Macroeconomics I

University of Tokyo

Lecture 13

The Neo-Classical Growth Model II:

Distortionary Taxes

LS Chapter 11.

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Environment

- Time is discrete: t = 0,1,2,...
- Agents: - *Consumers*: Representative agent.
  - *Firms*
  - *Government*
- Endowment: Households have *1* ut of labor (per period) and initial capital k_0.
- Preferences: The representative household's objective function is:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad \beta \in (0, 1). \tag{13.1}$$

where we assume that u is strictly increasing, strictly concave, differentiable, and satisfies the Inada condition $\left(\lim_{c \to 0} u'(c) = \infty\right)$.

- Technology: Production function is $F(k_t,n_t) = F(k_t, 1) \equiv f(k_t)$. Assume that $f(\ )$ satisfies the usual: $f(0) = 0, f' > 0, f'' < 0, \lim_{k\to\infty} f'(k) = 0, \lim_{k\downarrow 0} f'(k) = \infty$.
- Resource constraint (RC):

$$c_t + k_{t+1} - (1 - \delta)k_t + g_t \leq F(k_t, n_t), \quad (t = 0, 1, 2, \ldots), \; k_0 > 0 \text{ given.} \tag{13.2}$$

Competitive Equilibrium - Complete Markets

- Assume that markets are complete.
- Time 0 trading.
- Capital is owned by the household and rented out to firms.
- 3 physical commodities:
  - The good.
  - Capital services.
  - Labor.
- Let $\{q_t, r_t, w_t\}$ be the pre-tax Arrow-Debreu prices of these commodities.

Government and Taxes

- The government taxes the consumers to finance its expenditures.
  - Expenditures: Government consumption - g_t.
  - Revenues: - From distortionary taxes:
    - τ_{ct}: consumption tax.
    - τ_{nt}: labor income tax.
    - τ_{kt}: capital earnings tax.
    - τ_{it}: investment tax credit.

    - From non-distortionary taxes:
    - τ_{ht}: lump-sum tax.
- **Def**: A government policy consists of an expenditure plan $\{g_t\}_{t=0}^{\infty}$ and a tax plan $\{\tau_{ct}, \tau_{nt}, \tau_{kt}, \tau_{it}, \tau_{ht}\}_{t=0}^{\infty}$.
- **Def**: A government policy is budget feasible if it satisfies the government budget constraint (GBC):

$$\sum_{t=0}^{\infty} q_t g_t \leq \sum_{t=0}^{\infty} \left\{ \tau_{ct} q_t c_t - \tau_{it} q_t [k_{t+1} - (1-\delta) k_t] + r_t \tau_{kt} k_t + w_t \tau_{nt} n_t + q_t \tau_{ht} \right\}.$$

Households

- The household's problem is:

$$\max_{\{c_t, k^s_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{13.3}$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \left\{ q_t(1+\tau_{ct})c_t + (1-\tau_{it})q_t[k^s_{t+1} - (1-\delta)k^s_t] \right\}$$
$$\leq \sum_{t=0}^{\infty} \left\{ r_t(1-\tau_{kt})k^s_t + w_t(1-\tau_{nt})n^s_t - q_t\tau_{ht} \right\}. \tag{13.4}$$

$$c_t \geq 0,\ k^s_{t+1} \geq 0, \text{and } 0 \leq n^s_t \leq 1 \ (t = 0, 1, 2, \ldots).$$

$$k^s_0 = k_0 > 0 \text{ given.}$$

- As before, since agents get no utility from leisure: $n^s_t = 1$.

Household's Problem Optimal Conditions

- Let μ be the Lagrange multiplier associated with the budget const. (13.4).
- We can write the Lagrangian as:

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \mu \sum_{t=0}^{\infty} \Big\{ [r_t(1-\tau_{kt})k_t^s + w_t(1-\tau_{nt})n_t^s - q_t \tau_{ht}]$$

$$-q_t \left[(1+\tau_{ct})c_t + (1-\tau_{it})(k_{t+1}^s - (1-\delta)k_t^s) \right] \Big\}. \quad (13.5)$$

- FOCs:

$$c_t : \quad \beta^t u'(c_t) = \mu q_t (1+\tau_{ct}) \qquad (c_t > 0 \text{ due to Inada cond.}) \quad (13.6)$$

$$k_{t+1}^s : \quad r_{t+1}(1-\tau_{kt+1}) + q_{t+1}(1-\tau_{it+1})(1-\delta) \le q_t(1-\tau_{it}). \quad (13.7)$$

