Advanced Machine Learning - HW1

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Exercise 1 (5 points)

Show that the eigenvalues of a symmetric positive matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are all positive. (Hint: Recall that the eigenvalues of a symmetric matrix are real.)

If A is a real n-by-n symmetric matrix, then $A = A^{T}$.

Let λ be a (real) eigenvalue of A and V be the corresponding real eigenvector. Now, we have:

$$AV = \lambda V \tag{1.1}$$

Then we multiply by V^T on both sides (V^T is a transpose of V), the following equation would be: Note: The outcome of $V^T * V$ would be 1-by-1 vector.

$$V^T A V = \lambda V^T V = \lambda ||V||^2 \tag{1.2}$$

Because A is positive definite so the left hand side is positive, and V is an non-zero eigenvector. Also, the length $||V||^2$ must be positive, we can derive the eigenvalues of a symmetric positive matrix λ is positive. It follows that every eigenvalue λ of A is real.

$$\lambda = \frac{V^T A V}{||V||^2} > 0 \tag{1.3}$$

Exercise 2 (5 points)

Show that the determinant of an orthogonal matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is ± 1 . Next, for the rotation matrix $\mathbf{R} \in \mathbb{R}^{2 \times 2}$, show that the determinant equals 1.

Exercise 3 (5 points)

Show that $\mathbf{A}^{\top}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\top}$ have the same eigenvalues.

We know that λ is a eigenvalue of A, and V is a corresponding real eigenvector. To be precise, A is a n-by-n matrix, V is a non-zero n-by-1 vector, and λ is a scalar (real or complex number).

$$AV = \lambda V \tag{3.1}$$

Let's use A^TA be a matrix as a whole to replace A, the equation would be:

$$A^T A V = \lambda V \tag{3.2}$$

Then, we multiply by A on both sides, the equation would be:

$$AA^{T}AV = \lambda AV \tag{3.3}$$

Because A is a n-by-n matrix, V is a non-zero n-by-1 vector so the result of A*V is a n-by-1 value. In equation 3.2, A^TAV is equal to λV ; In equation 3.3, AA^TAV is equal to λAV . Therefore, we can find out there is a same eigenvalue λ between A^TA and AA^T .

Exercise 4 (5 points)

Show that the negative entropy function $f(x) = x \log x$ is convex for all x > 0. (Hint: If we know that a function is twice differentiable, that is, the Hessian exists for all values in the domain of x, then the function is convex if and only if the Hessian is positive semi-definite.)

Exercise 5 (5 points)

Show that the least-squares objective function $f(\mathbf{x}) = ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2$ is convex for any invertible matrix \mathbf{A} .

Exercise 6 (Extra credit)(5points)

Show that the spectral norm of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is its largest singular value σ_1 . This is

$$\max_{\mathbf{x}} \frac{||\mathbf{A}\mathbf{x}||_2}{||\mathbf{x}||} = \sigma_1$$

References

- https://yutsumura.com/positive-definite-real-symmetric-matrix-and-its-eigenvalues/
- $\bullet \ \ https://lpsa.swarthmore.edu/MtrxVibe/EigMat/MatrixEigen.html$
- https://www.wikihow.com/Find-Eigenvalues-and-Eigenvectors