Advanced Machine Learning - HW1

Yunting Chiu

12 September, 2021

Exercise 1 (5 points)
Show that the eigenvalues of a symmetric positive matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are all positive. (Hint: Recall that the eigenvalues of a symmetric matrix are real.)

Exercise 2 (5 points)
Show that the determinant of an orthogonal matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is ± 1. Next, for the rotation matrix $\mathbf{R} \in \mathbb{R}^{2 \times 2}$, show that the determinant equals 1.

Exercise 3 (5 points)
Show that $\mathbf{A}^\top \mathbf{A}$ and $\mathbf{A}\mathbf{A}^\top$ have the same eigenvalues.

Exercise 4 (5 points)
Show that the negative entropy function $f(x) = x \log x$ is convex for all $x > 0$. (Hint: If we know that a function is twice differentiable, that is, the Hessian exists for all values in the domain of x, then the function is convex if and only if the Hessian is positive semi-definite.)

Exercise 5 (5 points)
Show that the least-squares objective function $f(\mathbf{x}) = ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2$ is convex for any invertible matrix $\mathbf{A}$.

Exercise 6 (Extra credit)(5points)
Show that the spectral norm of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is its largest singular value σ_1. This is

$$\max_{\mathbf{x}} \frac{||\mathbf{A}\mathbf{x}||_2}{||\mathbf{x}||} = \sigma_1$$