(Holds with equality due to No-Arbitrage)

- From (12.21) in t and $t+1$:

$$u'(c_t) = \beta \frac{q_t}{q_{t+1}} \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} u'(c_{t+1}) \quad (13.8)$$

Firms

- Problem of the firm (no change from before):

$$\max_{\{n_t^d, k_t^d\}} \sum_{t=0}^{\infty} [q_t F(k_t^d, n_t^d) - r_t k_t^d - w_t n_t^d]. \tag{13.9}$$

  - Static problem:

$$\max_{n_t^d, k_t^d} [q_t F(k_t^d, n_t^d) - r_t k_t^d - w_t n_t^d]. \tag{13.10}$$

- FOCs (MP conditions):

$$n_t^d : \quad w_t = q_t F_n(k_t^d, n_t^d), \tag{13.11}$$

$$k_t^d : \quad r_t = q_t F_k(k_t^d, n_t^d). \tag{13.12}$$

Competitive Equilibrium

- **Def**: A competitive equilibrium with distortionary taxes is a sequence of Arrow-Debreu prices $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ and associated quantities $\{c_t, k_{t+1}^s, n_t^s\}_{t=0}^{\infty}$ and $\{k_t^d, n_t^d\}_{t=0}^{\infty}$ with $k_0^s = k_0$ and a gov. policy $\{g_t, \tau_{ct}, \tau_{nt}, \tau_{kt}, \tau_{it}, \tau_{ht}\}_{t=0}^{\infty}$ s.t.:

(i) Optimization: Given $k_0^s = k_0$, $\{q_t, r_t, w_t\}_{t=0}^{\infty}$ and $\{\tau_{ct}, \tau_{nt}, \tau_{kt}, \tau_{it}, \tau_{ht}\}_{t=0}^{\infty}$

- $\{c_t, k_{t+1}^s, n_t^s\}_{t=0}^{\infty}$ solve the HH's problem.
- $\{k_t^d, n_t^d\}$ solve the firm's problem.

(ii) Market clearing:

- Goods: $c_t + k_{t+1}^s - (1-\delta)k_t^s + g_t = F(k_t^d, n_t^d)$,
- Capital: $k_t^d = k_t^s$,
- Labor: $n_t^d = n_t^s \; (= 1)$.

(iii) GBC: $\sum_{t=0}^{\infty} q_t g_t \leq \sum_{t=0}^{\infty} \left\{ \tau_{ct} q_t c_t - \tau_{it} q_t [k_{t+1} - (1-\delta)k_t] + r_t \tau_{kt} k_t + w_t \tau_{nt} n_t + q_t \tau_{ht} \right\}$

No Arbitrage

- Rearrange the household's budget constraint as:

$$\sum_{t=0}^{\infty} q_t \left[(1+\tau_{ct})c_t\right] \leq \sum_{t=0}^{\infty} w_t(1-\tau_{nt})n_t - \sum_{t=0}^{\infty} q_t \tau_{ht} \tag{13.13}$$

$$+ \sum_{t=1}^{\infty} \left[r_t(1-\tau_{kt}) + q_t(1-\tau_{it})(1-\delta) - q_{t-1}(1-\tau_{it-1})\right] k_t$$

$$+ \left[r_0\left(1-\tau_{k0}\right) + (1-\tau_{i0})q_0(1-\delta)\right] k_0 - \lim_{T\to\infty}(1-\tau_{iT})q_T k_{T+1}.$$

- No-arbitrage condition:

$$q_t(1-\tau_{it}) = q_{t+1}(1-\tau_{it+1})(1-\delta) + r_{t+1}(1-\tau_{kt+1}) \tag{13.14}$$

