Ex 1 Cflfp Sol part 1

use ext

$$(f \circ id)\ x$$
$$= f\ (id\ x)$$
$$= f\ x$$

$$(id \circ f)\ x$$
$$= id\ (f\ x)$$
$$= f\ x$$

$$((f \circ g) \circ h)\ x \qquad (f \circ (g \circ h))\ x$$
$$= (f \circ g)\ (h\ x) \qquad = f\ ((g \circ h)\ x)$$
$$= f\ (g\ (h\ x)) \qquad = f\ (g\ (h\ x))$$

$= f(g(h\,x)) = f(g(h\,x))$

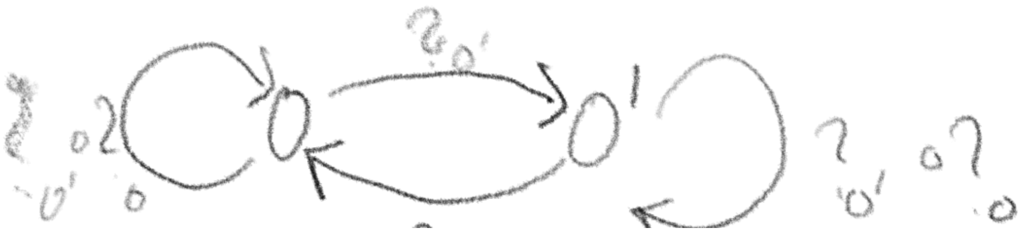
Ex 2

$$\underline{f \circ g = id} \qquad f \circ g' = id$$
$$g \circ f = id \qquad \underline{g' \circ f = id}$$

$$
\begin{aligned}
g &= id \circ g \\
&= (g' \circ f) \circ g \\
&= g' \circ (f \circ g) \\
&= g' \circ id = g'
\end{aligned}
$$

cf uniqueness of inverses in a group

Ex 3 Easy dualisation



$= id$ $\;\; ?_0 \;\;$ $= id$

by uniqueness $\;\;$ "

Q 4 $\quad$ initial 0

terminal no

Q 5 $\quad$ Neither

Only one object

there can only be 1 morphism

$\mathbb{C}(1,1)$ $\quad$ $\mathbb{C}(0,0)$

Only trivial monoid

has initial/terminal object

Ex 6 $\quad$ <u>$\mathbf{Set}^{\subseteq}$</u>

$$|\mathrm{Set}^{\cong}| = \mathrm{Set}$$

$$\mathrm{Set}^{\cong}(A, B) = A \cong B$$

$$= \{ f : A \to B \times g : B \to A \mid f \circ g = id, \; g \circ f = id \}$$

$$id_{\cong} = (id, id)$$ • iso $id \circ id = id$

$$(f, g) \circ (f', g') = (f \circ f', g' \circ g)$$

• iso:

$$f \circ \underbrace{f' \circ g'}_{= id} \circ g = f \circ g = id$$

$$g' \circ \underbrace{g \circ f} \circ f' = g' \circ f' = id$$

• laws

- $(f, g) \circ (id, id)$

$$= (f \circ id, \; id \circ g)$$

$$= (f, g)$$

$$= (id, id) \circ (f, g)$$

$$= (id \circ f, g \circ id)$$

$$= (f, g)$$

$$= ((f, g) \circ (f', g')) \circ (f'', g'')$$

$$= (f \circ f', g' \circ g) \circ (f'', g'')$$

$$= ((f \circ f') \circ f'', g'' \circ (g' \circ g))$$

$$\Leftarrow$$

$$(f, g) \circ ((f', g') \circ (f'' \circ g''))$$

$$= (f, g) \circ (f' \circ f'', g'' \circ g')$$

$$= (f \circ (f' \circ f''), (g'' \circ g') \circ g)$$

$\overline{ASS}$

$\overline{ass}$

Q 7

Pre

initial $(\emptyset, R_\emptyset)$ $xRy = \text{False}$
note all laws hold trivially

$?_0$ is a monoid morph

terminal $(1, R_1)$ $x R_1 y = \text{True}$

$!_1$ is a monoid morph

$xRy \to !x R_1 !y$ ✓

Mon

Null object

$0 = 1 = (\{e\}, e, \circ)$

$e \circ e = e$

$(A, i, *)$

$$!_\lambda a = e$$

$$!_\cdot i = e \checkmark$$

$$!_\cdot (a * b) = !_\cdot a \circ !_\cdot b = e \circ e = e$$

$$?_\cdot e = i \checkmark$$

$$?_\cdot (e \circ e) = ?_\cdot e = i = i * i$$
$$= ?_\cdot e * ?_\cdot e$$

<u>Ex 8</u>

inj $\rightarrow$ mono

$$\forall_{xy}.\ i x = i y \Rightarrow x = y$$

$$i \circ f = i \circ g \rightarrow f = g$$

$$(i \circ f)\, x = (i \circ g)\, x$$
$$\| \qquad\qquad \| \qquad \rightarrow f x = g x$$
$$i(fx) \qquad i(gx)$$

