Conjuntos y números 30/11

1. a) $\dfrac{1-i}{1+i} = \dfrac{(1-i)\cdot(1-i)}{(1+i)(1-i)} = \dfrac{1-1-2i}{2} = -i$, $\quad Re(-i) = 0$, $\quad Im(-i) = -1$

b) $\dfrac{(3-i)(2+i)}{(3+i)} = \dfrac{(3-i)^2(2+i)}{10} = \dfrac{(9-1-6i)(2+i)}{10} = \dfrac{16+8i-12i+6}{10}$

$$= \frac{11}{5} - \frac{2}{5}i$$

d) $\displaystyle\sum_{k=1}^{101} i^k$. Recordamos $\begin{cases} i^1 = i \\ i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \end{cases}$ $\quad i^k = \begin{cases} i & \text{si } k \equiv 1 \mod 4 \\ -1 & \text{si } k \equiv 2 \mod 4 \\ -i & \text{si } k \equiv 3 \mod 4 \\ 1 & \text{si } k \equiv 0 \mod 4 \end{cases}$

$$\left(\sum_{k=1}^{100} i^k\right) + i^{101} = 25\,(i + (-1) + (-i) + 1) + i^{101} = i^{101} = i \quad \blacksquare$$

$101 = 25\cdot 4 + 1$

2 a) $|(2+i)(1-i)^4| = |(2+i)|\cdot|(1-i)|^4 = \sqrt{5}\cdot(\sqrt{2})^4 = 4\sqrt{5}$.

c) $\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)^3 = \dfrac{1}{8}(1-\sqrt{3}i)^3 = \dfrac{1}{8}\left(1^3 + 3\cdot 1^2(-\sqrt{3}i) + 3\cdot 1\cdot(-\sqrt{3}i)^2 + (-\sqrt{3}i)^3\right)$

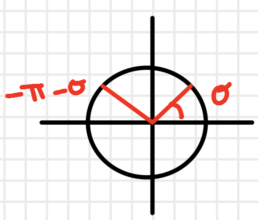
$$= \frac{1}{8}(1 - 3\sqrt{3}i - 9 + 3\sqrt{3}i) = \frac{-8}{8} = -1. \quad \blacksquare$$

3 $z, w \in \mathbb{C}$ tq. $|z| = |w| = 1$ y $z + w = 2$

$\begin{cases} z = \cos\sigma + i\,\text{sen}\,\sigma \\ w = \cos\varphi + i\,\text{sen}\,\varphi \end{cases} \Rightarrow$ Parte real: $\cos\sigma + \cos\varphi = 2$; Parte im.: $\text{sen}\,\sigma + \text{sen}\,\varphi = 0$

$\cos\sigma + \cos\varphi = 2 \Rightarrow \cos\sigma = \cos\varphi = 1 \Rightarrow \begin{cases} \sigma = 2k\pi \\ \varphi = 2k'\pi \end{cases} \Rightarrow z = w = 1.$

Si busco $z + w = 2$: $\quad \begin{cases} \cos\sigma + \cos\varphi = 1 \\ \text{sen}\,\sigma + \text{sen}\,\varphi = 0 \end{cases} \Rightarrow \text{sen}\,\sigma = \text{sen}(-\varphi)$



$\Rightarrow \begin{cases} -\varphi = \sigma - 2\pi \\ -\varphi = -\pi - \sigma \end{cases} \quad (\varphi \in [0, 2\pi))$

$$\Rightarrow \varphi = \begin{cases} 2\pi - \sigma \\ \pi + \sigma \end{cases}.$$

$\cos\sigma + \cos\varphi = 1$. $\quad \begin{cases} \cos(2\pi - \sigma) = \cos\sigma \to \cos\sigma + \cos\varphi = 2\cos\sigma = 1 \\ \cos(\pi + \sigma) = -\cos\sigma \to \cos\sigma + \cos\varphi = 0 \neq 1 \quad \times \end{cases}$