- No-Ponzi scheme condition:

$$-\lim_{T\to\infty}(1-\tau_{iT})q_T k_{T+1} = 0. \tag{13.15}$$

Equilibrium Conditions

- Factor market clearing conditions imply: $k_t = k_t^s = k_t^d, \; n_t = n_t^s = n_t^d$
- In equilibrium, $k_{t+1} > 0$ and $n_t = 1$ for all t.
- We can write $f(k_t) \equiv F(k_t, 1)$.
- The equilibrium conditions reduce to:

$$\text{(MP)} \qquad r_t = q_t f'(k_t), \quad w_t = q_t(f(k_t) - k_t f'(k_t)), \qquad (13.16), (13.17)$$

$$\text{(HH FOCs)} \qquad u'(c_t) = \beta \frac{q_t}{q_{t+1}} \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} u'(c_{t+1}),$$

$$r_{t+1}(1-\tau_{kt+1}) + q_{t+1}(1-\tau_{it+1})(1-\delta) = q_t(1-\tau_{it}),$$

$$\text{(RC)} \qquad c_t + k_{t+1} - (1-\delta)k_t + g_t = f(k_t). \qquad (13.18)$$

Note: The HH's BC is implied by the RC, MP an Euler's theorem.

- From the MP and HH's FOCs, we can derive the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1}) \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \left[\frac{(1-\tau_{it+1})}{(1-\tau_{it})}(1-\delta) + \frac{(1-\tau_{kt+1})}{(1-\tau_{it})} f'(k_{t+1}) \right] \qquad (13.19)$$

Equilibrium Conditions (Cont.)

- Using the resource constraint into the Euler equation we obtain:

$$
\begin{aligned}
0 = {} & \frac{u'(f(k_t) + (1-\delta)k_t - g_t - k_{t+1})}{(1+\tau_{ct})}(1-\tau_{it}) \\
& -\beta \frac{u'(f(k_{t+1}) + (1-\delta)k_{t+1} - g_{t+1} - k_{t+2})}{(1+\tau_{ct+1})} \times \qquad (13.20) \\
& [(1-\tau_{it+1})(1-\delta) + (1-\tau_{kt+1})f'(k_{t+1})] = 0.
\end{aligned}
$$

- This is a second order difference equation on capital.
- To solve it we need an initial condition, k_0, and a terminal condition, k_∞.

Steady State

- Assume that government policies converge, so that

$$\lim_{t\to\infty} z_t = z^* \quad , \text{where } z_t = [g_t \ \tau_{it} \ \tau_{kt} \ \tau_{ct}]'$$

- In the steady state $k_t = k^*,\ c_t = c^*\ \forall t$ and k^* and c^* can be found using:

$$1 = \beta[(1-\delta) + \frac{(1-\tau_k^*)}{(1-\tau_i^*)} f'(k^*)]. \tag{13.21}$$

Assuming that $\beta \equiv \dfrac{1}{1+\rho}$, we obtain:

$$(\rho+\delta)\frac{(1-\tau_i^*)}{(1-\tau_k^*)} = f'(k^*), \tag{13.22}$$

$$c^* = f(k^*) - \delta k^* - g^*. \tag{13.23}$$

Transition Experiments

- Study a foreseen one time change in gov. cons. or tax rates in period t =10.
- Functional forms:
  - Utility function: $u(c) = \frac{1}{(1-\gamma)} c^{1-\gamma}$
  - Production Function: $f(k) = k^{\alpha}$
- Parameters: $\alpha = .33, \delta = .2, \gamma = 2, \beta = .95$
- Initial values for exogenous variables:

