# Advanced Machine Learning - HW1

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**Exercise 1** (5 points)

Show that the eigenvalues of a symmetric positive matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are all positive. (Hint: Recall that the eigenvalues of a symmetric matrix are real.)

If $A$ is a real n-by-n symmetric matrix, then $A = A^T$.
Let $\lambda$ be a (real) eigenvalue of $A$ and $V$ be the corresponding real eigenvector. Now, we have:

$$AV = \lambda V \tag{1.1}$$

Then we multiply by $V^T$ on both sides ($V^T$ is a transpose of $V$), the following equation would be:
Note: The outcome of $V^T * V$ would be 1-by-1 vector.

$$V^T AV = \lambda V^T V = \lambda ||V||^2 \tag{1.2}$$

Because $A$ is positive definite so the left hand side is positive, and $V$ is a non-zero eigenvector. Also, the length $||V||^2$ must be positive, we can derive the eigenvalues of a symmetric positive matrix $\lambda$ is positive. It follows that every eigenvalue $\lambda$ of $A$ is real.

$$\lambda = \frac{V^T AV}{||V||^2} > 0 \tag{1.3}$$

**Exercise 2** (5 points)

Show that the determinant of an orthogonal matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is $\pm 1$. Next, for the rotation matrix $\mathbf{R} \in \mathbb{R}^{2 \times 2}$, show that the determinant equals 1.

**Exercise 3** (5 points)

Show that $\mathbf{A}^\top \mathbf{A}$ and $\mathbf{A}\mathbf{A}^\top$ have the same eigenvalues.

We know that $\lambda$ is a eigenvalue of $A$, and $V$ is a corresponding real eigenvector. To be precise, $A$ is a n-by-n matrix, $V$ is a non-zero n-by-1 vector, and $\lambda$ is a scalar (real or complex number).

$$AV = \lambda V \tag{3.1}$$

Let's use $A^T A$ be a matrix as a whole to replace $A$, the equation would be:

$$A^T AV = \lambda V \tag{3.2}$$

Then, we multiply by $A$ on both sides, the equation would be:

$$AA^TAV = \lambda AV \tag{3.3}$$

Because $A$ is a n-by-n matrix, $V$ is a non-zero n-by-1 vector so the result of $A * V$ is a n-by-1 value. In equation 3.2, $A^TAV$ is equal to $\lambda V$; In equation 3.3, $AA^TAV$ is equal to $\lambda AV$. Therefore, we can find out there is a same eigenvalue $\lambda$ between $A^TA$ and $AA^T$.

**Exercise 4** (5 points)
Show that the negative entropy function $f(x) = x \log x$ is convex for all $x > 0$. (Hint: If we know that a function is twice differentiable, that is, the Hessian exists for all values in the domain of $x$, then the function is convex if and only if the Hessian is positive semi-definite.)

**Exercise 5** (5 points)
Show that the least-squares objective function $f(\mathbf{x}) = ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2$ is convex for any invertible matrix $\mathbf{A}$.

**Exercise 6** (Extra credit)(5points)
Show that the spectral norm of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is its largest singular value $\sigma_1$. This is

$$\max_{\mathbf{x}} \frac{||\mathbf{A}\mathbf{x}||_2}{||\mathbf{x}||} = \sigma_1$$

## References


- https://yutsumura.com/positive-definite-real-symmetric-matrix-and-its-eigenvalues/
- https://lpsa.swarthmore.edu/MtrxVibe/EigMat/MatrixEigen.html
- https://www.wikihow.com/Find-Eigenvalues-and-Eigenvectors