Deducimos $\varphi = 2\pi - \sigma$, es decir $\cos\sigma = \frac{1}{2} \Rightarrow \underset{[0,2\pi)}{\sigma} = \pi/3$, $\varphi = \frac{5}{3}\pi$

$$\begin{cases} z = \cos \pi/3 + i \operatorname{sen} \pi/3 \\ w = \cos 5\pi/3 + i \operatorname{sen} 5\pi/3 \end{cases}$$ ∎

(4) $z = \dfrac{ai+b}{ci+d}$ tq. $ad-bc=1$. Dem.: • $\operatorname{Im}(z) = \dfrac{1}{c^2+d^2}$

$$z = \frac{(ai+b)(-ci+d)}{(ci+d)(-ci+d)} = \frac{adi + bd + ac - bci}{d^2+c^2} = \frac{(bd+ac) + (\overset{\overset{1}{\shortparallel}}{ad-bc})i}{d^2+c^2} \Rightarrow \operatorname{Im} z \overset{\shortparallel}{} \frac{1}{c^2+d^2}$$

• $\dfrac{|z-i|^2}{\operatorname{Im}(z)} = a^2+b^2+c^2+d^2-2.$

$i(z-\bar{z}) = i\,2\operatorname{Im}(z)\,i = -2\operatorname{Im}(z)$

$$|z-i|^2 = (z-i).(\bar{z}+i) = z\bar{z} + \overbrace{iz - i\bar{z}}^{\shortparallel} + 1 = |z|^2 - \frac{2}{c^2+d^2} + 1$$

$$\boxed{\frac{|z-i|^2}{\operatorname{Im}(z)} = |z|^2(c^2+d^2) - 2 + c^2 + d^2}$$

• $|z|^2 = \left(\dfrac{bd+ac}{c^2+d^2}\right)^2 + \left(\dfrac{1}{c^2+d^2}\right)^2 = \dots$

$$\dots = \frac{a^2c^2+b^2d^2+2abcd}{(c^2+d^2)^2} + \frac{(ad-bc)^2}{(c^2+d^2)^2}$$

$$= \frac{a^2c^2+b^2d^2+\cancel{2abcd}+a^2d^2+b^2c^2-\cancel{2abcd}}{(c^2+d^2)^2}$$

$$= \frac{a^2\cancel{(c^2+d^2)}+b^2\cancel{(c^2+d^2)}}{(c^2+d^2)^{\cancel{2}}} = \frac{a^2+b^2}{c^2+d^2}.$$

$$\frac{|z-i|^2}{\operatorname{Im}(z)} = \frac{(a^2+b^2)}{\cancel{(c^2+d^2)}} . \cancel{(c^2+d^2)} - 2 + c^2 + d^2 = a^2+b^2+c^2+d^2-2$$ ∎

(5) a) Dem. que si $n, m \in \mathbb{Z}^{\geq 0}$ son suma de dos cuadrados, entonces $n.m$ también lo es.

$$\begin{cases} n = a^2+b^2 = |a+bi|^2 \\ m = c^2+d^2 = |c+di|^2 \end{cases}, \quad n.m = |a+bi|^2\,|c+di|^2 = |(a+bi)(c+di)|^2$$

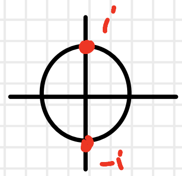
$$= |(ac-bd)+(bc+ad)i|^2$$

$$= (ac-bd)^2 + (bc+ad)^2$$

b) $13 = 2^2+3^2$, $29 = 2^2+5$

$377 = 13.29 = (2.2-3.5)^2 + (3.2+2.5)^2 = 11^2 + 16^2.$

(6) Expresar en forma polar:



a) $1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2}\, e^{i\pi/4}.$

b) $\frac{1}{2} - \frac{\sqrt{3}}{2}i = \underbrace{1}_{\text{mód.}} . \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \left(\cos\left(\frac{-\pi}{3}\right) + \operatorname{sen}\left(\frac{-\pi}{3}\right)i\right) = e^{-\pi/3\, i}.$