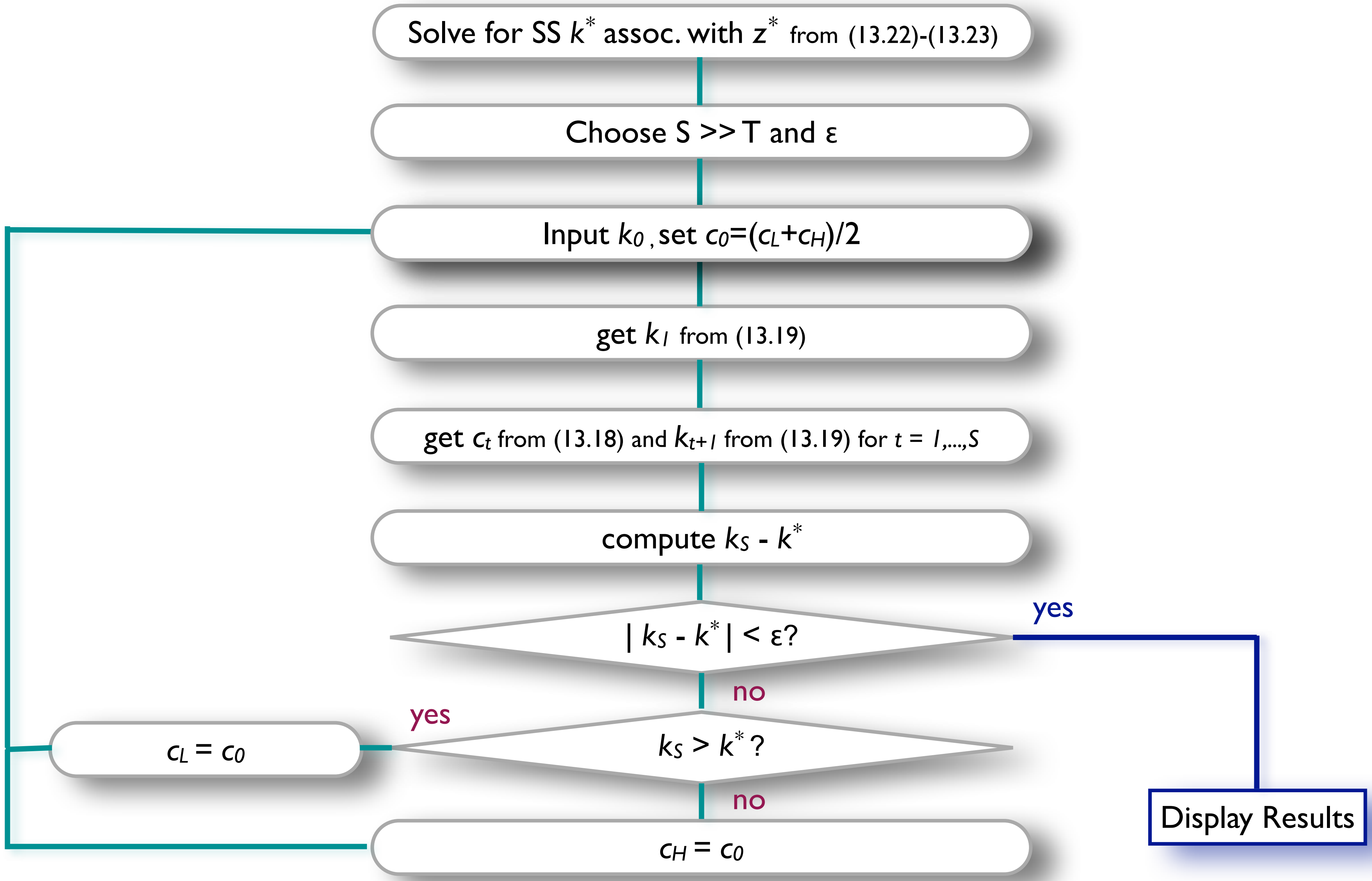
$$g_0 = 0.2, \tau_{c0} = 0, \tau_{n0} = 0, \tau_{k0} = 0, \tau_{i0} = 0.$$

- Experiment 1: Once-and-for-all increase in g at t = 10 from 0.2 to 0.4.
  - g is financed through lump sum taxes.
- Experiment 2: Once-and-for-all increase in τ_k at t = 10 from 0 to 0.05.
  - Tax revenues are rebated lump-sum to the consumers.

Computing the Equilibrium

- The equilibrium in this economy is computed using a forward shooting algorithm.
- The shooting algorithm solves a two-point boundary problem by finding the initial consumption, c_0, which makes the Euler equation (13.13) and the feasibility constraint (13.14) satisfy that $k_S \approx k^*$
  - S is a period far in the future - approximates for the SS.
  - k^* is the terminal SS for the policy analyzed.
  - Let T be the period of time after which z is constant.

Shooting Algorithm



Matlab Implementation - Preliminaries

- 1.- Set parameter values

```
% Iteration Parameters
S=100;             % number of periods in the shooting
niter=200;         % number of possible trials of the intial value
epsilon=10^(-5);% convergence criterion

% Model Parameters
alpha   = 0.33; % Share of capital in output rents
delta   = 0.2;  % Capital depreciation rate
gamma   = 2;    % Risk aversion parameter
beta    = 0.95; % Discount factor
```