$\ldots(f\,x) \quad \ldots(g\,x)$

mono $\to$ inj

$(1 \to A) \cong A$

$(\circ \mapsto a) \leftarrow\!\!| \; a$

$i\,x = i \circ y$

$\Rightarrow i \circ (\lambda\_ \to x) = i \circ (\lambda\_ \to y)$

$\Rightarrow \lambda\_ \to x = \lambda\_ \to y$

$\Rightarrow x = y$

- surj $\Rightarrow$ epi $\quad e : A \twoheadrightarrow B$

$\forall y : B\; \exists x : A.\; e\,x = y$

$f \circ e = g \circ e \to f = g$

$$f(\underset{\overset{\|}{y}}{e\,x}) = g(\underset{\overset{\|}{y}}{e\,x})$$

given $y : B$ choose $x : A$ s.t. $e\,x = y$

hence $\forall y.\, f y = g y$

$\underset{ext}{\Rightarrow} \quad f = g$

- epi $\Rightarrow$ surj ?

$f \circ e = g \circ e \to f = g$

? $\forall y : B\ \exists x : A .\ e x = y$

$$A \overset{e}{\twoheadrightarrow} B \underset{g}{\overset{f}{\rightrightarrows}} Prop$$

$f\ b = True$

$g\ b = \exists a : A .\ e a = b$

$g(ea) = True$

$f(ea) = True$

$\Rightarrow \quad f = g \Rightarrow \forall b : B\ \exists a : A .\ e a = b$

This uses $Prop_0 : Set_0$

$\Rightarrow$ impredicativity!

Ex 9: $f \circ r = id$

$$g \circ f = h \circ f$$

$$\Rightarrow \begin{array}{ccc} (g \circ f) \circ r & = & (h \circ f) \circ r \\ \| & & \| \\ g \circ (f \circ r) & & h \circ (f \circ r) \\ \| & & \| \\ id & & id \end{array}$$

$$\Rightarrow \quad g = h$$

Ex 10 Use epi $\Rightarrow$ surj

$$f : A \to B$$

$$\forall y : B \; \exists x : A. \; f x = y$$

$$f x = f y \to x = y$$

$$\Rightarrow \forall y : B \; \exists! x : A \; f x = y$$

$\Rightarrow \exists g : B \to A.\ \forall y : B.\ \underline{f(g\,y) = y}$

$$g(f\,x) = x$$

$$\Uparrow \qquad f \text{ is injective}$$

$$\underbrace{f(g}_{id}(f\,x)) = f\,x$$

Ex 11 $\quad i : \mathbb{N} \to \mathbb{Z}$

- mono in Set $\Rightarrow$ mono in $\underline{\text{Mon}}$
- epi $\quad \mathbb{N} \xrightarrow{i} \mathbb{Z} \underset{g}{\overset{f}{\rightrightarrows}} (M, e, *)$

$$f(i\,x) = g(i\,x)$$

$$f(-n) = g(-n)$$

$$f(-n) = f(-n) * \overset{e}{\overset{\shortparallel}{g(0)}}$$

$$g(0) = g(n + (-n)) = \ldots$$

$$gu * g(-u)$$
$$\|$$
$$\underbrace{\underbrace{\qquad fu}_{e}}_{g(-u)}$$

- An iso in Mon is also an iso in Set and $\mathbb{N}$ and $\mathbb{Z}$ are clearly not iso in Set

they are!

Fix it

$\boxed{\text{EX}}$

Ex 12

$$l \circ f = id$$
$$f \circ r = id$$

$$l = l \circ id$$
$$= \underline{l \circ f} \circ r$$

$= r$

$*$

$$\begin{tikzcd} \mathbb{N} \arrow[loop left] \arrow[r, bend left, "i"] & \mathbb{Z} \arrow[l, bend left, "g"] \arrow[loop right, "id"] \end{tikzcd}$$

if this is an iso

$\Rightarrow g(-n) = n$

because $i \circ g = id$

and inverses are unique

but $i \circ g(-1) = +1$ $\lightning$