c) $-\frac{\sqrt{3}}{2} - \frac{1}{2}i = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right).(-i) = e^{\frac{-\pi}{3}i} . e^{3\pi/2\, i} = e^{7\pi/6}$

d) $-2-2i = -2(1+i) = -2\sqrt{2}\, e^{i\pi/4}.$

(7) a) $e^{\pi i/2} = i$; $e^{\pi/3\, i} = \cos \pi/3 + i \operatorname{sen} \pi/3 = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$

b) $e^{-\pi/4\, i} = \cos(-\pi/4) + i \operatorname{sen}(-\pi/4) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$

c) $e^{2019\pi i} = e^{1009.2.\pi i} . e^{\pi i} = (e^{2\pi i})^{1009} . e^{\pi i} = -1.$

Recordar: $e^{2k\pi i} = 1$, $2019 = 1009.2+1$
$k \in \mathbb{Z}$

d) $e^{3^{2020}\frac{\pi}{2}i} = e^{(4k+1)\frac{\pi}{2}i} = \overbrace{e^{2\pi k i}}^{1} . e^{\pi/2\, i} = e^{\pi/2\, i} = i.$

$\rightarrow\ 3^{2020} \equiv (-1)^{2020} \equiv (1)^{1010} \equiv 1 \mod 4$

(8) a) $(1+i)^8 = (\sqrt{2}\, e^{i\pi/4})^8 = 2^4 e^{2\pi i} = 2^4.$

b) $\left(\cos\frac{\pi}{12} + i \operatorname{sen}\frac{\pi}{12}\right)^{20} \overset{D.M}{=} \cos\frac{20\pi}{12} + i \operatorname{sen}\frac{20\pi}{12} = \cos\frac{5\pi}{3} + i \operatorname{sen}\frac{5}{3}\pi$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}i.$

c) $\left(\frac{1}{1-i}\right)^{2020}+\left(\frac{1}{1+i}\right)^{2020}=\frac{(1+i)^{2020}+(1-i)^{2020}}{\left((1+i)(1-i)\right)^{2020}}=$

$$=\frac{\left(\sqrt{2}\,e^{i\pi/4}\right)^{2020}+\left(\sqrt{2}\,e^{-i\pi/4}\right)^{2020}}{2^{2020}}$$

$$=\frac{\sqrt{2}^{\,2020}}{2^{2020}}\left(e^{\pi i}+e^{-\pi i}\right)=-2\left(\frac{\sqrt{2}}{2}\right)^{2020}$$

$$=-2\left(\frac{1}{\sqrt{2}}\right)^{2020}$$

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9) $\displaystyle\sum_{n=-N}^{N} e^{inx}=\frac{\operatorname{sen}\left(\left(N+\frac{1}{2}\right)x\right)}{\operatorname{sen}\left(x/2\right)}$

Obs.: $e^{ikx}+e^{-ikx}=\cos kx+i\operatorname{sen} kx+\cos kx-i\operatorname{sen} kx=2\cos kx$

$$\sum_{n=-N}^{N} e^{inx}=1+2\sum_{n=1}^{N}\cos kx.$$

$$\sum_{n=-N}^{N} e^{inx}=\sum_{n=0}^{2N} e^{i(-N+n)x}=e^{-Nxi}\frac{\left(e^{i(2N+1)x}-1\right)}{e^{ix}-1}=\frac{e^{i(N+1)x}-e^{-Nxi}}{e^{ix}-1}$$

$$=\frac{\cancel{e^{ix/2}}\left(e^{i(N+\frac{1}{2})x}-e^{-i(N+\frac{1}{2})x}\right)}{\cancel{e^{ix/2}}\left(e^{ix/2}-e^{-ix/2}\right)}$$

$$=\frac{2i\operatorname{sen}(N+\frac{1}{2})x}{2i\operatorname{sen}(x/2)}=\frac{\operatorname{sen}(N+\frac{1}{2})x}{\operatorname{sen}(x/2)}$$

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