- 2.- Generate path of exogenous variables

```
% Initial values for g and taxes
g_ss0     = 0.2;  % Gov consumption
tauc_ss0  = 0;    % Consumption tax
taun_ss0  = 0;    % Labor income tax
tauk_ss0  = 0;    % Capital income tax
taui_ss0  = 0;    % Investment tax credit
% Path of Exogenous variables
g       = ones(S,1)*g_ss0;
tauk    = ones(S,1)*tauk_ss0;
if experiment ==1;
    g(11:S) = 0.4;
elseif experiment ==2;
    tauk(11:S) = 0.05;
end;
```

Matlab - Steady States & Initial Conditions

- 3.- Calculate initial and final steady states

```
% Initial SS
k_ss0=((1/alpha)*(1/beta-(1-delta))*((1-taui_ss0)/(1-tauk_ss0)))^(1/(alpha-1));
c_ss0=k_ss0^alpha-delta*k_ss0-g_ss0;

% Terminal SS
k_ss=((1/alpha)*(1/beta-(1-delta))*((1-taui_ss)/(1-tauk_ss)))^(1/(alpha-1));
c_ss=k_ss^alpha-delta*k_ss-g_ss;
```

- 4.- Initial conditions

```
% Initial condition for capital
k_0     = k_ss0;

% Initial condition for consumption
c_0   = c_ss0;            % Very initial guess for c_0
c_0_L = 0.001*c_ss;
c_0_H = (1-delta)*k(1)+(k(1)^alpha)-g_ss0; % Maximum consumption with no investment
```

Matlab - Shooting Algorithm

- 5.- Solve for k_{t+1} and c_t until convergence

```
% Start iteration
for j=1:niter;
  c_0=(c_0_L+c_0_H)/2;
% Generate paths for c and k given initial guess for c_0
  c(1)=c_0;
  for t=1:S-1;
    k(t+1) = k(t)^alpha+(1-delta)*k(t)-g(t)-c(t);  % Equation (13.18)
    if k(t+1)<0;        flag=1; break; end;
    if k(t+1)>2*k_ss;   flag=2; break; end
    c(t+1) = c(t)*((beta*((1+tauc(t))/(1+tauc(t+1)))*(((1-taui(t+1))/(1-taui(t)))...
          *(1-delta)+((1-tauk(t+1))/(1-taui(t)))*alpha*k(t+1)^(alpha-1)))^(1/gamma));
    end;                                            % Equation (13.19)
```

- 6.- Check for convergence

```
% Update the initial value if not convergence
  if t==S-1;
    if abs(k(S)-k_ss)/k_ss < epsilon && abs(c(S)-c_ss)/c_ss < epsilon;
    flag=10;disp('Converged!!');break;
```

- 7.- Update initial guess for c

```
    elseif (k(S)-k_ss)/k_ss<=epsilon;flag=1;
    elseif (k(S)-k_ss)/k_ss>=epsilon;flag=2;
    end;
  end;
  if flag==1; c_0_H=c_0; end;
  if flag==2; c_0_L=c_0; end;
end; % Ends iteration loop
```

Matlab - End Shooting and Compute other Var.

- **8.-** End shooting if max iterations has been reached

```
% End Shooting algorithm if reached max number of iterations
if j == niter;
    disp('Failure in Convergence!')
end;
```

- **9.-** Compute the values for the remaining variables

```
% Compute other variables
    q = (beta^t).*(c.^(-gamma))./(1+tauc);
    r_q = alpha.*k.^(alpha-1);
    w_q = k.^alpha - k.*(alpha.*k.^(alpha-1));
    s_q = (1-tauk).*(alpha.*k.^(alpha-1))+(1-delta);
    for t=1:S-1
        R(t) = ((1+tauc(t))/(1+tauc(t+1)))*(((1-taui(t+1))/(1-taui(t)))*(1-delta)...
               +((1-tauk(t+1))/(1-taui(t)))*alpha*(k(t+1)^(alpha-1)));
    end
```

Matlab - Show Results

- **10.-** Set number of displayed periods and initial and terminal conditions

```
% Periods for the time-path in the graph
TT = 50;
time = (0:1:TT);

% Initial and terminal conditions
k_s0     = ones(TT+1,1)*k_ss0;
c_s0     = ones(TT+1,1)*c_ss0;

k_s      = ones(TT+1,1)*k_ss;
c_s      = ones(TT+1,1)*c_ss;
```

