Conjuntos y números 10/12

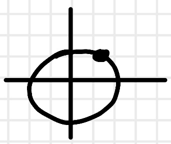
10 a) $1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = \sqrt{2}\, e^{\pi/4\, i}$, sea $z = r e^{\alpha i}$.

$$z^2 = r^2 e^{2\alpha i} = \sqrt{2}\, e^{\pi/4\, i} \iff r^2 = \sqrt{2} \text{ y } 2\alpha = \pi/4 + 2k\pi.$$

Cogemos la raíz positiva: $r = 2^{1/4}$, $\alpha = \frac{\pi}{8} + k\pi$ para $k = 0, 1$

Es decir $\sqrt{1+i} = \begin{cases} 2^{1/4} e^{\pi/8\, i} \\ 2^{1/4} e^{9/8\, \pi i} \end{cases}$.

c) $2+i = \sqrt{5}\left(\underbrace{2\frac{\sqrt{5}}{5}}_{\cos\theta} + \underbrace{\frac{\sqrt{5}}{5}}_{\text{sen}\,\theta} i\right)$, $\cos \theta/2 = \pm\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{\frac{\sqrt{5}+2}{2\sqrt{5}}}$



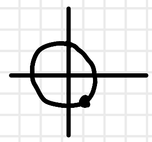
$$\text{sen}\, \theta/2 = \pm\sqrt{\frac{1-\cos\theta}{2}} = \pm\sqrt{\frac{\sqrt{5}-2}{2\sqrt{5}}}$$

Cogemos los dos con el mismo signo:

$$\sqrt{2+i} = \pm 5^{1/4}\left(\sqrt{\frac{\sqrt{5}+2}{2\sqrt{5}}} + \sqrt{\frac{\sqrt{5}-2}{2\sqrt{5}}}\, i\right)$$

11 a) $z^2 + 3iz - 3 + i$. La fórmula para pol. cuadráticos es válida en cualq. cuerpo con $2 \neq 0$ (por ej., no vale en $\mathbb{Z}/2$)

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



$$z = \frac{-3i \pm \sqrt{-9 - 4\cdot(-3+i)}}{2} = \frac{-3i \pm \sqrt{3-4i}}{2}$$

¿$\sqrt{3-4i}$?

$3 - 4i = 5\left(\underbrace{\frac{3}{5}}_{\cos\theta} - \underbrace{\frac{4}{5}}_{\text{sen}\,\theta} i\right)$

$$\cos\theta/2 = \pm\sqrt{\frac{1+\cos\theta}{2}} = \sqrt{8/10} = \frac{2}{\sqrt{5}}$$

$$\text{sen}\,\theta/2 = \pm\sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{2}{10}} = \frac{1}{\sqrt{5}}$$

$\text{sen}\,\theta/2$ y $\cos\theta/2$ van a tener signos opuestos:

$$\sqrt{3-4i} = \sqrt{5} \pm \left(\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}} i\right) = \pm(2-i)$$

$$z = \frac{-3i \pm (2-i)}{2} = \begin{cases} 1 - 2i \\ -1 - i \end{cases}$$

(12) a) $\sqrt[3]{-8}$, $\quad -8 = 8e^{\pi i} = \underset{z^3}{\underset{\shortparallel}{r^3 e^{3\alpha i}}} \iff r^3 = 8 \quad 3\alpha = \pi + 2k\pi$

$z = re^{\alpha i}$

$r = 2, \quad \alpha = \pi/3 + \frac{2}{3}k\pi \qquad k = 0,1,2 \qquad \sqrt[3]{-8} = \begin{cases} 2e^{\pi/3\, i} \\ 2e^{\pi i} = -2 \\ 2e^{5/3\,\pi i} \end{cases}$

d) $(1+i)^n + (1-i)^n \quad n \in \mathbb{N}$.

$$(\sqrt{2}e^{\pi/4 i})^n + (\sqrt{2}e^{-\pi/4 i})^n = (\sqrt{2})^n \left(e^{n\pi/4 i} + e^{-n\pi/4 i}\right)$$

