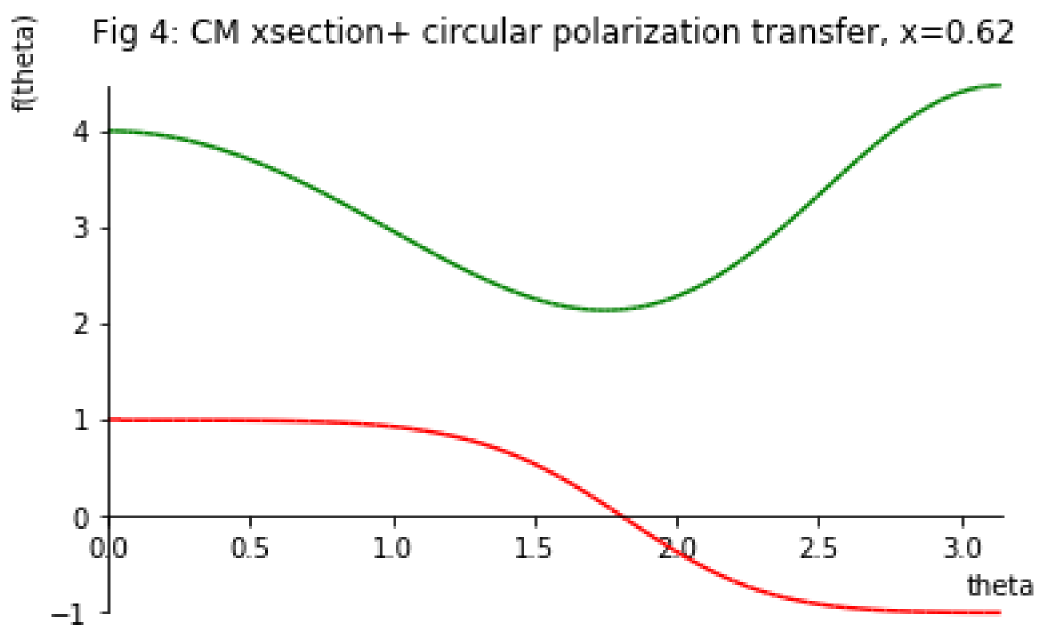
[94]:
```
lpdeg=simplify(lp/2/tsqavV1);lpdeg
```

[94]:

$$\frac{x\left(xy^3-3xy^2+4xy-2x+2y^3-4y^2+4y\right)}{x^2y^3-3x^2y^2+4x^2y-2x^2-4xy^2+4xy-4y^2}$$

[95]:
```
lpdegplot=lpdeg.subs(y,yCM)
```

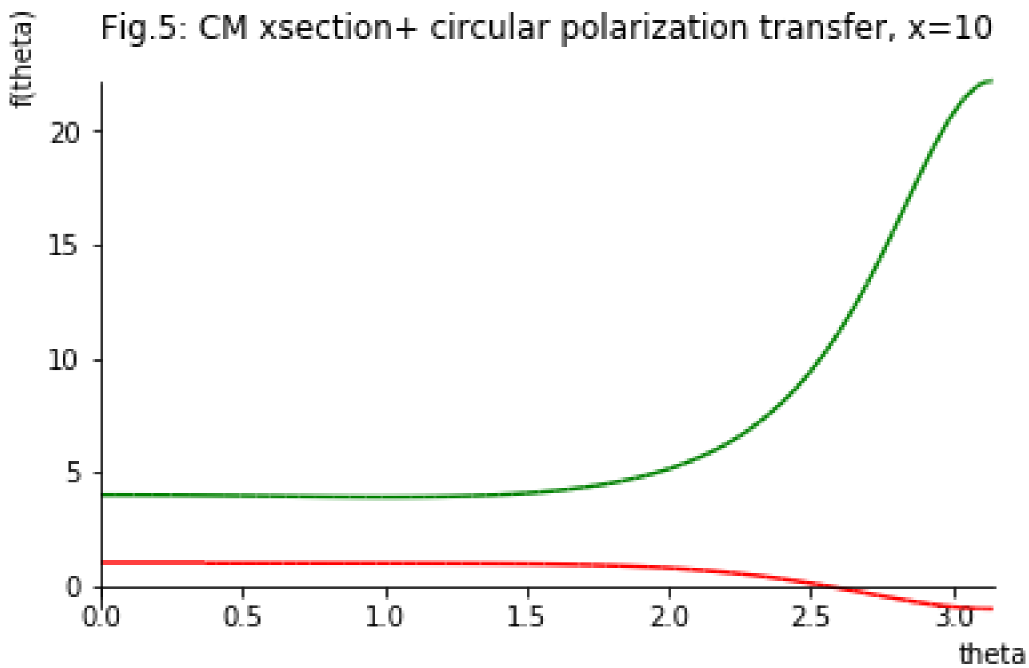
[96]:
```
p4=plot(lpdegplot.subs(x,0.62),sigplot.subs(x,0.62),(theta,0,pi),
title='Fig 4: CM xsection+ circular polarization transfer, x=0.62',show=False)
p4[0].line_color='r'
p4[1].line_color='g'
p4.show()
```



