Complex Numbers Vectors Problem Set I

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Q1. The curve of many loci can be found algebraically using the substitution $z = x + iy$. However, it is essential and oftentimes easier to consider the geometric conditions and properties, as opposed to algebra. For the following questions, consider some fixed $z_1, z_2 \in \mathbb{C}$.

a. It can be shown that the locus represented by $|z - z_1| = |z - z_2|$ is a linear curve.

   i. Explain geometrically why, detailing the position and nature of the curve in relation to the numbers z_1 and z_2.

   ii. Plot the locus represented by $|z + 3 + 2i| = |z - 1 + 4i|$.

b. It can be shown that the locus of the form $\arg(z - z_1) = \theta$, for some $\theta \in (-\theta, \theta]$ is a ray.

   i. Explain geometrically why, making specific note why there is an 'open circle' in our ray.

   ii. Sketch $\arg(z + 3 - 2i) = \frac{\pi}{4}$.

c. It can be shown that the locus of the form $|z - z_1| = r$, for some $r \in \mathbb{R}$, is a circle.

   i. Explain geometrically why, making specific reference to what the values z_1 and r represent.

   ii. Sketch $|z - 3 + 2i| = 3$.

d. It can be shown that the locus represented by $\arg(\frac{z-z_1}{z-z_2}) = \theta$ represents an arc, for some $\theta \in (-\pi, \pi]\backslash\{0, \pi\}$[1].

   i. Explain why, and note when the curve is a minor arc, a major arc, and a semicircle, noting whether the points z_1 and z_2 are open or closed 'circles.'

   ii. Sketch $\arg(\frac{z-3+2i}{z+2+i}) = \frac{\pi}{2}, \arg(\frac{z-2+i}{z+4+3i}) = \frac{\pi}{6}$ and $\arg(\frac{z-1-i}{z+1+i}) = \frac{2\pi}{2}$.

e. Nearly there! We have two special cases for the above; when $\theta \in \{0, \pi\}$.

   i. Explain what the the curve will be when $\theta = 0$. Then, sketch $\arg(\frac{z+2-3i}{z-3+2i}) = 0$.

   ii. Explain what the the curve will be when $\theta = \pi$. Then, sketch $\arg(\frac{z+2-3i}{z-3+2i}) = \pi$.

Q2. Consider $z \in \mathbb{C}$.

a. Plot the set of points that satisfy the equation $|z - 3 + 2i| = 1$.

b. Find the maximum and minimum value of $|z|$.

c. Find the maximum and minimum value of $|z + 4 - 3i|$.

d. Find the maximum and minimum value of $\arg(z)$.

e. Find the maximum and minimum value of $\arg(z + 4 - 3i)$.

Q3. (Cambridge)

a. Consider $z_1, z_2, z_3 \in \mathbb{C}$. Prove that z_1, z_2 and z_3 are collinear if,

$$\frac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R}.$$

[1] Scary notation!!! This is just any principal argument for θ, not including 0 and π (why?).

b. Thus, show that the points represented by the complex numbers $5+8i, 13+20i$ and $19+29i$ are collinear.

Q4. Consider some $z \in \mathbb{C}$ that satisfies the equations, $\arg(\frac{z-3}{z-3i}) = \frac{\pi}{2}$ and $\arg(z) = \frac{\pi}{4}$. Using geometrical methods, find z.

Q5.[2] Let $z = e^{i\theta}$, where $\theta \in (-\pi, \pi]$. Show that,

$$\frac{z^2-1}{2iz} = \sin(\theta).$$

Q6. Suppose $z \in \mathbb{C}$ satisfies the equation,

$$\arg\left(\frac{z-1+i}{z-1-i}\right) = \frac{\pi}{2}.$$

Find the maximum and minimum values of both $|z|$ and $\arg(z)$.

Q7. Suppose $z \in \mathbb{C}$ satisfies the equation,

$$|z-5+4i| \leq 3,$$

Find the possible values $\arg(z)$ can take (leave your answer in exact form).

Q8. (HSC 2011) On an Argand diagram, sketch the region described by the inequality,

$$\left|1+\frac{1}{z}\right| \leq 1.$$

Q9. On the Argand diagram, sketch the region defined by the inequalities,

$$\mathrm{Re}(z) \geq 0, \quad 0 \leq \mathrm{Im}(z) \leq 1, \quad |z| \leq 1, \quad -\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{4}.$$

Ensure you are explicit with the boundaries of the region (open and closed circles!).

Q10. Let $z_1, z_2, z_3 \in \mathbb{C}$ be complex numbers that are equally spaced apart on the unit circle and ordered anticlockwise in the Argand diagram.

a. If the points represented by each complex number are the vertexes of a shape, describe the shape formed by z_1, z_2, z_3.

b. Explain why $z_2 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) z_1$.

c. Hence, show that

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

d. Explain geometrically why the result in part c is holds. You may find drawing a diagram to be helpful.

[2]This is hardly a vectors question but oh well still a good question!