Conjuntos y números 28/09

16. $a_1 = 1$, $a_2 = a_3 = 7$, $a_{n+1} = 2a_n + a_{n-1} - 2a_{n-2}$

$n \geq 3$: $a_n = 1 + 2^n + 2(-1)^n$ $\leftarrow$

- Base de inducción: $\left[\begin{array}{l} n=1, \quad \overset{1}{\overset{\shortparallel}{a_1}} = 1 + 2^1 + 2(-1)^1 \\ n=2, \quad a_2 = 1 + 2^2 + 2(-1)^2 \\ \qquad\qquad\qquad\quad \overset{\shortparallel}{7} \\ n=3 \end{array}\right.$

$$\overset{7}{\overset{\shortparallel}{a_3}} = 1 + 2^3 + 2(-1)^3 = 1 + 8 - 2 = 7$$

- H.I. Para un determinado n,

 la fórmula se cumple para $n, n-1, n-2$

- $a_{n+1} = 2a_n + a_{n-1} - 2a_{n-2}$

$$\overset{(H.I)}{=} 2(1 + 2^n + 2(-1)^n) + (1 + 2^{n-1} + 2(-1)^{n-1})$$
$$- 2 \cdot (1 + 2^{n-2} + 2(-1)^{n-2})$$

$(-1)^n = (-1)^{n-2}$

$$= \cancel{2} + 2^{n+1} + 2^2\cancel{(-1)^n} + 1 + \cancel{2^{n-1}} + 2(-1)^{n-1} - \cancel{2} - \cancel{2^{n-1}} - 2^2\cancel{(-1)^{n-2}}$$

$$= 1 + 2^{n+1} + 2(-1)^{n-1} = 1 + 2^{n+1} + 2(-1)^{n+1}$$

$\blacksquare$

$$\boxed{(-1)^{n-1} = (-1)^{n-1} \cdot (-1)^2 = (-1)^{n+1}}$$

(23) • $\{e\} \subset \{e, \{e^2\}\}$, Verd. $\{e^2\} \in \{e, \{e^2\}\}$

$\{e\} \not\in \{e, \{e^2\}\}$ $\qquad$ $\{e^2\} \not\subset \{e, \{e^2\}\}$

$\{\{e^2\}\} \subset \{e, \{e^2\}\}$

• $\emptyset \subset \{e, \{\emptyset\}\}$

Verdadero siempre: $\emptyset \subset C$ para cualquier conjunto C

$\{e^2\} \to$ conj. con 1 elem.: e^2

$\{\{e^2\}\} \to$ conj. con 1 elem.: $\{e^2\}$

• $\emptyset \in \{e, \{\emptyset\}\}$. Falso ($\{\emptyset\} \in \{e, \{\emptyset\}\}$

($\emptyset \subset \{e, \{\emptyset\}\}$ sí es verdad)

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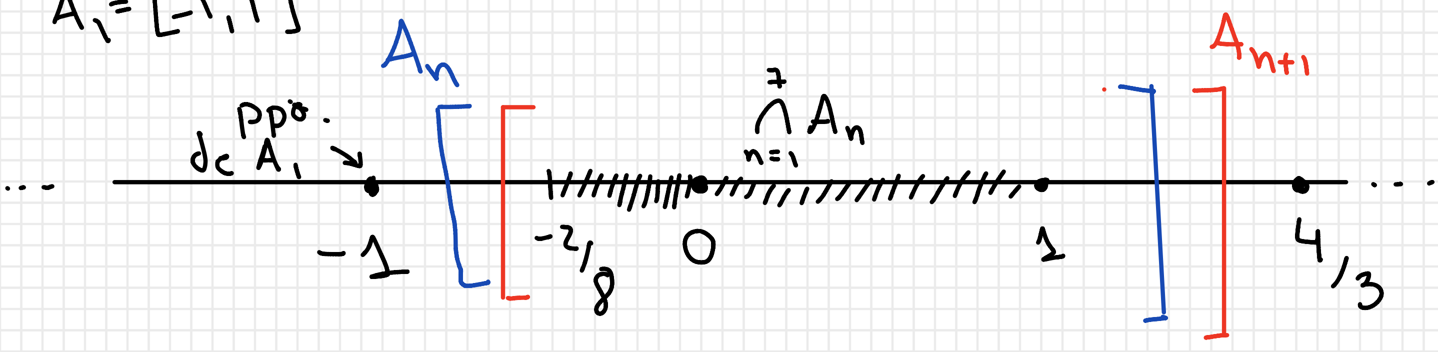
(21) a) $A_n = \left[\dfrac{-2}{n+1}, \dfrac{4n-1}{3n}\right]$ $\quad \displaystyle\bigcap_{n=1}^{7} A_n$?