In the nonrelativistic regime (Thomson scattering) lefthanded photons flip to righthanded photons in the backward direction. $x = 0.62$ corresponds to the scattering of laser photons by 20 GeV electrons at SLAC producing high energy photons in the backward direction. Our result for $\theta \to \pi$ remains valid because the Lorentz transformation is along the z-axis.

In the relativistic regime (e.g. $x = 10$, fig. 5) we also observe a flip of CM circular polarization in the backward direction where the cross section is largest. The flip again occurs at $\cos(\theta/2) \approx \sqrt{m^2/s_0} = \sqrt{1/x}$.

[97]:
```
p5=plot(lpdegplot.subs(x,10),sigplot.subs(x,10),(theta,0,pi),
title='Fig.5: CM xsection+ circular polarization transfer, x=10',show=False)
p5[0].line_color='r'
p5[1].line_color='g'
p5.show()
```



Compton scattering may be used to measure the polarization of the incoming electrons. For electrons with helicity -1/2 scattering off photons with helicity -1 one has $J_3 = 1/2$ and $\sigma_{1/2} \sim \sum T^2_{xx,--}$. For the scattering of positive helicity electrons $\sigma_{3/2} \sim \sum T^2_{xx,-+}$ holds.

[98]:
```
sigonehalf=simplify(T1sq+T2sq+T3sq+T4sq);sigonehalf
```

[98]:

$$\frac{8m^4t^2}{s_0^2u_0^2} + \frac{4m^2t}{u_0^2} - \frac{8m^2t}{s_0u_0} - \frac{4m^2t}{s_0^2} - \frac{4s_0}{u_0}$$

[99]:
```
sigonehalfv1=simplify(sigonehalf.subs(t,-y*s0).subs(u0,(y-1)*s0).
↪subs(s0,x*m**2));sigonehalfv1
```

[99]:

$$\frac{-4x^2y + 4x^2 + 4xy^3 - 8xy + 8y^2}{x^2\,(y^2 - 2y + 1)}$$

[100]:
```
sigthreehalf=simplify(T5sq+T6sq+T7sq+T8sq);sigthreehalf
```

[100]:

$$\frac{-8m^4t^3 - 8m^4t^2u_0 - 4m^2s_0t^3 + 8m^2s_0tu_0^2 - 4s_0^2u_0^3}{s_0^3u_0^2}$$

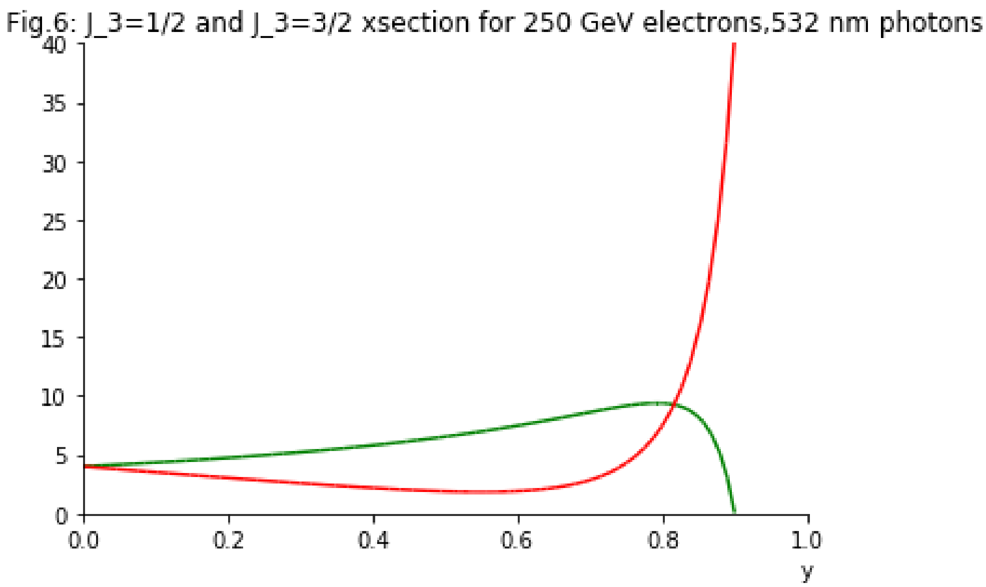
[101]:
```
sigthreehalfv1=simplify(sigthreehalf.subs(t,-y*s0).subs(u0,(y-1)*s0).
↪subs(s0,x*m**2));
sigthreehalfv1
```