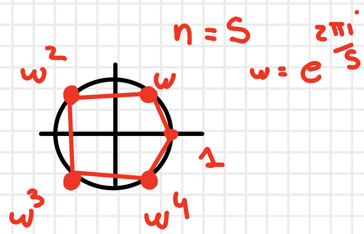
$$= (\sqrt{2})^n\, 2 \cdot \cos n\pi/4 .$$

Distinguir los casos mod 8. $\left(\cos n\pi/4 = \cos\left((n+8)\pi/4\right)\right)$ ■

(13) Raíces n-ésimas de 1: $\omega = e^{\frac{2\pi i}{n}}$, todas las raíces son:

$1, \omega, \omega^2, \omega^3, \dots, \omega^{n-1}$

$$\sum_{k=0}^{n-1} \omega^k = \frac{\omega^n - 1}{\omega - 1} = \frac{1-1}{\omega - 1} = 0 .$$



(14) $z = 2e^{2\pi i/5} + 1 + 2e^{-2\pi i/5}, \quad z^2 = 5.$

$$(2e^{2\pi i/5} + 1 + 2e^{-2\pi i/5}) \cdot (2e^{2\pi i/5} + 1 + 2e^{-2\pi i/5})$$

$$= 4e^{4\pi i/5} + 2e^{2\pi i/5} + 4 + 2e^{2\pi i/5} + 1 + 2e^{-2\pi i/5} + 4 + 2e^{-2\pi i/5} + 4e^{-4\pi i/5}$$

$$= 4e^{4\pi i/5} + 4e^{2\pi i/5} + 4e^{-2\pi i/5} + 4e^{-4\pi i/5} + 4 + 5$$

$$= 4\big(\underbrace{1}_{w^0} + \underbrace{e^{2\pi i/5}}_{w} + \underbrace{e^{4\pi i/5}}_{w^2} + \underbrace{e^{-4\pi i/5}}_{w^3} + \underbrace{e^{-2\pi i/5}}_{w^4}\big) + 5 = 5$$

$$2\left(e^{2\pi i/5} + e^{-2\pi i/5}\right) = z - 1 \implies \cos\frac{2\pi}{5} = \frac{z-1}{4} = x$$

$$x^2 + \frac{x}{2} - \frac{1}{4} = \frac{1}{16}(6 - 2z) + \frac{z}{8} - \frac{1}{8} - \frac{1}{4} = \frac{3}{8} - \frac{1}{8} - \frac{2}{8} = 0$$

$\downarrow \cdot 4$

$$2x^2 + x - 1/2 \rightsquigarrow x = \frac{-1 \pm \sqrt{1^2+4}}{4} = \frac{-1 \pm \sqrt{5}}{4}$$

Para $\cos\frac{2\pi}{5} > 0$, cogemos el valor $\frac{-1+\sqrt{5}}{4}$.

Hoja 7

(2) $P(x) = 3x^5 + 2x^3 + x + 1$, $Q(x) = 3x^2 + 1$

Sobre $\mathbb{Q}$:

$$\begin{array}{rrrr|l}
3x^5 & +2x^3 & +x & +1 & 3x^2+1 \\
\cline{5-5}
-3x^5 & +x^3 & & & x^3 + \frac{1}{3}x \\
\hline
 & x^3 & +x & +1 & \\
 & -x^3 & x/3 & & \\
\hline
 & & \frac{2}{3}x & +1 &
\end{array}$$

En $\mathbb{Z}/5$

$3^{-1} = 2$

$x^3 + 2x$

$x^3 + 2x$

$-x + 1$

$\frac{2}{3} = 2 \cdot 3^{-1}$

$$3x^5 + 2x^3 + x + 1 = (3x^2+1)\left(x^3 + \frac{1}{3}x\right) + \left(\frac{2}{3}x + 1\right)$$

$$3x^5 + 2x^3 + x + 1 = (3x^2+1)(x^3+2x) - x + 1$$