$A_1 = [-1, 1]$



$$\bigcap_{n=1}^{7} A_n = \left[-\frac{2}{8}, 1\right],$$

$A_1 = [-1, 1]$

$A_2 = [-2/3, 7/6]$

$\vdots$

$A_7 = \left[-2/8, \frac{27}{21}\right]$

(22.) S,T, U,T conj. arbitrarios

a) $(S\setminus T)\cup(T\setminus S) = (S\cup T)\setminus(S\cap T)$

$$x\in (S\setminus T)\cup(T\setminus S) \iff (x\in S \wedge x\notin T)\vee(x\in T\wedge x\notin S)$$

$$\iff (x\in S \wedge \neg(x\in T))\vee(x\in T \wedge \neg(x\in S))$$

$$\iff (x\in S \vee x\in T)\wedge \underbrace{(x\in S \vee \neg(x\in S))}_{V} \wedge \ldots$$

$$\ldots \wedge \underbrace{(\neg(x\in T)\vee x\in T)}_{V} \wedge (\neg(x\in T)\vee\neg(x\in S))$$

$$\iff (x\in S \vee x\in T)\wedge(\neg(x\in T)\vee\neg(x\in S))$$

De Morgan

$$\iff (x\in S\cup T)\wedge(\neg(x\in T\wedge x\in S))$$

$$\iff (x\in S\cup T)\wedge(x\notin T\cap S)$$

$$\iff x\in(S\cup T)\setminus(S\cap T)$$

$$(S\setminus T)\cup(T\setminus S) = (S\cap T^c)\cup(T\cap S^c)$$

$$= (S\cup T)\cap\cancel{(S\cup S^c)}\cap\cancel{(T\cup T^c)}$$

$$\cap\ (T^c\cup S^c)$$

Operando con conjuntos

$$= (S\cup T)\cap(T^c\cup S^c) = (S\cup T)\cap(T\cap S)^c$$

$$= (S\cup T)\setminus(T\cap S).$$

b) $(S \setminus (T \cup U)) \overset{?}{=} (S \setminus T) \cap (S \setminus U)$

$$(S \cap (T \cup U)^c) = S \cap T^c \cap U^c$$

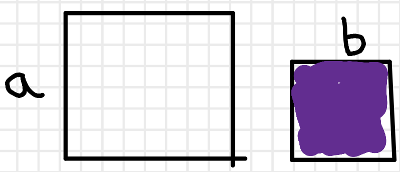
$$= (S \cap T^c) \cap (S \cap U^c)$$

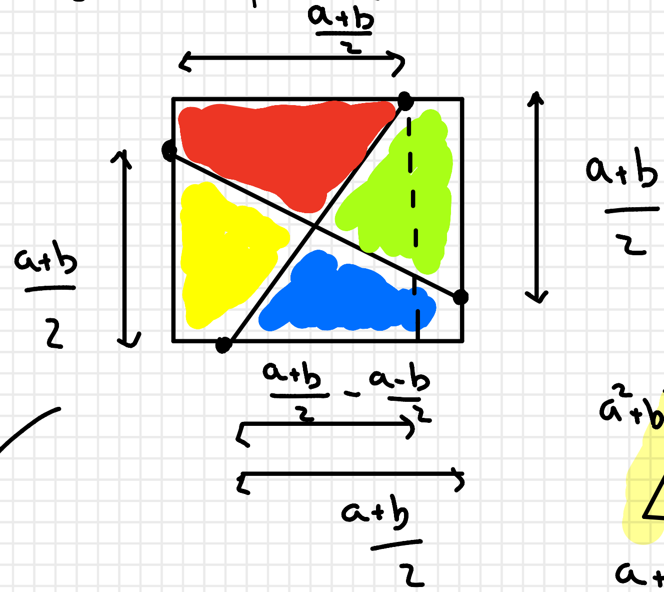
$$= (S \setminus T) \cap (S \setminus U). \quad \blacksquare$$

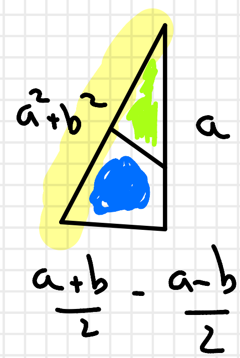
" Demostrar que dados n^2 cuadrados, los podemos cortar en trozos de manera que con esos trozos podamos formar otro cuadrado "