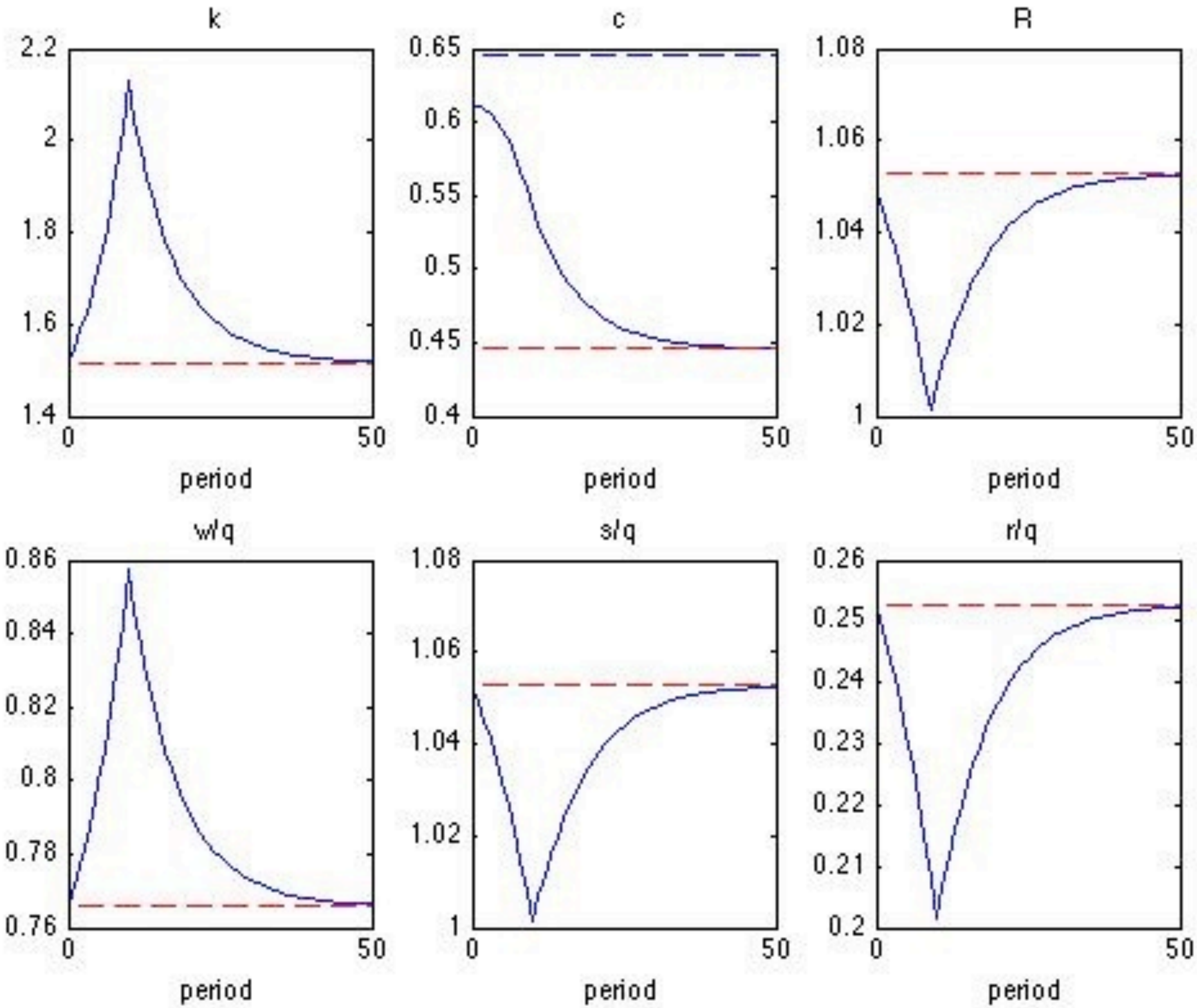
- **11.-** Plot the transition path to new steady state

```
% Plot the path of variables

figure;
subplot(2,3,1)
plot(time',k(1:TT+1),'b-',time',k_s0(1:TT+1),'b--',time',k_s(1:TT+1),'r--')
title('k');
xlabel('period')

subplot(2,3,2)
plot(time',c(1:TT+1),'b-',time',c_s0(1:TT+1),'b--',time',c_s(1:TT+1),'r--')
title('c');
xlabel('period')
```

Results for Experiment 1: Increase in g



Results for Experiment 2: Increase in τ_k