[101]:

$$-4y + 4 + \frac{4y^3}{x\,(y-1)^2} - \frac{8y}{x} + \frac{8y^2}{x^2\,(y-1)^2}$$

In the planned international linear collider ILC a beam of $E = 250$ GeV lefthanded electrons is foreseen. Its polarization may be determined by measuring the Compton cross section in backward direction (G. Mortgaat-Pick et al., Phys.Rept. 460, 131, 2008). A 532 nm laser beam ($\omega = 2.33$ eV) is directed along the negative z-axis versus the incoming electrons. Therefore $s_0 = 4\omega E$ and $y = \omega'/E \sin^2(\theta/2)$ which is approximately ω'/E for photon scattering angles θ near $\theta = \pi$. With $x = 8.9$ we have $y_{max} = 0.9$ resulting in photons of energy $\omega' = 225$ GeV or electrons of $E = 25$ GeV along the positive z-axis. Fig.6 shows $\sigma_{1/2}$ and $\sigma_{3/2}$ plotted versus $y = \omega'/E$ (again neglecting common factors needed to convert sqared amplitudes into cross sections). At high y a large difference between $\sigma_{1/2}$ and $\sigma_{3/2}$ is observed.

[102]:
```
p6=plot(sigonehalfv1.subs(x,8.9),sigthreehalfv1.subs(x,8.9),(y,0,0.9),ylabel=' ',
title='             Fig.6: J_3=1/2 and J_3=3/2 xsection for 250 GeV␣
 ↪electrons,532 nm photons ',
ylim=(0,40),xlim=(0,1),show=False)
p6[0].line_color='g'
p6[1].line_color='r'
p6.show()
```



y_{max} is reached for $\theta = \pi$. However, due to the enormous Lorentz-boost all scattering angle are close to π and therefor y depends very strongly on θ. This can be understood by squaring the fourmomentum conservation $k_i + p_i - k_f = p_f$ yielding

$$k_i \cdot p_i - k_i \cdot k_f - k_f \cdot p_i = 0 \ ,$$

which reads using the ILC kinematic variables (βE is the momentum of the incoming electron)

$$\omega E(1+\beta) - \omega\omega'(1-\cos\theta) - \omega' E(1+\beta\cos\theta) = 0 \ .$$

We set $\theta = \pi - \chi$ with a very small angle χ. It is safe to set $\beta = 1$ in the first term and $\cos\theta = -1$ in the second term. The 3rd term requires more care. Using $\beta \approx 1 - m^2/2E^2$ and $\cos\theta \approx -1 + \chi^2/2$ one gets $1 + \beta\cos\theta = m^2/2E^2 + \chi^2/2$ resulting in

$$y = \frac{\omega}{\omega + \frac{m^2}{4E} + \frac{E\chi^2}{4}}$$

or counting χ in multiples $n\chi_0$ of $\chi_0 = m/E$

$$y = \frac{1}{1 + \frac{1}{x} + \frac{n^2}{x}} = \frac{x}{1 + n^2 + x} \ .$$

Considering the extremely small value of χ_0 the outgoing photons and electrons travel along the z axis together with the electron beam. Note that for practical reasons the laser beam is directed at a small angele (say 10 mrad) with respect to the z axis. For measuring the Compton cross section it is probably easiest to magnetically separate the low energy electrons coresponding to the high y cross section.