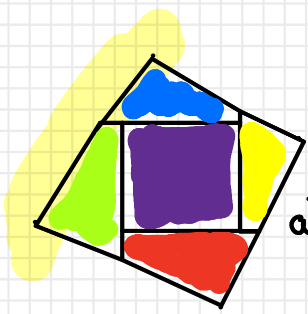
- Base de la inducción

$n = 2$









$$a^2 + \left(\frac{a+b}{2} - \frac{a-b}{2}\right)^2 = a^2 + b^2$$

Lado del nuevo cuadrado = lineas que atraviesan el otro

- $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}$ $(*)$

- Base de inducción:

$n = 1 \rightarrow \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$ ✓

H.I. Para un cierto n se cumple $\circledast$

Para $n+1$:

$$\frac{1}{n+2}+\frac{1}{n+3}+\dots+\frac{1}{2n+1}+\frac{1}{2n+2}$$

$$\|$$

$$\underbrace{\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}}_{\overset{\text{H.I.}}{=}\left(1-\frac{1}{2}+\dots+\frac{1}{2n-1}-\frac{1}{2n}\right)}+\frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1} = \dots$$

$$\dots = \left(1-\frac{1}{2}+\dots+\frac{1}{2n-1}-\frac{1}{2n}\right)+\frac{1}{2n+1}+\underbrace{\frac{1}{2n+2}-\frac{1}{n+1}}$$

$$\|$$

$$\frac{1}{2(n+1)}-\frac{1}{n+1}=\frac{1-2}{2(n+1)}=\frac{-1}{2(n+1)}$$

$$\|$$

$$\frac{-1}{2n+2} \quad \blacksquare$$

19. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right)$ para todo $n \in \mathbb{N}$

· Base de la inducción: $n=1$ $\quad 1 > 2\sqrt{2}-2$ ✓

· H.I. Para un cierto n se cumple la desigualdad

$$\cdot\ \underbrace{1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}}_{\text{Aplico H.I.}}+\frac{1}{\sqrt{n+1}} > 2(\sqrt{n+1}-1)+\frac{1}{\sqrt{n+1}} > \dots$$

(continúa en (γ))

Para cualquier K, $\frac{1}{\sqrt{K}+\sqrt{K}} > \frac{1}{\sqrt{K}+\sqrt{K+1}}$

$$\frac{1}{2\sqrt{K}} > \frac{1}{\sqrt{K}+\sqrt{K+1}} \iff \boxed{\frac{1}{\sqrt{K}} > \frac{2}{\sqrt{K}+\sqrt{K+1}}}$$

(γ)

$$\dots > 2(\sqrt{n+1}-1)+\frac{2}{\sqrt{n+2}+\sqrt{n+1}} = 2(\sqrt{n+1}-1)+\frac{2(\sqrt{n+2}-\sqrt{n+1})}{(n+2)-(n+1)}$$

$$= 2(\sqrt{n+1}-1)+2(\sqrt{n+2}-\sqrt{n+1})$$

$$= 2(\sqrt{n+2}-1) \quad \blacksquare$$

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20. $A \subset B \subset C$

· $A \setminus B$.

$A \setminus B \overset{?}{=} \emptyset$, tenemos que comprobar

$A \setminus B \subseteq \emptyset$, $\qquad A \subset B \Rightarrow B^c \subset A^c$

$x \in A \setminus B \iff x \in A \cap B^c \subset A \cap A^c = \emptyset \Rightarrow x \in \emptyset$, es decir

$A \setminus B \subseteq \emptyset$.

Con lo cual $A \setminus B = \emptyset$.

- $A \cap C \overset{?}{=} A$

Claramente $A \cap C \subseteq A$

Por otra parte, si $x \in A$, como $A \subset C$ tenemos que $x \in C$

$$\left\{\begin{array}{l} x \in A \\ \wedge \\ x \in C \end{array}\right. \Rightarrow x \in A \cap C.$$

Deducimos que $A \cap C = A$

- $A \cup B \overset{?}{=} B$

Claramente $B \subseteq A \cup B$

Por otra parte, si $x \in A \cup B \Leftrightarrow (x \in A) \vee (x \in B)$

si $x \in B$, hemos terminado.

si $x \in A$, como $A \subset B \Rightarrow x \in B$.

Con lo cual $A \cup B \subset B$

$\Rightarrow A \cup B = B$ $\blacksquare$

13 c

$1 \cdot 1! + 2 \cdot 2! + \dots + n\, n! = (n+1)! - 1$

- Base
- H.I

$$1 \cdot 1! + 2 \cdot 2! + \dots + n\, n! + (n+1)(n+1)! \overset{HI}{=} (n+1)! - 1 + (n+1)(n+1)!$$

$$= ((n+2)-1)(n+1)! + (n+1)! - 1$$

$$= (n+2)! - (n+1)! + (n+1)! - 1 = (n+2)! - 1 \quad \blacksquare$$