Optiver Prove It 2

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Question 1

Q1

Prove that for a random walk from 0 to n, the expected number of blocks is n^2.

Proof. Denote $m_{i,j}$ to be the expected number of blocks required to go from block i to block j in a symmetrical random walk. Of course, we seek a general expression for $m_{0,n}$. First, we begin with a simple recurrence relationship,

$$m_{0,n} = m_{0,n-1} + m_{n-1,n}, \tag{1}$$

that is, the expected number of blocks to go from 0 to n is simply the sum of the expected number of blocks to go from 0 to $n-1$ and expected number of blocks to go that extra step.

Now, let's consider $m_{i,n}$, for some integer $0 < i < n$, i.e., the expected number of steps to go from house i to house n. Because at any step, we can go either left $(i-1)$ or right $(i+1)$, we have that,

$$m_{i,n} = 1 + \frac{1}{2}m_{i+1,n} + \frac{1}{2}m_{i-1,n}. \tag{2}$$

Rearranging (2) and letting $a_i = m_{i,n}$ we get the recurrence relation

$$a_{i+1} = 2a_i - a_{i-1} - 2, \tag{3}$$

with boundary conditions $a_n = 0$ (the expected number of steps to reach house n from n is 0 because we are already at house n) and $a_0 = m_{0,n}$.

To solve (3), we first find the associated homogenous recurrence relation h_i, which can be found by considering the characteristic equation $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$. Thus, we have that

$$h_i = A(1)^i + Bi(1)^i = A + Bi,$$

for some constants A and B, where we multiplied B by i to maintain independence between the terms.

Now, for the particular solution p_i, we make a guess that it is a constant, and multiply it by i^2 to maintain independence among the terms, that is, we guess $p_i = Ci^2$. Substituting this into (3),

$$\begin{aligned} C(i+1)^2 - 2Ci^2 + C(i-1)^2 &= -2 \\ C(i^2+2i+1) - 2Ci^2 + C(i^2-2i+1) &= -2 \\ 2C &= -2 \\ C &= -1. \end{aligned}$$

So, $a_i = h_i + p_i = A + Bi - i^2$, and we can solve for A and B using the boundary conditions. Letting $i = 0$, we have

$$\begin{aligned} A + B(0) - 0^2 &= m_{0,n} \\ \implies A &= m_{0,n}, \end{aligned}$$

and letting $i = n$ we find

$$\begin{aligned} m_{0,n} + Bn - n^2 &= 0 \\ B &= \frac{n^2 - m_{0,n}}{n}, \end{aligned}$$

and thus,

$$a_i = m_{0,n} + i\frac{n^2 - m_{0,n}}{n} - i^2. \tag{4}$$

Now, recalling that $m_{n-1,n} = a_{n-1}$, we can apply (4) to (1),

$$\begin{aligned} m_{0,n} &= m_{0,n-1} + m_{0,n} + \frac{n-1}{n}(n^2 - m_{0,n}) - (n-1)^2 \\ \frac{n-1}{n} m_{0,n} &= m_{0,n-1} + n(n-1) - (n-1)^2 \\ m_{0,n} &= \frac{n}{n-1}(m_{0,n-1} + n - 1). \end{aligned} \tag{5}$$

With this, we can now prove the claim with induction. First, consider the base case when $n = 1$. Then, the expected number of steps is $m_{0,1} = 1 = 1^2$, as we are forced to go to the right from our starting home, and thus the claim is true for $n = 1$. Now, assume the claim is true for some positive integer n, that is, $m_{0,n} = n^2$. Then, using (5), we have

$$\begin{aligned} m_{0,n+1} &= \frac{n+1}{n}(m_{0,n} + n) \\ &= \frac{n+1}{n}(n^2 + n) \\ &= \frac{n+1}{n} n(n+1) \\ &= (n+1)^2, \end{aligned}$$

and thus the claim is true for $n+1$ given it is true for n. Thus, we must have that the claim is true for all integers n by mathematical induction, that is, the expected number of blocks for a random walk to get from 0 to n is n^2. $\square$

Question 2

Q2

What is the expected number of blocks to go from 0 to n if the coin is biased and is always twice as likely to land on tails?

Proof. First, we seek the probability of moving right given we are not at the best friend's or starting house, which is the probability of tossing a head. Given that the the coin is twice as likely to land on tails than it is heads, and we have that the coin landing on heads or tails occupies the entire sample space, we have

$$\begin{aligned} \mathbb{P}(\text{Heads}) + \mathbb{P}(\text{Tails}) &= 1 \\ \mathbb{P}(\text{Heads}) + 2 \cdot \mathbb{P}(\text{Heads}) &= 1 \\ \mathbb{P}(\text{Heads}) &= \frac{1}{3}. \end{aligned}$$

Thus, the probability of moving right is $1/3$, and the probability of moving left is $2/3$.

To solve this problem, we won't use an induction-style proof, but rather deriving a formula ourselves. Now, keeping the notation that $m_{i,j}$ represents the expected number of steps to go from i to j, we have that,

$$\begin{aligned} m_{i-1,i} &= 1 + \frac{1}{3} m_{i,i} + \frac{2}{3} m_{i-2,i} \\ &= 1 + \frac{1}{3}(0) + \frac{2}{3}(m_{i-2,i-1} + m_{i-1,i}) \\ \Rightarrow m_{i-1,i} &= 3 + 2m_{i-2,i-1}. \end{aligned} \tag{6}$$

We can solve (6) by first considering the associated homogeneous recurrence h_i, and as it is a simple first order recurrence relation, it must be $h_i = A2^i$, for some constant A.

Now, we can again guess the particular solution is a constant C, and substituting this into (6), we have that,

$$\begin{aligned} C &= 3 + 2C \\ C &= -3. \end{aligned}$$

Thus, we have $m_{i-1,i} = A2^{i-1} - 3$, and using the initial condition $m_{0,1} = 1$ again,

$$\begin{aligned} 1 &= A \cdot 2^0 - 3 \\ A &= 4. \end{aligned}$$

Thus, we have that our explicit formula for $m_{i-1,i}$ is

$$m_{i-1,i} = 2^{i+1} - 3. \tag{7}$$

Now, let's use this to get an explicit expression for $m_{0,n}$. Using the same idea in the previous section, we have that,

$$m_{0,n} = m_{0,1} + m_{1,2} + \cdots + m_{n-1,n} = \sum_{i=1}^{n} m_{i-1,i}.$$

Applying (7) to the above,

$$
\begin{aligned}
m_{0,n} &= \sum_{i=1}^{n} (2^{i+1} - 3) \\
&= \sum_{i=1}^{n} 2^{i+1} - 3n \\
&= 2^2 \frac{2^n - 1}{2 - 1} - 3n \\
&= 2^{n+2} - 4 - 3n.
\end{aligned}
$$

Thus, we have our solution - the expected number of blocks to go from 0 to n with the biased coin is $2^{n+2} - 4 - 3n$. $\square$

Simulations for Q2

Of course, we can and should check our solution is correct with a Monte Carlo Simulation. The results of 100,000 simulations for different values of n which were averaged to give an estimate which we shall denote $\hat{m}_{0,n}$ and the derived result for $m_{0,n}$, is summarised in Table 1.

n	$\hat{m}_{0,n}$	$m_{0,n}$
2	5.97618	6
3	19.05520	19
4	48.08596	48
5	109.33002	109
6	234.74280	234
7	487.67430	487
8	993.64778	996
9	2013.68590	2017
10	4061.72022	4062
11	8174.47614	8155

Table 1: Simulation Results

Not bad at all!