For unpolarized beams the Compton cross section may be written as $\sigma_U = (\sigma_{1/2} + \sigma_{3/2})/2$ whereas for partially polarized electron and photon beams the cross section is given by

$$\sigma_P = (1 - P_e P_\gamma)\sigma_U + P_e P_\gamma \sigma_{1/2}$$

or

$$\sigma_P = \sigma_U (1 + P_e P_\gamma A) \ ,$$

where the analyzing power A is given by

$$A = \frac{\sigma_{1/2} - \sigma_{3/2}}{\sigma_{1/2} + \sigma_{3/2}} \ .$$

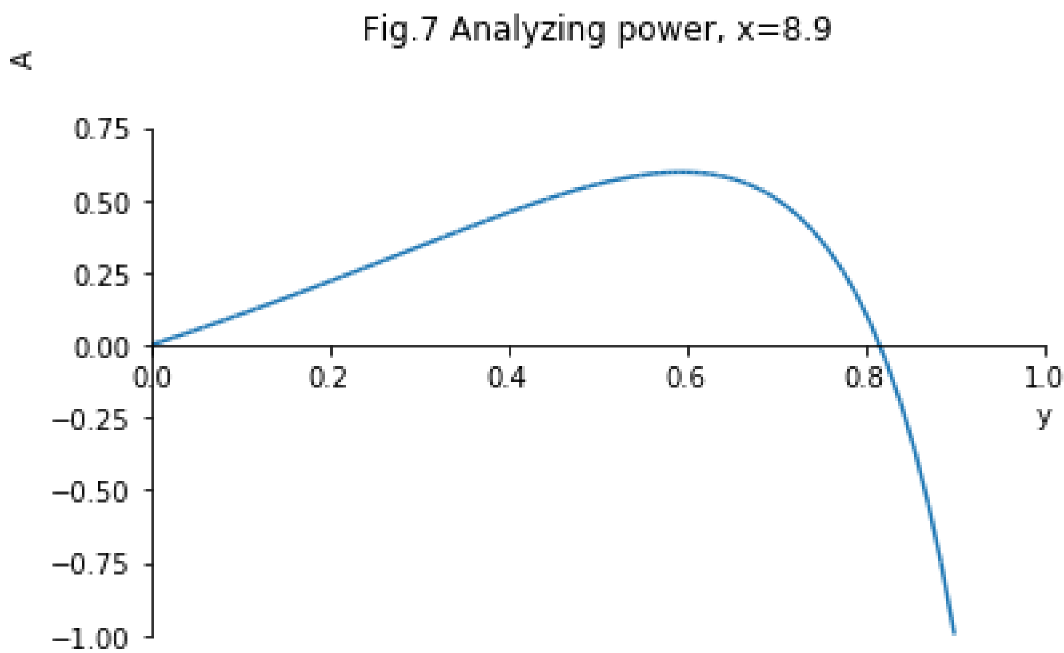
This quantity is plotted in fig.7 for the example of fig.6.

[103]:
```
A=simplify((sigonehalfv1-sigthreehalfv1)/(sigonehalfv1+sigthreehalfv1));A
```

[103]:

$$\frac{xy\left(xy^2 - 3xy + 2x + 2y^2 - 4y\right)}{-x^2y^3 + 3x^2y^2 - 4x^2y + 2x^2 + 4xy^2 - 4xy + 4y^2}$$

[104]:
```
p7=plot(A.subs(x,8.9),(y,0,0.9),ylim=(-1,1),xlim=(0,1),title=('Fig.7 Analyzing␣
↪power, x=8.9'),
ylabel=('A'))
```



6 Related cross sections

$\overline{\sum} T_{fi}^2$ for $\gamma\gamma \to e^- e^+$ can be obtained from the corresponing Compton scattering formula via the exchange $s \leftrightarrow t$ plus multiplication by -1 (using the abbreviatons $s_0 = s - m^2$ etc).

$$\overline{\sum} T_{fi}^2 = \left(\frac{2u_0}{t_0} + \frac{2t_0}{u_0} + \frac{8m^2 s}{t_0 u_0} - \frac{8m^4 s^2}{t_0^2 u_0^2} \right) \quad .$$

Multiplication with the kinematic factor $1/16\pi s(s - 4m^2)$ and the charge factor $e^4 = 16\pi^2\alpha^2$ yields the formula for the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(\gamma\gamma \to e^- e^+) = \frac{2\pi\alpha^2}{s(s-4m^2)} \left(\frac{u_0}{t_0} + \frac{t_0}{u_0} + \frac{4m^2 s}{t_0 u_0} - \frac{4m^4 s^2}{t_0^2 u_0^2} \right)$$

and finally

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(e^- e^+ \to \gamma\gamma) = \frac{2\pi\alpha^2}{s^2} \left(\frac{u_0}{t_0} + \frac{t_0}{u_0} + \frac{4m^2 s}{t_0 u_0} - \frac{4m^4 s^2}{t_0^2 u_0^2} \right) \quad .$$

[ ]: