Complex Numbers / Vectors Problem Set

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Q1. (Marist 2022) Let $z = e^{i\theta}, \theta \in (-\pi, \pi]$.

a. Show that $z^n + z^{-n} = 2\cos(n\theta)$.

b. Hence, or otherwise, determine the values of θ such that,

$$|e^{4i\theta} + 1| = \sqrt{3}.$$

This can be done geometrically, but an algebraic method is probably easier. You are encouraged to try both!

Q2. Consider some $z_1, z_2 \in \mathbb{C}$, where it is given that $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$. In the Argand diagram, the origin O, z_1, z_2 and $z_1 + z_2$ form a shape.

a. Name the shape, explaining the properties of such shape you used to reach this conclusion.

b. Show that the argument of $z_1 + z_2$ is the arithmetic mean of the arguments of our two complex numbers, that is,

$$\arg(z_1 + z_2) = \frac{\arg(z_1) + \arg(z_2)}{2}.$$

Q3. (BoS 2022) Let z be a complex number such that $|z + i|z|| = |z - 2i|z||$. Sketch the locus of z.

Q4. Consider the points $A(3, 5, -1), B(7, -2, 3)$. Denote $\mathcal{L}$ as the line that passes through A and B. Find the shortest distance between the line $\mathcal{L}$ and the origin O.

Q5. Consider the following two lines based upon the parameters $\lambda, \mu \in \mathbb{R}$,

$$\vec{r_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{r_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find the shortest distance between the two lines.

Q6. (Fitzpatrick) Show that the line between the points $(1, -1, 1$ and $(5, 3, 3)$ is perpendicular to the line between the points $(1, -1, 2)$ and $(4, -4, 6)$.

Q7. Consider the following two lines based upon the parameters $\lambda, \mu \in \mathbb{R}$,

$$\vec{r_1} = \begin{pmatrix} 7 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{r_2} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

a. Show that these lines are parallel, and explain why they do not intersect.

b. Hence, find the shortest distance between the two lines.